01: INTRODUTION TO DIFFERENTIAL EQUATION.

⇒ ordinary & partial diff.. eq =

A D.eq involving a single independent variable & hence only ordinary deri.. → ordinary D.eq (ODE)

eg → ① $(y^2+x)\dfrac{d^2y}{dx^2} + 2y\left(\dfrac{dy}{dx}\right)^2 = 7.$ → dependnt → x-നെ depent ആശ്രിതമാണ് y.

② $y'' + (8x+3)y' + e^y \sin x = 0$

③ $y\left(\dfrac{dy}{dt}\right)^2 + 2t\dfrac{dy}{dt} - y = 0$

* A D.eq involving more than 1 independent variable, & hence partial deri.. → partial diff.. eq (PDE)

eg → ① $t^2\dfrac{\partial^2 u}{\partial t^2} - x\left(\dfrac{\partial u}{\partial x}\right)^2 - \sin t\,\dfrac{\partial u}{\partial t} = 0$ — u → dependnt; t, x → independ

② $u_{xx} + u_{yy} + u_{zz} = 0$ → $\dfrac{\partial^2 u}{\partial x^2} = u_{xx}$

u → depndnt

x, y, z → indi..

* Rmrk

1) Leibniz notation = $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$

2) prime notation = $y', y'', y''', y^{(4)}, y^{(5)}$ — $y^{(4)}$ → 4th deri..

nth deri.. of f $= f^{(n)}$

⇒ order & degree of D.eq =

* The order of a D.eq is the order of highest deri.. occuring in the eq.

* The degree of a D.eq is the degree of the highest deri.. which occurs in it.

eg for order →

① $(y^2+x)\dfrac{d^2y}{dx^2} + 2y\left(\dfrac{dy}{dx}\right)^2 = 7$. ($\dfrac{d^2y}{dx^2}$ → 2nd deri, $\dfrac{dy}{dx}$ → 1st deri)

here order = 2.

Degree = 3 → ($\dfrac{d^2y}{dx^2}$ ... times)

② $(y'')^3 + 5xy' - 5xy = 8$

$O = 2$

$D = 3$

③ $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ $O = 2$, $D = 1$ (... power → 1)

④ $\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial^2 z}{\partial y^2}\right)^2 = 0$ $O = 2$, $D = 2$

* Rmrk

we can express n^{th} order ODE in 1 independent variable by the general form,

$$\boxed{F(x, y, y', y'', \dots, y^{(n)}) = 0} \text{ ── ①}$$

1st order ODE → $y' + x = 0$

2nd " → $y'' + x = 0$

F → real valued ( ) with $n+2$ variables.

($x, y, y', \dots, y^n = n+2$)

* The D.eq $\dfrac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$

f → real valued contin ( ) of $n+1$ variables. $x, y, y', \dots, y^{(n-1)}$ is referred to as normal form of eq ── ①

⇒ **Linear & Non-linear D.eq =**

$x' + 3 = 1$ → linear D.F.

$y'' + 3y + x = 0$ → L.DF.

$(y'')^3 + 3y' = 0$ → non-L.

$x + 3 = 1$ → linear

$x^2 + 3 = 1$?, $x^3 + 3 = 1$ → non-linear

* The ODE of order n $F(x, y, y', \dots, y^{(n)}) = 0$ is said to be linear if F is linear ( ) of variables $y, y', y'', \dots, y^{(n)}$.

* Any linear ODE of degree n can be return as,

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = \phi(x).$$

$a_n(x)y^n$

$a_2(x)y''$

$a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ & $\phi(x)$ are ( )s of independent variable x.

* 2 imp sple cases are linear 1st order & linear 2nd order ODE's,

1st o ← $a_1(x)\dfrac{dy}{dx} + a_0(x)y = \phi(x)$ ── ① &

2nd o ← $a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = \phi(x)$ ── ②

} format

* eg of linear D.eq →

1) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$. by ── ②

2) $\dfrac{d^4y}{dt^4} + t^2\dfrac{d^3y}{dt^3} + 5t^3\dfrac{d^2y}{dt^2} + 6\sin t\, y = te^t$

* eq for non-linear D.eq ⟶

1) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y^2 = 0$ (y ... y^2.)

2) $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^2 + 6y = 0$. [in general eq no whole square]

3) $\dfrac{d^2y}{dx^2} + 5y\dfrac{dy}{dx} + 6y = 0$

⇒ Solution of a D.eq =

Any ( ) ϕ defined on an interval 'I' & processing atleast n deri.. that are continu.. on I, which when substituted into an n^{th} order ODE reduces the eq to an identity, it is said to be a soln of the eq

Q
1) S.T the ( ) ϕ is defined by $\phi(t) = 2\sin t + 3\cos t$ is a soln of the follow.. D.eq for all real t. by,
$y'' + y = 0$.

$x-2=0$
$x = 0+2$
$x=2$
(or)
$x^2-2=0$
$2-2=0$

A) gvn $\phi(t) = 2\sin t + 3\cos t$

$y = \phi(t)$ $y = \phi(t) = 2\sin t + 3\cos t$ —①

$y' = \phi'(t) = 2\cos t - 3\sin t$

$y'' = \phi''(t) = -2\sin t - 3\cos t$ —②

②+①

$y''+y = -2\cancel{\sin t} - 3\cancel{\cos t} + 2\sin t + 3\cos t = 0$

hence $\phi(t)$ is a soln of gvn D.eq.

2) verify that the ( ) $y = xe^x$ is a soln of the D.eq by $y'' - 2y' + y = 0$

A)
$y = x \cdot e^x$

(product) $y' = x \cdot e^x + e^x = xe^x + e^x$

$y'' = x \cdot e^x + e^x + e^x$ or $xe^x + 2e^x$

$y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + xe^x$
$= xe^x + \cancel{2e^x} - 2xe^x - \cancel{2e^x} + xe^x$
$= x^{ex} - 2xe^x + xe^x$
$= 2xe^x - 2xe^x = 0$

hence y is the soln gvn D eq

Rmrk

* A soln of a D.eq that is identically 0 on an interval 'I' is said to be a trivial soln

3) Determine the value of r for which the D.eq $y''' - 3y'' + 2y' = 0$ has the soln of the form $y = e^{rx}$?

A) Since y satisfies the gvn D.eq

$y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$y''' - 3y'' + 2y' = 0 \Rightarrow r^3 e^{rx} - 3(r^2 e^{rx}) + 2(re^{rx}) = 0$
$= r^3 e^{rx} - 3r^2 e^{rx} + 2re^{rx} = 0$
$= e^{rx}[r^3 - 3r^2 + 2r] = 0$

Q soln ...

① ②
① 0 / ② ...
then only we gt 0 as answer
bt $e^x \neq 0$ (always), so

$\Rightarrow r^3 - 3r^2 + 2r = 0$

$\Rightarrow r[r^2 - 3r + 2] = 0$ $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
quad eq

$\Rightarrow r(r-2)(r-1) = 0$

$r = 0$ or $r-2 = 0$ or $r-1 = 0$

$r=0$ $r=2$ $r=1$

∴ r values 0, 1, 2

4) find the value of m so that $y=x^m$ is a sol^n of $x^2y''-7xy'+15y=0$.

A)

$y=x^m$

$y'=m\,x^{(m-1)}$

$y''=(m-1)m\,x^{((m-1)-1)}=(m^2-m)x^{(m-2)}$

$x^2y''-7xy'+15y \Rightarrow x^2\left[(m^2-m)\,x^{(m-2)}\right]-7x\left[m\,x^{(m-1)}\right]+15x^m=0$.

$=x^{((m-2)+2)}(m^2-m)-7x^{((m-1)+1)}m+15x^m=0$

$=x^m(m^2-m)-7x^m m+15x^m$

$=x^m\left[m^2-m-7m+15\right]=0$

$=x^m\left[m^2-8m+15\right]=0 \rightarrow \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=x^m\left[(x-3)(x-5)\right]=0$

$\therefore m=3,5$ $\quad \left.\begin{matrix} y_1=x^3 \\ y_2=x^5 \end{matrix}\right\} y=x^m$.

⇒ <u>Explicit & implicit sol^n</u> :-

A sol^n in which the dependent variable is expressed in terms of the independent variable & constants is said to be <u>Explicit sol^n</u>

(i.e) $\boxed{y=\phi x}$ ☆

A relation $\boxed{H(x,y)=0}$ is said to be an <u>implicit sol^n</u> of an ODE on an interval 'I' provided there exist at least 1 ( ) ϕ that satisfies the relation as well as the D.eq on 'I'.

eg → consider a relation $x^2+y^2=1$

diff w.r.t x, $\qquad \frac{d}{dx}(1)=0$.

$2x+2y\frac{dy}{dx}=0$. $\quad$ on $-1<x<1$

$\therefore$ $x^2+y^2=1$ is an implicit sol^n of

$2x+2y\frac{dy}{dx}=0$.

[illegible] → ex.

$x^2+y^2=1 \Rightarrow y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$.

let (i.e) $y=\sqrt{1-x^2}=\phi_1(x)$ &

$y=-\sqrt{1-x^2}=\phi_2(x)$

Saties the relation $x^2+(\phi_1(x))^2=1$ &

$x^2+(\phi_2(x))^2=1$

& are ex. sol^n defined on $(-1,1)$.

Q 1) verify that $-2x^2y+y^2=1$ is an implicit $H(x,y)=0$ sol^n of $2xy\,dx+(x^2-y)\,dy=0$. ② find ex. sol^n.

A) $-2x^2y+y^2=1$

diff w.r.t x, $\quad -2(x^2\cdot y)+y^2=1$

① $\cancel{-4xy+2y\frac{dy}{dx}=0}$ $\quad =-2\left[x^2\cdot\frac{dy}{dx}+y\cdot 2x\right]+2y\frac{dy}{dx}=0$

$=-2x^2\frac{dy}{dx}-4xy+2y\frac{dy}{dx}=0$

$=2\left[-x^2\frac{dy}{dx}-2xy+y\frac{dy}{dx}\right]=0$

$=-x^2\frac{dy}{dx}-2xy+y\frac{dy}{dx}=0$

$=-2xy+\frac{dy}{dx}\left[-x^2+y\right]=0$

$=-\left[2xy+\frac{dy}{dx}(x^2-y)\right]=0$

$= 2xy + \frac{dy}{dx}(x^2 - y) = 0$

$\times\, dx$ through out,

$= 2xy\,dx + (x^2 - y)\,dy = 0$

$\therefore\ -2x^2y + y^2 = 1$ is an imp. solⁿ of given D. eq.

② ex solⁿ $\rightarrow y = (\quad)$

To find ex. solⁿ consider $-2x^2y + y^2 = 1$

$y^2 - 2x^2y - 1 = 0$ (Qua. eq)

$a = 1 \quad b = 2x^2 \quad c = -1$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2x^2 \pm \sqrt{(2x^2)^2 - 4\times1\times-1}}{2\times1}$$

$$= \frac{2x^2 \pm \sqrt{4x^4 + 4}}{2} = \frac{2x^2 \pm \sqrt{4(x^4+1)}}{2}$$

$$= \frac{\cancel{2}x^2 \pm \cancel{2}\sqrt{x^4+1}}{\cancel{2}} = x^2 \pm \sqrt{x^4+1}$$

$\therefore\ y = x^2 + \sqrt{x^4+1} = \phi_1(x)$ &

$y = x^2 - \sqrt{x^4+1} = \phi_2(x)$

$y = (\quad)$ — ex. (),

$\phi_1(x)$ & $\phi_2(x)$ are the ex solⁿ of the relation $-2x^2y + y^2 = 1$

⇒ General & Particular Solⁿ =

Consider the eq $y' = \cos x$

$\therefore \int y' = \int \cos x \Rightarrow y = \sin x + C$ //

C-ന് ഏത് value കൊടുത്താലും ശരിയാകും.

∴ general eq.

$\therefore\ y = \sin x + C \rightarrow$ General solⁿ.

If $y = \sin x + 1$, $y = \sin x + 0.5$, $y = \sin x \rightarrow$ P. eq

* A solⁿ of a D. eq that is free of arbitrary paramtrs $\rightarrow$ particular solⁿ & obtained by giving a specific values to the paramtrs.

* If every solⁿ of an n^{th} order ODE on an interval 'I' can be obtained from an n-paramtr family $G(x, y, c_1, c_2 \ldots c_n) = 0$ by appropriate choices of the paramtr c_i, $i = 1, 2, 3 \ldots n$, we say that the family is the general solⁿ of the ODE.

⇒ System of D. eq =

System of ODE is 2/more eq involving

A system of ODE ()s of the deri.. of 2/more unknown variable. a single independent variable.

eg → Consider the system

$\frac{dx}{dt} = f(t, x, y)$ dependent. v → x, y

$\frac{dy}{dt} = g(t, x, y)$. i. n. v → t.

independent (ഒന്ന് മാത്രം)

D Varify that the pair of ()s $x = e^{-2t} + 3e^{6t}$, $y = -e^{-2t} + 5e^{6t}$ is a solⁿ of the D. eq

$\frac{dx}{dt} = x + 3y$ & $\frac{dy}{dt} = 5x + 3y$?

A) To ~~prove~~ ~~$\frac{dx}{dt} = x+3y = e^{-2t} + 3e^{6t}$~~ & $\frac{dy}{dt} =$

$\frac{dx}{dt} = x + 3y$.

$\frac{dx}{dt} = \frac{d}{dt}\left(e^{-2t} + 3e^{6t}\right)$

$= e^{-2t} \times -2 + 3e^{6t} \times 6$

$= -2e^{-2t} + 18e^{6t}$ — ①

(want to gt $x+3y$.)

eq-① നെ x & y ന്റെ terms ലേക്ക് ആക്കുക.

$\Rightarrow x+3y$

eg $5b = 3b+2b$

$x = \ldots$, $y = \ldots$

$x+3y = e^{-2t} + 3e^{6t} + 3\left(-e^{-2t} + 5e^{6t}\right)$

$= 1e^{-2t} + 3e^{6t} - 3e^{-2t} + 15e^{6t}$.

$1 - 3e^{-2t}$

$= -2e^{-2t} + 18e^{6t}$ — ②

$\therefore$ from ① & ②

$\frac{dx}{dt} = x + 3y$

$\frac{dy}{dt} = 5x + 3y$.

$\frac{dy}{dt} = \frac{d}{dt}\left(-e^{-2t} + 5e^{6t}\right)$

$= -e^{-2t} \times -2 + 5e^{6t} \times 6$

$= 2e^{-2t} + 30e^{6t}$ — ③

$5x+3y = 5\left(e^{-2t} + 3e^{6t}\right) + 3\left(-e^{-2t} + 5e^{6t}\right)$

$= 5e^{-2t} + 15e^{6t} - 3e^{-2t} + 15e^{6t}$.

$= 2e^{-2t} + 30e^{6t}$ — ④

③ & ④ $\therefore \frac{dy}{dt} = 5x + 3y$

The pair of eq is a sol^n

2) Varify that each of gvn (") is the sol^n of the D.eq.

a) $y'' - y = 0$; $y_1(t) = e^t$, $y_2(t) = \cosh t$.

$y'' - y = 0$ എന്ന sol^n, $y_1(t)$ & $y_2(t)$ ആണോ എന്ന് നോക്കുക. $=0$ ആണോ എന്ന് check ചെയ്യുക. ആണെങ്കിൽ sol^n.

A) Consider $y_1 = e^t$

$y_1' = e^t$

$y_1'' = e^t$

$y''-y=0$, $\therefore y_1'' - y = e^t - e^t = 0$ → sol^n.

$y_2 = \cosh t$

$y_2' = \sinh t$

$y_2'' = \cosh t$

$y_2'' - y_2 = \cosh t - \cosh t = 0$ → sol^n

b) $2t^2y'' + 3ty' - y = 0$; $y_1(t) = t/3$, $y_2(t) = t^{1/2}$.

A) $y_1 = t/3$

$y_1' = 1/3 \rightarrow \frac{1}{3} \times t \rightarrow \frac{1}{3} \times 1 = 1/3$.

$y_1'' = 0$

$2t^2y_1'' + 3ty_1' - y_1 = 2t^3 \times 0 + \cancel{3}t \times 1/\cancel{3} - t/3$

$= t - \frac{t}{3} = \frac{3t - t}{3} = \frac{2t}{3} \neq 0$

not a sol^n

$\frac{1}{2} - 1 = \frac{1-2}{2} = -\frac{1}{2}$

$y_2 = t^{1/2}$

$y_2' = \frac{1}{2}t^{\frac{1}{2}-1} = \frac{1}{2}t = \frac{1}{2}t^{-1/2}$

$-\frac{1}{2} - 1 = \frac{-1-2}{2} = \frac{-3}{2}$

$y_2'' = \frac{1}{2} \times -\frac{1}{2}t^{-\frac{1}{2}-1} = \frac{-1}{4}t^{-3/2}$

$2t^2y_2'' + 3ty_2' - y_2 = \cancel{2}t^2\left(-\frac{1}{\cancel{4}_2}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2}$

$= -\frac{1}{2}$

$= -\frac{1}{2} \times t^{2-3/2} + \frac{3}{2} t^{1-1/2} - t^{1/2}$ $\quad t^2 \cdot t^{-3/2}$, $2 + -3/2$, $\frac{2}{1} - \frac{3}{2} = \frac{3-4}{2} = -\frac{1}{2}$

$= -\frac{1}{2} t^{-1/2} + \frac{3}{2} t^{1/2} - t^{1/2}$

$= -\frac{1}{2} t^{-1/2} + t^{1/2} \left[\frac{3}{2} - 1\right]$ $\quad \frac{3}{2} - \frac{1}{1} = \frac{3-2}{2} = \frac{1}{2}$

$= -\frac{1}{2} t^{-1/2} + \frac{1}{2} t^{1/2} = \frac{1}{2}\left[-t^{-\frac{1}{2}+\frac{1}{2}}\right]$

$= \frac{1}{2} \times -1 = -1/2 \neq 0$ not a soln

~~find the~~

⇒ Initial value prbm = (IVP)

* A diff eq together with such an initial condi^n → an Initial value prbm

* The gvn IVP also → n^{th} order initial prbm.

$$\frac{d^n y}{dx^n} = f(x, y, y', y'' \dots y^{(n-1)})$$

Subject to $y(x_0) = y_0$, $y'(x_0) = y_1 \dots y^{(n-1)}(x_0) = y_{n-1}$

([illegible] D.E [illegible] condi^n → IVP)

* eg →

① Solve $\frac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$ ([illegible])

$y = \int 3y^{2/3} = \dots + c$, $y(0) = 0$

$x_0 = 0$

$y_0 = 0$.

② Solve $xy' = 2y$, $y(0) = 0$.

($y', x', \frac{d}{dx}$ → D.E)

Q

I

1) Solve $x = c_1 \cos 4t + c_2 \sin 4t$ is a 2 paramtr family of sol^ns of $x'' + 16x = 0$, using this family of sol^n find a sol^n of the IVP

$x'' + 16x = 0$, $x(\pi/2) = -2$, $x'(\pi/2) = 1$

∴ 2 condi..

$= 0$ → sol^n otherwise not a sol^n

A) $x = c_1 \cos 4t + c_2 \sin 4t$.

I

$x' = -c_1 \sin 4t \times 4 + c_2 \cos 4t \times 4$

$= -4c_1 \sin 4t + 4c_2 \cos 4t$.

$x'' = -4c_1 \cos 4t \times 4 - 4c_2 \sin 4t \times 4$

$= -16 c_1 \cos 4t - 16 c_2 \sin 4t$

$x'' + 16x = -16c_1 \cos 4t - 16c_2 \sin 4t + 16(c_1 \cos 4t + c_2 \sin 4t)$

$= -16\cancel{c_1 \cos 4t} - 16\cancel{c_2 \sin 4t} + 16c_1 \cancel{\cos 4t} + 16c_2 \cancel{\sin 4t}$

$x'' + 16x = 0$

∴ $x_{c_1 \cos 4t + c_2 \sin 4t}$ is a sol^n of the givn D.E

II

$x'' + 16x = 0$, $x(\pi/2) = -2$, $x'(\pi/2) = 1$

IVP used for constnt, here c_1 & c_2 are constants

using initial condi^n,

$x(\frac{\pi}{2}) = -2$ $\quad x(t) = 2$, $t = \pi/2$

∴ $x(\frac{\pi}{2}) = -2$ ⇒ $c_1 \cos 4 \cdot \frac{\pi}{2} + c_2 \sin 4 \cdot \pi/2 = -2$

⇒ $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$

⇒ $c_1 \times 1 + c_2 \times 0$

⇒ $c_1 = -2$

$x'(\pi/2) = 1$ ⇒ $-4c_1 \sin 4 \times \frac{\pi}{2} + 4c_2 \cos 4\frac{\pi}{2} = 1$

⇒ $-4c_1 \sin 2\pi + 4c_2 \cos 2\pi = 1$

⇒ $-4c_1 \times 0 + 4c_2 \times 1 = 1$

⇒ $4c_2 = 1$

$c_2 = \frac{1}{4}$

$\therefore \; x = -2\cos 4t + \frac{1}{4}\sin 4t$ is the

Sol^n of IVP

→ Therem [Existance & uniquickness theom]

Let 'R' be a □-lar region in the (x, y) plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior.

If $f(x, y)$ & $\frac{\partial f}{\partial y}$ are continu on 'R', then there exist some intrval I_0 such that $I_0 : (x_0 - h, x_0 + h)$, $h > 0$ contained in $[a, b]$ & a unique ( ) $y(x)$ defined on I_0 that is a sol^n of the 1st order IVP

$$\boxed{y' = f(x, y), \quad y(x_0) = y_0}$$