ASSIGNMENT

1) Given the frequency function,

$$f(x,\theta) = \frac{e^{-x^2/2\theta^2}}{\theta\sqrt{2\pi}}, \quad -\infty < x < \infty.$$

find the maximum likelihood estimator of θ?

A) $$L(\theta) = \left(\frac{1}{\theta\sqrt{2\pi}}\right)^n e^{-(x_1^2+x_2^2+\dots+x_n^2)/2\theta^2}$$

$$\log L = -n\log\theta - n\log\sqrt{2\pi} - \frac{1}{2\theta^2}(x_1^2+x_2^2+\dots x_n^2)$$

$$\frac{\partial \log L}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}(x_1^2+x_2^2+\dots x_n^2)$$

$$= 0.$$

$$\therefore \theta = \sqrt{\frac{x_1^2+x_2^2+\dots+x_n^2}{n}}$$

This is the maximum likelihood estimator of θ.

2) Let n be the no. of Bernoulli's trials, x the no. of success in a series of n trials with probability of success p for each trial, then show that (x/n) is a consistent estimator of p?

A) for large samples, $\frac{x}{n} \to N\left(p, \frac{\sqrt{pq}}{n}\right)$.

$$Z = \frac{(x/n) - p}{\sqrt{pq/n}} \longrightarrow N(0,1) \text{ as } n \to \infty.$$

$$\therefore \lim_{n\to\infty} P\left[\left|\frac{x}{n} - p\right| < \varepsilon\right] > 1 - \delta$$

$$\lim_{n\to\infty} P\left[\frac{(x/n) - p}{\sqrt{pq/n}} < \frac{\varepsilon}{\sqrt{pq/n}}\right] > 1 - \delta$$

$$\lim_{n\to\infty} P\left[|z| < \frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right] > 1 - \delta$$

$$\lim_{n\to\infty} P\left[-\frac{\varepsilon\sqrt{n}}{\sqrt{pq}} < z < \frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right] > 1 - \delta$$

$$\lim_{n\to\infty} P\left[-\frac{\varepsilon\sqrt{n}}{\sqrt{pq}} < z < \frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right] \longrightarrow$$

$$P[-\infty < z < \infty] \longrightarrow 1$$

Hence x/n is a consistent estimator of p.

3) For n Bernoullis trials with a probability of success p, $\sum_{i=1}^{n} x_i$ the no. of successes in the n trials is a sufficient estimator for p?

A) $f(x_i, p) = p^{x_i}(1-p)^{1-x_i}, \quad x_i = 0, 1.$

$$L_2(x_1, x_2 \dots x_n ; p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \times \dots \times p^{x_n}(1-p)^{1-x_n}$$

$$L = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

$$L = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i} \times 1$$

$$L_1(\Sigma x_i, p) = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

$$L_2(x_1, x_2 \dots x_n) = 1$$

$$\therefore \; L = L_1(\Sigma x_i, p) \cdot L_2(x_1, x_2, \dots x_n).$$

hence Σx_i is a sufficient estimator for p.

4) The hypothesis $H_0 : \theta = 2$ is accepted against $H_1 : \theta = 5$ if $x \leq 3$ when x has a exp distri... with mean θ, find type I type II error probabilities of the test?

A) $f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0, \; \theta > 0$

$$p(\text{type I error}) = p(\text{rejecting } H_0 / H_0)$$

$$= p(x > 3 \mid \theta = 2)$$

$$= \int f(x)\,dx \quad \text{when } \theta = 2$$

$$= \int_3^{\infty} \frac{1}{\theta} e^{-x/\theta} \quad \text{when } \theta = 2$$

$$= \int_3^{\infty} \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_3^{\infty}$$

$$= -\left[0 - e^{-3/2}\right] = \underline{\underline{e^{-3/2}}}$$

$$p(\text{type II error}) = p(\text{Accept } H_0 \mid H_1)$$

$$= p(x \leq 3 \mid \theta = 5)$$

$$= \int_0^3 \frac{1}{5} e^{-x/5}\,dx$$

$$= \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^3$$

$$= -\left[e^{3/5} - 1\right]$$

$$= \underline{\underline{1 - e^{-3/5}}}$$

5) A box is known to contain either 3 red & 5 black balls / 5 red & 3 black balls. 3 balls are to be drawn at random & it is concluded that the former is true if the number of red balls is less than 3 in the sample. Find α & β the probabilities of 2 types of errors?

A) $H_0 : \theta = 3$ against $H_1 : \theta = 5$

$$\alpha = p(\text{type I error})$$
$$= p(\text{Reject } H_0 \mid H_0 \text{ true})$$
$$= p(x \geq 3 \mid \theta = 3)$$
$$= 1 - p(x < 3 \mid \theta = 3)$$
$$= 1 - \{p(x=0) + p(x=1) + p(x=2)\} \quad \text{when } \theta = 3$$
$$= 1 - \left[\frac{\binom{5}{3}}{\binom{8}{3}} + \frac{\binom{3}{1}\binom{5}{2}}{\binom{8}{3}} + \frac{\binom{3}{2}\binom{5}{1}}{\binom{8}{3}}\right]$$
$$= 1 - \frac{10+30+15}{56} = 1 - \frac{55}{56}$$
$$= \underline{1/56}$$

$$\beta = p(\text{Accept } H_0 \mid H_1 \text{ true})$$
$$= 1 - p(\text{Reject } H_0 \mid H_1)$$

$= 1 - p(x \geq 3 \mid \theta = 5)$

$= 1 - p(x=0) + p(x=1) + p(x=2)$ when $\theta = 5$

$$= \frac{\binom{5}{0}\binom{3}{3}}{\binom{8}{3}} + \frac{\binom{5}{1}\binom{3}{2}}{\binom{8}{3}} + \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} = \frac{45}{56}$$

6) A machine in the long run puts out 16 imperfect articles for every 500 articles produced. After the machine is overhauled, it puts out 3 imperfect articles in a batch of 100. Has the machine improved in its performance?

A) $p_1' = \frac{16}{500} = 0.032$, $p_2' = \frac{3}{100} = 0.03$

$H_0 : p_1 = p_2$ against $H_1 : p_1 > p_2$.

let $\alpha = 0.05$, BCR $\omega = Z \geq 1.96$.

test statistic,

$$Z = \frac{p_1' - p_2'}{\sqrt{p^* q^* \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}.$$

$$p^* = \frac{n_1 p_1' + n_2 p_2'}{n_1 + n_2} = \frac{16+3}{500+100} = \frac{19}{600} = 0.03167$$

$$q^* = 1 - p^* = 1 - 0.03167 = 0.96833$$

$$\therefore Z = \frac{0.032 - 0.030}{\sqrt{0.03167 \times 0.96833 \left(\frac{1}{500} + \frac{1}{100}\right)}}$$

$$= \frac{0.002 \times 10}{\sqrt{0.03167 \times 0.96833 \times 1.2}}$$

$$= 0.1042$$

$$\therefore |Z| = 0.1042 < 1.96$$

Z lies in acceptance region, Ho is accepted. Thus machine has not improved its performance.

7) A sample of size 8 from a normal (p) is 6, 8, 11, 5, 9, 11, 10, 12. can such sample be regarded as drawn from a (p) with mean 7 at 2% level of significance?

A)

x_i:	6	8	11	5	9	11	10	12	72
x_i^2	36	64	121	25	81	121	100	144	692

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{72}{8} = 9$$

$$s^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{692}{8} - 9^2$$

$$= 86.5 - 81 = 5.5$$

$$\therefore s = \sqrt{5.5} = 2.345$$

$H_0: \mu = 7$ against $H_1: \mu \neq 7$

gvn $\alpha = 0.02$, the CR is,

$\omega = |t| \geq t_{\alpha/2}$. Frm t-table $t_{\alpha/2}$ for 7 df is 2.998.

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n-1}} = \frac{9-7}{2.345/\sqrt{7}} = \underline{\underline{2.256}}$$

$\therefore |t| = 2.256 < 2.998$

H_0 is accepted.

8) A random sample of 16 men from country A had a mean height of 68 inches & a sum of squares from the sample mean 132. A random sample of 25 men from country B had the corresponding values 66.5 inches & 165. Can the samples be regarded as drawn from the same normal (P)?

A) $n_1 = 16$, $n_2 = 25$, $\bar{x}_1 = 68$, $\bar{x}_2 = 66.5$

$\Sigma(x_i - \bar{x}_1)^2 = n_1 s_1^2 = 132$, $\Sigma(x_2 - \bar{x}_2)^2 = n_2 s_2^2 = 165$

$H_0: \mu_1 - \mu_2 = 0$ against $H_1: \mu_1 - \mu_2 \neq 0$.

Let $\alpha = 0.01$, BCR,

$\omega = |t| \geq t_{\alpha/2}$. $\qquad t_{\alpha/2} \rightarrow 2.705$.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{68 - 66.5}{\sqrt{\frac{132+165}{16+25-2}\left(\frac{1}{16}+\frac{1}{25}\right)}}$$

$= \underline{1.697}$

$\therefore |t| = 1.697 < 2.705$

H_0 is accepted

9) In a die throwing expt, the throw of 3 / 6 is reckoned as a success. Suppose 9000 times the die was thrown rslting in 3240 successes. Do you hue reasons to believe that the die is an unbiased one?

A) $H_0: P = \frac{1}{3}$ against $H_1: P \neq \frac{1}{3}$

H_0 : die is unbiased

H_1 : die is not unbiased.

Let $\alpha = 0.05$, BCR $\omega = |z| \geq 1.96$

test statistic,

$$t = \frac{\frac{3240}{9000} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3} + \frac{1}{3}}{9000}}} \Rightarrow \frac{0.36 - 0.35}{0.00496} = \underline{6.04}$$

$\therefore |z| = 6.04 > 1.96.$

H_0 is rejected

The die is not unbiased