04 : VECTOR INTEGRATION.

⇒ Line integrals :-

we integrate the integrand $f(x)$ from $x=a$ along x-axis to $x=b$.
In the line ∫ the interval $[a,b]$ is replaced by a curve C in plane/space described by a parametric eq. & integrand is a () $G(x,y,z)$ defined & bounded on this curve.
The resulting ∫ → line integral / a curvillinear ∫ / contour ∫.

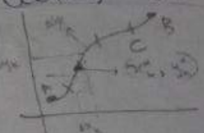
Suppose, C is a curve in the xy-plane parametrized by ~~x = f(t)~~
$x=f(t)$, $y=g(t)$, $a \le t \le b$ & A & B are points $(f(a), g(a))$ & $(f(b), g(b))$.
we call C the path of ∫-tion, A its initial point & B is terminal point.

* C is smooth curve, if f' & g' are contin... in closed intrval $[a,b]$ & not simultaneously 0 in open interval (a,b).

* C is piecewise smooth, if it consist of a finite no. of smooth curves $C_1, C_2 \ldots C_n$ joined end to end, (i.e) $C = C_1 \cup C_2 \cup \ldots \cup C_n$.

* C is a closed curve if $A=B$
* C is a simple closed curve if $A=B$ & the curve does not cross itself.

This same terminology carries ovr in a natural manner to curves in space.



→ Line ∫ in the plane :-

1) Let $Z = G(x,y)$ be defined in some region that contains the smooth curve C defined by $x=f(t)$, $y=g(t)$, $a \le t \le b$.
2) Divide C into n subarcs of length Δs_k according to the partition.
$$a = t_0 < t_1 < t_2 \ldots < t_n = b.$$
3) Let $\|P\|$ be the norm of partition / length of longest subarc.
4) Choose a sample point (x_k^*, y_k^*) on each subarc.
5) form the sum,
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\, \Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\, \Delta y_k,$$
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\, \Delta s_k.$$

Then the defn of line ∫ in the plane given below —

def → Let G be a ( ) of 2 variables x & y defined on a region of the plane containing a smooth curve C

a) line $\int$ of G along C from A to B with respect to x is,

$$\boxed{\int_C G(x,y)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k}$$

b) line $\int$ of G along C from A to B with respect to y,

$$\boxed{\int_C G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k}$$

c) line $\int$ of G along C from A to B with respect to arc length,

$$\boxed{\int_C G(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k}$$

⇒ Method of Evaluation – curve defined parametrically :-

$f'(t)\,dt$ or $g'(t)\,dt$ or $\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$

The exp., $ds = \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$ → differential of arc length.

$\int$-tion is carried out with respect to variable in usual manner:

$$\int_C G(x,y)\,dx = \int_a^b G[f(t), g(t)]\,f'(t)\,dt$$

$$\int_C G(x,y)\,dy = \int_a^b G[f(t), g(t)]\,g'(t)\,dt$$

$$\int_C G(x,y)\,ds = \int_a^b G[f(t), g(t)]\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$$

$y+x=0$ → Implicit
$y+x$ → Explicit

⇒ Methods of Evaluation – curve defined by an Explicit ( ) :-

If the curve 'C' is defined by an explicit ( ) $y = f(x)$, $a \le t \le b$, x as a paramtr,

$$dy = f'(x)\,dx \quad \& \quad ds = \sqrt{1+[f'(x)]^2}\,dx.$$

line $\int$ becomes,

$$\int_C G(x,y)\,dx = \int_a^b G(x, f(x))\,dx.$$

$$\int_C G(x,y)\,dy = \int G(x, f(x))\,f'(x)\,dx.$$

$$\int_C G(x,y)\,ds = \int G(x, f(x))\sqrt{1+(f'(x))^2}\,dx.$$

* line $\int$ along a piecewise-smooth curve C is defined as sum of $\int$ over various smooth curves whose union comprises C.

$$\int_C G(x,y)\,ds = \int_{C_1} G(x,y)\,ds + \int_{C_2} G(x,y)\,ds + \int_{C_3} G(x,y)\,ds.$$

Rmrk

* In many appli..., line $\int$ appear as a sum

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

* Common to write this sum as 1 $\int$ without parentheses,

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \quad \text{or} \quad \int_C P\,dx + Q\,dy.$$

* Line $\int$ along a ~~...~~ closed curve C is often,

$$\oint_C P\,dx + Q\,dy. \qquad \oint \to \text{closed } \int.$$

Q 1) Let C denote the quarter-circle defined by $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le \frac{\pi}{2}$, Evaluate,

a) $\int_C xy^2\,dx$ (b) $\int_C xy^2\,dy$ (c) $\int_C xy^2\,ds$ (arc length)

A. $x = 4\cos t \qquad y = 4\sin t$

$G(x,y) = xy^2$

$f(t) = x = 4\cos t \qquad a = 0$

$g(t) = y = 4\sin t \qquad b = \pi/2$.

dx ?

$dx = f'(t)\,dt$

$= -4\sin t\,dt$.

$dy = g'(t)\,dt$

$= 4\cos t\,dt$. $\qquad ds$?

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

$$= \sqrt{(-4\sin t)^2 + (4\cos t)^2} = \sqrt{16\sin^2 t + 16\cos^2 t}$$

$$\sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{16} = \underline{\underline{4\,dt}}.$$

a) $\int_C xy^2\,\underline{dx} = ?$

$$\boxed{\int_C G(x,y)\,\underline{dx} = \int_a^b G[f(t), g(t)]\,\underline{f'(t)}\,dt}$$

$\int xy^2$

$x \to f(t) \to 4\cos t$

$y^2 \to (g(t))^2 \to (4\sin t)^2$

$$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot (-4\sin t)\,dt$$

$$= -256\int_0^{\pi/2} \cos t \cdot \sin^3 t\,dt.$$

$$= -256\int_0^{\pi/2} \sin^3 t \cdot \cos t\,dt.$$

Substitution method

Let $u = \sin t$

$du = \cos t\,dt$

limit

$t = 0, \; u = 0 \; (\sin 0 = 0)$

$t = \pi/2, \; u = \sin(\pi/2) = 1 \qquad \begin{pmatrix} u = 0 \\ u = 1 \end{pmatrix}$

$$= -256\int_0^1 u^3\,du. = -256\left[\frac{u^4}{4}\right]_0^1$$

$$= -256 \cdot \frac{1}{4} \cdot 0 = \frac{-256}{4} = \underline{\underline{-64}}$$

b) $\int_C xy^2\,dy = ?$

$$\boxed{\int_C G(x,y)\,\underline{dy} = \int_a^b G[f(t), g(t)] \cdot \underline{g'(t)}\,dt}$$

$$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot 4\cos t\,dt$$

$$= 256\int_0^{\pi/2} \sin^2 t\,\cos^2 t\,dt.$$

$$= 256\int_0^{\pi/2} (\cos t\,\sin t)^2\,dt.$$

$\div 2^2$ & $\times 2$,

$= 64 \int_0^{\pi/2} (2\sin t \cos t)^2 \, dt$

$2\sin t \cos t = \sin 2t$

$= 64 \int_0^{\pi/2} (\sin 2t)^2 \, dt$

$= 64 \int_0^{\pi/2} \sin^2 2t \, dt$

$\boxed{\sin^2 t = \frac{1-\cos 2t}{2}}$

$= 64 \int_0^{\pi/2} \frac{1-\cos 4t}{2} \, dt$

$= \frac{64}{2} \int_0^{\pi/2} (1-\cos 4t) \, dt$

~~$\Rightarrow 32 \left[t + 4\sin 4t \cdot 4 \right]_0^{\pi/2}$~~

diff → chain rule → ×

$\int \rightarrow \div$

$\frac{d}{dt}(\cos t) = \sin t$

$= 32 \left[t - \frac{\sin 4t}{4} \right]_0^{\pi/2}$

$= 32 \left[\frac{\pi}{2} - \frac{\sin 4 \cdot \pi/2}{4} - 0 \right]$

$= 32 \left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} \right]$

$\boxed{\sin \pi = 0 \; ; \; \sin 2\pi = 0}$

$= 32 \left[\frac{\pi}{2} \right] = 16\pi$

c) $\int_C xy^2 \, ds =$

$\boxed{\int_C G(x,y) \, ds = \int_a^b G[f(t), g(t)] \cdot \sqrt{(f'(t))^2 + (g'(t))^2} \, dt}$

$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot \sqrt{(-4\sin t)^2 + (4\cos t)^2}$

$= \int_0^{\pi/2} 4\cos t \; 4^2 \sin^2 t \cdot \sqrt{16(\sin^2 t + \cos^2 t)}$

$= \int_0^{\pi/2} 4\cos t \cdot 16\sin^2 t \cdot 4 \, dt$

$= 256 \int_0^{\pi/2} \sin^2 t \cdot \cos t \, dt$ — ①

Sub. method

$u = \sin t$

$du = \cos t \, dt$

limit: $t = 0, u = 0$; $t = \frac{\pi}{2}, u = 1$

①

$= 256 \int_0^1 u^2 \cdot du$

$= 256 \left[\frac{u^3}{3} \right]_0^1 du = 256 \cdot \frac{1}{3} = \frac{256}{3}$

2) Let C denote the line segment $y = 2x+1$; $-1 \le x \le 0$ & $G(x,y) = 3x^2 + 6y^2$

find line ∫ -

a) $\int_C G(x,y) \, dx$ (b) $\int_C G(x,y) \, dy$ (c) $\int_C G(x,y) \, ds$.

A) $y = 2x+1 = f(x)$

$\boxed{dy = f'(x) dx} = 2 \, dx$.

$ds = \sqrt{1+(f'(x))^2} \, dx = \sqrt{1+2^2} \, dx = \sqrt{5} \, dx$

a) $\boxed{\int_C G(x,y) \, dx = \int_a^b G(x, f(x)) \, dx}$

$= \int_{-1}^0 3x^2 + 6(2x+1)^2 \, dx$

$(a+b)^2 = a^2 + 2ab + b^2$
$= 4x^2 + 4x + 1$

$3x^2 + 6y^2 \Rightarrow 3x^2 + 6(4x^2+4x+1)$

$= 3x^2 + 24x^2 + 24x + 6$
$= 27x^2 + 24x + 6$

$= \int_{-1}^0 27x^2 + 24x + 6 \, dx$

$= \left[27\frac{x^3}{3} + 24\frac{x^2}{2} + 6x \right]_{-1}^0$

$= \left[9x^3 + 12x^2 + 6x \right]_{-1}^0$

$= \left[0 - (-9 + 12 - 6) \right] = 3$

$-[3-6] = -[-3] = 3$

b) $\int_c G(x,y)\,dy = \int_a^b G(x, f(x))\,f'(x)\,dx$

$= \int_{-1}^{0} [3x^2 + 6(2x+1)^2]\,2\,dx$

$= 2\int_{-1}^{0} 27x^2 + 24x + 6\,dx$

$= 2\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$

$= 2\left(0 - (-9+12-6)\right) = 6$

c) $\int_c G(x,y)\,ds = \int_a^b G(x, f(x))\cdot\sqrt{1+(f'(x))^2}\,dx$

$= \int_{-1}^{0} [3x^2 + 6(2x+1)^2]\,\sqrt{5}\,dx$

$= \sqrt{5}\int_{-1}^{0} 27x^2 + 24x + 6\,dx$

$= \sqrt{5}\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$

$= \sqrt{5}\left[0 - (-9+12-6)\right] = 3\sqrt{5}$

3) Evaluate $\int_c xy\,dx + x^2\,dy$.

c is gvn by $y = x^3$, $-1 \le x \le 2$.

A) $y = x^3$

$dy = 3x^2\,dx$,

$\int_c xy\,dx + x^2\,dy = \int_{-1}^{2} x(x^3)\,dx + x^2(3x^2\,dx)$

$= \int_{-1}^{2} [x^4 + 3x^4]\,dx = \int_{-1}^{2} 4x^4\,dx$

$= 4\left[\frac{x^5}{5}\right]_{-1}^{2} = 4\left[\frac{32}{5} - \frac{-1}{5}\right] = \frac{132}{5}$

4) Evaluate $\oint_c x^2y^3\,dx - xy^2\,dy$.

c is the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ & $(-1,1)$ in the counter clockwise (dir)?

A) Let c_1, c_2, c_3 & c_4 be the sides of square c, joining $(-1,-1)$ to $(1,-1)$, $(1,-1)$ to $(1,1)$, $(1,1)$ to $(-1,1)$ & $(-1,1)$ to $(-1,-1)$

c is composed of smooth curves c_1, c_2, c_3, c_4.

(para) eq —

$c_1 = x = t$, $y = -1$, $-1 \le t \le 1$

$c_2 = x = 1$, $y = t$, $-1 \le t \le 1$

$c_3 = x = -t$, $y = 1$, $-1 \le t \le 1$

$c_4 = x = -1$, $y = -t$, $-1 \le t \le 1$

I On c_1 $x = t$, $y = -1$ & $-1 \le t \le 1$. Then

$dx = dt$, $dy = 0$ &

$\int_{c_1} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} t^2(-1)^3\,dt - t(-1)^2\cdot 0$

$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$

ii) On C_2, $x=1$, $y=t$, $-1 \le t \le 1$. Then

$dx = 0$, $dy = dt$.

$$\int_{C_2} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} 1\cdot t^3 \cdot 0 - 1\cdot t^2 \cdot dt$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

iii) On C_3, $x=-t$, $y=1$ & $-1 \le t \le 1$. Then

$dx = -dt$, $dy = 0$

$$\int_{C_3} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} (-t)^2(1)^3(-dt) - (-t)\cdot 1\cdot 0$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

iv) On C_4, $x=-1$, $y=-t$, $-1 \le t \le 1$. Then

$dx = 0$, $dy = -dt$

$$\int_{C_4} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} (-1)^2(-t)^3\cdot 0 - (-1)(-t)^2(-dt)$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

$$\therefore \oint_C x^2y^3\,dx - xy^2\,dy = \frac{-2}{3} + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)$$

$$= \frac{-8}{3}$$

Remark

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

Line $\int$ is independent of C.

⇒ Double integrals :-

- Let $Z = f(x,y)$ be defined in a closed & bounded region 'R' of 2 space.
- By means of a grid of vertical & horizontal lines ∥ to co-ordinates axis form a partition $P(R)$. n rectangular sub-regions r_K of areas ΔA_K, that lie entirely in R.
- Let $\|P\|$ be the norm of the partition or the length of longest diagonal of r_K.
- Choose a sample point (x_k^*, y_k^*) in each of sub-region r_K.
- Form the sum, $\boxed{\sum_{K=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$

$$\boxed{\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{K=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$$

- Let f be a ( ) of 2 variables defined on a closed region 'R' of 2 space, then the $\iint$ of f over R is given by.

$$\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{K=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$

$$\boxed{\text{Area, } A = \iint_R dA}$$

$$\boxed{\text{Volume},\ V = \iint_R f(x,y)\,dA}$$

Properties →

1) Constant multiple rule :-

$$\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

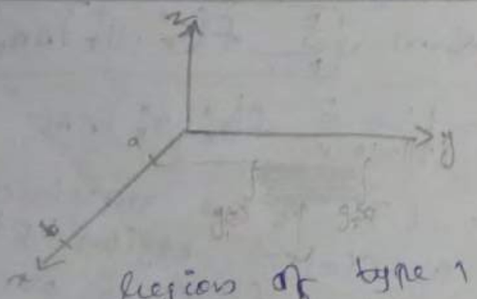
2) Sum & difference rule :-

$$\iint_R \left[f(x,y) \pm g(x,y)\right] dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA.$$

3) Additivity rule :-

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

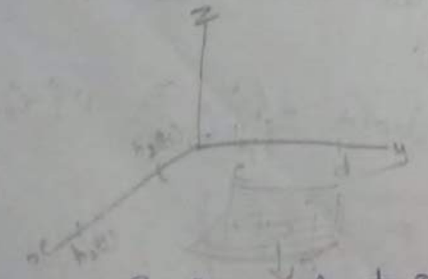
⇒ Evaluation of double $\iint$:-



Region of type 1

$R_1 :\ a \le x \le b$

$g_1(x) \le y \le g_2(x).$



Region of type II

$R_2 :\ c \le y \le d$

$h_1(y) \le x \le h_2(y).$

→ Fubini's Theorem :-

1) $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$, for type 1

2) $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$. for type II.

Q/1) If $f(x,y) = x^2y - 2xy$, $R: 0 \le x \le 3$, $-2 \le y \le 0$, then Evaluate $\iint_R f(x,y)\,dA$?

A)
$$\iint_R f(x,y)\,dA = \int_0^3 \int_{-2}^0 x^2y - 2xy\,dy\,dx$$

$$= \int_0^3 x^2\frac{y^2}{2} - \frac{2xy^2}{2}\,dx.$$

$$= \int_0^3 \left[x^2\frac{y^2}{2} - xy^2\right]_{-2}^0 dx.$$

$$= \int_0^3 \left[0 - \left(x^2\cdot 2 - x\cdot 4\right)\right] dx.$$

$$= \int_0^3 -(2x^2 - 4x)\,dx. \quad \text{(arrange)}$$

$$= \int_0^3 (-2x^2 + 4x)\,dx.$$

$$= \int_0^3 4x - 2x^2\,.dx.$$

$$= \left[4\cdot\frac{x^2}{2} - 2\cdot\frac{x^3}{3}\right]_0^3$$

$$= \left[2x^2 - 2\frac{x^3}{3}\right]_0^3$$

$$= 2\cdot 9 - 2\cdot\frac{27}{3} = 18 - 18 = 0$$

2) $\int_0^1 \left[\int_0^2 xy(x-y)\,dx\right] dy.$

A) $= \int_0^1 \left[\int_0^2 x^2y - xy^2\,dx\right] dy.$

$= \int_0^1 \left[\frac{x^3}{3}y - \frac{x^2}{2}y^2\right]_0^2 dy.$

$= \int_0^1 \left[\frac{8}{3}y - 2y^2\right] dy.$

$= \left[\frac{8}{3}\frac{y^2}{2} - 2\frac{y^3}{3}\right]_0^1$

$= \left[\frac{\cancel{8}^{4}}{3}\cdot\frac{1}{\cancel{2}} - 2\cdot\frac{1}{3}\right] = \frac{4}{3} - \frac{2}{3}$

$= \frac{4-2}{3} = \frac{2}{3}$

⇒ Line ∫-s in space :-

line ∫ of a f^n $G(x,y,z)$ of 3 variables along a curve c in space are defined in a way similar to line ∫ in plane.

let c be a smooth curve in 3space & (para) eq be $x=f(t)$, $y=(t)$ $z=h(t)$ $a \le t \le b$. Then line ∫ -

☆ $\int_C G(x,y,z)\,dx = \int_a^b G(f(t), g(t), h(t))\, f'(t)\,dt.$

☆ $\int_C G(x,y,z)\,dy = \int_a^b G(f(t), g(t), h(t))\, g'(t)\,dt$

☆ $\int_C G(x,y,z)\,dz = \int_a^b G(f(t), g(t), h(t))\, h'(t)\,dt.$

☆ $\int_C G(x,y,z)\,ds = \int_a^b G(f(t), g(t), h(t)) \cdot \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$

$x = f(t)$
$y = g(t)$
$z = h(t)$

Q

1) Let c denote curve defined by $x = \frac{1}{3}t^3$, $y = t^2$, $z = 2t$, $0 \le t \le 1$. evaluate

a) $\int_C 4xyz\,dx$ (b) $\int_C 4xyz\,dy$

(c) $\int_C 4xyz\,dz$ (d) $\int_C 4xyz\,ds.$

A) $G(x,y) = 4xyz$ $g(t) = t^2 = y$ $h(t) = 2t = z$

$f(t) = \frac{1}{3}t^3 = x$

$a = 0, \; b = 1$

$dx = f'(t)\,dt$ ~~$= t^2\,dt$~~ $\frac{1}{\cancel{3}}\cdot\cancel{3}t^2 = t^2\,dt$

$dy = g'(t)\,dt = 2t\,dt.$

$dz = h'(t)\,dt = 2\,dt.$

$ds = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} = \sqrt{t^4 + 4t^2 + 4}\,dt$

$= \sqrt{(t^2+2)^2} = (t^2+2)\,dt$

$(t^2+2)^2$ expansion

(a) $\int_C 4xyz\,dx = \int_C G(x,y,z)\,dx = \int_a^b G(f(t), g(t), h(t))\, f'(t)\,dt$
(3 variable)

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot t^2\,dt$

$= \int_0^1 \frac{8t^8}{3}\,dt = \frac{8}{3}\int_0^1 t^8\,dt.$

$$= \frac{8}{3}\left[\frac{t^9}{9}\right]_0^1 = \frac{8}{3}\cdot\frac{1}{9} = \frac{8}{27}$$

b) $\int_C 4xyz\, dy = \int_a^b G(f(t), g(t), h(t))\, g'(t)\, dt$

$$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot \cancel{x}\cdot dt$$

$$= \frac{16}{3}\int_0^1 t^7\, dt$$

$$= \frac{16}{3}\left[\frac{t^8}{8}\right]_0^1 = \frac{16}{3}\cdot\frac{1}{8} = \frac{16}{24} = \frac{2}{3}$$

c) $\int_C 4xyz\, dz = \int_a^b G(f(t), g(t), h(t))\, h'(t)\, dt$

$$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot 2\, dt$$

$$= \frac{16}{3}\int_0^1 t^6\, dt = \frac{16}{3}\left[\frac{t^7}{7}\right]_0^1$$

$$= \frac{16}{3}\cdot\frac{1}{7} = \frac{16}{21}$$

d) $\int_C 4xyz\, ds = \int_a^b G(f(t), g(t), h(t))\, ds$

$$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t\cdot(t^2+2)\, dt$$

$$= \frac{8}{3}\int_0^1 t^6(t^2+2)\, dt$$

$$= \frac{8}{3}\int_0^1 t^8 + 2t^6\, dt$$

$$= \frac{8}{3}\left[\frac{t^9}{9} + \frac{2t^7}{7}\right]_0^1 = \frac{8}{3}\left[\frac{1}{9}+\frac{2}{7}\right]$$

$$= \frac{200}{189}$$

2) $\int_C y\,dx + z\,dy + x\,dz$ $C_1 \to (0,0,0)\ (2,3,4)$, $C_2 \to (2,3,4)\ (6,8,5)$

'C' consists of line segment from $(0,0,0)$ to $(2,3,4)$ & from $(2,3,4)$ to $(6,8,5)$

A) Let $C_1 \to (0,0,0)$ to $(2,3,4)$. Then C is (subtract same component)
|| to vector.

$(2-0)i + (3-0)j + (4-0)k$ (i.e) $2i+3j+4k$ (b)
& passes through $(0,0,0)$, hence (para)
eq is $x = 0+2t$, $y = 0+3t$, $z = 0+4t$
$(0+2)t$ $(0+3)t$ $(0+4)t$
(i.e) $x = 2t$, $y = 3t$, $z = 4t$.

$(0,0,0)$ when $t=0$, & passes through $(2,3,4)$ when $t=1$.

(para) eq of C_1,

$x = 2t$ $y = 3t$ $z = 4t$ $0 \le t \le 1$

* Let $C_2 \to (2,3,4)$ to $(6,8,5)$. Then C_2 is (a)
|| vector

$(6-2)i + (8-3)j + (5-4)k$ (i.e) $4i+5j+k$
& passes through $(2,3,4)$. (para) eq,

$x = 2+4t$, $y = 3+5t$ $z = 4+t$
$(2+4)t$ $(3+5)t$ $(4+t)$
above lines passes through $(2,3,4)$ when $t=0$. & passes through $(6,8,5)$ when $t=1$

(para) eq,

$x = 2+4t$ $y = 3+5t$ $z = 4+t$ $0 \le t \le 1$

* On C_1, $x = 2t$ $y = 3t$ $z = 4t$ $0 \le t \le 1$
$dx = 2$ $dy = 3$ $dz = 4$

Then $dx = 2dt$, $dy = 3dt$.
$dz = 4dt$

$$\int_{c_1} y\,dx + z\,dy + x\,dz = \int_0^1 3t^2 \cdot 2\,dt + 4t \cdot 3dt + 2t \cdot 4\,dt$$

$$= \int_0^1 26t\,dt = 26\left(\frac{t^2}{2}\right)_0^1$$

$$= \left[13t^2\right]_0^1 = 13$$

* On c_2, $x = 2+4t$, $y = 3+5t$, $z = 4+t$, $0 \le t \le 1$

$dx = 4dt$, $dy = 5dt$, $dz = dt$.

$$\int_{c_2} y\,dx + z\,dy + x\,dz = \int_0^1 (3+5t)\cdot 4dt + (4+t)\cdot 5dt + (2+4t)\,dt$$

$$= \int_0^1 (34 + 29t)\,dt$$

$$= \left[34t + \frac{29}{2}t^2\right]_0^1 = \frac{97}{2}$$

Since c is composed of smooth curves c_1 & c_2, the $\int$ over c = sum of $\int$ over c_1 & c_2

$$\therefore \int_c y\,dx + z\,dy + x\,dz = \int_{c_1} y\,dx + z\,dy + x\,dz + \int_{c_2} y\,dx + z\,dy + x\,dz$$

$$= 13 + \frac{97}{2} = \frac{26+97}{2}$$

$$= \frac{123}{2}$$

Remark → we can use the concept of a (v) () of several variable to write a general line $\int$ in a compact fashion.

eg → $f(x,y) = p(x,y)i + q(x,y)j$ is defined along the curve $c: x = f(t)$, $y = g(t)$, $a \le t \le b$

& suppose $r(t) = f(t)i + g(t)j$ is position (v) of points on c, then $r'(t)$,

$$\frac{dr}{dt} = f'(t)i + g'(t)j$$

$$= \frac{dx}{dt}i + \frac{dy}{dt}j$$

to define, $dr = \frac{dr}{dt}dt = dx\,i + dy\,j$

Since $F(x,y)\cdot dr = p(x,y)\,dx + q(x,y)\,dy$,

$$\int_c p(x,y)\,dx + q(x,y)\,dy = \int_c F\cdot dr$$

$\approx$ for a line $\int$ on space curve,

$$\int_c p(x,y,z)\,dx + q(x,y,z)\,dy + R(x,y,z)\,dz$$

$$= \int_c F\cdot dr$$

where $F(x,y,z) = p(x,y,z)i + q(x,y,z)j + R(x,y,z)k$

& $dr = dx\,i + dy\,j + dz\,k$.

Q) If $F = 3xy\,i + y^2 j$, evaluate $\int_c F\cdot dr$

c is curve in xy-plane, $y = 2x^2$ from $(0,0)$ to $(1,2)$.

A) $y = 2x^2$

let $x = t$ then $y = 2t^2$

$dx = dt$ $dy = 4t\,dt$

$\boxed{dr = dx\,i + dy\,j}$ $\quad F = 3xy\,i - y^2\,j$, $0 \le t \le 1$

$\therefore \int_c F\cdot dr = \int_0^1 (3xy\,i - y^2 j)\cdot(dx\,i + dy\,j)$

$x = t$, $y = 2t^2$, $dx = dt$, $dy = 4t\,dt$

$= \int_0^1 (3xy\cdot dx) - (y^2 \cdot dy)$

$= \int_0^1 3\cdot t\cdot 2t^2\, dx - (2t^2)^2\, dy$

$= \int_0^1 3\cdot t \cdot 2t^2 \cdot dt - 4t^4 \cdot 4t\,dt$

$= \int_0^1 6t^3 - 16t^5\, dt$

$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$

$= 6\cdot\frac{1}{4} - 16\frac{1}{6} - 0$

$= \frac{6}{4} - \frac{16}{6} = \frac{3\times 3}{2\times 3} - \frac{8\times 2}{3\times 2}$

$= \frac{9-16}{6} = -\frac{7}{6}$

2) $F = (3x^2 + 6y)i - 14yz\,j + 20xz^2 k$.

evaluate $\int_c F\cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the path $x = t$, $y = t^2$, $z = t^3$

A) $dr = dx\,i + dy\,j + dz\,k$ $\quad 0 \le t \le 1$

$\therefore \int_c F\cdot dr = \int_0^1 [(3x^2 + 6y)i - 14yz\,j + 20xz^2 k]\cdot(dx\,i + dy\,j + dz\,k)$

$= \int_0^1 (3x^2 + 6y)dx - (14yz\,dy) + 20xz^2\,dz$

$= \int_0^1 (3t^2 + 6t^2)\,dt - (14t^2\cdot t^3\cdot 2t)\,dt + 20t(t^3)^2\cdot 3t^2\,dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$

$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$

$= (3 - 4 + 6) - 0 = 5$

$\Rightarrow$ path independence and conservative vector field :-

$\boxed{\int F\cdot dr.}$

Let F be a field defined on an open region R in space & suppose that for any 2 points A & B in R, the value of the line $\int$ from A to B is same for every path in R from A to B. Then the $\int F\cdot dr$ is path independent in R.

path is =

$\int_A^B F\cdot dr$

(ii) $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$

→ **Theorem :-**

Fundamental Theorem of line $\int$:-

[Relation b/w path independence & conservative v.f.]

Suppose C is a path in an open region R of xy plane & is defined by $r(t) = x(t)i + y(t)j$, $a \le t \le b$.

(pass) If $F(x,y) = P(x,y)i + Q(x,y)j$ is a (i) conservative v.f in R & (2) ϕ is a potential () for F, then

sol^n of $\int_C F \cdot dr = \int_{C\,(C \cdot v \cdot F)} \nabla\phi \cdot dr = \underset{\text{path ending point}}{\phi(B)} - \underset{\text{path starting point}}{\phi(A)}$

where $A = (x(a), y(a))$ & $B = (x(b), y(b))$

* **Conservative v.F :-** (or gradient v.F)

F is said to be conservative, if F can be written as the gradient of a scalar () ϕ.

$F(x,y) = yi + xj$

$$\boxed{F = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j}$$

$$= yi + xj = F(x,y).$$

Pf Let us prove the theorem for a smooth curve C,

Since F is conservative & ϕ is a potential () for F,

$$\boxed{F = \nabla\phi} = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j \text{ --- (1)}$$

path of $\int$-tion defined by,

$r(t) = x(t)i + y(t)j$

$\int F \cdot dr$ → $$\frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j \text{ --- (2)} \quad a \le t \le b$$

$$\int_C F \cdot dr = \phi(B) - \phi(A) \text{ → To prove}$$

Along the curve, ϕ is ~~defined by~~ differentiable () of t & by chain rule,

$$\cancel{\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{dt}} \quad \phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j$$

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial\phi}{\partial y} \cdot \frac{dy}{dt}$$

(split)

$$= \left[\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j\right] \cdot \left[\frac{dx}{dt} i + \frac{dy}{dt} j\right]$$

(1) $\nabla\phi$ dr/dt (2)

$$= \nabla\phi \cdot \frac{dr}{dt}$$

$$\frac{d\phi}{dt} = F \cdot \frac{dr}{dt} \text{ --- (3)}$$

wants to prove,

$$\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt}\, dt.$$

$$\text{(3) --- } = \int_a^b \frac{d\phi}{dt}\, dt.$$

both $\int$ to diff. to any one ~~diff.~~

$\therefore, \phi\left[x(t), y(t)\right]_a^b$ $\cancel{\int \frac{d\phi}{dt}\, dt} = \phi$.

definite $\int$ [illegible] limit [illegible].

$\Rightarrow \phi\left[x(b), y(b)\right] - \phi\left[x(a), y(a)\right]$

$$\boxed{\int_C F\cdot dr = \phi(B) - \phi(A)}$$

Q) Sketch the region of ~~f-tion~~ & evaluate

$\int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx$.

A) 

A) $= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$.

$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} - 0\right] dx$. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$

$= \int_0^1 \left[x^2 - x^3 + \frac{(1-3x+3x^2-x^3)}{3}\right] dx$.

$= \int_0^1 \left[\frac{3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3}{3}\right] dx$.

$= \int_0^1 1 - 4x^3 + 6x^2 - 3x\, dx$.

$= \frac{1}{3}\left[x - \cancel{4}\cdot\frac{x^4}{\cancel{4}} + \cancel{6}^{2}\cdot\frac{x^3}{\cancel{3}} - 3\cdot\frac{x^2}{2}\right]_0^1$

$= \frac{1}{3}\left[1 - 1 + 2 - \frac{3}{2}\right]$ ~~$= \frac{2}{1} + \frac{3}{2} = \frac{4+3}{2} = \frac{8}{2} = 4$~~ ~~$= 2 - \frac{3}{2} = \frac{4-3}{2} = \frac{1}{2}$~~

$= \frac{1}{3}\left[\frac{1}{2}\right] = \frac{1}{6}$

graph

$0 \le x \le 1$

$0 \le y \le 1-x$

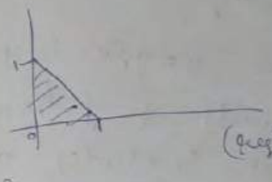
$y = 1-x$.

$x + y = 1$.

if $x=0$, $y=1$

if $y=0$, $x=1$

(1, 1)



(region)

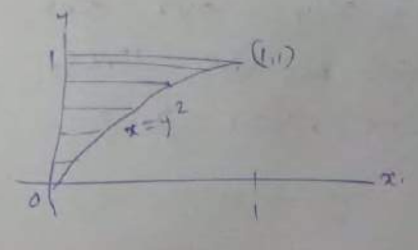
Q) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$.

$= \int_0^1 \left[3y^3 \frac{e^{xy}}{y}\right]_0^{y^2} dy$

$= \int_0^1 \left[3y^2 e^{y^3} - 3y^2\right] dy$ $\boxed{\int 3y^2 e^{y^3} dy = e^{y^3}}$

$= \left[e^{y^3} - y^3\right]_0^1 = e - 2$

$0 \le y \le 1$ $\quad 0 \le x \le y^2$



To evaluate $\iint_R f(x,y)\,dA$, $\int$-ing with respect to x & y.

1) Sketch the region of $\int$-tion & label the bounding curves.

2) Imagine a vertical line 'l' cutting through 'r' in dir of increasing y. Mark the y values, where 'l' enters & leaves the region 'r'.

Let these values be $y = g_1(x)$ & $y = g_2(x)$. This are the y limits of $\int$-tion.

3) Choose x-limits that includes all vertical lines through r.

4) Evaluate $\iint$ using the formula:

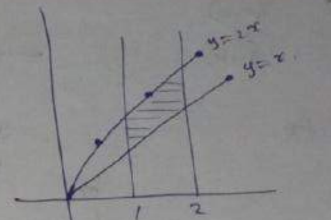
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

5) To evaluate the $\iint$ as an iterated with the order of $\int$-tion reversed, use horizontal lines instead of vertical lines, then required $\int$ is,

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

6) $\int$-ate $f(x,y) = x/y$ over the regions bounded by the lines,

$y = x$, $y = 2x$, $x = 1$, $x = 2$



$x = 1,\ x = 2 \Rightarrow \int_1^2$

$y = x,\ y = 2x \Rightarrow \int_x^{2x}$

$$= \int_1^2 \int_x^{2x} \frac{x}{y}\,dy\,dx$$

$$= \int_1^2 \left[x \cdot \log y\right]_x^{2x} dx \Rightarrow \int_1^2 \left[x \cdot \ln 2x - x \cdot \ln x\right] dx.$$

$\boxed{\ln A - \ln B = \ln (A/B)}$

$$= \int_1^2 x\left[\ln 2x - \ln x\right] dx.$$

$$= \int_1^2 x\left[\ln\left(\frac{2x}{x}\right)\right] dx = \int_1^2 x \cdot \ln 2 \cdot dx$$

$$= \ln 2 \int_1^2 x \cdot dx = \ln 2 \cdot \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{\ln 2}{2}\left[4 - 1\right] = \frac{\ln 2}{2}\left[3\right]$$

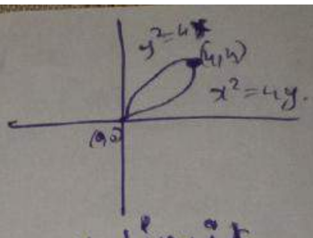
$$= \frac{3}{2}\ln 2$$

Q/ 1) Evaluate $\iint_R y\,dy\,dx$, where R is region bounded by parabola's

$y^2 = 4x$, $x^2 = 4y$

if $x=0$, $y=0$ $(0,0)$

if $x=4$, $y=4$ $(4,4)$



$$= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y \, dy \, dx.$$

y limit

$y^2 = 4x$

$y = \sqrt{4x} = 2\sqrt{x}.$

$\int_{x^2/4}^{2\sqrt{x}} \Rightarrow x^2 = 4y$

$y = x^2/4$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx.$$

$$= \int_0^4 \left[\frac{(2\sqrt{x})^2}{2} - \frac{(x^2/4)^2}{2}\right] dx. \quad \cancel{= \int_0^4 \left[\frac{4x - x^2}{2}\right] dx.}$$

$\left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$

$$= \frac{1}{2}\int_0^4 \left[4x - \frac{x^4}{16}\right] dx.$$

$$= \frac{1}{2}\int_0^4 \left[\frac{64x - x^4}{16}\right] dx.$$

$$= \frac{1}{32}\int_0^4 (64x - x^4)\, dx = \frac{1}{32}\left[64\frac{x^2}{2} - \frac{x^5}{5}\right] dx$$

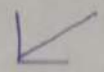
$$= \left[\frac{64}{32}\cdot\frac{x^2}{2} - \frac{x^5}{5\times 32}\right]_0^4 = \left[\cancel{2}\cdot\frac{x^2}{\cancel{2}} - \frac{x^5}{160}\right]_0^4$$

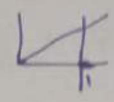
$$= \left[x^2 - \frac{x^5}{160}\right]_0^4 = 16 - \frac{1024}{160}$$

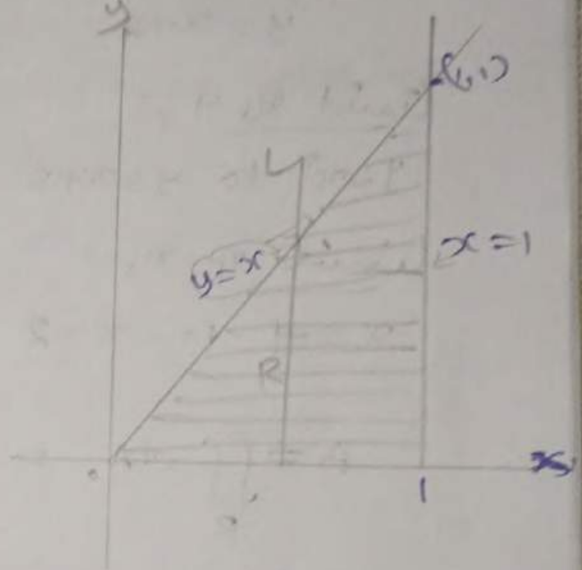
$$= \frac{2560 - 1024}{160} = \frac{\cancel{1536}^{\,768\ 384\ 192\ 96\ 48}}{\cancel{160}_{\,80\ 40\ 20\ 10\ 5}}$$

$$= \frac{48}{5}$$

2) Find the vol of prism whose base is the triangle in the xy-plane bounded by x-axis & lines $y=x$ & $x=1$ and whose top lines in the plane $z = f(x,y) = 3-x-y$.

A) $y = x \rightarrow$ 

$x = 1 \rightarrow$ 



limit

If $y=0$, $x=0$

$x=1$ $\quad \int_0^1 \int_1^x$

$$V = \iint_R f(x,y)\,dA$$

$$= \iint_R (3-x-y)\,dA$$

$$= \int_0^1 \int_1^x (3-x-y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_1^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2} - \left(3 - x - \frac{1}{2}\right)\right] dx$$

$$= \int_0^1 3x - \frac{x^2}{2} - 3 + x + \frac{1}{2} - \frac{x^2}{2}$$

$$= \int_0^1 \left[3x - \frac{3x^2}{2}\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$

$x^2 - \frac{x^2}{2}$

$= \frac{2x^2 - x^2}{2}$

$= \frac{x^2}{2}$

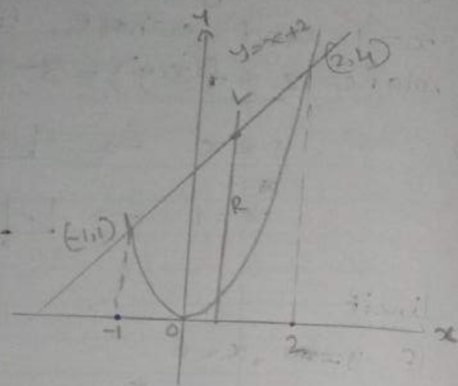
$3 + \frac{1}{2}$

$\frac{6+1}{2}$

$= \frac{7}{2}$

3) Find area of region enclosed by parabola $y = x^2$ & line $y = x+2$.

A) $A = \iint_R dA$



$y = x^2$

$y = x+2$

limit of y,

$y = x^2$ to $y = x+2$

limit of x,

$x = -1$ to $x = 2$ →

$$A = \iint_R dA = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right] dx$$

$$= \int_{-1}^{2} [y]_{x^2}^{x+2} dx = \int_{-1}^{2} \left[x+2-x^2\right] dx$$

$$= \left[\frac{x^2}{2} + 2x + \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

⇒ Laminas with variable Density :-

* If ρ is a constant density (mass per unit area) then mass of lamina coinciding with a region bounded by the graphs of $y = f(x)$, the x-axis & lines $x = a$ & $x = b$,

$$m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x^*_k)\, \Delta x_k.$$

$$\boxed{m = \int_a^b \rho f(x)\, dx}$$

* If a lamina to a region R has a variable density $\rho(x,y)$, ρ is non-ve & continous on R, then we define its mass m by double $\int$.

(mass). $m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x^*_k, y^*_k)\, \Delta A_k$

$$\boxed{m = \iint_R \rho(x,y)\, dA}$$

* Center of mass of lamina :

with x coordinates, $\boxed{\bar{x} = \frac{M_y}{m}}$

with y " , $\boxed{\bar{y} = \frac{M_x}{m}}$

M_y & M_x are moments,

$$\boxed{\begin{aligned} M_y &= \iint_R x\rho(x,y)\, dA \\ M_x &= \iint_R y\rho(x,y)\, dA \end{aligned}}$$

$\int$-s M_x & M_y also → 1^{st} moments of a lamina about x & y-axes.

2^{nd} moments of a lamina | moments of inertia about x & y-axes, → (I_x), (I_y)

$$\boxed{\begin{aligned} I_x &= \iint_R y^2 \rho(x,y)\, dA \\ I_y &= \iint_R x^2 \rho(x,y)\, dA \end{aligned}}$$

* Kinetic energy :-

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m (r\omega)^2 \qquad v = r\omega$$

$$= \frac{1}{2} (mr^2)\omega^2 \qquad I = mr^2 \text{ moment of inertia}$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

1) A lamina has the shape of the region in 1st quadrant that is bounded by graphs of $y = \sin x$, $y = \cos x$, b/w $x = 0$ & $x = \pi/4$. Find the center of mass if density is $\rho(x,y) = y$?

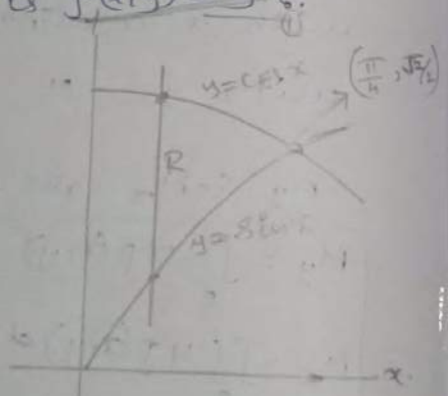
A) $y = \sin x$ ~~$\sin x$~~

$x = 0$

$y = \cos x$.

$x = \pi/4$.

$$\int_0^{\pi/4} \int_{\sin x}^{\cos x}$$



$$\boxed{m = \iint_R \rho(x,y)\, dA} \rightarrow$$ ~~center of mass~~.

$$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y \cdot dy\, dx = \int_0^{\pi/4} \left[\frac{y^2}{2}\right]_{\sin x}^{\cos x}$$

$$= \frac{1}{2} \int_0^{\pi/4} [\cos^2 x - \sin^2 x]\, dx.$$

$$= \frac{1}{2} \int_0^{\pi/4} \cos 2x\, dx. \qquad \boxed{\cos^2 x - \sin^2 x = \cos 2x}$$

$$= \frac{1}{2} \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \Rightarrow \frac{1}{2} \cdot \frac{1}{2} \left[\sin 2\,\pi/4 - 0\right]$$

$$= \frac{1}{4}\left[\sin \frac{\pi}{2}\right] = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

M_x, M_y ?

$$\boxed{M_x = \iint_R y\, \rho(x,y)\, dA} \rightarrow y \times y\, dy\, dx$$

$$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y^2\, dy\, dx.$$

$$= \int_0^{\pi/4} \left[\frac{y^3}{3}\right]_{\sin x}^{\cos x} dx. = \frac{1}{3} \int_0^{\pi/4} \left[y^3\right]_{\sin x}^{\cos x} dx.$$

$$= \frac{1}{3} \int_0^{\pi/4} [\cos^3 x - \sin^3 x]\, dx. \qquad \begin{aligned}\cos^2 x &\to 1 - \sin^2 x \\ \sin^2 x &\to 1 - \cos^2 x\end{aligned}$$

$$= \frac{1}{3} \int_0^{\pi/4} \left[\cos x(1 - \sin^2 x) - \sin x(1 - \cos^2 x)\right] dx.$$

$$(\cos^2 x + \sin^2 x = 1 \rightarrow \cos^2 x = 1 - \sin^2 x)$$

$$= \frac{1}{3} \int_0^{\pi/4} \cos x - \sin^2 x \cos x - \sin x + \sin x \cos^2 x.$$

$$= \frac{1}{3} \left[\int_0^{\pi/4} \cos x\, dx - \int \sin^2 x \cos x\, dx - \int \sin x\, dx + \int \sin x \cos^2 x\, dx.\right]$$

$$= \frac{1}{3} \left\{ \left[\sin x\right]_0^{\pi/4} - \frac{1}{3}\left[\sin^3 x\right]_0^{\pi/4} - \left[-\cos x\right]_0^{\pi/4} + \frac{1}{3}\left[-\cos^3 x\right]_0^{\pi/4} \right\}.$$

$$\boxed{\begin{aligned} &\cos x \rightarrow \sin x \\ &\sin^2 x \cos x \rightarrow \frac{\sin^3 x}{3} \\ &\sin x \rightarrow -\cos x. \\ &\sin x \cos^2 x \rightarrow -\frac{\cos^3 x}{3} \end{aligned}}$$

$$= \frac{1}{3}\left\{\left[\frac{1}{\sqrt{2}} - 0\right] - \frac{1}{3}\left[\frac{1}{2\sqrt{2}} - 0\right] - \left[1 - \frac{1}{\sqrt{2}}\right] + \frac{1}{3}\left[1 - \frac{1}{2\sqrt{2}}\right]\right\}$$

$$= \frac{1}{3}\left\{\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} + \frac{1}{3} - \frac{1}{6\sqrt{2}}\right\}. \quad (cm)$$

$$= \frac{6 - 1 - 6\sqrt{2} + 6 + 2\sqrt{2} - 1}{18\sqrt{2}} = \frac{10 - 4\sqrt{2}}{18\sqrt{2}}$$

$$\therefore M_x = \frac{5\sqrt{2} - 4}{18}$$

$$\boxed{M_y = \iint_R x\,\rho(x,y)\,dA} = \int_0^{\pi/4}\int_{\sin x}^{\cos x} xy\,dy\,dx.$$

$\rho \to \rho(x,y) = y$

$$= \int_0^{\pi/4}\left[x \cdot \frac{y^2}{2}\right]_{\sin x}^{\cos x} dx = \frac{1}{2}\int_0^{\pi/4}\left[x y^2\right]_{\sin x}^{\cos x} dx.$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[x \cdot (\cos^2 x - \sin^2 x)\right] dx.$$

$\boxed{\cos^2 x - \sin^2 x = \cos 2x}$

$$= \frac{1}{2}\int_0^{\pi/4}\left[x \cdot \cos 2x\right] dx.$$

(I.P. rule) $$= \frac{1}{2}\left[x \cdot \frac{\sin 2x}{2}\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \sin 2x\,dx.$$

integrating by parts.

$$= \frac{1}{2}\left\{\frac{\pi}{8} - \left[-\frac{1}{4}\cos 2x\right]_0^{\pi/4}\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{8} - \frac{1}{4}\right\} = \frac{\pi - 2}{16} \Rightarrow \frac{1}{2}\left[\frac{4\pi - 8}{32}\right] \Rightarrow \frac{4\pi - 8}{64} \frac{\div 4}{\div 4} = \frac{\pi - 2}{16}$$

hence coordinates of center of mass,

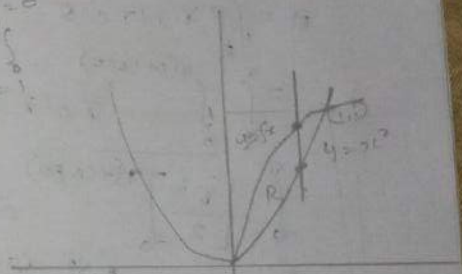
$$\bar{x} = \frac{M_y}{m} = \frac{(\pi - 2)/16}{1/4} = \frac{\pi - 2}{4} \approx 0.29$$

$$\bar{y} = \frac{M_x}{m} = \frac{(5\sqrt{2} - 4)/18}{1/4} = \frac{10\sqrt{2} - 8}{9} \approx 0.68$$

coordinates $(0.29, 0.68)$

2) find moment of inertia about x-axis of lamina that has shape of region bounded by graphs of $y = x^2$ & $y = \sqrt{x}$ & density $\rho(x,y) = x^2$?

A)



moment of inertia about x-axis. $\to$

$$I_x = \iint_R y^2 \rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} y^2 x^2\,dy\,dx.$$

$$= \int_0^1 x^2\left[\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 x^2\left[x^{3/2} - x^6\right] dx.$$

$y^3 \to (\sqrt{x})^3 \to (x^{1/2})^3 = x^{3/2}$

$y^3 \to (x^2)^3 \Rightarrow x^6$

$$= \frac{1}{3}\int_0^1\left[x^{7/2} - x^8\right] dx.$$

$x^2 \cdot x^{3/2}$, $x^{\frac{4+3}{2}}$ $\Rightarrow x^{7/2}$

$$= \frac{1}{3}\left[\frac{x^{9/2}}{9/2} - \frac{x^9}{9}\right] = \frac{1}{3}\left[\frac{2}{9} - \frac{1}{9}\right] = \frac{1}{27}$$

⇒ Area in polar coordinates :-

The area of a closed bounded R in the polar coordinates,

$$\boxed{A = \iint_R r\, dr\, d\theta}$$

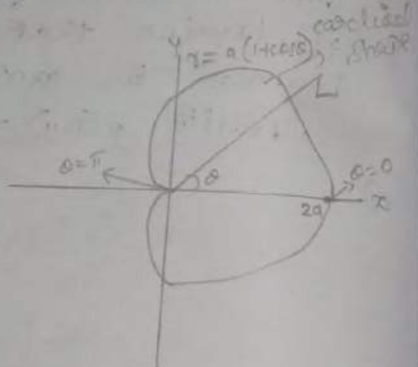
polar coordinates

want to evaluate $\iint_R f(r,\theta)\, dA$.

$$\boxed{\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta}$$

Q) find area enclosed by caroloid
$r = a(1+\cos\theta)$

A) $x = \pi \to 0$.



$A = \iint_R r\, dr\, d\theta$.

$= \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta$.

$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$.

$(a+b)^2$.

$= \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2}\, d\theta = \int_0^{\pi} \frac{a^2 + (\cos^2\theta + 2\cos\theta + 1)}{2}\, d\theta$.

$= a^2 \int_0^{\pi} \frac{\cos^2\theta + 2\cos\theta + 1}{2}\, d\theta = a^2 \int_0^{\pi} \frac{\cos^2\theta}{2} + \frac{\cancel{2}\cos\theta}{\cancel{2}} + \frac{1}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= a^2 \int_0^{\pi} \left[\frac{1+\cos 2\theta}{2\times 2} + \cos\theta + \frac{1}{2}\right] d\theta = a^2 \int_0^{\pi} \frac{1+\cos 2\theta}{4} + \cos\theta + \frac{1}{2}$

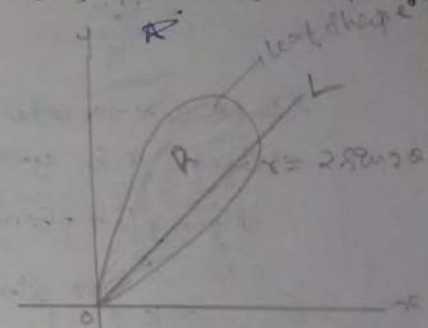
$= a^2 \left[\frac{3\theta}{4} + \frac{\sin 2\theta}{8} + \sin\theta\right]_0^{\pi} = a^2\left[\frac{3\pi}{4} + 0 - 0\right]$

$= \frac{3\pi a^2}{4}$

∴ Area enclosed by cardioid $=$
$2 \times R = \cancel{2} \cdot \frac{3\pi a^2}{\cancel{4}_2} = \frac{3\pi a^2}{2}$

2) find center of mass of lamina that corresponds to the region bounded by 1 leaf of rose $r = 2\sin 2\theta$ in the 1st quadriant if the density at a point P in the lamina is directly proportional to the distance from pole?

A) y limits
$0 \to 2\sin 2\theta$
x limits
$0 \to \frac{\pi}{2}$.



$m = \iint_R \rho(x,y)\, dA = \iint_R k|r|\, dA$

$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r \cdot r\, dr\, d\theta$.

$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r^2\, dr\, d\theta = k \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\sin 2\theta} d\theta$.

$= \frac{k}{3} \int_0^{\pi/2} 8\sin^3 2\theta\, d\theta = \frac{8k}{3} \int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta\, d\theta$.

let $\cos 2\theta = u$.
$-2\sin 2\theta\, d\theta = du$.
$\sin 2\theta\, d\theta = -du/2$

Limits

$\theta = 0 \;, u = 1$

$\theta = \pi/2 \;, u = \cos\pi = -1$.

$$\therefore\; m = \frac{8k}{3}\int_1^{-1}(1-u^2)\left(-\frac{du}{2}\right)$$

$$= \frac{4k}{3}\int_1^{-1}(1-u^2)\,du .$$

$$= \frac{4k}{3}\left[u - \frac{u^3}{3}\right]_1^{-1} = \frac{4k}{3}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

$$= \frac{16k}{9}.$$

Since $x = r\cos\theta$ & $y = r\sin\theta$, M_x & M_y about x & y-axes,

$$M_y = \iint_R x\,\rho(x,y)\,dA = \iint_R r\cos\theta \cdot k|r|\,dA.$$

$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2\cos\theta \cdot r\,dr\cdot d\theta$$

$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3\cos\theta\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\cos\theta\left[\frac{r^4}{4}\right]_0^{2\sin 2\theta} d\theta .$$

$$= 4k\int_0^{\pi/2}\sin^4 2\theta\cos\theta\,d\theta$$

$$= 4k\int_0^{\pi/2}(2\sin\theta\cos\theta)^4\cos\theta\,d\theta .$$

$$= 64k\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta .$$

$$= 64k\int_0^{\pi/2}\sin^4\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$$

$$= 64k\int_0^{\pi/2}\left[\sin^4\theta - 2\sin^6\theta + \sin^8\theta\right]\cos\theta\,d\theta$$

Sub. $\sin\theta = u$.
$\cos d\theta = du$
$\sin 0 = 0$
$\sin\pi/2 = 1$ } limit

$$= 64k\int_0^1\left[u^4 - 2u^6 + u^8\right] du$$

$$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]$$

$$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}\,k .$$

$$M_x = \iint_R y\,\rho(x,y)\,dA = \iint_R r\sin\theta \cdot k|r|\,dA$$

$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2\sin\theta \cdot r\cdot dr\,d\theta .$$

$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3\sin\theta\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\sin 2\theta}\sin\theta\cdot d\theta .$$

$$= 4k\int_0^{\pi/2}\sin^4 2\theta\,\sin\theta\,d\theta .$$

$$= 4k\int_0^{\pi/2}(2\sin\theta\cos\theta)^4\sin\theta\,d\theta$$

$$= 64k\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta .$$

$$= 64k\int_0^{\pi/2}\cos^4\theta\,(1-\cos^2\theta)^2\sin\theta\,d\theta .$$

$$= 64k\int_0^{\pi/2}\left[\cos^4\theta - 2\cos^6\theta + \cos^8\theta\right]\sin\theta\,d\theta$$

$\cos\theta = u$.
$-\sin d\theta = du$
$\sin d\theta = -du$

$$= 64k\int_0^1\left(u^4 - 2u^6 + u^8\right)(du)$$

$$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]_0^1$$

$$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}\,k$$

Centres of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{512k/315}{16k/9} = \frac{32}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{512k/315}{16k/9} = \frac{32}{35}$$

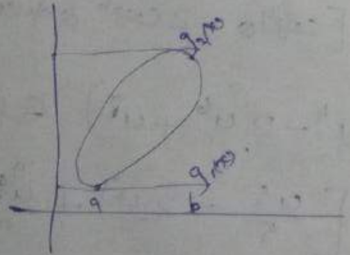
Coordinates $\left(\frac{32}{35}, \frac{32}{35}\right)$

⇒ Green's Theorem in the Plane:-

Suppose that 'c' is a piecewise smooth simple closed curve ($\oint$) bounding a simply connected region 'R', if $P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are contin- on R.

then,

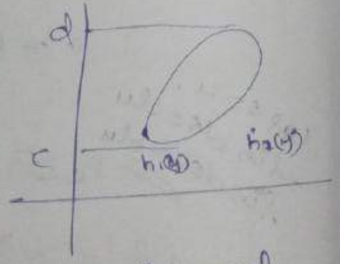
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



Type I

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$



Type II

$c \le y \le d$

$h_1(y) \le x \le h_2(y)$

$$\boxed{\int \frac{\partial P}{\partial y}\,dy = P(x,y)}$$

Type I

$$\iint_R \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_a^b \left[P(x,y)\right]_{g_1(x)}^{g_2(x)} dx$$

$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\, dx$$

$$= -\int_a^b P(x, g_1(x))\,dx + \int_a^b P(x, g_2(x))\,dx$$

$$= -\oint_C P(x, g(x))\,dx = -\oint P(x,y)\,dx \quad \text{(1)}$$

Type II

$$\iint_R \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \int_c^d \left[Q(x,y)\right]_{h_1(y)}^{h_2(y)} dy$$

$$= \int_c^d \left[Q(h_2(y), y) - Q(h_1(y), y)\right] dy$$

$$= -\int_c^d Q(h_1(y), y) + Q(h_2(y), y)\,dy$$

$$= \oint_C Q(x,y)\,dy. \longrightarrow \text{(2)}$$

from (1) & (2)

$$\Rightarrow \iint_R \frac{\partial P}{\partial y}\,dA - \iint_R \frac{\partial Q}{\partial x}\,dA$$

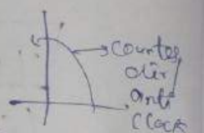
$$= -\oint P(x,y)\,dx - \oint Q(x,y)\,dy$$

$$\cancel{\Rightarrow \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA = \iint_R \left(Q(x,y) - P(x,y)\right) dA}$$

$$\iint_R \frac{\partial P}{\partial y}\,dA - \iint_R \frac{\partial Q}{\partial x}\,dA = \int P\,dx + \int Q\,dy$$

$$\int P\,dx + \int Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Q1) Verify Green's Theorem for line $\int$ $\oint_C (xy\,dx + x^2 dy)$ where C is the curve enclosed in the region bounded by the parabola & line $y = x^2$, $y = x$ in counter (clis)



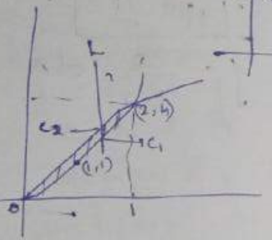
A)

$y = x^2$

(C1) $x = 0$, $y = 0$
$x = 1$, $y = 1$
$x = 2$, $y = 4$

$y = x$

(C2) $x = 0$ $y = 0$
$x = 1$ $y = 1$
$x = 2$ $y = 2$.



$C = C_1 \cup C_2$

(para) eq

on C_1 let $x = t$, $dx = dt$ (with respect to t) ; $y = t^2$, $dy = 2t\,dt$ → $y = t^2$, $\frac{dy}{dt} = 2t$, $dy = 2t\,dt$

$\cancel{\frac{dx}{dt} = \frac{dt}{dx}}$

$0 \le t \le 1$

Green's T.
$0 \to 1$
$1 \to 0$ (oppo).

on C_2 $y = t$ $\oint_C P\,dx + Q\,dy$ $1 \le t \le 0$.

$\therefore x = t$ $y = t$
$dx = dt$ $dy = dt$.

$P = xy$, $Q = x^2$. $C = C_1 \cup C_2$.

$\Rightarrow \int_{C_1 \cup C_2} xy\,dx + x^2 dy$

$\therefore = \int_{C_1} xy\,dx + \int_{C_2} x^2 dy$ $\cancel{\int_{C_2} xy\,dx + x^2 dy}$

$= \int_0^1 xy\,dx + \int_1^0 x^2 dy$ $\cancel{+ \int_1^0 xy\,dx + x^2 dy}$

$= \int_0^1 t \cdot t^2\,dt + \int_1^0 t^2 \cdot 2t\,dt$ $\cancel{+ \int_1^0 t \cdot t^2\,dt +}$

$= \int_0^1 t^3\,dt - \int_0^1 2t^3\,dt$

$\int_1^0 \Rightarrow 0 \ldots$ & $-\int_0^1$

$= \left[\frac{t^4}{4}\right]_0^1 - 2\left[\frac{t^4}{2}\right]_0^1$

$= \frac{1}{4} - 0 - \left(0 - \frac{1}{2}\right) = \frac{1}{4} \cancel{\left(\frac{-1}{2}\right)}$

$= \frac{1}{4} - \frac{1}{2} = \cancel{-2-4}$

2) using Green's theorem, evaluate $\oint_C x^2 e^y\,dx + y^2 e^x\,dy$ counter clock wise around the bounded in C, where $C \to$ ▭ with vertices $(0,0)$ $(2,0)$ $(2,3)$ $(0,3)$

$0 \le x \le 2$
$0 \le y \le 3$

A)

$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$

$= \oint P\,dx + Q\,dy$

$P = x^2 e^y$
$Q = y^2 e^x$

$\Rightarrow \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$

$\frac{\partial Q}{\partial x} = y^2 e^x$, $\Rightarrow \int_0^2 \int_0^3 (y^2 e^x - x^2 e^y)\,dy\,dx$

$\frac{\partial P}{\partial y} = x^2 e^y$. $\Rightarrow \int_0^2 \left[\frac{y^3}{3} e^x - x^2 e^y\right]_0^3 dx$.

$$= \int_0^2 \left[\frac{27}{3} e^x - x^2 e^3 + x^2\right] dx.$$

$$= \int_0^2 9e^x - x^2 e^3 + x^2 \, dx = \left(9e^x - \frac{x^3 e^3}{3} + \frac{x^3}{3}\right)_0^2$$

$$= 9e^2 - \frac{8e^3}{3} + \frac{8}{3} + \frac{9}{1}$$

$$= 9e^2 - \frac{8e^3}{3} - \frac{19}{3}$$

① → continous.

$$\oint_{C_1} P\,dx + Q\,dy \qquad 1 \le t \le 0$$

$P = xy$
$Q = x^2.$ $\qquad C = C_1 \cup C_2.$

$$= \int_{C_1 \cup C_2} xy\,dx + x^2\,dy.$$

$$= \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy$$

Sub

$$= \int_0^1 t \cdot t^2\,dt + t^2 \cdot 2t\,dt + \int_1^0 t \cdot t\,dt + t^2\,dt$$

$$= \int_0^1 t^3\,dt + 2t^3\,dt + \int_1^0 t^2\,dt + t^2\,dt$$

$$= \int_0^1 t^3 + 2t^3\,dt + \int_1^0 t^2 + t^2\,dt.$$

$$= \int_0^1 3t^3\,dt + \int_1^0 2t^2\,dt.$$

$$\cancel{= 3\left[\frac{t^4}{4}\right]_0^1 + 2\left[\frac{t^3}{3}\right]_1^0 = \frac{3}{4} + \frac{2}{3}}$$

$$= \int_0^1 3t^3\,dt - \int_0^1 2t^2\,dt.$$

$= \frac{1}{12}$

$$= \left[3\frac{t^4}{4} - 2\frac{t^3}{3}\right]_0^1$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{9-8}{12} = \frac{1}{12}$$

$y = x^2$, $y = x$

$x^2 \le y \le x.$

$P = xy$ $Q = x^2$

$\frac{x}{0 \to 1}$

$\frac{y}{x^2 \to x.}$

$$\frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = x.$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_0^1 \int_{x^2}^{x} (2x - x)\,dy\,dx.$$

$$= \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 \left[xy\right]_{x^2}^{x} dx.$$

$$\cancel{= \int_0^1 \frac{x^3}{2} - \frac{x^4}{4}\,dx} = \int_0^1 x \cdot x - x \cdot x^2\,dx.$$

$$= \int_0^1 x^2 - x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

LHS = RHS

3) $\oint_C y^2\,dx + x^2\,dy$

C is the boundary of Δ bounded

$x = 0$, $x + y = 1$, $y = 0$ in counter clockwise

(alis)

4) $y = 2x + 1$, $-1 \le x \le 0$

$G(x,y) = 3x^2 + 6y^2$ $\quad \int_C G(x,y)\,ds\,?.$

ii) $\int_C G(x,y)\, ds = \int_a^b G(x, f(x)) \sqrt{1+(f'(x))^2}\, dx$ —①

$f(x) = 2x+1$

$f'(x) = 2$ $\quad (f'(x))^2 = 4$

$1+(f'(x))^2 = 1+4 = 5$

$\sqrt{1+(f'(x))^2} = \sqrt{5}$

$G(x, f(x)) = G(x, 2x+1)$

$= 3x^2 + 6(2x+1)^2 \quad \leftarrow G(x,y) = 3x^2+6y^2$

$= 3x^2 + 6[(2x)^2 + 2\cdot 2x \cdot 1 + 1^2]$

$= 3x^2 + 6[4x^2 + 4x + 1]$

$= 3x^2 + 24x^2 + 24x + 6.$

$= 27x^2 + 24x + 6$

—①

$= \int_{-1}^{0} (27x^2 + 24x + 6) \sqrt{5}\, dx.$

$= \sqrt{5} \int_{-1}^{0} (27x^2 + 24x + 6\ \cancel{dx})\, dx.$

$= \sqrt{5} \left[\cancel{27}^{9} \frac{x^3}{\cancel{3}} + \cancel{24}^{12} \frac{x^2}{\cancel{2}} + 6x \right]_{-1}^{0}$

$= \sqrt{5} \left[9x^3 + 12x^2 + 6x \right]_{-1}^{0}$

$= \sqrt{5} \left[0 - \left[(9(-1)^3 + 12(-1)^2 + 6\times(-1) \right] \right]$

$= \sqrt{5} \left[9 - 12 + 6 \right] = \sqrt{5}\, [3]$

$= 3\sqrt{5}$

5) $x = \left(\frac{1}{3}\right) t^3, \quad y = t^2, \quad z = 2t \qquad 0 \le t \le 1$

$\int_C 4xyz\, dy$?

$dx = f'(t)\, dt = t^2$

$dy = g'(t)\, dt = 2t\, dt$

$dz = h'(t)\, dt = 2\, dt$

ii) $f(t) = \frac{1}{3} t^3 \qquad h(t) = 2t$

$g(t) = t^2$

$f'(t) = \frac{1}{3} \cdot 3t^2 = t^2$

$h'(t) = 2$

$g'(t) = 2t$

$\int_C 4xyz\, dy = \int_C G(x,y,z)\, dx = \int_a^b G(f(t), g(t), h(t))\, g'(t)\, dt.$

$= \int_0^1 4\left(\frac{t^3}{3}\right) \cdot t^2 \cdot 2t \cdot 2t\, dt$

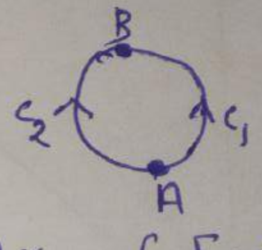
$= \frac{16}{3} \int_0^1 t^7\, dt = 16 \left[\frac{t^8}{8} \right]_0^1$

$= 2/3$

$= 2/3$

⇒ Integrals around closed paths :-

In an open connected region 'R' $\int_C F\cdot dr$ is independent of path, if and only if $\int_C F\cdot dr = 0$ for every closed path C in R.



$$\int_C F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr = 0$$

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

Pf/

Suppose that $\int_C F\cdot dr$ is independent of path of $\int$-tion C is R. Let C be any ~~∮ R~~ closed curve R, let A & B be a points in C. we get 2 paths from ~~A to~~ A to B say C_1

& C_2 along C. such that I say: C has the same orientation. then by the (pro) of line $\int$.

$\int_C F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr$.

$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$.

$\int_C F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr = 0$.

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Let us assume that the $\int$ around any closed curve C in R is 0. Give any 2 points → A & B, & C_1 & C_2 from A to B in R, we see that C_2 with orientation reversed & C_1 together,

$c = c_1 \cup (-c_2)$.

$$0 = \int_C F\cdot dr = \int_{C_1 \cup (-C_2)} F\cdot dr = \int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr$$

$$= \int_C F\cdot dr - \int_{C_2} F\cdot dr$$

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

If $\int_C F\cdot dr = 0$ for every closed path C in R line $\int$, $\int_C F\cdot dr$ is independent of path.

Q) $\int_C 4xyz\, ds$. $x = (1/3)t^3$ $y = t^2$ $z = 2t$

$0 \le t \le 1$

A) $\int_C 4xyz\, \underline{ds} = \int (f(t), g(t), h(t)) \sqrt{(f'(t))^2 (g'(t))^2 (h'(t))^2}$

$$= \int_0^1 \underbrace{4\cdot(t^3/3)\, t^2 \cdot 2t}_{4xyz} \cdot (t^2+2)\, dt.$$

$$= \int_0^1 \frac{4t^3}{3}\, 2t \cdot t^2\, (t^2+2)\, dt.$$

$$= \frac{8}{3} \int_0^1 t^3 \cdot t^1 \cdot t^2\, (t^2+2)\, dt$$

$$= \frac{8}{3} \int_0^1 t^6 (t^2+2)\, dt.$$

$$= \frac{8}{3} \int_0^1 t^8 + 2t^6\, dt.$$

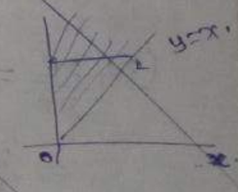
$$= \frac{8}{3} \left[\frac{t^9}{9} + 2\frac{t^7}{7} \right]_0^1$$

$$= \frac{8}{3} \left[\frac{1}{9} \cancel{\times} + \frac{2}{7} \right] = \frac{8}{3} \left[\frac{25}{63} \right]$$

$$= \frac{200}{189}$$

Q) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$.

R: $x \le y \le \infty$, $0 \le x \le \infty$.



A) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$.

$= \int_0^\infty \left[\int_x^\infty \right.$

⇒ Test for conservative field :-

let $f = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$ whose component ( ) hue continu... 1st partial derivation, then f is conservative if and only if,

(1) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (2) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ (3) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$f = Pi + Qj + Rk = \nabla \phi$.

$= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$.

$\frac{\partial \phi}{\partial x} \to P$, $\frac{\partial \phi}{\partial y} \to Q$, $\frac{\partial \phi}{\partial z} \to R$.

$\frac{\partial P}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}$.

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ proved

(2) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial R}{\partial y}$.

$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ //.

(3) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)$

$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

* Corollary-1

Suppose $f(x,y) = P(x,y)\,i + Q(x,y)\,j$ be a field whose compound ( ) hue continu... 1st partial derivatives, then f is conservative if and only if,

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

⇒ Physical eg of line $\int$:-

Total workdone $\longrightarrow W = \int_c F \cdot dr$

Q) find total work done in moving a particle in a force given by,
$F = 3xy\,i - 5z\,j + 10x\,k$ along the curve
$x = t^2+1$ $y = 2t^2$ $z = t^3$ from $t=1$ to $t=2$.

A) $F = 3xy\,i - 5z\,j + 10x\,k$ $1<t<2$

$dr = dx\,i + dy\,j + dz\,k$.

$x = t^2+1$ $y = 2t^2$ $z = t^3$

$dx = 2t\,dt$ $dy = 4t\,dt$. $dz = 3t^2\,dt$

$W = \int_c F \cdot dr = \int_c (3xy\,i - 5z\,j + 10x\,k) \cdot (dx\,i + dy\,j + dz\,k)$

$= (3xy \cdot dx)\,i - (5z \cdot dy)\,j + (10x \cdot dz)\,k$.

$= 3xy\,dx - 5z\,dy + 10x\,dz$.

Sub

$= \int_1^2 3 \cdot (t^2+1)\, 2t^2 \cdot 2t\,dt - 5t^3 \cdot 4t\,dt + 10(t^2+1) \cdot 3t^2\,dt$.

$= \int_1^2 3t^2 + 3 \cdot 2t^2 \cdot 2t\,dt - 20t^4\,dt + 10t^2 + 30t^2\,dt$.

$12t^5 + 12t^3$

$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$

$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$

$= (28+64+48+80) - (2+2+3+10)$

$= 303$ units of work

* Corollary-2 →

$F = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$.

is conservative if and only if $\text{curl } F = 0$

Pf curl $F = \nabla \times F$

$= \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \cdot \underline{\times} (Pi + Qj + Rk)$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$= i\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - j\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] +$

$k\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$.

$= \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] i + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right] j +$

$\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] k$.

* Rmrk →

F is conservative, we usually want to find a potential () for F. This requires solving the eq $\nabla\phi = F$.

$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = Pi + Qj + Rk$.

$\frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R$.

Q.1) S.T $F = [e^x \cos y + yz]\,i + [xz - e^x \sin y]\,j + (xy + z)\,k$

A) $P = e^x \cos y + yz$.

$Q = xz - e^x \sin y$

$R = (xy + z)$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y + yz)$

$= -e^x \sin y + z = \frac{\partial Q}{\partial x}$

$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xz - e^x \sin y) = z - e^x \cos y$.

$\frac{\partial R}{\partial x} = y = \frac{\partial P}{\partial z}$

Therefore by the component test for conservative field, F is conservative i.e hence there is a () ϕ such that $\nabla\phi = F$.

$\nabla\phi = F \Rightarrow \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = Pi + Qj + Rk$

~~$Pi + Qj + Rk$.~~

$\Rightarrow \frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R$.

$P = e^x \cos y + yz$

$\frac{\partial \phi}{\partial x} = e^x \cdot \cos y + yz$ — ①

$\frac{\partial \phi}{\partial y} = xz - e^x \sin y$ — ②

$\frac{\partial \phi}{\partial z} = xy + z$ — ③

$\int$-ing ① with respect to x, y & z constant.

$\phi(x,y,z) = e^x \cos y + xyz + g(y,z)$ — ④

the arbitrary constant of $f$tion $g(y,z)$ is free from x.

differenting — ④ with respect to y,

$$\frac{\partial\phi}{\partial y} = -e^{x}\sin y + xz + \frac{\partial g}{\partial y} \quad —⑤$$

Comparing ② & ⑤ $\qquad \frac{\partial\phi}{\partial y} = xz - e^{x}\sin y$

we gt $\frac{\partial g}{\partial y} = 0$

hence g is a ( ) of z,

let $g(y,z) = h(z)$

$$\phi(x,y,z) = e^{x}\cos y + xyz + h \quad —⑥$$

differentiating ⑥ w.r.to z

$$\frac{\partial\phi}{\partial z} = xy + \frac{dh}{dz}$$

Comparing ③ & ⑦

we gt $\frac{dh}{dz} = x \qquad h(z) = (z^2/2) + c$

then from — ⑥

$$\phi(x,y,z) = e^{x}\cos y + xyz + (z^2/2) + c.$$

2) S.T field $F = (z+y)i + zj + (y+x)k$ is not conservative.

A) $P = (z+y)$, $Q = z$, $R = y+x$.

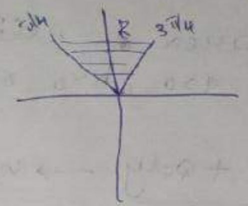
$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(z) = 0$

$\therefore \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

F is not conservative

3) find the workdone by the force, $F = (-16y + \sin x^2)i + (4e^{y} + 3x^2)j$ acting along simple closed curve C, boundary of the region enclosed by ◯, $x^2+y^2=1$ & lines $y=-x$ & $y=x$ above x axis in counter clockwise (fig)?

A) workdone, $\boxed{W = \oint_C F \cdot dr}$. force, displacement.



$F = (-16y + \sin x^2)i + (4e^{y} + 3x^2)j$

$\frac{\partial Q}{\partial x} = \cancel{4e^{y}}\, 6x$

$\frac{\partial P}{\partial y} = -16$

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dy = \int P\,dx + Q\,dy$$

$$= \iint_R 6x + 16\, dA.$$

eqn ◯ from area
$x^2+y^2=1$
$x = r\cos\theta \quad y = r\sin\theta$
$(r\cos\theta)^2 + (r\sin\theta)^2 = 1$
$r^2\cos^2\theta + r^2\sin^2\theta = 1$
$r^2(\cos^2\theta + \sin^2\theta) = 1$
$r^2 = 1$
$r = 1$

$0 \le r \le 1 \qquad \pi/4 \le \theta \le 3\pi/4$

$$W = \iint_R 6x + 16\, dA$$

$$= \int_{\pi/4}^{3\pi/4}\int_0^1 (6r\cos\theta + 16)\, r\,dr\,d\theta$$

$\iint 6r^2\cos\theta + 16r$

$\boxed{dA = r\,dr\,d\theta}$

$$= \int_{\pi/4}^{3\pi/4} \left[\frac{2\cancel{6}\, r^3}{\cancel{3}}\cos\theta + \overset{8}{\cancel{16}}\frac{r^2}{\cancel{2}}\right]_0^1 d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \left[2r^3\cos\theta + 8r^2\right]_0^1 d\theta = \int (2\cos\theta + 8)\, d\theta$$

$$= \left[2\sin\theta + 8\theta\right]_{\pi/4}^{3\pi/4} = \left[2\sin 3\pi/4 + 8 \cdot 3\frac{\pi}{4} - \left(2\sin\frac{\pi}{4} + \frac{8}{2}\pi/4\right)\right]$$

$= 2\sin 3\pi/4 + 8 \cdot 3\pi/4 - (\sin\frac{\pi}{2} + 2\pi)$

$= 6 \cdot 45 + 24 \cdot 45 - (1 + 2 \cdot 180)$

4) a) Let R be the region bounded by a piecewise smooth simple closed curve using green's T, P.T, $\frac{1}{2}\oint_c (-y\,dx + x\,dy)$ — ⓐ

= area of R.

b) using — ⓐ, find the area of ellipse $x = a\cos t$, $y = b\sin t$, $a>0$, $b>0$, $0 \le t \le 2\pi$

A) $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \underbrace{\int P\,dx + Q\,dy}_{①} \longrightarrow$ G.T.

a) $\frac{1}{2}\oint_c -y\,dx + x\,dy$ is in the form

$\oint P\,dx + Q\,dy.$ $\quad P = -y$, $Q = x$

$\therefore \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-y) = -1.$

$\frac{1}{2}\oint_c P\,dx + Q\,dy.$ — ⓐ

$\therefore \frac{1}{2}\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$

[① കാണാൻ ~ ആവശ്യപ്പെട്ടാൽ
ⓑ മാത്രം മതി.]

$\Rightarrow \frac{1}{2}\iint_R 1 - (-1)\, dA.$

$\Rightarrow \frac{1}{2}\iint_R 2\, dA.$

$= \iint_R dA.$ = area of region R

b) $\frac{1}{2}\oint_c (-y\,dx + x\,dy)$ — ellipse eq,

$x = a\cos t$ $\quad y = b\sin t$,

$dx = -a\sin t\,dt$ $\quad dy = b\cos t\,dt$.

$= \frac{1}{2}\oint_c -b\sin t\,dt\; a\cos\, dy.$

$= \frac{1}{2}\int_0^{2\pi} -b\sin \cdot -a\sin t\, dt + a\cos t \cdot b\cos t\, dt.$

$= \frac{1}{2}\int_0^{2\pi} (ab\sin^2 t\, dt + ab\cos^2 t)\, dt.$

$= \frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)\, dt = \frac{1}{2}\int_0^{2\pi} ab\, dt.$

$= \frac{1}{2} ab \int_0^{2\pi} 1\, dt. = \frac{1}{2} ab\,[t]_0^{2\pi}$

$= \frac{1}{2} ab\,[2\pi - 0] = \frac{1}{2} ab\,[2\pi]$

$= \pi ab$

⇒ Extension of Green's T to regions with holes :-

$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_{R_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint_{R_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$

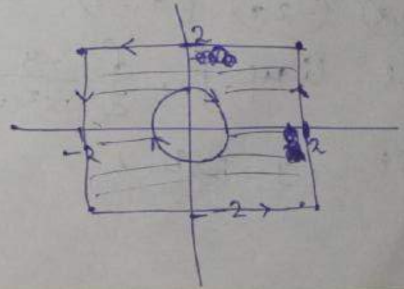
$\Rightarrow \int_c P\,dx + Q\,dy = \int_{c_1} P\,dx + Q\,dy + \int_{c_2} P\,dx + Q\,dy.$

i) Evaluate line $\int$, $\oint_c \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, $c = c_1 \cup c_2$, c_1 is □ with vertices $(2,-2), (2,2), (-2,2), (-2,-2)$ in the counter clock wise (clis), i.e. c_2 is the ○ $x^2+y^2=1$ in clock wise (clis).

counter c.w. dir

c.w dir

A) $x^2+y^2 = \frac{1}{r}$



$P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{x}{x^2+y^2}$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$

(Q. rule) $= \frac{(x^2+y^2)\cdot -1 - -y \cdot 2y}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2}$

by G.T.,

$\int \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \iint_R \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\,dA$

$= \iint_R \frac{-x^2+y^2}{(x^2+y^2)^2} - \frac{-x^2+y^2}{(x^2+y^2)^2}\,dA = 0$

$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ — (a)

$\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ — (b)

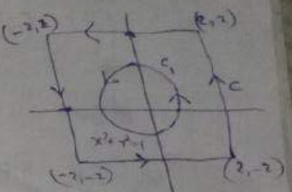
5) let C be □ with vertices $(2,-2)$ $(2,2)$ $(-2,2)$ $(-2,-2)$ in counter clock. w (bis)
Evaluate line $\int$, $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

A) $x = a\cos t$, $y = b\sin t$;

$P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{x}{x^2+y^2}$

(pg-187) $\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad \frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$

let C_1 denote $x^2+y^2=1$.



$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$

$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ — ①

$C_1 \rightarrow \bigcirc$, $r = 1$

$\therefore$ $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

on C_1, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.
$dx = -\sin t\,dt$, $dy = \cos t\,dt$.

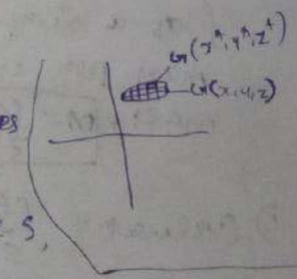
$\therefore \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$

$= \int_0^{2\pi} (-\sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$ $\quad (x^2+y^2=1)$ $\quad \cos^2 t + \sin^2 t = 1$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$

$\Rightarrow$ Surface $\int$:-

let G be a ( ) of 3 variables defined over a region of 3 space containing surface S, Then S.$\int$ of G over S



$$\boxed{\iint_S G(x,y,z)\,dS = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\,\Delta S_k}$$

* Methods of evaluation of S.I :-

(1) If $z=f(x,y)$ (2 variables $\rightarrow x,y$)

$\iint_{S} G(x,y,z)\,ds = \iint_R G(x,y,f(x,y))\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA$

$\left(ds = \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA\right)$

(2) If $y=g(x,z)$

$\iint_S G(x,y,z)\,ds = \iint_R G(x,g(x,z),z)\sqrt{1+g_x(x,z)^2+g_z(x,z)^2}\,dA$

(3) If $x=h(y,z)$

$\iint_S G(x,y,z)\,ds = \iint_R G(h(y,z),y,z)\sqrt{1+h_y(y,z)^2+h_z(y,z)^2}\,dA$

If $x=h(y,z)$ is the eq of the surface S that project into a region R of (y,z) plane & if G, f, h_y, h_z are continuous through out a region containing S.

⇒ Mass of a surface :-

Suppose $\rho(x,y,z)$ represents the ρ of a surface at any point/ mass per unit area, then

mass, $\boxed{M=\iint_S \rho(x,y,z)\,ds}$

density $(\rho)=\frac{\text{mass}}{\text{vol}}$

mass $=\int \rho \times \text{vol}$

$=\int \rho(x,y,z)\,ds$.

1) Evaluate $\iint_S xz\,ds$, where S is the part of the plane $x+y+z=1$ that lies in the 1st octant ?

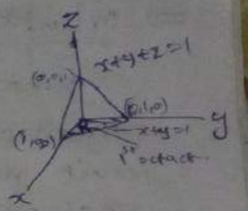
A) draw plane

$x+y+z=1$

$x=0,\ y=0,\ z=1$ $(0,0,1)$

$x=0,\ z=0,\ y=1$ $(0,1,0)$

$x=1,\ y=0,\ z=0$ $(1,0,0)$



R is the region bounded by lines $x=0, y=0, x+y=1$

$0\le y\le 1-x$, $0\le x\le 1$

$x+y=1$
$y=1-x$

$x+y+z=1$

$z=1-x-y$

$z=f(x,y)=1-x-y$. $G(x,y)=xz$

$f_x(x,y)=-1$

$f_y(x,y)=-1$

$\therefore$ eq $\rightarrow \iint_S G(x,y)\,ds = \iint_S G(x,y,z)$

$=\iint_R G(x,y,f(x,y))\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA$

$f(x,y)=1-x-y$

$=\int_0^1\int_0^{1-x} \underbrace{(1-x-y)x}_{xz}\sqrt{1+(-1)^2+(-1)^2}\,dy\,dx$

$=\iint x-x^2-yx\,\sqrt{3}\,dy\,dx$

$=\sqrt{3}\int_0^1\left[xy-x^2-\frac{y^2}{2}x\right]_0^{1-x}dx$

$=\sqrt{3}\int_0^1\left[x(1-x)-x^2(1-x)-\frac{1}{2}x(1-x)^2\right]dx$

$=\sqrt{3}\int_0^1\left[\frac{x}{2}-x^2+\frac{1}{2}x^3\right]dx$

$=\sqrt{3}\left[\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{8}\right]_0^1$

$=\sqrt{3}\left[\frac{1}{4}-\frac{1}{3}+\frac{1}{8}\right]=\frac{\sqrt{3}}{24}$

⇒ **orientation of a surface** :-

A smooth surface 's' is orientable / is an oriented surface if there exist a continuous unit normal vector (·) 'n' defined at each point (x,y,z) on the surface. The vector field n(x,y,z) → the orientation of 's'.

Since a unit normal to the surface 's' at (x,y,z) can be either n(x,y,z) (or) −n(x,y,z), an orientable surface has 2 orientation.

* A surface 's' is defined by z = f(x,y) has an upward orientation, when the unit normals directed upward (ie) have +ve k components & has downward orientation when the unit normals are directed downward (ie) have −ve k components.

$g(x,y,z) = 0$

(unit normal) → $\hat{n} = \frac{\nabla g}{\|\nabla g\|}$ ∇g

~~$\nabla g = \hat{n} \times \|\nabla g\|$~~

If a smooth surface 's' is defined by g(x,y,z) = 0 then a unit normal,

$$\hat{n} = \frac{\nabla g}{\|\nabla g\|}$$

(gradient of g) $\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k.$

* If 's' is defined by z = f(x,y) then we can use, $\boxed{\begin{matrix} g(x,y,z) = z - f(x,y) = 0 \\ g(x,y,z) = f(x,y) - z = 0 \end{matrix}}$ (1) (or) depending on the orientation of S.

eg →

Q) Consider the sphere of radius a > 0 defined by $x^2+y^2+z^2 = a^2$

A) $g(x,y,z) = x^2+y^2+z^2 - a^2$. by (1).

$$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k$$

$$= 2xi + 2yj + 2zk$$

$$\|\nabla g\| = \sqrt{(2x)^2+(2y)^2+(2z)^2} = \sqrt{4x^2+4y^2+4z^2}$$

$$= \sqrt{4\underbrace{(x^2+y^2+z^2)}_{a^2}} = \sqrt{4a^2} = 2a$$

$$\therefore n = \frac{\nabla g}{\|\nabla g\|} = \frac{2xi+2yj+2zk}{2a} = \frac{2xi}{2a} + \frac{2yj}{2a} + \frac{2zk}{2a}$$

$$n = \frac{xi+yi+zk}{a} = \frac{x}{a}i + \frac{y}{a}j + \frac{z}{a}k$$

and

$$-n = -\frac{x}{a}i - \frac{y}{a}j - \frac{z}{a}k$$

⇒ **flux** =

The total vol of a fluid passing through S per unit time — flux of 'f' through 'S'

& is gvn by, $\iint_S f \cdot n \, ds$

Suppose S is the orientable piecewise smooth closed surface bounded by the paraboloid

S_1, $z = x^2+y^2$ &

S_2, $z = 1$ then flux,

$$\boxed{\iint_{S_1} f\cdot n\, ds + \iint_{S_2} f\cdot n\, ds = \iint_S f\cdot n\, ds}$$

where S_1 oriented downward

where S_2 " upward.

7) Suppose that a curved lamina S with constant density $\delta(x,y,z) = k$ is the portion of the paraboloidal $z = x^2+y^2$ below the plane $z = 1$. Find mass of lamina?

A) $z = x^2+y^2$ we know that $z = f(x,y)$.

$\therefore f(x,y) = x^2+y^2$.

$$\text{mass} = \iint_S \delta(x,y,z)\, ds. \qquad z = f(x,y)$$

$$= \iint \delta(x,y,z)\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}$$

$f(x,y) = x^2+y^2$

$f_x = 2x$

$f_y = 2y$.

$$= \iint_R k\sqrt{1+(2x)^2+(2y)^2}\, dA.$$

$$= \iint_R k\sqrt{1+4x^2+4y^2}\, dA$$

$\int$ limit

$z = x^2+y^2$, $z = 1$

$x^2+y^2 = 1 \longrightarrow r = 1$

$0 \le r \le 1$ $\quad 0 \le \theta \le 2\pi$ $\quad$ (dA is $r\,dr\,d\theta$)

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 k\sqrt{1+4(r\cos\theta)^2+4(r\sin\theta)^2}\; r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 k\sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\; r\,dr\,d\theta.$$

$$\cancel{\int_0^{2\pi}\int_0^1 k\sqrt{r+4r^3\cos^2\theta+4r^3\sin^2\theta}}$$

$$k\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\left(\cos^2\theta+\sin^2\theta\right) r\,dr\,d\theta$$

$$k\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\,dr\,d\theta.$$

Rule method

put $1+4r^2 = u$

$r=0$, $r=1$

$\therefore u=0$, $r=5$

$8r\,dr = du$

$r\,dr = \frac{du}{8}$

$$k\int_0^{2\pi}\int_1^5 \sqrt{u}\,\frac{du}{8}\,d\theta.$$

$$\frac{k}{8}\int_0^{2\pi}\int_1^5 \sqrt{u}\,du\,d\theta$$

$\frac{u^{3/2}}{3/2} = \frac{2}{3}u^{3/2}$

$$\frac{k}{8}\int_0^{2\pi}\left[\frac{2}{3}u^{3/2}\right]_1^5 d\theta =$$

$$\frac{k}{8}\cdot\frac{2}{3}\int_0^{2\pi}\left[u^{3/2}\right]_1^5 d\theta$$

$$\frac{k}{12}\int_0^{2\pi} 5^{3/2}-1\; d\theta = \frac{k}{12}\int_0^{2\pi}(5\sqrt5-1)\,d\theta.$$

$$= \frac{k}{12}(5\sqrt5-1)\int_0^{2\pi} d\theta = \frac{k}{12}(5\sqrt5-1)(\theta)_0^{2\pi}$$

$$= \frac{k}{12}(5\sqrt5-1)\,2\pi = \frac{k}{6}(5\sqrt5-1)\pi$$

$$= \frac{k\pi}{6}(5\sqrt5-1)$$

7) find mass of surface of paraboloid $z = 1 + x^2 + y^2$ in the 1st octact for $1 \le z \le 5$, if the density at a point 'P' on the surface is directly proportional to its distance from the xy plane ?

⇒ Vector form of Green's Theorem :-

$F(x,y) = P(x,y)i + Q(x,y)j$

$F \cdot dr = (P(x,y)i + Q(x,y)j) \cdot (dx\,i + dy\,j)$

$= P(x,y)dx + Q(x,y)dy$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= i\left|\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}Q\right| - j\left|\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}P\right| + k\left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right).$$

$$= \underbrace{-\frac{\partial}{\partial z}i + \frac{\partial P}{\partial z}j}_{z\ [illegible]\ 0} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\boxed{\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_R \text{Curl} F \cdot k\, dA} \rightarrow \text{eq.}$$

→ Stoke's Theorem :-

let 'S' be a piecewise smooth orientable surface bounded by a piecewise smooth simple $\oint$ curve 'C'.

$F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be a v.field for which PQR are continu & hve continu 1st partial derivative in a region of space containing f. if C is traversed in the +ve (dir) then

$$\boxed{\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_S \text{curl} F \cdot n\, ds}$$

where n is a unit normal to S in the (dir) of the orientation of S.

7) let S be the part of cylinder $z = 1 - x^2$ for $0 \le x \le 1$, $-2 \le y \le 2$, varify Stock's therem for the v.field $f = xyi + yzj + xzk$. Assume S is oriented upward.

A) make 4 curves.

$c_1 \to (1,-2,0)$ to $(1,2,0)$ y limit change.

$c_2 \to (1,2,0)$ to $(0,2,1)$ x limit "

$c_3 \to (0,2,1)$ to $(0,-2,1)$ y limit "

$c_4 \to (0,-2,1)$ to $(1,-2,0)$ x limit "

$\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_S \text{curl} f \cdot n\, ds.$

$f = xyi + yzj + xzk$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix}$$

$$= i\left|\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz)\right| - j\left|\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(xy)\right|$$

(T [illegible])

$$+ k \left| \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right|$$

$$\iint_R \text{curl } F \cdot n \, ds$$

$$\text{curl } F = -y\,i - z\,j - x\,k.$$

$$g(x,y,z) = z + 1 - x^2 = 0 \qquad \therefore z = 1 - x^2 = 0$$

$$= z + 1 - x^2 = 0$$

(unit normal vector) $n = \dfrac{\nabla g}{\|\nabla g\|}$

$$\therefore \nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k.$$

$$= 2x\,i + 0j + 1k \quad = 2x\,i + 1k.$$

$$\|\nabla g\| = \sqrt{(2x)^2 + (k)^2} = \sqrt{4x^2 + 1^2} = 2x + 1$$

$$\therefore n = \frac{2x\,i + k}{\sqrt{4x^2+1}} = \frac{2x}{\sqrt{4x^2+1}} i + \frac{k}{\sqrt{4x^2+1}} j.$$

$$\text{curl } F \cdot n = \left(-y\,i - \underset{0}{z\,j} - x\,k\right) \cdot \left(\frac{2x}{\sqrt{4x^2+1}} i + \frac{k}{\sqrt{4x^2+1}} j\right)$$

$$= \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}}$$

$$\iint_S \text{curl } F \cdot n \, ds = ? \qquad \text{we know } z = f(x,y)$$

$$f(x,y) = 1 - x^2$$

$$f_x = 2x \qquad f_y = 0$$

$ds \to (x,y)$.

$$\iint_R \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}} \sqrt{1 + \underset{(2x)^2}{f_x(x,y)^2} + \underset{0}{f_y(x,y)^2}} \, dA.$$

$$\int_0^1 \int_{-2}^{2} \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}} \sqrt{1 + 4x^2} \, dA = \iint \frac{-2xy + -x}{\cancel{\sqrt{4x^2+1}}} \cancel{\sqrt{1+4x^2}} \, dA$$

$$= \int_0^1 \int_{-2}^{2} -2xy - x \, dy\,dx.$$

$$= \int_0^1 \left[-xy^2 - xy\right]_{-2}^{2} dx = \int_0^1 -4x \, dx$$

$$= \left[-2x^2\right]_0^1 = -2$$

$$\iint_S \text{curl}\, F \cdot n\, ds = -2 \quad \text{--- (1)}$$

$$\oint_C F \cdot dr = ?,$$

$y \to \int_{-2}^{2} \leftarrow C_1 :\ (1,-2,0)$ to $(1,2,0)$

$x \to \int_{1}^{0} \leftarrow C_2 :$ " to "

$y \to \int_{-2}^{2} \leftarrow C_3 :$ " to "

$x \to \int_{0}^{1} \leftarrow C_4$ " to "

$$\oint_C F \cdot dr = \oint_C (xy\,i + yz\,j + xz\,k) \cdot (dx\,i + dy\,j + dz\,k)$$

$$= \oint_C xy\,dx + yz\,dy + xz\,dz$$

$C_1 \to \int_{-2}^{2}$

$$= \int_{C_1} xy\,dx + yz\,dy + xz\,dz + \int_{C_2} xy\,dx + yz\,dy + xz\,dz$$

$C_2 \to \int_1^0$

$$+ \int_{C_3} xy\,dx + yz\,dy + xz\,dz + \int_{C_4} xy\,dx + yz\,dy + xz\,dz$$

$C_3 \to \int_{-2}^{2}$

$C_4 \to \int_0^1$

$$= \int_{-2}^{2} xy\,dx + yz\,dy + xz\,dz + \int_1^0 xy\,dx + yz\,dy + xz\,dz$$

$$+ \int_{-2}^{2} xy\,dx + yz\,dy + xz\,dz + \int_0^1 xy\,dx + yz\,dy\ xz\,dz. \quad \text{--- (a)}$$

before $\int$-tion

$C_1 \to x = 1,\ z = 0$
$dx = 0,\ dz = 0,$

$C_2 \to$ ~~x = ...~~ , $y = 2,\ z = 1 - x^2$
$dy = 0,\ dz = -2x\,dx$

$C_3 \to x = 0,\ z = 1$
$dx = 0,\ dz = 0$

$C_4 \to y = -2,\ z = 1 - x^2$
$dy = 0,\ dz = -2x\,dx.$

Rule x, y, z in --- (a),

$$= \int_{-2}^{2} y \cdot 0\,dy + \int_1^0 2x\,dx + [x(1-x^2) - 2x\,dx] +$$

$$\int_{-2}^{2} y \cdot 1\,dy + \int_0^1 -2x\,dx + x(1-x^2) \cdot -2x\,dx.$$

$$= 0 + \int_1^0 2x \cdot (x - x^3) \cdot -2x\,dx + \int_{-2}^{2} y\,dy + \int_0^1 -2x\,(x - x^3), -2x\,dx.$$

$$= \int_1^0 2x - 2x^2 + 2x^4\,dx + \int_{-2}^{2} y\,dy + \int_0^1 -2x - 2x^2 + 2x^4\,dx.$$

$$= \left[2\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{2x^5}{5}\right]_1^0 + \left[\frac{y^2}{2}\right]_{-2}^{2} + \left[-2\frac{x^2}{2} - \frac{2x^3}{3} + \frac{2x^5}{5}\right]_0^1$$

$$= 0 - \left(\frac{2}{2} - \frac{2}{3} + \frac{2}{5}\right) + \left(2 - \frac{(-2)^2}{2}\right) + \left(\frac{-2}{2} - \frac{2}{3} + \frac{2}{5}\right)$$

$2 - 2 = 0$

$$= -1 + \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}} + 0 - 1 - \cancel{\frac{2}{3}} + \cancel{\frac{2}{5}}$$

$$= -2 \quad \text{——} \; (2)$$

from (1) & (2) $\iint_S \text{curl} f \cdot n\,ds = \oint_C f \cdot dr$

Stock T varified

2) varify Stock T for the v. field $f = 2z\,i + 3x\,j + 5y\,k$, taking S to be the ~~...~~ portion of paraboloid $z = 4 - x^2 - y^2$ $z \geq 0$ with upward orientation & C to be tuely oriented ◯, $x^2 + y^2 = 4$ that forms the boundary of S in the xy plane.

3) use Stock's T to evaluate $\oint F \cdot dr$ $F = xz\,i + xy\,j + 3xz\,k$ & C is the boundary of portion of the plane $2x + y + z = 2$ in the 1st octanct traversed counter clock wise, as viewed from above &

A) $\oint_C F \cdot dr = \int xz\,i + \int xy\,j + \int 3xz\,k.$

$2x + y + z = 2$

curl f $= \nabla \times F$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xy & 3xz \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} 3xz - \frac{\partial}{\partial z} xy\right) - j\left(\frac{\partial}{\partial x} 3xz - \frac{\partial}{\partial z} xz\right) + k\left(\frac{\partial}{\partial x} xy - \frac{\partial}{\partial y}(xz)\right)$$

$$= 0 - j(3z - x) + k(y - 0)$$

$$= -(3z - x)j + yk = (x - 3z)j + yk$$

$-(3z - x) = -3z + x = (x - 3z)$

$g(x,y,z) = 2x + y + z - 2 = 0$ $\qquad n = \dfrac{\nabla g}{\|\nabla g\|}$

$$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k$$

$$= 2i + 1j + 1k$$

$$\|\nabla g\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$n = \frac{2i + j + k}{\sqrt{6}} = \frac{2}{\sqrt{6}} i + \frac{1}{\sqrt{6}} j + \frac{1}{\sqrt{6}} k$$

$$\text{curl} f \cdot n = (x - 3z)j + yk\left(\frac{2}{\sqrt{6}} i + \frac{1}{\sqrt{6}} j + \frac{1}{\sqrt{6}} k\right)$$

$$= \frac{(x - 3z)}{\sqrt{6}} j + \frac{y}{\sqrt{6}} k.$$

$$\text{curl} F \cdot n = \frac{1}{\sqrt{6}}(x + y - 3z)$$

To draw graph

$2x + y + z = 2.$

let $y = z = 0$; $2x = 2 \Rightarrow x = 1$ $\rightarrow (1, 0, 0)$

let $x = z = 0$; $y = 2$ $\rightarrow (0, 2, 0)$

let $x = y = 0$; $z = 2$ $\rightarrow (0, 0, 2)$

$2x+y+z-2=0$

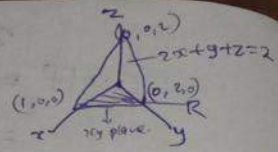
we know. $z=f(x,y)$

$z=2-2x-y$

$f(x,y)=2-2x-y$

$f_x=-2 \quad f_y=-1$

$0\le y\le 2-2x \qquad 0\le x\le 1$



$2-2x-y=0$

$2-2x=y$ (y)

$\oint_C F\cdot dr = \iint_R \text{curl} f\cdot n\, ds$

$=\int_0^1\int_0^{2-2x} \frac{1}{\sqrt{6}}\left(x+y-3\left(2-2x-y\right)\right)\sqrt{1+f_x(x,y)^2 + f_y(x,y)^2}\, dA$

$=\int_0^1\int_0^{2-2x} \frac{1}{\sqrt{6}}\left(x+y-6+6x+3y\right)\sqrt{1+4+1}\, dy\,dx$

$=\int_0^1\int_0^{2-2x} 7x+4y-6\, dy\,dx.$

$=\int_0^1 \left[7xy+\cancel{4}^{2}\frac{y^2}{2}-6y\right]_0^{2-2x} dx$

$2(4+8x-4x^2)$
$=8+16x+8x^2$

$=\int_0^1 \left[7x(2-2x)+2(2-2x)^2-6(2-2x)\right)-0$

$2(4-8x^2+4x^2)$
$=4+2\cdot2\cdot2x-(2x)^2=4+8x-4x^2 \quad dx.$

$=\int_0^1\left[14x-14x^2+\cancel{8}\cancel{-8x}+12-12x\right] dx.$

$-14+8=-6$

$=\int_0^1\left[14x-14x^2+8+16x+8x^2+12-12x\right] dx.$

$=\int_0^1\left[2x-6x^2+16x+20\right]\cdot dx.$

$\int_0^1 14x-14x^2+8+16x-8x^2+12-12x\, dx.$

$\int_0^1 7x(2-2x)+2(2-2x)^2-6(2-2x)\,dx.$

$14x-14x^2+2\left[4-8x+4x^2\right]+12+12x$

$14x-14x^2+8-16x+8x^2-12+12x$

$\int_0^1 10x-6x^2-4\, dx$

$=\left[\cancel{10}^{5}\frac{x^2}{\cancel{2}}-6\frac{x^3}{3}-4x\right]_0^1$

$=5\times1-\frac{\cancel{6}^{2}\times1}{\cancel{3}}-4\times1 = 5-2-4=-1$

5) using Stock's. T to evaluate $\oint_C f\cdot dr$ where c is the ○ $x^2+y^2=4,\ z=-3$ oriented counter clock wise as seen by a person standing at origin & with respect to the right handed cartition coordinate $f=yi+xz^3j-zy^3k$?.

A) curl $f=\nabla\times f$

$=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^3 & -zy^3\end{vmatrix}$

$=i\left(\frac{\partial}{\partial y}(-zy^3)-\frac{\partial}{\partial z}(xz^3)\right)-j\left(\frac{\partial}{\partial x}(-zy^3)-\frac{\partial}{\partial z}y\right)+k\left(\frac{\partial}{\partial x}xz^3-\frac{\partial}{\partial y}y\right)$

$=i(-3zy^2-3xz^2)-j(0)+k(z^3-1)$

$=-(3zy^2-3xz^2)i+(z^3-1)k$

$=-3(y^2z-xz^2)i+(z^3-1)k$

$n=k$ (ஒன்று மட்டும் ..., $z=k$ (component) ; $z=k$.

curl $f\cdot n=z^3-1 \Rightarrow z=-3$ Ⓐ $\Rightarrow(-3)^3-1=-28$

$\oint_C f\cdot dr=\iint_S \text{curl}\, f\cdot n\, ds.$

$z=-3$
$f(x,y)=3$
$f_x=0$
$f_y=0$

$=\iint -28\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA$

$$= -28 \underbrace{\iint_R dA}_{\text{area of } O} \quad \text{here } x^2+y^2=4 \xrightarrow{2\pi} 4\pi$$

$x^2+y^2=4$,
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$= -28 \cdot 4\pi = -112\pi$$