Module - III

The Laplace Transform ($\mathcal{L}$)

Let $f(t)$ be a () of t. defined for all non-ve values of t., then the (L.T) of $f(t)$ denoted by '$\mathcal{L}\,f(t)$' & is defined by

$$\boxed{\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt}$$

where $s \to$ parameter (real / complex)

[ans-m err s'nm $\to F(s)$]

;
$$\boxed{\begin{gathered}\mathcal{L}(f) = F(s) \\ f = \mathcal{L}^{-1}(F(s))\end{gathered}}$$

Note

1) clearly $\mathcal{L}(f)$ is a () of s say $F(s)$ then

✓ $$\boxed{\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt}$$

2) Inverse transform / inverse of $F(s)$ will be denoted by, $\mathcal{L}^{-1}(F(s))$

we shall write

✓ $\boxed{f = \mathcal{L}^{-1}(F)}$

$\Rightarrow$ Transforms of elemntry ()s =

1) $\boxed{\mathcal{L}(0) = 0}$

$\mathcal{L}(f(t)) = \mathcal{L}(0)$
$\therefore f(t) = 0$.

Pf

here $f(t) = 0$

$$\mathcal{L}\,f(t) = \int_0^{\infty} e^{-st} f(t)\,dt.$$

$f(t) = 0$.

$$= \int_0^{\infty} 0. \quad = 0$$

2) $\boxed{\mathcal{L}(1) = \frac{1}{s}}$, $s > 0$.

here $f(t) = 1$ $\mathcal{L}(1)$

Pf
$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt$$
$$= \int_0^{\infty} e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}.$$
$$= \frac{e^{-s\infty}}{-s} - \frac{e^0}{-s} = 0 - \frac{1}{s} = \frac{1}{s}$$

$e^{\infty} = \infty$
$e^{-\infty} = 0$

3) $\boxed{\mathcal{L}(t) = \frac{1}{s^2}}$, $s > 0$.

Pf
$f(t) = t$
$$\mathcal{L}(t) = \int_0^{\infty} e^{-st} f(t)\,dt$$
$$= \int_0^{\infty} e^{-st} \cdot t\,dt \qquad (\textstyle\int \text{ by parts})$$

(for easy)
$$= \int_0^{\infty} t \cdot e^{-st}\,dt$$
$$= t \cdot \int e^{-st} - \int \left[\frac{d}{dt}(t) \cdot \int e^{-st}\right] dt.$$
$$= \left[t \times \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1 \times \frac{e^{-st}}{-s}\,dt.$$
$$= 0 + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt$$

$\frac{e^{-st}}{e^{st}}) \rightarrow$
$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}\left[\frac{1}{-se^{\infty}} - \frac{1}{-se^0}\right]$$
$$= \frac{1}{s}\left[\frac{1}{s}\right] = \frac{1}{s^2}$$

4) $\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$ $n = 1, 2, 3 \ldots$, $s > 0$

Pf
$f(t) = t^n$.
$$\mathcal{L}(t^n) = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} e^{-st} \cdot t^n\,dt.$$
$$= \int_0^{\infty} t^n \cdot e^{-st}\,dt. \qquad (\textstyle\int \text{ by parts})$$

$$= \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \cdot \frac{e^{-st}}{-s}\, dt.$$

$$= 0 + \frac{n}{s} \underbrace{\int_0^\infty t^{n-1} e^{-st}\, dt}_{\text{difficult}}$$

$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt$

$\therefore \mathcal{L}(t^{n-1}) = \int_0^\infty t^{n-1} e^{-st} dt$

$$\boxed{\mathcal{L}(t^n) = \frac{n}{s} \mathcal{L}(t^{n-1})}$$

$\mathcal{L}(t^{n-1}) = \frac{n-1}{s} \mathcal{L}(t^{n-2})$

$\mathcal{L}(t^{n-2}) = \frac{n-2}{s} \mathcal{L}(t^{n-3})$

$\vdots$

$\mathcal{L}(t^3) = \frac{3}{s} \mathcal{L}(t^2)$

$\mathcal{L}(t^2) = \frac{2}{s} \mathcal{L}(t)$

$\mathcal{L}(t) = \frac{1}{s^2}$

$\therefore$ we gt $\mathcal{L}(t^n) = \frac{n}{s} \times \frac{(n-1)}{s} \times \frac{(n-2)}{s} \times \cdots \frac{3}{s} \times \frac{2}{s} \times \frac{1}{s^2}$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. \qquad \left(eg \to \mathcal{L}(t^5) = \frac{5!}{s^6}\right)$$

*) $\boxed{\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}}$

5) $\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$ $\quad s > a.$ $\qquad eg \to \mathcal{L}(e^{3t}) = \frac{1}{s-3}$

Pf/ $f(t) = e^{at}$

$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} \cdot e^{at}\, dt.$

$= \int_0^\infty e^{-st+at}\, dt = \int_0^\infty e^{at-st}$

$= \int_0^\infty e^{(a-s)t}\, dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty$

$= \frac{1}{a-s}\left[e^{-(s-a)t}\right]_0^\infty$ $\qquad$ for $\frac{e^{(s-a)t}}{e^{(s-a)t}}$ to gt ∞ to.

$= \frac{1}{a-s}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$

$= \frac{1}{a-s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = \frac{1}{a-s}\left[\frac{1}{\infty} - 1\right]$

$= \frac{1}{a-s}[0-1] = \frac{1}{a-s}[-1].$

$= -\frac{1}{a-s} = \frac{1}{-a+s}$

$= \frac{1}{s-a}$

6) $\boxed{\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}}$

7) $\boxed{\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}}$

Remark

$\mathcal{L}(e^{iat}) = \mathcal{L}(\cos at) + i\,\mathcal{L}(\sin at)$

8) $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$

9) $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$

$L(0) = 0$	$L(e^{at}) = \frac{1}{s-a}$
$L(1) = \frac{1}{s}$	$L(t^n) = \frac{n!}{s^{n+1}}$
$L(t) = \frac{1}{s^2}$	
$L(\sin at) = \frac{a}{s^2+a^2}$	$L(\sinh at) = \frac{a}{s^2-a^2}$
$L(\cos at) = \frac{s}{s^2+a^2}$	$L(\cosh at) = \frac{s}{s^2-a^2}$

→ **properties of Laplace transform** =

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Therom -1 [linearity property]

$$\boxed{\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))}$$

Therom -2 [1^{st} Shifting property]

If $\mathcal{L}(f(t)) = F(s)$ then,

$$\boxed{\mathcal{L}(e^{at} f(t)) = F(s-a)}$$

$\mathcal{L}(g(t)) = G(s). \Rightarrow \mathcal{L}(e^{at} g(t)) = G(s-a)$

Rmrk

$$\left[\begin{aligned} \mathcal{L}(1+5t) &= \mathcal{L}(1) + 5\mathcal{L}(t) \\ &= \frac{1}{s} + 5\frac{1}{s^2} \end{aligned}\right]$$

$s = s-a$

* $\mathcal{L}(e^{at} t^n) = \dfrac{n!}{(s-a)^{n+1}}$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

* $\mathcal{L}(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}$

$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

$(s = s-a)$

* $\mathcal{L}(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$

* $\mathcal{L}(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$

* $\mathcal{L}(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$

* $\mathcal{L}(e^{bt} \cos at) = \dfrac{s-b}{(s-b)^2 + a^2}$

A ( ) 'F' is said to be piecewise continous on an intrval $\alpha \le t \le \beta$, if the intrval can be partioned by a finite ~~pt~~ non of points $\alpha = t_0 < t_1 < t_2$, ~~f is~~ con so that,

1) f is continous on each (o.I) $t_{i-1} < t < t_i$

2) f approaches a finite limit as the end point of each sub-interval are approached from within the interval.

* Theorem [Existence therom for L. transform]

let $f(t)$ be a ( ) that is piecewise continous on every finite intrval in the range $t \ge 0$ & satisfies $|f(t)| \le Me^{kt}$ $\forall\, t \ge K$, for some real constnts M, k, K, then the L.transform,

$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ exists $\forall\, s > k$.

1) find L transform,

a) $2t + 6$

A) $\mathcal{L}(2t+6) = 2\mathcal{L}(t) + 6\mathcal{L}(1)$.

$= \dfrac{2}{s^2} + \dfrac{6}{s}$

$\mathcal{L}(t) = \dfrac{1}{s^2}$

$\mathcal{L}(1) = \dfrac{1}{s}$

b) te^{4t}

$\mathcal{L}(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}}$

$\therefore \mathcal{L}(t e^{4t}) = \dfrac{1!}{(s-4)^2}$

c) $4t^2 - 5\sin 3t$

A) $\mathcal{L}(4t^2 - 5\sin 3t) = 4\mathcal{L}(t^2) - 5\mathcal{L}(\sin 3t)$

$= 4\,\frac{2!}{s^3} - 5\,\frac{3}{s^2+3^2}$

$(2! = 2)$

$= \frac{8}{s^3} - \frac{15}{s^2+9}$

d) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

A) $\mathcal{L}(e^{2t}) + 4\mathcal{L}(t^3) - 2\mathcal{L}(\sin 3t) + 3\mathcal{L}(\cos 3t)$

$= \frac{1}{s-2} + 4\cdot\frac{3!}{s^4} - 2\cdot\frac{3}{s^2+9} + 3\cdot\frac{s}{s^2+9}$

$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$

e) $(t+1)^2$

A) $\mathcal{L}((t+1)^2) = \mathcal{L}[t^2+2t+1]$

$= \mathcal{L}(t^2) + 2\mathcal{L}(t) + \mathcal{L}(1)$

$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

f) $(t+1)^2 e^t$

A) $\mathcal{L}((t+1)^2 e^t) =$ $\frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}[(t^2+2t+1)e^t] = \mathcal{L}[e^t t^2 + e^t 2t + e^t]$

$= \frac{2}{(s-1)^{2+1}} + 2\times\frac{1!}{(s-1)^{1+1}} + \frac{1}{s-1}$

$\frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$ (Shifting property)

g) $\sinh 2t \cdot \sin 3t$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\therefore \sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$

A) $\sinh 2t \sin 3t$

$= \left[\frac{e^{2t} - e^{-2t}}{2}\right]\sin 3t$

$= \frac{1}{2}\left[e^{2t}\sin 3t - e^{-2t}\sin 3t\right]$

$\therefore \mathcal{L}(\sinh 2t \sin 3t) = \frac{1}{2}\left[\mathcal{L}(e^{2t}\sin 3t - e^{-2t}\sin 3t)\right]$

$= \frac{1}{2}\left[\mathcal{L}(e^{2t}\sin 3t) - \mathcal{L}(e^{-2t}\sin 3t)\right]$

$\boxed{\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}}$

$= \frac{1}{2}\left[\frac{3}{(s-2)^2+3^2} - \frac{3}{(s+2)^2+3^2}\right]$

$= \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$

2) Sketch the graph of (?), $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

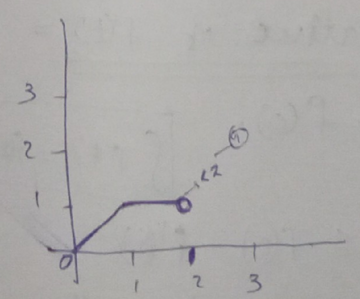
find L-transform?

A) $f(t)$ [illegible] 3 [illegible] → piecewise smooth curve.

$f(x) = x$

$f(x) = 1$

$1 \le t < 2$



[illegible] → piecewise conti~

clearly $f(t)$ is piecewise conti~ ( ) except 2.

∴ L.T for $f(t)$, (mean $\sim$ does formula patta (bcz interval))

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st} \times 1\,dt + 0$$

$\int$ by parts $$= \int_0^1 t\,e^{-st}\,dt + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \left[t\cdot\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 \left[1\cdot\frac{e^{-st}}{-s}\right] dt + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{-e^{-s}}{s} - \frac{1}{s^2}\left(\frac{e^{-s}}{s} - \frac{e^0}{s}\right) + \left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$$

Theorm =

I L. Transform of derivative of $f(t)$ =

$$\mathcal{L}(f'(t)) = s\cdot\mathcal{L}(f) - f(0).$$

$\boxed{\int f'(t) = f(t).}$

$$\mathcal{L}(f''(t)) = s^2\,\mathcal{L}(f) - s\cdot f(0) - f'(0)$$

$$\mathcal{L}(f'''(t)) = s^3\,\mathcal{L}(f) - s^2 f(0) - s\,f'(0) - f''(0)$$

II L.T of the $\int$ of $f(t)$ =

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\,F(s)$$