chapter : 2

partial D E (PDE)

* ordinary DE classified as linear & non-linear
* PDE is PDE in which dependent variable & partial deri- appear ~~to~~ only to the 1st power & are all multiplied together.
* U be the dependent, x & y be independent variables.
* G. form of linear 2nd order PDE is.

$$\boxed{A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+F=G}$$

when $G(x,y)=0$, → eq → homogeneous

otherwise → non-homogeneous.

* If $B^2-4AC>0$ → hyperbolic

If $B^2-4AC=0$ → parabolic

If $B^2-4AC<0$ → elliptic.

Q
1) classify the follow PDE

a) $3\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y}$

A) $3\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=0.$

here, $A = 3$, $B = C = D = F = G = 0$, $E = -$

$B^2 - 4AC \Rightarrow 0^2 - 3\times4\times0 = 0 = 0$

$\therefore$ parabolic

b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

A) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$

$A = 1$, $B = F = D = E = G = 0$, $C = -1.$

$B^2 - 4AC \Rightarrow 0 - 4\times1\times-1$

$= 4 > 0$

$\therefore$ Hyperbolic

c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$ $A = 1$, $C = 1$ $B^2 - 4AC \Rightarrow 0 - 4\times1\times1 = -4 < 0$

~~hyperbolic~~ eliptic

→ Sol^n of a PDE =

Sol^n of a PDE in some region ~~are~~ 'R' of the space of the independent variable is a (?) that has all the partial derivatives appearing in the eq in some domain containing R ie satisfies the eq every where in R (region)

~~→ Super principle~~

→ Super position principle =

If $u_1, u_2 \ldots u_k$ are sol^n of a homo- LPDE then the linear combination $u = c_1u_1 + c_2u_2 + \cdots + c_ku_k$, where $c_1, c_2 \ldots c_k$ are constants is also a sol^n.

Q)

1) varify that follow ( ) are sol^n of the 2D Laplace eq

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \longrightarrow \text{Laplace eq.}$$

a) $u = x^2 - y^2$.

A) $\frac{\partial u}{\partial x} = 2x$. , $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = 2y$, $\frac{\partial^2 u}{\partial y^2} = 2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Rightarrow 2 - 2 = 0$

b) $u = e^x \cos y$ (c) $u = \ln(x^2 + y^2)$

2) varify that ( ) $u = e^{2t} \cos x$ is a sol^n of the heat eq gvn below for a suitable value of c

(a) $\frac{\partial u}{\partial t} + c^2 \frac{\partial^2 u}{\partial x^2}$

(b) $u = x^3 + 3xt^2$

→ <u>Method of Separation of variables</u> =

$$\boxed{u(x,y) = X(x)\,Y(y)}.$$

$$\left.\begin{array}{l} \therefore \dfrac{\partial u}{\partial x} = X'Y \;,\; \dfrac{\partial^2 u}{\partial x^2} = X''Y \\ \dfrac{\partial u}{\partial y} = XY' \;,\; \dfrac{\partial^2 u}{\partial y^2} = XY'' \end{array}\right\} \leftarrow (a)$$

Q 1) Using the methode of seperation of variable find the product Sol^n of

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u$$

A) $u(x,y) = X(x)\,Y(y)$

$\longrightarrow u = XY$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u \quad \text{by } (a),$$

1) make into $\div$
2) $\frac{X'}{X}$
3) $\frac{Y'}{Y}$
4) X' ? → ∫.
$X' \to \ln x(x)$
5) Y' ? → ∫.
6) $u = XY$.

$$X'Y = \underline{2XY'} + XY$$

$\div 2XY$

$$\frac{X'Y}{2XY} = \frac{\cancel{2XY'}}{\cancel{2XY}} + \frac{XY}{2XY}$$

$$\frac{1}{2} = \frac{Y'}{Y} + \frac{X}{2X}.$$

$$\frac{X'}{2X} = \frac{Y'}{Y} + \frac{1}{2}.$$

$$\frac{X'}{2X} - \frac{1}{2} = \frac{Y'}{Y}$$

$$\frac{2x'-2x}{2x\cdot 2} = \frac{y'}{y}$$

Common

$$\frac{2(x'-x)}{2(x)\cdot 2} = \frac{y'}{y} \Rightarrow \frac{(x'-x)}{2x} = \frac{y'}{y} \quad \text{—①}$$

$$\frac{x'-x}{2x} = \frac{y'}{y} = \lambda \Rightarrow \frac{x'-x}{2x} = \lambda \quad \text{—ⓐ}$$

$\frac{y'}{y} = \lambda$ —ⓑ

—ⓐ

$x' - x = 2x\lambda$

$x' - x - 2x\lambda = 0$

$x' - x(1+2\lambda) = 0$

$x' = x(1+2\lambda)$

$\frac{x'}{x} = 1+2\lambda$

$x' = (1+2\lambda)x$

$\int x' = \int (1+2\lambda)x$

$\ln x(x) = (1+2\lambda)x + \ln c_1$

$x(x) = e^{(1+2\lambda)x} \cdot c_1$

—ⓑ $\int y' = \lambda y$

$\ln y(y) = \lambda y + \ln c_2$

$y(y) = e^{\lambda y} \cdot c_2$

$u = xy = c_1 e^{(1+2y)x} c_2 e^{\lambda y}$

$= c e^{(1+2y)x} e^{\lambda y}$

2) using m. of seperation of variable.

find product sol^n of $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$

A) $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$.

$X''Y = 4X Y'$

$\div 4XY$

$\frac{X''\cancel{Y}}{4X\cancel{Y}} = \frac{\cancel{4X}Y'}{\cancel{4X}Y}$

$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda. \Rightarrow \frac{X''}{4X} = -\lambda, \frac{Y'}{Y} = \lambda.$

$\frac{X''}{X} = 4\lambda. \quad \frac{Y'}{Y} = \lambda$

$\frac{X''}{4X} + \lambda = 0. \quad \frac{Y'}{Y} + \lambda = 0$

$X'' = 4X\lambda \quad , Y' = \lambda Y$

② — $X'' + 4X\lambda = 0 \quad , Y' + \lambda Y = 0$ — ③

let us consider 3 ~~conditions~~ cases:,

case 1, $\lambda = 0$.

eq ② & ③,

$X'' = 0$ & $Y' = 0$

by $\int$, $x = 0$, $X = c_1 x + c_2$, $y = c_3$

$u(x,y) = X(x)\, Y(y)$

$= (c_1 x + c_2 x)\, c_3 \longrightarrow \quad c_2 \cdot c_3 = A_1$
$c_1 \cdot c_3 = B_1$

$= A_1 + B_1 x.$

case 2 $\quad \lambda = -(\alpha^2)$

eq ② & ③, $X'' + 4 - (\alpha^2) x = 0$ — ④.

$Y' + 4 - (\alpha^2) = 0$ — ⑤

$X'' + (\alpha^2) \cdot x = 0$ — ⑥

2α.

$y' + y(\alpha^2) = 0 \quad \Rightarrow \cancel{8} y'$

$\cancel{x} = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x.$

$\mu = \alpha^2$

$y = c_6 \alpha^2 y$

$u = xy$

$= c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \cdot (c_6 \alpha^2 y)$

$= c_6 e^{\alpha^2 y} \cdot c_4 \cosh 2\alpha x + c_6 e^{\alpha^2 y} \sinh 2\alpha x.$

<u>case 3</u> : $\lambda = \alpha^2$

$x'' + 4\alpha^2 x = 0. \quad \rightarrow x'' A \alpha^2 \lambda = 0.$

$y' + 4 - \alpha^2 = 0. \quad \rightarrow y' + 4\alpha^2 = 0.$

$x = c_7 \cosh 2\alpha x + c_8 \sinh 2\alpha x.$

$m^2 + 4\alpha^2 x = 0$

$m = -4\alpha^2$

$m = 20\alpha.$

$y' + 4\alpha^2 = 0 \qquad y = c_9 e^{-\alpha^2 x}.$

$m + \alpha^2 = 0.$

$m = -\alpha^2.$

$u = xy$

$u = c_7(\cosh 2\alpha x + c_8 \sinh 2\alpha x) \; c_9 e^{-\alpha^2 y}$

$= c_9 e^{-\alpha x} c_7 \cos 2\alpha x + c_9 e^{-\alpha y} c \alpha \sinh 2\alpha$

① c) $\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot \frac{d}{dx}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(2) - (2x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)2 - (2y)(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} \Rightarrow \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

$$= 0$$

b) $u = e^x \cdot \cos y$.

$\frac{\partial u}{\partial x} = e^x \cos y.$ $\quad \frac{\partial^2 u}{\partial x^2} = e^x \cdot \cos y.$

$\frac{\partial u}{\partial y} = -e^x \sin x.$ $\quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$\Rightarrow 0$

(2) (a) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u = e^t \cos x.$

$\frac{\partial u}{\partial x} = 2e^{2t} \cos x.$

$\frac{\partial^2 u}{\partial x^2} = -e^t \cos u.$

$2e^{2t} \cos x = c^2 \cdot -e^{2t} \cos x.$

$\frac{2\cancel{e^{2t}\cos x}}{-\cancel{e^{2t}\cos x}} = c^2.$

$-2 = c^2$

$c = \pm\sqrt{2}\, i$

(b) $u = x^2 + 3xt^2$

$\frac{\partial u}{\partial x} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\cancel{\frac{\partial u}{\partial x} = 2x + 3t^2 \quad \frac{\partial^2 u}{\partial x^2} = 2}$

$\frac{\partial u}{\partial t} = 0 + 6tx, \quad \frac{\partial^2 u}{\partial t^2} = 6x.$

$6tx = c^2 \cdot 6x.$

$\frac{\partial^2 u}{\partial x} = 3x^2 + 3t^2, \quad \cancel{\frac{\partial^2 u}{\partial x^2} = 6x}.$

$\frac{6tx}{6x} = c^2$

$c^2 = t$

$c = \sqrt{t}$

* 2D Laplace eq $\rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

* 3D " eq $\rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

→ wave eq =

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$ *

$$a^2 = T/\rho.$$

→ Initial Condi~ & boundary condi~ =

→ Initial condi~ of heat eq :-

BVP $\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$

B.C – $u(0,t) = 0$
$u(L,t) = 0.$

I C, $u(x,0) = f(x).$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

B. C, $u(x,0) = 0$
$u(x,b) = f(x).$

Heat eq, $\quad k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$

Sol^n of heat conduction probm,

$$u(x,t) = \sum A_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin \frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

1) find temp $u(x,t)$ at any time in a metal rod 50 cm long insulated on the sides

which initially has a uniform temp of 20°C throughout, & whose ends are maintained at 0°C for all $t > 0$

A) Heat condn eq,

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad L = 0 < x < 25$$

$u(0,t) = 0$, $u(25,t) = 0$.

$u(x,0) = 20 = f(x)$

$$u(x,t) = \sum A_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx.$$

$$= \frac{2}{50}\int_0^{50} 20 \sin\frac{n\pi x}{50}\, dx.$$

$$= \frac{20}{25}\int_0^{50} \frac{\sin n\pi x}{50}\, dx$$

$$= \frac{20}{25}\left[\frac{-\frac{\cos n\pi x}{50}}{\frac{n\pi}{50}}\right]_0^{50} = \cancel{\frac{20}{25}\left[\frac{-\frac{\cos n\pi 50}{50}}{\frac{n\pi}{50}}\right]}$$

$$= \frac{4}{5} \times \frac{50}{n\pi}\left[-\frac{\cos\pi 50}{50} - -\frac{\cos\pi 0}{50}\right]$$

$$= \frac{4}{5} \times \frac{50}{n\pi}\left[-\cos n\pi + 1\right)$$

$$= \frac{4}{5} \times \frac{50}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{200}{5n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{40}{n\pi}\left(1 - (-1)^n\right)$$