Module II

Higher order DE.

A linear diffⁿ eq of order n is an eq of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \longrightarrow ①$$

where $a_0, a_1, \dots a_n$ & $g(x)$ are continuous real(s)

& eq ① $\longrightarrow$ non homogeneous if $g(x) \neq 0$

If $g(x) = 0 \longrightarrow$ Homogeneous LDE

If each (coe) in eq ① is a constant, then eq $\rightarrow$ LDE (linear diff eq) with constant (coe).

eq $\rightarrow$ $3y''' + 5y'' - y' + 7y = 0$

→ Initial value & boundary value prbms =

Theorem = [Existence of unique Solⁿ] $\left[\begin{array}{l} y(0)=1 \\ y(1)=2 \end{array}\right\}$ initial value

let $a_n(x), a_{n-1}(x) \dots a_1(x), a_0(x)$ & $g(x)$ be continuous on an interval 'I' & let $a_n(x) \neq 0$ for every x in the interval, if $x = x_0$ is any point in this interval, then the solⁿ $y(x)$ of initial value prbm,

$$\boxed{\begin{array}{l} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ \text{where } \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \dots \; y^{(n-1)}(x_0) = y_{n+1}. \end{array}}$$

(initial value p. (Starting point))

Exist on interval I & is unique.

→ Boundary Value Prbm = (BVP)

consider the prbm.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$y(a) = y_0$ & $y(b) = y_1 \longrightarrow$ 2 point boundary V.P.

where $y(a) = y_0$ & $y(b) = y_1 \rightarrow$ Boundary condi-

→ Differential operators = (D.O)

Symbol $\frac{dy}{dx} \rightarrow$ (D.O)

usually denoted D. & ( ) $y \longrightarrow$ operand.

eg → ① $\frac{d}{dx} = D$.

② $\frac{d}{dx}\sin x = \cos x$.

↳ $D(\sin x) = \cos x$.

③ $D(4x^3 + 5x^2) = 12x^2 + 10x$

$\frac{d^2}{dx^2}, \frac{d^3}{dx^3} \dots$ are denoted by $D^2, D^3 \dots$

In general, we define n^{th} order differential operator,

$$L = a_n(x)D^n + a_{n-1}D^{n-1} + \dots + a_1(x)D + a_0(x)$$

* Differential optor 'L' posesses the linearity (pro) (i.e)

$D[f(x) + g(x)] = Df(x) + Dg(x)$. & also.

$D[c\,f(x)] = c \cdot D\,f(x)$.

→ Solⁿ of homogeneous LDE =

Theorem = [Super position Principle]

Consider n^{th} order homogeneous DE

is the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

where $a_0(x), a_1(x), \dots a_n(x)$ are continuous real ()s on I, $a_n(x) \neq 0$.

Let $y_1, y_2 \dots y_k$ be soln of the eq on an interval I. Then the linear combination

$$\boxed{y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_k y_k(x)}$$

$c_1, c_2, \dots c_k$ are arbitory constnt, is also a soln on I.

⇒ linear dependence & linear independence of soln =

A set of ()s $f_1(x), f_2(x) \dots f_n(x)$ is said to be linearly dependent on a interval I if there exist constnts $c_1, c_2 \dots c_n$, not all zeroes such that $c_1 f_1(x) + c_2 f_2(x) \dots + c_n f_n(x) = 0$ for every x in the interval I.

If the set of all ()s is not linearly dependent on the interval, it is said to be linearly independent (i.e)

$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ ⇒

$\Rightarrow c_1 = c_2 = c_3 = \dots c_n = 0$

$x_1 \quad x_2$

$c_1 \quad c_2$

⇓

$c_1 x_1 + c_2 x_2$

linear combination

⇓

$c_1 x_1 + c_2 x_2 = 0$

linear eq

eg

L I

$c_1 x_1 + c_2 x_2 = 0$

if $c_1 = 0$ & $c_2 = 0$

(both 0)

L D

$c_1 x_1 + c_2 x_2 = 0$

if $c_1 = 3$ then c_2 is -3

are gt 0

(രണ്ടും ഒന്ന് 0 ആകണമെന്നില്ല)

1) Check whether the ()s $f_1(x) = 1+x$, $f_2(x) = x$, $f_3(x) = x^2$ are linearly indpnt (L I) on interval $(-\infty, \infty)$

A) 1) Linear eq ഉപയോഗിച്ച്

Let c_1, c_2 & c_3 are constnt

consider the linear eq,

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$c_1(1+x) + c_2 x + c_3 x^2 = 0$

$c_1 + c_1 x + c_2 x + c_3 x^2 = 0$

$c_1 + x(c_1 + c_2) + c_3 x^2 = 0 + 0x + 0x^2 + 0$

∴ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

∴ $f_1(x)$, $f_2(x)$ & $f_3(x)$ are L I.

→ wronskian method = (W)

Suppose of each of ()s $f_1(x), f_2(x) \dots f_n(x)$ possesses atleast n-1 derivatives.

The determinent,

3 () → 2 deri

$$W(f_1, f_2 \dots f_n) = \begin{vmatrix} f_1 & f_2 & \dots f_n \\ f_1' & f_2' & \dots f_n' \\ f_1'' & f_2'' & \dots f_n'' \\ \vdots & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots f_n^{(n-1)} \end{vmatrix}$$

is → W. of the ().

Theorum =

let $y_1, y_2 \dots y_n$ be n soln of a homogenous n^{th} order LDE on an interval I then the set of all soln is linearly Independent on I if & only if their wronskin is not identically 0. (i.e)

$LI \Leftrightarrow W \not\equiv 0$ ($\not\equiv$ → not identically)

1) Determine whether the gvn set of (), is LD / LI

a) $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$

A)

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 4x-3x^2 \\ 1 & 2x & 4-6x \\ 0 & 2 & -6 \end{vmatrix}$$

$$= x(-12x - 8 + 12x) - x^2(-6-0) + 4x - 3x^2(2-0)$$

$$= -8x + 6x^2 + 8x - 6x^2 = 0$$

$W = 0$, f_1, f_2, f_3 are LD

b) $f_1(x) = 0$ $f_2(x) = x$ $f_3(x) = e^x$

$$W(0, x, e^x) = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$$= 0[e^x - 0] - x[0-0] - e^x[0] = 0$$

LD

c) $f_1(x) = 5$ $f_2(x) = \cos^2 x$ $f_3(x) = \sin^2 x$

def →

Any set $y_1, y_2 \dots y_n$ of n **LI** Sol^n of the homogenous n^{th} order LDE on an I is said to be fundamental set of Sol^n on I.

* Therum =

let $y_1, y_2 \cdots y_n$ be a fundemental set of Sol^n of homogenous linear n^{th} order DE, with Continous (coe) on I, then the general Sol^n of the eq on I is,

$$\boxed{y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n}$$

$c_i \to i = 1, 2 \dots n$ are arbitory constnts.

Imp

1) Varify that the () $y_1 = x^3$, $y_2 = x^4$ form ① a fundamental set of Sol^n of DE

$x^2 y'' - 6xy' + 12y = 0$ on intrval $(0, \infty)$

② form the general Sol^n ?

A) ① y_1 satisfies eq — ⓪ / not. if $= 0 \to$ satisfies.

$y_1 = x^3$, $y_1' = 3x^2$, $y_1'' = 6x$

$\Rightarrow x^2 y_1'' - 6xy_1' + 12y_1$

$\Rightarrow x^2 6x - 6x\, 3x^2 + 12x^3$

$\Rightarrow 6x^3 - 18x^3 + 12x^3$

$\Rightarrow -12x^3 + 12x^3 = 0$

② $y_2 = x^4$, $y_2' = 4x^3$, $y_2'' = 4 \cdot 3x^2 = 12x^2$

$\Rightarrow x^2 y_2'' - 6xy_2' + 12y_2 =$

$\Rightarrow x^2 . 12x^2 - 6x . 4x^3 + 12x^4$

$\Rightarrow 12x^4 - 24x^4 + 12x^4$

$-12x^4 + 12x^4 \Rightarrow 0$

∴ Both y_1 & y_2 satisfies gvn DE

∴ they are solⁿ.

③ prove LI.

wronskin of the solⁿ,

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$$= 4x^3 \cdot x^3 - 3x^2 \cdot x^4$$

$$= 4x^6 - 3x^6 = x^6 \neq 0$$

$x^6 \to 0$ / not — bcz it is x, if non is instead of x^6 → can be LI.

(since $x \in (0,\infty)$ belongs to.)

∴ y_1 & y_2 are LI.

hence y_1 & y_2 form a f. set of solⁿ to the gvn solⁿ.

— ② general solⁿ,

$$y = c_1 y_1 + c_2 y_2$$

$= c_1 x^3 + c_2 x^4$ where c_1 & c_2 → arbitry constants

2) Varify that the (?) $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ form a f. set of solⁿ of DE $y''' - 6y'' + 11y' - 6y = 0$ on I $(-\alpha, \alpha)$ form general eq?

answer →

⇒ Method of reduction of order = $y = c_1 y_1 + c_2 y_2$

For a 2nd order LDE

$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$ —① → 2 deri → 2 solⁿ; 3 " → 3 "

The general solⁿ is,

$y = c_1 y_1 + c_2 y_2$, where y_1 & y_2 are solⁿ & both LI on same I.

Suppose we can find 1 non-zero solⁿ $y_1(x)$ of above homogneous LDE, so the method for finding y_2 → m.R of order

2.A) $y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$, $y_1''' = e^x$.

$y_1''' - 6y_1'' + 11y_1' - 6y_1 = e^x - 6e^x + 11e^x - 6e^x$

$= -5e^x + 5e^x = 0$

$y_2 = e^{2x}$, $y_2' = 2e^{2x}$, $y_2'' = 2 \cdot e^{2x} \cdot 2 = 4e^{2x}$,

$y_2''' = 4 \cdot e^{2x} \times 2 = 8e^{2x}$.

$y_2''' - 6y_2'' + 11y_2' - 6y_2 = 8e^{2x} - 6 \cdot 4e^{2x} + 11 \cdot 2e^{2x} - 6e^{2x}$

$= 8e^x - 24e^x + 22e^x - 6e^{2x}$

$= -16e^x + 16 = 0$

$y_3 = 3^{ex}$ ....

$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x} = 0$

y_1, y_2 & y_3 satisfies gvn DE. on $(-\alpha, \alpha)$

$$\omega(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$= e^x [18e^{5x} - 12e^{5x}] - e^{2x}[9e^{4x} - 3e^{4x}] + e^{3x}[4e^{3x} - 2e^{3x}]$

$= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0$

∴ Hence the sol^{n}s y_1, y_2 & y_3 are L.I.

general sol^n of given DE,

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

Consider 2nd order DE in standard form, $\boxed{y'' + p(x)y' + q(x)y = 0}$, where $p(x)$ & $q(x)$ are continuous on some I & $y_2(x)$

$$y_2(x) = u(x)\,y(x) = y_1(x)\int \frac{1}{y_1^2} e^{-\int p(x)dx}\,dx.$$

here $u = \int v\,dx = c_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}\,dx + c_2$

1) use the m. of reduction of order, find the general sol^n of DE $x^2y'' - 5xy' + 9y = 0$ on I $(0,\infty)$ given that $y = x^3$ is a sol^n.

A) $y = x^3 \longrightarrow y_1 = x^3$

y_2 ? $\longrightarrow$ m. of Reduction.

$x^2y'' - 5xy' + 9y = 0 \longrightarrow$ make this into (general form) ÷ throughout by x^2.

$$y'' - \frac{5xy'}{x^2} + \frac{9y}{x^2} = 0$$

$$y'' - \frac{5}{x}y' + \frac{9y}{x^2} = 0 \quad \text{—(1)} \Rightarrow \text{St. form} \checkmark$$

Choose $y_1 = x^3$

we have to find y_2.

$$y_2 = u(x)\,y_1(x) \longrightarrow u y_1 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}\,dx$$

$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$.
$y'' + p(x)y' + q(x)y = 0$ } ∴ $p(x) = -5/x$, $q(x) = \frac{9}{x^2}$

∴ $e^{-\int p(x)dx} = e^{-\int -5/x\,dx} = e^{\int 5/x\,dx} = e^{5\int 1/x\,dx}$

$= e^{5\ln x} = e^{\ln x^5} = x^5$

then only we can cut e & ln.

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}\,dx.$$

$$= x^3 \int \frac{1}{(x^3)^2} \cdot x^5\,dx = x^3 \int \frac{1}{x^6} x^5\,dx.$$

$$= x^3 \int \frac{1}{x}\,dx. = x^3 \ln x$$

(i.e) the general sol^n is $c_1 y_1(x) + c_2 y_2(x)$. =
$c_1 x^3 + c_2 x^3 \ln x$

⇒ Sol^n of homogenous LDE with constant (coe)=

Consider homogenous LDE with constant (coe),

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{—(1)}$$

let $\boxed{y = e^{mx}}$ be the possible sol^n of eq–(1)

$$\therefore \frac{dy}{dx} = me^{mx},\ \frac{d^2y}{dx^2} = m^2 e^{mx},\ \frac{d^3y}{dx^3} = m^3 e^{mx}, \dots$$

$$\frac{d^n y}{dx^n} = m^n e^{mx}$$

Sub in –(1),

$$\Rightarrow a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \dots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0\right) = 0 \quad \text{—(2)}$$

(e always ≠ 0)

$$\therefore \left(a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0\right) = 0 \quad \text{—(3)}$$

eq–(3) ⟶ Auxillary eq / characteristic eq of eq–(1)

let us denote the roots by $m_1, m_2 \ldots m_n$ where m_i's are need not all be distinct & real.
Then each () , $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ ... $y_n = e^{m_n x}$ is a solⁿ of DE — ①

while solving auxillary eq the following cases may occur.
1) All the roots are real & distinct
2) All roots are real & same.
3) All roots are not real (complex)

case I =

If $m_1, m_2 \ldots m_n$ are real & distinct the general solⁿ is,

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}}$$

case II =

If $m_1 = m_2 = \ldots = m_n$ are equal the g. solⁿ.

$$\boxed{y = (c_1 + c_2 x + c_3 x^2 + \ldots c_n x^{n-1}) e^{mx}}$$

case III =

If $\alpha + \beta i$ is 1 root, then $\alpha - \beta i$ must be another root.
∴ The g. solⁿ is given by,

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$ (after simplification)

$$\boxed{y = e^{\alpha x}[A \cos \beta x + B \sin \beta x] \quad \text{where } A = c_1 + c_2,\ B = i(c_1 - c_2)}$$

1) Solve : $y'' + 2y' + 5y = 0$

A) (1) homogeneous ✓ $=0$
2) constant multi ✓ $\underline{1}y'' + \underline{2}y' + \underline{5}y$)

corresponding auxillary eq,

$y'' + 2y' + 5y = 0$ — 2nd deri → m^2, 1 deri → m, y → 1.

set sub m → $m^2 + 2m + 5 = 0$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 4i}{2} = -\frac{2}{2} \pm \frac{4i}{2}$$

complex ∴ case III

$$= -1 \pm 2i$$

$$\Rightarrow -1 + 2i \ \& \ -1 - 2i$$

$\alpha + \beta i \rightarrow \alpha = -1, \beta = 2.$

here the roots not real,

∴ Solⁿ is $y = e^{\alpha x}[A \cos \beta x + B \sin \beta x]$

$$= e^{-x}[A \cos 2x + B \sin 2x]$$

($y'' + 2y' + 5y$
$y'' \rightarrow 2$ solⁿ, if $y''' \rightarrow 3$ solⁿ)

2) $4y'' + y' = 0$

A) homo... ✓
constant m ✓

auxillary eq,

$4y'' + y' = 0$

$4m^2 + m = 0.$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4 \times 4 \times 0}}{2 \times 4} = \frac{-1 \pm 1}{8}$$

$\therefore \frac{-1+1}{8}$ & $\frac{-1-1}{8}$

0 & $\frac{-2}{8} = \frac{-1}{4}$

$\therefore$ case II

$y = (c_1 + c_2 x) e^{mx}$

$= (c_1 + c_2 x) e^{-\frac{1}{4}x}$

3) $y'' + 8y' + 16y = 0$

A) auxillary eq,

$m^2 + 8m + 16 = 0$

$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 \pm 64}}{2}$

$= \frac{-8 \pm \sqrt{64 - 64}}{2} = \frac{-8}{2} = -4$

case II

$y = (c_1 + c_2 x) e^{mx}$

$= (c_1 + c_2 x) e^{-4x}$

4) $y'' - y' - 6y = 0$

A) a. eq,

$b = -1$
$c = -6$

$m^2 - m - 6 = 0$

$\therefore m = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2}$

$\cancel{\frac{1+5}{2} = \frac{6}{2} = 3}$ & $\cancel{\frac{1-5}{2} = \frac{-4}{2} = -2}$

$\frac{1+5}{2} = \frac{6}{2} = 3$, $\frac{1-5}{2} = \frac{-4}{2} = -2$

$\therefore$ g. sol^n, Case I

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{3x} + c_2 e^{-2x}$

5) $3y'' + 2y' + y = 0$

6) $y'' + 16y = 0$

7) $y'' + y' + 2y = 0$

(8) $y'' + 9y = 0$

5. A) $3y'' + 2y' + y = 0$.

$3m^2 + 2m + 1 = 0$.

$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{4 - 4 \times 3 \times 1}}{6}$

$= \frac{-2 \pm \sqrt{-8}}{6}$

$= \frac{-2 \pm 8i}{6} = 0$

6. A) $y'' + 16y = 0$

$m^2 + 16 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-16 \pm \sqrt{16^2 - 0}}{2}$

$= \frac{-16 \pm \sqrt{256}}{2}$

$= \frac{-16 \pm 16}{2}$

$= \frac{-16 + 16}{2} = 0$, $\frac{-16 - 16}{2} = -16$

$\therefore$ case II

$y = (c_1 + c_2 x) e^{mx}$

$= (c_1 + c_2 x) e^{-16x}$

8.A) $y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \frac{-9 \pm \sqrt{9^2 - 0}}{2}$

$= -\frac{9 \pm 9}{2}$, $\Rightarrow \frac{-9+9}{2} = 0$

$\Rightarrow \frac{-9-9}{2} = -9$

$\therefore$ case II

$y = (c_1 + c_2 x) e^{-9x}$

9) Solve $y''' - y'' - 6y' = 0$

A) 3 deri $\longrightarrow$ 3 solⁿ $\longrightarrow m_1, m_2, m_3$?

a. eq,

$\Rightarrow m^3 - m^2 - 6m = 0$

$\Rightarrow m(m^2 - m - 6) = 0$

$m^2 - m - 6 = 0$ $\quad b = -1$, $c = -6$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1 - 4 \times 1 \times -6}}{2} = \frac{1 \pm \sqrt{1 + 24}}{2}$

$= \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2}$

$= \frac{1+5}{2} = \frac{6}{2} = 3$, $\frac{1-5}{2} = \frac{-4}{2} = -2$

$\therefore m = 0$ or $m = 3$ or $m = -2$

3 values are diff — So case I

g. solⁿ,

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$= c_1 e^{0x} + c_2 e^{3x} + c_3 m^{-2x}$

~~$= c_1 + c_2 e + c_3 m$~~

10) Solve $\frac{d^4 y}{dx^4} + 8\frac{d^2 y}{dx^2} + 16y = 0$

A) a. eq,

$m^4 + 8m^2 + 16 = 0$ $\quad m^4 \longrightarrow 4$ solⁿ, $m_1\, m_2\, m_3\, m_4$?

Can be $\Rightarrow (m^2)^2 + 8m^2 + 4^2 = 0$

Can be $\Rightarrow (m^2)^2 + 2 \times 4 \times m^2 + 4^2 = 0$. $\longrightarrow (a+b)^2$

$\Rightarrow (m^2 + 4)^2 = 0$

$\Rightarrow m^2 + 4 = 0$ $\quad$ we need value of m ?

$m^2 = -4$

$m = \pm\sqrt{-4}$

~~m = 2~~ $\quad m = \pm 2i$.

$\underbrace{2i, -2i}_{\text{we get this}}$, $\underbrace{2i, -2i}_{2}$

but we need 4 no - so repeat this 2

both a case II & III,

here the solⁿ of eq are repeating bt not real (case II & 3 satisfied)

$m = \pm 2i \longrightarrow m = 0 \pm 2i$.

$\therefore \alpha = 0$, $\beta = 2$

$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

11) Solve $y'' + 4y = 0$, $y(0) = 3$, $y(\pi/2) = -3$

A) a. eq,

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$. $\rightarrow m = 0 \pm 2i$.

$\alpha = 0$, $\beta = 2$

case III

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

where $A = c_1 + c_2$ & $B = i(c_1 - c_2)$

$y = e^{0}[A\cos 2x + B\sin 2x]$

$y = A\cos 2x + B\sin 2x$ — (1)

here no; c_1 & c_2.

$y(0) = 3 \Rightarrow x = 0, y = 3$

$y = A\cos 2x + B\sin 2x$.

$3 = A\cos 0 + B\sin 0$

$3 = A$ $\therefore$ $A = 3$

$y(\pi/2) = -3 \Longrightarrow x = \pi/2, y = -3$

$y = A\cos 2x + B\sin 2x$.

$-3 = A\cos 2\times\pi/2 + B\sin 2\times\pi/2$

$-3 = A\cos\pi + B\sin\pi$ $\quad$ $\cos\pi = -1$, $\sin\pi = 0$

$-3 = -A$

$\Rightarrow A = 3$

B not get so,

$\therefore$ g sol^n,

on (1). $y = A\cos 2x + B\sin 2x$

$= 3\cos 2x + B\sin 2x$

Rmrk

Suppose that $m_1, m_2, m_3 \ldots m_n$ are roots of homogeneous LDE with constant (co),

case I $\rightarrow$

If m_1, m_2 are not real but $m_3 \ldots m_n$ are remaining real & distinct roots then the g. sol^n is, (combine case II & III)

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} \ldots c_n e^{m_n x}$.

case II $\rightarrow$

If there are 2 pairs of complex roots $\alpha + \beta i$ (m_1, m_2) & $\gamma + \delta i$ (m_3, m_4) then g. sol^n is, $(m_5 \ldots m_n)$

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x] + e^{\gamma x}[C\cos\delta x + D\sin\delta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} \ldots + c_n e^{m_n x}$.

1) Solve : $16y'' + 24y' + 9y = 0$

A) a. eq,

$16m^2 + 24m + 9 = 0$.

$m = \dfrac{-24 \pm \sqrt{24^2 - 4\times16\times9}}{2\times16}$

$= \dfrac{-24 \pm \sqrt{576 - 576}}{32} = \dfrac{-24}{32} = \dfrac{-12}{16}$

$= \dfrac{-6}{8}$

$= -3/4$

⇒ Solⁿ of non-homogenous LDE =

* Therum I =

consider non homogenous LDE,

$$a_n(x)y^{(n)}\ a_{n-1}(x)y^{(n-1)}+\dots+a_1(x)y'+a_0y=g(x)$$

where $a_0, a_1, \dots a_n$ & $g(x)$ are continuous real ( )s on an I with $a_n \neq 0$,

let y_p be any particular solⁿ of non homogenous LDE, eq-① on an I & let $y_1, y_2 \dots y_n$ be a fundamental set of solⁿ of the associated homogenous LDE,

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\dots+a_1(x)y'+a_0y=0 \text{ — ②}$$

on I, then g. solⁿ of eq — ① is,

$$\boxed{y=y_c(x)+y_p(x)} = \underbrace{c_1y_1(x)+c_2y_2(x)+\dots c_ny_n(x)}_{y_c}+y_p$$

where $c_1, c_2 \dots c_n$ are arbitary constnt.

Rmrks

* g. solⁿ of non-homogenous eq-① is
y = complemntry ( ) + any particular ( ).

* Therum - II (Super position principle)

let $y_{p_1}, y_{p_2} \dots y_{p_k}$ be k particular solⁿ of non homognous LDE eq-① on a I corresponding, inturns, to

k distant ( )s $g_1, g_2 \dots g_k$ (i.e)

$y_p(x) = y_{p_1}(x)+y_{p_2}(x)+\dots+y_{p_k}(x)$ is a particular solⁿ of

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\dots+a_1(x)y'+a_0y=g_1(x)+g_2(x)+\dots g_k(x).$$

where y_{p_i} is a particular solⁿ of

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\dots+a_1(x)y'+a_0y=g_i(x).$$

Q
1) Verify that $y_p(x)=x^3$ is a solⁿ of the DE: $y''-2y'+5y=5x^3-6x^2+6x$ — ①
find g. solⁿ ?

A) ① — $y = x^3$ or not

$$y''-2y'+5y=5x^3-6x^2+6x. \text{ — ①} \qquad y=x^3 \text{ — ②}$$

$y_p' = 3x^2$, $y_p'' = 6x$

Rule in ①

$$6x-2\times3x^2+5x^3=5x^3-6x^2+6x.$$
$$6x-6x^2+5x^3=5x^3-6x^2+6x. \text{ — ②}$$
$$① = ②.$$

it satisfies eq ① ∴ y_p is a solⁿ of eq-①

② g. solⁿ.

y_c ? corresponding homogenous eq ① is,

$$y''-2y'+5y=0 \text{ — ③}.$$

a. eq,

$$m^2-2m+5=0.$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4\times5\times1}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\therefore m = 1 \pm 2i$

$\therefore \alpha = 1, \beta = 2$

$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$.

$= e^{x}[A\cos 2x + B\sin 2x]$

$\therefore$ g. solⁿ,

$y = y_c + y_p$

$= e^{x}(A\cos 2x + B\sin 2x) + x^3$

2) varify that $y_p = -\cos 2x$. is a solⁿ of DE

$(D^2+3D-4)y = 8\cos 2x + 6\sin 2x$. — (1)

hence solue the initial value prbm

— (1), $y(0)=1$ & $y'(0) = -3$

A) $(D^2+3D-4)y = 8\cos 2x$ —

$\left(\frac{d^2}{dx^2} + 3\frac{d}{dx} - 4\right)y = 8\cos 2x$ —

$y'' + 3y' - 4y = 8\cos 2x + 6\sin 2x$. — (a)

$y_p = -\cos 2x$

$y_p' = -(-\sin 2x \times 2) = 2\sin 2x$

$y_p'' = 4\cos 2x$.

$\therefore 4\cos 2x + 6\sin 2x + 4\cos 2x = 8\cos 2x + 6\sin 2x$ — (b)

— (a) = — (b)

y_p satisfies eq — (a). hence it is a particular solⁿ.

y_c ?,

corresponding homogenous eq — (a),

$y'' + 3y' - 4y = 0$ — (c)

a. eq.

$m^2 + 3m - 4 = 0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{9-4\times1\times-4}}{2}$

$= \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2}$

$= \frac{-3+5}{2} = 1$, $\frac{-3-5}{2} = -4$

$m_1 = 1$

$m_2 = -4$

$\therefore$ 2 → diff → case I

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$y_c = \boxed{c_1 e^{x} + c_2 e^{-4x}}$

~~c_1 & c_2 ? $y(0) = 1$, $y'(0) = -3$~~

$\therefore$ g. solⁿ eq — (1),

$y = y_c + y_p$

$y = c_1 e^{x} + c_2 e^{-4x} - \cos 2x$. — (d)

guy $y(0) = 1$. $\Rightarrow 1 = c_1 e^{0} + c_2 e^{0} - \cos 0$.

$1 = c_1 + c_2 - 1$

$2 = c_1 + c_2$ — (e) $\Rightarrow 2 - c_2 = c_1$

$y'(0) = -3$

$y = c_1 e^{x} + c_2 e^{-4x} - \cos 2x.$

$y' = c_1 e^{x} - 4c_2 e^{-4x} + 2\sin 2x$

$y(0)=3 \Rightarrow 3 = c_1 e^{0} - 4c_2 e^{0} + 2\sin 0$

$\Rightarrow -3 = c_1 - 4c_2$ — (f)

Sub (e) in (f)

$-3 = 2 - c_2 - 4c_2$

$-3 = 2 - 5c_2$

$5c_2 = 3+2$

$5c_2 = 5$

$c_2 = 1$ ∴ $c_1 = 1$

(d) ∴ $y = e^{x} + e^{-4x} - \cos 2x$

⇒ **Method of undetermined (coe) = (To find y_p)**

Procedure for finding y_p is given,

($g(x)$-ന depend [illegible] y_p)

Terms in $g(x)$	choice of y_p
1) $k e^{\alpha x}$	$c e^{\alpha x}$ (exp)
2) $k x^n$ $(n=0,1,\ldots,n)$	$k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0$
3) $k\cos\beta x$ (or) $k\sin\beta x$	$k\cos\beta x + M\sin\beta x$. (trig)
4) $k e^{\alpha x}\cos\beta x$, $k e^{\alpha x}\sin\beta x$	$e^{\alpha x}[k\cos\beta x + M\sin\beta x]$. (exp+trig)
5) $x^n e^{\alpha x}$	$e^{\alpha x}[k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0]$.
6) $k x^n e^{\alpha x}\cos\beta x$, $k x^n e^{\alpha x}\sin\beta x$	$e^{\alpha x}\cos\beta x[k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0]$ + $e^{\alpha x}\sin\beta x[k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0]$.

• Find g. sol^n of $y'' - y' - 2y = 6e^{x}$

A) ① y_c?

g. sol^n → $y = y_c + y_p$

homogeneous DE is,

$y'' - y' - 2y = 0$ — ①

a. eq,

$m^2 - m - 2 = 0$

$m = \frac{1 \pm \sqrt{1 - 4 \times -2 \times 1}}{2} = \frac{1 \pm \sqrt{9}}{2}$

$= \frac{1+3}{2} = \frac{4}{2} = 2$, $\frac{1-3}{2} = \frac{-2}{2} = -1$

$m = 2, -1$

case I,

∴ $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$= c_1 e^{2x} + c_2 e^{-x}$ — ⓐ

(c_1, c_2 can't find here no initial condi—)

② y_p?

$y'' - y' - 2y = 6e^{x}$ → exp → so Ist in table.

Since $g(x) = 6e^{x}$

∴ choose y_p as, $y_p = c e^{\alpha x}$

$y_p = c e^{x}$

∴ y_p is sol^n of given DE ? c?

Substituting y_p for y in eq — ①

$y_p = ce^x$, $y_p' = ce^x$, $y_p'' = ce^x$

$y'' - y' - 2y = 6e^x$

$y_p'' - y_p' - 2y_p = 6e^x$

$ce^x - ce^x - 2ce^x = 6e^x$

$-2c\cancel{e^x} = 6\cancel{e^x}$

$-2c = 6$

$c = -6/2 = -3$

∴ particular sol^n,

$y_p = ce^x = -3e^x$

∴ g. sol^n,

$y = y_c + y_p$

$y = c_1e^{2x} + c_2e^{-x} - 3e^x$

2) Solve : $y'' - y = 2x^2$

$y'' - y = 2x^2$ — ①

A) y_c? $y'' - y = 0$

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$ case I

∴ $y_c = c_1e^{m_1x} + c_2e^{m_2x}$

$= c_1e^x + c_2e^{-x}$ — ②

y_p ?

$g(x) = 2x^2$ case II in table

$kx^n = k_nx^n + k_{n-1}x^{n-1} \dots$

$k_n\, k_{n-1}\, k_0$?

∴ $y_p = k_2x^2 + k_1x + k_0$

y_p satisfies eq - ①, sub y_p for y.

$y_p' = 2k_2x + k_1$

$y_p'' = 2k_2$

$y'' - y = 2x^2$

$y_p'' - y_p = 2x^2$

$\cancel{2k_2 - 2k_2x + k_1 = 2x^2}$

$2k_2 - [k_2x^2 + k_1x + k_0] = 2x^2$

$2k_2 - k_2x^2 - k_1x - k_0 = 2x^2$ (x^2 — 2 coeff. com.)

$-k_2\cancel{x^2} = 2\cancel{x^2}$

$-k_2 = 2 \Rightarrow k_2 = -2$

$2k_2 - k_2x^2 - k_1x - k_0 = 2x^2 + 0x + 0$ ($-k_1x$ ↓ $0x$ ↓ 0)

∴ $k_1 = 0$.

$2k_2\ \cancel{-k_2x^2 - k_1x}\ -k_0 = 2x^2 + 0x + 0$

$\cancel{k_0 = 0}$

($k_2 = -2$) $2k_2 - k_0 = 0$ (x … term …)

$2 \times -2 - k_0 = 0$

$-4 - k_0 = 0$

$-k_0 = 4$ $\Rightarrow k_0 = -4$

∴ $y_p = k_2x^2 + k_1x + k_0$.

$= -2x^2 - 4$

$y = y_c + y_p \Rightarrow c_1e^x + c_2e^{-x} - 2x^2 - 4$

3) Solve : $y'' - 3y' + 2y = 4t + e^{3t}$ — ①

A) $y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0$

$m = \dfrac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2} = \dfrac{3 \pm \sqrt{1}}{2}$

$= \dfrac{3+1}{2}, \dfrac{4}{2} = 2, \dfrac{3-1}{2} = 1$

case I

$y_c = c_1 e^{-t} + c_2 e^{-2t}$ → g(x) is in t term

$\therefore g(t) = 4t + e^{3t}$

means $y_p = k_1 t + k_0 + ce^{3t}$ k_1, k_0, c?

$y_p' = k_1 + 3ce^{3t}$, $y_p'' = 9ce^{3t}$

y_p is a sol^n of eq ①

$\Rightarrow y'' - 3y' + 2y = 4t + e^{3t}$

$y_p'' - 3y_p' + 2y_p = 4t + e^{3t}$

$9ce^{3t} - 3(k_1 + 3ce^{3t}) + 2(k_1 t + k_0 + ce^{3t}) = 4t + e^{3t}$

$\cancel{9ce^{3t}} - 3k_1 - \cancel{9ce^{3t}} + 2k_1 t + 2k_0 + 2ce^{3t} = 4t + e^{3t}$

$2ce^{3t} + 2k_1 t - 3k_1 + 2k_0 = 4t + e^{3t}$

$2c = 1$

$c = 1/2$

$2k_1 t - 3k_1 + 2k_0 = 4t$

$2k_1 = 4$

$k_1 = 2$

$-3k_1 + 2k_0 = 0$

$-3\times 2 + 2k_0 = 0 \Rightarrow -6 + 2k_0 = 0$

$2k_0 = 6$

$k_0 = 3$

$\therefore y_p = k_1 t + k_0 + ce^{3t}$

$= 2t + 3 + \frac{1}{2}e^{3t}$

$\therefore$ g. sol^n,

$y = y_c + y_p$

$= c_1 e^{-t} + c_2 e^{-2t} + 2t + 3 + \frac{1}{2}e^{3t}$

4) Solve: $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$

A) $y'' + 2y' - 35y = 0$

$m^2 + 2m - 35 = 0$

$\times 35$, $+2$: $7\times 5 = 35$, $7 - 5 = 2$

$m = 7, -5$

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{7x} + c_2 e^{-5x}$

$g(x) = 12e^{5x} + 37\sin 5x$

$y_p = ce^{5x} + k\cos 5x + M\sin 5x$

$y_p' = 5ce^{5x} - 5k\sin 5x + 5M\cos 5x$

$y_p'' = 25ce^{5x} - 25k\cos 5x - 25M\sin 5x$

$y_p \to y$

$y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$

$y_p'' + 2y_p' - 35y_p = 12e^{5x} + 37\sin 5x$

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 2[5ce^{5x} - 5k\sin 5x + 5M\cos 5x] - 35[ce^{5x} + k\cos 5x + M\sin 5x] = 12e^{5x} + 37\sin 5x$

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 10ce^{5x} - 10k\sin 5x + 10M\cos 5x - 35ce^{5x} - 35k\cos 5x - 35M\sin 5x = 12e^{5x} + 37\sin 5x$

$25c + 10c - 35c = 12$

$-20c = 12$

$c = \frac{-12}{20}$

$\cancel{-25M - 10K - 35M = 37}$

$\cancel{-60M - 10K = 37}$

$\Rightarrow 25ce^{5x} - 25k\cos 5x - 25M\sin 5x + 10ce^{5x} + 10k\sin 5x + 10M\cos 5x - 35ce^{5x} - 35k\cos 5x - 35M\sin 5x = 12e^{5x} - 37\sin 5x$

$\Rightarrow (-25+10-35)\cos 5x - (-25+10-35)\sin 5x + 15ce^{5x}$

5) Solve: $y'' + y = 10e^x \sin x$ — (1)

A) $y'' + y = 0$

$m^2 + 1 = 0$

$m^2 = -1 = i$

$m = \pm i$ case III

$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ $\alpha = 0$, $\beta = 1$

$= e^0 [A\cos x + B\sin x]$

$y_c = A\cos x + B\sin x$

y_p?

$g(x) = 10e^x \sin x$. (case 4 in table)

$y_p = e^{\alpha x}[K\cos\beta x + M\sin\beta x]$

$\therefore \alpha = 1$, $\beta = 1$

choose $y_p = e^x[K\cos x + M\sin x]$ K? M?

$y_p = e^x K\cos x + e^x M\sin x$

y_p is a sol^n eq (1)

$y \to y_p$.

$\cancel{y_p' = Ke^x\sin x + Me^x\cos x}$

$\cancel{y_p'' = -Ke^x\cos x + Me^x\sin x}$

$y_p = \underbrace{Ke^x\cos x}_{\text{p rule}} + \underbrace{Me^x\sin x}_{\text{p rule}}$

$y_p' = K[e^x \cdot -\sin x + \cos x e^x] + M[e^x\cos x + \sin x e^x]$

$= -Ke^x\sin x + Ke^x\cos x + Me^x\cos x + Me^x\sin x$

$y_p' = e^x\sin x[M-K] + e^x\cos x[K+M]$

$y_p'' = [e^x\cos x + \sin x e^x](M-K) + (K+M)[-e^x\sin x + \cos x e^x]$

$= (M-K)e^x\cos x + (M-K)\sin x e^x - (K+M)e^x\sin x + (K+M)e^x\cos x$.

$= e^x\cos x[(M-K)(K+M)] + e^x\sin x[(M-K)(K-M)]$

$= e^x\cos x[M - \cancel{K} + \cancel{K} + M]_{(2M)} + e^x\sin x[\cancel{M} - K - K - \cancel{M}]_{(-2K)}$

$y_p'' = 2Me^x\cos x - 2Ke^x\sin x$

$\therefore y'' + y = 10e^x\sin x$.

$y_p'' + y_p = 10e^x\sin x$

$2Me^x\cos x - 2Ke^x\sin x + Ke^x\cos x + Me^x\sin x = 10e^x\sin x$.

$= e^x\cos x(2M+K) + e^x\sin x(-2K+M) = 10e^x\sin x$

$= e^x\cos x(2M+K) + e^x\sin x(M-2K) = 10e^x\sin x$

$\Rightarrow 2M + K = 0$ & $(M - 2K) = 10$.

$K = -2M$. $M - 2K = 10$

$M - 2(-2M) = 10$.

$M + 4M = 10 \Rightarrow 5M = 10, M = 2$

$m = 2$

$K = -2M$

$\therefore K = -2 \times 2 = -4$

$K = -4$ & $m = 2$

$\therefore y_p = Ke^{x}\cos x + Me^{x}\sin x$
$= -4e^{x}\cos x + 2e^{x}\sin x.$

$\therefore$ g. sol^n,

$y = y_c + y_p$
$= A\cos x + B\sin x - 4e^{x}\cos x + 2e^{x}\sin x$

H.W.
6) Solve : $y''' + y'' = e^{x}\cos x$.

$\Rightarrow$ Method of variation of paramters =

Consider 2nd order LDE in standard form

$y'' + p(x)y' + q(x)y = f(x). \leftarrow (1)$

To find g. sol^n, follow the steps —

* find complementry (),
$\boxed{y_c = c_1 y_1(x) + c_2 y_2(x)}$

* compute wronskin,

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (not $\neq 0$ always)

* Assuming $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, to be particular sol^n

* cal u_1 & u_2 by,

$$\boxed{u_1(x) = \int \frac{-y_2(x) f(x)}{w(y_1, y_2)}\, dx}$$

$$\boxed{u_2(x) = \int \frac{y_1(x) f(x)}{w(y_1, y_2)}\, dx}$$

[illegible]

hence $y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$.

$\therefore$ g. sol^n,

$\boxed{y = y_c + y_p}$

) Solve : $y'' + y = \operatorname{cosec} x$ — (1).

A) not homogenous. ✓
undetermined () X, so method of variation ✓

y_c ?

$y'' + y = 0.$
$m^2 + 1 = 0.$
$m^2 = -1$
$m = \pm i$ case III

$\alpha = 0$
$\beta = 1$

$y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$

$y_c = c_1\cos x + c_2\sin x$

$y_c = c_1 y_1(x) + c_2 y_2(x)$
$\therefore y_1(x) = \cos x.$
$y_2(x) = \sin x.$

y_p ? $f(x) = \operatorname{cosec} x.$

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x$