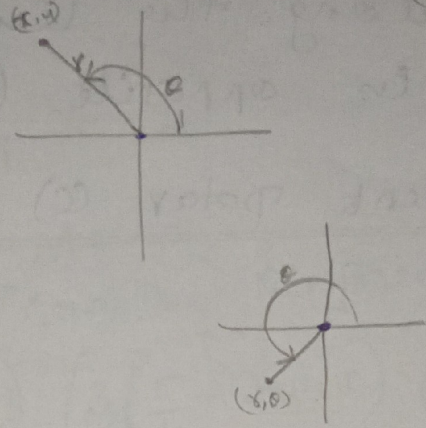
Module - I

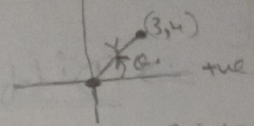
01: Polar Coordinates

$(x,y) \rightarrow$ cartesian coordinates

$(r,\theta) \rightarrow$ polar. C

(+ve to -ve)





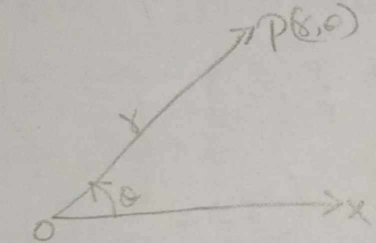
$r =$ dis to the point

$\theta = 5$

anticlock $\rightarrow$ +ve.

clockwise $\rightarrow$ -ve

1) consider the figure

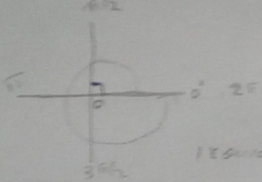


* we fix a point 'O' $\rightarrow$ pole / origin
* OX $\rightarrow$ polar axis / Initial ray / Initial line.
* let 'p' be any point in the plane the directed distance from O to P $\rightarrow$ radius vector. & denoted by 'r'.
* The directed angle from initial ray to ray OP $\rightarrow$ vectorial angle. (θ)
* P denoted by ordered pair (r,θ) where r & θ taken together $\rightarrow$ polar coordinates of P.

* x plane that is entered with a polar (C) system → $r\theta$ plane.
* r considered +ve if measured from '0' along the line, bounding the θ & -ve in opposite (dir).

⇒ **Different polar (C) of a point :-**

$\pi/3 = 60°$ $\pi/2 = 90°$
$\pi/4 = 45°$ $\pi = 180°$
$3\pi/2 = 270°$
$\pi/6 = 30°$ $2\pi = 360°$

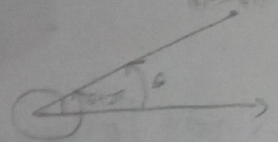


1 round = 360°

The point (r, θ) can also be written as $\boxed{(r, \theta + 2n\pi)}$

$n=1$ $\theta + 2\times1\pi \to \theta + 2\pi$ — 1 rotation

$n=2$ $\theta + 4\pi$ → 2 rotation



* If $r > 0$, if $P(r, \theta)$ is the terminal side of θ
* If $r < 0$, then $P(r, \theta)$ is on the opposite of the terminal side of θ, & at a distance of $|r|$ from the pole.
* The angle θ considered +ve if it is measured into counter clock wise (dis). & is -ve in clock wise (dis).
* The polar coordinate (r, θ) can also written as $\boxed{(-r, \theta + (2n+1)\pi)}$.

$n=1$
$-r, \theta + (2+1)\pi$
$= (-r, \theta + 3\pi)$

$n=2$
$(-r, \theta + 5\pi)$

Q/1) Find p.(C) of the points $(3, \frac{\pi}{4})$

A) The general representation of p.(C) is,
$(r, \theta + 2n\pi)$ & $(-r, \theta + (2n+1)\pi)$

$(3, \pi/4)$ $r = 3$
$\theta = \frac{\pi}{4}$.

for $n = 0, \pm1, \pm2 \ldots$
$(r, \theta + 2n\pi)$ can be represented by.

$n=0 \to \left(3, \frac{\pi}{4} + 2\times0\pi\right)$ $n=1$ $\left(3, \frac{\pi}{4} + 2\times1\times\pi\right)$ $n=2$ $\left(3, \frac{\pi}{4} + 2\times2\pi\right)$

$n=-1$ and $\left(3, \frac{\pi}{4} \times 2\times-1\times\pi\right)$ $\left(3, \frac{\pi}{4} \times 2\times-2\times\pi\right)$

after simplifying, we get.

$\to \left(3, \frac{\pi}{4}\right)$ $\left(3, \frac{\pi}{4} \pm 2\pi\right)$ $\left(3, \frac{\pi}{4} \pm 4\pi\right)$

$\left(3, \frac{\pi}{4} + 6\pi\right)$ .....

missing exam no. 1

$\to \left(-r, \theta + 2(n+1)\pi\right)$

$n=0$ $\left(-3, \frac{\pi}{4} + (2\times0+1)\times\pi\right)$ $n=1$ $\left(-3, \frac{\pi}{4} + (2\times1+1)\times\pi\right)$ $n=2$ $\left(-3, \frac{\pi}{4} + (2\times2+1)\pi\right)$ ....

and $\left(-3, \frac{\pi}{4} + (2\times-1+1)\times\pi\right)$ $n=-1$ $\left(-3, \frac{\pi}{4} + (2\times-2+1)\pi\right)$ ....

after simplifying

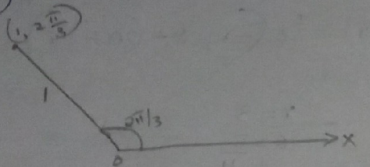
$\left(-3, \frac{\pi}{4} \pm \pi\right)$ $\left(-3, \frac{\pi}{4} \pm 3\pi\right)$ $\left(-3, \frac{\pi}{4} \pm 5\pi\right)$ ....

2) Plot the given points in (r, θ) plane.

a) $(1, \frac{2\pi}{3})$

$\frac{\pi}{3} = 60^\circ$

$\frac{2\pi}{3} = 2 \times 60 = 120^\circ$

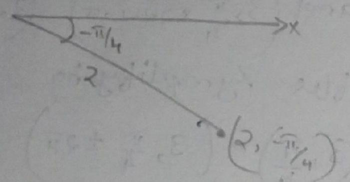


b) $(2, -\frac{\pi}{4})$

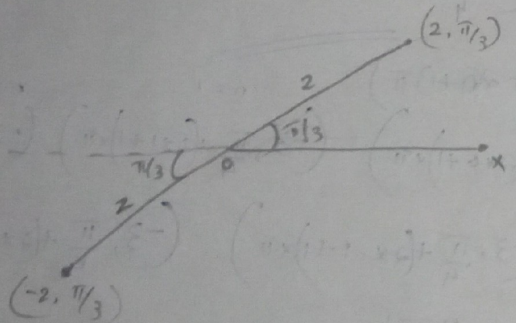
$\frac{\pi}{4} = 45^\circ$

-45°

(r always +ve)



c) $(-2, \frac{\pi}{3})$



3) Graph set of points whose p.c satisfies the condition

a) $0 \le r \le 1$

([illegible], angle [illegible] full angle [illegible])

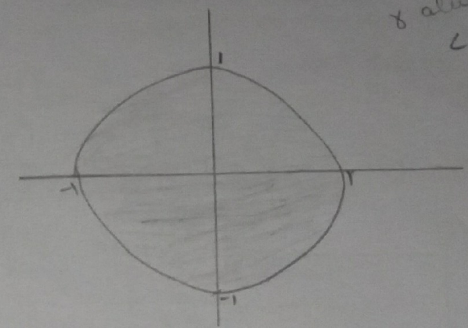
So draw

O, $r = 1$.

$0 \to 1$

①

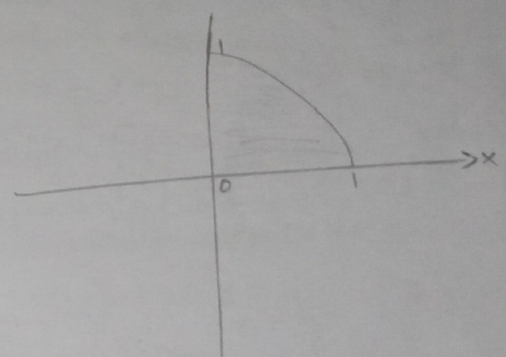
r always < 1.



b) $0 \le r \le 1$

$0 \le \frac{\pi}{2}$

$\pi/2 = 90^\circ$



$\pi = 180°$

$\pi/2 = 90°$

$\pi/3 = 60°$

$\pi/4 = 45°$

$3\pi/2 = 270°$

$2\pi = 360°$

$\pi/6 = 30°$

$\sin\theta \cos\theta = 1$

$\cos^2\theta + \sin^2\theta = 1$

$\cos\theta + \sin\theta = \sqrt{2}$

$\cos\theta - \sin\theta = 0$

$\cos(-x) = \cos x$

		$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
	$0°$	$30°$	$45°$	60	$90°$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—

cartition ~~co-or~~ coordinates & polar ~~co-or~~ coordinates:-

(x, y) denotes cartition coordinate

& $(r, \theta) \to$ polar (C).

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \to (r \,\&\, \theta \text{ given})$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}} \to (x \,\&\, y \text{ given})$$

(or) $\tan\theta = \frac{y}{x}$.

2)
1) The point $(4, \pi/6)$ is in polar (C). find its representation in $\square$ (C).

(r, θ)

rectangular (C) = cartition (C).

A) $(4, \pi/6)$ $\quad r = 4$, $\theta = \frac{\pi}{6}$

$\frac{\pi}{6} = 30°$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$x = r\cos\theta$

$= 4\cos(\pi/6)$

$= \cancel{4}^{2} \times \frac{\sqrt{3}}{\cancel{2}} = \underline{\underline{2\sqrt{3}}}$

$\sin\frac{\pi}{6} = \frac{1}{2}$

$y = r\sin\theta = 4\sin\left(\frac{\pi}{6}\right)$

$= 4 \times \frac{1}{2} = \underline{\underline{2}}$

$\therefore$ rectangular (C) is $(x, y) = (2\sqrt{3}, 2)$

2) The point $(-1,1)$ is given □'r (C). find its P. (C).

A) $(-1,1)$ $\quad x=-1$, $y=1$

$r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+(1)^2}=\sqrt{1+1}$

$\theta=\tan^{-1}\left(\frac{y}{x}\right)$

$=\tan^{-1}\left(\frac{-1}{1}\right)$

$\theta=\tan^{-1}(-1)$

$=\frac{3\pi}{4}$.

$\therefore (r,\theta)=\left(\sqrt{2},\frac{3\pi}{4}\right)$

3) Replace the following cartition eq into polar eq.

1) $y^2=4x$. $\quad x=r\cos\theta$, $y=r\sin\theta$

A) $(r\sin\theta)^2=4r\cos\theta$.

$r^2\sin^2\theta=4r\cos\theta$

$r\sin^2\theta=4\cos\theta$ (ans)

$r\sin^2\theta-4\cos\theta=0$

2) $(x-3)^2-(y+1)^2=4$

$(r\cos\theta-3)^2-(r\sin\theta+1)^2=4$

$(r\cos\theta)^2-2r\cos\theta\times 3+(3)^2-\left[(r\sin\theta)^2+2r\sin\theta\times 1+1^2\right]=4$

$r^2\cos^2\theta-6r\cos\theta+9-r^2\sin^2\theta-2r\sin\theta-1=4$.

$r^2\cos^2\theta-6r\cos\theta+9-r^2\sin^2\theta-2r\sin\theta-1-4=0$

$r^2\cos^2\theta-r^2\sin^2\theta-6r\cos\theta-2r\sin\theta+4=0$

3) $\underset{(a-b)^2}{(x-3)^2}+\underset{(a+b)^2}{(y+1)^2}=4$.

$x^2-2x\times 3+3^2+y^2+2\times y\times 1+1^2=4$

$x^2-6x+9+y^2+2y+1=4$.

$\Rightarrow x^2+y^2-6x+9+2y+1-4=0$.

$\Rightarrow x^2+y^2-6x+2y+6=0$

$x=r\cos\theta$, $y=r\sin\theta$

$\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2-6(r\cos\theta)+2(r\sin\theta)+6=0$

$r^2\cos^2\theta+r^2\sin^2\theta-6r\cos\theta+2r\sin\theta+6=0$

$r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}-6r\cos\theta+2r\sin\theta+6=0$

$r^2-6r\cos\theta+2r\sin\theta+6=0$

4) Replace the p. eq into cartition

1) $r\sin\theta = -1$ $\qquad y = r\sin\theta$

$\Rightarrow$ $\underline{y = -1}$

2) $r^2 = 4r\sin\theta$ $\qquad y = r\sin\theta$

$r^2 = 4y$ $\qquad r = \sqrt{x^2+y^2}$

$\qquad r^2 = x^2+y^2$

$x^2+y^2 = 4y$

$x^2+y^2-4y = 0$

3) $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$r(\sin\theta - 2\cos\theta) = 5$

$r\sin\theta - 2r\cos\theta = 5$

$y - 2x = 5$

$y - 2x - 5 = 0$

(polar axis $\to x$)

$\Rightarrow$ Graphing in polar coordinates :-

a) Symmetry :-

* Symmetry about the polar axis (

$y = f(x)$;, $\to$ $\boxed{r = f(\theta)}$

The graph of $r = f(\theta)$ is symmetric with respect of polar (x) , if $\boxed{f(\theta) = f(-\theta)}$ (x axis) $f(\theta) = f(-\theta)$ then we can say $r = f(\theta)$

In other words if the point (r, θ) lies on the graph, then the point $(r, \bar{\theta})$ or $(-r, \pi-\theta)$ lies on the graph

eg = $(2, \frac{\pi}{3}) \to (2, -\frac{\pi}{3})$ or $(-2, \pi - \frac{\pi}{3})$

* Symmetry about the vertical line :-

$\theta = \dfrac{\pi}{2}$ (y axis)

The graph of $r = f(\theta)$ is symmetric with respect vertical line $\theta = \frac{\pi}{2}$ if $\boxed{f(\theta) = f(\pi - \theta)}$ (y axis)

In other words if (r, θ) lies on the graph, the point $(r, \pi-\theta)$ or $(-r, -\theta)$ also lies on the graph.

* Symmetry about the pole (origin) :-

The graph of $r = f(\theta)$ is symmetric with respect to origin, if $\boxed{f(\theta) = f(\pi + \theta)}$

In other words if the point (r, θ) lies on the graph then, the points $(-r, \theta)$ or $(r, \pi+\theta)$ lies on the graph.

* 1) Slope of a (P) curve :-

slope of a (P) curve $r = f(\theta)$ is given by $\frac{dy}{dx}$ where $x = r\cos\theta$ &
$y = r\sin\theta$

$$\therefore \boxed{\frac{dy}{dx} = \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}}$$

Q/
i) find the slope of line tangent to the graph of $r = 3\cos^2 2\theta$ at $\theta = \pi/6$ $(30°)$

A) $r = 3\cos^2 2\theta \longrightarrow r = f(\theta)$

$f(\theta) = 3\cos^2 2\theta$.

then slope $= \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}$.

$f(\theta) = 3\cos^2 2\theta$

~~$f'(\theta) = 3$~~

$\cos^2 2\theta - 3(\cos 2\theta)^2$

$\frac{d}{dx}(\cos x) = -\sin x$

$(\cos x)^2 =$

$2\cos x \times -\sin x$

$= -2\cos x \sin x$

$(\cos 2x)^2$

$= 2(\cos 2x)$.

$-\sin 2x \cdot 2$

$\frac{d}{dx}(\cos 2x) =$

$-\sin 2x \cdot 2$

$= -\sin 2x$

$f'(\theta) = 3 \times (\cos 2\theta)^2$

$= 3 \times 2(\cos 2\theta) \times \cancel{-\sin 2\theta \cdot 2}\ \frac{d}{dx}(\cos 2\theta)$

$= 3 \cdot 2(\cos 2\theta) \times -\sin(2\theta) \cdot \frac{d}{dx}(2\theta)$

$= 3 \cdot 2(\cos 2\theta) \times -\sin 2\theta \cdot 2$

$= -12\cos 2\theta \times \sin 2\theta$

$\cos 2x = -\sin 2x \times 2$

$\sin(\cos x) = \cos(\cos x) \times -\sin x$

$\cos x^2 = -\sin(x^2) \times 2x$

$\tan\theta = \tan \pi/6$

$= 1/\sqrt{3}$

$\therefore \frac{dy}{dx} = \cancel{-12\cos 2\theta\ \sin 2\theta \times \frac{1}{\sqrt{3}} + 3\cos}$

$f'(\theta) = f'(\pi/6) = -12 \times \cos\left(2 \cdot \frac{\pi}{6}\right) \cdot \sin\left(2 \cdot \frac{\pi}{6}\right)$

$= -12 \cdot \cos\frac{\pi}{3} \cdot \sin\frac{\pi}{3}$

$= -\frac{12}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$.

$= -3\sqrt{3}$

$f(\theta) = f\left(\frac{\pi}{6}\right) = 3(\cos 2\theta)^2$

$= 3\left(\cos\left(2\frac{\pi}{6}\right)\right)^2$

$= 3\left(\cos\frac{\pi}{3}\right)^2 = 3\left(\frac{1}{2}\right)^2$

$= 3 \cdot \frac{1}{4} = \frac{3}{4}$

$$\frac{dy}{dx} = \frac{f'(\pi/6)\tan\frac{\pi}{6} + f\left(\frac{\pi}{6}\right)}{f'(\pi/6) - f\left(\frac{\pi}{6}\right)\tan\frac{\pi}{6}}$$

$$= \frac{-3\sqrt{3} \times \frac{1}{\sqrt{3}} + \frac{3}{4}}{-3\sqrt{3} - \frac{3}{4} \cdot \frac{1}{\sqrt{3}}}$$

$= \cancel{-3 + \frac{3}{4}}$

(12)

$$= \frac{-3 + \frac{3}{4}}{-3\sqrt{3} - \frac{3}{4\sqrt{3}}} = \frac{\frac{-12+3}{4}}{\frac{-3\sqrt{3}\cdot 4\sqrt{3} - 3}{4\sqrt{3}}}$$

$\sqrt{3}\cdot\sqrt{3} = 3$

$$= \frac{\frac{9}{4}}{\frac{-3\times 4\times 3 - 3}{4\sqrt{3}}}$$

$$= \frac{-9}{4} \cdot \frac{4\sqrt{3}}{-39} = \frac{-9(4\sqrt{3})}{4(-39)}$$

$$= \frac{-3\sqrt{3}}{-13} = +\frac{3\sqrt{3}}{+13}$$

$$\frac{-9(4\sqrt{3})}{-4(39)} = \frac{36\sqrt{3} \div 4}{156 \div 4} = \frac{9\sqrt{3} \div 3}{39 \div 3} = \frac{3\sqrt{3}}{13}$$

$$= \frac{3\sqrt{3}}{13}$$

2) find slope of line to the graph of $r = 3\sin\theta + \cos(\theta^2)$ at $\theta = 0$.

A) $$\frac{dy}{dx} = \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta} \quad \text{—(1)}$$

$f(\theta) = 3\sin\theta + \cos(\theta^2)$

At $\theta = 0$ $f(0) = 3\sin 0 + \cos(0^2)$ $\quad \cos 0 = 1$.

$= 3\times 0 + 1 = \underline{1}$

$f'(\theta) = 3\cos\theta + -\sin(\theta^2)\cdot\frac{d}{dx}(\theta^2)$

$= 3\cos\theta + -\sin(\theta^2)\cdot 2\theta$.

$= 3\cos\theta - 2\theta\sin(\theta^2)$.

At $\theta = 0$

$f'(0) = 3\cos 0 - 2\times 0\sin(0^2)$

$= 3\times 1 - 0 = \underline{3}$

$\tan\theta = \tan 0 = \underline{0}$

Rule —(1)

$$\frac{dy}{dx} = \frac{3\times 0 + 1}{3 - 1\times 0} = \underline{\frac{1}{3}}$$

3) Sketch the graph of the (P) eq $r = 2\cos(2\theta)$.

$\boxed{\cos(-x) = \cos x}$

A) $r = 2\cos(2\theta)$.

$f(\theta) = 2\cos(2\theta)$.

1st symmetry / not. (x, y).

$f(\theta) = f(\theta)$

$f(-\theta) = 2\cdot\cos(-2\theta)$

$= 2\cos 2\theta. = \underline{f(\theta)}$

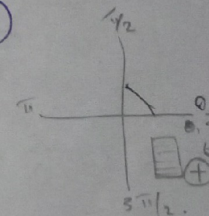
and $f(\pi - \theta) = 2\cos(2(\pi - \theta))$.

$(2\pi - 2\theta = 2\theta)$

$= 2\cos(2\pi - 2\theta)$

$= 2\cos 2\theta$

$= \underline{f(\theta)}$



Graph is symmetric about both. x axis & ~~y axis~~ vertical line $\theta = \frac{\pi}{2}$.

∴ To obtain the graph for $\underline{0 \le \theta \le \frac{\pi}{2}}$

Let us make a table of values of r & θ from 0 to $\pi/2$.

$0 - \frac{\pi}{2}$.

$0, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

$r = 2\cos(2\theta)$

At $\theta = 0$

$r = 2\cos(2 \times 0)$

$2\cos 0 = 2 \times 1 = 2$.

$\theta = \frac{\pi}{4}$

$2\cos 2\frac{\pi}{8_4}$

$2\cos\frac{\pi}{4} \to 2 \cdot \frac{1}{\sqrt{2}}$

$2 \to \sqrt{2} \times \sqrt{2}$. $\Rightarrow \frac{2}{\sqrt{2}} = \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{2}} = \sqrt{2}$

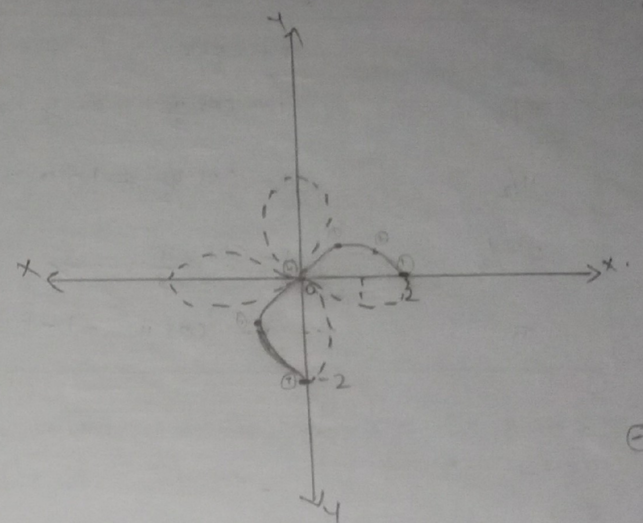
$2\cos\left(2\frac{\pi}{6_3}\right)$

$2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$

$2\cos\left(2\frac{\pi}{3}\right) = -1$

$\boxed{\begin{array}{l}\cos\pi = -1 \\ \sin\pi = 0\end{array}}$

θ	$r = 2\cos 2\theta$.
0	2.
$\frac{\pi}{8}$	$2\cos\left(2\frac{\pi}{8}\right) = \sqrt{2}$
$\frac{\pi}{6}$ (30°)	$2\cos\left(2\frac{\pi}{6}\right) = 1$
$\frac{\pi}{4}$ (45°)	$2\cos\left(2\frac{\pi}{4_2}\right)$ $2\cos\frac{\pi}{2} = 0$
$\frac{\pi}{3}$ (60°)	$2\cos\left(2\frac{\pi}{3}\right) = -1$
$\frac{3\pi}{8}$	$-\sqrt{2}$
$\pi/2$	-2



This graph $\to$ four leaved rose ($r = 2\cos 2\theta$)

4) Graph the curve $r = 1 - \cos\theta$

A) Let $r = f(\theta) = 1 - \cos\theta$

Sub $(-\theta)$,

$f(-\theta) = 1 - \cos(-\theta) = 1 - \cos\theta = f(\theta)$

(always +ve)

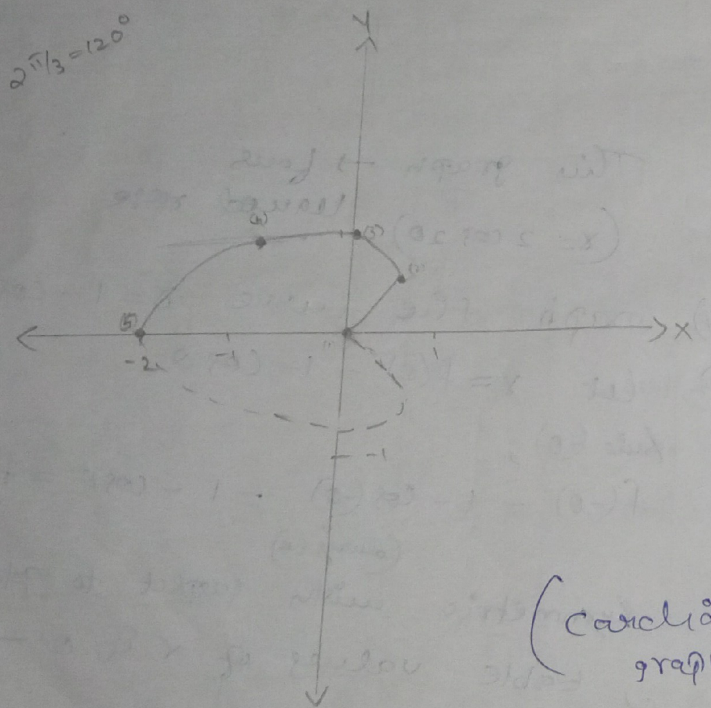
Symmetric with respect to polar axis.

$\therefore$ table values of r & θ —

θ	$r = 1-\cos\theta$
0	$1-\cos 0 = 1-1 = 0$
$\pi/3$ 60°	$1-\cos\frac{\pi}{3} = 1-\frac{1}{2} = \frac{1}{2}$
$\pi/2$	$1-\cos\frac{\pi}{2} = 1-0 = 1$
$2\pi/3$	$1-\cos\frac{2\pi}{3} = \frac{3}{2}$
π	$1-\cos\pi = 1-(-1) = \underline{\underline{2}}$

$\cos\pi = -1$

$2\pi/3 = 120^\circ$



(cardiac graph)

Q) graph the curve $r^2 = \sin 2\theta$ (leminiscate).

5) find the polar coordinate of $(x,y) = (5,-2)$ $r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}$

$\left(\sqrt{29}, \tan^{-1}\left(\frac{-2}{5}\right)\right) = (5.38, -0.38)$

6) find cartition coordinate $(r,\theta) = \left(2, \frac{\pi}{6}\right)$ $(\sqrt{3}, 1)$

7) Replace the following eq by equivalent (p) eq $\rightarrow x^2+xy+y^2 = 1$

$r^2(2+\sin 2\theta) = 2$

8) Replace the follow (p) eq by equivalent cartition eq $\rightarrow r = 1-2\cos\theta$

9) Find slope of the curve $r = 2-\sin\theta$ at $\theta = \pi/6$ $-1/\sqrt{3}$

8.A) $r = 1-2\cos\theta$. $\times r$ on LHS & RHS

$r\times r = 1-2\,\underline{r\cos\theta}$

$r^2 = 1-2x$

$x^2+y^2 = 1-2x$

$x^2+y^2-1+2x = 0$

$\underline{\underline{x^2+2x+y^2-1 = 0}}$