These are the slides of the lecture

Pattern Recognition
Winter term 2020/21
Friedrich-Alexander University of Erlangen-Nuremberg.

Erlangen, January 8, 2021
Prof. Dr.-Ing. Andreas Maier

Pattern Recognition (PR)

Prof. Dr.-Ing. Andreas Maier
Pattern Recognition Lab (CS 5), Friedrich-Alexander-Universität Erlangen-Nürnberg
Winter Term 2020/21

Pattern Recognition Basics

Classification of Simple Patterns

The system for the classification of simple patterns has the following generic structure

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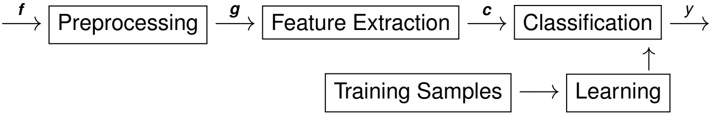

Classification of Simple Patterns (cont.)

- *Supervised learning:*
  m training samples include feature and associated class number

$$S = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), (\boldsymbol{x}_3, y_3), \ldots, (\boldsymbol{x}_m, y_m)\}$$

where $\boldsymbol{x}_i \in \mathscr{X}$ denotes the feature vector and $y_i \in Z$ denotes the class number of sample i. If nothing special is mentioned $\mathscr{X} \subseteq \mathbb{R}^d$.

Classification of Simple Patterns (cont.)

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  where $\boldsymbol{x}_i \in \mathscr{X}$ denotes the feature vector and $y_i \in Z$ denotes the class number of sample i. If nothing special is mentioned $\mathscr{X} \subseteq \mathbb{R}^d$.

- *Unsupervised learning:*
  m training samples just include features, no class assignments and even the number of classes is (not always) known

  $$S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots, \boldsymbol{x}_m\}$$

Bayesian Classifier

Notation:

$\boldsymbol{x} \in \mathbb{R}^d$: d-dimensional feature vector

y : class number
(usually $y \in \{0, 1\}$ or $y \in \{-1, +1\}$)

$p(y)$: prior probability of pattern class y

$p(\boldsymbol{x})$: evidence
(distribution of features in d-dimensional feature space)

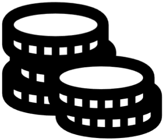
$p(\boldsymbol{x}, y)$: joint probability density function (pdf)

$p(\boldsymbol{x}|y)$: class conditional density

$p(y|\boldsymbol{x})$: posterior probability

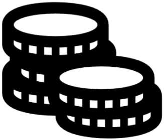
Bayesian Classifier (cont.)

	Heads	Tails
Evidence	18	33



Bayesian Classifier (cont.)

	Heads	Tails	Sum
Evidence	18	33	51



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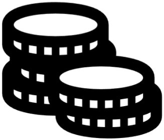

$$p(x = \text{“Heads“}) = \frac{18}{51} \approx 0{,}35$$

Bayesian Classifier (cont.)

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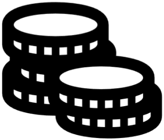

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	Heads	Tails	Sum
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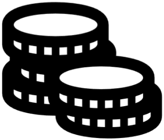
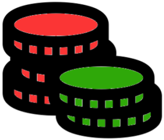


	Heads	Tails
Red coin	17	15
Green coin	1	18



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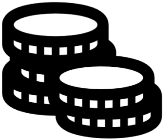

	Heads	Tails	Sum
Red coin	17	15	32
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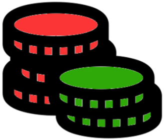

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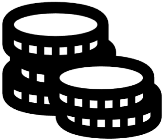

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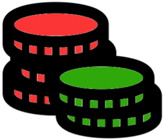

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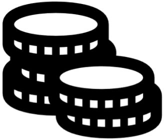

Bayesian Classifier (cont.)

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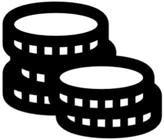
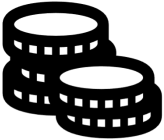


Bayesian Classifier (cont.)

$$p(x = \text{“Heads“}) = \frac{18}{51} \approx 0{,}35$$

$$p(x = \text{“Tails“}) = \frac{35}{51} \approx 0{,}65$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

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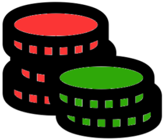
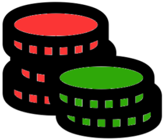


Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“}) = \frac{32}{51} \approx 0{,}63$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51

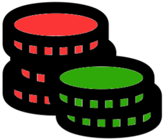


Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“}) = \frac{32}{51} \approx 0{,}63$$

$$p(y = \text{“Green coin“}) = \frac{19}{51} \approx 0{,}37$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

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Red coin	17	15	32
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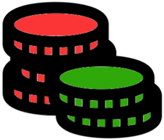
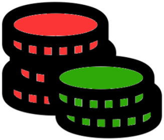


Bayesian Classifier (cont.)

$$p(x = \text{“Heads“}, y = \text{"Red coin"}) = \frac{17}{51} \approx 0{,}33$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

$$p(x = \text{“Heads“}, y = \text{"Red coin"}) = \frac{17}{51} \approx 0{,}33$$

$$p(x = \text{“Tails“}, y = \text{"Green coin"}) = \frac{18}{51} \approx 0{,}35$$

	Heads	Tails	Sum
Red coin	17	15	32
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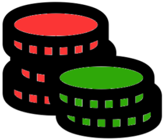

Bayesian Classifier (cont.)

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Red coin	17	15	32
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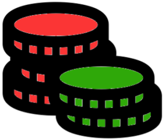
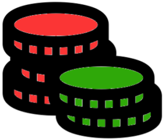


Bayesian Classifier (cont.)

$$p(x = \text{“Heads“} \mid y = \text{"Green coin"}) = \frac{1}{19} \approx 0{,}05$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

$$p(x = \text{“Heads“} \mid y = \text{"Green coin"}) = \frac{1}{19} \approx 0{,}05$$

$$p(x = \text{“Tails“} \mid y = \text{"Green coin"}) = \frac{18}{19} \approx 0{,}95$$

	Heads	Tails	Sum
Red coin	17	15	32
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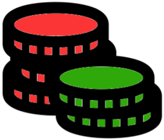

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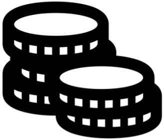
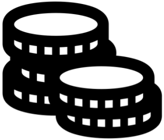


Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“} \mid x = \text{“Tails“}) = \frac{15}{33} \approx 0{,}45$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“} \mid x = \text{“Tails“}) = \frac{15}{33} \approx 0{,}45$$

$$p(y = \text{“Green coin“} \mid x = \text{“Tails“}) = \frac{18}{33} \approx 0{,}55$$

	Heads	Tails	Sum
Red coin	17	15	32
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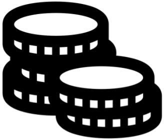

Bayesian Classifier (cont.)

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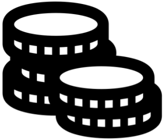
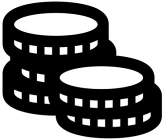


Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“} \mid x = \text{"Heads"}) = \frac{17}{18} \approx 0{,}94$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51

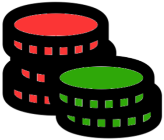


Bayesian Classifier (cont.)

$$p(y = \text{“Red coin“} \mid x = \text{"Heads"}) = \frac{17}{18} \approx 0{,}94$$

$$p(y = \text{“Green coin “} \mid x = \text{"Heads"}) = \frac{1}{18} \approx 0{,}06$$

	Heads	Tails	Sum
Red coin	17	15	32
Green coin	1	18	19
Sum	18	33	51



Bayesian Classifier (cont.)

Bayes rule:

$$\underbrace{p(\boldsymbol{x}, y)}_{\textit{joint pdf}} \quad =$$

Bayesian Classifier (cont.)

Bayes rule:

$$\underbrace{p(\boldsymbol{x}, y)}_{joint\ pdf} = \underbrace{p(y)}_{prior} \cdot \underbrace{p(\boldsymbol{x}|y)}_{class\ conditional\ pdf}$$

Bayesian Classifier (cont.)

Bayes rule:

$$\begin{aligned}\underbrace{p(\boldsymbol{x}, y)}_{joint\ pdf} &= \underbrace{p(y)}_{prior} \cdot \underbrace{p(\boldsymbol{x}|y)}_{class\ conditional\ pdf} \\ &= \underbrace{p(\boldsymbol{x})}_{evidence} \cdot \underbrace{p(y|\boldsymbol{x})}_{posterior}\end{aligned}$$

Bayesian Classifier (cont.)

Now we get the posterior as follows:

$$p(y|\boldsymbol{x}) \quad =$$

Bayesian Classifier (cont.)

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$$p(y|\boldsymbol{x}) = \frac{p(y) \cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})}$$

Bayesian Classifier (cont.)

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$$\begin{aligned} p(y|\boldsymbol{x}) &= \frac{p(y) \cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})} \\ &= \frac{p(y) \cdot p(\boldsymbol{x}|y)}{\sum_{y'} p(\boldsymbol{x}, y')} \end{aligned}$$

Bayesian Classifier (cont.)

Now we get the posterior as follows:

$$
\begin{aligned}
p(y|\boldsymbol{x}) &= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})} \\
&= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{\sum_{y'} p(\boldsymbol{x},y')} \\
&= \frac{p(y)\cdot p(\boldsymbol{x}|y)}{\sum_{y'} p(y')\cdot p(\boldsymbol{x}|y')}
\end{aligned}
$$

Bayesian Classifier (cont.)

Note:

$$p(\boldsymbol{x}) = \sum_y p(y) \cdot p(\boldsymbol{x}|y)$$

is a marginal of $p(\boldsymbol{x}, y)$.

- We get $p(\boldsymbol{x})$ by marginalizing $p(\boldsymbol{x}, y)$ over y.
- Accordingly we get $p(y)$ by marginalizing $p(\boldsymbol{x}, y)$ over $\boldsymbol{x}$, i. e.

$$p(y) = \int p(\boldsymbol{x}, y) \mathrm{d}\boldsymbol{x}$$

Did you notice: y is a discrete random variable whereas $\boldsymbol{x}$ is a continuous random vector (summation vs. integration).

Pattern Recognition Lab

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Next Time in

Pattern Recognition

Bayesian Classifier (cont.)

Now let us summarize the Bayesian decision rule:

We decide for the class y^* according to the decision rule

$$y^* \quad =$$

Bayesian Classifier (cont.)

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$$y^* = \underset{y}{\operatorname{argmax}}\, p(y|\boldsymbol{x})$$

Bayesian Classifier (cont.)

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We decide for the class y^* according to the decision rule

$$
\begin{aligned}
y^* &= \operatorname*{argmax}_y p(y|\boldsymbol{x}) \\
&= \operatorname*{argmax}_y \frac{p(y) \cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})}
\end{aligned}
$$

Bayesian Classifier (cont.)

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&= \operatorname*{argmax}_y p(y) \cdot p(\boldsymbol{x}|y)
\end{aligned}
$$

Bayesian Classifier (cont.)

Now let us summarize the Bayesian decision rule:

We decide for the class y^* according to the decision rule

$$
\begin{aligned}
y^* &= \underset{y}{\operatorname{argmax}}\, p(y|\boldsymbol{x}) \\
&= \underset{y}{\operatorname{argmax}}\, \frac{p(y) \cdot p(\boldsymbol{x}|y)}{p(\boldsymbol{x})} \\
&= \underset{y}{\operatorname{argmax}}\, p(y) \cdot p(\boldsymbol{x}|y) \\
&= \underset{y}{\operatorname{argmax}} \{\log p(y) + \log p(\boldsymbol{x}|y)\}
\end{aligned}
$$

Bayesian Classifier (cont.)

Notes:

- The key aspect in designing a classifier is to find a good model for the posterior $p(y|\boldsymbol{x})$.
- Feature vectors $\boldsymbol{x}$ usually have fixed dimensions d in simple classification schemes,
- but $\mathscr{X}$ is not necessarily a subset of $\mathbb{R}^d$:
  features of varying dimension, sequences and sets of features

Bayesian Classifier (cont.)

- Generative modeling:
  modeling and estimation of $p(y)$ and $p(\boldsymbol{x}|y)$.

- Discriminative modeling:
  straight modeling and estimation of $p(y|\boldsymbol{x})$.

Optimality of the Bayesian Classifier

Definition

$l(y_1, y_2)$ is the loss if a feature vector belonging to class y_2 is assigned to class y_1. The $(0, 1)$-loss function is defined by

$$l(y_1, y_2) = \begin{cases} 0 & , \textit{if } y_1 = y_2 \\ 1 & , \textit{otherwise} \end{cases}$$

Optimality of the Bayesian Classifier (cont.)

The best (or optimal) decision rule according to classification loss minimizes the average loss L:

$$\mathrm{AL}(\boldsymbol{x}, y) = \sum_{y'} l(y, y') p(y' | \boldsymbol{x})$$

Optimality of the Bayesian Classifier (cont.)

Using the $(0,1)$-loss function, the class decision is based on:

$$y^* = \operatorname*{argmin}_y \mathrm{AL}(\boldsymbol{x}, y)$$

Optimality of the Bayesian Classifier (cont.)

Using the $(0, 1)$-loss function, the class decision is based on:

$$
\begin{aligned}
y^* &= \underset{y}{\operatorname{argmin}} \, \mathrm{AL}(\boldsymbol{x}, y) \\
&= \underset{y}{\operatorname{argmin}} \sum_{y'} l(y, y') \cdot p(y' | \boldsymbol{x})
\end{aligned}
$$

Optimality of the Bayesian Classifier (cont.)

Using the $(0,1)$-loss function, the class decision is based on:

$$
\begin{aligned}
y^* &= \operatorname*{argmin}_y \mathrm{AL}(\boldsymbol{x}, y) \\
&= \operatorname*{argmin}_y \sum_{y'} l(y, y') \cdot p(y' | \boldsymbol{x}) \\
&= \operatorname*{argmax}_y p(y | \boldsymbol{x})
\end{aligned}
$$

Optimality of the Bayesian Classifier (cont.)

Conclusion:

- The optimal classifier w. r. t. the (0,1)-loss function applies the Bayesian decision rule.
- This classifier is called Bayesian classifier.

⚠ The loss function is **NOT** convex.

Lessons Learned

- General structure of a classification system

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- Supervised and unsupervised learning

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- Basics on probabilities (probability, pdf, Bayes rule, etc.)

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- General structure of a classification system
- Supervised and unsupervised learning
- Basics on probabilities (probability, pdf, Bayes rule, etc.)
- Optimality of Bayes classifier and the role of the loss function
- Discriminative and generative approach to model a posteriori probability

Pattern Recognition Lab

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Next Time in

Pattern Recognition

Further Readings

- Heinrich Niemann:
  Pattern Analysis,
  Springer Series in Information Sciences 4, Springer, Berlin, 1982.
- Heinrich Niemann:
  Klassifikation von Mustern,
  Springer Verlag, Berlin, 1983.
- Richard O. Duda, Peter E. Hart, David G. Stork:
  Pattern Classification, 2nd Edition,
  John Wiley & Sons, New York, 2000.