Module IV

orthogonal ( )s and fourier series

→ Inner product =

$u = u_1 i + u_2 j + u_3 k$,
$v = v_1 i + v_2 j + v_3 k$ then

$\underbrace{\langle u, v \rangle}_{\text{i.product }(u,v)} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i$

* 4 properties →

1) $\langle u, v \rangle = \langle v, u \rangle$
2) $\langle k u, v \rangle = k \langle u, v \rangle$
3) $\langle u, v \rangle = 0 \Rightarrow u = 0$ i.e $\langle u, v \rangle > 0$, → $u \neq 0$
4) $\langle u+v, w \rangle = \langle u, w \rangle \langle v, w \rangle$.

* Inner product ( )s :-

$\langle f_1, f_2 \rangle$ 2 ( )s on an $[a, b]$.

- $\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx$.. ~~2 ( )s fi~~
- If $\int_a^b f_1(x) f_2(x)\, dx = 0$. then it said to be orthogonal on $[a, b]$

Q
1) The ( )s $f_1(x) = x^2$, $f_2(x) = x^3$ are orthognal on $[-1, 1]$. using i.p ( ) ?.

A) $\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx$.

$= \int_{-1}^{1} x^2 \cdot x^3\, dx = \int_{-1}^{1} x^5\, dx = \left[\frac{x^6}{6}\right]_{-1}^{1}$

$= \frac{1}{6} - \frac{1}{6} = 0$ → orthognal

2) $f_1(x) = \cos x$, $f_2(x)\ \sin^2 x$ with $[0, \pi]$

$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx.$ $\qquad u = \sin x$, $du = \cos x\, dx$

$= \int_0^\pi \cos x \sin^2 x\, dx.$

$= \int_0^\pi u^2\, du = \left[\frac{u^3}{3}\right]_0^\pi \Rightarrow \left[\frac{1}{3}\sin^3 x\right]_0^\pi$

$= 0$

→ orthogonal set =

A set of real valued ( )s $\phi_0(x), \phi_1(x), \phi_2(x) \ldots$ is said to be orthognal on $[a, b]$ if

$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x) \phi_n(x)\, dx = 0$, $m \neq n$

→ orthonormal =

$\|\phi_n(x)\|^2 = \langle \phi_n(x), \phi_n(x) \rangle$ $\qquad \|u^2\| = \langle u, u \rangle$

$= \int_a^b \phi_n^2(x)\, dx.$ $\qquad \|u\| = \langle \sqrt{u}, \sqrt{u} \rangle.$

$\|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)\, dx}$

Q

1) S.T the set $\{1, \cos x, \cos 2x \ldots\}$ is orthognal on $[-\pi, \pi]$

A) $\{1, \cos x, \cos 2x \ldots\}$ $\qquad \phi_0(x) = 1$

$\{\phi_0(x), \phi_1(x), \phi_2(x) \ldots\}$ $\qquad \phi_1(x) = \cos x.$

$\phi_2(x) = \cos 2x$

$\vdots$

$\phi_n(x) = \cos nx.$

$\int_{-\pi}^{\pi} \phi_0(x) \phi_n(x)\, dx.$

$= \int_{-\pi}^{\pi} 1 \cdot \cos nx\, dx. = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}$ $\qquad \sin \pi = 0$

$= \frac{1}{n}\left[\sin n\pi - \sin n(-\pi)\right] = 0$

$$\int_{-\pi}^{\pi} \phi_n(x)\, \phi_m(x)\, dx = \int_{-\pi}^{\pi} \cos n x \ldots$$

$\cos A \cos B = \frac{\cos(A+B)}{2} + \frac{\cos(A-B)}{2}$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x + \cos(m-n)x \, dx.$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\, 0 + 0 \,\right] = 0$$

* A set of real valued f's $\phi_0(x), \phi_1(x), \phi_2(x) \ldots$ is said to be orthogonal w.r.t weight f $w(x)$ on an interval $[a,b]$, if

$$\int_a^b w(x)\, \phi_m(x)\, \phi_n(x)\, dx = 0 \quad , m \neq n.$$

→ **Fourier series** = (eqs)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{P} x + b_n \sin \frac{n\pi}{P} x$$

✓ $a_0 = \frac{1}{P}\int_{-P}^{P} f(x)\, dx.$

✓ $a_n = \frac{1}{P}\int_{-P}^{P} f(x) \cos \frac{n\pi}{P} x \, dx$

✓ $b_n = \frac{1}{P}\int_{-P}^{P} f(x) \sin \frac{n\pi}{P} x \, dx.$

Q) Find the fourier series $f(x)$ is defined by $f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$

A) $P = (-\pi, \pi)$

$\int_{-P}^{P} \to \int_{-\pi}^{\pi}$ $\therefore P = \pi$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} (\pi - x)\, dx.\right.$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$\cancel{\frac{\pi^2}{\pi}} = \cancel{\frac{\pi^2}{2\pi}} = \frac{1}{\pi}\left[\frac{2\pi^2}{2} - \frac{\pi^2}{2}\right]$

$= \frac{1}{\cancel{\pi}} \cdot \frac{\pi^{\cancel{2}}}{2} = \frac{\pi}{2}$

$\boxed{u\int v - \int du \int v}$ — ①

$a_n = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} 0 \cos\frac{n\pi}{\pi}x\,dx}_{0} + \int_0^{\pi}(\pi - x)\cos\frac{n\pi}{\pi}x\,dx\right]$

$p = \pi$

$= \frac{1}{\pi}\left[\int_0^{\pi}\underset{(u)}{(\pi - x)} \cdot \underset{(v)}{\cos n\frac{\cancel{\pi}x}{\cancel{\pi}}}\,dx\right]$ — by ①

$= \frac{1}{\pi}\left[\underbrace{(\pi - x)\left(\sin\frac{n\pi}{\pi}x\right)_0^{\pi}}_{0} + \frac{1}{n}\left[\int_0^{\pi}\sin nx\right]dx\right]$

$= \cancel{\frac{1}{\pi}}\left[\frac{1}{n}\int_0^{\pi}\sin nx\right]dx.$

$= \frac{-1}{n\cancel{\pi}}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-1}{n^2\cancel{\pi}}\left[\underset{(-1)^n}{\cos n\pi} - \underset{1}{\cos 0}\right]$

$= \frac{-1}{n^2\cancel{\pi}}(-1)^n - 1.$

$b_n = \frac{1}{p}\left[\int_{-p}^{p} f(x) \cdot \sin\frac{n\pi}{p}x\right]dx.$

$= \frac{1}{\pi}\underbrace{\int_{-\pi}^{0} 0 \sin\frac{n\pi}{\pi}x\,dx}_{0} + \int_0^{\pi}(\pi - x)\sin\frac{n\cancel{\pi}}{\cancel{\pi}}x\,dx.$

$= \int_0^{\pi}\underset{(u)}{(\pi - x)}\underset{(v)}{\sin nx}\,dx.$

$= \cancel{\pi x - \frac{x^2}{2}}\,(\pi - x)\int_0^{\pi}\sin nx\,dx - \int\frac{d}{dx}(\pi - x) \cdot \int\sin nx\,dx$

$= (\pi - x)\left[\frac{-\cos nx}{n}\right]_0^{\pi} - \int(-1)\cdot\int\sin nx\,dx.$

$= (\pi - x)\left[\frac{-\cos n\pi}{n} - \frac{\cos n0}{n}\right] + \left[\frac{-\cos nx}{n}\right]$

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx.$$

$$\frac{1}{p}\int_{-p}^{p} f(x) \cdot \sin \frac{n\pi x}{p} \cdot dx$$

$$\frac{1}{p} = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot \frac{\sin n\pi x}{0} \cdot dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} (\pi - x) \cdot \frac{\cos nx}{n} + \int_0^{\pi} \cos nx \, dx.\right.$$

$$= \frac{1}{\pi}\left[\pi + \left(\frac{\sin nx}{n}\right)_0^{\pi}\right].$$

$$= \frac{1}{n\cancel{\pi}}(\cancel{\pi}) = \frac{1}{n}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{n^2\pi}((-1)^n - 1) \cos\frac{n\pi}{\pi} + \frac{1}{n}\sin\frac{n\pi}{\pi}x.$$

2) find the F. series of $f(x) = \begin{cases} 0 & , -2 < x < 0 \\ 1 & , 0 < x < 2 \end{cases}$

A) $p = [2,2)$ $\quad \int_{-p}^{p} \longrightarrow \int_{-2}^{2} \quad \therefore p = 2$

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0} 0 \, dx + \int_0^2 1 \, dx\right].$$

$$= \frac{1}{2}[x]_0^2 = \frac{1}{\cancel{2}} \times \cancel{2} = 1$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} 0 \cos\frac{n\pi}{2}x \, dx + \int_0^2 1 \cdot \cos\frac{n\pi}{2}x \cdot dx.\right].$$

$$= \frac{1}{2}\int_0^2 \cos\frac{n\pi}{2}x \, dx = \frac{1}{2}\left[\frac{\sin\frac{n\pi}{2}x.}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2}{n\pi} \cdot \sin\frac{n\pi}{2}x\right]_0^2 = \frac{1}{2}\left[\frac{2}{n\pi} \cdot \underbrace{\sin\frac{n\pi}{\cancel{2}} \cdot \cancel{2}}_{0}\right]$$

$$= 0$$

$$b_n = \frac{1}{2}\left[\underbrace{\int_{-2}^{0} 0 \cos\frac{n\pi}{2}x \, dx}_{0} + \int_0^2 \sin\frac{n\pi}{2}x \, dx\right]$$

$$= \frac{1}{2}\left[\frac{-\cos\frac{n\pi}{2}}{\frac{n\pi}{2}}\right]_0^2 = \frac{1}{2}\left[\frac{-2}{n\pi} \cdot \cos\frac{n\pi}{2}x\right]_0^2$$

$= \frac{1}{\cancel{2}}\left[\frac{-2}{n\pi}\left(\cos\frac{n\pi}{\cancel{2}}\cdot 2 - \cos 0\right)\right]$

$= \frac{1}{n\pi}\left[(-1)^n - 1\right]$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x.$

$= \frac{1}{2} + \sum_{n=1}^{\infty} 0\cdot\cos\frac{n\pi}{2}x + \frac{-1}{n\pi}\left[(-1)^n - 1\right]\sin\frac{n\pi}{2}x.$

$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi}\left[(-1)^n - 1\right]\sin\frac{n\pi}{2}x.$

$= \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \dots\right]$

$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{2}}{2}$

3) find F.S $f(x) = \begin{cases} 0, & -3 < x \le -1 \\ 1, & -1 < x \le 1 \\ 0, & 1 < x < 3 \end{cases}$

A) $p = 3$

$a_0 = \frac{1}{p}\left(\int_{-p}^{p} f(x)\,dx\right)$ ~~$= \frac{1}{3}\left[\int_{-3}^{-1} 0\,dx + \int_{-1}^{1} 0\,dx + \int_{1}^{3} 0\,dx\right]$~~

~~$= 0$~~

~~$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx.$~~

~~$a_0 = \frac{1}{3}\int_{-3}^{-1} 0\cos\frac{n\pi}{p}x\,dx +$~~

$a_0 = \frac{1}{3}\left[\int_{-3}^{-1} 0\,dx + \int_{-1}^{1} 1\cdot dx + \int_{1}^{3} 0\,dx\right]$

$= \frac{1}{3}\left[[x]_{-1}^{1}\right] = \frac{1}{3}\cdot[1+1] = 2/3$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx.$$

$$= \frac{1}{3}\left[\int_{-3}^{-1} 0\cdot\cos\frac{n\pi}{3}x\,dx + \int_{-1}^{1} 1\cdot\cos\frac{n\pi}{3}x\,dx + 0\right]$$

$$= \frac{1}{3}\left[\int_{-1}^{1}\cos\frac{n\pi}{3}x\,dx\right] = \frac{1}{3}\left[\frac{\sin\frac{n\pi}{3}x}{\frac{n\pi}{3}}\right]_{-1}^{1}$$

$$= \frac{1}{3}\left[\frac{3}{n\pi}\left(\sin\frac{n\pi}{3} + \sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{1}{\cancel{3}}\left[\frac{\cancel{3}}{n\pi}\left(\sin\frac{n\pi}{3} + \sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{1}{n\pi}\left[2\sin\frac{n\pi}{3}\right]$$

~~$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\cdot dx.$~~

~~$= \frac{1}{3}\left[\int_{-1}^{1}\sin\frac{n\pi}{3}x\,dx\right] = \frac{1}{3}\left[\frac{\cos\frac{n\pi}{3}x}{\frac{n\pi}{3}}\right]_{-1}^{1}$~~

~~$= \frac{1}{\cancel{3}}\left[\frac{3}{n\pi}\left(\cos\frac{n\pi}{3}x\right)\right]_{-1}^{1}$~~

~~$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{3} + \frac{n\pi}{3}\right]$~~

~~$= \frac{1}{n\pi}\left[2\sin\frac{n\pi}{3}\right]$~~

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\cdot dx.$$

$$= \frac{1}{3}\int_{-1}^{1}\sin\frac{n\pi}{3}x\,dx = \frac{1}{3}\left[\frac{-\cos\frac{n\pi}{3}}{n\pi/3}\right]_{-1}^{1}$$

$$= \frac{1}{3}\left[\frac{3}{n\pi}\left(-\cos\frac{n\pi}{3}\right)\right]_{-1}^{1}$$

3)

A)

$= \frac{1}{\cancel{3}} \frac{\cancel{3}}{n\pi}\left(-\cos\frac{n\pi}{3} + \cos\frac{n\pi}{3}\right)$ $\quad \cos(-x) = \cos x$

$= \frac{1}{n\pi}\left[\frac{1}{n\pi}\left[-\frac{\cos n\pi x}{3}\right]_{-1}^{1}\right.$

$\frac{1}{n\pi}\left[-\cos\frac{n\pi}{3} - \left[-\cos\left(\frac{n\pi \times -1}{3}\right)\right]\right]$

$\frac{1}{n\pi}\left[-\cos\frac{n\pi}{3} - \left(-\cos\frac{n\pi}{3}\right)\right]$

$\frac{1}{n\pi}\left[-\cos\frac{n\pi}{3} + \cos\frac{n\pi}{3}\right] = 0$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p} x + b_n \sin\frac{n\pi}{p} x.$

$= \frac{\frac{2}{3}}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\sin\frac{n\pi}{3}\right] \cos\frac{n\pi}{3} x + 0.$

$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\sin\frac{n\pi}{3}\right] \frac{\cos n\pi}{3}$

$= \frac{1}{3} + \left[\frac{2}{\pi} \sin\frac{\pi}{3} \cos\frac{\pi}{3} + \frac{2}{2\pi} \sin\frac{2\pi}{3} \cdot \cos\frac{2\pi}{3} \times 2 + \frac{2}{3\pi} \sin\frac{3\pi}{3} \cos\frac{3\pi}{3} \times 3 \cdots\right]$

(continous) $\rightarrow$

$\Rightarrow$ <u>conditions of convergnce</u> =

let f & f' be piecewise continous on [-p, p], let f & f' be continous except at a finite no. of points in the interval & have only finite discontinuties at this points, then for all x in the interval [-p, p] the f. series of F converges to f(x) at a point of continuity & at the point of discontinuity the f. series converges to avg $\frac{f(x^+) + f(x^-)}{2}$

where $f(x^+)$ & $f(x^-)$ denote the limits of f at x from the right.

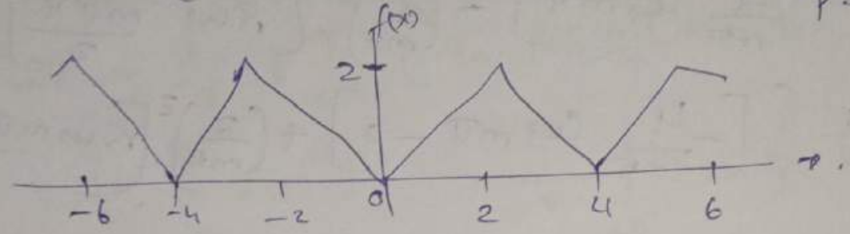
H.W

1) Sketch graph of f,

$f(x) = |x|$, $-2 \le x \le 2$, $f(x+4) = f(x)$.

A) $f(x) = \begin{cases} -x & , -2 \le x < 0 \\ x & , 0 \le x < 2 \end{cases}$, $f(x+4) = f(x)$.

$p = 2$.



$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = \frac{1}{2}\left[2+2\right] = 2$$

$$a_m = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{m\pi x}{p}\right)dx = \frac{1}{2}\int_{-2}^{2} -x\cos\left(\frac{m\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{m\pi x}{2}\right)dx.$$

$$= \frac{1}{2}\left\{\left[\frac{-2}{m\pi}x\sin\frac{m\pi x}{2}\right]_{2}^{0} - \int_{-2}^{0}\left[\frac{-2}{m\pi}\sin\frac{m\pi x}{2}\right]dx\right\} +$$

$$\frac{1}{2}\left\{\left[\frac{2}{m\pi}\sin\frac{m\pi x}{2}\right]_{0}^{2} - \int_{0}^{2}\left[\frac{2}{m\pi}\sin\frac{m\pi x}{2}\right]dx\right\}$$

$$= \frac{1}{2}\left\{0 - \left(\frac{2}{m\pi}\right)^2\cdot\left[\cos\frac{m\pi x}{2}\right]_{-2}^{0}\right\} + \frac{1}{2}\left\{0 + \left(\frac{2}{m\pi}\right)^2\left[\cos\frac{m\pi x}{2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left\{-\left(\frac{2}{m\pi}\right)^2\left[1 - \cos m\pi\right]\right\} + \frac{1}{2}\left\{\left(\frac{2}{m\pi}\right)^2\left[\cos m\pi - 1\right]\right\}.$$

$$= \frac{4}{m^2\pi^2}\left[\cos m\pi - 1\right] = \frac{4\left[(-1)^m - 1\right]}{m^2\pi^2}$$

$$b_m = \frac{1}{P}\int_{-P}^{P} f(x)\sin\left(\frac{m\pi x}{P}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{0}(-x)\sin\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{m\pi x}{2}\,dx.$$

$$= \frac{1}{2}\left\{\left[\frac{2}{m\pi}x\cdot\cos\frac{m\pi x}{2}\right]_{-2}^{0} - \int_{0}^{2}\frac{2}{m\pi}\cdot\cos\frac{m\pi x}{2}\,dx\right\}$$

$$+ \frac{1}{2}\left\{\left[-\frac{2}{m\pi}x\cos\frac{m\pi x}{2}\right]_{0}^{2} - \int_{0}^{2}\left[\frac{-2}{m\pi}\cos\frac{m\pi x}{2}\right]dx\right\}$$

$$= \frac{1}{2}\left\{\left[0+\frac{4}{m\pi}\cos m\pi\right] - \left(\frac{2}{m\pi}\right)^2\left[\sin\frac{m\pi x}{2}\right]_{-2}^{0}\right\}$$

$$+ \frac{1}{2}\left\{\left[-\frac{4}{m\pi}\cos m\pi - 0\right] + \left(\frac{2}{m\pi}\right)^2\left[\sin\frac{m\pi x}{2}\right]_{0}^{2}\right\}$$

$$= 0$$

$$f(x) = \frac{2}{2} + \sum_{m=1}^{\infty}\left[\frac{4[(-1)^m-1]}{m^2\pi^2}\cdot\cos\frac{m\pi x}{2}\right]$$

$$= 1 - \frac{8}{\pi^2}\cos\frac{\pi x}{2} + 0 - \frac{8}{3^2\pi^2}\cdot\cos\frac{3\pi x}{2} + 0 - \frac{8}{5^2\pi^2}\cdot\cos\frac{5\pi x}{2} + \cdots$$

$$= 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right].$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}.$$

⇒ periodic ( ) =

the gvn ( ) is periodic with period $T = 2P$ (i.e) $\boxed{f(x+T) = f(x)}$, when f is piecewise continous & left hand & right hand derivatives exist at $x = -P$ & $x = P$, then the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{P} + b_n\sin\frac{n\pi x}{P}\right]$$

converges to avg $\frac{f(p^-)+f(p^+)}{2}$. At this end points is to this value extended periodically to $\pm 3p, \pm 5p, \pm 7p$ is so on.

Q
1) find F. series representing $f(x) = x^2$ on $(-\pi, \pi)$ deduced that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \frac{\pi^2}{6}$? 10.

A) $f(x) = x^2$ $\therefore p = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right].$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi} \left[\pi^3 - (-\pi)^3 \right] \quad \cancel{\frac{1}{3\pi}\left[\pi^3+\pi\right]} \quad \cancel{\frac{1}{3\pi}\left[\pi^2\right]}$$

$$= \frac{1}{3\pi} \left[\pi^3 + \pi^3 \right]$$

$$= \frac{1}{3\pi} (2\pi^3) = \frac{2\pi^{3}}{3\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\,dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \cos \frac{n\pi}{p} x. \quad \longrightarrow \left(\int \text{by parts} \right)$$

$$= \frac{1}{\pi} \left[x^2 \cdot \frac{\sin \frac{n\pi}{\pi} x}{\frac{n\pi}{\pi}} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \cdot \frac{\sin \frac{n\pi}{\pi} x}{\frac{n\pi}{\pi}}$$

$$= \frac{1}{\pi} \left[x^2 \cdot \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \cdot \frac{\sin nx}{n}$$

$$= \frac{1}{\pi} \left[\pi^2 \cdot 0 - (\pi^2) \frac{\sin(\pi)}{n} \right] + \cancel{\left[x^2 \cdot \frac{\cos nx}{n} \right]_{-\pi}^{\pi}}$$

$$-2 \left[x \cdot \frac{\cos nx}{n} - \int_{-\pi} -\frac{\cos nx}{n}\,dx. \right]$$

$= 0 - \left[2 \left[\pi - \frac{\cos n\pi}{n} - \left(-\pi - \frac{\cos n(\pi)}{n} \right) \right] - \right.$

$\underbrace{- \frac{1}{n} - \frac{\sin nx}{n} \Big]_{-\pi}^{\pi}}_{0}$

$= \frac{2}{n\pi} \left[2\left(\pi - \frac{\cos n\pi}{n} \right) + \pi - \frac{\cos n\pi}{n} \right].$

$= \frac{2}{n\pi} \left[\frac{2\pi \ (-1)^n}{n} \right]$

$b_n = \frac{1}{P} \int_{-P}^{P} f(x) \sin \frac{n\pi}{P} x \cdot dx.$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \sin \frac{n\pi}{\cancel{\pi}} x \cdot dx.$ ($\int$ by parts)

$= \frac{1}{\pi} \left[x^2 \cdot \frac{-\cos \frac{n\pi x}{\cancel{\pi}}}{\frac{n\pi}{\cancel{\pi}}} - \int 2x \cdot \frac{-\cos \frac{n\pi x}{\pi}}{\frac{n\pi}{\cancel{\pi}}} \right.$

$= \frac{1}{\pi} \left(x^2 \cdot - \frac{\cos nx}{n} \right]_{-\pi}^{\pi} - \int 2x \cdot - \frac{\cos nx}{n}$

$= \frac{1}{\pi} \left[\pi^2 \cdot - \frac{\cos n\pi}{n} - (-\pi)^2 - \frac{\cos n(-\pi)}{n} \right] - \Big]$

$2 \left[x \cdot - \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} x^2 \cdot \int - \frac{\sin nx}{n}$

$= \frac{1}{\pi} \left[\cancel{\pi^2} - \frac{\cancel{(-1)^n}}{\cancel{n}} - \cancel{\pi^2} - \frac{\cancel{(-1)^n}}{\cancel{n}} \right] - 2 \left[\pi - \frac{\sin n\pi}{n} - \right.$

$\left. (-\pi) - \frac{\sin n(-\pi)}{n} \right] - 2 \left[x - \frac{\cos nx}{n} \right]$

$= 0$

$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}.$

$f(x = \pi)$

$\frac{f(\pi^-) + f(\pi^+)}{2}$

$f(x) = x^2$

$$= \frac{1}{2}\left[\lim_{h\to 0} f(\pi - h) + \lim_{h\to 0} f(\pi + h)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} (\pi - h)^2 + \lim_{h\to 0} (\pi + h)^2\right]$$

$$f(\pi) = \frac{1}{2}\left[\pi^2 + \pi^2\right] = \frac{1}{\cancel{2}} \cancel{2}\pi^2 = \pi^2$$

$$f(x) = \pi^2$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n^2}.$$

$$= \frac{\pi^2}{3} + 4\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}}$$ (expand)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2 - \frac{\pi^2}{3}}{4} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right].$$

$$\frac{\cancel{2}\pi^2}{\cancel{12}_{6}} = \text{"}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$

A) $f(x)$ wor same in last q.

$$f(x) = \frac{1}{2}\left[\pi^2 + \pi^2\right] = \frac{1}{\cancel{2}} \cancel{2}\pi^2 = \pi^2$$

→ odd
→ even

Fourier for even & odd ( )s =

$F(-x) = f(x)$ even
$f(-x) = -f(x)$ odd

They have the (pro) —

a) product of 2 even ( ) → even.
b) " of 2 odd ( ) → even
c) " of even ( ) & odd ( ) → odd
d) sum / difference of 2 even ( ) → even
e) " " of 2 odd ( ) → odd.

$$* \int_{-L}^{L} f(x)\,dx = \begin{cases} 2\int_0^L f(x)\,dx, & \text{even} \\ 0, & \text{odd.} \end{cases}$$

* **F. series for even ( ) =**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{P}, \quad a_0 = \frac{2}{P}\int_0^P f(x)\,dx,$$

Also → fourier cosine series, $a_n = \frac{2}{P}\int_0^P f(x)\cos\frac{n\pi x}{P}\,dx$.

* **F. series for odd ( ) =**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{P}, \quad b_n = \frac{2}{P}\int_0^P f(x)\sin\frac{n\pi x}{P}\,dx$$

Also → fourier sin series.

ined

Q
1) Expand the follow ( ) in a F. series

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$\xrightarrow{\text{consider}}$ $-x \in (\pi, 0)$

A) $x \in (-\pi, 0)$

$\therefore f(x) = -1 \rightarrow -f(x)$. (odd) (even (?))

$f(x) = 1 \rightarrow f(x).$

$(-\pi, \pi).$

$f(x) \begin{cases} -1, & -\pi < x < 0 \rightarrow \text{odd} \\ 1, & 0 < x < \pi \rightarrow \text{even} \end{cases}$

(1)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{P} x.$$

$P = \pi$

$$b_n = \frac{1}{P} \int_{-P}^{P} f(x) \sin \frac{n\pi}{P} x \cdot dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} -1 \cdot \sin \frac{n\pi x}{\pi} + \int_{0}^{\pi} 1 \cdot \sin \frac{n\pi x}{\pi}$$

$=$

$$b_n = \frac{2}{P} \int_{0}^{P} f(x) \sin \frac{n\pi x}{P} \, dx.$$

$$= \frac{2}{\pi} \int_{0}^{\pi} 1 \cdot \sin \frac{n\pi x}{\pi} \, dx.$$

$$= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_{0}^{\pi} = \frac{2}{\pi} \left[-(-1)^n - 1 \right]$$

$$= \frac{2}{\pi} \left[1 - (-1)^n \right]$$

$$f(x) = \frac{2}{\pi} \sum \frac{1 - (-1)^n}{n} \sin nx.$$

⇒ Half range expansion =

I cosine Series :

even periodic expansion $F(x)$ of $f(x)$ defined by $F(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0. \end{cases}$

$f(x + 2L) = f(x)$

cosine Series expansion $\rightarrow$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L} \cdot \int_0^L f(x)\, dx.$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L}\, dx.$$

II Sin Series =

odd periodic expansion =

$g(x)$ of $f(x)$ defined by

$$g(x) = \begin{cases} f(x) & , 0 < x < L \\ 0 & , x = 0, L \\ -f(x) & , -L < x < 0 \end{cases}$$

$$g(x+2L) = g(x).$$

Sine Series expansion $[0, L] \rightarrow$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi x}{L}\, dx.$$

$$= \frac{2}{L} \int_0^L f(x) \cdot \sin\frac{n\pi x}{L}\, dx.$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{P},$$

$P = L$

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx.$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos\frac{2n\pi x}{L}\, dx.$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} \, dx.$$