Module - III

The Laplace Transform ($\mathcal{L}$)

Let $f(t)$ be a () of t. defined for all non-ve values of t., then the (L.T) of $f(t)$ denoted by '$\mathcal{L}\, f(t)$' is is defined by

$$\boxed{\mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)\,dt}$$ where $s \to$ parameter (real / complex)

[ans-m ere s' m $\to F(s)$]

$\therefore$
$$\boxed{\begin{aligned}\mathcal{L}(f) &= F(s)\\ f &= \mathcal{L}^{-1}(F(s))\end{aligned}}$$

Note

1) clearly $\mathcal{L}(f)$ is a () of s say $F(s)$ then

✓ $$\boxed{\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt}$$

2) Inverse transform / inverse of $F(s)$ will be denoted by, $\mathcal{L}^{-1}(F(s))$

we shall write

✓ $\boxed{f = \mathcal{L}^{-1}(F)}$

$\Rightarrow$ Transforms of elementry ()s =

1) $\boxed{\mathcal{L}(0) = 0}$ $\qquad \mathcal{L}(f(t)) = \mathcal{L}(0)$ $\therefore f(t) = 0$.

Pf

here $f(t) = 0$

$$\mathcal{L}\, f(t) = \int_0^{\infty} e^{-st} f(t)\,dt. \qquad f(t) = 0.$$

$$= \int_0^{\infty} 0. \quad = 0$$

2) $\boxed{\mathcal{L}(1) = \frac{1}{s}}$, $s > 0$. $\qquad$ $\mathcal{L}(1)$ here $f(t) = 1$

Pf
$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt$$
$$= \int_0^{\infty} e^{-st}\,dt. = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}.$$

$\boxed{\begin{aligned}\infty &= \infty\\ e^{\infty} &= \infty\\ e^{-\infty} &= 0\end{aligned}}$

$$= \frac{e^{-s\infty}}{-s} - \frac{e^{0}}{-s} = 0 - \frac{1}{s} = \frac{1}{s}$$

3) $\boxed{\mathcal{L}(t) = \frac{1}{s^2}}$, $s > 0$.

Pf
$f(t) = t$
$$\mathcal{L}(t) = \int_0^{\infty} e^{-st} f(t)\,dt$$
$$= \int_0^{\infty} e^{-st} \cdot t\,dt \qquad (\textstyle\int \text{ by parts})$$

(for easy)
$$= \int_0^{\infty} t \cdot e^{-st}\,dt$$
$$= t \cdot \int e^{-st} - \int \left[\frac{d}{dt}(t) \cdot \int e^{-st}\right] dt.$$
$$= \left[t \times \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1 \times \frac{e^{-st}}{-s}\,dt.$$
$$= 0 + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt$$

$\frac{e^{-st}}{e^{st}}) \rightarrow$
$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}\left[\frac{1}{-se^{\infty}} - \frac{1}{-se^{0}}\right]$$
$$= \frac{1}{s}\left[\frac{1}{s}\right] = \frac{1}{s^2}$$

4) $\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$ $\qquad n = 1, 2, 3.., s > 0$

Pf
$f(t) = t^n$.
$$\mathcal{L}(t^n) = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} e^{-st} \cdot t^n\,dt.$$
$$= \int_0^{\infty} t^n \cdot e^{-st}\,dt. \qquad (\textstyle\int \text{ by parts})$$

$$= \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \cdot \frac{e^{-st}}{-s}\, dt.$$

$$= 0 + \frac{n}{s}\underbrace{\int_0^\infty t^{n-1} e^{-st}\, dt}_{\text{difficult.}}$$

$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt$

$\therefore \mathcal{L}(t^{n-1}) = \int_0^\infty t^{n-1} e^{-st} dt$

$$\boxed{\mathcal{L}(t^n) = \frac{n}{s}\mathcal{L}(t^{n-1})}$$

$$\mathcal{L}(t^{n-1}) = \frac{n-1}{s}\mathcal{L}(t^{n-2})$$

$$\mathcal{L}(t^{n-2}) = \frac{n-2}{s}\mathcal{L}(t^{n-3})$$

$\vdots$

$$\mathcal{L}(t^3) = \frac{3}{s}\mathcal{L}(t^2)$$

$$\mathcal{L}(t^2) = \frac{2}{s}\mathcal{L}(t)$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$\therefore$ we gt $\mathcal{L}(t^n) = \frac{n}{s} \times \frac{(n-1)}{s} \times \frac{(n-2)}{s} \times \cdots \frac{3}{s} \times \frac{2}{s} \times \frac{1}{s^2}$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. \quad \left(\text{eg} \to \mathcal{L}(t^5) = \frac{5!}{s^6}\right)$$

*) $\boxed{\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}}$

5) $\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$ $\quad s > a.$ $\quad$ eg $\to \mathcal{L}(e^{3t}) = \frac{1}{s-3}$

Pf/ $f(t) = e^{at}$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} \cdot e^{at}\, dt.$$

$$= \int_0^\infty e^{-st+at}\, dt = \int_0^\infty e^{at-st}$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

for $\frac{e^{(s-a)t}}{e^{(s-a)t}}$ to gt $\infty / 0$.

$$= \frac{1}{a-s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = \frac{1}{a-s}\left[\frac{1}{\infty} - 1\right]$$

$$= \frac{1}{a-s}[0-1] = \frac{1}{a-s}[-1].$$

$$= -\frac{1}{a-s} = \frac{1}{-a+s}$$

$$= \frac{1}{s-a}$$

6) $\boxed{\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}}$

7) $\boxed{\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}}$

Remark

$$\mathcal{L}(e^{iat}) = \mathcal{L}(\cos at) + i\,\mathcal{L}(\sin at)$$

8) $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$

9) $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$

$L(0) = 0$	$L(e^{at}) = \frac{1}{s-a}$
$L(1) = \frac{1}{s}$	$L(t^n) = \frac{n!}{s^{n+1}}$
$L(t) = \frac{1}{s^2}$	
$L(\sin at) = \frac{a}{s^2+a^2}$	$L(\sinh at) = \frac{a}{s^2-a^2}$
$L(\cos at) = \frac{s}{s^2+a^2}$	$L(\cosh at) = \frac{s}{s^2-a^2}$

→ **properties of Laplace transform =**

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

Therom - 1 [linearity property]

$$\boxed{\mathcal{L}\left(a f(t) + b g(t)\right) = a\,\mathcal{L}(f(t)) + b\,\mathcal{L}(g(t))}$$

Therom - 2 [1^{st} Shifting property]

If $\mathcal{L}(f(t)) = F(s)$ then,

$$\boxed{\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)}$$

$\mathcal{L}(g(t)) = G(s) \Rightarrow \mathcal{L}\left(e^{at} g(t)\right) = G(s-a)$

Rmrk

$$\left[\mathcal{L}(1+5t) = \mathcal{L}(1) + 5\,\mathcal{L}(t) = \frac{1}{s} + 5\,\frac{1}{s^2}\right]$$

$s = s-a$

* $\mathcal{L}(e^{at} t^n) = \dfrac{n!}{(s-a)^{n+1}}$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

* $\mathcal{L}(e^{at} \sin bt) = \dfrac{b}{(s-a)^2+b^2}$

$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

$(s = s-a)$

* $\mathcal{L}(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$

* $\mathcal{L}(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$

* $\mathcal{L}(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$

* $\mathcal{L}(e^{bt} \cos at) = \dfrac{s-b}{(s-b)^2+a^2}$

A ( ) F is said to be piecewise continous on an interval $\alpha \le t \le \beta$, if the interval can be partitioned by a finite ~~pe~~ no. of points so that,
$\alpha = t_0 < t_1 < t_2$, ~~f is con~~

1) f is continous on each (o.I) $t_{i-1} < t < t_i$
2) f approaches a finite limit as the end point of each sub-interval are approached from within the interval.

* Theorem [Existence theorem for L. transform]

let $f(t)$ be a ( ) that is piecewise continous on every finite interval in the range $t \ge 0$ & satisfies $|f(t)| \le Me^{kt}$ $\forall\, t \ge k$, for some real constants M, k, K, then the L. transform,

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt \quad \text{exists } \forall\, s > k.$$

1) find L. Transform,

a) $at + b$

A) $\mathcal{L}(at+b) = a\mathcal{L}(t) + b\mathcal{L}(1)$

$= \dfrac{a}{s^2} + \dfrac{b}{s}$

$\mathcal{L}(t) = \dfrac{1}{s^2}$

$\mathcal{L}(1) = \dfrac{1}{s}$

b) te^{4t}

$\mathcal{L}(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}}$

$\therefore \mathcal{L}(t\, e^{4t}) = \dfrac{1!}{(s-4)^2}$

c) $4t^2 - 5\sin 3t$

A) $\mathcal{L}(4t^2 - 5\sin 3t) = 4\mathcal{L}(t^2) - 5\,\mathcal{L}(\sin 3t)$

$= 4\,\frac{2!}{s^3} - 5\,\frac{3}{s^2+3^2}$

$(2! = 2)$

$= \frac{8}{s^3} - \frac{15}{s^2+9}$

d) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

A) $\mathcal{L}(e^{2t}) + 4\mathcal{L}(t^3) - 2\,\mathcal{L}(\sin 3t) + 3\,\mathcal{L}(\cos 3t)$

$= \frac{1}{s-2} + 4\cdot\frac{3!}{s^4} - 2\cdot\frac{3}{s^2+9} + 3\cdot\frac{s}{s^2+9}$

$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$

e) $(t+1)^2$

A) $\mathcal{L}((t+1)^2) = \mathcal{L}[t^2+2t+1]$

$= \mathcal{L}(t^2) + 2\,\mathcal{L}(t) + \mathcal{L}(1)$

$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

f) $(t+1)^2 e^t$

A) $\mathcal{L}((t+1)^2 e^t) =$ [illegible]

$\frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}[(t^2+2t+1)\,e^t] = \mathcal{L}[e^t t^2 + e^t 2t + e^t]$

$= \frac{2!}{(s-1)^{2+1}} + 2\times\frac{1!}{(s-1)^{1+1}} + \frac{1}{s-1}$

$\frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$ (Shifting property)

g) $\sinh 2t \cdot \sin 3t$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\therefore \sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$

A) $\sinh 2t \sin 3t$

$= \left[\frac{e^{2t} - e^{-2t}}{2}\right]\sin 3t$

$= \frac{1}{2}\left[e^{2t}\sin 3t - e^{-2t}\sin 3t\right]$

$\therefore \mathcal{L}(\sinh 2t \sin 3t) = \frac{1}{2}\left[\mathcal{L}(e^{2t}\sin 3t - e^{-2t}\sin 3t)\right]$

$= \frac{1}{2}\left[\mathcal{L}(e^{2t}\sin 3t) - \mathcal{L}(e^{-2t}\sin 3t)\right]$

$\boxed{\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}}$

$= \frac{1}{2}\left[\frac{3}{(s-2)^2+3^2} - \frac{3}{(s+2)^2+3^2}\right]$

$= \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$

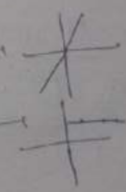
2) Sketch the graph of (?) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

find L-transform?

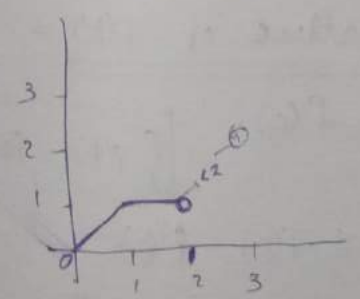
A) $f(t)$ — [illegible] 3 [illegible] → piecewise smooth curve.

$f(x) = x$ →

$f(x) = 1$ →



$1 \le t < 2$



[illegible] → piecewise conti~

clearly $f(t)$ is piecewise conti~ () except 2.

∴ L.T for f(t), (most नोट्स formula (bcz interval → pattern)

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}\times 1\,dt + 0$$

$\int$ by parts

$$= \int_0^1 t\,e^{-st}\,dt + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \left[t\cdot\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1\left[1\cdot\frac{e^{-st}}{-s}\right]dt + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{-e^{-s}}{s} - \frac{1}{s^2}\left(\frac{e^{-s}}{s} - \frac{e^{0}}{s}\right) + \left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$$

Theorm =

I. L. Transform of derivative of f(t) =

$$\mathcal{L}(f'(t)) = s\cdot\mathcal{L}(f) - f(0).$$ $\boxed{\int f'(t) = f(t)}$

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f) - s\cdot f(0) - f'(0)$$

$$\mathcal{L}(f'''(t)) = s^3\mathcal{L}(f) - s^2 f(0) - s f'(0) - f''(0)$$

II. L.T of the $\int$ of f(t) =

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{1}{s}F(s)$$

Note

$$\boxed{\begin{aligned}\mathcal{L}(t f(t)) &= -F'(s)\\ \mathcal{L}(t^n f(t)) &= (-1)^n F'(s)\\ \mathcal{L}\left(\frac{f(t)}{t}\right) &= \int_s^\infty F(u)\,du.\end{aligned}}$$ } for prbms ☆

Q) find L.T of t cos ωt

$\mathcal{L}(f(t)) = \mathcal{L}(t(\cos\omega t))$ ∴ f(t)

A) [illegible] eq-n [illegible], so use $\mathcal{L}(t f(t)) = -F'(s)$

Now, $f(t) = \cos\omega t$

$$\mathcal{L}(t f(t)) = -F'(s) \quad \text{—(1)}$$

$F'(s)$?

we know, $\mathcal{L}(f(t)) = F(s)$.

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+a^2} = \frac{s}{s^2+\omega^2} = F(s)$$

$$F'(s) = \frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) \quad \text{(Q-rule)}$$

$$= \frac{(s^2+\omega^2)\cdot 1 - s(2s+0)}{(s^2+\omega^2)^2}$$

$$= \frac{s^2+\omega^2 - 2s^2}{(s^2+\omega^2)^2}$$

$$= \frac{-s^2+\omega^2}{(s^2+\omega^2)^2} = \frac{\omega^2-s^2}{(s^2+\omega^2)^2} \quad \text{—(2)}$$

(2) in (1)

$$\mathcal{L}(t f(t)) = -F'(s)$$

$$= -\left[\frac{\omega^2-s^2}{(s^2+\omega^2)^2}\right] = \frac{-\omega^2+s^2}{(s^2+\omega^2)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

2) find $\mathcal{L}(t^2 \sin\omega t)$

A) by $\mathcal{L}(t^n f(t)) = (-1)^n F'(s)$. $n = 2$. $f(t) = \sin\omega t$

$\mathcal{L} f(t) = F(s)$

$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$

$F'(s) = \frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$ (Q-rule)

$= \frac{(s^2+\omega^2)\times 0 - \omega\,(2s+0)}{(s^2+\omega^2)^2}$

$= \frac{-2\omega s}{(s^2+\omega^2)^2}$

$\mathcal{L}(t^2 \sin\omega t) = (-1)^n F'(s)$

$= (-1)^2 \cdot \frac{-2\omega s}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2}$

3) find $\mathcal{L}\left(\frac{1-e^t}{t}\right)$

A) by $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(u)\,du$, $f(t) = 1-e^t$

F(u) $\mathcal{L}(f(t)) = F(s) = \mathcal{L}(1-e^t) = \mathcal{L}(1) - \mathcal{L}(e^t)$

$\Rightarrow \frac{1}{s} - \frac{1}{s-1}$

$\therefore \int_s^{\infty} F(s)\,ds = \int_s^{\infty} \frac{1}{s} - \frac{1}{s-1}\,ds$

$= \left[\ln|s| - \ln|s-1|\right]_s^{\infty}$

direct ∞ assign not defined [illegible]

$= \left[\ln\left(\frac{s}{s-1}\right)\right]_s^{\infty}$ ([illegible]) $\ln a - \ln b = \ln\frac{a}{b}$

$= \left[\ln\left(\frac{s}{s(1-1/s)}\right)\right]_s^{\infty} = \left[\ln\left(\frac{1}{1-1/s}\right)\right]_s^{\infty}$

$= \left[\ln\left(\frac{1}{1-0}\right) - \ln\left(\frac{1}{1-\frac{1}{s}}\right)\right] = \ln 1 - \ln\frac{1}{1-\frac{1}{s}}$

$= 0 - \ln\frac{1}{1-1/s}$

$\Rightarrow$ **Inverse L. Transform** = [[illegible]]

If $\mathcal{L}(f(t)) = F(s)$

$$\boxed{\mathcal{L}^{-1}(F(s)) = f(t).}$$

Rslts $\rightarrow$

1) $\mathcal{L}^{-1}(0) = 0$
2) $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$
3) $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$
(4) $\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
(5) $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
(6) $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
7) $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
(8) $\mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
(9) $\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

$\rightarrow$ **linearity property** =

$\mathcal{L}^{-1}[a F(s) + b G(s)] = a\,\mathcal{L}^{-1}(F(s)) + b\,\mathcal{L}^{-1}(G(s))$

Rslt $\rightarrow$ [1^{st} shifting property] =

If $\mathcal{L}(f(t)) = F(s)$ then

$\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$.

<u>Rslt - 2</u>

If $\mathcal{L}[F(s)] = f(t)$ & $f(0) = 0$ then

— ⓐ $\mathcal{L}^{-1}(s\,F(s)) = f'(t)$

In general

$\mathcal{L}^{-1}(s^n F(s)) = f^{(n)}(t)$. , if $f(0) = f'(0) = f''(0) \ldots = f^{(n-1)}(0) = 0$.

<u>Rslt - 3</u>

— ⓑ $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$.

<u>Rslt - 4</u>.

$\mathcal{L}^{-1}(F'(s)) = -t\,f(t)$

$\mathcal{L}^{-1}(F^{(n)}(s)) = (-1)^n t^n f(t)$.

$\mathcal{L}^{-1}\left[\int_s^\infty F(u)\,du\right] = \frac{f(t)}{t}$

<u>Rslt - 5</u>

Denominator	Partial Fraction
1) $s-a$	$\frac{A}{s-a}$ A?
2) $(s-a)^r$	$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \ldots + \frac{A_r}{(s-a)^r}$. $A_1, A_2, A_3, \ldots A_r$?
3) s^2+as+b	$\frac{As+B}{s^2+as+b}$

1) find Inverse L.F — $s \longrightarrow t$

1ⓐ $\frac{3s+7}{s^2-2s-3}$

A) cleuo → s^2-2s-3, [illegible]

I $s^2-2s-3 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4\times1\times-3}}{2}$

$= \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2}$

$= \frac{2+4}{2} = \frac{6}{2} = 3$, $\frac{2-4}{2} = \frac{-2}{2} = -1$

$\therefore s = 3, -1$

B

$s^2-2s-3 = (s-3)(s+1)$

$\therefore \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)}$

II Now take

$\frac{As+B}{s^2+as+b} \Rightarrow \frac{3s+7}{(s-3)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+1)}$ — ①

A, B ?

$= \frac{A}{(s-3)} + \frac{B}{(s+1)} = \frac{A(s+1) - B(s-3)}{(s-3)(s+1)}$

$\frac{3s+7}{\cancel{(s-3)}\cancel{(s+1)}} = \frac{A(s+1) - B(s-3)}{\cancel{(s-3)}\cancel{(s+1)}}$

ⓐ $3s+7 = A(s+1) - B(s-3)$

(A-ന്റെ value [illegible] $B \to 0$.) $B(s-3)$ if $\boxed{s=3}$ $B \to 0$.

Putting $s=3 \Rightarrow 3\times3+7 = A(3+1)$ — ⓪

$16 = 4A$

$16/4 = A$ $\Rightarrow \underline{4 = A}$

(b) B-கான காணால் $A \to 0$.

$A(s+1) \to$ if $s = 7$, $A \to 0$.

putting $s = -1 \Rightarrow 3 \times -1 + 7 = 0 + B(-1-3)$

$\Rightarrow 4 = -B4$

$1 = -B$

$\therefore B = -1$

Sub in ①.

$$\frac{3s+7}{s^2-2s-3} = \frac{A}{(s-3)} + \frac{B}{(s+1)} = \frac{4}{(s-3)} + \frac{-1}{(s+1)}$$

$$\therefore \frac{3s+7}{s^2-2s-3} = \left(\frac{4}{s-3}\right) + \left(\frac{-1}{s+1}\right)$$

taking inverse L.T

$$\mathcal{L}^{-1}\left[\frac{4}{s-3} - \frac{1}{s+1}\right] = \mathcal{L}^{-1}\left(\frac{4}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= 4\,\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-(-1)}\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-(-1)}\right) = e^{-t}$$

$$= 4e^{3t} - e^{-t}$$

2) $\dfrac{s^2}{(s-2)^3}$

A) Sub $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \frac{s^2}{(s-2)^3} = \frac{(s-2)^2 + 4(s-2) + 4}{(s-2)^3}$$

$$= \frac{\cancel{(s-2)^2}}{(s-2)^{\cancel{3}}} + \frac{4\cancel{(s-2)}}{(s-2)^{\cancel{3}\,2}} + \frac{4}{(s-2)^3}$$

$$= \frac{1}{(s-2)} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

[we want to get in the form of $\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3}$]

$$\therefore \mathcal{L}^{-1}\left(\frac{s^2}{(s-2)^3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + 4\,\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) + 4\,\mathcal{L}^{-1}\left(\frac{1}{(s-2)^3}\right)$$

$$= e^{2t} + 4\,\mathcal{L}^{-1}\left(\frac{1}{(s-2)^{1+1}}\right) + 4\,\mathcal{L}^{-1}\left(\frac{1}{(s-2)^{2+1}}\right)$$

$$= e^{2t} + 4 \cdot e^{2t}\frac{t^1}{1!} + \overset{2}{\cancel{4}} \cdot \frac{e^{2t}t^2}{\cancel{2!}}$$

$$= e^{2t} + 4te^{2t} + 2e^{2t}t^2$$

$$\boxed{\begin{aligned} &\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \\ &\mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}} \\ &e^{at}t^n = \mathcal{L}^{-1}\left(\frac{n!}{(s-a)^{n+1}}\right) \\ &n! \text{ - constant.} \\ &\frac{e^{at}t^n}{n!} = \mathcal{L}^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) \end{aligned}}$$

3) $\dfrac{s^2}{(s^2+\omega^2)^2}$

A) we know, $\mathcal{L}(t\cos\omega t) = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$$\therefore \mathcal{L}^{-1}\left(\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right) = t\cos\omega t \quad \text{—①}$$

we need $s^2-\omega^2$ by change the numerator in Q $\to$ only s^2.

Sub. $s^2 = \frac{1}{2}\left[s^2+\omega^2-\omega^2+s^2\right]$

$$\therefore \frac{s^2}{(s^2+\omega^2)^2} = \frac{1}{2}\left[\frac{s^2+\omega^2-\omega^2+s^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{(s^2+\omega^2)}}{(s^2+\omega^2)^{\cancel{2}}} + \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+\omega^2} + \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{(s^2+\omega^2)^2}\right) = \frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) + \mathcal{L}^{-1}\left(\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right)\right.$$

$\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin\omega t$

$$= \frac{1}{2}\left[\frac{\sin\omega t}{\omega} + t\cos\omega t\right] \quad (\text{1st written})$$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega t}{\omega}$

4) $\frac{4}{s^3-2s^2}$

A) $\frac{4}{s^3-2s^2} = \frac{4}{s^2(s-2)}$

by rslt − 3, $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$.

$\mathcal{L}^{-1}\left(\frac{4}{s-2}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 4e^{2t}$ $\quad F(s) = \frac{4}{s-2}$

are hue,

$$\mathcal{L}^{-1}\left(\frac{4}{s(s-2)}\right) = \int_0^t \mathcal{L}^{-1}(F(s))\,dt \quad \left(f(t) = \mathcal{L}^{-1}(F(s))\right)$$

$$= \int_0^t \mathcal{L}^{-1}\left(\frac{4}{s-2}\right)dt$$

$$= \int_0^t 4e^{2t}\,dt = 4\int_0^t e^{2t}\,dt.$$

$$= \cancel{4}^{2}\,\frac{e^{2t}}{\cancel{2}} = \left[2e^{2t}\right]_0^t.$$

$$= 2\left[e^{2t} - 2e^{0}\right] = 2e^{2t} - 2$$

∴ again applying the above rslt $\quad s^2 = s \cdot s$

$$\mathcal{L}^{-1}\left(\frac{4}{s^2(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\left(\frac{4}{s(s-2)}\right)\right) \quad f(s) = \frac{4}{s(s-2)}$$

$$= \int_0^t f(t)\,dt = \int_0^t \mathcal{L}^{-1}(F(s))\,dt.$$

$$= \int_0^t \mathcal{L}^{-1}\left(\frac{4}{s(s-2)}\right)dt.$$

$$= \int_0^t 2e^{2t} - 2\,dt.$$

$$= 2 \cdot \frac{e^{2t}}{2} - 2t$$

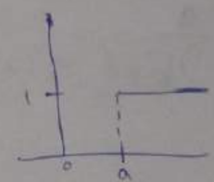
$$= \left[e^{2t} - 2t\right]_0^t.$$

$$= \left[e^{2t} - 2t - (e^{0} - 0)\right]$$

$$= e^{2t} - 2t - 1$$

⇒ **Unit Step ( ) =**

$$U(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$



* L.T of Unit stp ( )

$$\boxed{\begin{aligned} \mathcal{L}\left[U(t-a)\right] &= \frac{e^{-as}}{s} \\ \mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right) &= U(t-a) \end{aligned}}$$

* Consider the ( ), $f(t) = \begin{cases} 2, & 0 \le t < 2 \\ -1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

can be written as $f(t) = 2 - 1 \times 3 \times U(t-2) + U(t-3)$

* In general,

$$f(t) = \begin{cases} g(t); & 0 \le t < a \\ h(t), & t \ge a \end{cases} \Rightarrow f(t) = g(t) - g(t)\cdot U(t-a) + h(t)\cdot U(t-a)$$

* Consider the ( ) $g(t) = \begin{cases} 0, & t < a \\ f(t-a), & t \ge a \end{cases}$

$\Rightarrow g(t) = f(t-a)\, u(t-a)$.

I **Therom** [2nd shifting Theorm] =

If $F(s) = \mathcal{L}(f(t))$ exist for $s > k \geq 0$ & if a is a non -ve constnt then,

$$\boxed{\begin{aligned}\mathcal{L}\left[f(t-a)\cdot u(t-a)\right] &= e^{-as}\,\mathcal{L}(f(t)) \\ &= e^{-as}\,F(s)\end{aligned}}$$

conversly if $f(t) = \mathcal{L}^{-1}(F(s))$ then,

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\, u(t-a)$$

Q

1) find the L.t of $f(t) = \begin{cases} 2, & 0 \leq t < 0 \\ -1, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$ stp ()

A) $f(t)$ can be writtn as unit stp (),

$f(t) = 2 - 3\,u(t-2) + u(t-3)$

$\mathcal{L}(f(t)) = \mathcal{L}(2) - \mathcal{L}(3\,u(t-2)) + \mathcal{L}(u(t-3))$

$= \frac{2}{s} - 3 \times \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$

$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$

2) find inverse L.T of $\frac{e^{-3s}}{(s-1)^4}$ $\rightarrow \frac{e^{-as}}{s}$

A) we know that $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\, u(t-a)$ ①

$F(s) = \frac{1}{(s-1)^4}$ $\quad f(t)$? $\rightarrow \mathcal{L}^{-1}(F(s)) = f(t)$

$a = 3$ $\quad \mathcal{L}^{-1}\left(\frac{1}{(s-1)^4}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^{3+1}}\right)$

$= e^t \cdot \frac{t^3}{3!} = f(t)$ $\quad \mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$ $n=3$

$\therefore f(t) = \frac{e^t t^3}{6}$

$f(t-a) = \frac{e^{(t-a)}(t-a)^3}{6}$

$a = 3$ ~~$f(t-3)$~~

$\mathcal{L}^{-1}\left(e^{-3s} \times \frac{1}{(s-1)^4}\right) = f(t-3)\, u(t-3)$ eq ①

$= \frac{e^{t-3}(t-3)^3}{6} \cdot u(t-3)$

→ **Dirac Delta** () =

let us consider the impulse of a force acting only for an instnt, we define a piecewise ()

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \frac{1}{2a}, & t_0 - a < t < t_0 + a \\ 0, & t_0 + a \leq t \end{cases}$$

hence the () $\delta_a(t-t_0) \rightarrow$ a unit impulse.

unit impulse $\delta(t-t_0) \rightarrow$ Dirac delta () $\left(\int_0^{\infty} \delta_a(t-t_0)\,dt = 1\right)$

* L.T of a Dirac delta () —

$$\mathcal{L}\left(\delta(t-t_0)\right) = e^{-st_0}$$

* D. Delta () in terms of unit step () —

$$\delta_a(t-t_0) = \frac{1}{2a}\left[u(t-(t_0-a)) - u(t-(t_0+a))\right]$$

=> Method of convolution = (using *)

* The convolution of f & g is defined by

$$\boxed{(f*g)(t) = \int_0^t f(u)\, g(t-u)\, du}$$

Pro —

*) $f*g = g*f$

*) $f*(g_1+g_2) = f*g_1 + f*g_2$

*) $f*(g*h) = (f*g)*h$

*) $f*0 = 0*f = 0$

(* → norm convolution
2 term-nd product
norm convolution)

→ Convolution Theorem = (L.T)

If f(t) & g(t) are piecewise continuous & of exponential order then,

$$\boxed{L((f*g)(t)) = L(f(t)) \cdot L(g(t))}$$

equivalently,

if $F(s) = L(f(t))$ &
$G(s) = L(g(t))$ then,

$$\boxed{L^{-1}(F(s)\, G(s)) = (f*g)(t) = \int_0^t f(u)\, g(t-u)\, du}$$

Q
1) Evaluate $L(e^t * \sin t)$

$\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$

A) $L(e^t * \sin t) = L(e^t) \cdot L(\sin t)$

$= \frac{1}{s-1} \cdot \frac{1}{s^2+1}$

2) $L(1*t^3) = L(1) \cdot L(t^3) = \frac{1}{s} \cdot \frac{3!}{s^4}$

3) $L(4t * 3t^2) = 4\,L(t) \cdot 3\,L(t^2)$

$L(t) = \frac{1}{s^2}$

$= \frac{4}{s^2} \cdot 3 \cdot \frac{2!}{s^3}$

$= \frac{4}{s^2} \cdot \frac{6}{s^3} = \frac{24}{s^5}$

4) $L\left(\int_0^t \sin u\, du\right)$

$(f*g)$ same as, so $\int \sin t \times 1$

$= L\left(\int_0^t \sin u \times 1\, du\right)$ — ①

$f*g = \int_0^t f(u)\, g(t-u)\, du$ — ②

compare ① & ②,

$f(u) = \sin u$

$g(t-u) = 1$

$\therefore = L((f*g)(t)) \Rightarrow L(\sin u * 1)$

$\Rightarrow L(\sin u) \cdot L(1)$

$\Rightarrow \frac{1}{s^2+1} \cdot \frac{1}{s}$

5) $L\left(\int_0^t (t-u)\, e^u\, du\right) =$

$L\left(\int_0^t e^u (t-u)\, du\right)$

$\therefore f(u) = e^u$
$g(t-u) = (t-u)$

Problem
correspond
$u \to t$
$f(t) = e^t$
$g(t) = t$

$\therefore L\left[\int_0^t f(u)\, g(t-u)\, du\right] = L((f*g)(t))$

$L((f*g)(t)) = L(e^t * t)$

$= L(e^t) \cdot L(t)$

$= \frac{1}{s-1} \cdot \frac{1}{s^2}$

6) $\mathcal{L}\left(\int_0^t (t-u)^2 \sin u \, du\right)$ $f(u) = \sin u$
$f(t) = \sin t$
$g(t-u) = (t-u)^2$
$g(t) = t^2$

$\therefore \mathcal{L}\left(\int_0^t (t-u)^2 \sin u \, du\right) = \mathcal{L}((f*g)(t))$

$\mathcal{L}((f*g)(t)) = \mathcal{L}(\sin t) \cdot \mathcal{L}(t^2)$

$= \frac{1}{s^2+1} \cdot \frac{2!}{s^3}$

7) $\mathcal{L}\left(\int_0^t \sin(t-u) \cos u \, du\right)$

$f(u) = \cos u$
$f(t) = \cos t$
$g(t-u) = \sin(t-u)$
$g(t) = \sin t$

$\mathcal{L}\left(\int_0^t \sin(t-u) \cos u \, du\right) = \mathcal{L}(f*g)(t)$

$= \mathcal{L}(\cos t) \cdot \mathcal{L}(\sin t)$

$= \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$

$= \frac{s}{(s^2+1)^2}$

3) use the method of convolution find inverse-

a) $\frac{1}{(s^2+1)^2}$

A) $\boxed{\mathcal{L}^{-1}(F(s)\, G(s)) = (f*g)(t) = \int_0^t f(u)\, g(t-u)\, du}$

So we need $\mathcal{L}() \to F(s)\ \&\ G(s)$
convolution

$\frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)} \times \frac{1}{(s^2+1)}$.

$F(s) = \frac{1}{s^2+1}$ $G(s) = \frac{1}{s^2+1}$

so, $g(t)$?

$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$(f*g)(t)$ $\mathcal{L} \to \int_0^t f(u)\, g(t-u)\, du$.

$f(t) = \sin t$
$\sin u \to f(u) = \sin u$

$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = (f*g)(t)$

$= (\sin t * \sin t)(t) = \int_0^t \sin u \cdot \sin(t-u)\, du$.

$= \frac{1}{2}\int_0^t \left[\cos(u-(t-u)) - \cos(u+t-u)\right] du$

$\boxed{\cos(x-y) - \cos(x+y) = 2\sin x \sin y}$

$\boxed{\therefore \frac{\cos(x-y)-\cos(x+y)}{2} = \sin x \sin y}$

$= \frac{1}{2}\int_0^t \left[\cos(u-t+u) - \cos(t)\right] du$.

$= \frac{1}{2}\int_0^t \left[\cos(2u-t) - \cos(t)\right] du$. $\to$ cost $\to$ constant

$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - \cos t \int 1\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - \cos(t)\, u\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin(2t-t)}{2} - t\cos t - \frac{\sin(-t)}{2} - 0\right]$

$= \left[\frac{1}{4}\sin t - \frac{1}{2}t\cos t + \frac{\sin t}{4}\right]$

$= 2 \cdot \frac{1}{4}\sin t - \frac{1}{2}t\cos t$.

$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

b) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

A) $\mathcal{L}^{-1}(F(s)\, G(s)) = (f*g)(t) = \int_0^t f(u)\, g(t-u)\, du$.

$F(s) = \frac{s}{s^2+a^2}$

$G(s) = \frac{s}{s^2+b^2}$

$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)} \times \frac{s}{(s^2+b^2)}$

$\boxed{\frac{s}{(s^2+a^2)(s^2+b^2)}\ \frac{s^2}{s^2+a^2} \cdot \frac{1}{s^2+b^2}}$

$f(t) = \mathcal{L}^{-1}(F(s)) = \cos at$

$g(t) = \mathcal{L}^{-1}(G(s)) = \cos bt$

$$(f*g)(t) = \int_0^t f(u)\, g(t-u)\, du = \int_0^t \cos au \cdot \cos(b(t-u))\, du.$$

$$= \int_0^t \cos \underbrace{au}_{x} \cdot \cos \underbrace{(bt-bu)}_{y}\, du$$

$$\boxed{\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]}$$

$$= \frac{1}{2}\int_0^t \left[\cos(au-(bt-bu)) + \cos(au+bt-bu)\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au-bt+bu) + \cos(au+bt-bu)\right] du.$$

$$= \frac{1}{2}\left[\frac{\sin(au-bt+bu)}{a-0+b} + \frac{\sin(au+bt-bu)}{a+0-b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(au-bt+bu)}{a+b} + \frac{\sin(au+bt-bu)}{a-b}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(at-bt+bt)}{a+b} + \frac{\sin(at+bt-bt)}{a-b}\right) - \left(\frac{\sin(-bt)}{a+b} + \frac{\sin(bt)}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at)}{a+b} + \frac{\sin(at)}{a-b} + \frac{\sin(bt)}{a+b} - \frac{\sin(bt)}{a-b}\right]$$

3) Solve the follow $\int$ eq, $y(t) = t + \int_0^t \underbrace{y(u)\sin(t-u)\,du}_{(f*g)(t)}$.

A) $y(t) = t + (\underbrace{y * \sin t}_{g})(t)$ — ①

taking Laplace T on both side,

$L(y(t)) = L(t) + L(y * \sin t)(t))$

$y(s) = \frac{1}{s^2} + \underbrace{L((y*\sin t)(t))}_{g}$ — ②

$\therefore L((y*\sin t)(t)) = L(y) \times L(\sin t)$ (using conv- therm)

$= L(y(s)) \times \frac{1}{s^2+1}$ — ③

③ in ②

$y(s) = \frac{1}{s^2} + \{y(s)\} \times \frac{1}{s^2+1}$

$y(s) - y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$

$y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2} \Rightarrow y(s)\left(\frac{s^2+1-1}{s^2+1}\right) = \frac{1}{s^2}$

$y(s)\left(\frac{s^2}{s^2+1}\right) = \frac{1}{s^2} \Rightarrow y(s) = \frac{s^2+1}{s^2 \times s^2}$

$\Rightarrow y(s) = \frac{s^2+1}{s^4}$ want $y(t)$ $\therefore$ Inverse

$y(t) = L^{-1}(y(s)) = L^{-1}\left(\frac{s^2+1}{s^4}\right) = L^{-1}\left(\frac{s^2}{s^4} + \frac{1}{s^4}\right)$

$= L^{-1}\left(\frac{1}{s^2} + \frac{1}{s^4}\right)$

$= L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^4}\right)$

$= t + \frac{t^3}{3!}$

$L^{-1}\left(\frac{1}{s^4}\right) \Rightarrow$

$L(t^3) = \frac{3!}{s^{3+1}}$

$L(t^3) = \frac{3!}{s^4}$

$\Rightarrow L\left(\frac{t^3}{3!}\right) = \frac{1}{s^4}$

→ Laplace transform of a periodic () =

If a periodic () f has period T, T > 0 then $f(t+T) = f(t)$.

Therom –

If f(t) is piecewise contin., of exponential order & periodic with period T, then

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\, dt}$$

→ Applications to DE :-

Formulas :

1) $\mathcal{L}(y'(t)) = s\,\mathcal{L}(y) - \cancel{\mathcal{L}(0)}\, y(0)$

2) $\mathcal{L}(y''(t)) = s^2 \mathcal{L}(y) - s\, y(0) - y'(0)$

3) $\mathcal{L}(y'''(t)) = s^3 \mathcal{L}(y) - s^2 y(0) - s\, y'(0) - y''(0)$

Q

1) Solve $y'' + y = t$, $y(0) = 1$, $y'(0) = -2$ using L.T.

A) $y = ?$, So;

taking L.T on both side.

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(t)$ (arala, so that itself)

$\Rightarrow s^2 \mathcal{L}(y) - s\, y(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s^2}$ $\quad \because \begin{cases} y(0) = 1 \\ y'(0) = -2 \end{cases}$

$\Rightarrow s^2 \mathcal{L}(y) - s \times 1 + 2 + \mathcal{L}(y) = \frac{1}{s^2}$

$\Rightarrow s^2 \mathcal{L}(y) - s + 2 + \mathcal{L}(y) = \frac{1}{s^2}$

$\Rightarrow s^2 \mathcal{L}(y) + \mathcal{L}(y) - s + 2 = 1/s^2$

$\Rightarrow \mathcal{L}(y)(s^2+1) - s + 2 = \frac{1}{s^2}$

$\Rightarrow \mathcal{L}(y)(s^2+1) = \frac{1}{s^2} + s - 2$

$\Rightarrow \mathcal{L}(y) = \frac{\frac{1}{s^2} + s - 2}{(s^2+1)} = \frac{\frac{1}{s^2}}{(s^2+1)} + \frac{(s-2)}{s^2+1}$

$\cancel{\frac{1}{s^2+1}\left[\frac{1}{s^2} + s - 2\right]} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$

$\mathcal{L}(y) = \underbrace{\frac{1}{s^2(s^2+1)}}_{\text{[illegible]}} + \underbrace{\frac{s}{s^2+1}}_{\text{[illegible]}} - \underbrace{\frac{2}{s^2+1}}_{\text{[illegible]}}$

$(+s^2 \text{ \& } -s^2)$

$= \frac{(1+s^2) - s^2}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$

$= \frac{(1+s^2)}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$

$\mathcal{L}(y) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$

$y = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$

$= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$

$= t - 3\sin t + \cos t$

2) $y'' - 3y' + 2y = 4e^{2t}$; $y(0) = -3$, $y'(0) = 5$

$\mathcal{L}(y'') - 3\mathcal{L}(y') + 2\mathcal{L}(y) = 4\mathcal{L}(e^{2t})$.

$\cancel{\Rightarrow s^2\mathcal{L}(y) - s\, y(0) - y'(0) - 3\, s\, \mathcal{L}(y) - \mathcal{L}(0) + 2\mathcal{L}(y) = 4\frac{2!}{s^2}}$

$\cancel{s^2\mathcal{L}(y) - s\,y(0) - y'(0)] - 3[s\mathcal{L}(y) - y(0)}$

$s^2\mathcal{L}(y) - s\,y(0)\,y'(0) - 3[s\,\mathcal{L}(y) - y(0)] + 2\mathcal{L}(y) = 4 \cdot \frac{1}{(s-2)}$

$s^2\mathcal{L}(y) - s\,y(0)\,y'(0) - 3[s\,\mathcal{L}(y) - y(0)] + 2\mathcal{L}(y) = \frac{4}{s-2}$

$s^2\mathcal{L}(y) + s \times 3 - 5 - 3[s\,\mathcal{L}(y) + 3] + 2\mathcal{L}(y) = \frac{4}{s-2}$

$s^2\mathcal{L}(y) + 3s - 5 - 3[s\,\mathcal{L}(y) + 3] + 2\mathcal{L}(y) = \frac{4}{s-2}$

$\cancel{s^2\mathcal{L}(y) - 3s\,\mathcal{L}(y) + 2\mathcal{L}(y) + 3s - 5 + 3 = \frac{4}{s-2}}$

$\cancel{\mathcal{L}(y)(s^2 - 3s + 2)}$

$s^2\mathcal{L}(y) + 3s - 5 - 3s\,\mathcal{L}(y) - 9 + 2\mathcal{L}(y) = \frac{4}{s-2}$

$\mathcal{L}(y)(s^2 - 3s + 2) + 3s - 14 = \frac{4}{s-2}$

$\mathcal{L}(y)(s^2 - 3s + 2) = \frac{4}{s-2} + 14 - 3s$

$\mathcal{L}(y) = \frac{4}{(s^2-3s+2)(s-2)} + \frac{14 - 3s}{(s^2-3s+2)}$

$\mathcal{L}(y) = \frac{4}{(s-1)(s-2)(s-2)} + \frac{14}{(s-1)(s-2)} - \frac{3s}{(s-1)(s-2)}$

[$\mathcal{L}^{-1}$ direct [illegible] 180 [illegible] [illegible] $\boxed{\ }$ — [illegible] same [illegible]]

$= \frac{4}{(s-1)(s-2)(s-2)} + \frac{14(s-2)}{(s-1)(s-2)(s-2)} - \frac{3s(s-2)}{(s-1)(s-2)(s-2)}$

$= \frac{4 + 14(s-2) - 3s(s-2)}{(s-1)(s-2)^2} = \frac{4 + 14s - 28 - 3s^2 + 6s}{(s-1)(s-2)^2}$

$\mathcal{L}(y) = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$ — (f) (denom partial (?) [illegible]) on table.

$\dfrac{\ }{(s-3)(s-1)} = \dfrac{A}{s-3} + \dfrac{B}{s-1}$

$a-3$

$\dfrac{\ }{(s-a)^2} = \dfrac{A}{s-a} + \dfrac{B}{(s-a)^2}$

y

$\Rightarrow \dfrac{-3s^2+20s-24}{(s-1)(s-2)^2}$

2 terms

(A, B, C we don't know)

$= \dfrac{A}{s-1} + \dfrac{B}{(s-2)^2}$

$= \dfrac{A}{s-1} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-2)^2}$. A, B, C ?

$\Rightarrow \dfrac{-3s^2+20s-24}{(s-1)(s-2)^2} = \dfrac{A}{s-1} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-2)^2}$ — (b)

(1) = (2). So, denom same [illegible].

$= \dfrac{A(s-2)^2}{(s-1)(s-2)^2} + \dfrac{B(s-1)(s-2)}{(s-1)(s-2)^2} + \dfrac{C(s-1)}{(s-1)(s-2)^2}$

$\Rightarrow \dfrac{-3s^2+20s-24}{\cancel{(s-1)(s-2)^2}} = \dfrac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{\cancel{(s-1)(s-2)^2}}$

$\Rightarrow -3s^2+20s-24 = \underbrace{A(s-2)^2}_{A\to 0 \text{ if } s=2} + B(s-1)(s-2) + C(s-1)$ — (c)

put $s=2$,

$\Rightarrow -3\times2^2+20\times2-24 = A(2-2)^2 + B(2-1)(2-2) + C(2-1)$

$\Rightarrow -12+40-24 = C$

$\Rightarrow 4 = C$

$\boxed{C=4}$

from — (c) put $s=1$.

$B(s-1)(s-2)$

$-3+20-24 = A(1-2)^2 + 0 + 0$

$-7 = A$

$\boxed{A=-7}$

next B ?

expand — (c).

$-3s^2+20s-24 = A(s^2-4s+4) + B(s^2-3s+2) + C(s-1)$.

$-3s^2+20s-24 = As^2-4sA+4A+Bs^2-3sB+2B+C(s-1)$

equating (coe) from LHS & RHS. of s^2.

$A=-7$

$-3 = A+B$

$-3 = -7+B$.

$-3+7 = B$

$4 = B$

$\boxed{B=4}$

Sub values of A, B, C in — (b),

$L(y) = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{(s-2)^2}$

$L(y) = \dfrac{-7}{s-1} + \dfrac{4}{s-2} + \dfrac{4}{(s-2)^2}$

$y = -7L^{-1}\left(\dfrac{1}{s-1}\right) + 4L^{-1}\left(\dfrac{1}{s-2}\right) + 4L^{-1}\left(\dfrac{1}{(s-2)^2}\right)$

$= -7e^t + 4e^{2t} + 4te^{2t}$

$L^{-1}\left(\dfrac{1}{s^2}\right) = t$

$\therefore L^{-1}\dfrac{1}{(s-2)^2} =$

$t^n e^{at} = \dfrac{n!}{(s-a)^{n+1}}$

3) find the inverse L.T of $\ln\left(\frac{s+a}{s+b}\right)$

A) we have to find $L^{-1}\left(\ln\left(\frac{s+a}{s+b}\right)\right) = f(t)$

we know, $\boxed{\text{if } f(t) = L^{-1}(F(s)) \text{ then } L^{-1}(F'(s)) = -t f(t)}$

$\therefore \; -t f(t) = L^{-1}(F'(s))$

$= L^{-1}\left[\frac{d}{ds} \ln\left(\frac{s+a}{s+b}\right)\right]$

$\cancel{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

$\ln\frac{a}{b} = \ln a - \ln b$

$= L^{-1}\left[\frac{d}{ds}\left[\ln(s+a) - \ln(s+b)\right]\right]$

$= L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$= L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$

$-t f(t) = e^{-at} - e^{-bt}$

$f(t) = \frac{-1}{t}\left[e^{-at} - e^{-bt}\right]$

4) Solve DE $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4$, $y(0) = 2$, $y'(0) = 3$ using L.T

A) $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4$

$y'' - 3y' + 2y = 4$ L. on both side,

$L(y'') - 3L(y') + 2L(y) = 4L(1)$

$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) - 3\left[s L(y) - y(0)\right] + 2L(y) = 4 \times \frac{1}{s}$

$\Rightarrow s^2 L(y) - s \times 2 - 3 - 3\left[s L(y) - 2\right] + 2L(y) = 4/s$

$\Rightarrow s^2 L(y) - 2s - 3 - 3s L(y) + 6 + 2L(y) = \frac{4}{s}$

$\Rightarrow L(y)(s^2 - 3s + 2) - 2s - \cancel{3+6} = \frac{4}{s}$

$\Rightarrow L(y)(s^2 - 3s + 2) = \frac{4}{s} + 2s - 3$

$\Rightarrow L(y) = \frac{4}{s(s^2 - 3s + 2)} + \frac{2s}{s^2 - 3s + 2} - \frac{3}{s^2 - 3s + 2}$

$= \frac{4}{s(s-1)(s-2)} + \frac{2s}{(s-1)(s-2)} \times \frac{s}{s} - \frac{3}{(s-1)(s-2)} \times \frac{s}{s}$

$\Rightarrow \frac{4 + 2s^2 - 3s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

(check once done)

$\Rightarrow \frac{4 + 2s^2 - 3s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)}{s(s-1)(s-2)} + \frac{B(s-2)s}{(s-1)(s-2)s} + \frac{C(s-1)s}{(s-2)(s-1)s}$

$\Rightarrow 4 + 2s^2 - 3s = A(s-1)(s-2) + B(s-2)s + C(s-1)s$

put $s = 2$

$4 + 8 - 6 = A(2-1) \times 0 + 0 + 2C$

$6 = 2C$

$6/2 = C$ // $\Rightarrow \underline{C = 3}$

Put $s=1$,

$4+2-3 = B(1-2)1$

$3 = -B \Rightarrow B = -3$

$4+2s^2-3s = A(s^2-2s-s+2) + B(s^2-2s) + C(s^2-s)$

$4+2s^2-3s = As^2-2sA-As+2A+Bs^2-2sB+Cs^2-Cs$

$2s^2 = As^2+Bs^2+Cs^2$

$2 = A+B+C.$

$2 = A-3+3 \Rightarrow A=2$

$L(y) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} = \frac{2}{s} - \frac{3}{s-1} + \frac{3}{s-2}$

$y = 2L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s-1}\right) + 3\left(\frac{1}{s-2}\right)$

$y = 2 - 3e^{t} + 3e^{2t}$