Question 1

Part (a)

The given series is: $\frac{4}{1} - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

This is a geometric series with:

- First term, $a = \frac{4}{1} = 4$
- Common ratio, $r = \frac{5}{4}$

To show that the series converges to 1, we can use the formula for the sum of a geometric series: $S = \frac{a}{1-r}$

Plugging in the values, we get: $S = \frac{4}{1 - \frac{5}{4}} = \frac{4}{\frac{-1}{4}} = -4$

Therefore, the series converges to -4.

Part (b)

To find the limit using L'Hopital's rule: $\lim_{n \to \infty} \frac{2n^2}{5n} = \lim_{n \to \infty} \frac{4n}{5} = \frac{4}{5}$

Question 2

Part (a)

To investigate the convergence of the series $\sum_{n=0}^{\infty} \frac{2^n}{n^3 + 5}$ using the Ratio test:

Let $a_n = \frac{2^n}{n^3 + 5}$ Then, $\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^3 + 5}}{\frac{2^n}{n^3 + 5}} = \frac{2}{1 + \frac{5}{n^3} - \frac{3}{n^2} + \frac{1}{n}}$

As $n \to \infty$, the limit of $\frac{a_{n+1}}{a_n}$ is: $\lim_{n \to \infty} \frac{2}{1 + \frac{5}{n^3} - \frac{3}{n^2} + \frac{1}{n}} = 2$

Since the limit is equal to 2, which is greater than 1, the series diverges.

Part (b)

To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ using the Ratio test:

Let $a_n = \frac{1}{n(n+1)}$ Then, $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \frac{n}{n+2}$

As $n \to \infty$, the limit of $\frac{a_{n+1}}{a_n}$ is: $\lim_{n \to \infty} \frac{n}{n+2} = 1$

Since the limit is equal to 1, the series converges by the Ratio test.

Question 3

Part (a)

To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2n}}$ using the Root test:

Let $a_n = \frac{1}{\sqrt{n^2 + 2n}}$ Then, $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{\sqrt{n^2 + 2n}}} = \frac{1}{\sqrt[2n]{n^2 + 2n}}$

As $n \to \infty$, the limit of $\sqrt[n]{a_n}$ is: $\lim_{n \to \infty} \frac{1}{\sqrt[2n]{n^2 + 2n}} = \frac{1}{\sqrt{1}} = 1$

Since the limit is equal to 1, the series converges by the Root test.

Part (b)

To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n n^2}$ using the Root test:

Let $a_n = \frac{1}{2^n n^2}$ Then, $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{2^n n^2}} = \frac{1}{2\sqrt[n]{n^2}}$

As $n \to \infty$, the limit of $\sqrt[n]{a_n}$ is: $\lim_{n \to \infty} \frac{1}{2\sqrt[n]{n^2}} = \frac{1}{2} > 0$

Since the limit is less than 1, the series converges by the Root test.

Question 4

Part (a)

To find the angle between the vectors $\vec{A} = i + j + k$ and $\vec{B} = 2i - 2j + 2k$, we can use the formula: $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$

First, let's calculate the dot product: $\vec{A} \cdot \vec{B} = (1)(2) + (1)(-2) + (1)(2) = 2 - 2 + 2 = 2$

Next, let's calculate the magnitudes of the vectors: $|\vec{A}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ $|\vec{B}| = \sqrt{2^2 + (-2)^2 + 2^2} = \sqrt{12}$

Finally, we can calculate the angle: $\cos \theta = \frac{2}{\sqrt{3}\sqrt{12}} = \frac{1}{\sqrt{3}}$ $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$

Part (b)

To find $\vec{A} \cdot \vec{B}$ and $\vec{B} \cdot \vec{A}$, we can use the dot product formula: $\vec{A} \cdot \vec{B} = (2)(2) + (-3)(3) + (1)(1) = 4 - 9 + 1 = -4$ $\vec{B} \cdot \vec{A} = (2)(-2) + (3)(3) + (1)(2) = -4 + 9 + 2 = 7$

Question 5

Part (a)

To evaluate the double integral $\int_0^6 \int_0^y xdxdy$, we can use the following steps:

1. The region of integration is the triangle bounded by the lines $y = 0$, $y = x$, and $y = 6$.
2. We can use the substitution $u = x$ and $v = y$ to simplify the integral.
3. The limits of integration become:
   - $\int_0^6 \int_0^y xdxdy = \int_0^6 \int_0^v ududv$
4. Evaluating the inner integral:
   - $\int_0^v udud = \left[\frac{u^2}{2}\right]_0^v = \frac{v^2}{2}$
5. Evaluating the outer integral:
   - $\int_0^6 \frac{v^2}{2}dv = \left[\frac{v^3}{6}\right]_0^6 = \frac{6^3}{6} - 0 = 72$

Therefore, the value of the double integral is 72.

Part (b)

To evaluate the triple integral $\int_0^1 \int_0^1 \int_0^{\sqrt{2-x^2-y^2}} dzdydx$, we can use the following steps:

1. The region of integration is the first quadrant of the unit circle in the xy-plane, with the z-coordinate being bounded by the equation $z = \sqrt{2 - x^2 - y^2}$.
2. We can use the substitution $u = x$, $v = y$, and $w = z$ to simplify the integral.
3. The limits of integration become:
   - $\int_0^1 \int_0^1 \int_0^{\sqrt{2-x^2-y^2}} dzdydx = \int_0^1 \int_0^1 \int_0^{\sqrt{2-u^2-v^2}} dwdvdu$
4. Evaluating the inner integral:
   - $\int_0^{\sqrt{2-u^2-v^2}} dwdv = \left[w\right]_0^{\sqrt{2-u^2-v^2}} = \sqrt{2-u^2-v^2}$
5. Evaluating the middle integral:
   - $\int_0^1 \sqrt{2-u^2-v^2}dv = \left[-\frac{2}{3}(2-u^2-v^2)^{3/2}\right]_0^1 = \frac{2}{3}(2-u^2)^{3/2}$
6. Evaluating the outer integral:
   - $\int_0^1 \frac{2}{3}(2-u^2)^{3/2}du = \left[-\frac{2}{9}(2-u^2)^{5/2}\right]_0^1 = \frac{2}{9}(2^{5/2} - 0) = \frac{8\sqrt{2}}{9}$

Therefore, the value of the triple integral is $\frac{8\sqrt{2}}{9}$.

Question 6

To find the area of the region R bounded by the curves $y = x$ and $y = x^2$ in the first quadrant, we can use the following steps:

1. The region of integration is the area between the curves $y = x$ and $y = x^2$ in the first quadrant.
2. We can set up the integral as follows:
   - $\int_{0}^{1} (x^2 - x)dx$
3. Evaluating the integral:
   - $\int_{0}^{1} (x^2 - x)dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{0}^{1} = \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$

Therefore, the area of the region R is $\frac{1}{6}$ square units.

Question 7

To evaluate the limit $\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^2 + y^2}$, we can use the following steps:

1. Observe that the limit is in the form of $\frac{0}{0}$, which is an indeterminate form.
2. We can use L'Hôpital's rule to evaluate the limit:
   - $\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \frac{2xy(x^2 + y^2) - 2x^3y}{2x(x^2 + y^2)}$
3. Simplifying the expression:
   - $\lim_{(x,y) \to (0,0)} \frac{2xy(x^2 + y^2) - 2x^3y}{2x(x^2 + y^2)} = \lim_{(x,y) \to (0,0)} \frac{xy - x^3y}{x(x^2 + y^2)}$
4. Applying L'Hôpital's rule again:
   - $\lim_{(x,y) \to (0,0)} \frac{xy - x^3y}{x(x^2 + y^2)} = \lim_{(x,y) \to (0,0)} \frac{y - 3x^2y}{x^2 + y^2 - 2x^2}$
5. Simplifying the expression further:
   - $\lim_{(x,y) \to (0,0)} \frac{y - 3x^2y}{x^2 + y^2 - 2x^2} = \lim_{(x,y) \to (0,0)} \frac{y(1 - 3x^2)}{x^2 + y^2 - 2x^2} = 0$

Therefore, the limit is 0.