Module II

Higher order DE.

A linear diff^n eq of order n is an eq of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \quad \text{--- (1)}$$

where $a_0, a_1 \dots a_n$ & $g(x)$ are continuous real(s)

& eq (1) $\longrightarrow$ non homogeneous if $g(x) \neq 0$

If $g(x) = 0$ $\longrightarrow$ Homogenous LDE

If each (coe) in eq (1) is a constnt, then eq $\rightarrow$ LDE (linear diff^n eq) with constnt (coe).

eq $\rightarrow$ $3y''' + 5y'' - y' + 7y = 0$

→ Initial value & boundary value prbms =

Theorem = [Existence of unique Sol^n]

$\left.\begin{array}{l} y(0)=1 \\ y(1)=2 \end{array}\right\}$ initial value

Let $a_n(x), a_{n-1}(x) \dots a_1(x), a_0(x)$ & $g(x)$ be continuous on an intrval 'I' & let $a_n(x) \neq 0$ for every x in the intrval, if $x = x_0$ is any point in this intrval, then the sol^n $y(x)$ of initial value prbm,

$$\boxed{\begin{array}{l} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ \text{where } \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \dots \; y^{(n-1)}(x_0) = y_{n+1}. \end{array}}$$

(initial value p. (Starting point))

Exist on intrval I & is unique.

→ Boundary Value Prbm = (BVP)

consider the prbm,

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$y(a) = y_0$ & $y(b) = y_1$ $\longrightarrow$ 2 point boundary v.p.

where $y(a) = y_0$ & $y(b) = y_1$ $\longrightarrow$ Boundary condi~

→ Differential operators = (DO)

Symbol $\frac{dy}{dx} \rightarrow$ (DO)

usually denoted D. & ( ) $y \longrightarrow$ operand.

eg → ① $\frac{d}{dx} = D$.

② $\frac{d}{dx}\sin x = \cos x$.

$\hookrightarrow D(\sin x) = \cos x$.

③ $D(4x^3 + 5x^2) = 12x^2 + 10x$

$\frac{d^2}{dx^2}, \frac{d^3}{dx^3} \dots$ are denoted by $D^2, D^3 \dots$

In general, we define n^{th} order differential operator,

$$L = a_n(x)D^n + a_{n-1}D^{n-1} + \dots + a_1(x)D + a_0(x)$$

* Differtial oprtor 'L' posesses the linearity (pro) (i.e)

$D[f(x) + g(x)] = Df(x) + Dg(x)$. & also,

$D[c\,f(x)] = c \cdot D\,f(x)$.

→ Sol^n of homogeneous LDE =

Theorem = [Super position Principle]

Consider n^{th} order homognous DE

is the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

where $a_0(x), a_1(x), \dots a_n(x)$ are continuous real ( )s on I, $a_n(x) \neq 0$.

let $y_1, y_2 \dots y_k$ be soln of the eq on an interval I. Then the Linear combination

$$\boxed{y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_k y_k(x)}$$

$c_1, c_2, \dots c_k$ are arbitory constnt, is also a soln on I.

⇒ linear dependence & linear independence of soln =

A set of ( )s $f_1(x), f_2(x) \dots f_n(x)$ is said to be linearly dependent on a interval I if there exist constnts $c_1, c_2 \dots c_n$, not all zeroes such that $c_1 f_1(x) + c_2 f_2(x) \dots + c_n f_n(x) = 0$ for every x in the interval I.

If the set of all ( )s is not linearly dependent on the interval, it is said to be linearly independent (i.e)

$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ ⇒

$\Rightarrow c_1 = c_2 = c_3 = \dots c_n = 0$

$x_1 \quad x_2$
$c_1 \quad c_2$
⇓
$c_1 x_1 + c_2 x_2$
linear combination
⇓
$c_1 x_1 + c_2 x_2 = 0$
linear eq

eg

L I

$c_1 x_1 + c_2 x_2 = 0$
if $c_1 = 0$ & $c_2 = 0$
(full 0)

L D

$c_1 x_1 + c_2 x_2 = 0$
if $c_1 = 3$ then c_2 is -3
we gt 0
([illegible])

1) Check whether the ( )s $f_1(x) = 1+x$, $f_2(x) = x$, $f_3(x) = x^2$ are linearly indpnt (L I) on, interval $(-\alpha, \alpha)$

A) 1) Linear eq [illegible]

Let c_1, c_2 & c_3 are constnt consider the linear eq,

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$.
$c_1 (1+x) + c_2 x + c_3 x^2 = 0$
$c_1 + c_1 x + c_2 x + c_3 x^2 = 0$
$c_1 + x(c_1 + c_2) + c_3 x^2 = 0x + 0x^2 + 0$

∴ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

∴ $f_1(x)$, $f_2(x)$ & $f_3(x)$ are L I.

→ wronskian method = (W)

Suppose of each of ( )s $f_1(x), f_2(x) \dots f_n(x)$ possesses atleast n-1 derivatives.

The determinent,

3 ( ) → 2 deri

$$W(f_1, f_2 \dots f_n) = \begin{vmatrix} f_1 & f_2 & \dots f_n \\ f_1' & f_2' & \dots f_n' \\ f_1'' & f_2'' & \dots f_n'' \\ \vdots & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots f_n^{(n-1)} \end{vmatrix}$$

is → W. of the ( ).

Theorem =

let $y_1, y_2 \dots y_n$ be n soln of a homogenous n^{th} order LDE on an interval I then the set of all soln is linearly Independent on I if & only if their wronskin is not identically 0. (i.e)

$LI \iff W \not\equiv 0$ ($\not\equiv$ → not identically)

1) Determine whether the gvn set of f_i's is LD / LI

a) $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$

A)

$$W(f_1,f_2,f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & 4x-3x^2 \\ 1 & 2x & 4-6x \\ 0 & 2 & -6 \end{vmatrix}$$

$$= x(-12x - 8 + 12x) - x^2(-6-0) + (4x - 3x^2)(2-0)$$

$$= -8x + 6x^2 + 8x - 6x^2 = 0$$

$W = 0$, f_1, f_2, f_3 are LD

b) $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = e^x$

$$W(0, x, e^x) = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$$= 0[e^x - 0] - x[0-0] - e^x[0] = 0$$

c) $f_1(x) = 5$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$

LD

def →

Any set $y_1, y_2 \cdots y_n$ of n LI sol^n of the homogenous n^{th} order LDE on an I is said to be fundamental set of sol^n on I.

* Therum =

let $y_1, y_2 \cdots y_n$ be a fundemental set of sol^n of homogenous linear n^{th} order DE, with continous (coe) on I, then the general sol^n of the eq on I is,

$$\boxed{y = c_1y_1 + c_2y_2 + \cdots + c_ny_n}$$

$c_i \to i = 1, 2 \ldots n$ are arbitory constnts.

Imp

1) Varify that the f $y_1 = x^3$, $y_2 = x^4$ form ① a fundamental set of sol^n of DE $x^2y'' - 6xy' + 12y = 0$ on intrval $(0,\infty)$ ② form the general sol^n ?.

A) ① y_1 satisfies eq — ⓪ / not. if $= 0$ → satisfies.

$y_1 = x^3$, $y_1' = 3x^2$, $y_1'' = 6x$

$\Rightarrow x^2y'' - 6xy_1' + 12y_1$

$\Rightarrow x^2\,6x - 6x\,3x^2 + 12x^3$

$\Rightarrow 6x^3 - 18x^3 + 12x^3$

$\Rightarrow -12x^3 + 12x^3 = 0$

② $y_2 = x^4$, $y_2' = 4x^3$, $y_2'' = 4 \cdot 3x^2 = 12x^2$

$\Rightarrow x^2y_2'' - 6xy_2' + 12y_2 =$

$\Rightarrow x^2 \cdot 12x^2 - 6x \cdot 4x^3 + 12x^4$

$\Rightarrow 12x^4 - 24x^4 + 12x^4$

$-12x^4 + 12x^4 \Rightarrow 0$

∴ Both y_1 & y_2 satisfies gvn DE

∴ they are solⁿ.

③ prove LI.
wronskin of the solⁿ,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$$= 4x^3 \cdot x^3 - 3x^2 \cdot x^4$$

$$= 4x^6 - 3x^6 = x^6 \neq 0$$

$x^6 \to 0$ / not → [illegible]
bcz it is x, if non
is instead of x^6 → can be
LI.

(Since $x \in (0, \infty)$)
belongs to.

∴ y_1 & y_2 are LI.

hence y_1 & y_2 form a f. set of solⁿ
to the gvn. solⁿ.

— ② general solⁿ,

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 x^3 + c_2 x^4 \quad \text{where } c_1 \text{ \& } c_2 \to \text{arbitry constnts}$$

2) Varify that the (?) $y_1 = e^x$, $y_2 = e^{2x}$
$y_3 = e^{3x}$ form a f. set of solⁿ of DE
$y''' - 6y'' + 11y' - 6y = 0$ on I $(-\alpha, \alpha)$ form
general eq?

answer →

⇒ Method of reduction of order = $y = c_1 y_1 + c_2$

For a 2nd order LDE
$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$ — ① → 2 deri.. → 2 solⁿ
3 " → 3 "
The general solⁿ is,
$y = c_1 y_1 + c_2 y_2$, where y_1 & y_2 are solⁿ & both LI. on some I.

Suppose we can find 1 non-zero solⁿ $y_1(x)$ of above homognous LDE, so the method for finding y_2 → m. R. of order

2.A) $y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$, $y_1''' = e^x$.

$$y''' - 6y_1'' + 11y_1' - 6y_1 = e^x - 6e^x + 11e^x - 6e^x.$$
$$= -5e^x + 5e^x = 0$$

$y_2 = e^{2x}$, $y_2' = 2e^{2x}$, $y_2'' = 2 \cdot e^{2x} \cdot 2 = 4e^{2x}$,
$y_2''' = 4 \cdot e^{2x} \cdot 2 = 8e^{2x}$.

$$y_2''' - 6y_2'' + 11y_2' - 6y_2 = 8e^{2x} - 6 \cdot 4e^{2x} + 11 \cdot 2e^{2x} - 6e^{2x}$$
$$= 8e^x - 24e^x + 22e^x - 6e^{2x}$$
$$= -16e^x + 16 = 0$$

$y_3 = 3^{ex}$ ....

$$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x} = 0$$

y_1, y_2 & y_3 satisfies gvn DE. on $(-\alpha, \alpha)$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x[18e^{5x} - 12e^{5x}] - e^{2x}[9e^{4x} - 3e^{4x}] + e^{3x}[4e^{3x} - 2e^{3x}]$$

$$= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0$$

∴ Hence the sol^ns y_1, y_2 & y_3 are L.I.

general soln of givn D.E,

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$