Pattern Recognition (PR)

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This is a printable version of the slides of the lecture

Pattern Recognition (PR)
Winter term 2020/21
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Prof. Dr.-Ing. Andreas Maier

Pattern Recognition Basics

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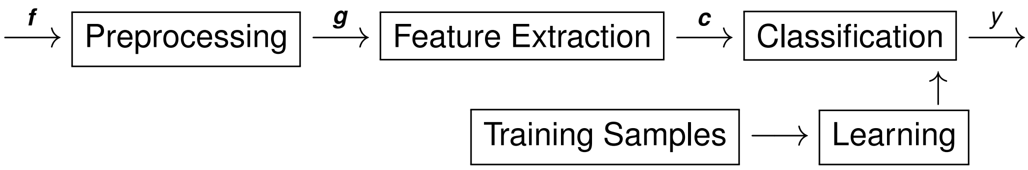
Classification of Simple Patterns

The system for the classification of simple patterns has the following generic structure



Classification of Simple Patterns (cont.)

- *Supervised learning:*
  m training samples include feature and associated class number

$$S = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), (\boldsymbol{x}_3, y_3), \ldots, (\boldsymbol{x}_m, y_m)\}$$

  where $\boldsymbol{x}_i \in \mathscr{X}$ denotes the feature vector and $y_i \in Z$ denotes the class number of sample i. If nothing special is mentioned $\mathscr{X} \subseteq \mathbb{R}^d$.

- *Unsupervised learning:*
  m training samples just include features, no class assignments and even the number of classes is (not always) known

$$S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_m\}$$

Bayesian Classifier

Notation:

$\boldsymbol{x} \in \mathbb{R}^d$: d-dimensional feature vector

y : class number
(usually $y \in \{0, 1\}$ or $y \in \{-1, +1\}$)

$p(y)$: prior probability of pattern class y

$p(\boldsymbol{x})$: evidence
(distribution of features in d-dimensional feature space)

$p(\boldsymbol{x}, y)$: joint probability density function (pdf)

$p(\boldsymbol{x}|y)$: class conditional density

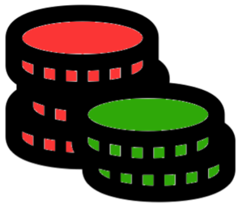
$p(y|\boldsymbol{x})$: posterior probability

Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“} \mid x = \text{"Heads"}) = \frac{17}{18} \approx 0{,}94$$

$$p(y = \text{“Green coin “} \mid x = \text{"Heads"}) = \frac{1}{18} \approx 0{,}06$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

Bayes rule:

$$\underbrace{p(\boldsymbol{x}, y)}_{\textit{joint pdf}} = \underbrace{p(y)}_{\textit{prior}} \cdot \underbrace{p(\boldsymbol{x} \mid y)}_{\textit{class conditional pdf}}$$

$$= \underbrace{p(\boldsymbol{x})}_{\textit{evidence}} \cdot \underbrace{p(y \mid \boldsymbol{x})}_{\textit{posterior}}$$

Bayesian Classifier (cont.)

Now we get the posterior as follows:

$$
\begin{aligned}
p(y|\boldsymbol{x}) &= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})} \\
&= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{\sum_{y'} p(\boldsymbol{x},y')} \\
&= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{\sum_{y'} p(y')\cdot p(\boldsymbol{x}|y')}
\end{aligned}
$$

Bayesian Classifier (cont.)

Note:

$$p(\boldsymbol{x}) = \sum_{y} p(y)\cdot p(\boldsymbol{x}|y)$$

is a marginal of $p(\boldsymbol{x},y)$.

- We get $p(\boldsymbol{x})$ by marginalizing $p(\boldsymbol{x},y)$ over y.
- Accordingly we get $p(y)$ by marginalizing $p(\boldsymbol{x},y)$ over $\boldsymbol{x}$, i. e.

$$p(y) = \int p(\boldsymbol{x},y)\mathrm{d}\boldsymbol{x}$$

Did you notice: y is a discrete random variable whereas $\boldsymbol{x}$ is a continuous random vector (summation vs. integration).

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Next Time in Pattern Recognition

Bayesian Classifier (cont.)

Now let us summarize the Bayesian decision rule:

We decide for the class y^* according to the decision rule

$$
\begin{aligned}
y^* &= \operatorname*{argmax}_y p(y|\boldsymbol{x}) \\
&= \operatorname*{argmax}_y \frac{p(y) \cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})} \\
&= \operatorname*{argmax}_y p(y) \cdot p(\boldsymbol{x}|y) \\
&= \operatorname*{argmax}_y \{\log p(y) + \log p(\boldsymbol{x}|y)\}
\end{aligned}
$$

Bayesian Classifier (cont.)

Notes:

- The key aspect in designing a classifier is to find a good model for the posterior $p(y|\boldsymbol{x})$.
- Feature vectors $\boldsymbol{x}$ usually have fixed dimensions d in simple classification schemes,
- but $\mathscr{X}$ is not necessarily a subset of $\mathbb{R}^d$:
  features of varying dimension, sequences and sets of features

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Bayesian Classifier (cont.)

- Generative modeling:
  modeling and estimation of $p(y)$ and $p(\boldsymbol{x}|y)$.
- Discriminative modeling:
  straight modeling and estimation of $p(y|\boldsymbol{x})$.

Optimality of the Bayesian Classifier

Definition

$l(y_1, y_2)$ is the loss if a feature vector belonging to class y_2 is assigned to class y_1. The $(0, 1)$-loss function is defined by

$$l(y_1, y_2) = \begin{cases} 0 & , \textit{if } y_1 = y_2 \\ 1 & , \textit{otherwise} \end{cases}$$

Optimality of the Bayesian Classifier (cont.)

The best (or optimal) decision rule according to classification loss minimizes the average loss L:

$$\mathrm{AL}(\boldsymbol{x}, y) = \sum_{y'} l(y, y') p(y' | \boldsymbol{x})$$

Optimality of the Bayesian Classifier (cont.)

Using the $(0,1)$-loss function, the class decision is based on:

$$
\begin{aligned}
y^* &= \operatorname*{argmin}_y \mathrm{AL}(\boldsymbol{x}, y) \\
&= \operatorname*{argmin}_y \sum_{y'} l(y, y') \cdot p(y' | \boldsymbol{x}) \\
&= \operatorname*{argmax}_y p(y | \boldsymbol{x})
\end{aligned}
$$

Optimality of the Bayesian Classifier (cont.)

Conclusion:

- The optimal classifier w. r. t. the (0,1)-loss function applies the Bayesian decision rule.
- This classifier is called Bayesian classifier.

⚠ The loss function is **NOT** convex.

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Lessons Learned

- General structure of a classification system
- Supervised and unsupervised learning
- Basics on probabilities (probability, pdf, Bayes rule, etc.)
- Optimality of Bayes classifier and the role of the loss function
- Discriminative and generative approach to model a posteriori probability

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Next Time in

Pattern Recognition



Further Readings

- Heinrich Niemann:
  Pattern Analysis,
  Springer Series in Information Sciences 4, Springer, Berlin, 1982.
- Heinrich Niemann:
  Klassifikation von Mustern,
  Springer Verlag, Berlin, 1983.
- Richard O. Duda, Peter E. Hart, David G. Stork:
  Pattern Classification, 2nd Edition,
  John Wiley & Sons, New York, 2000.