Pattern Recognition (PR)

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Erlangen, January 8, 2021
Prof. Dr.-Ing. Andreas Maier

Duality in Optimization

The Primal Problem

- Consider the *primal optimization problem*:

$$\begin{array}{ll} \text{minimize} & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_i(\boldsymbol{x}) = 0, \quad i = 1, 2, \dots, p \end{array}$$

  with variable $\boldsymbol{x} \in \mathbb{R}^n$.

- The function $f_0(\boldsymbol{x})$ is not required to be convex.

The Lagrangian

Lagrangian

The *Lagrangian L* of the aforementioned problem is defined as

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i h_i(\boldsymbol{x})$$

- λ_i is the Lagrange multipliers associated with the i-th inequality constraint $f_i(\boldsymbol{x}) \leq 0$.
- ν_i is the Lagrange multiplier associated with the i-th equality constraint $h_i(\boldsymbol{x}) = 0$.
- The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are called *Lagrange multiplier vectors* or simply *dual variables*.

Langrange Dual Function

Lagrange dual function

The *Lagrange dual function* is defined as the infimum of the Lagrangian over $\boldsymbol{x}$

$$\begin{aligned} g(\boldsymbol{\lambda}, \nu) &= \inf_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \\ &= \inf_{\boldsymbol{x}} \left(f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i h_i(\boldsymbol{x}) \right) \end{aligned}$$

Note:

- The Lagrange dual function is a pointwise affine function in the dual variables.
- The Lagrange dual function is concave (even if the original problem is not convex).

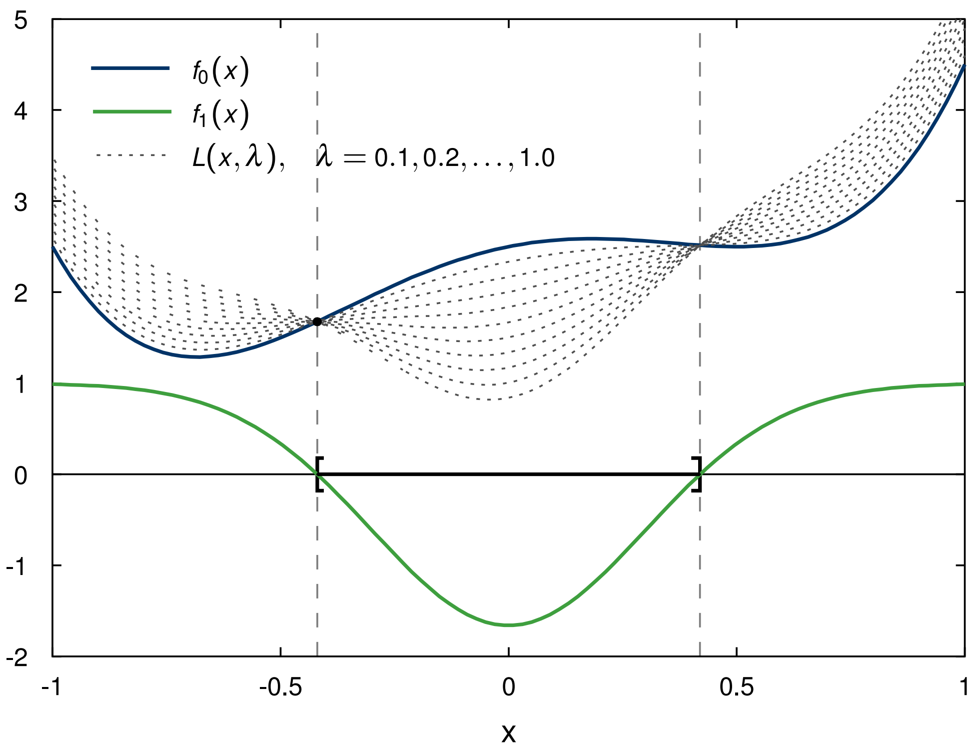
Optimal Value and Lower Bound

Lemma

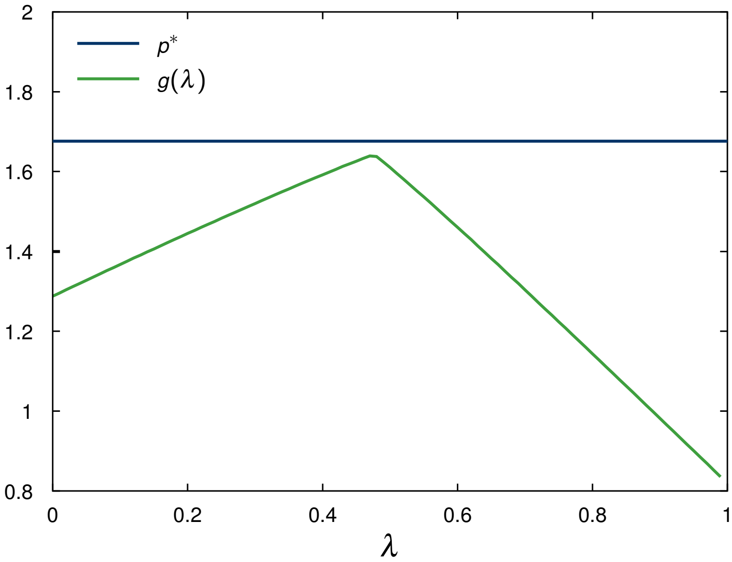
Let p^* be the optimal value of the optimization problem.
For any $\boldsymbol{\lambda} \succeq 0$ and any $\boldsymbol{\nu}$ the following bound is valid:

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq p^*$$

Optimal Value and Lower Bound (cont.)



Optimal Value and Lower Bound (cont.)



- Neither $f_0(x)$ nor $f_1(x)$ is convex,
- but the dual function $g(\lambda)$ is concave!

Optimal Value and Lower Bound (cont.)

Let $\tilde{\boldsymbol{x}}$ be a feasible point of the optimization problem.

If $\boldsymbol{\lambda} \succeq 0$, we have due to the m inequality and p equality constraints:

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{\boldsymbol{x}}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{\boldsymbol{x}}) \leq 0 \ ,$$

Thus we have

$$L(\tilde{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\tilde{\boldsymbol{x}}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{\boldsymbol{x}}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{\boldsymbol{x}}) \leq f_0(\tilde{\boldsymbol{x}}) \ .$$

Optimal Value and Lower Bound (cont.)

Using the definition of the dual function we get:

$$g(\boldsymbol{\lambda}, \nu) = \inf_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \leq L(\tilde{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \leq f_0(\tilde{\boldsymbol{x}})$$

The inequality $g(\boldsymbol{\lambda}, \nu) \leq f_0(\tilde{\boldsymbol{x}})$ holds for every feasible point $\tilde{\boldsymbol{x}}$.

Consequently, the dual function $g(\boldsymbol{\lambda}, \nu)$ is also smaller or equal to the optimal value p^*:

$$g(\boldsymbol{\lambda}, \nu) \leq p^*$$

$\square$

The Lagrange Dual Problem

Problem: how to find the best lower bound for the primal problem

The Lagrange dual problem is given by the optimization problem:

$$\begin{array}{ll} \text{maximize} & g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \\ \text{subject to} & \boldsymbol{\lambda} \succeq 0 \end{array}$$

The Lagrange Dual Problem (cont.)

Optimal duality gap

Let p^* be the optimal value of the primal problem and d^* the optimal value of the Lagrange dual problem.

- The difference $p^* - d^*$ is the *optimal duality gap*.
- If $p^* = d^*$, the duality gap is zero.
  In this case we talk about *strong duality*.
- If $p^* > d^*$, we have *weak duality*.

Slater's Condition

Theorem

Given a *convex* primal optimization problem:

$$
\begin{array}{ll}
\text{minimize} & f_0(\boldsymbol{x}) \\
\text{subject to} & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \dots, m \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

with $f_0, f_1, \dots, f_m$ being convex.

If there exists an $\boldsymbol{x} \in \operatorname{relint} \left\{ \mathscr{D} = \cap_{i=0}^{m} \operatorname{dom}(f_i) \right\}$ with

$$
\begin{array}{l}
f_i(\boldsymbol{x}) < 0, \quad i = 1, \dots, m \\
\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

then strong duality holds.

Refinement of Slater's Condition

Theorem

Given a *convex* primal optimization problem.

If the first k constraint functions $f_1, \dots, f_k$ are *affine*, and
if there exists an $\boldsymbol{x} \in \operatorname{relint} \mathscr{D}$ with

$$\begin{aligned} &f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, k && \text{(affine constraints)} \\ &f_i(\boldsymbol{x}) < 0, \quad i = k+1, \dots, m && \text{(convex constraints)} \\ &\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

then strong duality holds.

Note: the refined Slater's condition reduces to feasibility when the constraints are all linear equalities and inequalities, and $\operatorname{dom}(f_0)$ is open.

Karush-Kuhn-Tucker Optimality Conditions

Let $\boldsymbol{x}^*$ be a primal and $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ dual optimal points with zero duality gap.

For the primal optimal point $\boldsymbol{x}^*$, the gradient with respect to $\boldsymbol{x}$ of $L(\boldsymbol{x}, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ is 0:

$$\nabla L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) = \nabla f_0(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\boldsymbol{x}^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(\boldsymbol{x}^*) = 0$$

Karush-Kuhn-Tucker Optimality Conditions

The following four conditions are called KKT conditions:

1. Primal constraints:
   - $f_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \ldots, m$
   - $h_i(\boldsymbol{x}) = 0, \quad i = 1, 2, \ldots, p$
2. Dual constraints: $\boldsymbol{\lambda} \succeq 0$
3. Complementary slackness: $\lambda_i f_i(\boldsymbol{x}) = 0$
4. Gradient of the Lagrangian L is zero:

$$\nabla L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \nabla f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i \nabla h_i(\boldsymbol{x}) = 0$$

If strong duality holds and if $\boldsymbol{x}^*$ and $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ are optimal points, then the KKT conditions hold.

Karush-Kuhn-Tucker Optimality Conditions

Complementary slackness: $\lambda_i^* \cdot f_i(\boldsymbol{x}^*) = 0$

$$\begin{aligned}
\boxed{f_0(\boldsymbol{x}^*)} &= g(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) \\
&= \inf_{\boldsymbol{x}} \left(f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i^* f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i^* h_i(\boldsymbol{x}) \right) \\
&\overset{=}{\cancel{\leq}} f_0(\boldsymbol{x}^*) + \underbrace{\boxed{\sum_{i=1}^{m} \underbrace{\lambda_i^* f_i(\boldsymbol{x}^*)}_{=0}}}_{\overset{=}{\cancel{\leq}} 0} + \underbrace{\sum_{i=1}^{p} \nu_i^* h_i(\boldsymbol{x}^*)}_{=0} \\
&\overset{=}{\cancel{\leq}} \boxed{f_0(\boldsymbol{x}^*)}
\end{aligned}$$

□

Karush-Kuhn-Tucker Optimality Conditions (cont.)

Conclusions (Boyd 2004, Sec. 5.5.3)

- For *any* optimization problem with differentiable objective and constraint functions for which strong duality obtains, any pair of primal and dual optimal points must satisfy the KKT conditions.
- For any *convex* optimization problem with differentiable objective and and constraint functions, any points that satisfy the KKT conditions are primal and dual optimal, and have zero duality gap.
- If a *convex* optimization problem with differentiable objective and constraint functions satisfies Slater's condition, then the KKT conditions provide necessary and sufficient conditions for optimality.

Lessons Learned

- Formalization of the primal problem using the Lagrangian
- Lagrange dual function
- Duality gap
- Karush-Kuhn-Tucker optimality conditions

Next Time in

Pattern Recognition



Further Readings

- S. Boyd, L. Vandenberghe:
  Convex Optimization,
  Cambridge University Press, 2004.
  ☞ `http://www.stanford.edu/~boyd/cvxbook/`
- Jorge Nocedal, Stephen Wright:
  Numerical Optimization,
  Springer, New York, 1999.

Comprehensive Questions

- What is the Lagrangian of a constrained objective function?
- What is the Lagrange dual function?
- What is the duality gap?
- What are the Karush-Kuhn-Tucker optimality conditions?