ENUMERATION OF SIMPLE, CONNECTED, PLANAR, GRAPHS WITH EACH CORNER HAVING POSITIVE CORNER CURVATURE, AND ENUMERATION OF SIMPLE, CONNECTED, PLANAR GRAPHS WITH EACH EDGE HAVING POSITIVE FORMAN CURVATURE (DRAFT)

Abstract. We enumerate all the 116 2-connected, simple, planar graphs such that every edge has positive Forman curvature, by translating them into 4-regular graphs. We enumerate all the 22 2-connected, simple, planar graphs such that every corner has positive corner curvature.

1. Introduction

Let $\mathcal{G}$ be the set of connected, simple, planar 4-regular graph $G = (V, E, F)$ such that both of face degrees and vertex degrees are at least 3 and finite. For $G = (V, E, F) \in \mathcal{G}$ and an edge $e \in E$, *Forman-Ricci curvature* of e is defined by:

$$Ricc(e) = 16 - (|x_1| + |x_2| + |f_1| + |f_2|) \tag{1}$$

where e is $\{x_1, x_2\}$ and shared by the two faces f_1 and f_2. Here $|x_i|$ is the degree of a vertex x_i, and $|f_i|$ is the facial degree of a face f_i. Forman-Ricci curvature of a graph is always an integer and is defined for the edges. It is unlike combinatorial curvature. We are concerned with discrete analogue of Bonnet-Myers theorem [1] for Forman-Ricci curvature. Let

$$\mathcal{FC}_{>0} := \{G \in \mathcal{G} \mid G \text{ has positive Forman-Ricci curvature everywhere}\}.$$

If G is in $\mathcal{FC}_{>0}$ then the dual G^* is too. On the other hand, the 7-gonal prism $Prism_7$ has positive combinatorial curvature everywhere, but the dual, the 7-gonal bipyramid, has negative curvature at the apexes. We will prove that $\#\mathcal{FC}_{>0} < \infty$, by using combinatorial curvature. Then, we will enumerate them such $\mathcal{FC}_{>0}$.

To relate Forman-Ricci curvature with combinatorial curvature, we translate a $G \in \mathcal{G}$ to a so-called *meridian graph*, which is 4-regular.

Definition 1. *By a* meridian graph *of G, we mean $m(G) = (V', E', F') \in \mathcal{G}$ such that*

1. *$V' = E$,*
2. *$\{e, e'\} \in E' \iff e$ and e' are adjacent edges of a common face, and*
3. *$F' = F \cup V$ where (a) $\{e, e'\} \in E'$ is incident to $f \in F$ in $m(G) \iff e$ and e' are both incident to f in G, (b) $\{e, e'\} \in E'$ is incident to $v \in V$ in $m(G) \iff v$ is incident to e and to e' in G.*

For example, the meridian graph of tetrahedron is the graph of the octahedron, and the meridian graph of octahedron (cube, resp.) is the graph of the cuboctahedron, where the cuboctahedron is an Archimedean solid. An edge $e \in G$ is

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translated to a vertex of $m(G)$, and the data $|x_i|$ and $|f_i|$ to define the Forman-Ricci curvature (1) of an edge $e \in G$ is translated as the environment of a vertex e of $m(G)$.

We will first verify this translation is at most two-to-one.

Lemma 1. *If $G' = (V', E', F') \in \mathcal{G}$ and $F' \neq \emptyset$, there is $G \in \mathcal{G}$ such that $m(G) = G'$. G is unique up to the duality.*

Proof. Let $G' = (V', E', F')$. Take a $f_0 \in F'$.

- Let V be the least subset of F' such that
  - $f_0 \in V$.
  - If $f \in V$, $f' \in F'$, and there is a unique vertex $v \in V'$ such that incident to f and to f' in G', then $f' \in V$.
- Let F be $F' \setminus V$.
- Let E be the set of $\{f, f'\} \subseteq V$ such that $f \neq f'$ and there is a unique vertex $v \in V'$ incident to f and to f'.
- Let $v \in V,\ e = \{f_1, f_2\} \in E,\ f \in F$. Then the incidence relation $\prec_G$ of G is defined as follows:
  - $v \prec_G e :\iff v = f_1$ or $v = f_2$.
  - $v \prec_G f :\iff v$ is adjacent to f in G'.
  - $e \prec_G f :\iff f_i$ is adjacent to f $(i = 1, 2)$.

Because G' is 4-regular and planar, for each $f \in F'$, (1) there is a unique $f' \in F'$ such that (*) there is only one $v' \in V'$ incident to f and to f', and (2) there are two $f' in F'$ such that $(*)$ does not hold. Hence, V and F are nonempty.

We can prove that $m(G) = G'$.

□

Suppose a 4-degree vertex $v \in V$ of G is shared by p_1-gon, p_2-gon, p_3-gon and p_4-gon where $p_1 \leq p_2 \leq p_3 \leq p_4$. Let us call (p_1, p_2, p_3, p_4) the *vertex pattern* of v. Let us define

$$\Phi(v) = 16 - p_1 - p_2 - p_3 - p_4. \tag{2}$$

Let $G = (V, E, F)$ be a connected, simple, planar 4-regular graph such that both of face degrees and vertex degrees are at least 3, and $\Phi(v) > 0$ for every $v \in V$. Suppose $v \in V$ has vertex pattern $(p_1, p_2, p_3, 7)$. Then $p_1 + p_2 + p_3 = 5$ which is impossible because face degrees are at least three. Hence, we have $p_4 \leq 6$.

Lemma 2. *$m(\mathcal{FC}_{>0}) := \{m(G) \mid G \in \mathcal{FC}_{>0}\}$ is the* finite *set of $G' \in \mathcal{G}$ such that any vertex pattern is one of*

(3)
$$(3,3,3,5),\ (3,3,3,4),\ (3,3,3,3),\ (3,3,4,4),\ (3,3,4,5),\ (3,4,4,4),\ (3,3,3,6).$$

Moreover, the number of vertices of G' is greater than or equal to 6 and less than or equal to 24.

Proof. By the definition of Forman-Ricci curvature and that of meridian graph, the vertex types are exactly as (3).

As to the finiteness, for any $G \in \mathcal{FC}_{>0}$, observe that the seven vertex types have combinatorial curvature greater than or equal to 1/12. The total combinatorial curvature is at most two by [1]. □

We enumerate the finite set $m(\mathcal{FC}_{>0})$ by a computer program. The program is a modification of Brinkmann-MacKay's `plantri`. `Plantri` can enumerates all of $\mathcal{G}$ such that the number of vertices is at most 24. By modifying `plantri`, for all the graphs $m(G)$ with the vertex patterns being one of (3), we computed an inverse meridian graph G.

As a result, we found that $m(\mathcal{FC}_{>0})$ consists of 73 $m(G)$'s. By examining an inverse meridian graph G of the 73 $m(G)$'s, we found 9 G's are self-dual. Hence, $\mathcal{FC}_{>0}$ consists of $30 + (73 - 30) \times 2 = 116$ tessellations.

To describe the members of $\mathcal{FC}_{>0}$, we fix notations. ......

References

[1] M. DeVos and B. Mohar. An analogue of the Descartes-Euler formula for infinite graphs and Higuchi's conjecture. *Trans. Amer. Math. Soc.*, 359(7):3287–3300, 2007.

[2] M.-M. Deza and M. Dutour Sikiríc. Geometry of Chemical Graphs: Polycycles and Twofaced Maps. Encyclopedia of Mathematics, Cambridge University Press, 2008.

Appendix A. All 116 2-connected, simple, planar graphs such that any edge has positive Forman curvature

Among the 116, there are one 6-edge self-dual graph, one 8-edge self-dual graph, 3 10-edge self-dual graphs, 6 12-edge self-dual graphs, 4 14-edge self-dual graphs, 6 16-edge self-dual graphs, 4 18-edge self-dual graphs, 3 20-edge self-dual graphs, one 22-edge self-dual graph, and one 24-edge self dual graph.

#E	#	#V	fig	#V	the dual
6	1	4			
8	1	5			
9	1	6		5	
10	1	6			
10	2	6			
10	3	6			
11	1	7		6	
11	2	7		6	

#E	#	#V	fig	#V	the dual
12	1	7			
12	2	7			
12	3	7			
12	4	7			
12	5	7			
12	6	7			
12	7	8			
6					
12	8	7		7	
12	9	7		7	

#E	#	#V	fig	#V	the dual
13	1	7		8	
13	2	7		8	
13	3	7		8	
13	4	8		7	
13	5	8		7	
13	6	7		8	
13	7	8		7	

#E	#	#V	fig	#V	the dual
14	10	8			
14	1	8			
14	2	7		9	
14	3	8		8	
14	4	8			
14	5	8		8	
14	6	8			
14	7	8		8	
14	8	8		8	
14	9	9		7	

#E	#	#V	fig	#V	the dual
15	1	8		9	
15	2	8		9	
15	3	9		8	
15	4	8		9	
15	5	8		9	
15	6	9		8	
15	7	9		8	
15	8	10		7	

#E	#	#V	fig	#V	the dual
16	1	9			
16	2	9		9	
16	3	9			
16	4	9			
16	5	9			
16	6	9			
16	7	9			
16	8	10		8	
16	9	8		10	

Appendix B. All 22 2-connected, simple, planar graphs such that any edge has positive corner curvature

Among the 22, there are one 4-edge self-dual graph, one 8-edge self-dual graph, one 9-edge self-dual graphs, and one 10-edge self dual graph. See Table 1 for Schlegel diagrams or 3d representations of the 22 graphs.

#E	#	#V	fig	#V	the dual
17	1	10		9	
17	2	10		9	
17	3	9		10	
17	4	9		10	

#E	#	#V	fig	#V	the dual
18	1	10			
18	2	10			
18	3	10		10	
18	4	9		11	
18	5	10			
18	6	10			
18	7	9		11	
18	8	10		10	

#E	#	#V	fig	#V	the dual
19	1	11		10	
19	2	10		11	
19	3	10		11	
20	1	11			
20	2	11			
20	3	11			

#E	#	#V	fig	#V	the dual
21	1	12		11	
22	1	12			
24	1	14		12	
24	2	13			

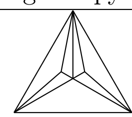
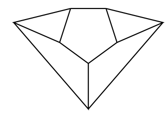
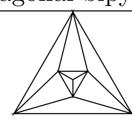
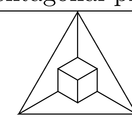
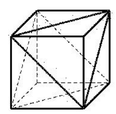
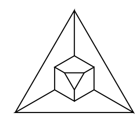
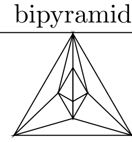
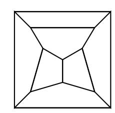
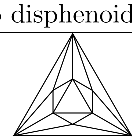
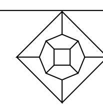
#E	#	#V	graph	#V	the dual	PFC?
4	1	4	Regular tetrahedron	4	Regular tetrahedron	YES
8	1	5	Square pyramid	5	Square pyramid	YES
9	1	5	Triangular bipyramid	5	Triangular bipyramid	YES
10	1	6	Pentagonal pyramid	6	Pentagonal pyramid	YES
12	1	6	 Biaugmented tetrahedron	8		NO
12	1	6	Regular octahedron	8	Cube	YES
15	1	7	Pentagonal bipyramid	10	Pentagonal prism	YES
15	2	7		10		NO
18	1	8	 Gyroelongated triangular bipyramid	12	 Snub $Prism_3$ [2, p.20]	NO
18	2	8	 Snub disphenoid (J_{84})	12		NO
21	1	9	 Triaugmented triangular prism (J_{51})	13	3D-representation	NO
24	1	10	(3d) representation Gyroelongated square bipyramid (J_{17})	16	 Snub $Prism_4$ [2, p.20]	NO
30	1	12	Regular icosahedron	20	Regular dodecahedron	NO

Table 1. All the 22 2-connected, simple, planar graphs such that every corner curvature is positive.