Module-4

01: Inverse of a Metrix.

(m) = metrix
(c) = cofactor

* A square metrix 'A' of order 'n' is invertible / non singular if there 'x' & exist.
* A square metrix 'B' of the same order such that $\boxed{AB = I = BA}$

Inverse of A, $\boxed{A^{-1} = B}$ $\boxed{(A^{-1})^{-1} = A}$

→ Adjoint Metrix :-

Let 'A' = $[a_{ij}]$ be a □ metrix of order 'n' & let $[C_{ij}]$ be the cofactor of

then the (m) $[C_{ij}]$ of cofactors of elements of 'A' is → Cofactor metrix of A. & its transpose (crossing of A & B...)

→ the adjoint (m) of A.

$$A = \begin{bmatrix} a_{12} & a_{12} \cdots \\ a_{21} & a_{22} \cdots \\ \vdots & \\ a_{n1} & a_{n2} \cdots \end{bmatrix}$$

$$\text{Co factor} = \begin{bmatrix} C_{11} & C_{12} \cdots \\ C_{21} & C_{22} \cdots \\ \vdots & \\ C_{n1} & C_{n2} \cdots C_{nn} \end{bmatrix}$$

(C) and adjoint

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} \cdots \\ C_{12} & C_{22} \cdots \\ C_{1n} & C_{2n} \cdots C_{nn} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{\text{Adj } A}{|A|}}$$

⇒ Algorithum for finding inverse of a □ metrix 'A' :-

* find |A|. , If |A| = 0 → A is not invertible,
  If |A| ≠ 0 → A exist.
* find cofactor of each of the elemnt 'A'.
* write down the (c) (m) of 'A'
* find the transpose of (c) (m) to get the Adj A.
* ÷ Adj A by |A| to get A^{-1}.

1) find Adj of (m) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A) To find A, want to find C.

$c_{11} = (-1)^{1+1}\, m$

$= d$

$c_{12} = (-1)^{1+2}\, m$

$= -c$

$c_{21} = (-1)^{2+1}\, m$

$= -b$

$c_{22} = (-1)^{2+2}\, m$

$= a$

$$\begin{pmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{pmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \downarrow \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$C_{11} \to C_{12}$ (a.d.)
$C_{11} \to C_{21}$ (c.f.)
viseversa

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2) find the inverse of (A). $(A^{-1} = ?)$

$$A = \begin{bmatrix} 1_{11} & 2_{12} & -2_{13} \\ -1_{21} & 3_{22} & 0_{23} \\ 0_{31} & -2_{32} & 1_{33} \end{bmatrix} \quad 3\times 3$$

$(-, +)$

A) $1\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + -2\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$

$1(3\times1-(-2\times0)) - 2(-1\times1-0) + -2(-1\times-2-0\times3)$

$= 3+2-4 = 1$

$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}$

$= 3$

$C_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$

$= 1$

$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$ $(-1\times-2)-(0\times3)$

$= 2$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix}$ $(2-4) = -2$

$-\times-2 = 2$

$= 2$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$= 1$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix}$ $(-1\times-2)-(3)$ $= 2$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6$ $\quad 2\times0-(3\times-2)$ $\quad 0 - -6 = 6$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = \cancel{1\times0-(-1\times2)} = 0+2 = 2$

$= -2$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

$$\text{Co factor} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & -2 & 5 \end{bmatrix}$$

$$\text{Adj } A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & -2 \\ 2 & 2 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & -2 \\ 2 & 2 & 5 \end{bmatrix}}{1} = \begin{pmatrix} \overset{+}{3} & \overset{-}{2} & \overset{+}{6} \\ \overset{-}{1} & \overset{+}{1} & \overset{-}{-2} \\ \overset{+}{2} & \overset{-}{2} & \overset{+}{5} \end{pmatrix}$$

3) $A = \begin{bmatrix} 3_{11} & -2_{12} & 3_{13} \\ 2_{21} & 1_{22} & -1_{23} \\ 4_{31} & -3_{32} & 2_{33} \end{bmatrix}$

A) $|A| =$

$$3\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}$$

$$3\left(1\times2-(-3\times-1)\right)+2\left(2\times2-(4\times-1)\right)+3\left(-3\times2-4\times1\right)$$

$$3(2-3)+2(4+4)+3(-6-4)$$

$$-3+16-30 = -17$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 2-(3\times-1) = -1$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 4-(4\times-1) = 4+4 = -8$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = -6-(4) = -10$$

$$\cancel{C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 2-(-3\times-1) = -1}$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} = -4-(-3\times3)$$
$$= -1(-4+9)$$
$$= -1\times5 = -5$$

(odd component = minus)

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} = \cancel{4+4=8}$$
$$= \cancel{6\times12}\ 6-12$$
$$= -6$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} = (3\times-3)-(4\times-2)$$
$$= -9-(-8) = -1(-9+8)$$
$$= -1(-1)$$
$$= 1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix}$$
$$= (-2\times-1)-3 = -1$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} = -3-6 = -9 = 9$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3+4 = 7$$

$$\text{co factor} = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}}{-17}$$

→ Row operation Method To find the Inverse :-

(I_n → identity (m))

If an $\frac{n}{n}$ 'A' can be transformed into the $\frac{n}{n}$ metrix, I_n, by a sequence of elementry row operations then 'A' is non singular. The same sequence of operation that transform "A" into I_n will also transform I_n into A^{-1}

Q/1) find inverse the (m).

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} = A \qquad A \times I_n$$

A)

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 5R_1$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 3 & 5 \\ 0 & 5 & 17/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \\ 5/2 & 0 & 1 \end{bmatrix}$$

$R_2 = \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 5 & 17/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 5/2 & 0 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 1 & 17/10 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1/30 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 1/6 & -1/3 & \frac{1}{5} \end{bmatrix}$$

$\frac{17}{2} \times \frac{1}{5} = \frac{17}{10}$

$\frac{5}{2} \times \frac{1}{5} = 1/2$

$1 \times \frac{1}{5} = 1/5$

$\frac{17}{10} - \frac{5}{3} = \frac{1}{30}$

$\frac{1}{2} - \frac{1}{3} = 1/6$

$R_3 \to 30R_3$.

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 5 & -10 & 6 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{2} R_3$

$R_2 \to R_2 - \frac{5}{3} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

2) $\begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$

A) $$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{-1} = -1$, $\frac{2}{-1} = -2$

~~$R_1 \to -R_1$~~

$R_1 \to -1 \cdot R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~$R_2 \to R_2 + 2R_1$~~

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 7 \\ -1 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 7 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 7/2 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 3/2 & 1/2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 7/2 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 3/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 7/2 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 1/2 & 0 \\ 3/2 & 1/2 & 0 \\ -\frac{1}{2} & 0 & 1/2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 7/2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 1/2 & 0 \\ 3/2 & 1/2 & 0 \\ -2 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$R_3 \longrightarrow -\frac{2}{5} R_3$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 7/2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 3/2 & 1/2 & 0 \\ 4/5 & \frac{1}{5} & -1/5 \end{bmatrix}$$

$R_1 \to R_1 - \frac{3}{2} R_3$

$R_2 \to R_2 - \frac{7}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -7/10 & \frac{1}{5} & \frac{3}{10} \\ -13/10 & -1/5 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -7/10 & 1/5 & 3/10 \\ -13/10 & -1/5 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

⇒ Using the inverse to solve systems of linear eq.

$$\boxed{X = A^{-1}B}$$

Q.
1) Solve the follow eq by finding inverse of coefficient (M).

$$x + 2y + 3z = 14$$
$$3x + y + 2z = 11$$
$$2x + 3y + z = 11$$

$Ax = B$

eq form

A)
$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

multiply A^{-1} on both side

$A^{-1}Ax = BA^{-1}$

$$Ax = B$$
$$A^{-1}Ax = BA^{-1}$$
$$Ix = BA^{-1}$$
$$\boxed{x = BA^{-1}}$$

$$A^{-1}A = I$$
$$I \times x = x.$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$3-4=-1$

$$= 1(1\times1 - 2\times3) - 2(3\times1 - 2\times2) + 3(3\times3 - 2\times1)$$
$$= -5 + 2 + 21 = \underline{18} \neq 0$$

adj A

1st find cofactor,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1\times1 - 3\times2 = \underline{-5}$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -(3-4) = -(-1) = \underline{1}$$

$(-1)^3$ → odd → -ve

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 9 - 2 = \underline{7}$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -(2-9) = -(-7) = \underline{7}$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = \underline{-5}$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3-4) = -(-1) = \underline{1}$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4-3 = \underline{\underline{1}}$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2-9) = -(-7) = \underline{\underline{7}}$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = \underline{\underline{-5}}$$

$$C = \begin{bmatrix} -5 & 1 & 7 \\ 7 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}}{18}$$

$$A^{-1} = \frac{1}{18}\begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$$

$$X = A^{-1} B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{vmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{vmatrix}\begin{vmatrix} 14 \\ 11 \\ 11 \end{vmatrix}$$

then multip.

$$= \frac{1}{18}\begin{bmatrix} -5\times 14 + 7\times 11 + 1\times 11 \\ 1\times 14 + -5\times 11 + 7\times 11 \\ 7\times 14 + 1\times 11 + -5\times 11 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} -70 + 77 + 11 \\ 14 - 55 + 77 \\ 98 + 11 - 55 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 18 \\ 36 \\ 54 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore x = 1 \quad y = 2 \quad z = 3$

Remrk 1

when $|A| \neq 0$, the solⁿ of the system $Ax = B$ is unique

X_1, X_2 are 2 different solⁿ vectors

$AX_1 = B \quad , \quad Ax_2 = B$ both $= B$.

$$Ax_1 = Ax_2$$

$$\underline{x_1 = x_2}$$

(or)
using A^{-1}

$$xA^{-1} = (A^{-1}A)x_1 = (A^{-1}A)x_2$$

$$Ix_1 = Ix_2$$

$$\underline{x_1 = x_2}$$

Remrk 2,

* A homogenous system of eq. can be written $AX = 0$
* A homogenous system always forces the trivial soln, $X = 0$ & possibly an infinite no. of non trivial soln.

If A is non singular, by remrk 1 the system $Ax = 0$ has a unique sol.

$$Ax = 0$$

$$x = A^{-1}0$$

$$\underline{x = 0}$$

→ Theorem 1:-

A homogenous system of a linear eq in 'n' variables $Ax = 0$ has only the trivial soln if an only if 'A' is non-singular (i.e) $|A| \neq 0$

→ Theorem 2:-

A homogenous system of 'n' linear eq in 'n' variables $Ax = 0$ has a non-trivial soln if an only if 'A' is singular.

→ properties of inverse :-

1) $(A^{-1})^{-1} = A$
2) $(AB)^{-1} = B^{-1} \cdot A^{-1}$
3) $(A^T)^{-1} = (A^{-1})^T$ T= Transpose.

A & B are non-singular (M).

Remrk
The (pro) -② can be extended to any no. of non-singular (M).

$A_1, A_2 \dots A_m$ are m non-singular (M) of same order

$$(A_1, A_2 \cdots A_m)^{-1} = A_m^{-1} \cdot A_{m-1}^{-1} \cdots A_2^{-1} A_1^{-1}.$$

using pro-②

(i.e) The A^{-1} of a product of non-singular (M) is the product of their inverses in reverse order.

9) If $A = \begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}$ $B = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$.

S.T $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

A) $A^{-1} = ?$.

$A^{-1} = \dfrac{\text{adj } A}{|A|}$.

$|A| = 5\,|2| - 3\,|2| = 10 - 6 = 4 \neq 0$.

Adj $A = ?$.

$C_{11} = (-1)^{1+1}\ 2 = 2$

$C_{12} = (-1)^{1+2}\ 2 = -2$

$C_{21} = (-1)^{2+1}\ 3 = -3$

$C_{22} = (-1)^{2+2}\ 5 = 5$

$C = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$.

Adj $A = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}$

$A^{-1} = \dfrac{\begin{matrix} 2 & -3 \\ -2 & 5 \end{matrix}}{4} = \dfrac{1}{4} \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}$

$B^{-1} = \dfrac{\text{adj } B}{|B|}$.

$|B| = 7\,|3| - 5\,|4| = 21 - 20 = 1 \neq 0$

adj $B = ?$.

$C_{11} = (-1)^{1+1}\ 3 = 3$

$C_{12} = (-1)^{1+2}\ 4 = -4$

$C_{21} = (-1)^{2+1}\ 5 = -5$

$C_{22} = (-1)^{2+2}\ 7 = 7$

$C = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$

Adj $B = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$

$B^{-1} = \dfrac{\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}}{1} = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$

$B^{-1} \cdot A^{-1} = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix} \cdot \dfrac{1}{4} \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}$

$= \cancel{\dfrac{1}{4} \begin{bmatrix} 6 & 10 \\ 12 & \end{bmatrix}}$

$= \cancel{\dfrac{1}{4} \begin{bmatrix} 6 + 10 \end{bmatrix}}$.

$$= \frac{1}{4}\begin{bmatrix} 3\times2 + -5\times-2 & 3\times-3 + -5\times5 \\ -4\times2 + 7\times-2 & -4\times-3 + 7\times5 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6 + 10 & -9 - 25 \\ -8 - 14 & 12 + 35 \end{bmatrix}$$

$$B^{-1}\cdot A^{-1} = \frac{1}{4}\begin{bmatrix} 16 & -34 \\ -22 & 47 \end{bmatrix}$$

$(AB)^{-1} = ?$. $\rightarrow$ $AB = ?$.

$$A\cdot B = \begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}\cdot\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}.$$

$$= \begin{bmatrix} 5\times7 + 3\times4 & 5\times5 + 3\times3 \\ 2\times7 + 2\times4 & 2\times5 + 2\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 35 + 12 & 25 + 9 \\ 14 + 8 & 10 + 6 \end{bmatrix}$$

$$A\cdot B = \begin{bmatrix} 47_{11} & 34_{12} \\ 22_{21} & 16_{22} \end{bmatrix}$$

$(AB)^{-1} = \frac{adj\,AB}{|AB|}$.

$$|AB| = 47|16| - 34|22|$$

$$= 752 - 748 = 4$$

~~$(AB)^{-1} = ad$~~

$adj\,AB = ?$.

$C_{11} = (-1)^{1+1}\,16 = 16$

$C_{12} = (-1)^{1+2}\,22 = -22$

$C_{21} = (-1)^{2+1}\,34 = -34$

$C_{22} = (-1)^{2+2}\,47 = 47$

$$C = \begin{bmatrix} 16 & -22 \\ -34 & 47 \end{bmatrix}$$

$$adj\,AB = \begin{bmatrix} 16 & -34 \\ -22 & 47 \end{bmatrix}$$

$$(AB)^{-1} = \frac{\begin{bmatrix} 16 & -34 \\ -22 & 47 \end{bmatrix}}{4} = \frac{1}{4}\begin{bmatrix} 16 & -34 \\ -22 & 47 \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 1/2 \\ 2 & 3/2 \end{bmatrix}$

$$A^{-1} = \frac{adj\,A}{|A|}.$$

$$|A| = 1|3/2| - 1/2|2| = \frac{3}{2} - \frac{2}{2}$$

~~$6 - 4$~~

$$= \frac{3-2}{2} = \frac{1}{2}$$

$\frac{1}{2}\times\frac{2}{1}$

$\frac{2}{2} = 1$

$\operatorname{adj} A = ?$

$C_{11} = (-1)^{1+1} \frac{3}{2} = 3/2$

$C_{12} = (-1)^{1+2}\, 2 = -2$

$C_{21} = (-1)^{2+1}\, 1/2 = -\frac{1}{2}$

$C_{22} = (-1)^{2+2}\, 1 = 1$

$$C = \begin{bmatrix} \frac{3}{2} & -2 \\ \frac{-1}{2} & 1 \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$\frac{\frac{3}{2}}{1/2} = \frac{3}{2} \cdot \frac{2}{1} = \frac{6}{2} = 3$

$$A^{-1} = \frac{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}}{1/2} = \frac{1}{2}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

$\frac{-2}{1/2}$

$\frac{-2}{1} \times \frac{2}{1} = -4$

$\frac{1}{1/2} = \frac{1}{1} \cdot \frac{2}{1} = 2$

3) $A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$A^{-1} = \frac{\operatorname{adj} A}{|A|}$.

$|A| = 1\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 0$

$= 1|0-2| + 1|3-2| + 0$

$= 1|-2| + 1|1| = -2+1$

$= -1$

$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 0-2 = -2$

$C_{12} = -1$

$C_{13} = 3$

$C_{21} = 1$

$C_{22} = 1$

$C_{23} = -2$

$C_{31} = -2$

$C_{32} = -2$

$C_{33} = 3$

$$C = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 1 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

$$\operatorname{Adj} A = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 1 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} -2 & 1 & -2 \\ -1 & 1 & -2 \\ 3 & -2 & 3 \end{bmatrix}}{-1}$$

$$= \begin{bmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{bmatrix}$$

→ Eigen value prbm :-

* Eigen values & vectors -

(E) →

Let A be an n × n (M).

A no. λ is said to be an (E) value [characteristic root] of A, if there exist an non zero soln vector K of linear system; $\boxed{AK = \lambda K}$.

The soln vector 'K' is said to be an (E) vector corresponding to the (E) value λ. we can write the eq $AK = \lambda K$ in the alternative form $(A - \lambda I)K = 0$ ——① , where I is the multiplicative Identity.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

eq

$$\boxed{|A - \lambda I| = 0}$$

$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{bmatrix}$

~~(A − λ)K~~

$(A - \lambda I)K \Rightarrow \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \cdots \\ 0 & 0 & \lambda & \cdots \end{bmatrix}\right) \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (a_{11}-\lambda)k_1 + a_{12}k_2 \cdots + a_{1n}k_n = 0 \\ a_{21}k_1 + (a_{22}-\lambda)k_2 \cdots + a_{2n}k_n = 0 \\ \vdots \\ a_{n1}k_1 + a_{n2}k_2 \cdots + (a_{nn}-\lambda)k_n = 0 \end{bmatrix} \text{——②}$$

$$|(A - \lambda I)K| = 0.$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

The (E) values of A are the roots of the characteristic eq ~~A~~ $|A-\lambda I| = 0$.

* To find an (E) vector corresponding an (E) value λ, we simply solve the system of eq $(A-\lambda I)k = 0$, by applying Gaussian Elimination / Gauss Jordan method.

Q) find the (E) values of (M)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

A) $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$$

def →

(cross)

$\Rightarrow (1-\lambda)\left[(-1-\lambda)(-1-\lambda)-(-2\times 0)\right] - 2\left[6(-1-\lambda)\right.$
$\left.(-1\times 0)\right] + 1\left[6\times -2 - (-1(-1-\lambda))\right] = 0$

$\Rightarrow (1-\lambda)(1+\lambda)^2 - 2\left[-6-6\lambda\right] + \left[-12-(1+\lambda)\right] = 0$

$\Rightarrow (1-\lambda)(1+\lambda)^2 \cancel{+12} + 12\lambda \cancel{-12} - (1+\lambda) = 0$

$\Rightarrow (1-\lambda)\left[1+2\lambda+\lambda^2\right] + 12\lambda - (1+\lambda) = 0$

$\Rightarrow \cancel{1} + 2\lambda + \lambda^2 - \lambda - 2\lambda^2 - \lambda^3 + 12\lambda - \cancel{1} - \lambda = 0$

$-\lambda^3 - \lambda^2 + 12\lambda = 0$ (avoid -ve to +ve)

$\lambda^3 + \lambda^2 - 12\lambda = 0$.

$\lambda(\lambda^2 + \lambda - 12) = 0$ or $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1^2-4\times 1\times -12}}{2\times 1}$

$\lambda = 0\ (\lambda+4)(\lambda-3) = 0$

$= \frac{-1 \pm \sqrt{49}}{2}$

$\lambda = 0 \quad \lambda+4 = 0 \quad \lambda - 3 = 0$

$\lambda = 0 \quad \lambda = -4 \quad \lambda = 3$

$= \frac{-1 \pm 7}{2}$

$\lambda = 0, -4, 3$

$\Rightarrow \frac{-1+7}{2} = \frac{6}{2} = 3 \qquad \frac{-1-7}{2} = \frac{-8}{2} = -4$

2) S.T the ~~eig~~ (E) values of a diagonal (M) are the same as its diagonal elements.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & & 0 \\ 0 & 0 & a_{33} & \cdots & a_{nn} \end{bmatrix}$$

A) $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & a_{nn} \end{bmatrix}$

$\boxed{|A - \lambda I| = 0} \longrightarrow$ characteristic eq.

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & \cdots & & & a_{nn}-\lambda \end{vmatrix} = 0$$

• $|\,| \to ?$

$(a_{11}-\lambda)\ (a_{22}-\lambda) \cdots (a_{nn}-\lambda) = 0$

$a_{11} - \lambda = 0 \qquad a_{22} - \lambda = 0 \qquad a_{nn} - \lambda = 0$

$a_{11} = \lambda \qquad a_{22} = \lambda \qquad a_{nn} = \lambda$

Remrk

In a similar manner we can S.T the (E) values of a Δ'ar (M) are also ~~Δ'ar~~ diagonal elemnts.

3) find (E) values & vectors of (M)

$A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$. form. $\to$ (a×d b×c)

A) $|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$

$\underset{(a-b)}{(5-\lambda)}\ \underset{(c-d)}{(2-\lambda)} - (-2 \times -2)$ = ac − ad − bc + bd.

$\Rightarrow 5\times 2 - 5\lambda - \lambda 2 + \lambda^2 - 4$

~~$\Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 + 4 = 0$~~

$\Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow 6 - 7\lambda + \lambda^2 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{7 \pm \sqrt{7^2 - 4\times 1\times 6}}{2\times 1}$

$= \dfrac{7 \pm \sqrt{49 - 24}}{2} = \dfrac{7 \pm \sqrt{5}}{2}$

$\Rightarrow \dfrac{7+5}{2}, \quad \dfrac{7-5}{2}$

$\Rightarrow \dfrac{12}{2} = 6, \quad \dfrac{2}{2} = 1$

$\lambda = 6, 1$

(E) vector

when $\lambda = 1$

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix}$$

$$\lambda = 1 \Rightarrow = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} \rightarrow \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4k_1 - 2k_2 = 0$

$-2k_1 + k_2 = 0$

$-2 \; , \; 1$

wants to 0. (deno).

$$\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} \rightarrow R_1 \rightarrow \frac{1}{2} R_1$$

$$\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} \Rightarrow R_2 \rightarrow R_2 + R_1.$$

$2+-2 = 0$

$-1+1 = 0$

$$\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}$$

rank = 1 [non zero rows = 1].

n = 2 [no-rows]

System has $(n-r) = 2-1 = 1$ soln.

$(A-\lambda I) K_1 = 0.$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(then multiply)

$2k_1 - k_2 = 0$

$2k_1 = k_2.$

$2 \times 1 = k_2$

Choose $k_1 = 1$ then $k_2 = 2$

$$K_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

when $\lambda = 6$,

$(A - 6I) K_2 = 0$

$$\begin{bmatrix} 5-6 & -2 \\ -2 & 2-6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-k_1 - 2k_2 = 0$

$-2k_1 - 4k_2 = 0$

$-2 \times -1 = 2.$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \quad R_1 \rightarrow -R_1$$

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \quad R_2 \rightarrow R_2 + 2R_1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

$(n-r) = 2-1 = 1$ soln.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$k_1 + 2k_2 = 0$

$k_1 = 2k_2$

Choose $k_2 = 1$ then $k_1 = -2$

$$K_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

4) find (E) values & (E) vectors of (M).

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

A) $|A - \lambda I| = 0$.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$= (8-\lambda)\left[(7-\lambda)(3-\lambda) - (-4\times -4)\right] - -6\left(-6\times(3-\lambda) - -4\times 2\right) + 2\left(-6\times -4 - (7-\lambda)\times 2\right).]$

$= (8-\lambda)\left[(7\times 3 - 7\times\lambda - \lambda\times 3 - \lambda\times\lambda - 16) + 6(-18+6\lambda+8) + 2(24-14+2\lambda)\right]$

$= (8-\lambda)\left[21 - 7\lambda - 3\lambda + \lambda^2 - 16) + 6(-10+6\lambda) + 2(10+2\lambda)\right].$

$= (8-\lambda)\left[(5 - 10\lambda + \lambda^2) + 6(-10+6\lambda) + 2(10+2\lambda)\right]$

$\cancel{(8-\lambda)}[$

$= \cancel{40} - \cancel{80}\lambda + \cancel{8\lambda^2} - \cancel{5}\lambda + \cancel{10\lambda^2} - \lambda^3 - \cancel{60} + 3\cancel{6}\lambda + \cancel{20} + 4\cancel{\lambda}$

$= -45\lambda + 18\lambda^2 - \lambda^3 \quad \div(-)$

$\Rightarrow \lambda^3 + 18\lambda^2 + 45\lambda$

$\lambda(\lambda^2 + 18\lambda + 45) = 0$

$\lambda = 0 \qquad \lambda^2 - 18\lambda + 45 = 0 \qquad \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

~~(or)~~ factorise

$\dfrac{18 \sqrt{18^2 - 4\times 1\times 45}}{2\times 1}$

$\dfrac{18 \pm \sqrt{324 - 180}}{2} = \dfrac{18 \pm \sqrt{144}}{2}$

$= \dfrac{18 \pm 12}{2}$

$\Rightarrow \dfrac{18+12}{2} \qquad \Rightarrow \dfrac{18-12}{2}$

$= 15 \qquad = 3$

$\therefore \lambda = 0, 15, 3 \rightarrow$ (E) values.

(E) vector

$\lambda = 0. \qquad |A - \lambda I| k_1 = 0.$

$|A - 0I| k_1 = 0.$

$A - 0I = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \qquad k_1 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8k_1 - 6k_2 + 2k_3 = 0$$
$$-6k_1 + 7k_2 - 4k_3 = 0$$
$$2k_1 - 4k_2 - 3k_3 = 0.$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad R_1 \to \frac{R_1}{8}$$

$$\begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad R_2 \to R_2 + 6R_1$$

$\to 7 + 6 \times -\frac{3}{4}$

$\frac{7}{1} - \frac{18}{4} = \frac{28-18}{4} = \frac{10}{4} = \frac{5}{2}$

$\to -4 + \frac{6}{4} = \frac{-16+6}{4} = \frac{-10}{4} = -5/2$

$$\begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 2 & -4 & 3 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$\to -4 - 2 \times -\frac{3}{4}$

$-4 + \frac{6}{4} = \frac{-16+6}{4} = \frac{-10}{4} = -5/2$

$$\begin{bmatrix} 1 & -3/4 & \frac{1}{4} \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix}$$

$\to 3 - \frac{2}{4} \Rightarrow \frac{12-2}{4} = \frac{10}{4} = 5/2$

~~$R_1 \to R_1 + \frac{3}{4}$~~

$$\begin{bmatrix} 1 & -3/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & -5/2 & 5/2 \end{bmatrix} \quad R_2 \to R_2 \cdot \frac{2}{5}$$

$$\begin{bmatrix} 1 & -3/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \quad R_3 \to R_3 + \frac{5}{2} R_2$$

$\frac{5}{2} \cdot \frac{2}{5} = \frac{10}{10} = 1$

$-\frac{5}{2} \cdot \frac{2}{5} = \frac{-10}{10} = -1$

$\frac{5}{2} + \frac{5}{2} \cdot -1$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \quad R_1 \to R_1 + \frac{3}{4} R_2.$$

Remove 1 row

$\frac{1}{4} + \frac{3}{4} \cdot -1$

$\frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = \frac{-1}{2}$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$R_3 \to \frac{R_3}{-10}$~~

~~$R_1 \to R_1$~~

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence (Co) of (M) for above (M)

rank = 2 < 3 i.e. no. of unknown

∴ this eq passes non trivial i.e the system has ~~non trivial soln~~.

$(n-r) = 3-2 = \underline{1}$

$\frac{1}{4} + 3/4 \times -1$

$\frac{1}{4} - 3/4 = -\frac{2}{4} = -1/2$

$5/2 + -5/2 = 0$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$k_1 - \frac{1}{2} k_3 = 0$ $\quad$ $k_2 - k_3 = 0$

$k_2 = k_3.$

$k_1 - \frac{1}{2} \times 1 = 0$

If $k_2 = 1$, $k_3 = 1$

$k_1 = \frac{1}{2}.$

$\therefore$ (fraction not, only whole no)

$$\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = k_1$$

$\lambda = 3$

$(A - 3I)\, k_2 = 0$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad k_1 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$8k_1 - 6k_2 + 2k_3 = 0$~~

$$(A - 3I) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$(A - 3I) = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5k_1 - 6k_2 + 2k_3 = 0$

$-6k_1 + 4k_2 - 4k_3 = 0$

$2k_1 - 4k_2 = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{5}$$

$$\begin{bmatrix} 1 & \frac{-6}{5} & 2/5 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \quad R_2 \to R_2 + 6R_1.$$

$4 + 6 \cdot -\frac{6}{5}$

$4 - \frac{36}{5} = \frac{20-36}{5} = \frac{-16}{5}$

$$\begin{bmatrix} 1 & -6/5 & 2/5 \\ 0 & -16/5 & -8/5 \\ 2 & -4 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$-4 + 6 \cdot \frac{2}{5}$

$-4 + \frac{12}{5} = \frac{-20+12}{5} = -8/5$

$\frac{-6}{5} \quad -\frac{8}{5}$

$4 - \frac{36}{5}$

$-16/5$

$$\begin{bmatrix} 1 & -6/5 & 2/5 \\ 0 & 14/5 & -8/5 \\ 0 & -8/5 & -4/5 \end{bmatrix}$$

$-4 + 6 \cdot \frac{2}{5}$

$-4 + \frac{12}{5}$

$-4 - 2 \times \frac{-6}{5}$

$-4 + \frac{12}{5}$

$0 - 2 \times 2/5$

$-4/5$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix} \qquad R_1 \to \frac{1}{2} R_1$$

$\frac{1}{2} \cdot -4 = -2$

$$\begin{vmatrix} 1 & -2 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{vmatrix} \qquad \begin{array}{l} R_2 \to R_2 + 6R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 4 & 2 \end{vmatrix} \qquad R_2 \to -\frac{1}{8} R_2.$$

$-\frac{1}{8} \cdot -4 = \frac{4}{8} = \frac{2}{4} = \frac{1}{2}$

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 4 & 2 \end{vmatrix} \qquad R_3 \to R_3 - 4R_2.$$

$2 - 4 \cdot \frac{1}{2} = 0$

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{vmatrix}$$

rank = 2

$(n-r) = 3-2 = \underline{1}$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$k_1 - 2k_2 = 0$ — (1)

$k_2 + \frac{1}{2} k_3 = 0$ — (2)

from (1)

$k_1 = 2k_2$ if $k_2 = 1$, $k_1 = 2$

from (2)

$1 + \frac{1}{2} k_3 = 0$

$\frac{1}{2} k_3 = -1$

$k_3 = \frac{-1}{1/2} = \frac{-2}{1}$

$\therefore k_3 = \underline{-2}$

$k_3 = -2$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$\underline{\lambda = 15}$

$$A = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} \quad \begin{vmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$|A-15I| = \begin{vmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7k_1 - 6k_2 + 2k_3 = 0$$
$$-6k_1 - 8k_2 - 4k_3 = 0$$
$$2k_1 - 4k_2 - 12k_3 = 0.$$

$$\begin{vmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{vmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{vmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{vmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$\begin{vmatrix} 1 & -2 & -6 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + 6R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\begin{vmatrix} 1 & -2 & -6 \\ 0 & -20 & -16 \\ 0 & -20 & -40 \end{vmatrix} \quad R_2 \to \frac{-1}{20} R_2$$

$$\begin{vmatrix} 1 & -2 & -6 \\ 0 & 1 & 4/5 \\ 0 & -20 & -40 \end{vmatrix} \quad R_3 \to R_3 + 20R_2$$

$$\begin{vmatrix} 1 & -2 & -6 \\ 0 & 1 & 4/5 \\ 0 & 0 & [illegible] \end{vmatrix}$$

$k_1 - 2k_2 - 6 = 0$ $\qquad$ $k_2 + 4/5\, k_3 = 0$

$$\therefore\ X_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

(5) find (E) values & (E) vectors –

$$M = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

A) - $|M - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & 2-\lambda \end{vmatrix} = 0.$$

$= (1-\lambda)\left[(1-\lambda)(-2-\lambda)-4\right] - (4)\left[\cancel{8} -4(-2-\lambda) - (2\cdot -2)\right] + 2\left[8-2+2\lambda\right] = 0$

$\cancel{\lambda^3 \; 27}$

$= (1-\lambda)\left[(1-\lambda)(2-\lambda)-4\right] + 4\left[8+4\lambda+4\right] + 2\left[8-2+2\lambda\right] = 0$

$\Rightarrow -\lambda^3 + 27\lambda + 54 = 0$

$\Rightarrow (6-\lambda)(\lambda^2+6\lambda+9) = 0$

$\lambda = 6, -3, -3.$

$\lambda = 6$

$(M - 6I)\, k_1 = 0.$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \begin{bmatrix} \cancel{x} \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & -2 & -8 \\ -4 & -5 & -2 \\ -5 & -4 & 2 \end{bmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$= \begin{vmatrix} 1 & -1 & -4 \\ -4 & -5 & -2 \\ -5 & -4 & 2 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 5R_1. \end{array}$$

$$= \begin{vmatrix} 1 & -1 & -4 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{vmatrix} \quad R_2 \to -\frac{1}{9} R_2. \quad -5 + 4 \times -1 = -9$$

$$= \begin{vmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & -9 & -18 \end{vmatrix} \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 9R_2. \end{array}$$

$$= \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

$(n - r) = 3 - 2 = 1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_1 - 2k_3 = 0 \qquad k_2 + 2k_3 = 0.$

$k_3 = 1 \qquad k_2 = -2$

$k_1 = 2$

$$k_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$\lambda = 3$

$(M + 3I)\, k_2 = 0.$

$$\begin{vmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$k_1 + 2k_2 - 3k_3 = 0$

$2k_1 + 4k_2 - 6k_3 = 0$

$-k_1 - 2k_2 + 3k_3 = 0.$

$$\begin{vmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{vmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$\begin{vmatrix} 2 & -2 & 1 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - \frac{1}{2} R_1 \end{array}$$

$$\begin{vmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$(n - r) = 3 - 1 = 2$

$$\begin{pmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$2k_1 - 2k_2 + k_3 = 0$

$k_3 = -2k_1 + 2k_2$

If $k_1 = 1$, $k_2 = 1$

$\to k_1 = 0$, $k_2 = 1$

$$k_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\approx$ $\lambda = -3$

$$k_3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

H/w
1) find (E) values ~~of (E) vectors~~

a) $A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$

A) $|A - \lambda I| = 0.$

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$= (6-\lambda)(4-\lambda) + 5$

$= 24 - 6\lambda - 4\lambda + \lambda^2 + 5$

$= 29 - 10\lambda + \lambda^2$

$\Rightarrow \lambda^2 - 10\lambda + 29 = 0$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{10 \pm \sqrt{10^2 - 4\times1\times29}}{2}$

$= \frac{10 \pm \sqrt{100-116}}{2} = \frac{10 \pm \sqrt{16}}{2}$

$= \frac{10 \pm 4}{2} \Rightarrow \frac{10+4}{2}, \frac{10-4}{2}$

$\Rightarrow 7, 3$

$\lambda = 7, 3$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$|A - \lambda I| = 0.$

$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$

$= (1-\lambda)[(-1-\lambda)(-1-\lambda) - 0] - 2[6\times(-1-\lambda) - 0]$

$+ 1[-12 - (1+\lambda)] = 0$

$= (1-\lambda)[(-1-\lambda)^2 - 0] - 2[-6-6\lambda-0] + 1$

$[-12 - (1+\lambda)] = 0$

$= -\lambda^3 - \lambda^2 + 12\lambda = 0.$

$\Rightarrow \lambda(\lambda+4)(\lambda-3) = 0$

$\lambda = 0, -4, 3$

c) (E) values & (E) vectors.

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0.$

$= (-2-\lambda)[-\lambda(1-\lambda) - \lambda - 12] - 2[-2\lambda - 6] - 3$

$[-4 + 1 - \lambda] = 0$

$= -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$

$\Rightarrow (5-\lambda)(\lambda^2 + 6\lambda + 9) = 0$ $\quad \lambda = 5, -3, -3$

$\underline{\lambda = 5}$

$|A - 5I| k_1 = 0.$

$$\begin{vmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}.$$

$$-7k_1 + 2k_2 - 3k_3 = 0$$
$$2k_1 - 4k_2 - 6k_3 = 0$$
$$-k_1 - 2k_2 - 5k_3 = 0.$$

$$\begin{vmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{vmatrix} \quad R_1 \leftrightarrow R_3.$$

$$\begin{vmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{vmatrix}. \quad \begin{matrix} R_1 \to -R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{vmatrix} \quad R_2 \to -\frac{1}{8} R_2$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 16R_2. \end{matrix}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

$(n - r) = 3 - 2 = \underline{1}$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$k_1 + k_3 = 0 \qquad k_2 + 2k_3 = 0$

If $k_3 = -1$, $k_1 = 1$ & $k_2 = 2$.

$$k_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\underline{\lambda = -3}$

$|A + 3I| k_2 = 0$

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$k_1 + 2k_2 - 3k_3 = 0$$
$$2k_1 + 4k_2 - 6k_3 = 0$$
$$-k_1 - 2k_2 + 3k_3 = 0$$

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{vmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad r = 1$$

$(n-r) = 3 - 1 = 2$

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \\ k_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$k_1 + 2k_2 - 3k_3 = 0$$

If $k_2 = 1$, $k_3 = 0$.
$k_2 = 0$, $k_3 = 1$

$$k_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\simeq \lambda = -3 \rightarrow k_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

→ Complex (E) values :-

Theorem → Let A be a □ (M) with real entries, If $\lambda = \alpha + i\beta$, $\beta \neq 0$, is a complex (E) value of A, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also an (E) value of A.
If k is an (E) vector to λ, then its conjugate $\bar{k}$ is an (E) vector to $\bar{\lambda}$.

Proof
Since A is a (M). with real entities. the c. eq → $\det(A - \lambda I) = 0$ is a polynomial eq of degree n.
from algebra, complex roots of such ~~entities~~ eq appear in conjugate pairs.
If $\lambda = \alpha + i\beta$ is a root of c. eq, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also a root of c. eq.
Hence if $\lambda = \alpha + i\beta$, $\beta \neq 0$ to (E) value λ. Then by definition $\boxed{Ak = \lambda k.}$
Taking complex conjugate of both sides of a eq, —

$$\overline{Ak} = \overline{\lambda k} \quad \text{(i.e)} \quad \bar{A}\,\bar{k} = \bar{\lambda}\,\bar{k}$$

$A \to$ (m) $\bar{A} = A$

$\therefore A\bar{k} = \bar{\lambda}\,\bar{k}$.

$\bar{k} \to$ (E) ~~val~~ vector of A to $\bar{\lambda}$

Q) find (E) value & (E) vector. $A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$

A) $\det(A - \lambda I) = 0$.

$$\lambda = \frac{-(-10) \pm \sqrt{10^2 - 4 \times 29}}{2} = \frac{10 \pm 4i}{2}$$
$$= 5 \pm 2i$$

(E) vectors of A,

$\lambda_1 = 5 + 2i$ $\quad\quad \lambda_2 = 5 - 2i$

(E) vectors to $\lambda_1 = 5 + 2i$,

$$[A - (5+2i)I]\, k = 0$$

$$\begin{vmatrix} 1-2i & -1 \\ 5 & -1-2i \end{vmatrix} \begin{vmatrix} k_1 \\ k_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$(1-2i)k_1 - k_2 = 0$ —(1)

$5k_1 - (1+2i)k_2 = 0$ —(2)

from (1)

we get, $k_2 = (1-2i)k_1$

from (2)

we get, $k_2 = \frac{5}{1+2i} k_1$

$$= \frac{5(1-2i)}{(1+2i)(1-2i)} k_1 = \frac{5(1-2i)}{1+4} k_1$$

$$= (1-2i)k_1$$

Choosing $k_1 = 1$, (E) vector to $\lambda_1 = 5+2i$,

$$k_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

(E) vector to $\lambda_2 = 5 - 2i = \overline{\lambda_1}$,

$$k_2 = \overline{k_1} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

→ (E) values & singular (M) :-

let A be n×n (M) & $\lambda = 0$ be (E) value the eq $(A - \lambda I)k = 0$ (i.e) $Ak = 0$ has a nonzero solⁿ.

Then the homogenous linear system of eq to (M) $Ak = 0$ has nontrivial solⁿ.

A homogenous system of n eq in n variables posses a non trivial solⁿ if & only if the coefficint (M) A is singular.

* Theorem :-

let A be n×n (M). Then the no. $\lambda = 0$ is an (E) value of A if and only if A is singular.

A (M) A is non singular if & only if the no. 0 is not an (E) value of A.

Proof

A is n×n (M). c. eq $\det(A - \lambda I) = 0$ is polynomial eq of degree n.

Let n roots be $\lambda_1, \lambda_2 \ldots \lambda_n$. By factor Theorem of algebra, c. polynomial $\det(A-\lambda I)$.

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda)$$

putting $\lambda = 0$,

$$\det A = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n.$$

Hence for any (M) A, $|A|$ is the product of its (E) values. Then it follows any $n \times n$ (M) A, the no. $\lambda = 0$ is an (E) value of A if and only if $\det A = 0$, that is if and only if A is singular.

→ (E) values & (E) vectors of A^{-1} :-

Theorem :-

Let A be a nonsingular (M). If λ is an (E) value of A with (E) vector k, then I/λ is an (E) value of A^{-1} with same corresponding (E) vector k.

Proof

Inverse of A, A^{-1} exists.

$$AA^{-1} = I = A^{-1}A.$$

A is nonsingular all (E) values of A are non zero. Let λ be an (E) value of A with (E) vector k,

$$Ak = \lambda k.$$

× both side with A^{-1}.

$$A^{-1}(Ak) = A^{-1}(\lambda k).$$

$$(A^{-1}A)k = \lambda A^{-1}k$$

(by associativity of (M) ×)

$$Ik = \lambda A^{-1}k.$$

we get, $k = \lambda A^{-1}k$ also so

$$A^{-1}k = (I/\lambda)k.$$

∴ I/λ is an (E) value of A^{-1} & k is (E) vector.

⇒ Cayley Hamilton theorem =

It is imp to compute A^{-1} | A^m, m is a +ve intgr of $n \times n$ (M) A.

$A^2 = A \cdot A$,
$A^3 = A \cdot A^2$.
$A^4 = A \cdot A^3$ ....

To sketch an alternative method for computing A^{-1} & A^m by means of follow therem → C.H.T.

$\boxed{\text{(i)} \rightarrow \text{An } n\times n \text{ (m) A satisfies its own characteristic eq.}}$

→ Finding inverse using C.H.T =

Let A be an $n\times n$ □ (m). The c.eq of A, $\det(A-\lambda I)=0$.
which is a polynomial eq in λ of degree n. Let c.eq be,

$a_0 + a_1\lambda + a_2\lambda^2 \dots a_n\lambda^n = 0.$

by C.H.T ; $\lambda = A$,

$a_0 I + a_1 A + a_2 A^2 \dots a_n A^n = 0$ — (1)

If A is non singular, A^{-1} exists

× (1) by A^{-1}.

$a_0 A^{-1} + a_1 I + a_2 A + \dots a_n A^{n-1} = 0.$

$A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \dots \frac{a_n}{a_0} A^{n-1}.$

Q.1) Find c.eq of (m) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$
& varify C.H.T &
hence obtain A^{-1} ?

A) $\det|A-\lambda I| = 0$.

$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0.$

$(2-\lambda)\left[(2-\lambda)^2 - 1\right] - (-1)\left[(\lambda-2+1)\right] + 1\left[1-(2-\lambda)\right] = 0$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

by C.H.T,

$\lambda = A$.

$\therefore A^3 - 6A^2 + 9A - 4I = 0$ — (2)

$A^3 = ?$
$A^2 = ?$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$A^2 = A \cdot A$.

$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$(2\times2) + (-1\times-1) + (1\times1) = 4+1+1 = 6$
$(2\times-1) + (-1\times2) + (1\times-1) = -2+-2+-1 = -5$ } 1st row
$(2\times1) + (-1\times-1) + (1\times2) = 2+1+2 = 5$
$(-1\times2) + (2\times-1) + (-1\times1) = -2+-2+-1 = -5$
$(-1\times-1) + (2\times2) + (-1\times-1) = 1+4+1 = 6$ } 2nd row
$(-1\times1) + (2\times-1) + (-1\times2) = -1+-2+-2 = -5$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A \cdot A^2$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, sub ②.

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$\Rightarrow \begin{vmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{vmatrix} - 6 \underbrace{\begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix}}_{\times 6} + 9 \underbrace{\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}}_{\times 9}$$

$$- 4 \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{\times 4} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{vmatrix} - \begin{vmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{vmatrix} + \begin{vmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{vmatrix}$$

$$- \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

$22 - 36 + 18 - 4 = 0$

$-21 + 30 + (-9) + 0 = 0 \Rightarrow 0$

$21 - 30 + 9 - 0 = 0$

$\therefore A^3 - 6A^2 + 9A - 4I = 0$ ③

Thus (m) satisfies c.eq. Hence C.H.T verified.

$$\boxed{\begin{matrix} A \cdot A^{-1} = I \\ I \cdot A^{-1} = A^{-1} \end{matrix}}$$

$A^{-1} = ?$

$\times A^{-1}$ in ③

$$A^3 \cdot A^{-1} - 6A^2 \cdot A^{-1} + 9A \cdot A^{-1} - 4I \cdot A^{-1} = 0$$

$$\Rightarrow A^3 \cdot A^{-1} - 6A^2 \cdot A^{-1} + 9A \cdot A^{-1} - 4I \cdot A^{-1} = 0$$

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$-4A^{-1} = -A^2 + 6A - 9I.$$

$$A^{-1} = \frac{1}{4}\left[-A^2 + 6A - 9I\right] \quad \text{(Avoid -ve)}$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$A^2 - 6A + 9I \Rightarrow$

$$\begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} + 9 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$A^2 - 6A + 9I = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

→ finding powers of a (m) using C.H.T =

let A be n×n (m).

$\det(A - \lambda I) = 0$

$a_0 + a_1\lambda + a_2\lambda^2 \dots a_n\lambda^n = 0$

by C.H.T, $\lambda = A$.

$\lambda^n = b_0 + b_1\lambda + b_2\lambda^2 \dots b_n\lambda^{n-1}$ — ①

$A^n = b_0 + b_1 A + b_2 A^2 \dots b_n A^{n-1}$ — ①

let $\lambda^m = C_0 + C_1\lambda + C_2\lambda^2 + \dots C_{n-1}\lambda^{n-1}$

from — ②

$A^m = C_0 I + C_1 A + C_2 A^2 \dots C_{n-1} A^{n-1}$

Q/1) $A = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}$ $A^m = ?$ for any integr $m \geq 2$ & find A^{10}

A) $\det(A - \lambda I) = 0$.

$\begin{pmatrix} -2-\lambda & 4 \\ -1 & -3-\lambda \end{pmatrix} = 0$.

$(-2-\lambda)(-3-\lambda) + 4 = 0$.

$-6 + 2\lambda - 3\lambda + \lambda^2 + 4 = 0$.

$-2 - \lambda + \lambda^2 = 0$

$\lambda^2 - \lambda - 2 = 0$. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = 2, -1$.

$\lambda_1 = 2 \quad \lambda_2 = -1$

by C.H.T,

$A^m = C_0 I + C_1 A \dots$ — ①

$\lambda^m = C_0 + C_1\lambda \dots$ — ②

$\lambda = 2$ & $\lambda = -1$ in eq — ②

$C_0 + 2C_1 = 2^m$ — ③

$C_0 - C_1 = (-1)^m \dots$ — ④ $\times 2$

we get, $3C_0 = 2^m + 2(-1)^m$

so $C_0 = \frac{1}{3}\left[2^m + 2(-1)^m\right]$.

Sub — ④, $C_1 = \frac{1}{3}\left[2^m - (-1)^m\right]$.

$$A^m = c_0 I + c_1 A$$

$$= \frac{1}{3}\left[2^m + 2(-1)^m\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{3}\left[2^m - (-1)^m\right]\begin{bmatrix}-2 & 4\\ -1 & 3\end{bmatrix}$$

$$A^m = \begin{bmatrix} \frac{1}{3}\left[-2^m + 4(-1)^m\right] & \frac{4}{3}\left[2^m - (-1)^m\right] \\ -\frac{1}{3}\left[2^m - (-1)^m\right] & \frac{1}{3}\left[2^{m+2} - (-1)^m\right] \end{bmatrix}$$

$$\therefore A^{10} = \begin{bmatrix} \frac{1}{3}\left[-2^{10} + 4(-1)^{10}\right] & \frac{4}{3}\left[2^{10} - (-1)^{10}\right] \\ -\frac{1}{3}\left[2^{10} - (-1)^{10}\right] & \frac{1}{3}\left[2^{12} - (-1)^{10}\right] \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} -340 & 1364 \\ -341 & 1365 \end{bmatrix}$$

Theorem :-

Let A be symmetric (m) with real entities. Then E. values of A are real.

Pf Let λ be E. value of A & k be the E. vector,

$$Ak = \lambda k.$$

Taking conjugate on both sides,

$$\overline{Ak} = \overline{\lambda k} \Rightarrow \bar{A}\,\bar{k} = \bar{\lambda}\,\bar{k}$$

$A \rightarrow$ real (m), $\bar{A} = A$

$$\therefore A\bar{k} = \bar{\lambda}\bar{k} \quad \text{—— (1)}$$

Taking Transpose of both side,

$$(A\bar{k})^T = (\bar{\lambda}\bar{k})^T \Rightarrow \bar{k}^T A^T = \bar{\lambda}\bar{k}^T$$

$A \rightarrow$ symmetric, $A^T = A$ so $\bar{k}^T A = \bar{\lambda}\bar{k}^T$ —— (2)

× both side by k,

$$\bar{k}^T A k = \bar{\lambda}\, \bar{k}^T k.$$

if we × $Ak = \lambda k$ by $\bar{k}^T$, we get.

$$\bar{k}^T (Ak) = \bar{k}^T (\lambda k) \Rightarrow \bar{k}^T A k = \lambda \bar{k}^T k \text{ — (3)}$$

from (2) & (3),

$$\bar{\lambda}\, \bar{k}^T k = \lambda \bar{k}^T k. \text{ — (4)}$$

k is E.vector of A, k is nonzero $n \times 1$ (m).

Let $k = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$ Then $\bar{k}^T = (\bar{k}_1, \bar{k}_2 \ldots \bar{k}_n)$

$$\therefore \bar{k}^T k = (\bar{k}_1, \bar{k}_2 \ldots \bar{k}_n) \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

$$= (|k_1|^2 + |k_2|^2 + \cdots + |k_n|^2) \neq 0.$$

from (4) we get $\bar{\lambda} = \lambda$. $\Rightarrow \lambda$ is real &

hence λ is an arbitrary constant.

⇒ Orthogonal E.vectors :-

(o) →

R^n → inner product / dot product.

$x = (x_1, x_2 \ldots x_n)$ & $y = (y_1, y_2 \ldots y_n)$,

$$x \cdot y = x_1 y_1 + x_2 y_2 \cdots x_n y_n. \text{ — (1)}$$

Now if x & y are $n \times 1$ column vectors.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

(1) → $x \cdot y = x^T y = x_1 y_1 + x_2 y_2 \ldots x_n y_n$

column vectors, $x^T y = y^T x$

2 column (v) x & y said to be (o) if

$$\boxed{x \cdot y = 0}$$

The norm of a column (v),

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$$

Theorem :-

Let A be $n \times n$ (m), Then E(v) corresponding to distinct E.values are (o).

Pf let λ_1 & λ_2 → 2 distinct E.values of A.

k_1 & k_2 → E.(v) of A.

$$Ak_1 = \lambda_1 k_1 \quad \& \quad Ak_2 = \lambda_2 k_2.$$

$$Ak_1 = \lambda_1 k_1 \Rightarrow (Ak_1)^T = (\lambda_1 k_1)^T \Rightarrow k_1^T A^T = \lambda_1 k_1^T$$

$(A^T = A)$

$$\Rightarrow k_1^T A = \lambda_1 k_1^T \Rightarrow k_1^T A k_2 = \lambda_1 k_1^T k_2.$$

$\hookrightarrow \times k_2$.

$$\Rightarrow k_1^T \lambda_2 k_2 = \lambda_1 k_1^T k_2 \quad (Ak_2 = \lambda_2 k_2)$$

$$\Rightarrow \lambda_2 k_1^T k_2 = \lambda_1 k_1^T k_2$$

$$\Rightarrow (\lambda_2 - \lambda_1) k_1^T k_2 = 0 \Rightarrow k_1^T k_2 = 0$$

$(\lambda_2 \neq \lambda_1)$

hence k_1 & k_2 are (o)

eg →

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$\lambda_1 = 0 \quad \lambda_2 = 3 \quad \lambda_3 = 15$

$k_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$ $k_2 = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$ $k_3 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

all E.values are distinct, it follows above theorem,

$k_1^T k_2 = \begin{pmatrix}1 & 2 & 2\end{pmatrix}\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 1\cdot(-2) + 2\cdot(-1) + 2\cdot 2 = 0.$

$k_1^T k_3 = \begin{pmatrix}1 & 2 & 2\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix} = 1\cdot 2 + 2\cdot(-2) + 2\cdot 1 = 0$

$k_2^T k_3 = \begin{pmatrix}-2 & -1 & 2\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix} = (-2)\cdot 2 + (-1)\cdot(-2) + 2\cdot 1 = 0$

⇒ Orthogonal matrix :- (O.M)

A □ (m) A is said to (O) if

$$\boxed{A^{-1} = A^T}$$

$AA^T = A^T A = I$

eg-: $A = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$.

$A^T = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

$A \cdot A^T = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$

$= \begin{bmatrix}\cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$

$= I$

$\approx A^T A = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = I$

hence $A^T = A^{-1}$ i.e A is (O) for all values of θ.

⇒ Theorem [Criterion for an O. matrix] =

An $n \times n$ (m) A is O., if and only if its column vectors $x_1, x_2, \dots x_n$ form an O. set.

problem 01

i) $A = \begin{bmatrix}1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3\end{bmatrix}$

A) $x_1 = \begin{pmatrix}1/3\\-2/3\\2/3\end{pmatrix}$ $x_2 = \begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}$ $x_3 = \begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix}$

$x_1^T x_2 = \begin{pmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix} = \frac{1}{3}\cdot\frac{2}{3} + \left(-\frac{2}{3}\right)\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} = 0$

$x_1^T x_3 = \begin{pmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix} = \frac{1}{3}\cdot\left(-\frac{2}{3}\right) + (-2/3)\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3} = 0$

$x_2^T x_3 = \begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix} = \frac{2}{3}\cdot\left(-\frac{2}{3}\right) + \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot 2/3 = 0$

hence column (v), x_1, x_2, x_3 of (m) A are (O).

Also, $\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$

$\|x_2\| = \sqrt{x_2^T x_2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$

$\|x_3\| = \sqrt{x_3^T x_3} = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$

2) If A & B are n-rowed (o)·(m), then both AB & BA are (o).

A) Since A & B are (o).

$\rightarrow A^{-1} = A^T$ & $B^{-1} = B^T$.

Since A & B are invertible, both are nonsingular & hence $|A| \neq 0$ & $|B| \neq 0$.

hence $|AB| = |A| \cdot |B| \neq 0$ &

$|BA| = |B|\,|A| \neq 0$, Thus both AB & BA are nonsingular.

$(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T$

$(BA)^{-1} = A^{-1}B^{-1} = A^T B^T = (BA)^T$.

∴ AB & BA are (o).

3) S.T determinant of every (o) (m) is ± 1

A) A be (o) (m).

$A \Rightarrow AA^T = I \Rightarrow |AA^T| = |I|$

$\Rightarrow |A|\,|A^T| = 1$. $\quad (|AB| = |A|\,|B|$ & $|I| = 1)$

$= |A| \cdot |A| = 1 \quad (|A| = |A^T|)$

$|A|^2 = 1 \Rightarrow |A| = \pm 1$

4) Construct an (o)(m) from E·vectors of

$$M = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

A) Since $M^T = M$, M is a symmetric (m)

Show that E·(v) of M are ~~...~~

$\lambda_1 = 6$, $\lambda_2 = \lambda_3 = -3$, & corresponding

E·(v) →

$k_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad k_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad k_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

observe that vector k_1 is (o) to k_2 & to k_3. But k_2 & k_3, e·(v) corresponding to repeated E·value $\lambda_2 = \lambda_3 = -3$ are not (o) since $k_2 \cdot k_3 = 1 \neq 0$. $\quad [1\ 1\ 0]\begin{bmatrix}0\\1\\2\end{bmatrix} = 1 \neq 0$

we use Gram-Schmidt orthogonalization process to transform the set $\{k_2, k_3\}$ into an (o) set.

Let $v_2 = k_2 = (1\ \ 1\ \ 0)^T$ &

$v_3 = k_3 - \frac{k_3 \cdot v_2}{v_2 \cdot v_2} v_2$,

$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{0\cdot1 + 1\cdot1 + 0\cdot0}{1\cdot1 + 1\cdot1 + 0\cdot0}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$V_3 = \begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\2\end{pmatrix}$$

$0-\frac{1}{2}\cdot 1 = -\frac{1}{2}$, $1-\frac{1}{2}\cdot 1 = \frac{1}{2}$, $2-\frac{1}{2}\cdot 0 = 2$

The set $\{v_2, v_3\}$ is an (o) set of (v).
the set $\{k_1, v_2, v_3\}$ is an (o) set of E.(v).

norm

$$\|k_1\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

$$\|v_2\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\|v_3\| = \sqrt{(-\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + 2^2} = \tfrac{3}{2}\sqrt{2}$$

$= \sqrt{\frac{1}{4}+\frac{1}{4}+4} = \sqrt{\frac{1}{2}+4} = \sqrt{\frac{1+8}{2}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = 3\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{3}{2}\sqrt{2}$ to avoid $\sqrt{\ }$

$\div$ (v) k_1, v_2 & v_3 by their norms,

$$= \begin{pmatrix}\frac{2}{3}\\-2/3\\1/3\end{pmatrix} \quad \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} \quad \begin{bmatrix}-1/3\sqrt{2}\\1/3\sqrt{2}\\4/3\sqrt{2}\end{bmatrix}$$

$\frac{-\frac{1}{2}}{\frac{3}{2}\sqrt{2}} = \frac{-1}{2\cdot\frac{3}{2}\sqrt{2}} = \frac{-1}{3\sqrt{2}}$, $\frac{\frac{1}{2}}{\frac{3}{2}\sqrt{2}} = \frac{1}{2\cdot\frac{3}{2}\sqrt{2}} = \frac{1}{3\sqrt{2}}$

& o) (m) $\rightarrow$

$$P = \begin{bmatrix}\frac{2}{3} & 1/\sqrt{2} & -1/3\sqrt{2}\\ \frac{-2}{3} & 1/\sqrt{2} & 1/3\sqrt{2}\\ 1/3 & 0 & 4/3\sqrt{2}\end{bmatrix}$$

$\Rightarrow$ **Diagonalization :-**

An $n\times n$ (m) said to be similar to an $n\times n$ (m) A if,

$$\boxed{M = P^{-1}AP}$$

for some nonsingular $n\times n$ (m) P, This transformation which gives M from A

$\rightarrow$ Similarity transformation.

Similarity transformation are imp since they preserve E. values. if M is similar to A, then M & A have the same characteristic eq & hence the same E. values.

A (m) A is said to (d) if it is similar to a diagonal (m).

Def $\rightarrow$ If a non singular $n\times n$ (m) P can be found so that,

$$\boxed{P^{-1}AP = D}$$

is a diagonal (m), then we say that the $n\times n$ (m) A can be diagonalized & that P diagonalizes A.

eg $\rightarrow$

$$D = \begin{bmatrix}d_{11} & 0 & 0\\ 0 & d_{22} & 0\\ 0 & 0 & d_{33}\end{bmatrix}.$$

If P_1, P_2 & $P_3 \rightarrow$ columns of P, by def of (m) multiplication, the eq $AP = PD$ is,

$$(AP_1 \quad AP_2 \quad AP_3) = (d_{11}P_1 \quad d_{22}P_2 \quad d_{33}P_3)$$

→ Theorum 1 :- [Sufficient condin for (D)]

If an $n \times n$ (m) A has a linearly independent E. vectors $k_1, k_2 \ldots k_n$, then A is diagonalizable i.e (m) P with these E. vectors $k_1, k_2 \ldots k_n$ as column (v) diagonalizes A.

→ Theorum 2 :- [criteria for (D)].

An $n \times n$ (m) A is (D) if & only if A has n linearly independent E.(v).

→ Theorum 3 :- [Sufficient condin for (D)]

If an $n \times n$ (m) A has n distinct E. values, then A is (D).

Q 1) (D) the (m),

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

A) $|A - \lambda I| = 0$.

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$6 - 7\lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 6) = 0$

$\lambda = 1, 6 \rightarrow$ E. values.

k_1 to $\lambda = 1$,

$(A - I) k_1 = 0$.

$$\lambda = 1 \quad \begin{vmatrix} 5-1 & 4 \\ 1 & 2-1 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix}$$

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$4k_1 - 4k_2 = 0$

$k_1 + k_2 = 0$.

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{by } R_1 \rightarrow \frac{1}{4} R_1$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \quad \text{by } R_2 \rightarrow R_2 - R_1.$$

$(n - r) = 2 - 1 = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$k_1 + k_2 = 0$

If we choose $k_1 = 1$ then $k_2 = -1$

$1 + k_2 = 0$

$k_2 = -1$

$$\therefore k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$k_2 \rightarrow \lambda = 6$

$(A - 6I) k_2 = 0$.

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$-k_1 + 4k_2 = 0$

$k_1 - 4k_2 = 0$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} \quad R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$(n-r) = 2-1 = 1$

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$k_1 - 4k_2 = 0$

If $k_2 = 1$ then $k_1 = 4$

$\therefore k_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Thus A has 2 independent E.V k_1 & k_2. Hence A is (D). Eq

(m) P ~~of~~ (D) A is,

$$P = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$\det P = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \cancel{1+4}\ 5 \quad 1+4 = 5 \neq 0$

then co-factor,

$C(a_{11}) = (-1)^{1+1} \cdot 1 = 1$

$C(a_{22}) = (-1)^{2+2} \cdot 1 = 1$

$C(a_{12}) = (-1)^{1+2} \cdot -1 = 1$

$C(a_{21}) = (-1)^{2+1} \cdot 4 = -4$

$C\ a = \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix}$

$\text{adj } P = \begin{vmatrix} 1 & -4 \\ 1 & 1 \end{vmatrix}$

By interchanging the diagonal elements & changing the signs of off-diagonal elements,

$$\text{adj } P = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

$|P| = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = 1-(-4) = 1+4 = 5$

$$P^{-1} = \frac{1}{|P|} \text{adj } P$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{5} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5-4 & 20+4 \\ 1-2 & 4+2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 24 \\ -1 & 6 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1+4 & 24-24 \\ 1-1 & 24+6 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

2) $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

A) $(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

$(3-\lambda)^2 - 4 = 0$

$\lambda^2 - 6\lambda + 5 = 0$ $\quad (\lambda-1)(\lambda-5) = 0,$

$\lambda = 1, 5.$

$\underline{\lambda = 1}$

$(A - I) K_1 = 0.$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2k_1 + 2k_2 = 0$

$2k_1 + 2k_2 = 0.$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$(n-r) = 2-1 = \underline{1}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$k_1 + k_2 = 0.$

If $k_1 = 1$ then $k_2 = -1.$

$K_1 = \underline{(1, -1)^T}$

$\underline{\lambda = 5}$

$(A - 5I) K_2 = 0.$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2k_1 + 2k_2 = 0$

$2k_1 - 2k_2 = 0$

$k_1 - k_2 = 0.$

If $k_1 = 1$ then $k_2 = 1.$

$K_2 = \underline{(1, 1)^T}$

$A \to 2$ independt E.(V), K_1 & K_2.

$A \to$ (D) & (m) P is (D) A. is,

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1+1 = \underline{2}$$

by interchanging diagonal elements,

co-factor = ...

$\text{adj } P = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$\boxed{P^{-1} = \frac{1}{|P|} \text{adj } P}$

$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\boxed{P^{-1}AP} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3-2 & 3+2 \\ 2-3 & 2+3 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1+1 & 5-5 \\ 1-1 & 5+5 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

3) (D) →

$A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$

a) $(A - \lambda I) = 0$

$\begin{vmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{vmatrix} = 0$

$(3-\lambda)(7-\lambda) + 4 = 0$

$\lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda - 5)^2 = 0$

$\lambda_1 = \lambda_2 = 5.$

$\lambda = 5$

$-2k_1 + 4k_2 = 0$

$-k_1 + 2k_1 = 0.$

$k_1 = 2$

$k_2 = 1$

$k = (2, 1)^T$

~~$k_1 - 2k_2 = 0$~~

~~$k_1 = 2$~~ ~~$k_2 = 1$~~

A is a 2×2 (m) has only 1 independent e.v.

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4) (D) → $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$

a) $|A - \lambda I| = 0$

$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$

$-\lambda [(-\lambda)(1-\lambda) - 0] - 1[(1-\lambda) - 0] + 0 = 0$

$-\lambda^3 + \lambda^2 + \lambda - 1 = 0$

$-(\lambda + 1)(\lambda - 1)^2 = 0.$ $\lambda = -1, 1, 1$

$\lambda = -1$

$(A + I)k_1 = 0.$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$, $k_1 + k_2 = 0$

$k_1 + k_2 = 0$

$2k_3 = 0$

$k_3 = 0$, $k_1 + k_2 = 0$.

If we choose $k_1 = 1$ then $k_2 = -1$

$$k_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1}$

$(A - I) k_2 = 0$.

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad , -k_1 + k_2 = 0 \quad , k_1 - k_2 = 0$$

choosing $k_3 = 0$ $k_1 = 1$ & $k_3 = 1$, $k_1 = 0$

$$k_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A is (D) i.e (m) P which (D) A,

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, $|P| = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) - 1(-1-0) + 0$

$= \underline{\underline{2}}$

Co-factors

$C_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\qquad C_{12} = (-1)^3 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$C_{13} = (-1)^5 \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0$ $\qquad C_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$

$C_{22} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\qquad C_{23} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$ $\qquad C_{32} = (-1)^5 \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 0$

$C_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

$$\therefore \text{adj } P = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{adj } P = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

hence

$$P^{-1} A P = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ Orthogonal Diagonalizability :-

(O) we can find n mutually (O) E. vectors, we can use an (O) (M) P to diagonalize A. A symmetric (M) is said to be orthogonally (D).

eg →

$$M = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

$|M - \lambda I| = 0.$

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(1-\lambda)(-\lambda-2) - 4] + 4[8 + 4\lambda + 4] + 2[8 - 2 + 2\lambda] = 0.$

$-\lambda^3 + 27\lambda + 54 = 0$

$(6-\lambda)(\lambda^2 + 6\lambda + 9) = 0.$

$\lambda = 6, -3, -3$

$\lambda = 6$

$(M - 6I) k_i = 0.$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x - 4y + 2z = 0$

$-4x - 5y - 2z = 0$

$2x - 2y - 8z = 0.$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -8 \\ -4 & -5 & -2 \\ -5 & -4 & 2 \end{bmatrix} \quad \text{by } R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -4 \\ -4 & -5 & -2 \\ -5 & -4 & 2 \end{bmatrix} \quad \text{by } R_1 \to \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{bmatrix} \quad R_2 \to R_2 + 4R_1,\ R_3 \to R_3 + 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & -9 & -18 \end{bmatrix} \quad R_2 \to -\tfrac{1}{9} R_2.$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + R_2,\ R_3 \to R_3 + 9R_2$$

$(n-r) = 3-2 = 1$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$k_1 - 2k_3 = 0$ $\quad$ $k_2 + 2k_3 = 0$.

choose $k_3 = 1$ then $k_1 = 2$ & $k_2 = -2$.

$$k_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_1 + 2k_2 - 3k_3 = 0$

$2k_1 + 4k_2 - 6k_3 = 0$

$-k_1 - 2k_2 + 3k_3 = 0$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{2}R_1$$

$(n-r) = 3-2 = 1$.

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2k_1 - 2k_2 + k_3 = 0$

$k_3 = -2k_1 + 2k_2$

choosing $k_1 = 1$, $k_2 = 1$ & $k_1 = 0$, $k_2 = 1$

$$K_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = -3$ $\quad$ $k_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

hence linear combi... of k_2 & k_3 are E.(v) to E. value $\lambda = -3$.

from these independent E.(v) we construct the follo... set of (o) set by using Gram-schmidt orthogonalization process :-

$$\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$$

we can use these (v) as columns to construct an (o) (m) that (P) M,

$$|P| = \frac{2}{3}\left[\frac{1}{\sqrt{2}} \times \frac{4}{3\sqrt{2}} - 0\right] - \frac{1}{\sqrt{2}}\left[-\frac{2}{3} \cdot \frac{4}{3\sqrt{2}} - \frac{1}{3} \cdot \frac{1}{3\sqrt{2}}\right] - \frac{1}{3\sqrt{2}}\left[\frac{-2}{3} \times 0 - \frac{1}{3} \cdot \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{3}\left[\frac{4}{\cancel{6}_{3}}\right] - \frac{1}{\sqrt{2}}\left[\frac{-8}{9\sqrt{2}} - \frac{1}{9\sqrt{2}}\right] - \frac{1}{3\sqrt{2}} \cdot \frac{-1}{3\sqrt{2}}$$

$$= \frac{4}{9} - \frac{1}{2}[-1] + \frac{1}{18}$$

$$= \frac{4}{9} + \frac{1}{2} + \frac{1}{18} = \frac{9}{18} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

co-factor,

$$C_{11} = (-1)^2\left[\frac{1}{\sqrt{2}} \cdot \frac{4}{3\sqrt{2}}\right] = \frac{4}{6} = \frac{2}{3}$$

$$C_{12} = (-1)^3\left[\frac{-2}{3} \cdot \frac{4}{3\sqrt{2}} - \frac{1}{3\sqrt{2}} \cdot \frac{1}{3}\right] = -\left[\frac{-8}{9\sqrt{2}} - \frac{1}{9\sqrt{2}}\right] = 1$$

$$C_{13} = (-1)^4\left[-\frac{1}{3} \cdot \frac{1}{\sqrt{2}}\right] = \frac{-1}{3\sqrt{2}}$$

$$C_{21} = (-1)^3\left[\frac{1}{\sqrt{2}} \cdot \frac{4}{3\sqrt{2}}\right] = -\left[\frac{4}{6}\right]$$

$$C_{22} = (-1)^4\left[\frac{2}{3} \cdot \frac{4}{3\sqrt{2}} - \frac{1}{3} \cdot \frac{-1}{3\sqrt{2}}\right]$$

$$= \left[\frac{8}{9\sqrt{2}} + \frac{1}{9\sqrt{2}}\right] = 1/\sqrt{2}$$

$$C_{23} = (-1)^5\left[\frac{-1}{3\sqrt{2}}\right] = \frac{1}{3\sqrt{2}}$$

$$C_{31} = (-1)^4\left[\frac{1}{6} + \frac{1}{6}\right] = \frac{12}{36} = \frac{2}{6} = \frac{1}{3}$$

$$C_{32} = (-1)^5\left[\frac{2}{3} \cdot \frac{1}{3\sqrt{2}} - \frac{-2}{3} \cdot \frac{-1}{3\sqrt{2}}\right] - \left[\frac{2}{9\sqrt{2}} + \frac{-2}{9\sqrt{2}}\right] = 0$$

$$C_{33} = (-1)^6\left[\frac{2}{3} \cdot \frac{1}{\sqrt{2}} - \frac{-2}{3} \cdot \frac{1}{\sqrt{2}}\right]$$

$$= \left[\frac{2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}\right] = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\text{c.factor} = \begin{bmatrix} \frac{2}{3} & 1 & -1/3\sqrt{2} \\ -4/6 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 1/3 & 0 & 2\sqrt{2}/3 \end{bmatrix}$$

$$\frac{2 \cdot 3\sqrt{2} + 2 \cdot 3\sqrt{2}}{18} = \frac{6\sqrt{2}+6\sqrt{2}}{18} = \frac{6(\sqrt{2}+\sqrt{2})}{6 \cdot 3} = \frac{2\sqrt{2}}{3}$$

$$\text{Adj } P = \begin{bmatrix} \frac{2}{3} & -2/3 & 1/3 \\ 1 & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & \frac{1}{3\sqrt{2}} & 2\sqrt{2}/3 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } P}{|P|} = \begin{bmatrix} \frac{2}{3} & -2/3 & 1/3 \\ 1 & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & 1/3\sqrt{2} & 2\sqrt{2}/3 \end{bmatrix}$$

$$P^{-1}MP = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1 & 1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & 1/3\sqrt{2} & 2\sqrt{2}/3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ -2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 1/3 & 0 & 4/3\sqrt{2} \end{bmatrix}$$

$$MP = \begin{bmatrix} \frac{2}{3} + -4\cdot\frac{-2}{3} + 2\cdot\frac{1}{3} & \frac{1}{2\sqrt2} - \frac{4}{\sqrt2} & \frac{-1}{3\sqrt2} \cdot \frac{-4}{3\sqrt2} + \frac{8}{3\sqrt2} \\ -4\cdot\frac{2}{3} + 1 \times \frac{-2}{3} + -2 \times \frac{1}{3} & -4\cdot\frac{1}{\sqrt2} + \frac{1}{\sqrt2} & \frac{4}{3\sqrt2} + \frac{1}{3\sqrt2} - \frac{8}{3\sqrt2} \\ \frac{4}{3} + \frac{4}{3} - \frac{2}{3} & \frac{2}{\sqrt2}\ \frac{-2}{\sqrt2} & \frac{-2}{3\sqrt2}\ \frac{-2}{3\sqrt2}\ \frac{-8}{3\sqrt2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{8}{3} + \frac{2}{3} & \frac{-3}{\sqrt2} & \frac{3}{3\sqrt2} \\ \frac{-8}{3} - \frac{2}{3}\ \frac{-2}{3} & \frac{-3}{\sqrt2} & \frac{-3}{3\sqrt2} \\ 6/3 & 0 & \frac{-12}{3\sqrt2} \end{bmatrix}$$

$$MP = \begin{bmatrix} 4 & -3/\sqrt2 & 3/3\sqrt2 \\ -4 & -3/\sqrt2 & -1/\sqrt2 \\ 2 & 0 & -4/\sqrt2 \end{bmatrix}$$

$$P^{-1}MP = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1 & 1/\sqrt2 & 0 \\ 1/3\sqrt2 & 1/3\sqrt2 & \frac{2\sqrt2}{3} \end{bmatrix} \begin{bmatrix} 4 & -3/\sqrt2 & 3/3\sqrt2 \\ -4 & -3/\sqrt2 & 1/\sqrt2 \\ 2 & 0 & -4/\sqrt2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \times 4 + \frac{-2}{3} \times -4 + \frac{1}{3} \times 2 & \frac{2}{3} \times \frac{-3}{\sqrt2} - \frac{2}{3} \times \frac{-3}{\sqrt2} & \frac{2}{3} \cdot \frac{3}{3\sqrt2} - \frac{2}{3}\cdot\frac{1}{\sqrt2} + \frac{1}{3}\cdot\frac{-4}{\sqrt2} \\ 4-4 & \left(\frac{-3}{\sqrt2} - \frac{3}{\sqrt2}\right)\frac{1}{\sqrt2} & \frac{3}{3\sqrt2} - \frac{1}{\sqrt2} \\ \frac{-4}{3\sqrt2} - \frac{4}{3\sqrt2} + \frac{2}{3\sqrt2} & + \frac{3}{6} - \frac{3}{6} & \frac{-3}{18} - \frac{1}{6} + \frac{8}{\sqrt2} \cdot \frac{\sqrt2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{3} + \frac{8}{3} + \frac{2}{3} & \frac{-6}{3\sqrt2} + \frac{6}{3\sqrt2} & \frac{6}{9\sqrt2} + \frac{2}{3\sqrt2} - \frac{4}{3\sqrt2} \\ 0 & \frac{-6}{\sqrt2} \times \frac{1}{\sqrt2} & 0 \\ 0 & 0 & -\frac{1}{6} - \frac{1}{6} + \frac{8}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & \frac{6}{9\sqrt2} - \frac{2}{3\sqrt2} \\ 0 & \frac{-6}{\sqrt2}\,\frac{1}{\sqrt2} & 0 \\ 0 & 0 & -\frac{2}{6} + \frac{8}{3} \end{bmatrix}$$

$$P^{-1}MP = P^TMP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

⇒ <u>Application of Diagonalization</u> :-

A center conic section (hyperbola / ellipse) with centre at origin has eq,

$$\boxed{ax^2 + 2bxy + cy^2 = d,} \quad \text{——} ①$$

$a, b, c, d \to$ constant.

* An algebric exp of the form $ax^2 + 2bxy + cy^2$

→ <u>quadratic form</u>. → (M) form ⇒ X^TAX.

In (m) form eq ① be,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d \quad \text{(or)} \quad X^TAX = d$$

~~$X^TAX = d$~~

where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \end{bmatrix}$

let the axes (x', y') be rotated by

some angle θ from (x,y), Then (x,y) & (x',y') coordinates of a point are related by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix} \quad — ③$$

where $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is (o) mtrix.

Taking transpose of —③,

$$[x\ y] = [x'\ y'] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$[x\ y] = [x'\ y'] \quad P^T = (x'\ y')\ P^{-1} \quad — ④$$

(as) P is an (o) (m).

Sub ③ – ④ into ②,

$$(x'\ y')\ P^{-1}AP \begin{pmatrix} x' \\ y' \end{pmatrix} = d \quad — ⑤$$

If P is (m) which (D) A, then —⑤ is equation of conic relative to its principal axes.

$ax^2 + 2bxy + cy^2 = d$

Q

1) Identify the conic $2x^2 + 4xy - y^2 = 1$.

A) In (m) form, $(a = 2,\ b = 2,\ c = -1$ & $d = 1)$

$4xy$

$2bxy \to 2 \cdot ② xy$

$$(x\ y)\ A \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

c. eq of (m) A,

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (2-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow -2 - \lambda + \lambda^2 - 4 = 0.$

$(2-\lambda)(-1-\lambda) - 4 = 0$, $\lambda^2 - \lambda - 6 = 0.$

$(\lambda - 3)(\lambda + 2) = 0.$

$\lambda = -2,\ \lambda = 3 \to$ E. values

let P be (o) (m), we know that P (D) A

so, $P^{-1}AP = D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}.$

hence eq of given conic relative to principal axes is,

$$\boxed{(x'\ y')\ P^{-1}AP \begin{pmatrix} x' \\ y' \end{pmatrix} = d} \Rightarrow (x'\ y') \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 1.$$

$$\Rightarrow (x'\ y') \begin{bmatrix} -2x' \\ 3y' \end{bmatrix} = 1 \Rightarrow -2x'^2 + 3y'^2 = 1$$

which is standard form of hyperbola

2) Find eq of quadric surface

$$x^2 + y^2 + z^2 + 4xy + 2xz - 2yz = 12$$

relative to p. axes?

$$x^2+y^2+z^2+4xy+2xz-2yz=12$$

$$\Rightarrow x^2+2xy+xz+2xy+y^2-yz+xz-yz+z^2=12.$$

$$\Rightarrow x(x+2y+z)+y(2x+y-z)+z(x-y+z)=12$$

$$\Rightarrow (x\ y\ z)\begin{bmatrix}1&2&1\\2&1&-1\\1&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=12$$

In (m) form,

$$(x\ y\ z)\ A\begin{pmatrix}x\\y\\z\end{pmatrix}=12 \qquad A=\begin{bmatrix}1&2&1\\2&1&-1\\1&-1&1\end{bmatrix}$$

A should be symmetric

$$|A-\lambda I|=0.$$

$$\begin{vmatrix}1-\lambda&2&1\\2&1-\lambda&-1\\1&-1&1-\lambda\end{vmatrix}=0$$

$-2[2(1-\lambda)+1]=-2[2-2\lambda+1]$
$=4+4\lambda+2=6+4\lambda$

$-2-(1-\lambda)$
$-2-1+\lambda$
$-3+\lambda$

$$(1-\lambda)[(1-\lambda)^2-1]-2[2(1-\lambda)+1]+[-2-(1-\lambda)]=0$$

$$(1-\lambda)[\lambda^2-2\lambda]-6+4\lambda-3+\lambda=0$$

$$-\lambda^3+3\lambda^2+3\lambda-9=0 \qquad -2\lambda+\lambda^2+2\lambda^2-\lambda^3+5\lambda-9=0$$

$$(\lambda-3)(\lambda^2-3)=0 \qquad -\lambda^3+3\lambda^2+3\lambda-9=0$$

give $\lambda=1,2 \neq 0$

$$\lambda=3,\ \sqrt{3},\ -\sqrt{3}$$

So $(\lambda-3)(\lambda-\sqrt{3})(\lambda+\sqrt{3})=0$

$\lambda=3,\sqrt{3},-\sqrt{3}$

Let P be (o) (m), we know that P (D) A

$$P^{-1}AP=D=\begin{bmatrix}3&0&0\\0&\sqrt{3}&0\\0&0&-\sqrt{3}\end{bmatrix}.$$

q. surface relative to p. axes is,

$$(x'\ y'\ z')\ P^{-1}AP\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=12$$

$$(x'\ y'\ z')\begin{bmatrix}3&0&0\\0&\sqrt{3}&0\\0&0&-\sqrt{3}\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=12$$

$$(x'\ y'\ z')\begin{bmatrix}3x'\\\sqrt{3}y'\\-\sqrt{3}z'\end{bmatrix}=12$$

$$3x'^2+\sqrt{3}y'^2-\sqrt{3}z'^2=12$$

Rmrk. In last eq, if the order of e. values in D is changed, we get anthr eq for conic / q. surface relative to the p. axes. This amounts simply to interchanging the x' & y' or x', y' & z' axes.