02: FIRST ORDER DE.

$\left(\frac{d}{dx} \text{ or } y'\right)$

⇒ Direction fields =

The method (also) field is a graphical method for displaying the general shapes & behaviour of solns of DE of the form $\frac{d}{dx} = f(x,y)$, where f is a gvn () of a 2 variables x & y

⇒ Seperable diffn eq =

The diffn eq can be reduced to the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow h(y)dy = g(x)dx.$$

$$\Rightarrow \int h(y)\,dy = \int g(x)\,dx + c.$$

(a) [seperate → ∫ → sol]

1) Solve the DE $y' = (1+x)(1+y^2)$

A) $y' = (1+x)(1+y^2)$ by (a)

here $\frac{dy}{dx} = (1+x)(1+y^2)$

(dy & dx → numrtr)

$y' = \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{1+y^2} = (1+x)\,dx.$$

$$\therefore \int \frac{dy}{1+y^2} = \int (1+x)\,dx$$

$$\tan^{-1} y = x + \frac{x^2}{2} + c$$

$$\Rightarrow y = \tan\left(x + \frac{x^2}{2} + c\right)$$

$\left(\frac{dy}{1+y^2} = \tan^{-1} y\right)$

$\frac{dx}{1+x^2} = \tan^{-1} x$

3) find the sol^n of $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$ using seperation variables.

A) (take common out side)

$$\frac{dy}{dx} = \frac{y(x+2)-(x+2)}{y(x-3)+(x-3)} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$$

$$\frac{(y+1)}{(y-1)}\,dy = \frac{x+2}{x-3}\,dx$$

$$\int \frac{y+1}{y-1}\,dy = \int \frac{x+2}{x-3}\,dx$$

[denomtr-ൽ periority ആയിട്ടാണ് → y-1. denomtr-ൽ numertr-ൽ y+1 → +1-ഉം, ഉള്ളത് y-1 → -1 ആക്കണം ⇒ $\frac{y-1}{y-1} \to \frac{y-1+2}{y-1}$]

$$\Rightarrow \int \frac{y-1+2}{y-1}\,dy = \int \frac{x-3+5}{x-3}\,dx$$

$$\int \frac{y-1}{y-1} + \frac{2}{y-1}\,dy = \int \frac{x-3}{x-3} + \frac{5}{x-3}\,dx$$

$$\int 1 + \frac{2}{y-1}\,dy = \int 1 + \frac{5}{x-3}\,dx$$

$$\int y + 2\ln|y-1| = x + 5\ln|x-3| + C$$

$\frac{2}{y-1}$

$2 \times \frac{1}{y-1}$

$2 \times \log(y-1)$

2 ln(y-1)

3) Solve the inertial value prbm. (last +c → c ആണ്)

$y' = -2xy$, $y(0)=1$ & also determine the interval in which sol^n is defined.

A) $\frac{dy}{dx} = -2xy \Rightarrow \frac{1}{y}\,dy = -2x\,dx$

$$\int \frac{1}{y}\,dy = \int -2x\,dx$$

$$\ln y = -2\frac{x^2}{2} + C$$

$$\ln y = -x^2 + C \quad \text{—(1)}$$

To find $y = \ldots$ cancel ln with e.

gvn $y(0)=1$, $\to x=0$.

Sub in —(1)

$\ln y = 0 + C$

$\ln 1 = C$

$0 = C \Rightarrow \underline{C = 0}$

$\therefore \ln y = -x^2 + C$

$\ln y = -x^2$

$e^{\ln y} = e^{-x^2}$

$\underline{y = e^{-x^2}}$

Since y is defined for all x, the interval of above sol^n is the entire real line. [exp-നുള്ളതിന് ഒരു non-zero ആയിരിക്കും]

4) find the sol^n of $xy' = 4y$

A) $x\frac{dy}{dx} = 4y$.

$\Rightarrow \frac{dy}{y} = 4\frac{dx}{x}$.

$\int \frac{dy}{y} = 4\int \frac{dx}{x} \Rightarrow \ln y = 4\ln x + C$

$e^{\ln y} = e^{\ln x^4} + C$

$y = e^{\ln x^4}$

$\underline{y = x^4 + C}$

$\boxed{\ln a^b = b\ln a \quad \ln x^4 = 4\ln x}$

⇒ Eq reducible to seperable form =

Type 1 → Homogenous DE.

A DE $y' = f(x,y)$ is said to be a homogenous DE, if $f(x,y)$ can be expressed as a ( ) of the ratio y/x only (i.e) $\boxed{f(x,y) = F(y/x)}$

(Homogeneous eq: y/x form)

Homo- eq can be transformed into seperable eq by $\boxed{y/x = v}$

1) Solve $(x-y)\,dx + x\,dy = 0$

A) Step 1 → check if it seperable. $(x-y)$ not possible (difficult)
Step 2 → so Homo- DE.

$(x-y)\,dx + x\,dy = 0$.

$x\,dy = -(x-y)\,dx$.

$x\frac{dy}{dx} = \frac{-x+y}{x}$

$x\frac{dy}{dx} = y-x$

$\frac{dy}{dx} = \frac{y-x}{x}$ — ①

$\cancel{\frac{dy}{dx} = \frac{y}{x} - 1}$

↳ we get y/x form

Clearly this is a homo- DE

Rule $u = y/x$

$u = \frac{y}{x}$

$y = ux$

diff. w.r.t x,

$\frac{dy}{dx} = u\times 1 + x\frac{du}{dx}$.

$\frac{dy}{dx} = u + x\frac{du}{dx}$ — ②

frm ① & ②

$\frac{dy}{dx} = \frac{dy}{dx}$

$\frac{y-x}{x} = \frac{ux-x}{x} = u + x\frac{du}{dx}$

$\Rightarrow \frac{ux}{x} - \frac{x}{x} = u + x\frac{du}{dx}$

$\Rightarrow u - 1 = u + x\frac{du}{dx}$

$\Rightarrow \cancel{u} - 1 - \cancel{u} = x\frac{du}{dx}$

$\Rightarrow -1 = x\frac{du}{dx}$

$\Rightarrow -\frac{1}{x}\,dx = du$

$\Rightarrow \int -\frac{1}{x}\,dx = \int du$

$\Rightarrow -\ln|x| = u + c$ (i.e)

$u = -\ln|x| + c$

2) Solve the initial value prbm $2xy\frac{dy}{dx} - y^2 + x^2 = 0$ $y(1) = 1$.

A) dy/dx form

$2xy\frac{dy}{dx} = y^2 - x^2$

$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$\frac{(ux)^2 - x^2}{2x(ux)}$ we can write it as so homothe dE is homogeneous

~~Suppose~~ It is a homon eq, it can be written by $\frac{dy}{dx} = f(y/x)$

$\therefore$ Substitute $u = y/x$. $y = ux$.

$$\frac{dy}{dx} = \frac{(ux)^2 - x^2}{2x(ux)}$$

$$\frac{dy}{dx} = \frac{(ux)^2 - x^2}{2ux^2} \quad \text{--- (1)}$$

Since $y = ux$, diff w.r.t x.

$$\frac{dy}{dx} = u + x\frac{du}{dx} \quad \text{--- (2)}$$

from (1) & (2)

(LHS eq equal $\frac{dy}{dx}$)

$$\frac{ux^2 - x^2}{2ux^2} = u + x\frac{du}{dx}$$

send x-term one side & u another side,

$$\frac{u^2x^2 - x^2}{2ux^2} = u + x\frac{du}{dx}$$

$$\frac{u^2-1}{2u} - u = x\frac{du}{dx}$$

$$\frac{u^2-1-2u^2}{2u^2} = x\frac{du}{dx}$$

$$\frac{-u^2-1}{2u} = x\frac{du}{dx}$$

$$\frac{dx}{x} = \frac{-2u}{(u^2+1)}du$$

(dx, dy are numerator & denomo.)

$$\int \frac{dx}{x} = -\int \frac{2u}{u^2+1}du.$$

$$\Rightarrow \ln x = -\ln|u^2+1| + c \quad \text{--- (1)}$$

$$\Rightarrow \ln x + \ln|u^2+1| = c$$

$$\Rightarrow \ln(x(u^2+1)) = c$$

$$e^{\ln(x(u^2+1))} = e^c$$

$$x(u^2+1) = e^c = k \ (\text{const})$$

$$xu^2 + x = k$$

$y(1) = 1$

here no y, $\rightarrow u = y/x$.

$$x\left(\frac{y}{x}\right)^2 + x = k.$$

$$x\left(\frac{y^2}{x^2}\right) + x = k$$

$$\frac{y^2}{x} + x = k.$$

$y(1)=1$, $y=1$, $x=1$

$$1 + 1 = k \Rightarrow 2 = k.$$

$$\Rightarrow 2 = e^c \Rightarrow \boxed{\ln 2 = c} \quad \text{--- (2)}$$

(2) in (1)

$$\ln x = -\ln|u^2+1| + \ln 2.$$

homogenous

meaning,

(1) $\frac{dy}{dx} = \frac{y^2+x^2}{2+x}$

(2) $\frac{y'}{x'}$

(3) $\frac{y-x'}{x^2}$

(4) $\frac{y'x+x'}{x^2}$

(5) $\frac{x^2-xy}{x^2+y^2}$

(6) $\frac{x^2+xy+y^2}{x^2}$

$\Rightarrow$ Type II

$$\boxed{\frac{dy}{dx} = \frac{ax+by+l}{k(ax+by)+m}}$$

Above eq can be converted into separable eq by substituting

$$\boxed{ax+by = u}$$

=> Linear differential eq =

Standard form of a Linear eq —

$$\boxed{\frac{dy}{dx} + P(x)y = f(x)}$$

This form LDE

x^2, y^2

Both ( )s $p(x)$ & $f(x)$ are continous

eg → 1) $\frac{dy}{dx} + y\underbrace{\tan x}_{x \text{ term } p(x)} = \cos^3 x$. ✓

2) $x\frac{dy}{dx} - 4y = x^6 e^x$ ✗

($\frac{dy}{dx}$)

÷ throughout by x,

$$\frac{dy}{dx} - \frac{4}{x}y = \frac{x^6 e^x}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \checkmark$$

$p(x) = -4/x$, $f(x) = x^5 e^x$

def —

* A 1^{st} order DE of the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)} \text{ — (1)}$$

(general)

is said to be linear eq in the dependent variable 'y'.

$g(x) = 0 \Rightarrow$ homogenous
or
$g(x) \neq 0$, non-homogenous

* when $g(x) = 0$, the linear eq (1) is said to be homogenous otherwise it is non-homogenous.

→ Standard form :-

The standard form of a linear eq

$$\boxed{\frac{dy}{dx} + p(x)y = f(x)} \text{ — (2)}$$

where $p(x)$ & $f(x)$ are continous.

General Sol^n :

The DE (2) has the property that its sol^n is the sum of 2 sol^n,

$$\boxed{y = y_c + y_p} \text{ — (3)}$$

y_c & y_p } 2 sol^n.

where y_c is the sol^n of the associated homogenous eq

$$\frac{dy}{dx} + p(x)y = 0.$$

standard form — homogenous — last 0 — (i.e) $\frac{dy}{dx} + p(x)y = 0$

$y_c = c\,y_1(x)$

$$\boxed{y_c = c\,e^{-\int p(x)dx}} \text{ — (4)}$$

$\frac{dy}{dx} + p(x)y = 0$
y_c

next we have to find a particular sol^n (P) y_p by a procedure → variation of paramters

$y_p = u(x)\,y_1(x)$

$$\boxed{y_p = e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)\,dx} \text{ — (5)}$$

Sub (4) & (5) in (3)

~~$y = y_c + y_p =$~~

$$\boxed{y = y_c + y_p = c\,e^{-\int p(x)dx} + e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)\,dx}$$

(general form of a DE — write this full)

=> Method of sol^n :-

Procedure to solve 1^{st} order DE →

1) write the gvn DE in standard form. Identify $p(x)$ & $f(x)$

2) find the integrating factor (IF).

$e^{\int p(x)dx}$

3) Multiply both side of the DE by integrating factor.

(i.e) $e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}f(x)$

p. rule ..., so

$$\frac{d}{dx}\left(y e^{\int p(x)dx}\right) = e^{\int p(x)dx}f(x)$$

integrating on both side,

$$y e^{\int p(x)dx} = \int e^{\int p(x)dx} f(x)dx + c$$

$$y = \frac{\int e^{\int p(x)dx} f(x)dx + c}{e^{\int p(x)dx}}$$

$$y = e^{-\int p(x)dx}\int e^{\int p(x)dx} f(x)dx + c$$

Q

1) Solve $\frac{dy}{dx} + y\tan x = \cos^3 x$ — ①

if $3\frac{dy}{dx} + y\tan x = \cos^3 x$ → ÷ 3 throughout

A) ① standard form,

$$\frac{dy}{dx} + y\tan x = \cos^3 x.$$

$p(x) = \tan x.$, $f(x) = \cos^3 x$, (both p(x), f(x) are continous.

② integrating factor, $e^{\int p(x)}$

$$e^{\int p(x)dx} = e^{\int \tan x\, dx}$$

$$= e^{\ln|\sec x|}$$

$$= |\sec x|$$

$\int \tan x = \ln|\sec x|$

$\cos^2 x = \frac{1+\cos 2x}{2}$

IF ⇒ $|\sec x|$

③ multiply IF by $e^{\int p(x)dx} \rightarrow \sec x$.

$$\sec x \frac{dy}{dx} + y \tan x \sec x = \sec x \cos^3 x.$$

① deri-② + ② deri of ①.

$$\frac{d}{dx}(y \cdot \sec x) = \sec x \cos^3 x$$

$$= \frac{1}{\cos x} \cos^3 x. = \cos^2 x.$$

$$\frac{d}{dx}(y \sec(x)) = \cos^2 x.$$

$$= \frac{1 + \cos 2x}{2}. = \frac{1}{2} + \frac{\cos 2x}{2}.$$

$\int$ integ,

$$y \sec x = \int \left(\frac{1}{2} + \frac{\cos 2x}{2}\right) dx.$$

$$= \int \frac{1}{2} + \frac{1}{2} \int \cos 2x \cdot dx.$$

$$= \frac{1}{2} x + \frac{1}{2} \frac{\sin 2x}{2}.$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x.$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + c.$$

$\therefore$, $$y \sec x = \frac{1}{2} x + \frac{1}{4} \sin 2x + c$$

⇒ Error function :- (erf)

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt}$$

→ Complimentry error () :- (erfc)

$$\boxed{erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt}$$

$$\boxed{erf(x) + erfc(x) = 1}$$

1) Solve the intial value probm.

$$y' + 6x^2 y = \frac{e^{-2x^3}}{x^2}, \quad y(1) = 0.$$

A)

① 1st check stand form → $\frac{dy}{dx} + p(x)y = f(x)$.

✓. $y' + 6x^2y = \underbrace{\frac{e^{-2x^3}}{x^2}}_{\text{only } x \text{ term}}$

$p(x) = 6x^2$

$f(x) = \frac{e^{-2x^3}}{x^2}$

(integrating factor) ② $IF = e^{\int p(x)dx} = e^{\int 6x^2 dx} = e^{\frac{6x^3}{3}} = e^{2x^3}$

[p(x), f are contin on $(-\infty, 0)$ & $(0, \infty)$]

③ × IF,

$$\underbrace{e^{2x^3} \cdot y'}_{\text{p rule}} + \underbrace{y \cdot 6x^2 \cdot e^{2x^3}}_{\text{p rule.}} = \frac{e^{-2x^3}}{x^2} \times e^{2x^3}$$

$$= \frac{1}{x^2}$$

$$\frac{d}{dx}\left[e^{2x^3} y\right] = \frac{1}{x^2}$$

$\int$ on both sides,

$$e^{2x^3} y = \int \frac{1}{x^2}\, dx.$$

$$e^{2x^3} y = -\frac{1}{x} + c \quad \text{—①}$$

$\frac{1}{x^2} = x^{-2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$

to find c, $y(1) = 0$ ②.

$$e^{2x^3} y = -\frac{1}{x} + c$$

$$c = e^{2x^3} y + \frac{1}{x}$$

gvu $y(1) = 0$ ⇒ $e^2 \cdot 0 + 1 = 0 + 1 = 1$.
$x=1, y=0$

∴ $c = 1$ —③

Sub-③ in -①

$$e^{2x^3} y = -\frac{1}{x} + 1.$$

$$y = \frac{-1}{e^{2x^3} x} + \frac{1}{e^{2x^3}}$$

⇒ Bernoulli's eq :-

① → $$\boxed{\frac{dy}{dx} + p(x)y = f(x)y^a}$$

$a \to 1, 2 \ldots$ [from eq ... B. eq]

p(x) & f(x) are ()s & a is any real no –
is → B. eq.

Note
eq →

* if $a = 0$ or $a = 1$, the above eq is linear
otherwise non-linear.

* ÷ing eq-① by y^a.

$$\Rightarrow \frac{dy}{dx} \frac{1}{y^a} + p(x) \frac{y}{y^a} = f(x).$$

$$\Rightarrow y^{-a} \frac{dy}{dx} + p(x)\, y' y^{-a} = f(x).$$

$$\Rightarrow \boxed{y^{-a} \frac{dy}{dx} + p(x)\, y^{1-a} = f(x)} \quad \text{—②}$$

Put $u = y^{1-a}$ then

$$\frac{du}{dx} = (1-a)\, y^{1-a-1} \times \frac{dy}{dx}$$

$$\frac{du}{dx} = (1-a)\, y^{-a} \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{1-a} \frac{du}{dx} = y^{-a} \frac{dy}{dx} \quad \text{—③}$$

Sub —③ in —②

$$y^{-a} \frac{dy}{dx} + p(x) y^{1-x} = f(x).$$

$$\frac{1}{1-a} \frac{du}{dx} + p(x) u = f(x)$$

× throughout by $1-a$,

$$\boxed{\frac{du}{dx} + (1-a)\, p(x)\, u = \underbrace{(1-a) f(x)}_{\text{constant}}}$$

standard form.

2) Solve $x \frac{dy}{dx} - 4y = x^6 e^x$.

A) Stand eq → $\frac{dy}{dx} + p(x) y = f(x)$.

$$x \frac{dy}{dx} - 4y = x^6 e^x.$$

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$$p(x) = -\frac{4}{x}, \quad f(x) = x^5 e^x.$$

$p(x)$ & $f(x)$ are contin.. on $(0, \infty)$

$$IF = e^{\int p(x) dx} = e^{\int -\frac{4}{x} dx} = e^{-4 \int \frac{1}{x} dx}$$

$$= e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4}.$$

× IF throughout,

$$x^{-4} \frac{dy}{dx} - 4x^{-5} y = x e^x$$

$$\frac{d}{dx}\left[x^{-4} y \right] = x e^x.$$

$\int$ both sides of above eq,

$$x^{-4} y = \int x e^x dx = x e^x - \int e^x dx.$$

$$= x e^x - e^x + c.$$

× throughout by x^4.

$$y = x^5 e^x - x^4 e^x + c x^4$$

3) Solve $y\, dx - x\, dy + \ln x\, dx = 0$.

A) $y\, dx - x\, dy + \ln x\, dx = 0$

$$\Rightarrow y - x \frac{dy}{dx} + \ln x = 0.$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} \ln x$$

$$p(x) = -\frac{1}{x} \qquad f(x) = \frac{\ln x}{x}.$$

both p & f are contin.. $(0, \infty)$

$$IF = e^{\int p(x) dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

× IF on both sides,

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x} \times \frac{1}{x} = \frac{\ln x}{x} \times \frac{1}{x}.$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{x} \times y \right] = \frac{\ln x}{x^2}$$

$\int$ on both sides,

$$\frac{1}{x} y = \int \frac{\ln x}{x^2} \, dx.$$

$$= \frac{1}{x} \int \frac{\ln x}{x} \, dx.$$

put $\ln x = t$
$e^{\ln x} = e^t$
$x = e^t$
$\frac{dx}{x} = dt$

$$= \int e^{-t} \, t \, dt.$$

$$= t(-e^{-t}) - \int 1 \times (-e^{-t}) + c.$$

$\int$ by parts,

$$= -t e^{-t} - e^{-t} + c$$

$$= -e^{-t}(1+t) + c.$$

$$= \frac{-1}{x}(1 + \ln x) + c.$$

$\times$ throughout by x,

$$\boxed{y = cx - (1 + \ln x)}$$

D) Solve $xy' + y = xy^3$

1st check bernolli.

A) $\div$ throughout by y^3,

$$\frac{x}{y^3} \frac{dy}{dx} + \frac{y}{y^3} = x$$

$$x y^{-3} \frac{dy}{dx} + y^{-2} = x. \quad \text{rule.} \quad \text{— (1)}$$

put $u = y^{-2}$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$\frac{-1}{2} \frac{du}{dx} = y^{-3} \frac{dy}{dx}$$

$$\Rightarrow -\frac{x}{2} \cdot \frac{du}{dx} + u = x$$

[illegible] statard form eq
[illegible] $\rightarrow \frac{dy}{dx} + P(y) = f(x)$.

$\times \frac{-2}{x}$ throughout.

$$\frac{du}{dx} - \frac{2u}{x} = -2 \quad \text{— (2)}$$

eq – (2) is in standard form,

[illegible]

$\therefore P(x) = \frac{-2}{x}$, $f(x) = -2$

$$IF = e^{\int P(x) dx} = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}$$

$\times$ IF with eq – (2)

$$\Rightarrow x^{-2} \frac{du}{dx} - \frac{2}{x} u x^{-2} = -2x^{-2}$$

$$\frac{1}{x^2} \frac{du}{dx} - \frac{2u}{x} \frac{1}{x^2} = \frac{-2}{x^2}$$

$$\frac{1}{x^2} \frac{du}{dx} - \frac{2u}{x^3} = \frac{-2}{x^2}$$

(p rule)

$$\frac{d}{dx}\left[\frac{1}{x^2} \times u\right] = \frac{-2}{x^2}$$

$\int$,

$$\frac{1}{x^2} \times u = \int \frac{-2}{x^2} \, dx.$$

$$\frac{1}{x^2} u = -2 \int \frac{1}{x^2} \, dx. = -2 \frac{x^{-2+1}}{-2+1} + c$$

$$= -2 \frac{x^{-1}}{-1} + c$$

$$= \frac{2}{x} + c$$

[illegible]

$$\frac{1}{x^2} y^{-2} = \frac{2}{x} + c$$

$$y^{-2} = \frac{2x^2}{x} + cx^2$$

$$y^{-2} = 2x + cx^2$$

2) Solve $y(2xy+e^{x})dx - e^{x}\,dy = 0$,

(1) $\frac{dy}{dx}$ [illegible].

(2) general form, $\frac{dy}{dx} + p(x)y = f(x)$.

A) $y(2xy+e^{x})dx = e^{x}dy$

$\Rightarrow 2xy^2 + e^{x}y = e^{x}\frac{dy}{dx}$

$e^{x}\frac{dy}{dx} - e^{x}y = 2xy^2$

$\longrightarrow$ Bernoulli eq ✓

$\div$ th.out by y^2,

$\frac{e^{x}}{y^2}\frac{dy}{dx} - \frac{e^{x}y}{y^2} = 2x$

$e^{x}y^{-2}\frac{dy}{dx} - e^{x}y^{-1} = 2x$

put $u = y^{-1}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{du}{dx} = y^{-2}\frac{dy}{dx}$

$-e^{x}\frac{du}{dx} - e^{x}u = 2x$.

$\div$ th.out by $-e^{x}$,

$\frac{du}{dx} + u = \frac{-2x}{e^{x}}$ — (1) ✓

$\longrightarrow$ Standard form

$\therefore p(x) = 1$, $f(x) = \frac{-2x}{e^{x}}$

$IF = e^{\int p(x)dx} = e^{\int 1 dx} = e^{x}$,

$\times$ IF, on — (1)

$e^{x}\frac{du}{dx} + e^{x}u = \frac{-2x}{\cancel{e^{x}}} \times \cancel{e^{x}}$

$e^{x}\frac{du}{dx} + ue^{x} = -2x$

$\frac{d}{dx}$ P-rule $\to e^{x}\times u$.

$\Rightarrow \frac{d}{dx}\left[e^{x}\times u\right] = -2x$.

$\int$, $e^{x}u = \int -2x$.

$= -2\int x = -\cancel{2}\frac{x^2}{\cancel{2}} = -x^2 + c$

we know, $u = y^{-1}$.

$\therefore e^{x}y^{-1} = -x^2 + c$.

$y^{-1} = \frac{-x^2}{e^{x}} + \frac{c}{e^{x}}$

$y^{-1} = \frac{1}{e^{x}}(-x^2+c)$

⇒ Exact DE :-

A necessary condition that a DE,

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

$dx \to M$ $\quad dy \to N$

be exact is that,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

([illegible] eq, ~~exact DE~~ exact [illegible])

1) Determine whether the gun eq is exact,

a) $(2x-1)dx + (3y+7)dy = 0$

A) $M = 2x-1$, $N = 3y+7$

$\frac{\partial M}{\partial y} = 0$ $\quad \frac{\partial N}{\partial x} = 0$

clearly $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ✓ $\therefore$ it is exact

b) $(5x+4y)dx + (4x-8y^3)dy = 0$.

A) $M = 5x+4y$, $N = 4x-8y^3$

$\frac{\partial M}{\partial y} = 4$ $\quad \frac{\partial N}{\partial x} = 4$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(Exact)

3) $xy' + y + 4 = 0$

A) convert this to $M\,dx + N\,dy = 0$.,

$$x\frac{dy}{dx} + y + 4 = 0.$$

$$x\frac{dy}{dx} = -y - 4$$

$$x\,dy = (-y-4)\,dx \Rightarrow \cancel{x\,dy + y + 4dx = 0}$$

$$\Rightarrow (-y-4)\,dx - x\,dy = 0$$

$\therefore M = -y-4$, $N = -x$

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{exact}$$

4) find the value of k, so that the given DE, $(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$ is exact?

A) $M = y^3 + kxy^4 - 2x$. $\qquad N = 3xy^2 + 20x^2y^3$

$$\frac{\partial M}{\partial y} = 3y^2 + 4kxy^3 \qquad \frac{\partial N}{\partial x} = 3y^2 + 20\times 2x\,y^3 = 3y^2 + 40xy^3$$

(comparing a-ans)

$$\therefore \cancel{3y^2} + 4kxy^3 = \cancel{3y^2} + 40xy^3$$

$$4k\cancel{xy^3} = 40\cancel{xy^3}$$

$$4k = 40$$

$$k = \frac{40}{4} = 10$$

⇒ **Method of Solⁿ of exact DE :-**

If $M\,dx + N\,dy = 0$ is exact, it can be solved by following method —

① 1st $\int$-ate M w.r.t x, regarding 'y' as a constant, ② then $\int$-ate N w.r.t y those terms in N which donot involve x.

③ The sum of 2 exp, thus obtained equated to a constant is the required solⁿ. $(\ldots + \ldots = c)$

In other words the solⁿ of an exact DE –

$$\boxed{\underset{y\text{-constant}}{\int M\,dx} + \underset{\text{terms not having } x}{\int N\,dy} = c}$$

1) S.T the eq $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$ is exact & solve it?

A) $M = 1 + 4xy + 2y^2$ $\qquad N = 1 + 4xy + 2x^2$

$$\frac{\partial M}{\partial y} = 4x + 4y \qquad \frac{\partial N}{\partial x} = 4y + 4x.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{exact}$$

Solve

1) $\int M = \int 1 + 4xy + 2y^2\,dx$

$= x + 2x^2y + 2y^2x$ — ①

② $\int$-ate N w.r.t y donot involve x. (i.e)

$N = \underbrace{1}_{\text{no } x \text{ terms}} + 4xy + 2x^2 \rightarrow \int 1\,dy = y$ — ②

hence solⁿ of DE from ① & ② is,

$$x + 2x^2y + 2y^2x + y = c$$

2) S.T $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$ is exact & solve it?

A) $M = 2xy + y - \tan y$ $\qquad N = x^2 - x\tan^2 y + \sec^2 y$

$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y$ $\qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$

(a -ൽ exact ആണോ എന്ന് നോക്കണം.)

$\cancel{\tan^2 y - \sec^2 y = 1}$

$\tan^2 y = 1 + \sec^2 y$

$\sec^2 y - \tan^2 y = 1$

$-\tan^2 y = 1 - \sec^2 y$

$\cancel{\sec^2 y = 1 + \tan^2 y}$

$\cancel{\frac{\partial N}{\partial x} = 2x - (1 + \tan^2 y)}$

$\therefore \frac{\partial N}{\partial x} = 2x - \tan^2 y$

$= 2x + 1 - \sec^2 y$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$2u - \tan^2 y$

$2u - (\sec^2\theta - 1)$

$2u - \sec^2\theta + 1$

$2u + 1 - \sec^2\theta$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

Solve

$\int M dx = \int 2xy + y - \tan y \, dx$

$= x^2 y + yx - x\tan y$ — ⓐ

$N = x^2 - x\tan^2 y + \sec^2 y + 2$

$\Rightarrow \sec^2 y + 2$

$\Rightarrow \int \sec^2 y + 2 \, dy$

$\Rightarrow \tan y + 2y$ — ⓑ

from ⓐ & ⓑ

$x^2 y + yx - x\tan y + \tan y + 2y = c$

3) Solve $(3x^2 y + e^y)dx + (x^3 + xe^y - 2y)dy = 0$

A) $M = 3x^2 y + e^y$ $\qquad N = x^3 + xe^y - 2y$

$\frac{\partial M}{\partial y} = 3x^2 + e^y$ ... (1) $\qquad \frac{\partial N}{\partial x} = 3x^2 + e^y$ ... (2)

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

$\int M dx = \int 3x^2 y + e^y dx = x^3 y + xe^y$

$N = x^3 + xe^y - 2y$

$\therefore \Rightarrow \int -2y = -2\int y = -2 \times \frac{y^2}{2} = -y^2$

$\therefore x^3 y + xe^y - y^2 = c$

Sometime a gvn DE $M(x,y)dx + N(x,y)dy = 0$ — (1)

$\cancel{\rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$ ~~exact~~

is not exact, bt it can be exact by multipling it by a suitable ( ) & this ( ) → IF of eq — (1)

1) $\boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$ is a ( ) of x along, then $e^{\int f(x)dx}$ is an IF. of — (1)

2) $\boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$ is a ( ) of y along, then $e^{\int f(y)dy}$ is an IF of — (1)

Q

1) Solve DE: $(x^2 - 2x + 2y^2)dx + 2xy \, dy = 0$ — (1)

A) $M = x^2 - 2x + 2y^2$ $\qquad N = 2xy$

$\frac{\partial M}{\partial y} = 4y$ $\qquad \frac{\partial N}{\partial x} = 2y$

not exact

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(4y - 2y)$

$= \frac{1}{2xy}(2y) = \frac{1}{x}$ (only x)

$\therefore \; f(x) = \frac{1}{x}$ (form of x alone)

$\therefore$ IF is,

$e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

× ⓐ in ①,

$x\left(x^2 - 2x + 2y^2\right) dx + x(2xy)dy = 0.$

$\left(x^3 - 2x^2 + 2y^2 x\right) dx + \left(2x^2 y\right) dy = 0$ —②

exact DE

Now solve,

$M_1 = x^3 - 2x^2 + 2y^2 x \quad , \quad N_1 = 2x^2 y$

$\int M_1 \, dx = \int x^3 - 2x^2 + 2y^2 x \, dx.$

$= \frac{x^4}{4} - 2\frac{x^3}{3} + 2y^2 \frac{x^2}{2}$

$= \frac{x^4}{4} - \frac{2x^3}{3} + y^2 x^2 \ldots$

$N_1 = 2x^2 y$ w.r.t x, is 0

no x, so 0

$\therefore$ Sol^n is,

$\frac{x^4}{4} - 2\frac{x^3}{3} + y^2 x^2 = C$

2) Solve: $\left(xy^3 + y\right) dx + 2\left(x^2 y^2 + x + y^4\right) dy = 0.$ —①

$\rightarrow$ $M = xy^3 + y \qquad N = 2x^2y^2 + 2x + 2y^4$

$\frac{\partial M}{\partial y} = 3xy^2 + 1 \qquad \frac{\partial N}{\partial x} = 4xy^2 + 2$

not exact

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x^2y^2 + 2x + 2y^4}\left(\left(3xy^2 + 1\right) - \left(4xy^2 + 2\right)\right)$

$= \frac{1}{2x^2y + 2x + 2y^4}\left(-xy^2 - 1\right)$ not only x, y also there so ×

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3 + y}\left(\left(4xy^2 + 2\right) - \left(3xy^2 + 1\right)\right)$

$= \frac{1}{xy^3 + y}\left(xy^2 + 1\right) = \frac{\cancel{xy^2+1}}{y\left(\cancel{xy^2+1}\right)} = \frac{1}{y}$

$\therefore f(y) = \frac{1}{y}$ (y alone)

$\therefore$ IF is,

$e^{\int f(y)dy} = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$ —ⓐ

× ⓐ by ①

$y\left(xy^3 + y\right) dx + \left(2x^2y^2 + 2x + 2y^4\right) y \, dy = 0.$

$\Rightarrow \left(xy^4 + y^2\right) dx + \left(2x^2y^3 + 2xy + 2y^5\right) dy = 0.$ $\rightarrow N_1$

exact ✓

Solve,

$\int M_1 \, dx = \int xy^4 + y^2 \, dx = \frac{x^2}{2} y^4 + xy^2.$

$N_1 = 2x^2y^3 + 2xy + 2y^5$ w.r.t x.

$\Rightarrow \int 2y^5 = 2\int y^5 = 2\frac{y^6}{6}$

$\therefore$ Sol^n,

$y^4 \frac{x^2}{2} + xy^2 + 2\frac{y^6}{6} = C.$

⇒ Euler's Method =

In this method we use the property that in a small interval a curve can be treated as a straight line & apply eq of the tangent at (x_0, y_0), then continue the procedure of using successive tangent line → E. Method

$\boxed{y_1 = y_0 + h\, f(x_0, y_0)}$ $\boxed{y_2 = y_1 + h\, f(x_1, y_1)}$ . . . .

(f → slope of the curve).

$\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$

[2 ans (error) [illegible]] (for f → differentiate, h → difference.)

1) Apply E. method to solve $y' = x + y$, $y(0) = 0$, choosing $h = 0.2$ & computing y_1, y_2, y_3, y_4, y_5 ?.

A)

$x_n = x_0 + nh$ —①

$h = 0.2$

$y(0) = 0$

$x_0 = 0,\ y_0 = 0$

$y_1 = y_0 + h\, f(x_0, y_0)$

$y(0) = 0 \Rightarrow x_0 = 0,\ y_0 = 0.$

$f(x_0, y_0) = y' = x_0 + y_0$

$f(x, y) = x + y$

$f(x_0, y_0) = x_0 + y_0$

∴ $y_1 = y_0 + h\, f(x_0, y_0)$

$x_0 = 0,\ y_0 = 0$

$x_0 + y_0 = 0 + 0 = 0$

$= 0 + 0.2(x_0 + y_0) = \cancel{0.2x + 0.2y}$

$y_1 = 0$

by ①.

$x_1 = x_0 + nh$

$= 0 + 1 \times 0.2 = 0.2$

$f(x, y) = x + y$

$f(x_1, y_1) = x_1 + y_1$

$= 0.2 + 0$

$= 0.2$

∴ $y_2 = y_1 + h\, f(x_1, y_1)$

$= 0 + 0.2 \times 0.2$

$= 0.04$

by ①

$x_2 = x_0 + nh$

$= 0 + 2 \times 0.2 = 0.4$

$x_2 + y_2$

$= 0.4 + 0.04$

$= 0.44$

∴ $y_3 = y_2 + h\, f(x_2, y_2)$

$= 0.04 + 0.2 \times 0.44$

$= 0.128$

by ①

$x_3 = x_0 + nh = 0 + 3 \times 0.2 = 0.6$

∴ $y_4 = y_3 + h\, f(x_3, y_3)$

$x_3 + y_3$

$= 0.6 + 0.128$

$= 0.728$

$= 0.128 + 0.2 \times 0.728$

$= 0.2736$

by ①

$x_4 = x_0 + nh = 0 + 4 \times 0.2 = 0.8$

∴ $y_5 = y_4 + h\, f(x_4, y_4)$

$x_4 + y_4$

$= 0.8 + 0.2736$

$= 1.0736$

$= 0.2736 + 0.2 \times$

$= 0.488$