02: FIRST ORDER DE.

($\frac{d}{dx}$ or y')

⇒ Direction fields =

The method (dir) field is a graphical method for displaying the general shapes & behaviour of solns of DE of the form $\frac{d}{dx} = f(x,y)$, where f is a fun of a 2 variables x & y

⇒ Seperable diffn eq =

The diffn eq can be reduced to the form

$\frac{dy}{dx} = \frac{g(x)}{h(y)}$ ⇒ $h(y)dy = g(x)dx$.

(y-നെ ഒരു side-ലേക്കും x-നെ " ") ⇒ $\int h(y)\, dy = \int g(x)\, dx + C$.

(deri- ആണെങ്കിൽ $\int$ ചെയ്യണം)

(a) [ഒരു diff - eq -നെ seperate ചെയ്താൽ എളുപ്പത്തിൽ solve ചെയ്യാം] (seprte → $\int$ → soln)

1) Solve the DE $y' = (1+x)(1+y^2)$

A) $y' = (1+x)(1+y^2)$ by (a)

here $\frac{dy}{dx} = (1+x)(1+y^2)$

(dy & dx → numrtr ആക്കണം)

$y' = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{1+y^2} = (1+x)\,dx$.

$\therefore \int \frac{dy}{1+y^2} = \int (1+x)\,dx$

$\tan^{-1} y = x + \frac{x^2}{2} + c$

$\Rightarrow y = \tan\left(x + \frac{x^2}{2} + c\right)$ //

($\int \frac{dy}{1+y^2} = \tan^{-1} y$)

$\int \frac{dx}{1+x^2} = \tan^{-1} x$

2) find the soln of $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$ using seperation variables.

A) (take common out side)

$$\frac{dy}{dx} = \frac{y(x+2)-(x+2)}{y(x-3)+(x-3)} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$$

$$\frac{(y+1)}{(y-1)}\,dy = \frac{x+2}{x-3}\,dx.$$

$$\int \frac{y+1}{y-1}\,dy = \int \frac{x+2}{x-3}\,dx.$$

[denomtr-ന് periority ആക്കണം → y-1. denomtr-ൽ numtr-ൽ y+1 → +1-നെ, denomtr-ൽ ഉള്ള y-1 → -1 ആക്കണം ⇒ $\frac{y-1}{y-1} \to \frac{y-1+2}{y-1}$.]

$$\Rightarrow \int \frac{y-1+2}{y-1}\,dy = \int \frac{x-3+5}{x-3}\,dx.$$

$$\int \frac{y-1}{y-1} + \frac{2}{y-1}\,dy = \int \frac{x-3}{x-3} + \frac{5}{x-3}\,dx.$$

$$\int 1 + \frac{2}{y-1}\,dy = \int 1 + \frac{5}{x-3}\,dx.$$

$$\int y + 2\ln|y-1| = x + 5\ln|x-3| + C$$

$\frac{2}{y-1}$.
$2 \times \frac{1}{y-1}$
$2 \times \log(y-1)$
+ constant
ans - ആയ
so $y-1$
$2\ln(y-1)$.

3) Solve the inertial value prbm.
(last $+c \to c$ ആണോ)

$y' = -2xy$, $y(0)=1$ & also determine the interval in which soln is defined.

A) $\frac{dy}{dx} = -2xy \Rightarrow \frac{1}{y}\,dy = -2x\,dx$

$$\int \frac{1}{y}\,dy = \int -2x\,dx.$$

$$\ln y = -2\frac{x^2}{2} + c.$$

$$\ln y = -x^2 + c \quad —①$$

To find $y = \sim$ cancel ln with e.

gvn $y(0)=1$, $\hookrightarrow x=0$.
Sub in ①

$\ln y = 0 + c$
$\ln 1 = c$
$0 = c \Rightarrow \underline{c=0}$

$\therefore \ln y = -x^2 + c$
$\ln y = -x^2$
$e^{\ln y} = e^{-x^2}$
$\underline{y = e^{-x^2}}$

Since y is defined for all x, the interval of above soln is the entire real line. [exp-ന്റെ എല്ലാ non-ve ആകും]

4) find the soln of $xy' = 4y$

A) $x\frac{dy}{dx} = 4y.$

$$\Rightarrow \frac{dy}{y} = 4\frac{dx}{x}.$$

$$\int \frac{dy}{y} = 4\int \frac{dx}{x} \Rightarrow \ln y = 4\ln x + c$$

$$e^{\ln y} = e^{\ln x^4} + c$$

$$y = e^{\ln x^4}$$

$$\underline{y = x^4 + c}$$

$\boxed{\ln a^b = b\ln a \quad \ln x^4 = 4\ln x.}$

⇒ Eq reducible to seperable form

★ **Type 1 → Homogenous DE.**

★ A DE $y' = f(x,y)$ is said to be a homogenous DE, if $f(x,y)$ can be expressed as a ( ) of the ratio y/x only (i.e) $\boxed{f(x,y) = F(y/x)}$

(comm eq-nn y/x -nn form-ð [illegible] homo)

★ Homo eq can be transfored into seperable eq by $\boxed{y/x = v}$

1) Solve $(x-y)\,dx + x\,dy = 0$

A) Step 1 → check if it seperable. $(x-y)$ not possible (difficult)
Step 2 → so Homo DE.

$(x-y)\,dx + x\,dy = 0$.

$x\,dy = -(x-y)\,dx$.

$x\dfrac{dy}{dx} = \dfrac{-x+y}{x}$

$x\dfrac{dy}{dx} = y - x$

$\dfrac{dy}{dx} = \dfrac{y-x}{x}$ —①

$\cancel{\dfrac{dy}{dx} = \dfrac{y}{x} - 1}$

we get y/x form

(homo [illegible])

Clearly this is a homo-DE.

Sub $u = y/x$

$u = \dfrac{y}{x}$

$y = ux$

diff. w.r.t x,

$\dfrac{dy}{dx} = u \times 1 + x\dfrac{du}{dx}$.

$\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$ —②

frm ① & ②

$\dfrac{dy}{dx} = \dfrac{dy}{dx}$

$\dfrac{y-x}{x} = \dfrac{ux-x}{x} = u + x\dfrac{du}{dx}$ ($y = ux$)

$\Rightarrow \dfrac{ux}{x} - \dfrac{x}{x} = u + x\dfrac{du}{dx}$

$\Rightarrow u - 1 = u + x\dfrac{du}{dx}$.

$\Rightarrow \cancel{u} - 1 - \cancel{u} = x\dfrac{du}{dx}$

$\Rightarrow -1 = x\dfrac{du}{dx}$

$\Rightarrow -\dfrac{1}{x}\,dx = du$

$\Rightarrow \int -\dfrac{1}{x}\,dx = \int du$

$\Rightarrow -\ln|x| = u + c$ (i.e)

$u = -\ln|x| + c$

2) Solve the initial value prbm $2xy\dfrac{dy}{dx} - y^2 + x^2 = 0$, $y(1) = 1$.

A) (dy/dx form [illegible])

$2xy\dfrac{dy}{dx} = y^2 - x^2$

$\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$ → $\dfrac{(y/x)^2 - x^2}{2x(y/x)}$ we can write it as so homo-the dE is homogenous

~~Suppose~~ It is a homon eq, it can be written by $\frac{dy}{dx} = f(y/x)$

$\therefore$ Substitute, $u = y/x$. $y = ux$.

$$\frac{dy}{dx} = \frac{(ux)^2 - x^2}{2x(ux)}$$

$$\frac{dy}{dx} = \frac{(ux)^2 - x^2}{2ux^2} \quad \text{--- (1)}$$

Since $y = ux$, diff w.r.t x.

$$\frac{dy}{dx} = u + x\frac{du}{dx} \quad \text{--- (2)}$$

from (1) & (2)

$$\frac{ux^2 - x^2}{2ux^2} = u + x\frac{du}{dx}$$

x one side & u another side,

$$\frac{u^2x^2 - x^2}{2ux^2} = u + x\frac{du}{dx}$$

$$\frac{u^2-1}{2u} - u = x\frac{du}{dx}$$

$$\frac{u^2 - 1 - 2u^2}{2u^2} = x\frac{du}{dx}$$

$$\frac{-u^2-1}{2u} = x\frac{du}{dx}$$

$$\frac{dx}{x} = \frac{2u}{-(u^2+1)}du$$

dx, dy are numerator, denomo.

$$\int \frac{dx}{x} = -\int \frac{2u}{u^2+1}du.$$

$$\Rightarrow \ln x = -\ln|u^2+1| + c \quad \text{--- (1)}$$

$$\Rightarrow \ln x + \ln|u^2+1| = c$$

$$\Rightarrow \ln(x(u^2+1)) = c$$

$$e^{\ln(x(u^2+1))} = e^c$$

$$x(u^2+1) = e^c = k \text{ (constant)}$$

$y(1) = 1$

$$xu^2 + x = k$$

here no y, $\rightarrow u = y/x$.

$$x\left(\frac{y}{x}\right)^2 + x = k.$$

$$x\left(\frac{y^2}{x^2}\right) + x = k$$

$$\frac{y^2}{x} + x = k.$$

$y(1) = 1$, $y = 1$, $x = 1$

$$1 + 1 = k \Rightarrow 2 = k.$$

$$\Rightarrow 2 = e^c \Rightarrow \boxed{\ln 2 = c} \quad \text{--- (2)}$$

(2) in (1)

$$\ln x = -\ln|u^2+1| + \ln 2.$$

$\Rightarrow$ Type II

$$\boxed{\frac{dy}{dx} = \frac{ax+by+l}{k(ax+by)+m}}$$

Above eq can be converted into separable eq by substituting $\boxed{ax+by = u}$

homogenous

means,

(1) $\frac{dy}{dx} = \frac{y^2+x^2}{2+x}$

(2) $\frac{y}{x}$

(3) $\frac{y-x}{x}$

(4) $\frac{1+x+x}{x}$

(5) $\frac{x^2-xy}{x^2+y^2}$

(6) $\frac{x^2+xy+y^2}{x^2}$

⇒ Linear differential eq =

Standard form of a Linear eq —

$$\boxed{\frac{dy}{dx} + P(x)y = f(x)}$$

This form LDE

x^2, y^2 [illegible]

Both ( )s $p(x)$ & $f(x)$ are continous

eg → 1) $\frac{dy}{dx} + y\underbrace{\tan x}_{\substack{x \text{ term} \\ p(x)}} = \cos^3 x$. ✓

2) $x\frac{dy}{dx} - 4y = x^6 e^x$ ✗

($\frac{dy}{dx}$ [illegible])

÷ throughout by x,

$$\frac{dy}{dx} - \frac{4}{x}y = \frac{x^{6} e^{x}}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \checkmark$$

$p(x) = -4/x$, $f(x) = x^5 e^x$.

$\frac{dy}{dx}$ →

* A 1st order DE of the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)} \quad (1) \qquad \text{(general)}$$

is said to be linear eq in the dependent variable 'y'.

$g(x) = 0 \Rightarrow$ homogenous or $g(x) \neq 0$. non-homogenous

* when $g(x) = 0$, the linear eq (1) is said to be homogenous otherwise it is non-homogenous.

→ Standard form :-

The standard form of a linear eq

$$\boxed{\frac{dy}{dx} + p(x)y = f(x)} \quad (2)$$

where $p(x)$ & $f(x)$ are continous.

General Solⁿ :

The DE (2) has the property that its solⁿ is the sum of 2 solⁿ,

$$\boxed{y = y_c + y_p} \quad (3) \qquad y_c \text{ \& } y_p \} \text{ 2 sol}^n.$$

where y_c is the solⁿ of the associated homogenous eq

standard form-[illegible] homogenous [illegible] last 0 [illegible] (i.e) $\frac{dy}{dx} + p(x)y = 0$

$$\frac{dy}{dx} + p(x)y = 0.$$

$$y_c = c\,y_1(x)$$

$$\boxed{y_c = c\,e^{-\int p(x)dx}} \quad (4)$$

$\frac{dy}{dx} + p(x)y = 0$ → y_c.

next we hve to find a particular solⁿ (P) y_p by a procedure → variation of paramters

$$y_p = u(x)\,y_1(x)$$

$$\boxed{y_p = e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)\,dx.} \quad (5)$$

Sub (4) & (5) in (3)

~~$y = y_c + y_p =$~~

$$\boxed{y = y_c\,y_p = c\,e^{-\int p(x)dx} + e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)\,dx}$$

(general form of a DE — [illegible] exam [illegible] write this full)

⇒ Method of solⁿ :-

Procedure to solve 1st order DE →

1) write the gvn DE in standard form. Identify $p(x)$ & $f(x)$

2) find the integrating factor (IF).

$e^{\int p(x)dx}$

3) Multiply both side of the DE by integrating factor.

(i.e) $\underbrace{e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y}_{\text{p. rule ആണ്, so}} = e^{\int p(x)dx}f(x)$

$$\frac{d}{dx}\left(y e^{\int p(x)dx}\right) = e^{\int p(x)dx}f(x)$$

integrating on both side,

$$y e^{\int p(x)dx} = \int e^{\int p(x)dx} f(x)dx + c$$

$$y = \frac{\int e^{\int p(x)dx} f(x)dx + c}{e^{\int p(x)dx}}$$

$$y = e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)dx + c$$

Q

1) Solve $\frac{dy}{dx} + y\tan x = \cos^3 x$ — (1)

if $3\frac{dy}{dx} + y\tan x = \cos^2 x$ → ÷3 throughout

A) (1) standard form,

$$\frac{dy}{dx} + y\tan x = \cos^3 x.$$

$p(x) = \tan x$, $f(x) = \cos^3 x$, (both p & f are continous,)

(2) integrating factor, $e^{\int p(x)}$

$$e^{\int p(x)dx} = e^{\int \tan x\,dx}$$

$$= e^{\ln|\sec x|}$$

$$= |\sec x|$$

$\boxed{\int \tan x = \ln|\sec x| \quad \cos^2 x = \frac{1+\cos 2x}{2}}$

IF = $|\sec x|$.

③ multiply IF by $e^{\int p(x)dx} \rightarrow \sec x$.

$$\sec x \frac{dy}{dx} + y\tan x \sec x = \sec x \cos^3 x.$$

① deri-② + ② deri of ①.

$$\frac{d}{dx}(y \cdot \sec x) = \sec x \cos^3 x$$

$$= \frac{1}{\cos x}\cos^{\cancel{3}2} x. = \cos^2 x.$$

$$\frac{d}{dx}(y\sec(x)) = \cos^2 x.$$

$$= \frac{1+\cos 2x}{2}. = \frac{1}{2} + \frac{\cos 2x}{2}.$$

$\int$ integ,

$$y\sec x = \int\left(\frac{1}{2} + \frac{\cos 2x}{2}\right)dx.$$

$$= \int\frac{1}{2} + \frac{1}{2}\int \cos 2x \cdot dx.$$

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 2x}{2}.$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x.$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c.$$

$\therefore$

$$y\sec x = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

⇒ Error function :- (erf)

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.}$$

→ Complimentry error ( ) :- (erfc)

$$\boxed{erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt}$$

$$\boxed{erf(x) + erfc(x) = 1}$$

1) Solve the intial value prbm.

$$y' + 6x^2 y = \frac{e^{-2x^3}}{x^2}, \quad y(1) = 0.$$

A) ① 1st check stand form → $\frac{dy}{dx} + p(x)y = f(x)$.

$$y' + 6x^2 y = \underbrace{\frac{e^{-2x^3}}{x^2}}_{\text{only } x \text{ term}} \checkmark$$

$p(x) = 6x^2$, $f(x) = \frac{e^{-2x^3}}{x^2}$

(integrating factor) ② $IF = e^{\int p(x)dx} = e^{\int 6x^2 dx} = e^{6\frac{x^3}{3}} = e^{2x^3}$

[p & f are contin on $(-\infty,0)$ & $(0,\infty)$]

③ × IF,

$$\underbrace{e^{2x^3} \cdot y'}_{\text{p rule}} + \underbrace{y \cdot 6x^2 \cdot e^{2x^3}}_{\text{p rule}} = \frac{e^{-2x^3}}{x^2} \times e^{2x^3}$$

$$= \frac{1}{x^2}$$

$$\frac{d}{dx}\left[e^{2x^3} y\right] = \frac{1}{x^2}$$

$\int$ on both sides,

$$e^{2x^3} y = \int \frac{1}{x^2}\, dx.$$

$\frac{1}{x^2} = x^{-2}$, $\frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$$e^{2x^3} y = -\frac{1}{x} + c \quad \text{—①}$$

to find c, $y(1) = 0$ ②.

$$e^{2x^3} y = -\frac{1}{x} + c$$

$$c = e^{2x^3} y + \frac{1}{x}$$

given $y(1) = 0$ ⇒ $e^{2}\cdot 0 + 1 = 0 + 1 = 1$.
$x=1, y=0$

∴ $c = 1$ —③

Sub-③ in —①

$$e^{2x^3} y = -\frac{1}{x} + 1. \checkmark$$

$$y = \frac{-1}{x e^{2x^3}} + \frac{1}{e^{2x^3}}$$

⇒ Bernoulli's eq :-

$a \to 1, 2 \sim$

① → $$\boxed{\frac{dy}{dx} + p(x)y = f(x)y^a}$$

[form eq [illegible] form — [illegible] B.eq]

p(x) & f(x) are c's & a is any real no —

is → B. eq.

Note →

* if a=0 or a=1, the above eq is linear otherwise non-linear.

* ÷ ing eq-① by y^a.

$$\Rightarrow \frac{dy}{dx}\frac{1}{y^a} + p(x)\frac{y}{y^a} = f(x).$$

$$\Rightarrow y^{-a}\frac{dy}{dx} + p(x)\, y\, y^{-a} = f(x).$$

$$\Rightarrow \boxed{y^{-a}\frac{dy}{dx} + p(x)\, y^{1-a} = f(x)} \quad \text{—②}$$

Put $u = y^{1-a}$ then

$$\frac{du}{dx} = (1-a)\, y^{1-a-1} \times \frac{dy}{dx}$$

$$\frac{du}{dx} = (1-a)\, y^{-a} \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{1-a} \frac{du}{dx} = y^{-a} \frac{dy}{dx} \quad \text{—③}$$

Sub —③ in —②

$$y^{-a} \frac{dy}{dx} + p(x)\, y^{1-x} = f(x).$$

$$\frac{1}{1-a} \frac{du}{dx} + p(x)u = f(x)$$

× throughout by $1-a$,

$$\boxed{\frac{du}{dx} + (1-a)\, p(x)\, u = \underbrace{(1-a) f(x)}_{\text{Costant nots.}}}$$

standard form.

2) Solve $x \frac{dy}{dx} - 4y = x^6 e^x$.

A) stand eq → $\frac{dy}{dx} + p(x)y = f(x)$.

$$x \frac{dy}{dx} - 4y = x^6 e^x.$$

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$p(x) = -\frac{4}{x}$, $f(x) = x^5 e^x$.

$p(x)$ & $f(x)$ are contin.. on $(0,\alpha)$

$$IF = e^{\int p(x)dx} = e^{\int -\frac{4}{x} dx} = e^{-4\int \frac{1}{x} dx}$$

$$= e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4}.$$

× IF throughout,

$$x^{-4} \frac{dy}{dx} - 4x^{-5} y = x e^x.$$

$$\frac{d}{dx}\left[x^{-4} y\right] = x e^x.$$

$\int$ both sides of above eq,

$$x^{-4} y = \int x e^x dx = x e^x - \int e^x dx.$$

$$= x e^x - e^x + c.$$

× throughout by x^4.

$$y = x^5 e^x - x^4 e^x + c x^4$$

3) Solve $y\,dx - x\,dy + \ln x\, dx = 0.$

A) $y\,dx - x\,dy + \ln x\, dx = 0$

$$\Rightarrow y - x \frac{dy}{dx} + \ln x = 0.$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} \ln x$$

$p(x) = -\frac{1}{x}$ $f(x) = \frac{\ln x}{x}$.

both p & f are contin.. $(0,\alpha)$

$$IF = e^{\int p(x)dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

× IF on both sides,

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x} \times \frac{1}{x} = \frac{\ln x}{x} \times \frac{1}{x}.$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{x} \times y\right] = \frac{\ln x}{x^2}$$

$\int$ on both sides,

$$\frac{1}{x} y = \int \frac{\ln x}{x^2}\,dx.$$

put $\ln x = t$
$e^{\ln x} = e^t$
$x = e^t$
$\frac{dx}{x} = dt$

$$= \frac{1}{x}\int \frac{\ln x}{x}\,dx.$$

$$= \int e^{-t}\, t\, dt.$$

$$= t(-e^{-t}) - \int (1 \times (-e^{-t})) + c.$$

($\int$ by parts)

$$= -te^{-t} - e^{-t} + c$$

$$= -e^{-t}(1+t) + c.$$

$$= \frac{-1}{x}(1+\ln x) + c.$$

$\times$ throughout by x,

$$y = cx - (1 + \ln x)$$

) Solve $xy' + y = xy^3$ 1st check bernalli.

A) $\div$ throughout by y^3,

$$\frac{x}{y^3}\frac{dy}{dx} + \frac{y}{y^3} = x$$

$$xy^{-3}\frac{dy}{dx} + y^{-2} = x. \quad \text{— ① Rule.}$$

put $u = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$① \quad \frac{-1}{2}\frac{du}{dx} = y^{-3}\frac{dy}{dx}$$

$$\Rightarrow -\frac{x}{2}\cdot\frac{du}{dx} + u = x$$

ഇനി statard form eq [illegible] $\frac{dy}{dx} + P(y) = f(x)$.

$\times \frac{-2}{x}$ throut.

$$\frac{du}{dx} - \frac{2u}{x} = -2 \quad \text{— ②}$$

eq – ② is in standard form.

ഇനി [illegible],

$\therefore P(x) = \frac{-2}{x}$, $f(x) = -2$

$$IF = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$\times$ IF with eq – ②

$$\Rightarrow x^{-2}\frac{du}{dx} - \frac{2}{x}ux^{-2} = -2x^{-2}$$

$$\frac{1}{x^2}\frac{du}{dx} - \frac{2u}{x}\frac{1}{x^2} = \frac{-2}{x^2}$$

$$\frac{1}{x^2}\frac{du}{dx} - \frac{2u}{x^3} = \frac{-2}{x^2}$$

(p rule)

$$\frac{d}{dx}\left[\frac{1}{x^2} \times u\right] = \frac{-2}{x^2}$$

$\int$,

$$\frac{1}{x^2} \times u = \int \frac{-2}{x^2}\,dx.$$

$$\frac{1}{x^2}u = -2\int \frac{1}{x^2}\,dx. = -2\,\frac{x^{-2+1}}{-2+1} + c$$

$$= -2\,\frac{x^{-1}}{-1} + c$$

$$= \frac{2}{x} + c$$

u-ന്റെ [illegible], and – ൽ y-ന്റെ [illegible],

$$\frac{1}{x^2} y^{-2} = \frac{2}{x} + c.$$

$$y^{-2} = \frac{2x^2}{x} + cx^2$$

$$y^{-2} = 2x + cx^2$$

2) Solve $y(2xy+e^x)\,dx - e^x\,dy = 0$,

① $\frac{dy}{dx}$ ഉണ്ടെങ്കിൽ.

(2) general form, $\frac{dy}{dx} + p(x)\underline{y} = f(x)$.

A) $y(2xy+e^x)\,dx = e^x\,dy$

$\Rightarrow 2xy^2 + e^x y = e^x \frac{dy}{dx}$

$e^x \frac{dy}{dx} - e^x y = 2xy^2$

$\longrightarrow$ Bernoulli eq ✓

$\div$ th.out by y^2,

$\frac{e^x}{y^2}\frac{dy}{dx} - \frac{e^x y}{y^2} = 2x$

$e^x y^{-2}\frac{dy}{dx} - e^x \underline{y^{-1}} = 2x$

put $u = y^{-1}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{du}{dx} = y^{-2}\frac{dy}{dx}$

$-e^x \frac{du}{dx} - e^x u = 2x$.

$\div$ th.out by $-e^x$,

$\frac{du}{dx} + u = \frac{-2x}{e^x}$ — ① ✓

$\longrightarrow$ Standarm form

$\therefore p(x) = 1$, $f(x) = \frac{-2x}{e^x}$

$IF = e^{\int p(x)dx} = e^{\int 1 dx} = e^x$,

$\times$ IF, on ①

$e^x \frac{du}{dx} + e^x u = \frac{-2x}{\cancel{e^x}} \times \cancel{e^x}$

$e^x \frac{du}{dx} + ue^x = -2x$

P.rule → $e^x \times u$,

$\Rightarrow \frac{d}{dx}\left[e^x \times u\right] = -2x$.

$\int$, $e^x u = \int -2x$.

$= -2\int x = -\cancel{2}\frac{x^2}{\cancel{2}} = -x^2 + c$

we know, $u = y^{-1}$.

$\therefore e^x y^{-1} = -x^2 + c$.

$y^{-1} = \frac{-x^2}{e^x} + \frac{c}{e^x}$

$y^{-1} = \frac{1}{e^x}(-x^2 + c)$

★ $\Rightarrow$ Exact DE :-

A necessary condition that a DE,

$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$

$dx \to M$ $\quad$ $dy \to N$

be exact is that,

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

(തന്ന eq, ~~exact~~ exact ആണോ, അല്ലേ)

1) Determine whether the gvn eq is exact,

a) $(2x-1)\,dx + (3y+7)\,dy = 0$

A) $M = 2x-1$, $N = 3y+7$

$\frac{\partial M}{\partial y} = 0$ $\quad$ $\frac{\partial N}{\partial x} = 0$

clearly $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ✓ $\therefore$ it is exact

b) $(5x+4y)\,dx + (4x-8y^3)\,dy = 0$.

A) $M = 5x+4y$, $N = 4x - 8y^3$

$\frac{\partial M}{\partial y} = 4$ $\quad$ $\frac{\partial N}{\partial x} = 4$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(Exact)

3) $xy' + y + 4 = 0$

A) convert this to $M\,dx + N\,dy = 0$.,

$x \frac{dy}{dx} + y + 4 = 0.$

$x \frac{dy}{dx} = -y - 4$

$x\,dy = (-y-4)\,dx \Rightarrow \cancel{x\,dy + y + 4x = 0}$

$\Rightarrow (-y-4)dx - x\,dy = 0$

$\therefore M = -y-4$, $N = -x$

$\frac{\partial M}{\partial y} = -1$ $\qquad \frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

Q) find the value of k, so that the gvn DE, $(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$ is exact?

A) $M = y^3 + kxy^4 - 2x$. $\qquad N = 3xy^2 + 20x^2y^3$

$\frac{\partial M}{\partial y} = 3y^2 + 4kxy^3$ $\qquad \frac{\partial N}{\partial x} = 3y^2 + 20\times 2x\,y^3$

$= 3y^2 + 40xy^3$

(condition a-ans)

$\therefore \cancel{3y^2} + 4kxy^3 = \cancel{3y^2} + 40xy^3$

$4k\cancel{xy^3} = 40\cancel{xy^3}$

$4k = 40$.

$k = \frac{40}{4} = 10$

⇒ Method of solⁿ of exact DE :-

If $M\,dx + N\,dy = 0$ is exact, it can be solved by following method —

① 1st ∫-ate M w.r.t x, regarding 'y' as a constant, ② then ∫-ate N w.r.t y those terms in N which do not involve x.

③ The sum of 2 exp, thus obtained equated to a constant is the required solⁿ. $(\ldots + \ldots = c)$

In other words the solⁿ of an exact DE —

$$\boxed{\underset{y \to \text{constant}}{\int M\,dx} + \underset{\text{terms not having } x}{\int N\,dy} = c}$$

1) S.T the eq $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$ is exact & solve it?

A) $M = 1 + 4xy + 2y^2$ $\qquad N = 1 + 4xy + 2x^2$

$\frac{\partial M}{\partial y} = 4x + 4y$ $\qquad \frac{\partial N}{\partial x} = 4y + 4x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

Solve

1) $\int M = \int 1 + 4xy + 2y^2\,dx$

$= x + 2x^2y + 2y^2x$ — ①

② ∫-ate N w.r.t y donot involve x. (i.e)

$N = 1 + 4xy + 2x^2$ (1 no x terms) $\rightarrow \int 1\,dy = y$ — ②

hence solⁿ of DE frm ① & ② is,

$x + 2x^2y + 2y^2x + y = c$

2) S.T $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$ is exact & solve it?

A) $M = 2xy + y - \tan y$ $\qquad N = x^2 - x\tan^2 y + \sec^2 y$

$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y$ $\qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$

(a - [illegible] exact [illegible])

~~$\tan^2 y - \sec^2 y = 1$~~

$\tan^2 y = 1 + \sec^2 y$

$\sec^2 y - \tan^2 y = 1$

$-\tan^2 y = 1 - \sec^2 y$

~~$\sec^2 y = 1 + \tan^2 y$~~

~~$\frac{\partial N}{\partial x} = 2x - (1 + \tan^2 y)$~~

$\therefore \frac{\partial N}{\partial x} = 2x - \tan^2 y$

$= 2x + 1 - \sec^2 y$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$2u - \tan^2 y$

$2u - (\sec^2\theta - 1)$

$2u - \sec^2\theta + 1$

$2u + 1 - \sec^2\theta$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

Solve

$\int M dx = \int 2xy + y - \tan y \, dx$

$= x^2 y + yx - x\tan y$ — (a)

$N = x^2 - x\tan^2 y + \sec^2 y + 2$

$\Rightarrow \sec^2 y + 2$

$\Rightarrow \int \sec^2 y + 2 \, dy$

$\Rightarrow \tan y + 2y$ — (b)

from — (a) & (b)

$x^2 y + yx - x\tan y + \tan y + 2y = c$

3) Solve $(3x^2 y + e^y)dx + (x^3 + xe^y - 2y)dy = 0$

A) $M = 3x^2 y + e^y$ $\qquad N = x^3 + xe^y - 2y$

$\frac{\partial M}{\partial y} = 3x^2 + e^y$ ... (1) $\qquad \frac{\partial N}{\partial x} = 3x^2 + e^y$ ... (2)

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

$\int M dx = \int 3x^2 y + e^y dx = x^3 y + xe^y$

$N = x^3 + xe^y - 2y$

$\therefore \Rightarrow \int -2y = -2\int y = -2 \times \frac{y^2}{2} = -y^2$

$\therefore x^3 y + xe^y - y^2 = c$

Sometime a gvn DE $M(x,y)dx + N(x,y)dy = 0$ — (1)

~~$\rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ exact~~

is not exact, bt it can be exact by multipling it by a suitable ( ) & this ( ) $\rightarrow$ IF of eq — (1)

1) $\boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$ is a ( ) of x along, then $\underset{f(x)}{\downarrow}$

$e^{\int f(x)dx}$ is an IF. of — (1)

2) $\boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$ is a ( ) of y along, then $\underset{f(y)}{\downarrow}$

$e^{\int f(y)dy}$ is an IF of — (1)

Q

1) Solve DE : $(x^2 - 2x + 2y^2)dx + 2xy \, dy = 0$ — (1)

A) $M = x^2 - 2x + 2y^2$ $\qquad N = 2xy$

$\frac{\partial M}{\partial y} = 4y$ $\qquad \frac{\partial N}{\partial x} = 2y$

not exact

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(4y - 2y)$

$= \frac{1}{2xy}(2y) = \frac{1}{x}$ (only x)

$\therefore f(x) = \frac{1}{x}$ (form of x alone)

$\therefore$ IF is,
$e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$\times$ ⓐ in ①,

$x(x^2 - 2x + 2y^2)\, dx + x(2xy)\, dy = 0$.

$(x^3 - 2x^2 + 2y^2 x)\, dx + (2x^2 y)\, dy = 0$ ——②

exact DE

Now solve,

$M_1 = x^3 - 2x^2 + 2y^2 x$, $N_1 = 2x^2 y$

$\int M_1\, dx = \int x^3 - 2x^2 + 2y^2 x.\, dx$.

$= \frac{x^4}{4} - 2\frac{x^3}{3} + \cancel{2}y^2 \frac{x^2}{\cancel{2}}$

$= \frac{x^4}{4} - \frac{2x^3}{3} + y^2 x^2$ .....

$N_1 = 2x^2 y$ w.r.t x, is 0

no x, so, 0

$\therefore$ Sol^n is,

$$\frac{x^4}{4} - \frac{2x^3}{3} + y^2 x^2 = c$$

2) Solve: $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0$. ——①

$M = xy^3 + y$ $N = 2x^2y^2 + 2x + 2y^4$.

$\frac{\partial M}{\partial y} = 3xy^2 + 1$ $\frac{\partial N}{\partial x} = 4xy^2 + 2$

not exact //

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x^2y^2 + 2x + 2y^4}\left((3xy^2 + 1) - (4xy^2 + 2)\right)$

$= \frac{1}{2x^2y^2 + 2x + 2y^4}\left(-xy^2 - 1\right)$ not only x y also there so ✗

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3 + y}\left((4xy^2 + 2) - (3xy^2 + 1)\right)$

$= \frac{1}{xy^3 + y}\left(xy^2 + 1\right) = \frac{\cancel{xy^2+1}}{y(\cancel{xy^2+1})} = \frac{1}{y}$

$\therefore f(y) = \frac{1}{y}$ (y alone)

$\therefore$ IF is,

$e^{\int f(y)dy} = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$ ——ⓐ

$\times$ ⓐ by ①

$y(xy^3 + y)\, dx + (2x^2y^2 + 2x + 2y^4)y\, dy = 0$.

$\Rightarrow (xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$.

exact ✓ $\longrightarrow N_1$.

Solve,

$\int M_1\, dx = \int xy^4 + y^2\, dx = \frac{x^2}{2} y^4 + x y^2$.

$N_1 = 2x^2y^3 + 2xy + \underline{2y^5}$. w.r.t x.

$\Rightarrow \int 2y^5 = 2\int y^5 = 2\frac{y^6}{6}$

$\therefore$ Sol^n,

$$y^4 \frac{x^2}{2} + xy^2 + 2\frac{y^6}{6} = c.$$