Module II

Higher order DE.

A linear diff^n eq of order n is an eq of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \quad \text{—①}$$

where $a_0, a_1, \dots a_n$ & $g(x)$ are continuous real(s)

& eq —① ⟶ non homogeneous if $g(x) \neq 0$

If $g(x) = 0$ ⟶ Homogenous LDE

If each (coe) in eq —① is a constant, then eq ⟶ LDE (linear diff^n eq) with constant (coe).

eq ⟶ $3y''' + 5y'' - y' + 7y = 0$

→ Initial value & boundary value prbms =

Theorem = [Existence of unique sol^n] $\left.\begin{array}{l} y(0)=1 \\ y(1)=2 \end{array}\right\}$ initial value

Let $a_n(x), a_{n-1}(x) \dots a_1(x), a_0(x)$ & $g(x)$ be continuous on an interval 'I' & let $a_n(x) \neq 0$ for every x in the interval, if $x = x_0$ is any point in this interval, then the sol^n $y(x)$ of initial value prbm,

$$\boxed{\begin{array}{l} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ \text{where } y(x_0) = y_0,\ y'(x_0) = y_1, \dots y^{(n-1)}(x_0) = y_{n+1}. \end{array}}$$

(initial value p. (Starting point))

Exist on interval I & is unique.

→ Boundary Value Prbm = (BVP)

consider the prbm,

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$y(a) = y_0$ & $y(b) = y_1$ ⟶ 2 point boundary v.p.

where $y(a) = y_0$ & $y(b) = y_1$ ⟶ Boundary condi—

→ Differential operators = (D.O)

Symbol $\frac{dy}{dx}$ ⟶ (D.O)

usually denoted D. & ( ) y ⟶ operand.

eg ⟶ ① $\frac{d}{dx} = D$.

② $\frac{d}{dx}\sin x = \cos x$.

↳ $D(\sin x) = \cos x$.

③ $D(4x^3 + 5x^2) = 12x^2 + 10x$

$\frac{d^2}{dx^2}, \frac{d^3}{dx^3} \dots$ are denoted by $D^2, D^3 \dots$

In general, we define n^{th} order differential operator,

$$L = a_n(x)D^n + a_{n-1}D^{n-1} + \dots + a_1(x)D + a_0(x)$$

* Differential optor 'L' posesses the linearity (pro) (i.e)

$D[f(x) + g(x)] = Df(x) + Dg(x)$. & also.

$D[c\,f(x)] = c \cdot D f(x)$.

→ Sol^n of homogenous LDE =

Theorem = [Super position Principle]

Consider n^{th} order homogenous DE

is the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

where $a_0(x), a_1(x), \dots a_n(x)$ are continuous real ( )s on I, $a_n(x) \neq 0$.

Let $y_1, y_2 \dots y_k$ be Solⁿ of the eq on an interval I. Then the Linear Combination

$$\boxed{y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_k y_k(x).}$$

$c_1, c_2, \dots c_k$ are arbitory constnt, is also a Solⁿ on I.

⇒ linear dependence & linear independence of Solⁿ =

A set of ( )s $f_1(x), f_2(x) \dots f_n(x)$ is said to be linearly dependent on a interval I if there exist constnts $c_1, c_2 \dots c_n$, not all zeroes such that $c_1 f_1(x) + c_2 f_2(x) \dots + c_n f_n(x) = 0$ for every x in the interval I.

If the set of all ( )s is not linearly dependent on the interval, it is said to be linearly independent (i.e)

$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ ⇒

$\Rightarrow c_1 = c_2 = c_3 = \dots c_n = 0$

$x_1 \quad x_2$
$c_1 \quad c_2$
⇓
$c_1 x_1 + c_2 x_2$
linear combination
⇓
$c_1 x_1 + c_2 x_2 = 0$
linear eq

eg

(L I)
$c_1 x_1 + c_2 x_2 = 0$
if $c_1 = 0$ & $c_2 = 0$
(both 0)

(L D)
$c_1 x_1 + c_2 x_2 = 0$
if $c_1 = 3$ then c_2 is -3
are gt 0
([illegible])

1) Check whether the ( )s $f_1(x) = 1 + x$, $f_2(x) = x$, $f_3(x) = x^2$ are linearly indpnt (L I) on, interval $(-\infty, \infty)$

A) 1) Linear eq Consisdered

Let c_1, c_2 & c_3 are constnt consider the linear eq,

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$c_1(1+x) + c_2 x + c_3 x^2 = 0$

$c_1 + c_1 x + c_2 x + c_3 x^2 = 0$

$c_1 + x(c_1 + c_2) + c_3 x^2 = 0 + 0x + 0x^2 + 0$

∴ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

∴ $f_1(x)$, $f_2(x)$ & $f_3(x)$ are L I.

⇒ wronskian Method = (W)

Suppose of each of ( )s $f_1(x), f_2(x) \dots f_n(x)$ possesses atleast n-1 derivatives.

The determinent,

$$W(f_1, f_2 \dots f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ f_1'' & f_2'' & \dots & f_n'' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

3 ( ) → 2 deri

is → W. of the ( ).

Theorum =

let $y_1, y_2 \dots y_n$ be n Solⁿ of a homogenous nth order LDE on an interval I then the set of all Solⁿ is linearly Independent on I if & only if their wronskin is not identically 0. (i.e)

$LI \Leftrightarrow W \not\equiv 0$ ($\not\equiv$ → not identically)

1) Determine whether the gvn set of (), is LD/LI

a) $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$

A)

$$W(f_1,f_2,f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 4x-3x^2 \\ 1 & 2x & 4-6x \\ 0 & 2 & -6 \end{vmatrix}$$

$$= x(-12x - 8 + 12x) - x^2(-6-0) + 4x - 3x^2(2-0)$$

$$= -8x + 6x^2 + 8x - 6x^2 = 0$$

$W = 0$, f_1, f_2, f_3 are LD

b) $f_1(x) = 0$ $f_2(x) = x$ $f_3(x) = e^x$

$$W(0, x, e^x) = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$$= 0[e^x - 0] - x[0-0] - e^x[0] = 0$$

LD

c) $f_1(x) = 5$ $f_2(x) = \cos^2 x$ $f_3(x) = \sin^2 x$

def →

Any set $y_1, y_2 \dots y_n$ of n LI soln of the homogenous n^{th} order LDE on an I is said to be fundamental set of soln on I.

* Therum =

let $y_1, y_2 \dots y_n$ be a fundemental set of soln of homogeneous linear n^{th} order DE, with continous (coe) on I, then the general soln of the eq on I is,

$$\boxed{y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n}$$

$c_i \to i = 1, 2 \dots n$ are arbitory constnts.

Imp

1) Varify that the () $y_1 = x^3$, $y_2 = x^4$ form
① a fundamental set of soln of DE
$x^2 y'' - 6xy' + 12y = 0$ on interval $(0, \infty)$
② form the general soln ?

A) ① ⓐ y_1 satisfies eq — ⓐ / not. if $= 0$ → satisfies.

$y_1 = x^3$, $y_1' = 3x^2$, $y_1'' = 6x$

$\Rightarrow x^2 y_1'' - 6xy_1' + 12y_1$

$\Rightarrow x^2 6x - 6x\, 3x^2 + 12x^3$

$\Rightarrow 6x^3 - 18x^3 + 12x^3$

$\Rightarrow -12x^3 + 12x^3 = 0$

ⓑ $y_2 = x^4$, $y_2' = 4x^3$, $y_2'' = 4 \cdot 3x^2 = 12x^2$

$\Rightarrow x^2 y_2'' - 6xy_2' + 12y_2 =$

$\Rightarrow x^2 .12x^2 - 6x.4x^3 + 12x^4$

$\Rightarrow 12x^4 - 24x^4 + 12x^4$

$-12x^4 + 12x^4 \Rightarrow 0$

∴ Both y_1 & y_2 satisfies gvn DE

∴ they are solⁿ.

③ ~~to~~ prove L.I.

wronskin of the solⁿ,

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$$= 4x^3 \cdot x^3 - 3x^2 \cdot x^4$$

$$= 4x^6 - 3x^6 = x^6 \neq 0$$

[$x^6 \to 0$ / not →
bcz it is x, if non.
is instead of x^6 → can be L.I.]

(Since $x \in (0, \alpha)$)
belongs to.

∴ y_1 & y_2 are L.I.

hence y_1 & y_2 form a f. set of solⁿ to the gvn solⁿ.

— ② general solⁿ,

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 x^3 + c_2 x^4 \quad \text{where } c_1 \,\&\, c_2 \to \text{arbitry constants}$$

2) Varify that the (?) $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ form a f. set of solⁿ of DE $y''' - 6y'' + 11y' - 6y = 0$ on I $(-\alpha, \alpha)$ form general eq?

answer →

⇒ Method of reduction of order = $y = c_1 x + c_2$

For a 2ⁿᵈ order LDE ~~~~

$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$ —① → 2 deri → 2 solⁿ ; 3 " → 3 "

The general solⁿ is,

$y = c_1 y_1 + c_2 y_2$, where y_1 & y_2 are solⁿ & both L.I. on some I.

Suppose we can find 1 non-zero solⁿ $y_1(x)$ of above homogneous LDE, so the method for finding y_2 → m.R. of order

2.A) $y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$, $y_1''' = e^x$.

$$y_1''' - 6y_1'' + 11y_1' - 6y_1 = e^x - 6e^x + 11e^x - 6e^x.$$
$$= -5e^x + 5e^x = 0$$

$y_2 = e^{2x}$, $y_2' = 2e^{2x}$, $y_2'' = 2 \cdot e^{2x} \cdot 2 = 4e^{2x}$,
$y_2''' = 4 \cdot e^{2x} \times 2 = 8e^{2x}$.

$$y_2''' - 6y_2'' + 11y_2' - 6y_2 = 8e^{2x} - 6 \cdot 4e^{2x} + 11 \cdot 2e^{2x} - 6e^{2x}.$$
$$= 8e^x - 24e^x + 22e^x - 6e^{2x}$$
$$= -16e^x + 16 = 0$$

$y_3 = 3^{ex}$ ....

$$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x} = 0$$

y_1, y_2 & y_3 satisfies gvn DE. on $(-\alpha, \alpha)$

$$\omega(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x[18e^{5x} - 12e^{5x}] - e^{2x}[9e^{4x} - 3e^{4x}] + e^{3x}[4e^{3x} - 2e^{3x}]$$

$$= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0$$

∴ Hence the sol^{n}s y_1, y_2 & y_3 are L.I.

general Sol^n of givn DE,

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Consider 2nd order DE in standard form, $\boxed{y'' + p(x)y' + q(x)y = 0}$, where $p(x)$ & $q(x)$ are continuous on some I & $y_2(x)$

$y_2(x) = u(x)\, y(x) = y_1(x) \int \frac{1}{y^2} e^{-\int p(x)dx}\, dx$.

here $u = \int v\, dx = c_1 \int \frac{1}{y_1^2} e^{-\int p(x) dx}\, dx + c_2$

1) Use the M. of reduction of order find the general Sol^n of DE $x^2y'' - 5xy' + 9y = 0$ on I $(0,\infty)$ given that $y = x^3$ is a Sol^n.

A) $y = x^3 \longrightarrow y_1 = x^3$

y_2 ? → M. of Reduction.

$x^2y'' - 5xy' + 9y = 0$ → make this into (General form)

÷ throughout by x^2.

$y'' - \frac{5xy'}{x^2} + \frac{9y}{x^2} = 0$

$y'' - \frac{5}{x}y' + \frac{9y}{x^2} = 0$ —① ⇒ St. form ✓

Choose $y_1 = x^3$

we hue to find y_2.

$y_2 = u(x)\, y_1(x) \rightarrow u y_1 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}\, dx$

$\left.\begin{array}{l} y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0. \\ y'' + p(x)y' + q(x)y = 0 \end{array}\right\}$ ∴ $p(x) = -5/x$, $q(x) = \frac{9}{x^2}$

∴ $e^{-\int p(x)dx} = e^{-\int -5/x\, dx} = e^{\int 5/x\, dx} = e^{5\int 1/x\, dx}$

$= e^{5\ln x} = e^{\ln x^5} = x^5$

then only we can cut e & ln.

$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}\, dx$.

$= x^3 \int \frac{1}{(x^3)^2} \cdot x^5\, dx = x^3 \int \frac{1}{x^6} x^5\, dx$.

$= x^3 \int \frac{1}{x}\, dx = x^3 \ln x$

(i.e) the general Sol^n is $c_1 y_1(x) + c_2 y_2(x) =$

$c_1 x^3 + c_2 x^3 \ln x$

⇒ Sol^n of homogenous LDE with constnt Coe=

Consider homogenous LDE with constant Coe,

$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0$ —①

let $\boxed{y = e^{mx}}$ be the possible Sol^n of eq-①

∴ $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2 e^{mx}$, $\frac{d^3y}{dx^3} = m^3 e^{mx}$, ...

$\frac{d^n y}{dx^n} = m^n e^{mx}$

Sub in ①,

⇒ $a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \dots + a_1 m e^{mx} + a_0 e^{mx} = 0$

⇒ $e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0) = 0$ —②

(e always ≠ 0)

∴ $(a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0) = 0$ —③

eq-③ → Auxillary eq / Characteristic eq of eq-①

let us denote the roots by $m_1, m_2 \ldots m_n$ where m_i's are need not all be distinct & real.
Then each (), $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ ... $y_n = e^{m_n x}$ is a sol^n of DE — ①

while solving auxillary eq the following cases may occur.
1) All the roots are real & distinct.
2) All roots are real & same.
3) All roots are not real (complex-i)

case I =

If $m_1, m_2 \ldots m_n$ are real & distinct the general sol^n is,

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}}$$

case II =

If $m_1 = m_2 = \ldots = m_n$ are equal the g. sol^n.

$$\boxed{y = (c_1 + c_2 x + c_3 x^2 + \ldots c_n x^{n-1}) e^{mx}}$$

case III =

If $\alpha + \beta i$ is 1 root, then $\alpha - \beta i$ must be another root.
∴ The g. sol^n is gvn by,

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$ (after simplification)

$$\boxed{y = e^{\alpha x}[A \cos \beta x + B \sin \beta x] \quad \text{where } A = c_1 + c_2,\ B = i(c_1 - c_2)}$$

1) Solve : $y'' + 2y' + 5y = 0$

A) (1) homogeneous ✓ 2) constant multi- ✓ $1y'' + 2y' + 5y = 0$)

corresponding auxillary eq,

$y'' + 2y' + 5y = 0$ 2 deri → m^2, 1 deri → m, $y \to 1$.

let sub- m → $m^2 + 2m + 5 = 0$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 4i}{2} = -\frac{2}{2} \pm \frac{4i}{2}$$

complex ∴ case III

$$= -1 \pm 2i$$

$$\Rightarrow -1 + 2i \ \& \ -1 - 2i$$

$\alpha + \beta i \to \alpha = -1, \beta = 2.$

here the roots not real,

∴ Sol^n is $y = e^{\alpha x}[A \cos \beta x + B \sin \beta x]$

$$= e^{-x}[A \cos 2x + B \sin 2x]$$

($y'' + 2y' + 5y$
$y'' \to 2$ sol^n, if $y''' \to 3$ sol^n)

2) $4y'' + y' = 0$

A) homo ✓ constant m ✓

auxillary eq,

$4y'' + y' = 0$

$4m^2 + m = 0.$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4 \times 4 \times 0}}{2 \times 4} = \frac{-1 \pm 1}{8}$$

$\therefore \frac{-1+1}{8}$ & $\frac{-1-1}{8}$

0 & $\frac{-2}{8} = \frac{-1}{4}$

$\therefore$ case II

$$y = (c_1 + c_2 x) e^{mx}$$
$$= (c_1 + c_2 x) e^{-\frac{1}{4}x}$$

3) $y'' + 8y' + 16y = 0$

A) auxillary eq,

$$m^2 + 8m + 16 = 0$$
$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 \pm 64}}{2}$$
$$= \frac{-8 \pm \sqrt{64 - 64}}{2} = \frac{-8}{2} = -4$$

case II

$$y = (c_1 + c_2 x) e^{mx}$$
$$= (c_1 + c_2 x) e^{-4x}$$

4) $y'' - y' - 6y = 0$

A) a. eq,

$m^2 - m - 6 = 0$ $\quad b = -1$, $c = -6$.

$$\therefore m = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2}$$

$\cancel{\frac{1+5}{2} = \frac{6}{2} = 3}$ & $\cancel{\frac{1-5}{2} = \frac{-4}{2} = -2}$

$$\frac{1+5}{2} = \frac{6}{2} = 3, \quad \frac{1-5}{2} = \frac{-4}{2} = -2$$

$\therefore$ g. sol^n, Case I

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
$$= c_1 e^{3x} + c_2 e^{-2x}$$

5) $3y'' + 2y' + y = 0$

6) $y'' + 16y = 0$

7) $y'' + y' + 2y = 0$

(8) $y'' + 9y = 0$

5. A) $3y'' + 2y' + y = 0$.

$$3m^2 + 2m + 1 = 0.$$
$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{4 - 4 \times 3 \times 1}}{6}$$
$$= \frac{-2 \pm \sqrt{-8}}{6}$$
$$= \frac{-2 \pm 8i}{6} = 0$$

6. A) $y'' + 16y = 0$

$$m^2 + 16 = 0$$
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-16 \pm \sqrt{16^2 - 0}}{2}$$
$$= \frac{-16 \pm \sqrt{256}}{2}$$
$$= \frac{-16 \pm 16}{2}$$
$$= \frac{-16 + 16}{2} = 0, \quad \frac{-16 - 16}{2} = -16$$

$\therefore$ case II

$y = (c_1 + c_2 x) e^{mx}$

$= (c_1 + c_2 x) e^{-16x}$

8.A) $y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \frac{-9 \pm \sqrt{9^2 - 0}}{2}$

$= \frac{-9 \pm 9}{2}$, $\Rightarrow \frac{-9+9}{2} = 0$

$\Rightarrow \frac{-9-9}{2} = -9$

$\therefore$ case II

$y = (c_1 + c_2 x) e^{-9x}$

9) Solve $y''' - y'' - 6y' = 0$

A) 3 deri → 3 solⁿ → m_1, m_2, m_3?

a. eq,

$\Rightarrow m^3 - m^2 - 6m = 0$

$\Rightarrow m(m^2 - m - 6) = 0$

$m^2 - m - 6 = 0$ $\quad b = -1$, $c = -6$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1 - 4 \times 1 \times \bar{6}}}{2} = \frac{1 \pm \sqrt{1 + 24}}{2}$

$= \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2}$

$= \frac{1+5}{2} = \frac{6}{2} = 3$, $\frac{1-5}{2} = \frac{-4}{2} = -2$

$\therefore m = 0$ or $m = 3$ or $m = -2$

3 values are diff → so case I

g. solⁿ,

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$= c_1 e^{0x} + c_2 e^{3x} + c_3 m^{-2x}$

10) Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

A) a. eq,

$m^4 + 8m^2 + 16 = 0$ $\quad m^4 \rightarrow 4$ solⁿ, $m_1\, m_2\, m_3\, m_4$?

can be $\Rightarrow (m^2)^2 + 8m^2 + 4^2 = 0$

can be. $\Rightarrow (m^2)^2 + 2 \times 4 \times m^2 + 4^2 = 0$. $\quad (a+b)^2$

$\Rightarrow (m^2 + 4)^2 = 0$ $\quad$ we need value of m?

$\Rightarrow m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$.

$2i, -2i$ $\quad 2i, -2i$

we get this 2 — but we need 4no. so repeat this

both case II & III,

here the solⁿ of eq are repeating but not real (case II & 3 satisfied)

$m = \pm 2i \rightarrow m = 0 \pm 2i$.

$\therefore \alpha = 0$, $\beta = 2$

$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

11) Solve $y'' + 4y = 0$, $y(0) = 3$, $y(\pi/2) = -3$

A) a. eq,

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i \rightarrow m = 0 \pm 2i$.

$\alpha = 0$, $\beta = 2$

case III

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

where $A = c_1 + c_2$ & $B = i(c_1 - c_2)$

$y = e^{0}[A\cos 2x + B\sin 2x]$

$y = A\cos 2x + B\sin 2x$ —— (1)

here no; c_1 & c_2.

$y(0) = 3 \Rightarrow x = 0$, $y = 3$

$y = A\cos 2x + B\sin 2x$.

$3 = A\cos 0 + B\sin 0$

$3 = A$ ∴ $A = 3$

$y(\pi/2) = -3 \implies x = \pi/2$, $y = -3$

$y = A\cos 2x + B\sin 2x$.

$-3 = A\cos \cancel{2}\times\frac{\pi}{\cancel{2}} + B\sin \cancel{2}\times\frac{\pi}{\cancel{2}}$

$-3 = A\cos\pi + B\sin\pi$

$\cos\pi = -1$

$\sin\pi = 0$

$-3 = -A$

$\Rightarrow A = 3$

B not get so,

∴ g sol^n,

on (1). $y = A\cos 2x + B\sin 2x$

$= 3\cos 2x + B\sin 2x$

Rmrk

Suppose that $m_1, m_2, m_3 \dots m_n$ are roots of homogeneous LDE with constant (cc),

case I →

If m_1, m_2 are not real but $m_3 \dots m_n$ are remaining real & distinct roots then the g. sol^n is, (combine case II & III)

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} \dots c_n e^{m_n x}$.

case II →

If there are 2 pairs of complex roots $\alpha + \beta i$ (m_1, m_2) & $\gamma + \delta i$ (m_3, m_4) then g. sol^n is, $(m_5 \dots m_n)$

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + e^{\gamma x}[C\cos\delta x + D\sin\delta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} \dots + c_n e^{m_n x}$.

1) Solve : $16y'' + 24y' + 9y = 0$

A) a, eq,

$16m^2 + 24m + 9 = 0$.

$m = \frac{-24 \pm \sqrt{24^2 - 4\times 16\times 9}}{2\times 16}$

$= \frac{-24 \pm \sqrt{576 - 576}}{32} = \frac{-24}{32} = \frac{-12}{16}$

$= \frac{-6}{8}$

$= -3/4$

⇒ Sol^n of non-homogeneous LDE =

*** Theorem I =**

consider non homogeneous LDE,

$$a_n(x)\, y^{(n)}\ a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0 y = g(x)$$

where $a_0, a_1, \dots a_n$ & $g(x)$ are continuous real ( )s on an I with $a_n \neq 0$,

let y_p be any particular sol^n of non homogeneous LDE, eq-① on an I & let $y_1, y_2 \dots y_n$ be a fundamental set of sol^n of the associated homogeneous LDE,

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0 y = 0 \quad \text{②}$$

on I, then g. sol^n of eq-① is,

$$\boxed{y = y_c(x) + y_p(x)} = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \dots c_n y_n(x)}_{y_c} + y_p$$

where $c_1, c_2 \dots c_n$ are arbitrary constant.

Rmrks

* g. sol^n of non-homogeneous ~~eq~~ eq-① is

y = complementary ( ) + any particular ( ).

*** Theorem - II (Super position principle)**

let $y_{p_1}, y_{p_2} \dots y_{p_k}$ be k particular sol^n of non homogeneous LDE eq-① on a I corresponding, inturn, to k distinct ( )s $g_1, g_2 \dots g_k$ (i.e)

$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$ is a particular sol^n of

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0 y = g_1(x) + g_2(x) + \dots g_k(x).$$

where y_{p_i} is a particular sol^n of

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0 y = g_i(x).$$

Q

1) Verify that $y_p(x) = x^3$ is a sol^n of the DE: $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$ — ⓐ

find g. sol^n?

A) ① — ⓐ $= x^3$ or not

$y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$. — ① $\qquad y = x^3$ — ⓐ

$y_p' = 3x^2$, $y_p'' = 6x$

Sub in ①

$6x - 2 \times 3x^2 + 5x^3 = 5x^3 - 6x^2 + 6x$

$6x - 6x^2 + 5x^3 = 5x^3 - 6x^2 + 6x$. — ②

① = ②.

it satisfies eq-① ∴ y_p is a sol^n of eq-①

② g. sol^n.

y_c? corresponding homogeneous eq-① is,

$y'' - 2y' + 5y = 0$. — ③

a. eq,

$m^2 - 2m + 5 = 0$.

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4\times5\times1}}{2}$

$= \frac{2 \pm \sqrt{-16}}{2}$

$= \frac{2 \pm 4i}{2} = 1 \pm 2i$

$\therefore m = 1 \pm 2i$

$\therefore \alpha = 1, \ \beta = 2$

$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$.

$= e^{x}[A\cos 2x + B\sin 2x]$

$\therefore$ g. solⁿ,

$y = y_c + y_p$

$= e^{x}[A\cos 2x + B\sin 2x] + x^3$

2) varify that $y_p = -\cos 2x$. is a solⁿ of DE
$(D^2+3D-4)y = 8\cos 2x + 6\sin 2x$. — (1)
hence solve the initial value prblm
— (1), $y(0)=1$ & $y'(0)=-3$

A) $(D^2+3D-4)y = 8\cos 2x$ —

$\left(\frac{d^2}{dx^2} + 3\frac{d}{dx} - 4\right)y = 8\cos 2x$ —

$y'' + 3y' - 4y = 8\cos 2x + 6\sin 2x$. — (a)

$y_p = -\cos 2x$

$y_p' = -(-\sin 2x \times 2) = 2\sin 2x$

$y_p'' = 4\cos 2x$.

$\therefore 4\cos 2x + 6\sin 2x + 4\cos 2x = 8\cos 2x + 6\sin 2x$ — (b)

— (a) = — (b)

y_p satisfies eq — (a). hence it is a particular solⁿ.

y_c ?,
corresponding homogenous eq — (a),

$y'' + 3y' - 4y = 0$ — (c)

a, eq,
$m^2 + 3m - 4 = 0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{9-4\times1\times-4}}{2}$

$= \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2}$

$= \frac{-3+5}{2} = 1$, $\frac{-3-5}{2} = -4$

$m_1 = 1$
$m_2 = -4$

$\therefore$ 2 → diff → case I

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$y_c = \boxed{c_1 e^{x} + c_2 e^{-4x}}$

~~c_1 & c_2 ? $y(0)=1$, $y'(0)=-3$~~

$\therefore$ g. solⁿ eq — (1),

$y = y_c + y_p$

$y = c_1 e^{x} + c_2 e^{-4x} - \cos 2x$. — (d)

given $y(0)=1$. $\Rightarrow 1 = c_1 e^{0} + c_2 e^{0} - \cos 0$.

$1 = c_1 + c_2 - 1$

$2 = c_1 + c_2 \Rightarrow 2 - c_2 = c_1$ — (e)

$y'(0) = -3$

$y = c_1 e^{x} + c_2 e^{-4x} - \cos 2x$

$y' = c_1 e^{x} - 4c_2 e^{-4x} + 2\sin 2x$

$y(0)=3 \Rightarrow -3 = c_1 e^{0} - 4c_2 e^{0} + 2\sin 0$

$\Rightarrow -3 = c_1 - 4c_2$ —— (f)

Sub (e) in (f)

$-3 = 2 - c_2 - 4c_2$

$-3 = 2 - 5c_2$

$5c_2 = 3+2$

$5c_2 = 5$

$c_2 = 1$ $\therefore c_1 = 1$

(d) $\therefore y = e^{x} + e^{-4x} - \cos 2x$

⇒ Method of undetermined (coe) = (To find y_p)

Procedure for finding y_p is given,

($g(x)$-നെ depend ചെയ്താണ് ഉണ്ടാക്കുന്നത് y_p)

Terms in $g(x)$	choice of y_p	
1) $k e^{\alpha x}$	$c e^{\alpha x}$	(exp)
2) $k x^n$ $(n=0,1,\dots,n)$	$k_n x^n + k_{n-1} x^{n-1} + \dots + k_1 x + k_0$	(x^n)
3) $k \cos\beta x$ (or) $k \sin\beta x$	$k\cos\beta x + M\sin\beta x$	(trig)
4) $k e^{\alpha x}\cos\beta x$, $k e^{\alpha x}\sin\beta x$	$e^{\alpha x}[k\cos\beta x + M\sin\beta x]$	(exp+trig)
5) $x^n e^{\alpha x}$	$e^{\alpha x}[k_n x^n + k_{n-1}x^{n-1} + \dots + k_1 x + k_0]$	(exp+x^n)
6) $k x^n e^{\alpha x}\cos\beta x$, $k x^n e^{\alpha x}\sin\beta x$	$e^{\alpha x}\cos\beta x[k_n x^n + k_{n-1}x^{n-1} + \dots + k_1 x + k_0] + e^{\alpha x}\sin\beta x[k_n x^n + k_{n-1}x^{n-1} + \dots + k_1 x + k_0]$	

Find g. sol^n of $y'' - y' - 2y = 6e^{x}$

A) ① y_c?

g. sol^n → $y = y_c + y_p$

homogeneous DE is,

$y'' - y' - 2y = 0$ —— ①

a. eq,

$m^2 - m - 2 = 0$

$m = \frac{1 \pm \sqrt{1 - 4 \times -2 \times 1}}{2} = \frac{1 \pm \sqrt{9}}{2}$

$= \frac{1+3}{2} = \frac{4}{2} = 2$, $\frac{1-3}{2} = \frac{-2}{2} = -1$

$m = 2, -1$

case I,

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{2x} + c_2 e^{-x}$ —— ⓐ

(c_1, c_2 can't find here no initial condi..)

② y_p?

$y'' - y' - 2y = 6e^{x}$ → exp → so Ist in table.

Since $g(x) = 6e^{x}$

$\therefore$ choose y_p as, $y_p = c e^{\alpha x}$

$\alpha = 1$, $6e^{x}$

$y_p = c e^{x}$

$\therefore y_p$ is sol^n of given DE

c?

Substituting y_p for y in eq – (1)

$y_p = ce^x$
$y_p' = ce^x$
$y_p'' = ce^x$

$y'' - y' - 2y = 6e^x$

$y_p'' - y_p' - 2y_p = 6e^x$

$\cancel{ce^x} - \cancel{ce^x} - 2ce^x = 6e^x$

$-2c\cancel{e^x} = 6\cancel{e^x}$

$-2c = 6$

$c = \frac{-6}{2} = -3$

∴ particular sol^n,

$y_p = ce^x = -3e^x$

∴ g. sol^n,

$y = y_c + y_p$

$y = c_1e^{2x} + c_2e^{-x} - 3e^x$

2) Solve : $y'' - y = 2x^2$

$y'' - y = 2x^2$ — (1)

A) y_c? $y'' - y = 0$

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$ case I

∴ $y_c = c_1e^{m_1x} + c_2e^{m_2x}$

$= c_1e^x + c_2e^{-x}$ — (2)

y_p ?

$g(x) = 2x^{2}$ case II in table

k_n, k_{n-1}, k_0?

$kx^n = k_nx^n + k_{n-1}x^{n-1}$

∴ $y_p = k_2x^2 + k_1x + k_0$

y_p satisfies eq – (1), sub y_p for y.

$y'' - y = 2x^2$

$y_p' = 2k_2x + k_1$

$y_p'' = 2k_2$

$y_p'' - y_p = 2x^2$

~~$2k_2 - 2k_2x + k_1 - 2x^2$~~

$2k_2 - [k_2x^2 + k_1x + k_0] = 2x^2$

$2k_2 - k_2x^2 - k_1x - k_0 = 2x^2$

(x^2 – 2 side coeff.)

$-k_2\cancel{x^2} = 2\cancel{x^2}$

$-k_2 = 2 \Rightarrow k_2 = -2$

$2k_2 - k_2x^2 - k_1x - k_0 = 2x^2 + 0x + 0$

($-k_1x$ = $0x$, 0)

∴ $k_1 = 0$.

$2k_2$ ~~$-k_2x^2 - k_1x$~~ $-k_0 = 2x^2 + 0x + 0$

($k_2 = -2$) $2k_2 - k_0 = 0$ (x [illegible] term [illegible])

$2 \times -2 - k_0 = 0$

$-4 - k_0 = 0$

$-k_0 = 4$ $\Rightarrow k_0 = -4$

∴ $y_p = k_2x^2 + k_1x + k_0$

$= -2x^2 - 4$

$y = y_c + y_p \Rightarrow c_1e^x + c_2e^{-x} - 2x^2 - 4$

3) Solve : $y'' - 3y' + 2y = 4t + e^{3t}$ — (1)

A) $y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0$

$m = \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2} = \frac{3 \pm \sqrt{1}}{2}$

$= \frac{3+1}{2}, \frac{4}{2} = 2, \frac{3-1}{2} = 1$

case I

$y_c = c_1 e^{-t} + c_2 e^{-2t}$ ⟶ g(x) is in t term

$\therefore g(t) = 4t + e^{3t}$

choose $y_p = k_1 t + k_0 + c e^{3t}$ $\quad k_1, k_0, c$?

$y_p' = k_1 + 3c e^{3t}$, $y_p'' = 9c e^{3t}$

y_p is a sol^n of eq — ①

$\Rightarrow y'' - 3y' + 2y = 4t + e^{3t}$.

$y_p'' - 3y_p' + 2y_p = 4t + e^{3t}$

$9ce^{3t} - 3(k_1 + 3ce^{3t}) + 2(k_1 t + k_0 + ce^{3t}) = 4t + e^{3t}$

$\cancel{9ce^{3t}} - 3k_1 - \cancel{9ce^{3t}} + 2k_1 t + 2k_0 + 2ce^{3t} = 4t + e^{3t}$

$2ce^{3t} + 2k_1 t - 3k_1 + 2k_0 = 4t + e^{3t}$.

$2c = 1$

$c = 1/2$

$2k_1 t - 3k_1 + 2k_0 = 4t$

$2k_1 = 4$

$k_1 = 2$

$-3k_1 + 2k_0 = 0$.

$-3\times 2 + 2k_0 = 0 \Rightarrow -6 + 2k_0 = 0$.

$2k_0 = 6$

$k_0 = 3$

$\therefore y_p = k_1 t + k_0 + c e^{3t}$

$= 2t + 3 + \frac{1}{2} e^{3t}$.

$\therefore$ g. sol^n,

$y = y_c + y_p$

$= c_1 e^{-t} + c_2 e^{-2t} + 2t + 3 + \frac{1}{2} e^{3t}$

4) Solve : $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$

A) $y'' + 2y' - 35y = 0$

$m^2 + 2m - 35 = 0$. $\quad \begin{matrix} \times 35 \\ +2 \end{matrix} \Big\} \begin{matrix} 7\times 5 = 35 \\ 7 - 5 = 2 \end{matrix}$

$m = 7, -5$

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{7x} + c_2 e^{-5x}$

$g(x) = 12e^{5x} + 37\sin 5x$.

~~$y_p \to y$.~~

~~$y'' + 2y' - 35y$~~

$y_p = ce^{5x} + k\cos 5x + M\sin 5x$.

$y_p' = 5ce^{5x} - 5k\sin 5x + 5M\cos 5x$.

$y_p'' = 25ce^{5x} - 25k\cos 5x - 25M\sin 5x$.

$y_p \to y$.

$y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$.

$y_p'' + 2y_p' - 35y_p = 12e^{5x} + 37\sin 5x$.

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 2[5ce^{5x} - 5k\sin 5x + 5M\cos 5x] - 35[ce^{5x} + k\cos 5x + M\sin 5x] = 12e^{5x} + 37\sin 5x]$

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 10ce^{5x} - 10k\sin 5x + 10M\cos 5x] - 35ce^{5x} - 35k\cos 5x - 35M\sin 5x = 12e^{5x} + 37\sin 5x$.

$25c + 10c - 35c = 12$

$-20c = 12$

$c = \frac{-12}{20}$

$\cancel{-25M - 10K - 35M = 37}$

$\cancel{-60M - 10K = 37}$

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 10ce^{5x} + 10k\sin 5x + 10M\cos 5x - 35ce^{5x} - 35k\cos 5x - 35M\sin 5x = 12e^{5x} - 37\sin 5x$

$\Rightarrow (-25+10-35)\cos 5x - (-25+10-35)\sin 5x + 15ce^{5x}$

5) Solve: $y''+y = 10e^{x}\sin x$ — (1)

A) $y''+y=0$

$m^2+1=0$

$m^2=-1 = i$

$m = \pm i$ case III

$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ $\alpha = 0$, $\beta = 1$

$= e^{0}[A\cos x + B\sin x]$

$y_c = A\cos x + B\sin x$

y_p?

$g(x) = 10e^{x}\sin x$ (case 4 in table)

$y_p = e^{\alpha x}[K\cos\beta x + M\sin\beta x]$

$\therefore \alpha = 1$, $\beta = 1$

choose $y_p = e^{x}[K\cos x + M\sin x]$ K? M?

$y_p = e^{x}K\cos x + e^{x}M\sin x$

y_p is a sol^n eq (1)

$y \longrightarrow y_p$.

$\cancel{y_p' = Ke^{x}\sin x + Me^{x}\cos x}$

$\cancel{y_p'' = -ke^{x}\cos x + Me^{x}\sin x}$

$y_p = \underbrace{ke^{x}\cos x}_{\text{p rule}} + \underbrace{Me^{x}\sin x}_{\text{p rule}}$

$y_p' = k[e^{x}\cdot -\sin x + \cos x\, e^{x}] + M[e^{x}\cos x + \sin x\, e^{x}]$

$= -ke^{x}\sin x + ke^{x}\cos x + Me^{x}\cos x + Me^{x}\sin x$

$y_p' = e^{x}\sin x\,[M-k] + e^{x}\cos x\,[k+M]$

$y_p'' = [e^{x}\cos x + \sin x\, e^{x}](M-k) + (k+M)[-e^{x}\sin x + \cos x\, e^{x}]$

$= (M-k)e^{x}\cos x + (M-k)\sin x\, e^{x} - (k+M)e^{x}\sin x + (k+M)e^{x}\cos x$

$= e^{x}\cos x\,[(M-k)+(k+M)] + e^{x}\sin x\,[(M-k)-(k+M)]$

$= e^{x}\cos x\,[\underbrace{M-\cancel{k}+\cancel{k}+M}_{(2M)}] + e^{x}\sin x\,[\underbrace{\cancel{M}-k-k-\cancel{M}}_{-2k}]$

$y_p'' = 2Me^{x}\cos x - 2ke^{x}\sin x$

$\therefore y''+y = 10e^{x}\sin x$

$y_p''+y_p = 10e^{x}\sin x$

$2Me^{x}\cos x - 2ke^{x}\sin x + ke^{x}\cos x + Me^{x}\sin x = 10e^{x}\sin x$

$= e^{x}\cos x(2M+k) + e^{x}\sin x(-2k+M) = 10e^{x}\sin x$

$= e^{x}\cos x\underset{0}{\underset{\downarrow}{(2M+k)}} + e^{x}\sin x(M-2k) = 10e^{x}\sin x$

$\Rightarrow 2M+k=0$ & $(M-2k)=10$

$k=-2M$ $M-2k=10$

$M-2(-2M)=10$

$M+4M=10$ $\Rightarrow 5M=10, M=2$

$K = -2M$ $\qquad m = 2$

$\therefore K = -2 \times 2 = -4$

$K = -4$ & $m = 2$

$\therefore y_p = Ke^x \cos x + Me^x \sin x$

$= -4e^x \cos x + 2e^x \sin x$

$\therefore$ g. sol^n,

$y = y_c + y_p$

$= A\cos x + B\sin x - 4e^x\cos x + 2e^x \sin x$

H.W.

6) Solve : $y''' + y'' = e^x \cos x$.

$\Rightarrow$ Method of variation of parameters =

Consider 2^{nd} order LDE in standard form

$y'' + p(x)y' + q(x)y = f(x)$. ← ①

To find g. sol^n, follow the steps —

* find complementary (),

$\boxed{y_c = c_1 y_1(x) + c_2 y_2(x)}$

* compute wronskin,

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (not $\neq 0$ always)

* Assuming $y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$, to be a particular sol^n

* cal u_1 & u_2 by,

$$\boxed{u_1(x) = \int \frac{-y_2(x) f(x)}{w(y_1, y_2)} dx}$$

$$\boxed{u_2(x) = \int \frac{y_1(x) f(x)}{w(y_1, y_2)} dx}$$

[illegible]

hence $y_p = u_1(x)y_1(x) + u_2(x) y_2(x)$.

$\therefore$ g. sol^n,

$\boxed{y = y_c + y_p}$

) Solve : $y'' + y = \operatorname{cosec} x$ — ①.

A) ⓐ — not homogeneous ✓

ⓑ — undetermined () X, so method of variation ✓

y_c ?

$y'' + y = 0$.

$m^2 + 1 = 0$.

$m^2 = -1$

$m = \pm i$ $\qquad$ case III

$y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$ $\qquad \alpha = 0, \beta = 1$

$y_c = c_1 \cos x + c_2 \sin x$ $\qquad y_c = c_1 y_1(x) + c_2 y_2(x)$

$\therefore y_1(x) = \cos x$.
$y_2(x) = \sin x$.

y_p ? $\qquad f(x) = \operatorname{cosec} x$.

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$

$= 1$.

$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$.

$u_1(x) = \int \frac{-y_2(x) f(x)}{w(y_1, y_2)} = \int \frac{-\sin x \cdot \operatorname{cosec} x}{1} dx$

$= -\int \sin x \operatorname{cosec} x \, dx = -\int \sin x \cdot \frac{1}{\sin x} dx$

$= -\int 1 \, dx = -x$

$$u_2(x) = \int \frac{y_1(x) f(x)}{W(y_1, y_2)} dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx.$$

$$= \int \cos x \cdot \operatorname{cosec} x \, dx = \int \cos x \cdot \frac{1}{\sin x} dx.$$

$$= \int \frac{\cos x}{\sin x} = \int \frac{1}{u} du.$$

$$= \ln|u| \Rightarrow \ln|\sin x|$$

$\int \frac{\cos x}{\sin x} dx$

Sub. method

let $u = \sin x$

$\frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

$\therefore \int \frac{1}{u} du$

$\Rightarrow \ln|u|$

$\therefore y_p(x) = -x \cos x + \sin x \ln|\sin x|$

$\therefore y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|$

2) find g. sol^n of $y'' - 3y' + 2y = \sin e^{-x}$ —(1).

A) —(1) x.

—(1)

$\therefore$ method of v-- of paranter ✓

y_c ?.

$y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0.$

$(m-1)(m-2) = 0$

$m = 1, \; m = 2$ case I

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$

$y_c = c_1 e^{x} + c_2 e^{2x}.$ $\therefore y_1(x) = e^{x}$, $y_2(x) = e^{2x}.$

y_p ?. $f(x) = \sin e^{-x}.$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}$$

$= 2e^{2x} \cdot e^{x} - e^{2x} \cdot e^{x}$

$= e^{x} e^{2x}(2-1) = e^{x} e^{2x} \Rightarrow e^{3x}$

$a^b \cdot a^c = a^{b+c}$, $e^{x} \cdot e^{2x} = e^{3x}$

$y_p = u_1 y_1 + u_2 y_2$

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{e^{2x} \sin(e^{-x})}{e^{3x}} dx.$$

$\cancel{= -\int \frac{e^{2x}}{e^{3x}} \cdot \frac{\sin e^{-x}}{e^{3x}} dx = -\int \frac{1}{e^{x}} \sin}$

$$= \int \frac{\sin(e^{-x})}{e^{x}} dx. = -\int e^{-x} \cdot \sin(e^{-x}) dx.$$

let $u = e^{-x}$

$du = -e^{-x} dx$

$u_1 = \int \sin u \, du. = -\cos u \Rightarrow -\cos(e^{-x})$

$$u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^{x} \sin e^{-x}}{e^{3x}}$$

$$= \int \frac{\sin(e^{-x})}{e^{2x}} = \int e^{-2x} \cdot \sin(e^{-x})$$

split into $e^{-x} \, e^{-x}$

$= \int e^{-x} \cdot e^{-x} \sin(e^{-x}) dx.$

put. $u = e^{-x}$

$du = -e^{-x} dx.$

$-du = e^{-x} dx$

$= \int u \sin u \cdot -du.$

$= -\int u \cdot \sin u \, du$

by $\int$ by parts $u = u$, $v = \sin u$.

$1^{st} \cdot \int 2^{nd} - \int \left(\frac{d}{dx} 1^{st} \cdot \int 2^{nd}\right)$

$= -\left[\left(u \cdot \int \sin x - \left[\int 1 \times \int \sin x\right]\right)\right]$ ILATE → u, v selection.

$= -u \cdot -\cos u + \int -\cos u \, du$

$= u\cos u - \int \cos u \, du$

$u_2 = u\cos u - \sin u \Rightarrow e^{-x}\cos(e^{-x}) - \sin(e^{-x})$

$y_p = -\cos(e^{-x}) \cdot e^{x} + \left[e^{-x}\cos(e^{-x}) - \sin e^{-x}\right] \cdot e^{2x}$

$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{2x} - e^{x}\cos(e^{-x}) + \left[e^{-x}\cos(e^{-x}) - \sin e^{-x}\right] \cdot e^{2x}$

$\Rightarrow$ Cauchy-Euler eq's =

Any linear differential eq,

g.form $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$.

where $a_0, a_1, \dots a_n$ are constnts.

$g(x) = 0$ = homo-
$g(x) \neq 0$ = non-ho

Consider 2nd order C-E eq,

$$\boxed{a x^2 \frac{d^2y}{dx^2} + b x \frac{dy}{dx} + cy = g(x)} \quad -(1)$$

To solve the DE, we consider $y = x^m$ is a soln of DE,

$$\boxed{\frac{dy}{dx} = m x^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1) x^{m-2}} \quad (2)$$

hence a. eq,

Sub (2) in (1).

$a x^2 \cdot m(m-1) x^{m-2} + b x' \cdot m x^{m-1} + c x^m = 0$

$\Rightarrow a m(m-1) x^m + b m x + c x^m = 0$

$x^m = y$
a.eq $\to y = 1$

$\therefore$ a. eq,

$$\boxed{a m(m-1) + b m + c = 0} \quad m?$$

~~$am^2 - m + bm + c = 0$~~

There are 3 diffrnt cases — (homogeneous)

case I : (Real & distinct) $(m_1 \neq m_2)$.

$y_1 = x^{m_1}, \quad y = x^{m_2}$

$\therefore$ g. soln,

$y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$$

case II : (Repeated real roots) $(m_1 = m_2)$

$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_1} \ln x}$$

case III : (Conjugate complex roots)

$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$

$y_1 = x^{\alpha}\cos(\beta \ln x), \quad y_2 = x^{\alpha}\sin(\beta \ln x)$

then g. soln,

$$\boxed{y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]}$$

1) Solve : $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$. ✓ C-E eq

A) $x^{(2)} \frac{d^{(2)}y}{dx^2} - 2x^{(1)} \frac{d^{(1)}y}{dx} - 4y = 0$.

Since it is C-E eq, corresponding a. eq,

$am(m-1)+bm+c=0.$ $\quad a=1$, $b=-2$, $c=-4$

$m(m-1)-2m-4=0$

$m^2-m-2m-4=0.$

$m^2-3m-4=0$

$m=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{3\pm\sqrt{9-4\times-4}}{2}$

$=\dfrac{3\pm\sqrt{9+16}}{2}=\dfrac{3\pm\sqrt{25}}{2}=\dfrac{3\pm5}{2}$

$=\dfrac{3+5}{2}=4$, $\dfrac{3-5}{2}=-1$ $\quad m_1=4.$ $m_2=-1$

Case I

g.soln,

$y=c_1x^{m_1}+c_2x^{m_2}$

$=c_1x^4+c_2x^{-1}$

2) Solve : $4x^2y''+8xy'+y=0$ $\quad$ c.e ✓

A) $a=4$, $b=8$, $c=1$

a. eq,

$am(m-1)+bm+c=0.$

$4m(m-1)+8m+1=0.$

$4m^2-4m+8m+1=0$

$4m^2+4m+1=0$

$m=\dfrac{-4\pm\sqrt{16-4\times4\times1}}{8}=\dfrac{-4}{8}=\dfrac{-2}{4}=\dfrac{-1}{2}$

Only 1 value, so, $m_1=\frac{-1}{2}$, $m_2=\frac{-1}{2}$
(repeated real roots)

Case II.

$y=c_1x^{m_1}+c_2x^{m_1}\ln x.$

$=c_1x^{-1/2}+c_2x^{-1/2}\ln x.$

3) Solve: $4x^2y''+17y=0$ $\quad$ c.e ✓

A) $a=4$, $c=17$, $b=0$

a. eq,

$am(m-1)+bm+c=0$

$4m(m-1)+0+17=0.$

$4m^2-4m+17=0.$

$m=\dfrac{4\pm\sqrt{16-4\times4\times17}}{8}=\dfrac{4\pm\sqrt{-256}}{8}$

$=\dfrac{4\pm16i}{8}$ $\quad$ $\frac{4}{8}=\frac{1}{2}$

$=\frac{1}{2}\pm2i$ $\quad m_1=\frac{1}{2}+2i$, $m_2=\frac{1}{2}-2i$

$\therefore \alpha=\frac{1}{2}$, $\beta=2$

Case III

$y=x^{\alpha}\left[c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right].$

$=x^{1/2}\left[c_1\cos(2\ln x)+c_2\sin(2\ln x)\right].$

4) Solve: $4x^2y''+24xy'+25y=0.$

A) $a=4$, $b=24$, $c=25.$

a.eq,

$am(m-1)+bm+c=0.$

$4m(m-1)+24m+25=0.$

$4m^2-4m+24m+25=0.$

$4m^2+20m+25=0.$

$$m = \frac{-20 \pm \sqrt{400 - 4 \times 4 \times 25}}{8}$$

$$= \frac{-20}{8} = \frac{-5}{2} \qquad m_1 = \frac{-5}{2}, \ m_2 = \frac{-5}{2}$$

case II

$$y = c_1 x^{m_1} + c_2 x^{m_2} \ln x$$
$$= c_1 x^{-5/2} + c_2 x^{-5/2} \ln x$$

Non-homogeneous Q. =>

1) Solve: $x^2y'' + xy' - 4y = x^2$

A) non-homo, C.E eq ✓

g. solⁿ

$$y = y_c + y_p$$

y_c?, $x^2y'' + xy' - 4y = 0$. $\quad a=1, b=1, c=-4$

$$am(m-1) + bm + c = 0$$
$$m(m-1) + m - 4 = 0$$
$$m^2 - m + m - 4 = 0$$
$$m^2 - 4 = 0$$
$$m^2 = 4 \Rightarrow m = \pm 2, \quad m_1 = 2, m_2 = -2$$

case I

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$
$$= c_1 x^2 + c_2 x^{-2}$$

$$y_1 = x^2, \ y_2 = x^{-2}$$

y_p?

C-E → normal DE.

x^2y'' → cancel x^2 ✓

$$x^2y'' + xy' - 4y = x^2 \quad \text{— ①}$$

① by $\frac{1}{x^2}$: $\frac{x^2}{x^2}y'' + \frac{x}{x^2}y' - \frac{4y}{x^2} = \frac{x^2}{x^2}$

$$y'' + \frac{y'}{x} - \frac{4}{x^2}y = 1 \quad \text{— ②}$$

② → standard form ✓

$$\boxed{y_p = u_1(x)y_1(x) + u_2(x)y_2(x)} \quad \text{— ③}$$

$f(x) = 1$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$$
$$= -2x^{-1} - 2x^{-1} = -4x^{-1} = \frac{-4}{x}$$

$$u_1(x) = -\int \frac{y_2(x) f(x)}{w(y_1, y_2)} dx$$
$$= -\int \frac{x^{-2}}{-4/x} dx$$
$$\cancel{= \int \frac{x^{-2} \cdot x}{4} dx}$$
$$= \int \frac{x}{4x^2} dx = \frac{1}{4}\int \frac{1}{x} dx$$
$$= \frac{1}{4} \ln x$$

$$u_2(x) = \int \frac{y_1(x) f(x)}{w} dx \quad \cancel{= \int \frac{x^2}{-4/x} dx}$$

$\int \frac{x^2}{\frac{-4}{x}}$, $\int \frac{x^2}{1} \times \frac{x}{4}$, $\int \frac{x^3}{4}$

$$\cancel{= -\int \frac{x}{4} = -\frac{1}{4}\int x}$$
$$= -\int \frac{x^2 \times x}{4} dx = -\frac{1}{4}\int x^3 dx$$
$$= -\frac{1}{4} \frac{x^4}{4} = \frac{-x^4}{16}$$

$$\therefore y_p = \frac{1}{4} \ln x \, x^2 + \frac{-x^4}{16} x^{-2}$$
$$= \frac{x^2}{4} \ln x - \frac{x^2}{16}$$

$$y = y_c + y_p \Rightarrow c_1 x^2 + c_2 x^{-2} + \frac{x^2}{4} \ln x - \frac{x^2}{16}$$