Application of Integrals.

* Area b/w graphs :-

If $g(x) \le f(x)$ for all x in $[a,b]$

then, the area b/w the graph of f & g

on $[a,b]$ = $\boxed{\int_a^b (f(x) - g(x))\ .dx.}$

Q 1) Sketch graph. & find the area of the region b/w the graphs x^2 & $x+3$ on $[-1,1]$

A) $y = x^2 \rightarrow$ parabola (U)

x	0	1	2	-1	-2
y	0	1	4	1	4

$y = x+3 \rightarrow$ straight line

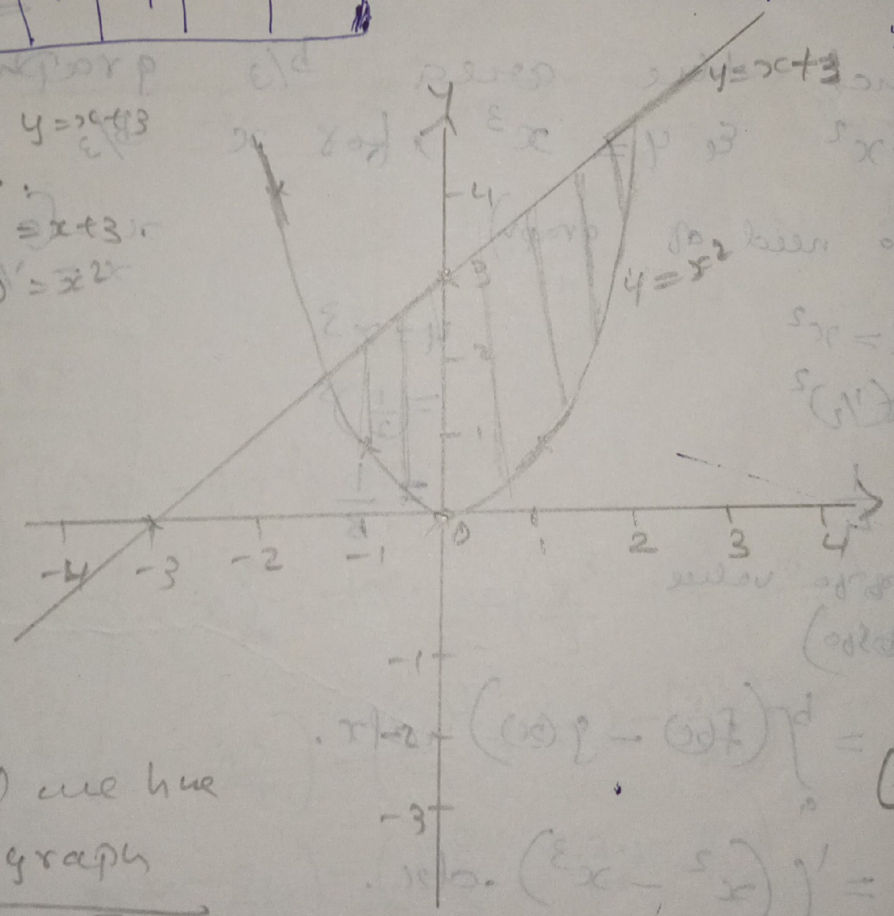
x=0 assign then y=0 assign

x	0	-3
y	3	0

$x=0$
$y=0.$

$y=0$
$y=x+3$
$0=x+3$
$x=-3$

here, $y=x+3$ is >.
∴ $f(x) = x+3$.
$g(x) = x^2$



Instead we have no graph

$x = (-1, 1)$

$x = -1, 0, 1$
take centre one.

$y = x^2$
$= 0$

$y = x+3$
$y = 3$
$3 \rightarrow$ biggest.

$[-1, 1]$
↓ a ↓ b

$f(x) = x+3$
$g(x) = x^2$

$$\text{Area} = \int_a^b (f(x) - g(x))\,dx.$$

$$= \int_{-1}^{2} ((x+3) - x^2)\,dx.$$

$$= \left(\frac{x^2}{2} + 3x - \frac{x^3}{3}\right)_{-1}^{2}$$

$$= \left(\frac{1}{2} + 3 - \frac{1}{3}\right) - \left(\frac{(-1)^2}{2} - 3 - \frac{(-1)^3}{3}\right)$$

$-\frac{1}{3} + \frac{-1}{3}$
$= \frac{-2}{3} = 2\frac{1}{3}$

$$= \cancel{\frac{1}{2}} + 3 - \frac{1}{3} - \cancel{\frac{1}{2}} + 3 - \frac{1}{3}$$

$$= 6 - 2\frac{1}{3} = \frac{6}{1} - \frac{2}{3}$$

$$= \frac{18-2}{3} = \frac{16}{3}$$

2) find the area b/w graph $y = x^2$ & $y = x^3$, for x b/w 0 & 1.

A) (No need of graph)

$x = [0\ 1]$
$x = 1/2$.

$y = x^2$
$= (1/2)^2$
$= \frac{1}{4}$

$y = x^3$
$= \left(\frac{1}{2}\right)^3$
$= \frac{1}{8}$

cake eg
cake -> 4 pieces
or
cake -> 8 pieces
4 pieces is big
so 4 is big

(Zero - value)

$$\text{Area} = \int_a^b (f(x) - g(x))\,dx.$$

$$= \int_0^1 (x^2 - x^3)\,dx.$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \left(\frac{1}{3} \times \frac{1}{4}\right) - 0$$

$$= \frac{4-3}{12} = \frac{1}{12}$$

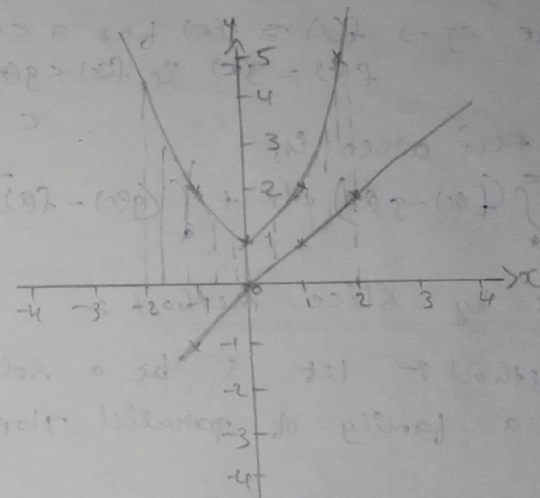
3) Sketch & find the area of the region b/w the graph x & x^2+1 on $[-2, 2]$.

A) $y = x$

x	0	1	2	-1
y	0	1	2	-1

$y = x^2+1$.

x	0	1	2	-1
y	1	2	5	2



$f(x) = x^2+1$
$g(x) = x$.

$$A = \int_a^b (f(x) - g(x))\,dx.$$

$$\int ((x^2+1) - x)\,dx. = \left(\frac{x^3}{3} + x - \frac{x^2}{2}\right)_{-2}^{2}$$

$$= \left(\frac{2^3}{3} + 2 - \frac{2^2}{2}\right) - \left(\frac{(-2)^3}{3} + (-2) - \frac{(2)^2}{2}\right)$$

$$= \left(\frac{8}{3} + \frac{2\times3}{1\times3} - \frac{2\times3}{1\times3}\right) - \left(\frac{-8}{3} - \frac{2\times3}{1\times3} - \frac{2\times3}{1\times3}\right)$$

$$= \frac{8+6-6}{3} - \left(\frac{-8-6-6}{3}\right) = \frac{8}{3} + \frac{20}{3}$$

$$= \frac{28}{3}$$

=> Area b/w intersecting graphs :-

To find the area of the region b/w the graphs of f & g and b/w $x=a$ & $x=b$, 1st plot the graphs and locate points where $f(x) = g(x)$.

Suppose eg → $f(x) \geq g(x)$ for $a \leq x \leq c$, $f(c) = g(c)$ & $f(x) \leq g(x)$ for $c \leq x \leq b$

then the area is,

$$A = \int_a^c (f(x) - g(x))\,dx + \int_a^b (g(x) - f(x))\,dx.$$

-> volumes by slice method :-

* Slice method :- Let 's' be a solid & 'P_x' be a family of parallel planes such that

a) S lies b/w P_a & P_b.

b) The area of slice of S cut by P_x is $A(x)$, then the volume of S =

$$\int_a^b A(x)\,dx$$ (Area)

Q

1) find the vol of a ball of radius 'r'

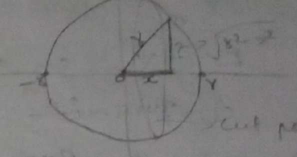
A) solid → ball → circle. (cut section)

Area of circle $= \pi r^2$

$Alt^2 = hyp^2 - base^2$

$Alt^2 = r^2 - x^2$

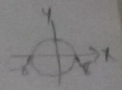
$Alt = \sqrt{r^2 - x^2}$



Ar of $\bigcirc = \pi r^2$

$= \pi\left(\sqrt{r^2 - x^2}\right)^2$

$= \pi(r^2 - x^2)$



∴ vol of ball $= \int_a^b A(x)\,dx$

$= \int_{-r}^{r} (\pi r^2 - \pi x^2)\,dx.$

$= \int_{-r}^{r} \pi(r^2 - x^2)\,dx.$

π & r → constant.

$\int 1 = x$

$\int x^2 = \frac{x^3}{3}$

$(-r)^3 = -r^3$

$= \int \pi r^2 (1 - x^2)$

$= \left(\pi r^2 x - \frac{\pi x^3}{3}\right)_{-r}^{r}$

$= \pi r^2 \cdot r - \pi\frac{r^3}{3} - \left(\pi r^2 \cdot (-r) - \frac{\pi(-r)^3}{3}\right)$

$= \pi r^3 - \frac{\pi r^3}{3} + \pi r^3 + \frac{\pi r^3}{3}$

$= \pi r^3 - \frac{\pi r^3}{3} + \pi r^3 - \frac{\pi r^3}{3}$

$= \frac{2\pi r^3}{1} - \frac{2\pi r^3}{3}$

$= \frac{6\pi r^3 - 2\pi r^3}{3} = \frac{4\pi r^3}{3}$

2) A Pyramid 3mtr height has a □ base (i.e) 3mtr on a side. The cross section of the pyramid ⊥ to the altitute

Xm down from the vertex is a Xm on a side. find the vol

□ Xm on a side. find the vol of pyramid?

A) vol of pyramid $= \int_a^b A(x)\,dx$.

$A(x) = x^2$

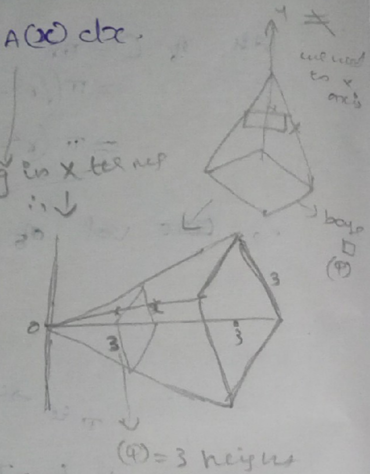
eg in x terms

$\therefore$ Vol $= \int_0^3 x^2 \cdot dx$.

$= \left(\frac{x^3}{3}\right)_0^3$

$= \frac{3^3}{3} - 0$

$= \frac{27}{3} = 9$



⇒ Vol of Solid of Revolution :-

* Rotation about X-axis :-

The vol of solid generated by revolving about the X axis, the region b/w the X-axis & the graph of contin () $y = f(x)$, $a \le x \le b$. is

$$\boxed{V = \int_a^b \pi (f(x))^2\,dx.}$$

* Rotation about Y-axis :-

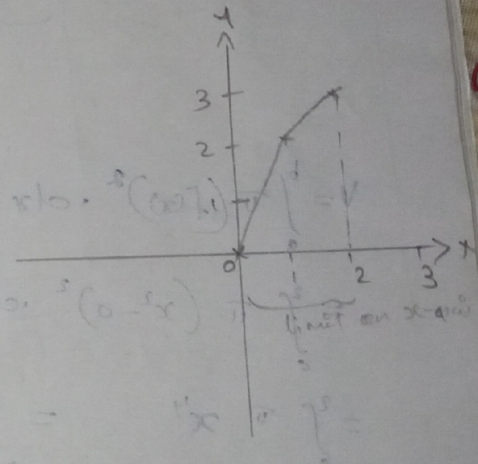
vol of solid generated by revolving about Y-axis, the region b/w the Y axis & the graph of contin () . $x = f(y)$ $c \le y \le d$ is

$$\boxed{V = \int_c^d \pi (f(y))^2 \cdot dy}$$

Q 1) The region b/w the curve $y = 2\sqrt{x}$ $0 \le x \le 2$ & the x-axis is revolved about the x-axis to generate a solid, find its Vol ? $v = \int_a^b \pi (f(x))^2\,dx$

A) $y = 2\sqrt{x}$. $2\sqrt{2}$ (calculate)

x	0	1	2
y	0	2	2.8



$V = \int_a^b \pi (f(x))^2 \cdot dx$

$= \int_0^2 \pi (2\sqrt{x})^2 \cdot dx$

$= \int_0^2 \pi\, 4x \cdot dx$

$= 4\pi \int_0^2 x \cdot dx$

$\int x = \frac{x^2}{2}$

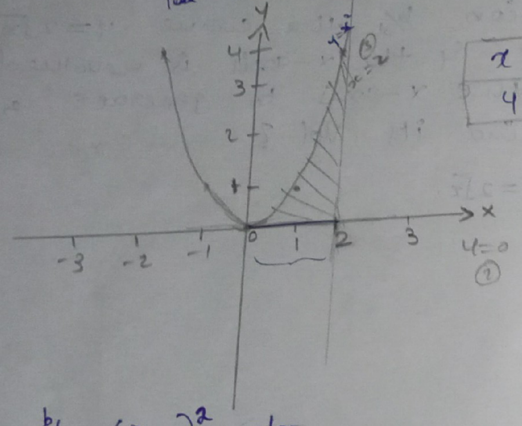
$= 4\pi \left(\frac{x^2}{2}\right)_0^2 = 4\pi \left[\frac{2^2}{2} - 0\right]$

$= 4\pi \left(\frac{4}{2}\right) = 4\pi \times 2 = 8\pi$

2) find the vol of solid generated by revolving the region bounded by curve $y = x^2$ & line $y=0$, $x=2$ about x axis.

A)

$y = x^2$ ①, $y=0$ ②, $x=2$ ③;



$y = x^2$

x	0	1	2	-1	-2
y	0	1	4	1	4

$(x^2 - 0)$

$0 = y = 0.$

$$V = \int_a^b \pi (f(x))^2 \cdot dx.$$

$$= \int_0^2 \pi (x^2 - 0)^2 \cdot dx.$$

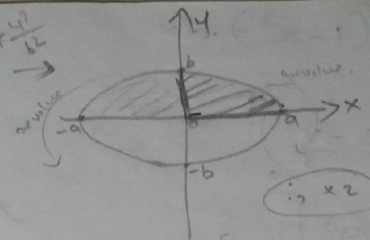
$$= \int_0^2 \pi x^4 \quad = \pi \int_0^2 x^4 \cdot dx.$$

$$= \pi \left(\frac{x^5}{5}\right)_0^2$$

$$= \pi \left[\frac{2^5}{5} - 0\right] = \frac{32\pi}{5}$$

3) find the vol of solid generated by revolving the elips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x axis?

A) $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ → elipse →



⇒ 4 sides equal. So find 1 side

$\times 2$

$f(x)$ ⇒ we need all in x terms. So find y. then sub y in x.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = b^2 \left(\frac{a^2 - x^2}{a^2}\right)$$

$$y = \sqrt{b^2 \left(\frac{a^2 - x^2}{a^2}\right)}$$

$$y = \sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$$ '-', so $\sqrt{\ }$ needed.

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}$$

$$V = \int_a^b \pi (f(x))^2 \, dx.$$

$$= \int_0^a \pi \left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx.$$

$$= \int_0^a \pi \frac{b^2}{a^2}(a^2 - x^2)$$

constant out $= \frac{\pi b^2}{a^2} \int_0^a (a^2 - x^2) \cdot dx.$

$= \frac{\pi b^2}{a^2} \quad a^2 \int (1 - x^2) \, dx$

$= \frac{\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3}\right)_0^a$

$= \frac{\pi b^2}{a^2} \left[a^3 - \frac{a^3}{3}\right] - 0.$

$= \frac{\pi b^2}{a^2} \left[\frac{3a^3 - a^3}{3}\right]$ $\qquad 3a^3 - a^3 = 2a^3$

$= \frac{\pi b^2}{a^2} \quad \frac{2a^3}{3}$

$V = \frac{2\pi b^2 a}{3}$

$\therefore$ required vol $= 2 \cdot V.$

$= 2 \cdot \frac{2\pi b^2 a}{3}$

$= \frac{4\pi b^2 a}{3}$

Q) find the area b/w the graphs $x = y^2 - 2$ & $y = x$

A)

$x = y^2 - 2 \qquad y = x$

$x = y$

$\therefore \; y^2 - 2 = y$

$y^2 - 2 - y = 0$

$\Rightarrow y^2 - y - 2 = 0.$ (qua eq)

$a \cdot b = -2$ $\qquad$ (a.b, a+1)

$a + b = -1$

$\therefore \quad -1 \times 2 = 2 \qquad -1 + 2 = 2$

$1 \times -2 = -2 \qquad 1 - 2 = -1$

$(1, -2)$

$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0.$

$y = -1, 2$

($f(x) - g(x)$ → [illegible] $x = y^2 - 2$ [illegible] $y = x$ [illegible] big value [illegible])

$y = [-1, 2] \quad \to \quad -1, 0, 1, 2.$

let $y = 0$

$x = y^2 - 2$ $\qquad x = y$

$= 0 - 2$ $\qquad x = 0$ true

$x = -2$

$f(y) = y \qquad g(y) = y^2 - 2$

all terms in y.

$A(y) = \int_a^b (f(y) - g(y)) \, dy$

$= \int_{-1}^{2} (y - (y^2 - 2)) \cdot dy.$

$= \int_{-1}^{2} (y - y^2 + 2) \cdot dy.$

$\int) \quad = \left(\frac{y^2}{2} - \frac{y^3}{3} + 2y\right)_{-1}^{2}$

$= \left[\frac{2^2}{2} - \frac{2^3}{3} + 2 \times 2\right] - \left[\frac{(-1)^2}{2} - \frac{(-1)^3}{3} + 2 \times (-1)\right]$

$= \left[\frac{4 \times 3}{2 \times 3} - \frac{8 \times 2}{3 \times 2} + \frac{4 \times 6}{1 \times 6}\right] - \left[\frac{1 \times 3}{2 \times 3} + \frac{1 \times 2}{3 \times 2} - \frac{2 \times 6}{1 \times 6}\right]$

$= \cancel{\left[2 - \frac{8}{3} + 4\right] - \left[\frac{1}{2} + \left(\frac{1-6}{3}\right)\right].}$

$= \cancel{\frac{6-8}{3} + 4 - \left(\frac{1}{2} - \frac{5}{3}\right)}$

$$= \frac{-2}{3} + \frac{4}{1} - \left(\frac{1}{2} - \frac{5}{3}\right)$$

$$= \frac{-2+12}{3} - \left(\frac{1}{2} - \frac{5}{3}\right)$$

$$= \frac{10}{3} - \frac{1^{\times 3}}{2_{\times 3}} + \frac{5^{\times 2}}{3_{\times 2}}$$

$$= \frac{10}{3} - \frac{3+10}{6}$$

$$= \frac{10\times 2}{3\times 2} - \frac{13}{6} \qquad = \frac{20+13}{6} \qquad = \frac{87}{6}$$

$$= \frac{10}{3} - \frac{7}{6} = \frac{60-218}{18} = \frac{39}{18}$$

$$= \left[\frac{12-16+24}{6}\right] - \left[\frac{3+2-12}{6}\right]$$

$$= \frac{20}{6} - \left[\frac{-7}{6}\right]$$

$$= \frac{20}{6} + \frac{7}{6} = \frac{27^{\div 3}}{6_{\div 3}} = \frac{9}{2}$$

~~⟹ L' Hopital's Rule~~