0.8 : Conditional probability

let A and B be any 2 events, the (prob) of the event A given that the event B denoted by $P(A|B)$ is given by defined as.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad P(B) \neq 0$$

≈ the conditional (prob) of B given A is defined as,

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}} \quad P(A) \neq 0.$$

* Rmrk →

1) For $P(B) > 0$, $P(A|B) \leq P(A)$
2) $P(A|B)$ is not defined if $P(B) = 0$
3) $P(B|B) = 1$

→ Therum :- For a fixed B with $P(B) > 0$, $P(A|B)$ is a (prob) ( ) (or (prob) measure)

by using 3 Axioms,

1) $0 \leq P(A) \leq 1$ → $0 \leq P(A|B) \leq 1$
2) $P(S) = 1$ → $P(S|B) = 1$
3) $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) \dots + P(A_n)$.

1) To prove $P(A|B) \geq 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0 \quad \therefore P(A|B) \geq 0 //$$

2) To prove $P(S|B) = 1$

$$P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3) To prove $P(A \cup C/B) \overset{(split)}{=} P(A/B) + P(C/B)$

$$= \frac{P(A \cup C) \cap B}{P(B)}$$

$$= \frac{P[(A \cap B) \cup (C \cap B)]}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(C \cap B)}{P(B)}$$

$$= P(A|B) + P(C|B)$$

∴ Conditional (prob) satisfies all the axioms of (prob),

∴ $P(A|B)$ is (prob) ( ).

→ Multiplication Law of (prob) :-

Therum → For any 2 events A and B $P(A) > 0$, $P(B) > 0$.

$$\boxed{\begin{aligned} P(A \cap B) &= P(A) \cdot P(B|A), \\ P(A \cap B) &= P(B) \cdot P(A|B) \end{aligned}}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Pf

N = total sample point

N_A = No. of favourable cases in event A.

N_B = No. ... f. cases in event B

N_{AB} = No. .. f cases in event $A \cap B$.

$$P(A) = \frac{N_A}{N} \quad P(B) = \frac{N_B}{N} \quad P(A \cap B) = \frac{N_{AB}}{N}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{N_{AB}/N}{N_B/N} = \frac{N_{AB}}{N_B} //$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{N_{AB}/N}{N_A/N} = \frac{N_{AB}}{N_A}$$

$P(A \cap B)$ = we can write this as $\frac{N_{AB}}{N} \Rightarrow \frac{N_{AB}}{N_A} \cdot \frac{N_A}{N} \to P(B/A) \cdot P(A)$

on thought

$$P(A \cap B) = \frac{N_{AB}}{N_B} \cdot \frac{N_B}{N}$$

$$P(A \cap B) = P(A/B) \cdot P(B)$$

(want to get $P(B/A) \cdot P(A)$ and $P(A/B) \cdot P(B)$ so we write like this)

Note

For 3 events → A, B, C

$$P(A \cap B \cap C) = P(A/B \cap C) \cdot P(B \cap C) \text{ by (1)}$$

$$= P(A/B \cap C) \cdot P(B/C) \cdot P(C).$$

in 2 events,

$P(A \cap B) = P(B) \cdot P(A/B)$ — (1)

→ **Independent events :-**

An event A is said to be independent of event B, then

$P(A/B) = P(A)$ (A, B independent)

$P(B/A) = P(B)$.

hence, the events A & B are independent if, $P(A \cap B) = P(A) \cdot P(B/A)$

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Pairwise and mutual independent (I) :-

A set of events $A_1, A_2 \ldots A_n$ are said to be pairwise (I) if every pair of diff. event are (I). (i.e)

$P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ ∀ i and j, i≠j

* A set of events $A_1 A_2 \ldots A_n$ are said to be mutual (I) if

$P(A_i \cap A_j \cap \ldots \cap A_r) = P(A_i) \cdot P(A_j) \ldots P(A_r)$ ∀ every subset of $(A_i A_j \ldots A_r)$ of $A_1, A_2 \ldots A_r$.

→ multiplication Theorem (Independent events) :-

If A and B 2 (I) events

$$P(A \cap B) = P(A) \cdot P(B).$$

we've for any 2 events A & B.

$$P(A \cap B) = P(A) \cdot P(B/A)$$

$$= P(A) \cdot P(B) \quad \therefore (P(B/A) = P(B))$$

Note →

If A & B are independent the addition theorem can be stated as

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B).$$

Q 1) If A & B are 2 events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{8}$

find $P(A/B)$ and $P(A/B')$?

A) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/4} = \frac{1}{8} \cdot \frac{4}{1}$

$= \frac{1}{2}$

$$P(A/B') = \frac{P(A \cap B')}{P(B')}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{1/3 - 1/8}{1 - \frac{1}{4}} = \frac{\frac{8-3}{24}}{\frac{4-1}{4}}$$

$$= \frac{5/24}{3/4} = \frac{5}{24} \cdot \frac{4}{3} = \frac{5}{18}$$

2) Let A & B be 2 events associated with an expt & suppose $P(A) = 0.5$ while $P(A \text{ or } B) = 0.8$ [$P(A \cup B)$]. Let $P(B) = p$ for wht values of p are

(a) A and B mutually exclusive

(b) A and B are (I).

$p = ?$

A) a) $\boxed{\text{mutually exclusive} \rightarrow A \cap B = \phi}$ (disjoint · no common).

$P(A \text{ or } B) = P(A \cup B) = 0.8$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 0$

$\boxed{\begin{array}{l} \text{or} \rightarrow \cup \\ \text{and} \rightarrow \cap \end{array}}$

$\boxed{\begin{array}{l} A \cap B = \phi \\ \therefore P(A \cap B) = 0 \end{array}}$

$p = ?$

$0.8 = 0.5 + p$

$p = 0.8 - 0.5 = \underline{0.3}$

b) $P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}\ P(A) \cdot P(B)$ (A & B are independent)

$p = ?$

$0.8 = 0.5 + p - 0.5 \cdot p$

$0.8 - 0.5 = p - 0.5p$

$0.3 = p(1 - 0.5)$

$0.3 = p(0.5)$

$p = \frac{0.3}{0.5} = \underline{\underline{0.6}}$

→ **Baye's Theorum:-**

Let S be a sample space partitient into n mutually exclusive events $B_1, B_2 \ldots B_n$ such that $P(B_i) > 0$, $i = 1, 2, \ldots n$

Let A be any event of test $P(A) \neq 0$ then the (prob) for the event B_i $(i = 1, 2 \ldots n)$ gn the event A is,

$$\boxed{P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)}}$$

just in more than 1 condition [illegible] choose ... like...

eg→ $i=3$ $P(B_3|A) = \frac{P(B_3) \cdot P(A|B_3)}{P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3)}$

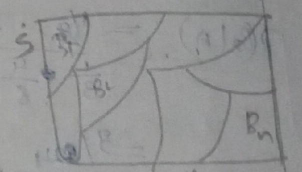
Pf $P(B_i|A) = \frac{P(A \cap B_i)}{P(A)}$

$= \frac{P(B_i) \cdot P(A|B_i)}{P(A)}$ — (1)

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(A) \cdot P(B|A)$

$(=) = P(B) \cdot P(A|B)$



$S = \bigcup_{i=1}^{n} B_i$

$A = A \cap S$

$= A \cap \left(\bigcup_{i=1}^{n} B_i\right)$

$A = \bigcup_{i=1}^{n} A \cap B_i$

$\sum$ (outside) $\cup$

$P(A) = P\left(\bigcup_{i=1}^{n} A \cap B_i\right)$

$= \sum_{i=1}^{n} P(A \cap B_i)$

$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)$ — (2)

Rule (2) in (1)

$$P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)}$$

Q) The contents of 3 urns are given below.

Urn I — 4 black balls, 4 red balls

Urn II → 3 blck " , 5 red "

Urn III → 5 blck " , 3 red "

An urn is chosen at random & a ball is drawn from it. If the choosen ball is red find the (prob) that it is from Urn III ?

A)

B_1 = Urn I

B_2 = Urn II

B_3 = Urn III

A = Choosen ball is red.

we've to find $P(B_3|A)$

$$P(B_3|A) = \frac{P(B_3)\cdot P(A|B_3)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)} \quad -①$$

$P(B_1) = \frac{1}{3}$ $P(B_2) = \frac{1}{3}$ $P(B_3) = 1/3$

$P(A|B_1) = \frac{4}{8}$ $\frac{4 \text{ R ball}}{4+4}$ $P(A|B_2) = 5/8$ $P(A|B_3) = 3/8$

Sub ①

$$P(B_3|A) = \frac{1/3 \cdot 3/8}{\frac{1}{3}\cdot\frac{4}{8} + \frac{1}{3}\cdot\frac{5}{8} + \frac{1}{3}\cdot\frac{3}{8}} = \frac{3/24}{\frac{4}{24}+\frac{5}{24}+\frac{3}{24}}$$

$$= \frac{3/24}{12/24} = \frac{3}{24}\cdot\frac{24}{12}$$

$$= \frac{1}{4}$$

2) 3 machines A, B, C produce 60, 30, 10 respectivly of total production of a factory it is estimated that A produces 2% deffective, B produces 3% (d) & C produces 4% (d) in theis production. an item chosen randomly from the total production is found to be deffective. wht is the (prob) that it has come from machine A ? or machine B ?, or machine C ?.

A)

B_1 → machine A

B_2 → M. B

B_3 → M. C

A → item drawn from the production is deffective.

A →

$$P(B_1|A) = \frac{P(B_1)\cdot P(A|B_1)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)} \quad ①$$

$P(B_1) = \frac{60}{100} \rightarrow \%$ $= 0.60$. $P(B_2) = \frac{30}{100} = 0.30$

$P(B_3) = \frac{10}{100} = 0.10$.

$P(A|B_1) = \frac{2}{100} = 0.02$ $P(A|B_2) = \frac{3}{100} = 0.03$

$P(A|B_3) = \frac{4}{100} = 0.04$.

Sub ①

$$P(B_1|A) = \frac{0.60 \cdot 0.02}{0.60\cdot 0.02 + 0.30\cdot 0.03 + 0.10\cdot 0.04}$$

$$= \frac{0.012}{0.012 + 0.009 + 0.004} = \frac{0.012}{0.025}$$

$$= 0.48$$

$$P(B_2|A) = \frac{P(B_2)\cdot P(A|B_2)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)}$$

$$= \frac{0.30 \cdot 0.03}{0.025} = \frac{0.009}{0.025}$$

$$= 0.36$$

$$P(B_3|A) = \frac{0.10 \cdot 0.04}{0.025} = 0.16$$

→ Theorem (on total probability):-

Let $B_1, B_2 \ldots B_n$ be a partition of the sample space 'S' into n mutually exclusive events such that $P(B_i) > 0$ for $i = 1, 2, \ldots n$. Then for any arbitrary event A.

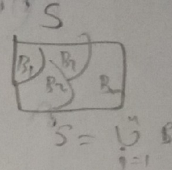
$$\boxed{P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)}$$

Pf/

$S = \bigcup_{i=1}^{n} B_i$

$A \cap S \to A$

$A \cup S \to S$



$A = A \cap S$

$= A \cap \left(\bigcup_{i=1}^{n} B_i\right)$

$A = \bigcup_{i=1}^{n} A \cap B_i$

$P(A \cap B) = P(A) \cdot P(B/A)$

$= P(B) \cdot P(A/B)$

$P(A) = P\left(\bigcup_{i=1}^{n} A \cap B_i\right)$

$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$

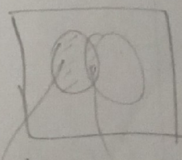
$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$

(Total prob Theorem)

Note →

$P(A) = P(A \cap B) + P(A \cap B')$

$= P(B) \cdot P(A/B) + P(B') \cdot P(A/B')$



($B' \to B$-ന്റെ പുറത്തുള്ളത്)

$A \to A$-യും common ഉള്ളത്

Theorem →

If A & B are 2 independent event then

1) ~~A and B'~~ are independent
2) ~~A'~~ & B are "
3) A' & B' are "

Pf/ Since A & B are independent,

$P(A/B) = P(A)$ $\quad$ $P(B/A) = P(B)$

$P(A \cap B) = P(A) \cdot P(B)$

(1) $P(A \cap B') = P(A) \cdot P(B'/A)$

$= P(A) \cdot P(1 - P(B/A)) = P(A) \cdot P(1 - P(B))$

$P(A \cap B') = P(A) \cdot P(B')$

A & B' are (I).

want to get.

$P(A \cap B) = P(A) \cdot P(B)$

$\therefore P(A \cap B') = P(A) \cdot P(B')$

$P(A \cap B) = P(A) \cdot P(B/A)$

$= P(B) \cdot P(A/B)$

$P(A') = 1 - P(A)$

$\therefore P(B'/A) = 1 - P(B$

$1 - P(B) = P(B$

(2) $P(A' \cap B) = P(B) \cdot P(A'/B)$

$= P(B) \cdot P(1 - P(A/B)) = P(B) \cdot P(1 - P(A))$

$P(A' \cap B) = P(B) \cdot P(A')$ $\quad$ $\therefore A'$ & B are (I).

(3) $P(A' \cap B') = P(A') \cdot P(B')$ (To prove)

$P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A) \cdot P(B)$ (Take out)

$= 1 - P(A) - P(B)[1 - P(A)]$

$= 1 - P(A) [1 - P(B)]$

$P(A' \cap B') = P(A') \cdot P(B')$

$\therefore A'$ & B' are (I)

* Result :-

If A & B are (I) & mutually exclusive events then either $P(A) = 0$ or $P(B) = 0$

Pf/ A & B are (I).

$A \cap B = \phi$ $\longrightarrow$ mutually exclusive.

$P(A \cap B) = P(\phi)$ $\quad$ $P(A \cap B) = P(A) \cdot P(B)$ (I).

$P(A \cap B) = 0$

$P(A) \cdot P(B) = 0$ $\quad$ $a \cdot b = 0$

$P(A) = 0$ (or) $\quad$ $\Rightarrow a = 0$ or $b = 0$.

$P(B) = 0$

* Result $\rightarrow$

Assume $0 < P(A) < 1$ & $0 < P(B) < 1$, the quantity
$$\boxed{r(A, B) = \frac{P(A \cap B) - P(A) \cdot P(B)}{\sqrt{[P(A)(1 - P(A))][P(B)(1 - P(B))]}}}$$
will be called the coefficient of correlation b/w events A & B lies b/w -1 and 1

$\rightarrow r(A, B) = 0$, if A and B are (I)

$\rightarrow r(A, B) = 1$, if $A = B$

$\rightarrow r(A, B) = -1$, if $A = B'$

$\Rightarrow$ Boole's inequality :-

For n events $A_1, A_2 \dots A_n$,

(i) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$ $\quad$ (ii) $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

Pf/ (i) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$

$\rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$.

Suppose that (i) is true for $n = m$, we have

$P\left(\bigcap_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} P(A_i) - (m-1)$

$$P\left(\bigcap_{i=1}^{m+1} A_i\right) = P\left(\bigcap_{i=1}^{m} A_i \cap A_{m+1}\right) \geq P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1}) - 1$$

$$\geq \sum_{i=1}^{m} P(A_i) - (m-1) + P(A_{m+1}) - 1$$

(i.e) $$P\left(\bigcap_{i=1}^{m+1} A_i\right) \geq \sum_{i=1}^{m+1} P(A_i) - m.$$

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

(ii) Applying boole's (i) to events $A_1^c, A_2^c \dots A_n^c$,

$$P(A_1^c \cap A_2^c \dots A_n^c) \geq \sum_{i=1}^{n} P(A_i^c) - (n-1)$$

$$= P(A_1^c) + P(A_2^c) \dots + P(A_n^c) - (n-1)$$

$$= 1 - P(A_1) + 1 - P(A_2) + \dots + 1 - P(A_n) - n - 1$$

$$= n - \{P(A_1) + P(A_2) + \dots P(A_n)\} - n + 1$$

(i.e) $$P(A_1) + P(A_2) + \dots + P(A_n) \geq 1 - P(A_1^c \cap A_2^c \cap \dots)$$

$$= 1 - P(A_1 \cup A_2 \cup \dots \cup A_n)^c$$

$$= P(A_1 \cup A_2 \cup \dots \cup A_n)$$

$$\Rightarrow P(A_1 \cup A_2 \cup \dots \cup A_n) \leq P(A_1) + P(A_2) + \dots + P$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$