Chapter : 03

## Mathematical Expectation. (g R.v)

E → Expectation

$E(x^2) = \sum x^2 f(x)$.

$E(x) = \sum x f(x)$

$E(x) = \int x f(x)$

$E(y) = \sum y f(y)$

$= \int y f(y)$.

$E(g(x,y)) = \sum_x \sum_y g(x,y) f(x,y)$

$= \int_x \int_y g(x,y) f(x,y) dx dy$

Let $g(x,y)$ be a real () such that $E(g(x,y)) < \infty$. In the case of

discreate R.v $x$ & $y$, $\cancel{E\, g(x,y)}$

$$\boxed{E(g(x,y)) = \sum_x \sum_y g(x,y) f(x,y)}$$

contin R.v $x$ & $y$ with joint pdf,

$$\boxed{E(g(x,y)) = \int_x \int_y g(x,y) \cdot f(x,y) dx dy.}$$

→ Addition Theorem on (E) :-

$E(x) < \infty$ & $E(y) < \infty$

$$\boxed{E(x+y) = E(x) + E(y)}$$

① — $E(x) = \sum x f(x)$

② — $E(y) = \sum y f(y)$

③ $f_1(x) = \sum_y f(x,y)$

$= \int_y f(x,y) dy$

④ $f_2(x) = \sum_x f(x,y)$

$= \int_x f(x,y)$

**Proof**

(dis)

$E(x+y) = \sum_x \sum_y (x+y) \cdot f(x,y)$

$= \sum_x \sum_y (x \cdot f(x,y) + y \cdot f(x,y))$

$= \sum_x \sum_y x \cdot f(x,y) + \sum_x \sum_y y \cdot f(x,y)$

$= \sum_x x \underbrace{\sum_y f(x,y)}_{③} + \sum_y y \cdot \underbrace{\sum_x f(x,y)}_{④}$

$= \sum_x x f_1(x) + \sum_y y f_2(y)$

$= \underbrace{\sum_x x f(x)}_{①} + \underbrace{\sum_y y f(y)}_{②}$

$E(x+y) = E(x) + E(y)$

**Proof**

contin

* If $x$ & $y$ are contin with j.pdf.

$E(x+y) = \int_x \int_y (x+y) f(x,y) dx dy$.

$= \int_x \int_y x f(x,y) + y \cdot f(x,y) dx dy$

$= \int_x \int_y x \cdot f(x,y) + \int_x \int_y y \cdot f(x,y) dx dy$

$= \int_x x \left[\int_y f(x,y) dy\right] dx + \int_y y \left[\int_x f(x,y) dx\right] dy$

$E(x) = \int x f(x) dx$

$E(y) = \int y f(y) dy$

$= \int_x x f_1(x) dx + \int_y y f_2(y) dy$.

$E(x+y) = E(x) + E(y)$.

★ The (pr.) of additivity extends immediately to any finite no. of R.V with finite (E)

$E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) \dots + E(X_n)$

$$\boxed{E\left(\sum_{i=1}^{n} X_i\right) = \sum E(X_i)} \qquad (\Sigma \to \text{add / sum})$$

→ Multiplication Theorem :-

If $x$ & $y$ are independent. R.V with $E(x) < \infty$ & $E(y) < \infty$ ∴ & $E(xy) < \infty$.

$$\boxed{E(xy) = E(x) \cdot E(y)}.$$

proof

$E(xy) = \sum_x \sum_y (xy) \, f(x,y)$ (independent) $f(x,y) = f_1(x) \cdot f_2(y)$

(dis.) → $= \sum_x \underline{\sum_y (xy)} \, f_1(x) \cdot f_2(y)$

$= \sum_x x \, f_1(x) \sum_y y \, f_2(y)$

$E(xy) = E(x) \cdot E(y)$ $\qquad \sum x \, f(x) = E(x)$

$\sum y \, f(y) = E(y)$

proof

$E(xy) = \int_x \int_y (xy) \cdot f(x,y) \, dx \, dy.$

(contin) → $= \int_x \int_y (xy) \cdot f_1(x) \cdot f_2(y) \, dx \, dy$

$\cancel{= \int_x x \int_y (x, y)}$

$= \int_x x \cdot f_1(x) \cdot dx \cdot \int_y y \, f_2(y) \cdot dy.$

$E(xy) = E(x) \cdot E(y)$

★ This result can be extended to a finite no. of independent R.V,

If $X_1, X_2 \dots X_n$ are $n$ independent R.V then,

$E(X_1, X_2, X_3) \dots = E(X_1) \cdot E(X_2) \cdot E(X_3) \dots$

$$\boxed{E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)} \qquad (\Pi \to \text{product})$$

→ Theorem :-

If $x$ & $y$ are 2 indepnt R.V then,

$$\boxed{M_{x+y}^{(t)} = M_x^{(t)} \cdot M_y^{(t)}}$$

M → Moment generating (). (mgf)

$M_x^{(t)} = E(e^{tx})$

$M_y^{(t)} = E(e^{ty})$.

proof

$M_{x+y}^{(t)} = E(e^{t(x+y)})$ $\qquad e^{a+b} = e^a \cdot e^b$

$= E(e^{tx+ty})$

$= E(e^{tx} \cdot e^{ty})$

$= E(e^{tx}) \cdot E(e^{ty})$

$M_{x+y}^{(t)} = M_x^{(t)} \cdot M_y^{(t)}$ //

→ **Theorem:-**

If $x_1, x_2 \ldots x_n$ are $n$ indepnt R.v then

$$\boxed{M^{(t)}_{\sum_{i=1}^{n} x_i} = \prod_{i=1}^{n} M^{(t)}_{x_i}}$$

**Proof**

$$M^{(t)}_{\sum_{i=1}^{n} x_i} = E\left(e^{t\sum_{i=1}^{n} x_i}\right)$$

$$= E\left(e^{\sum_{i=1}^{n} t x_i}\right)$$

$$= E\left(\prod_{i=1}^{n} e^{t x_i}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{t x_i}\right)$$

$$M_{\sum_{i=1}^{n} X_i} = \prod_{i=1}^{n} M^{(t)}_{x_i}$$

$e^{x+y} = e^{x} \cdot e^{y}$

$e^{\sum x_i} = \prod e^{t x_i}$

$M^{(t)}_{x} = E(e^{tx})$

$M^{(t)}_{x_i} = E(e^{t x_i})$

* **Result 1** →

For a pair of R.v $(x, y)$ the covariance b/w $x$ & $y$ is defined as,

$$\mathrm{cov}(x, y) = E\left[(x - E(x)) \cdot (y - E(y))\right]$$

* **Result 2** →

$$\mathrm{cov}(x, y) = E(x, y) - E(x) \cdot E(y).$$

**Proof**

$$\mathrm{cov}(x, y) = E\left[(x - E(x)) \cdot (y - E(y))\right]$$

$$= E\left[xy - x E(y) - y E(x) + E(x) \cdot E(y)\right]$$

$$= E(xy) - E(x\,E y) - E(y\,E(x)) + E(E(x) \cdot E(y))$$

$$= E(xy) - E(x) \cdot E(y) - \cancel{E(y) E(x)} + \cancel{E(x) \cdot E(y)}$$

$$\mathrm{cov}(x, y) = E(x \cdot y) - E(x) \cdot E(y)$$

* **Result 3 :-**

If $x$ & $y$ are indepndt R.v then

$\mathrm{cov}(x, y) = 0$

$f(x, y) = f(x) \cdot f(y)$

$E(x, y) = E(x) \cdot E(y)$

**Proof**

$$\mathrm{cov}(x, y) = E(x, y) - E(x) \cdot E(y).$$

$$= E(x) \cdot E(y) - E(x) \cdot E(y)$$

$$\mathrm{cov}(x, y) = 0$$

* **Result 4 :-**

The correlation (coe) b/w the 2 R.v $x$ & $y$ is defined as,

$$\boxed{\rho_{xy} = \frac{\mathrm{cov}(x, y)}{\sqrt{V(x)} \cdot \sqrt{V(y)}}}$$

$$\mathrm{cov}(x, y) = E(x, y) - E(x) \cdot E(y)$$

$V(x) = E(x - E(x))^2$ (or) $V(x) = E(x^2) - (E(x))^2$

$V(y) = E(y - E(y))^2$ (or) $V(y) = E(y^2) - (E(y))^2$

* Result 5 :-

For any 2 r.v $x$ & $y$

$$[E(xy)]^2 \leq E(x^2) \cdot E(y^2)$$

↳ [Schwartz's Inequality]

Proof when $\lambda$ is constant real no., the variable $(\lambda x - y)^2 \geq 0$ (non -ve).

$\lambda \to$ real no.

$$(\lambda x - y)^2 \geq 0$$

$$\therefore E(\lambda x - y)^2 \geq 0$$

$(a-b)^2$

$$\Rightarrow E[\lambda^2 x^2 - 2\lambda xy + y^2] \geq 0$$

$$\Rightarrow E(\lambda^2 x^2) - E(2\lambda xy) + E(y^2) \geq 0$$

$$\Rightarrow \lambda^2 E(x^2) - 2\lambda E(xy) + E(y^2) \geq 0$$

$$\Rightarrow \underbrace{E(x^2)}_{a} \lambda^2 - \underbrace{2E(xy)}_{b} \lambda + \underbrace{E(y^2)}_{c} \geq 0$$

This is quadratic $(ax^2 + bx + c = 0 \to x = \lambda)$ (quadratic eq) in $\lambda \geq 0$

sub this eq in $b^2 - 4ac \to$ discriminent must be -ve

$$b^2 - 4ac \leq 0$$

$$(2E(xy))^2 - 4E(x^2) \cdot E(y^2) \leq 0$$

$$4E(xy)^2 - 4E(x^2) \cdot E(y^2) \leq 0$$

$$\cancel{4} E(xy)^2 \leq \cancel{4} E(x^2) \cdot E(y^2)$$

$$(E(xy))^2 \leq E(x^2) \cdot E(y^2)$$

* Result 6 :-

If $(x, y)$ is a pair of R.v then

$$-1 \leq \rho_{xy} \leq 1$$

↳ corr (coe)

(R-4)

Proof

$$\rho_{xy} = \frac{\operatorname{cov}(x,y)}{\sqrt{V(x)}\sqrt{V(y)}}$$

$$= \frac{E[(x - E(x))(y - E(y))]}{\sqrt{E(x - E(x))^2} \cdot \sqrt{E(y - E(y))^2}}$$

$V(x) = E(x^2) - (E(x))^2$

$= E(x - E(x))^2$

$V(y) = E(y - E(y))^2$

let $x = x - E(x)$

$y = y - E(y)$.

$$\therefore \rho_{xy} = \frac{E(x \cdot y)}{\sqrt{E(x^2)} \cdot \sqrt{E(y^2)}}$$

Squaring both sides,

$$\rho^2_{xy} = \frac{(E(x,y))^2}{(\sqrt{E(x^2)})^2 (\sqrt{E(y^2)})^2}$$

$$\rho^2_{xy} = \frac{(E(x,y))^2}{E(x^2) \cdot E(y^2)} \leq 1$$

$$\rho^2_{xy} \leq 1$$

also $|\rho_{xy}| \leq 1$

$$-1 \leq \rho_{xy} \leq 1$$

$|x| \leq 1$

$\to -1 \leq x \leq 1$

$|y| \leq 1$

$\to -1 \leq y \leq 1$

* **Result 7** :-

$\rho_{xy} = \pm 1$ only if there exist a ( ) connecting $x$ & $y$ of the form

$y = ax + b$

$a$ & $b$ are constant.

* **Result 8** :-

If $x$ & $y$ are 2 R.V

$$V(ax+by) = a^2 V(x) + b^2 V(y) + 2ab \operatorname{cov}(x,y)$$

Proof

$$V(ax+by) = E[ax+by - E(ax+by)]^2$$

$V(x) = E(x - E(x))^2$

$$= E[ax + by - E\,a(x) - E\,b(y)]^2$$

$$= E[ax + by - aE(x) - bE(y)]^2$$

$$= E[\underbrace{a(x-E(x))}_{(a+b)^2} + \underbrace{b(y-E(y))}_{b}]^2$$

$$= E[a^2(x-E(x))^2 + b^2(y-E(y))^2 + 2ab\,(x-E(x)).(y-E(y))]$$

$$= a^2 E(x-E(x))^2 + b^2 E(y-E(y))^2 + 2ab\,E[(x-E(x)).(y-E(y))]$$

$$V(ax+by) = a^2 V(x) + b^2 V(y) + 2ab \operatorname{cov}(x,y)$$

Note,

when $x$ & $y$ are independent,

a) $V(ax+by) = a^2 V(x) + b^2 V(y)$. (since $\operatorname{cov}(x,y) = 0$. (independent).)

b) when $a=1$, $b=1$

$V(x+y) = V(x) + V(y) + 2 \operatorname{cov}(x,y)$

c) when $a=1$ $b=-1$

$V(x-y) = V(x) + V(y) - 2 \operatorname{cov}(x,y)$.

So $x$ & $y$ are independent,

$\therefore$ $V(x+y) = V(x) + V(y)$

$V(x-y) = V(x) + V(y)$.

* **Result 9** :-

**Mgf of the R.V**

Let $(x,y)$ be a pair of R.V, Then for any 2 real no. $t_1$ & $t_2$ the mgf of $(x,y)$ is defined as,

$$M_{x,y}(t_1,t_2) = E(e^{t_1x+t_2y}) \quad \text{as long as } E \text{ exist.}$$

(dis) $\rightarrow$ $= \sum_x \sum_y e^{t_1x+t_2y} . f(x,y)$ $\quad$ $E(x) = \sum x\, f(x,y)$

(contin) $\rightarrow$ $= \int_x \int_y e^{t_1x+t_2y} . f(x,y).dxdy$

→ Conditional (E) :-

Let $(x,y)$ be a pair of R.V with j pdf $f(x,y)$. Then the conditional (E) of $x$ given $Y=y$ is defined as,

$E(X|Y=y) = \sum_x x f(x|y)$. → (dis)

$= \int_x x f(x|y) \cdot dx$ → contin

≈, conditional (E) of $Y$ given $X=x$ is defined as,

$E(Y|X=x) = \sum_y y \, f(y|x)$ → dis

$= \int_y y \, f(y|x) \, dy$ → contin

→ Conditional variance :-

C.V of X given $Y=y$ is givn by,

$V(X|Y=y) = E(x^2/Y=y) - (E(x|Y=y))^2$

$V(x) = E(x^2) - (E(x))^2$

≈ C.V of Y givn $X=x$ is givn by

$V(Y|X=x) = E(y^2|X=x) - (E(Y|X=x))^2$

$V(Y) = E(Y) - (E(Y))^2$

Q.1) If $x$ & $y$ hve J pdf,

$f(x,y) = \frac{x+y}{21}$, $x=1,2,3$, $y=1,2$.

obtain (a) corre (coe) $\rho_{xy}$

(b) $E(X|Y=2)$ & $V(X|Y=2)$.

A) $f(x,y) = \frac{x+y}{21}$.

a) $\rho_{xy} = \frac{\text{cov}(x,y)}{\sqrt{V(x)}\sqrt{V(y)}}$ — ①

$\text{cov}(x,y) =$, $\sqrt{V(x)} =$, $\sqrt{V(y)} = $ ?

$\text{cov}(x,y) = E(xy) - E(x) \cdot E(y)$ — ②

$E(x) =$, $E(y) =$, $E(xy) =$ ?

$E(x) = \sum x \, f_1(x)$ — ③

$E(y) = \sum y \, f_2(y)$ — ④

$E(xy) = \sum xy \, f(x,y)$. — ⑤

$f_1(x)$, $f_2(y) = $ ?

$f_1(x) = \sum_y f(x,y)$

$= \sum_{y=1,2} \frac{x+y}{21}$

$= \frac{x+1}{21} + \frac{x+2}{21} = \frac{x+1+x+2}{21}$

$f_1(x) = \frac{2x+3}{21}$.

$f_2(y) = \sum_x f(x,y)$

$= \sum_{x=1,2,3} \frac{x+y}{21}$.

~~If x & y are joint pdf given by~~

$$f_2(y) = \frac{1+y + 2+y + 3+y}{21} = \frac{3y+6}{21}.$$

(3) $E(x) = \sum_{x=1,2,3} x \; f_1(x).$

$$= \sum_{x=1,2,3} x\left(\frac{2x+3}{21}\right)$$

$$= 1\left(\frac{2+3}{21}\right) + 2\left(\frac{4+3}{21}\right) + 3\left(\frac{6+3}{21}\right)$$

$$= 1\left(\frac{5}{21}\right) + 2\left(\frac{7}{21}\right) + 3\left(\frac{9}{21}\right).$$

$$= \frac{5+14+27}{21} = \frac{46}{21}$$

$$E(y) = \sum_{y=1,2} y \; f_2(y)$$

$$= \sum_{y=1,2} y\left(\frac{3y+6}{21}\right)$$

$$= 1\left(\frac{3+6}{21}\right) + 2\left(\frac{6+6}{21}\right)$$

$$= \frac{9+24}{21} = \frac{33}{21}$$

$\cancel{E(xy) = E(x) - E(y)}$

$$E(xy) = \sum_{\substack{x=1,2,3\\ y=1,2}} xy \; f(x,y)$$

$$= \sum xy\left(\frac{x+y}{21}\right)$$

$$= 1\cdot1\left(\frac{1+1}{21}\right) + 1\cdot2\left(\frac{1+2}{21}\right) +$$

$$2\cdot1\left(\frac{2+1}{21}\right) + 2\cdot2\left(\frac{2+2}{21}\right) +$$

$$3\cdot1\left(\frac{3+1}{21}\right) + 3\cdot2\left(\frac{3+2}{21}\right).$$

$$= \frac{2+6+6+16+12+30}{21}$$

$x \to$ 1 2 3

$y \to$ 1 2

$$E(xy) = \frac{72}{21}$$

(2) $\mathrm{cov}(x,y) = E(xy) - E(x) \cdot E(y).$

$$= \frac{72}{21} - \frac{46}{21}\cdot\frac{33}{21}$$

$$= \frac{72\times21}{21\times21} - \frac{1518}{441}.$$

$$= \frac{1512 - 1518}{441} = \frac{-6}{441}$$

$\sqrt{V(x)} = ?$

$V(x) = E(x^2) - (E(x))^2$ — (6)

$$E(x^2) = \sum_{x=1,2,3} x^2 \cdot f_1(x)$$

$$= \sum_{x=1,2,3} x^2 \cdot \left(\frac{2x+3}{21}\right)$$

$$= 1\cdot\left(\frac{2+3}{21}\right) + 4\cdot\left(\frac{4+3}{21}\right) + 9\cdot\left(\frac{6+3}{21}\right)$$

$$= \frac{5 + 28 + 81}{21} = \frac{114}{21}$$

$$E(x^2) = \frac{114}{21}$$

ⓘ $$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{114}{21} - \left(\frac{46}{21}\right)^2$$

$$= \frac{114 \times 21}{21 \times 21} - \frac{2116}{441} = \frac{2394 - 2116}{441}$$

$$V(x) = \frac{278}{441}$$

$$V(y) = E(y^2) - (E(y))^2.$$

$$E(y^2) = \sum_{y=1,2} y^2 \cdot f_2(y)$$

$$= \sum_{y=1,2} y^2 \cdot \left(\frac{3y+6}{21}\right)$$

$$= 1 \cdot \left(\frac{3+6}{21}\right) + 4 \cdot \left(\frac{6+6}{21}\right)$$

$$= \frac{9 + 48}{21} = \frac{57}{21}$$

$$\cancel{V(y^2) = \frac{57}{21}}$$

$$V(y) = E(y^2) - (E(y))^2$$

$$= \frac{57}{21} - \left(\frac{33}{21}\right)^2$$

$$= \frac{57 \times 21}{21 \times 21} - \frac{1089}{441}$$

$$V(y) = \frac{1197 - 1089}{441} = \frac{108}{441}$$

$$\rho_{xy} = \frac{\mathrm{cov}(x,y)}{\sqrt{V(x)}\sqrt{V(y)}}.$$

$$= \frac{-6/441}{\sqrt{\frac{278}{441}}\sqrt{\frac{108}{441}}}$$

$$\cancel{= \frac{-6}{441} \times \frac{\sqrt{278}\cdot\sqrt{108}}{21 \cdot 21}}$$

$$\cancel{= \frac{-6}{441} \cdot \frac{441}{\sqrt{278}\cdot\sqrt{108}}}$$

$$\cancel{= \frac{\left(\frac{-6}{441}\right)}{\frac{\sqrt{278}}{21}\cdot\frac{\sqrt{108}}{21}}}$$

$$= \frac{-0.0136}{0.793 \cdot 0.494} = \frac{-0.0136}{0.3924}$$

$$= -0.0346$$

b) $$E(x \mid y=2)$$

$$= \sum_x x\, f(x \mid y)$$

$$= \sum_x x \cdot \frac{f(x,y)}{f_2(y)}$$

$$\cancel{= \sum_x x \cdot \frac{f(x, y=2)}{f_2(y=2)}}$$

$$f_2(y) = \frac{3y+6}{21}$$

$$f_2(y=2) = \frac{3\times 2+6}{21} = \frac{6+6}{21} = \frac{12}{21}$$

$$f(x,y) = \frac{x+y}{21}$$

$$f(x, y=2) = \frac{x+2}{21}$$

$$\therefore E(x|y=2) = \sum_{x=1,2,3} x \frac{\left(\frac{x+2}{\cancel{21}}\right)}{\frac{12}{\cancel{21}}}$$

$$= \sum_{x=1,2,3} x\left(\frac{x+2}{12}\right)$$

$$= 1\cdot\left(\frac{1+2}{12}\right) + 2\left(\frac{2+2}{12}\right) + 3\cdot\left(\frac{3+2}{12}\right)$$

$$= \frac{3+8+15}{12} = \frac{26}{12}$$

$$V(x|y=2) = ?$$

$$= E(x^2|y=2) - \left(E(x|y=2)\right)^2$$

$$E(x|y=2) = \frac{26}{12}$$

$$E(x^2|y=2) = \sum x^2 f(x|y=2)$$

$$= \sum x^2 \frac{x+2}{12}$$

$$= 1\times\frac{3}{12} + 4\times\frac{4}{12} + 9\times\frac{5}{12}$$

$$= \frac{64}{12}$$

$$\therefore V(x|y=2) = E(x^2|y=2) - \left[E(x|y=2)\right]^2$$

$$= \frac{64}{12} - \left(\frac{26}{12}\right)^2$$

$$= \frac{64\times 12}{12\times 12} - \frac{676}{144}$$

$$= \frac{768-676}{144}$$

$$= \frac{92}{144}$$

$$= \frac{23}{36}$$

2) The pro. density () of 2 R.V. $(x,y)$,

$$f(x,y) = \begin{cases} 2, & 0<x<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

find the conditional mean & variance of X & Y=y.

A) Condit. mean $\Rightarrow E(X|Y=y) = ?$

variance of x & y $\Rightarrow V(X|Y=y)$

$$E(X|Y=y) = \int_x x\, f(x|y)\cdot dx$$

$$= \int_0^y \frac{f(x,y)}{f_2(y)}\cdot dx$$

$$= \int_0^y x \frac{2}{f_2(y)}\cdot dx. \quad -(1)$$

$$f_2(y) = \int_x f(x,y)\cdot dx.$$

$$= \int_0^y 2\cdot dx. \quad = [2x]_0^y$$

$$f_2(y) = 2y - 0 \quad = 2y$$

in (1)

$$= \int_0^y x \frac{\cancel{2}}{\cancel{2}y}\cdot dx.$$

$$= \int_0^y x\cdot \frac{1}{y}\cdot dx.$$

$$= \frac{1}{y}\int_0^y x\cdot dx.$$

$$= \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y$$

$$= \frac{1}{y}\left[\frac{y^2}{2} - 0\right] = \frac{y^{\cancel{2}}}{2\cancel{y}}, \quad x<y<1$$

$$E(X|Y=y) = \frac{y}{2}, \quad x<y<1$$

Condition. V.

$$V(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2 \quad (4)$$

$$E(X^2|Y=y) = \int x^2 f(x|y)\cdot dx.$$

$$= \int_0^y x^2 \frac{f(x,y)}{f_2(y)}\cdot dx.$$

$$= \int_0^y x^2 \frac{\cancel{2}}{\cancel{2}y}\cdot dx.$$

$$= \frac{1}{y}\int_0^y x^2\cdot dx.$$

$$= \frac{1}{y}\left[\frac{x^3}{3}\right]_0^y.$$

$$= \frac{1}{y}\left[\frac{y^3}{3} - 0\right]$$

$$= \frac{y^{\cancel{3}2}}{3\cancel{y}} = \frac{y^2}{3}, \quad x<y<1$$

(4): $V(X|Y=y) = \frac{y^2}{3} - \left(\frac{y}{2}\right)^2$

$$= \frac{y^2}{3\times 4}\times 4 - \frac{y^2}{4\times 3}\times 3$$

$$= \frac{4y^2 - 3y^2}{12} = \frac{y^2}{12}, \quad x<y<1$$

3) If $x$ & $y$ have the J pdf has follows —

$(x,y)$ : $(0,0)$ $(0,1)$ $(0,2)$ $(1,0)$, $(1,1)$ $(1,2)$

$f(x,y)$ : $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{12}$, $\frac{1}{12}$

find covariance of $(x,y)$?

A) $\text{cov}(x,y) = E(xy) - E(x) \cdot E(y)$.

$x = 0,1$ $\quad y = 0,1,2$

| y \ x | 0 | 1 | $f_2(y)$ |
|---|---|---|---|
| 0 | $\frac{1}{12}$ | $\frac{1}{3}$ | $4/12$ |
| 1 | $1/6$ | $\frac{1}{12}$ | $3/12$ |
| 2 | $1/4$ | $\frac{1}{12}$ | $4/12$ |
| $f_1(x)$ | $6/12$ | $6/12$ | 1 |

$$f_1(x) = \frac{1}{12} + \frac{1\times2}{6\times3} + \frac{1\times3}{4\times3} = \frac{6}{12}$$

$$= \frac{1\times4}{3\times4} + \frac{1}{12} + \frac{1}{12} = \frac{6}{12}$$

$$f_2(y) = \frac{1}{12} + \frac{1\times4}{3\times4} = \frac{4}{12}$$

$$= \frac{1\times2}{6\times2} + \frac{1}{12} = \frac{3}{12}$$

$$= \frac{1\times3}{4\times3} + \frac{1}{12} = 4/12$$

$E(x) = \cancel{E(x)} \sum x \, f_1(x)$

$$= 0 \cdot \frac{6}{12} + 1 \cdot \frac{6}{12} = \frac{6}{12}$$

$E(y) = \sum y \, f_2(y)$

$$= 0 \cdot \frac{4}{12} + 1 \cdot \frac{3}{12} + 2 \cdot \frac{4}{12}$$

$$= \frac{11}{12}$$

$x$: 0 1

$y$: 0 1 2

$E(x,y) = \sum xy \, f(x,y)$

$$= 0 \cdot 0 \cdot \frac{1}{12} + 0 \cdot 1 \cdot \frac{1}{6} + 0 \cdot 2 \cdot \frac{1}{4} + 1 \cdot 0 \cdot \frac{1}{3} + 1 \cdot 1 \cdot \frac{1}{12} + 1 \cdot 2 \cdot \frac{1}{12}$$

$$= 0 + 0 + 0 + 0 + \frac{1}{12} + \frac{2}{12}$$

$$= \frac{3}{12}$$

$\text{cov}(x,y) = E(xy) - E(x) \cdot E(y)$.

$$= \frac{3}{12} - \frac{6}{12} \cdot \frac{11}{12}$$

$$= \frac{3 \times 12}{12 \times 12} - \frac{66}{144}$$

$$= \frac{36 - 66}{144} = \frac{-30}{144}$$

q) If x & y have J.p.d.f has follows
a) find cov(x,y).  $x = 0, 3, 9$
b) Are x & y independent.  $y = 2, 4, 6$

| y \ x | 0 | 3 | 9 | $f_2(y)$ |
|---|---|---|---|---|
| 2 | 1/8 | 1/24 | 1/12 | 6/24 |
| 4 | 1/4 | 1/4 | 0 | 2/4 |
| 6 | 1/8 | 1/24 | 1/12 | 6/24 |
| $f_1(x)$ | 4/8 | 8/24 | 2/12 | 1 |

$f_1(x) = \frac{1}{8} + \frac{1\times2}{4\times2} + \frac{1}{8} = \frac{1+2+1}{8} = \frac{4}{8}$

$= \frac{1}{24} + \frac{1\times6}{4\times6} + \frac{1}{24} = \frac{1+6+1}{24} = \frac{8}{24}$

$= \frac{1}{12} + \frac{1}{12} = \frac{2}{12}$

$\Rightarrow \frac{4\times3}{8\times3} + \frac{8}{24} + \frac{2\times2}{12\times2}$

$\Rightarrow \frac{12+8+4}{24} = \frac{24}{24} = 1$

$f_2(y) =$

$= \frac{1\times3}{8\times3} + \frac{1}{24} + \frac{1\times2}{12\times2} = \frac{3+1+2}{24} = \frac{6}{24}$

$= \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$

$= \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{6}{24}$ $\Rightarrow \frac{6}{24} + \frac{2\times6}{4\times6} + \frac{6}{24} = 1$

$cov(x,y) = E(xy) - E(x) \cdot E(y)$.

$E(x) = \sum x \; f_1(x)$

$= 0 \cdot \frac{4}{8} + 3 \cdot \frac{8}{24} + 9 \cdot \frac{2}{12}$

$= 0 + \frac{1\times12}{1\times12} + \frac{18}{12}$

$= \frac{12+18}{12} = \frac{36}{12} = 3$

$E(y) = \sum y \; f_2(y)$.

$= 2 \cdot \frac{6}{24} + 4 \cdot \frac{2}{4} + 6 \cdot \frac{6}{24}$

$= \frac{12}{24} + \frac{8\times6}{4\times6} + \frac{36}{24}$

$= \frac{12+48+36}{24} = \frac{96}{24} = 4$

$E(xy) = \sum_x \sum_y xy \; f(x,y)$

$= 0\cdot2 \cdot \frac{1}{8} + 0\cdot4 \cdot \frac{1}{4} + 0\cdot6 \cdot \frac{1}{8} +$
$3\cdot2 \cdot \frac{1}{24} + 3\cdot4 \cdot \frac{1}{4} + 3\cdot6 \cdot \frac{1}{24} +$
$9\cdot2 \cdot \frac{1}{12} + 9\cdot4 \cdot 0 + 9\cdot6 \cdot \frac{1}{12}$

$= 0 + 0 + 0 + \frac{6}{24} + 3 + \frac{18}{24} + \frac{18}{12} +$

$0 + \frac{54\times2}{12\times2}$

$6 + 72 + 18 + 36 +$

$\frac{108}{24}$

$= \frac{288}{24} = 12$

$cov(x,y) = 12 - 12$

$= 0$

b) $P(X=3, y=2) = \frac{1}{24} \neq \frac{8}{24} \times \frac{6}{24}$.

$\therefore$ x & y are not independent

3) let x & y hue J pdf $f(x,y) = x+y$, $0<x<y<1$, then find $E(x^2y^3)$ & $E(x+y)^2$.

($E(x)$ ... common must be no. 1st $\int$ must be no. and $\int$ take normal limit)

a) $f(x,y) = x+y$

$E(x^2y^3) = \int_x \int_y (x^2y^3) \cdot f(x,y)\, dx\, dy$.

$= \int_0^1 \int_x^1 (x^2y^3) \cdot (x+y) \cdot dx\, dy$.

$= \int_0^1 \int_x^1 (x^3y^3 + x^2y^4)\, dy\, dx$.

$= \int_0^1 \left(x^3 \cdot \frac{y^4}{4} + x^2 \cdot \frac{y^5}{5}\right)_x^1 dx$.

$= \int_0^1 \left(\frac{x^3}{4} + \frac{x^2}{5} - \left(\frac{x^3 \cdot x^4}{4} + \frac{x^2 \cdot x^5}{5}\right)\right) dx$

$= \int_0^1 \frac{x^3}{4} + \frac{x^2}{5} - \left(\frac{x^7}{4} + \frac{x^7}{5}\right) dx$.

$= \int_0^1 \left(\frac{x^3}{4} + \frac{x^2}{5} - \frac{x^7}{4} - \frac{x^7}{5}\right) dx$.

$= \frac{1}{4}\int x^3 + \frac{1}{5}\int x^2 - \frac{1}{4}\left(x^7 - \frac{1}{5}\int x^7\right)$

$= \frac{1^4}{4} + \frac{1}{5} - \frac{1}{4} - \frac{1}{5} - 0$.

$= \left(\frac{1}{4} \cdot \frac{x^4}{4} + \frac{1}{5} \cdot \frac{x^3}{3} - \frac{1}{4} \cdot \frac{x^8}{8} - \frac{1}{5} \cdot \frac{x^8}{8}\right)_0^1$

$= \frac{1}{16} + \frac{1}{15} - \frac{1}{32} - \frac{1}{40} - 0$.

$= \frac{2-1}{32} + \frac{1}{15} - \frac{1}{40}$

$= \frac{1}{32} + \frac{1}{15} - \frac{1}{40} = \frac{15+32}{480} - \frac{1}{40}$

$= \frac{47}{480} - \frac{1}{40}$

$= \frac{1880 - 480}{19200}$

$= \frac{1400}{19200} = \frac{14}{192}$

$= \frac{7}{96}$

b) $E(x+y)^2 = \int_0^1 \int_x^1 (x+y)^2 \cdot f(x,y)\, dy\, dx$.

$= \int_0^1 \int_x^1 \underbrace{(x+y)^2}_{(a+b)^2} (x+y)\, dy\, dx$.

$= \int_0^1 \int_x^1 (x^2 + 2xy + y^2)(x+y)\, dy\, dx$.

$= \int_0^1 \int_x^1 (x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3)\, dy\, dx$

$$= \int_0^1 \int_x^1 \left(x^3 + 3x^2 y + 3xy^2 + y^3\right) \underline{dy}\, dx.$$

$$\cancel{= \int_0^1 \left(\frac{x^4}{4} + 6x\right)}.$$

$$= \int_0^1 \left(x^3 y + 3x^2 \cdot \frac{y^2}{2} + 3x \cdot \frac{y^3}{3} + \frac{y^4}{4}\right)_x^1 dx.$$

$$= \int_0^1 x^3 + \frac{3x^2}{2} + \frac{3x}{3} + \frac{1}{4} - \left(x^4 + \frac{3x^4}{2} + x^4 + \frac{x^4}{4}\right)\Big)\, dx.$$

$$= \int_0^1 x^3 + \frac{3x^2}{2} + \frac{3x}{3} + \frac{1}{4} - x^4 - \frac{3x^4}{2} - x^4 - \frac{x^4}{4}\Big)\, dx.$$

$$= \left(\frac{x^4}{4} + \frac{3}{2} \cdot \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{4}x - \frac{x^5}{5} - \frac{3}{2} \cdot \frac{x^5}{5} - \frac{x^5}{5} - \frac{1}{4}\frac{x^5}{5}\right)_0^1.$$

$$= \frac{1\times5}{4\times5} + \frac{1\times10}{2\times10} + \frac{1\times10}{2\times10} + \frac{1\times5}{4\times5} - \frac{1\times4}{5\times4} - \frac{3\times2}{10\times2} - \frac{1\times4}{5\times4} - \frac{1}{20} - 0$$

$$= \frac{5 + 10 + 10 + 5 - 4 - 6 - 4 - 1}{20}$$

$$= \frac{15}{20} = \frac{3}{4}$$

6) Let $x$ & $y$ have J pdf $\cancel{f(x,y) = \begin{cases} \frac{3}{4} \end{cases}}$

$$f(x,y) = \begin{cases} \frac{3}{4}x, & 0 < x < y < 2 \\ 0, & \text{elsewhere} \end{cases}$$

find (a) conditional pdf of $y$ given $X = x$

(b) the conditional mean & variance of $x \mid y = y$.

a) $E(x \mid y=y) = ?$ $\quad f(y \mid x=x) = ?$

$V(x \mid y=y) = ?$

a) $E(y \mid x=x) = \dfrac{f(x,y)}{f_1(x)}$

$$f_1(x) = \int f(x,y)\, dy.$$

$$= \int_x^2 \frac{3}{4} x \, dy.$$

$$= \frac{3}{4} x \int_x^2 1 \, dy \Rightarrow \frac{3}{4} x (y)_x^2.$$

$$= \frac{3}{4} x \cdot 2 - \frac{3}{4} x \cdot x.$$

$$= \cancel{\frac{3}{4} \cdot 2x - \frac{3}{4} x^2 \Rightarrow \frac{6x}{4} - \frac{3x^2}{4}}$$

$$\Rightarrow \cancel{\frac{6x - 3x^2}{4}}$$

$$\frac{3x(2-x)}{4} \longrightarrow = \frac{3}{4} x (2-x), \quad 0 < x, y.$$

$$f(y \mid x=x) = \frac{\frac{3}{4} x}{\frac{3}{4} x (2-x)}$$

$$= \frac{x}{x(2-x)} \Rightarrow \frac{1}{x(2-x)}$$

b) $E(X|Y=y) = \int_x x \cdot f(x|y)\, dx$

$= \int_0^y x \, \frac{f(x,y)}{f_2(y)}\, dx$ —(1)

$f_2(y) = \int_x f(x,y)\, dx$

$= \int_0^y \frac{3}{4} x \, dx$

$= \frac{3}{4} \left(\frac{x^2}{2}\right)_0^y$

$\Rightarrow \frac{3}{4} \cdot \frac{y^2}{2} - 0 \Rightarrow \frac{3y^2}{8}$

in (1)

$\Rightarrow \int_0^y x \cdot \frac{3/4\, x}{\frac{3}{8} y^2} \Rightarrow \int_0^y \frac{x^2}{y^2/2}\, dx$

$\Rightarrow \int_0^y \frac{2x^2}{y^2}\, dx \Rightarrow \frac{2}{y} \int_0^y x^2 \, dx$

$\Rightarrow \frac{2}{y^2} \left(\frac{x^3}{3}\right)_0^y$

$\Rightarrow \frac{2}{y^2} \cdot \frac{y^3}{3} - 0 \Rightarrow \frac{2y}{3}$

$V(X|Y=y) = E(X^2|Y=y) - \left(E(X|Y=y)\right)^2$

$E(X^2|Y=y) = \int_x x^2 f(x|y)\, dx$

$= \int_0^y x^2 \frac{f(x,y)}{f_2(y)}\, dx$

$= \int_0^y x^2 \frac{3/4\, x}{\frac{3y^2}{8}} \Rightarrow \int_0^y \frac{x^3}{y^2/2}\, dx$

$\Rightarrow \int_0^y \frac{2x^3}{y^2}\, dx$

$\Rightarrow \frac{2}{y^2} \int_0^y x^3 \Rightarrow \frac{2}{y^2} \left(\frac{x^4}{4}\right)_0^y$

$= \frac{2}{y^2} \cdot \frac{y^4}{4} - 0$

$= \frac{2y^4}{4y^2} = \frac{2y^2}{4} = \frac{y^2}{2}$

$\therefore V(X|Y=y) = \frac{y^2}{2} - \left(\frac{2y}{3}\right)^2$

$= \frac{y^2}{2} - \frac{4y^2}{9}$

$= \frac{9y^2 - 8y^2}{18} = \frac{y^2}{18}$

7) If $x$ & $y$ are 2 R.V having J.pdf

$$f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

find $\rho_{xy}$?

A) $f(x,y) = 2-x-y$

$f_1(x) = \int_y f(x,y)\, dy$

$= \int_0^1 (2-x-y)\, dy$

$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$

$f_1(x) = \frac{3-2x}{2}$ , $0 < x < 1$.

$f_2(y) = \frac{3-2y}{2}$ , $0 < y < 1$.

$E(x) = \int_x x \, f_1(x) \, dx$.

$= \frac{1}{2} \int_0^1 x(3-2x) \, dx$.

$= \frac{1}{2} \int_0^1 (3x - 2x^2)$

$= \frac{1}{2} \left( \frac{3x^2}{2} - \frac{2x^3}{2} \right)_0^1 = \cancel{\frac{1}{2}\left[\frac{3x}{2} - \frac{2}{2} - 0\right]}$

$= \cancel{\frac{1}{2}\left[\frac{3}{2} - 1\right]}$

$E(x) = \frac{5}{12}$

$E(y) = \int_y y \, f_2(y) \, dy$.

$= \frac{5}{12}$.

$E(x^2) = \int_x x^2 \, f(x) \, dx$.

$= \frac{1}{2} \int_0^1 x^2 (3-2x) \, dx$.

$= \frac{1}{2} \int_0^1 (3x^2 - 2x^3) \, dx$.

$= \frac{1}{2} \left[ \frac{3x^3}{3} - \frac{2x^4}{4} \right]$.

$= \frac{1}{4}$.

$V(x) = E(x^2) - (E(x))^2$

$= \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$

$\approx$ $V(y) = \frac{11}{144}$.

$E(xy) = \int_x \int_y xy(2-x-y) \, dx \, dy$

$= \int_0^1 \int_0^1 xy(2-x-y) \, dx \, dy$.

$= \int_0^1 y \int_0^1 (2x - x^2 - xy) \, dx \, dy$

$= \int_0^1 y \left[ \frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 dy$.

$= \int_0^1 y \left( 1 - \frac{1}{3} - \frac{y}{2} \right) dy$.

$= \int_0^1 \left( y - \frac{y}{3} - \frac{y^2}{2} \right) dy$.

$= \left( \frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6} \right)_0^1$.

$= \frac{1}{6}$

$\therefore$ $\mathrm{Cov}(x,y) = E(xy) - E(x) \cdot E(y)$

$= \frac{1}{6} - \frac{5}{12} - \frac{5}{12} = \frac{-1}{144}$

$\therefore$ $\rho_{xy} = \frac{\mathrm{Cov}(x,y)}{\sqrt{V(x)}\sqrt{V(y)}} = \frac{-1/144}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-1}{11}$