Stat 107: Introduction to Business and Financial Statistics
Class 9: Portfolios, Part II

Adding Developer Tab to Excel

How to add Developer Tab into Excel 2010 and 2013 Ribbon:

1. Click the File tab;
2. Click the Options at the left to enter into Excel Option window;
3. Click the Customize Ribbon at the left;
4. At the right, select the Main Tabs from Customize The Ribbon drop down box;
5. Check the Developer item;

More items...

How to add Developer Tab into Microsoft Excel 2010 and 2007 Ribbon?
www.addintools.com/documents/**excel**/how-to-add-**developer-tab**.html

Adding Solver to Excel

The Solver Add-in is a Microsoft Office Excel add-in program that is available when you install Microsoft Office or Excel.

To use the Solver Add-in, however, you first need to load it in Excel.

1. In Excel 2010 and later goto **File** > **Options**

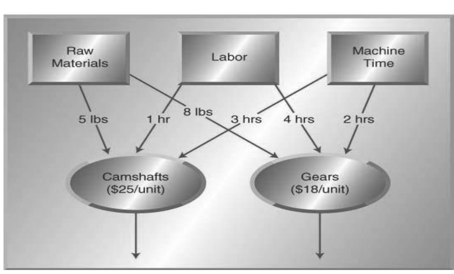
   NOTE: For Excel 2007 click the **Microsoft Office Button**, and then click **Excel Options**.
2. Click **Add-Ins**, and then in the **Manage** box, select **Excel Add-ins**.
3. Click **Go**.
4. In the **Add-Ins available** box, select the **Solver Add-in** check box, and then click **OK**.
   a. **Tip** If the **Solver Add-in** is not listed in the **Add-Ins available** box, click **Browse** to locate the add-in.
   b. If you get prompted that the Solver Add-in is not currently installed on your computer, click **Yes** to install it.
5. After you load the Solver Add-in, the **Solver** command is available in the **Analysis** group on the **Data** tab.

https://support.office.com/en-us/article/Load-the-Solver-Add-in-612926fc-d53b-46b4-872c-e24772f078ca

Example: Product Mix Decision

- DJJ Enterprises makes automotive parts, Camshafts & Gears
- Unit Profit: Camshafts $25/unit, Gears $18/unit
- Resources needed: Steel, Labor, Machine Time. In total, 5000 lbs steel available, 1500 hours labor, and 1000 hours machine time.
- Camshafts need 5 lbs steel, 1 hour labor, 3 hours machine time.
- Gears need 8 lbs steel, 4 hours labor, 2 hours machine time.
- How many camshafts & gears to make in order to maximize profit?

Understanding the Problem



- Formulation
  - Decision Variables: Number of camshafts to make, number of gears to make
  - Objective Function: Maximize profit
  - Constraints: Don't exceed amounts available of steel, labor, and machine time.

Algebraic Formulation

- Decision Variables
  - C = number of camshafts to make
  - G = number of gears to make
- Objective Function
  - Maximize 25C + 18G (profit in $)
- Constraints
  - 5C + 8G <= 5000 (steel in lbs)
  - 1C + 4G <= 1500 (labor in hours)
  - 3C + 2G <= 1000 (machine time in hours)
  - C >= 0, G >= 0 (non-negativity)

Important Concepts

- Linear Program: The objective function and constraint are linear functions of the decision variables. Therefore, this is a Linear Program.
- Feasibility
  - Feasible Solution. A solution is feasible for an LP if *all* constraints are satisfied.
  - Infeasible Solution. A solution is infeasible if *one or more* constraints is violated.
- Optimal Solution. The optimal solution is the feasible solution with the largest (for a max problem) objective value (smallest for a min problem).

Brute Force

```
mprofit=0
profit=NULL
for(c in 1:500)
 for(g in 1:500) {
  if((5*c+8*g<=5000) && (c+4*g<=1500)
     && (3*c+2*g<=1000)) {
     cprofit=25*c+18*g
     if (cprofit>mprofit) {
        bestc=c
        bestg=g
        mprofit=cprofit
        profit = c(profit,cprofit)
     }
}
}
print(paste("Max Profit =",mprofit,"C = ",bestc,"G = ",bestg))
```

"Max Profit = 8800 C = 100 G = 350"

Solving Linear Programming Problems

- Trial and error: possible for very small problems; virtually impossible for large problems.
- Simplex Method. This is a mathematical approach developed by George Dantzig. Can solve small problems by hand.
- Computer Software. Most optimization software actually uses the Simplex Method to solve the problems. Excel's Solver Add-In is an example of such software.
- Solver can solve LPs of up to 200 variables. Enhanced versions of Solver are available from Frontline Systems (http://www.solver.com).

Spreadsheet Model

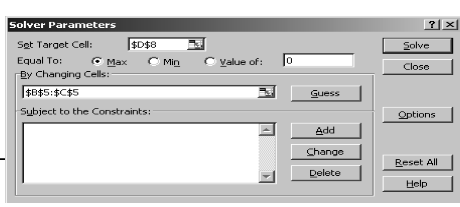
	A	B	C	D	E	F	G
1	**Example B.1**						
2	**DJJ Enterprises Production Planning**						
3							
4	**Decision Variables**	**Camshafts**	**Gears**				
5	Units to Make	75	200				
6							
7	**Objective**			**Total**			
8	Profit	$25	$18	**$5,475**			
9							
10	**Constraints**			**Used**		**Available**	
11	Steel (lbs)	5	8	1975	<=	5000	
12	Labor (hrs)	1	4	875	<=	1500	
13	Machine Time (hrs)	3	2	625	<=	1000	

D8: =B8*B$5+C8*C$5
(copied to D11:D13)

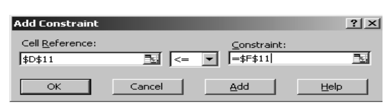
- C=75, G=200 (cells B5:C5) entered as trial values. This is a feasible, but not the optimal, solution.
- Note close relationship to algebraic formulation.
- Note that only one distinct formula needed to be entered; once entered in Cell D8, it was copied to Cells D11:D13.
- This is possible because the coefficients were stored separately in a specific structure.

Solver Settings: Target Cell and Changing Cells

- Specify Target Cell: D8
  - Equal to: Max
- Changing Cells (decision variables)
  - B5:C5



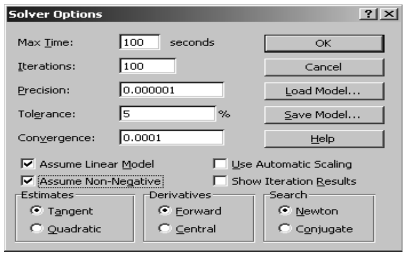
Solver Settings: Constraints



- Click "Add" to add constraints
- Select LHS Cell (D11), relationship (<=), and RHS Cell (F11).
  - LHS Cell should contain a formula which computes the LHS Value of the constraint.
  - Typically, RHS Cell should contain a fixed value, but this is not absolutely required.
- Repeat for the other two constraints (for labor and machine time).

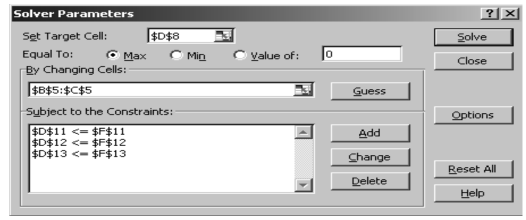
Solver Options

- Check "Assume Linear Model"
  - Tells Solver to use the Simplex Method, which is faster and more reliable than Solver's default nonlinear optimization method.
- Check "Assume Non-Negative"
  - Tells Solver that the decision variables (B5:C5, representing the number of Camshafts and Gears) must be ≥0 in any feasible solution.
- Leave other settings at their defaults.

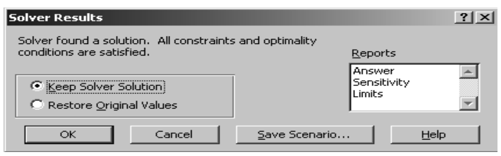


Completed Solver Box

- Click "Solve" to tell Solver to find the Optimal Solution.



Solver Results Box



- Be sure to read message of box. The one shown indicates the optimal solution has been found. There are others that indicate other possible ending points (covered later).
- Click on Reports desired before clicking OK. Reports covered later.

Solved Spreadsheet (Optimal Solution)

- Optimal Solution: Make 100 camshafts, 350 gears.
- Optimal Objective Value: $8800 profit.
- Both pieces of information are important. Knowing the optimal objective value is useless without knowing *how* that value can be attained.

	A	B	C	D	E	F
1	**Example B.1**					
2	**DJJ Enterprises Production Planning**					
3						
4	**Decision Variables**	**Camshafts**	**Gears**			
5	Units to Make	100	350			
6						
7	**Objective**			**Total**		
8	Profit	$25	$18	**$8,800**		
9						
10	**Constraints**			**Used**		**Available**
11	Steel (lbs)	5	8	3300	<=	5000
12	Labor (hrs)	1	4	1500	<=	1500
13	Machine Time (hrs)	3	2	1000	<=	1000

Expected Return of a Portfolio

The Expected Return on a Portfolio is simply the weighted average of the expected returns of the individual assets that make up the portfolio:

$$E(R_p) = \sum_{i=1}^{n}[w_i \times E(R_i)]$$

The portfolio weight of a particular security is the percentage of the portfolio's total value that is invested in that security.

Expected Return of a Portfolio

Suppose *$E(R_A)$= 14%, $E(R_B)$= 6%,*
w_A = weight of security A = 28.6%
w_B = weight of security B = 71.4%

$$E(R_p) = \sum_{i=1}^{n}[w_i \times E(R_i)] = (.286 \times 14\%) + (.714 \times 6\%)$$
$$= 4.004\% + 4.284\% = 8.288\%$$

Range of Returns

- In a two asset portfolio, simply by changing the weight of the constituent assets, different portfolio returns can be achieved.
- Because the expected return on the portfolio is a simple weighted average of the individual expected returns of the assets, you can achieve portfolio returns bounded by the highest and the lowest individual asset returns.

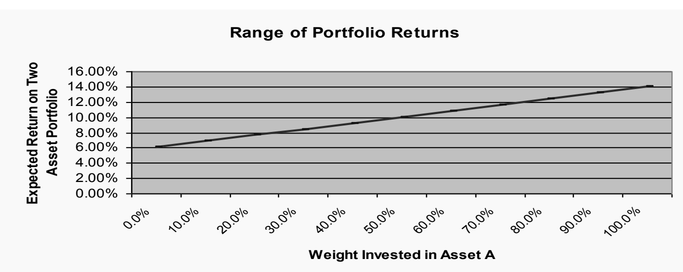
Range of Returns in a Two Asset Portfolio

Expected return on Asset A =		14.0%
Expected return on Asset B =		6.0%

Weight of Asset A	Weight of Asset B	Expected Return on the Portfolio
0.0%	100.0%	6.0%
10.0%	90.0%	6.8%
20.0%	80.0%	7.6%
30.0%	70.0%	8.4%
40.0%	60.0%	9.2%
50.0%	50.0%	10.0%
60.0%	40.0%	10.8%
70.0%	30.0%	11.6%
80.0%	20.0%	12.4%
90.0%	10.0%	13.2%
100.0%	0.0%	14.0%

Range of Returns in a Two Asset Portfolio

$E(r)_A = 14\%$, $E(r)_B = 6\%$



Risk For Portfolios

Standard Deviation of a Two-Asset Portfolio using Covariance

$$\sigma_p = \sqrt{(w_A)^2(\sigma_A)^2 + (w_B)^2(\sigma_B)^2 + 2(w_A)(w_B)(COV_{A,B})}$$

Risk of Asset A adjusted for weight in the portfolio

Risk of Asset B adjusted for weight in the portfolio

Factor to take into account comovement of returns. This factor can be negative.

Risk For Portfolios

Standard Deviation of a Two-Asset Portfolio using Correlation

$$\sigma_p = \sqrt{(w_A)^2(\sigma_A)^2 + (w_B)^2(\sigma_B)^2 + 2(w_A)(w_B)(\rho_{A,B})(\sigma_A)(\sigma_B)}$$

Factor that takes into account the degree of comovement of returns. It can have a negative value if correlation is negative.

	A	B	C	D	E	F
1	**CALCULATING THE MEAN AND STANDARD DEVIATION OF A PORTFOLIO**					
2	Asset returns	**WMT**	**TGT**			
3	Mean return	1.59%	0.46%			
4	Variance	0.93%	0.52%			
5	Standard deviation	9.63%	7.19%			
6	Covariance	0.0038				
7						
8	Proportion of WMT	0.5	<-- In the data table below this is varied from -0.5 to 1.5			
9						
10	Portfolio mean return	1.02%	<-- =B8*B3+(1-B8)*C3			
11	Portfolio return variance	0.0036	<-- =B8^2*B4+(1-B8)^2*C4+2*B8*(1-B8)*B7			
12	Portfolio return standard deviation	6.01%	<-- =SQRT(B11)			

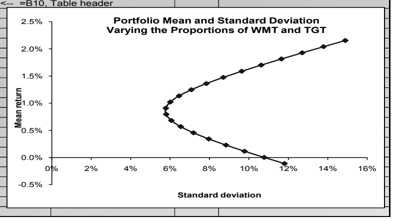
Portfolio expected return $E(r_p) = x_{WMT}E(r_{WMT}) + x_{TGT}E(r_{TGT})$

Portfolio variance $\sigma_p^2 = w_{WMT}^2\sigma_{WMT}^2 + w_{TGT}^2\sigma_{TGT}^2 + 2 * w_{WMT} * w_{TGT}Cov(r_{WMT}, r_{TGT})$

Note that $x_{TGT} = 1 - x_{WMT}$

In Excel

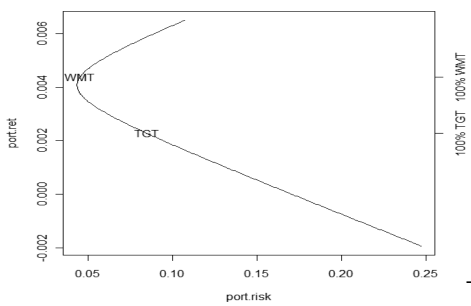
	A	B	C	D	E	F	G
14	Data table: Varying the proportion of WMT						
15		Portfolio standard deviation	Porfolio mean return				
16	Proportion of WMT	6.01%	1.02%	<-- =B10, Table header			
17	-0.5	11.81%	-0.11%				
18	-0.4	10.77%	0.00%				
19	-0.3	9.78%	0.12%				
20	-0.2	8.84%	0.23%				
21	-0.1	7.96%	0.34%				
22	0	7.19%	0.46%				
23	0.1	6.54%	0.57%				
24	0.2	6.06%	0.68%				
25	0.3	5.80%	0.80%				
26	0.4	5.78%	0.91%				
27	0.5	6.01%	1.02%				
28	0.6	6.46%	1.14%				
29	0.7	7.08%	1.25%				
30	0.8	7.84%	1.36%				
31	0.9	8.70%	1.48%				
32	1	9.63%	1.59%				
33	1.1	10.62%	1.70%				
34	1.2	11.65%	1.82%				
35	1.3	12.71%	1.93%				
36	1.4	13.79%	2.04%				
37	1.5	14.89%	2.16%				



In R (but Excel is fine for this too)

```
myport = function(ticker1,ticker2) {

s1 = getSymbols(ticker1,auto.assign=FALSE)
s2 = getSymbols(ticker2,auto.assign=FALSE)

r1 = monthlyReturn(Ad(s1))
r2 = monthlyReturn(Ad(s2))

w=seq(-1.5,1.5,.01)
port.ret = w*mean(r1)+(1-w)*mean(r2)

port.risk=sqrt(w^2*var(r1)+(1-w)^2*var(r2)+2*w*(1-w)*cov(r1,r2))
plot(port.risk,port.ret,type="l")

text(sd(r1),mean(r1),ticker1)
text(sd(r2),mean(r2),ticker2)
axis(side=4,at=c(mean(r1),mean(r2)),labels=c(paste("100%",ticker1),past
e("100%",ticker2)))

}
```


The Output (more recent data)



Example: The Caffeine Portfolio

- Consider Coke(KO) and Starbucks (SBUX)

```
> getSymbols("SBUX")
[1] "SBUX"
> getSymbols("KO")
[1] "KO"
> r1=monthlyReturn(SBUX)
> r2=monthlyReturn(KO)
> cov(r1,r2)
                monthly.returns
monthly.returns     0.001526567
> cor(r1,r2)
                monthly.returns
monthly.returns       0.2649362
> mean(r1)
[1] 0.004292857
> mean(r2)
[1] 0.006915436
> var(r1)
                monthly.returns
monthly.returns      0.01141654
> var(r2)
                monthly.returns
monthly.returns     0.002908136
>
```


The 50/50 Portfolio by hand (argh!)

Average portfolio return

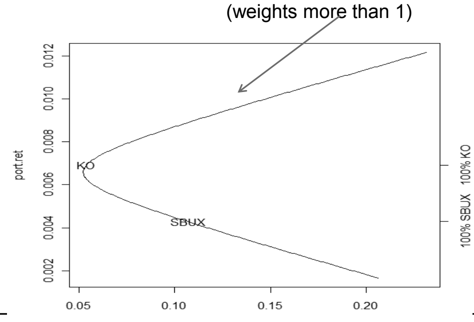
$$\bar{R}_P = 0.5\bar{R}_{SBUX} + 0.5\bar{R}_{KO} = 0.5(.0043) + (0.5)(0.0069) = 0.056$$

$$s^2_{R_P} = (0.5)^2 s^2_{R_{SBUX}} + (0.5)^2 s^2_{R_{KO}} + 2(0.5)(0.5)s_{R_{SBUX},R_{KO}} = (0.5)^2(.01141)^2 + (0.5)^2(.0029)^2 + 2(0.5)(0.5)(.0015) = 0.0043$$

$$s_{R_P} = \sqrt{s^2_{R_P}} = \sqrt{0.0043} = 0.0655$$ ← portfolio standard deviation

The Graph

Some leveraged coke portfolios (weights more than 1)



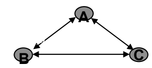
Portfolio Riskiness

- The riskiness of a portfolio that is made of different risky assets is a function of three different factors:
  - the riskiness of the individual assets that make up the portfolio
  - the relative weights of the assets in the portfolio
  - the degree of comovement of returns of the assets making up the portfolio
- The standard deviation of a two-asset portfolio may be measured using the Markowitz model:

$$\sigma_p = \sqrt{\sigma_A^2 w_A^2 + \sigma_B^2 w_B^2 + 2 w_A w_B \rho_{A,B} \sigma_A \sigma_B}$$

Risk of a Three-Asset Portfolio

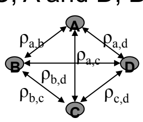
- The data requirements for a three-asset portfolio grows dramatically if we are using Markowitz Portfolio selection formulae.
- We need 3 (three) correlation coefficients between A and B; A and C; and B and C.



$$\sigma_p = \sqrt{\sigma_A^2 w_A^2 + \sigma_B^2 w_B^2 + \sigma_C^2 w_C^2 + 2 w_A w_B \rho_{A,B} \sigma_A \sigma_B + 2 w_B w_C \rho_{B,C} \sigma_B \sigma_C + 2 w_A w_C \rho_{A,C} \sigma_A \sigma_C}$$

Risk of a Four-asset Portfolio

- The data requirements for a four-asset portfolio grows dramatically if we are using Markowitz Portfolio selection formulae.
- We need 6 correlation coefficients between A and B; A and C; A and D; B and C; C and D; and B and D.



Portfolio Standard Deviation Formula

$$\sigma_{\text{port}} = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{i=1}^{n} w_i w_j \text{Cov}_{ij}}$$

where:

σ_{port} = the standard deviation of the portfolio

W_i = the weights of the individual assets in the portfolio, where weights are determined by the proportion of value in the portfolio

σ_i^2 = the variance of rates of return for asset i

Cov_{ij} = the covariance between the rates of return for assets i and j,

where $\text{Cov}_{ij} = r_{ij} \sigma_i \sigma_j$

You probably don't want to hear it, but these formulas look a lot nicer using matrix notation. Note also the curse of dimensionality!

Portfolio Standard Deviation Calculation

- The portfolio standard deviation is a function of:
  - The variances of the individual assets that make up the portfolio
  - The covariances between all of the assets in the portfolio
- The larger the portfolio, the more the impact of covariance and the lower the impact of the individual security variance

Importance of Correlation

- Correlation is important because it affects the degree to which diversification can be achieved using various assets.
- Theoretically, if two assets returns are perfectly positively correlated, it is possible to build a riskless portfolio (sadly though, that portfolio may not have a positive expected return!).

Diversification Potential

- The potential of an asset to diversify a portfolio is dependent upon the degree of co-movement of returns of the asset with those other assets that make up the portfolio.
- In a simple, two-asset case, if the returns of the two assets are perfectly negatively correlated it is possible (depending on the relative weighting) to eliminate all portfolio risk.
- This is demonstrated through the following series of spreadsheets, and then summarized in graph format.

Example of Portfolios and Correlation

Asset	Expected Return	Standard Deviation	Correlation Coefficient
A	5.0%	15.0%	1
B	14.0%	40.0%	

Perfect Positive Correlation – no diversification

Portfolio Components: Weight of A	Weight of B	Portfolio Characteristics: Expected Return	Standard Deviation
100.00%	0.00%	5.00%	15.0%
90.00%	10.00%	5.90%	17.5%
80.00%	20.00%	6.80%	20.0%
70.00%	30.00%	7.70%	22.5%
60.00%	40.00%	8.60%	25.0%
50.00%	50.00%	9.50%	27.5%
40.00%	60.00%	10.40%	30.0%
30.00%	70.00%	11.30%	32.5%
20.00%	80.00%	12.20%	35.0%
10.00%	90.00%	13.10%	37.5%
0.00%	100.00%	14.00%	40.0%

Both portfolio returns and risk are bounded by the range set by the constituent assets when ρ=+1

Example of Portfolios and Correlation

Asset	Expected Return	Standard Deviation	Correlation Coefficient
A	5.0%	15.0%	0.5
B	14.0%	40.0%	

Positive Correlation – weak diversification potential

Portfolio Components: Weight of A	Weight of B	Portfolio Characteristics: Expected Return	Standard Deviation
100.00%	0.00%	5.00%	15.0%
90.00%	10.00%	5.90%	15.9%
80.00%	20.00%	6.80%	17.4%
70.00%	30.00%	7.70%	19.5%
60.00%	40.00%	8.60%	21.9%
50.00%	50.00%	9.50%	24.6%
40.00%	60.00%	10.40%	27.5%
30.00%	70.00%	11.30%	30.5%
20.00%	80.00%	12.20%	33.6%
10.00%	90.00%	13.10%	36.8%
0.00%	100.00%	14.00%	40.0%

When ρ=+0.5 these portfolio combinations have lower risk – expected portfolio return is unaffected.

Example of Portfolios and Correlation

Asset	Expected Return	Standard Deviation	Correlation Coefficient
A	5.0%	15.0%	0
B	14.0%	40.0%	

No Correlation – some diversification potential

Portfolio Components: Weight of A	Weight of B	Portfolio Characteristics: Expected Return	Standard Deviation
100.00%	0.00%	5.00%	15.0%
90.00%	10.00%	5.90%	14.1%
80.00%	20.00%	6.80%	14.4%
70.00%	30.00%	7.70%	15.9%
60.00%	40.00%	8.60%	18.4%
50.00%	50.00%	9.50%	21.4%
40.00%	60.00%	10.40%	24.7%
30.00%	70.00%	11.30%	28.4%
20.00%	80.00%	12.20%	32.1%
10.00%	90.00%	13.10%	36.0%
0.00%	100.00%	14.00%	40.0%

Portfolio risk is lower than the risk of either asset A or B.

Example of Portfolios and Correlation

Asset	Expected Return	Standard Deviation	Correlation Coefficient
A	5.0%	15.0%	-0.5
B	14.0%	40.0%	

Negative Correlation – greater diversification potential

Portfolio Components: Weight of A	Weight of B	Portfolio Characteristics: Expected Return	Standard Deviation
100.00%	0.00%	5.00%	15.0%
90.00%	10.00%	5.90%	12.0%
80.00%	20.00%	6.80%	10.6%
70.00%	30.00%	7.70%	11.3%
60.00%	40.00%	8.60%	13.9%
50.00%	50.00%	9.50%	17.5%
40.00%	60.00%	10.40%	21.6%
30.00%	70.00%	11.30%	26.0%
20.00%	80.00%	12.20%	30.6%
10.00%	90.00%	13.10%	35.3%
0.00%	100.00%	14.00%	40.0%

Portfolio risk for more combinations is lower than the risk of either asset

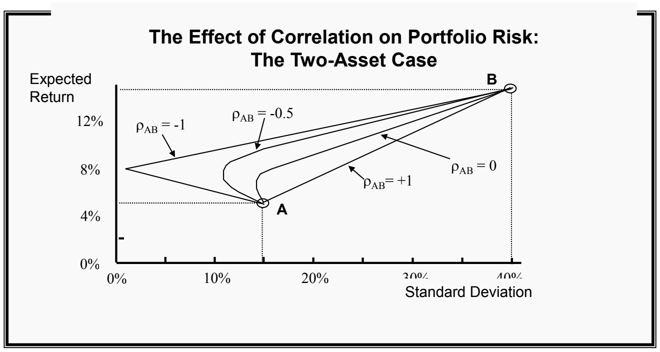
Example of Portfolios and Correlation

Asset	Expected Return	Standard Deviation	Correlation Coefficient
A	5.0%	15.0%	-1
B	14.0%	40.0%	

Perfect Negative Correlation – greatest diversification potential

Portfolio Components: Weight of A	Weight of B	Portfolio Characteristics: Expected Return	Standard Deviation
100.00%	0.00%	5.00%	15.0%
90.00%	10.00%	5.90%	9.5%
80.00%	20.00%	6.80%	4.0%
70.00%	30.00%	7.70%	1.5%
60.00%	40.00%	8.60%	7.0%
50.00%	50.00%	9.50%	12.5%
40.00%	60.00%	10.40%	18.0%
30.00%	70.00%	11.30%	23.5%
20.00%	80.00%	12.20%	29.0%
10.00%	90.00%	13.10%	34.5%
0.00%	100.00%	14.00%	40.0%

Risk of the portfolio is almost eliminated at 70% invested in asset A



Zero Risk Portfolio

- We can calculate the portfolio that removes all risk.
- When ρ = -1, then

$$\sigma_p = \sqrt{(w_A)^2(\sigma_A)^2 + (w_B)^2(\sigma_B)^2 + 2(w_A)(w_B)(\rho_{A,B})(\sigma_A)(\sigma_B)}$$

- Becomes:

$$\sigma_p = w\sigma_A - (1-w)\sigma_B$$

- Solve this equation for 0.

The Zero Risk Portfolio

- As you can see from the previous slide, if you can find two stocks that have a correlation of -1, you can build a portfolio with 0 risk!
- Mathematically it can be shown that this will happen with

$$w_1 = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

- However, will the portfolio have positive return? Unfortunately, usually no!

Example

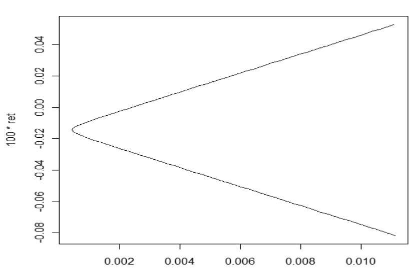
- Consider
  - RYURX (rydex ursa mutual fund)
  - VFINX (Vangaurds S&P 500 mutual fund)

```
cor(vfinx,ryurx)
                RYURX.Adjusted
VFINX.Adjusted      -0.9968647
```


No Free Lunch

- Darn!

As is typical, the zero risk portfolio has a negative expected return!



An Exercise using T-bills, Stocks and Bonds

Base Data:

	Stocks	T-bills	Bonds
Expected Return(%)	12.73383	6.151702	7.0078723
Standard Deviation (%)	0.168	0.042	0.102

Correlation Coefficient Matrix:

	Stocks	T-bills	Bonds
Stocks	1	-0.216	0.048
T-bills	-0.216	1	0.380
Bonds	0.048	0.380	1

Portfolio Combinations:

	Weights			Portfolio		
Combination	Stocks	T-bills	Bonds	Expected Return	Variance	Standard Deviation
1	100.0%	0.0%	0.0%	12.7	0.0283	16.8%
2	90.0%	10.0%	0.0%	12.1	0.0226	15.0%
3	80.0%	20.0%	0.0%	11.4	0.0177	13.3%
4	70.0%	30.0%	0.0%	10.8	0.0134	11.6%
5	60.0%	40.0%	0.0%	10.1	0.0097	9.9%
6	50.0%	50.0%	0.0%	9.4	0.0067	8.2%
7	40.0%	60.0%	0.0%	8.8	0.0044	6.6%
8	30.0%	70.0%	0.0%	8.1	0.0028	5.3%
9	20.0%	80.0%	0.0%	7.5	0.0018	4.2%
10	10.0%	90.0%	0.0%	6.8	0.0014	3.8%

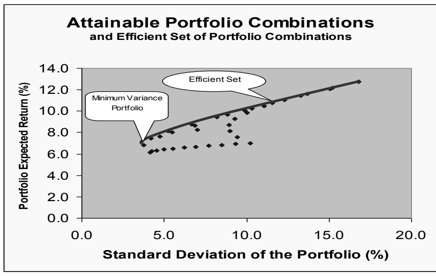
Historical averages for returns and risk for three asset

Historical correlation coefficients between the asset

Each achievable portfolio combination is plotted on expected return, risk (σ) space, found on the following slide.

each combination

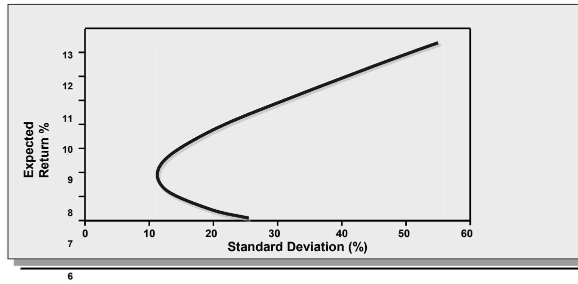
Achievable Portfolios



The efficient set is that set of achievable portfolio combinations that offer the highest rate of return for a given level of risk. The solid blue line indicates the efficient set.

The plotted points are attainable portfolio combinations.

Achievable Two-Security Portfolios

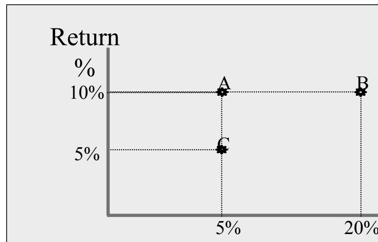


This line represents the set of portfolio combinations that are achievable by varying relative weights and using two non-correlated securities.

Dominance

- It is assumed that investors are rational, wealth-maximizing and risk averse.
- If so, then some investment choices dominate others.

Investment Choices



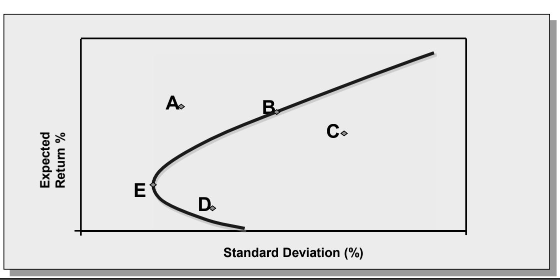
A dominates B because it offers the same return but for less risk.

A dominates C because it offers a higher return but for the same risk.

To the risk-averse wealth maximizer, the choices are clear, A dominates B, A dominates C.

Efficient Frontier

The Two-Asset Portfolio Combinations



A is not attainable

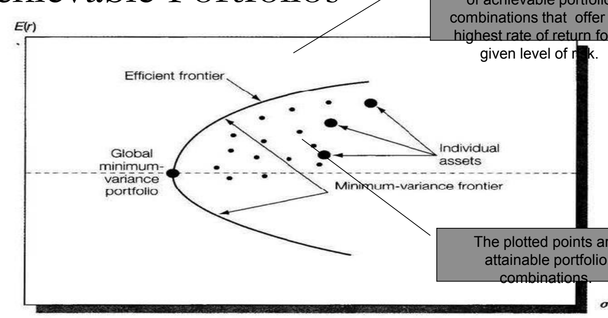
B,E lie on the efficient frontier and are attainable

E is the minimum variance portfolio (lowest risk combination)

C, D are attainable but are dominated by superior portfolios that line on the line above **E**

Achievable Portfolios

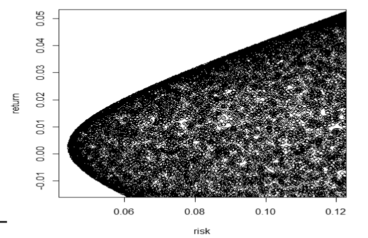
The efficient frontier is that set of achievable portfolio combinations that offer the highest rate of return for a given level of risk.



The plotted points are attainable portfolio combinations.

See port_simu.txt

- Example of portfolios



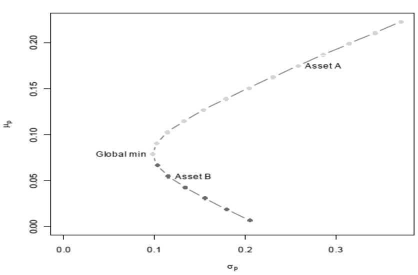
Example in R

- The asset values

μ_A	μ_B	σ_A^2	σ_B^2	σ_A	σ_B	σ_{AB}	ρ_{AB}
0.175	0.055	0.067	0.013	0.258	0.115	-0.004875	-0.164

- Recall efficient portfolios: those portfolios that have the highest expected return for a given level of risk as measured by portfolio variance. These are the portfolios that investors are most interested in holding.

See frag1_lec10.txt



Efficient portfolios are those with the highest expected return for a given level of risk. These portfolios are colored green.

Inefficient portfolios are then portfolios such that there is another feasible portfolio that has the same risk (σ) but a higher expected return (μ). These portfolios are colored red.

Note that the inefficient portfolios are the feasible portfolios that lie below the global minimum variance portfolio, and the efficient portfolios are those that lie above the global minimum variance portfolio.

- The implication for investment decisions is that some portfolios are "inefficient" in the mean variance sense
- Looking at the previous graph we see there are two portfolios which have the same risk, but one has a greater return
- All points below the minimum risk portfolio are said to be inefficient
- All points above the minimum risk portfolio are said to be efficient and make up the efficient frontier.
- The calculation of the efficient frontier is artificially simple when we only have 2 assets in the portfolio
- For the case of 3 or more assets we need to use a constrained optimisation tool like Solver in Excel

The Global Minimum Portfolio

- To find the global minimum portfolio we need to solve the following optimization problem:

$$\min_{x_A, x_B} \sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}$$
$$s.t. \quad x_A + x_B = 1.$$

- This is called a constrained optimization problem, and is easily solved by substitution, which converts this problem into a one-variable unconstrained problem.

The Global Minimum Portfolio

- Straightforward calculations show that

$$x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \quad x_B^{\min} = 1 - x_A^{\min}.$$

$$x_A^{\min} = \frac{0.01323 - (-0.004866)}{0.06656 + 0.01323 - 2(-0.004866)} = 0.2021, \quad x_B^{\min} = 0.7979.$$

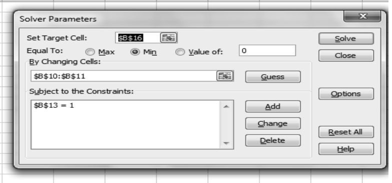
$$\begin{aligned}
\mu_p &= (0.2021) \cdot (0.175) + (0.7979) \cdot (0.055) = 0.07925 \\
\sigma_p^2 &= (0.2021)\hat{}2 \cdot (0.067) + (0.7979)\hat{}2 \cdot (0.013) \\
&\quad +2 \cdot (0.2021)(0.7979)(-0.004875) \\
&= 0.00975 \\
\sigma_p &= \sqrt{0.00975} = 0.09782
\end{aligned}$$

Using SOLVER in Excel

	A	B	C
1		Stock A	Stock B
2	Mean	17.500%	5.500%
3	Variance	6.656%	1.323%
4			
5	Cov(r_A,r_B)	-0.004875	
6			
7			
8			
9	Portfolio proportions		
10	x_A	0.5000	
11	x_B	0.5000	
12			
13	constraint	1.0000	<-- =SUM(B10:B11)
14	Market portfolio statistics		
15	Mean	0.115000	<-- =B10*B2+B11*C2
16	Variance	0.017510	<-- =B10^2*B3+B11^2*C3+2*B10*B11*B5
17	Sigma	0.132325	<-- =SQRT(B16)
18			

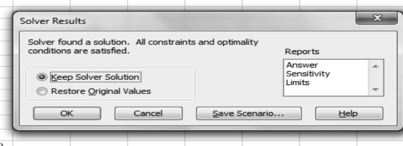
Using SOLVER

	A	B	C
1		Stock A	Stock B
2	Mean	17.500%	5.500%
3	Variance	6.656%	1.323%
4			
5	Cov(r_A,r_B)	-0.004875	
6			
7			
8			
9	Portfolio proportions		
10	x_A	0.5000	
11	x_B	0.5000	
12			
13	constraint	1.0000	<-- =SUM(B10:B11)
14	Market portfolio statistics		
15	Mean	0.115000	<-- =B10*B2+B11*C2
16	Variance	0.017510	<-- =B10^2*B3+B11^2*C3+2*B10*B11*B5
17	Sigma	0.132325	<-- =SQRT(B16)



The SOLVER Solution

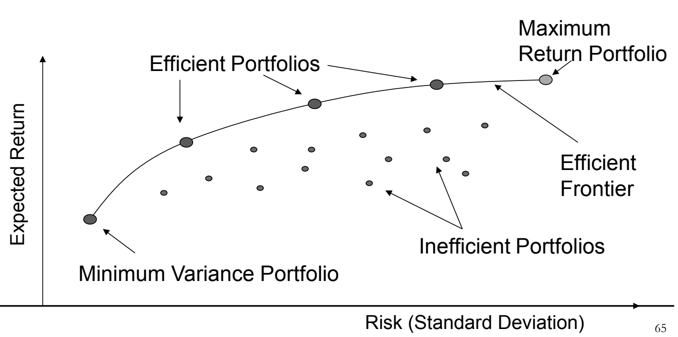
	A	B	C
1		Stock A	Stock B
2	Mean	17.500%	5.500%
3	Variance	6.656%	1.323%
4			
5	Cov(r_A,r_B)	-0.004875	
6			
7			
8			
9	Portfolio proportions		
10	x_A	0.2022	
11	x_B	0.7978	
12			
13	constraint	1.0000	<-- =SUM(B10:B11)
14	Market portfolio statistics		
15	Mean	0.079264	<-- =B10*B2+B11*C2
16	Variance	0.009569	<-- =B10^2*B3+B11^2*C3+2*B10*B11*B5
17	Sigma	0.097822	<-- =SQRT(B16)



Optimising Portfolios, More Than 2 Assets

- For portfolios containing 3 assets or more there are no simple graphical representations
- The two asset case is simple because we only have one choice variable (i.e. investment in asset A, the investment in the second asset is then implied)
- We must rely entirely on mathematics to solve problems of 3 assets or more
- Many of the insights we observed for the 2 assets are true for all frontiers
- We will used Excel's solver to calculate the optimal portfolios and the efficient frontier

An Efficient Frontier



Review: Using Excel to find Eff fr

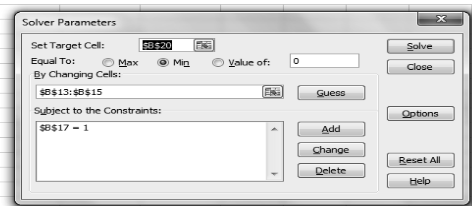
- We will use file mike_3_try.xls to play with.
- We will work with monthly returns of AAPL, RIMM and JNJ.
- We will allow sort sales (negative weights), though this can be changed via the SOLVER constraints.

The Spreadsheet

	A	B	C	D	E
1		Univariate Statistics			
2		aapl	rmm	jnj	
3	Average	0.03469	-0.01273	0.00071	
4	Standard Deviation	0.10605	0.19759	0.04465	
5	Variance	0.01125	0.03904	0.00199	
6					
7	Covraiance matrix		aapl	rimm	jnj
8		aapl	0.01125	0.01096	0.00161
9		rimm	0.01096	0.03904	0.00204
10		jnj	0.00161	0.00204	0.00199
11					
12					
13	w1	0.3000			
14	w2	0.4000			
15	w3	0.3000			
16					
17	constraint	1			
18					
19	port mean	0.005528			
20	port variance	0.0108478			
21	port sd	0.104152772			

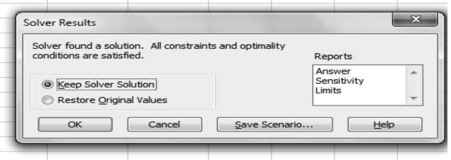
Find the Global MinimumVariance

12		
13	w1	0.3000
14	w2	0.4000
15	w3	0.3000
16		
17	constraint	1
18		
19	port mean	0.0055
20	port variance	0.0108
21	port sd	0.1042
22		
23		
24		



The Solver Output

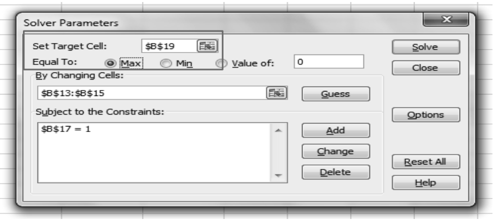
12		
13	w1	0.0511
14	w2	-0.0142
15	w3	0.9631
16		
17	constraint	1
18		
19	port mean	0.0026
20	port variance	0.0020
21	port sd	0.0444
22		
23		
24		



The minimum variance portfolio has mean 0.0026 and standard deviation 0.0444

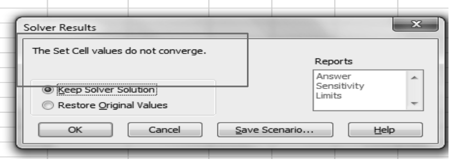
Find the Maximum Return Portfolio

12		
13	w1	0.0511
14	w2	-0.0142
15	w3	0.9631
16		
17	constraint	1
18		
19	port mean	0.0026
20	port variance	0.0020
21	port sd	0.0444
22		
23		
24		
25		



Find the Maximum Return Portfolio

12		
13	w1	############
14	w2	############
15	w3	############
16		
17	constraint	0.999999985
18		
19	port mean	############
20	port variance	############
21	port sd	############
22		
23		



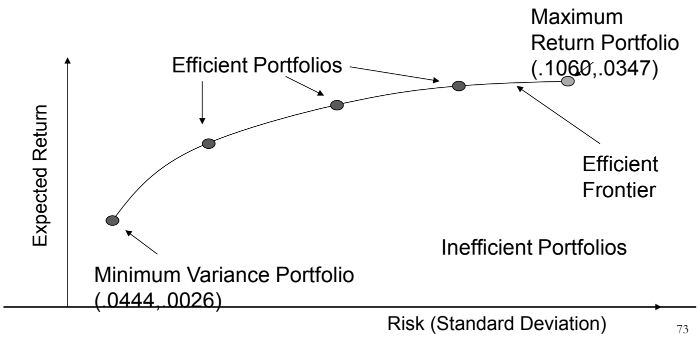
Why no maximum return portfolio?

What to use for Maximum Return?

- It is customary and an easy solution to use (when the weights can be negative) the maximum of the individual security returns as the maximum return portfolio.

1		Univariate Statistics		
2		aapl	rmm	jnj
3	Average	0.03469	-0.01273	0.00071
4	Standard Deviation	0.10605	0.19759	0.04465
5	Variance	0.01125	0.03904	0.00199

What we know so far



How do we find other points?

- We know the returns have to range between 0.0026 and 0.0347.
- So we make a table of returns in this range, and use solver to find the portfolio standard deviation that has a specified return.

That is, we want to fill in this table

risk	return
0.0444	0.0026
	0.0072
	0.0118
	0.0164
	0.0209
	0.0255
	0.0301
0.1060	0.0347

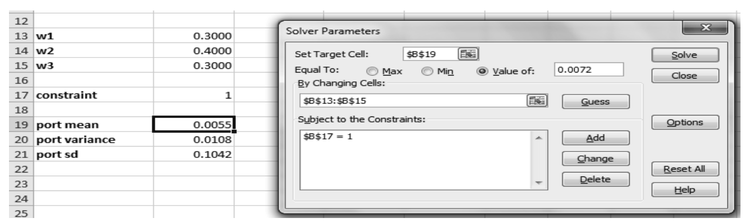
=0.0026+(1*(0.0347-0.0026)/7)

=0.0026+(6*(0.0347-0.0026)/7)

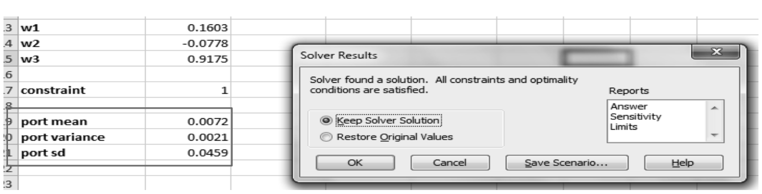
Arbitrary grid of 7 values

Finding the table values

- Need to use Solver like this



The Solution



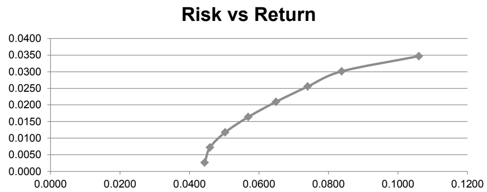
So the point (0.0459,0.0072) is on the efficient frontier.

Do this for all 7 grid points

- We obtain the following filled in table:

risk	return
0.0444	0.0026
0.0459	0.0072
0.0503	0.0118
0.0569	0.0164
0.0649	0.0209
0.0740	0.0255
0.0838	0.0301
0.1060	0.0347

Can then plot this in Excel



Although this method of computing the efficient frontier (hopefully!) makes it clear what is going on, it is a complete pain to do!

Some R functions

- portfolio.txt code on course web site
- Will compute
  - Efficient portfolio for a target return
  - The efficient frontier
  - Global Minimum Portfolio
  - Tangency Portfolio (discussed today)

Example: Minimum Portfolio

- R Code

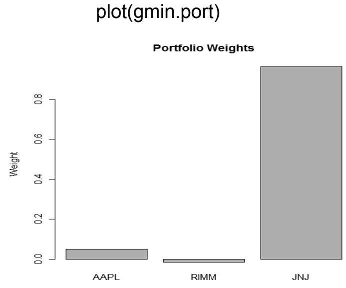
```
gmin.port <- globalMin.portfolio(er, covmat)
attributes(gmin.port)
print(gmin.port)
plot(gmin.port)
```


R output for GMV

```
> er
    AAPL     RIMM      JNJ
 0.03469 -0.01273  0.00071
> covmat
        AAPL    RIMM     JNJ
AAPL 0.01125 0.01096 0.00161
RIMM 0.01096 0.03904 0.00204
JNJ  0.00161 0.00204 0.00199
> gmin.port=globalMin.portfolio(er,covmat)
> print(gmin.port)
Call:
globalMin.portfolio(er = er, cov.mat = covmat)

Portfolio expected return:     0.002638269
Portfolio standard deviation:  0.04438313
Portfolio weights:
   AAPL    RIMM     JNJ
 0.0511 -0.0142  0.9631
```


R output for GMV

plot(gmin.port)

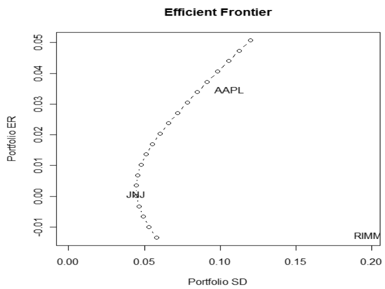


The efficient frontier in R

- The R code

```
> ef=efficient.frontier(er,covmat)
> plot(ef,plot.assets=T)
```

The Output



Can do this with many stocks

- Loaded in the first 20 stocks of the Nasdaq 100

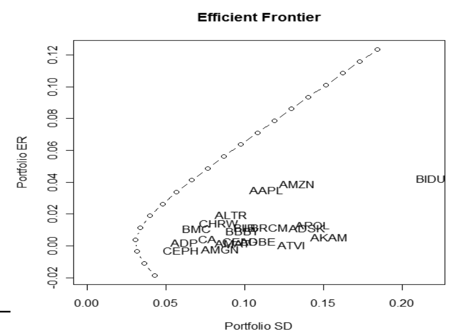
```
 [1] "ATVI" "ADBE" "AKAM" "ALTR" "AMZN" "AMGN" "APOL"
"AAPL" "AMAT" "ADSK"
[11] "ADP"  "BIDU" "BBBY" "BIIB" "BMC"  "BRCM" "CHRW"
"CA"   "CELG" "CEPH"
```


The GMV

- Easy for R to compute

```
Portfolio expected return:     0.001796427
Portfolio standard deviation:  0.03050987
Portfolio weights:
   ATVI    ADBE    AKAM    ALTR    AMZN    AMGN    APOL    AAPL    AMAT    ADSK
 0.0238  0.0485  0.0784  0.1209  0.0063  0.0647  0.0593 -0.0272  0.1280 -0.1455
    ADP    BIDU    BBBY    BIIB     BMC    BRCM    CHRW      CA    CELG    CEPH
 0.3335 -0.0498 -0.0788 -0.0585  0.2273 -0.0514  0.1792 -0.2517  0.1484  0.2446
```


The Efficient Fronteir

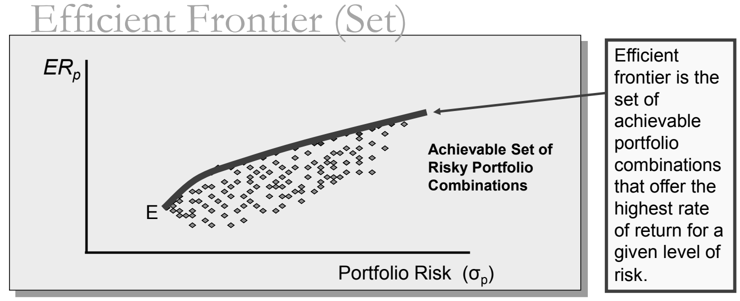


What now?

- The efficient frontier displays all the feasible portfolios possible, given any number of assets.
- However, one can do even better by introducing a risk-free asset into the mix.
- We will now describe the new, or super, efficient frontier.

Recall: Achievable Portfolio Combinations

Efficient Frontier (Set)



A Risk-Free Asset

- Now consider a risk-free asset
- Such an asset is generally consider a T-bill or Treasury Bill, or money in the bank.
- Let r_f denote the return of the risk-free asset
- Then $Var(r_f)$ =0 (its risk-free!).

Risk-free Investing

- When we introduce the presence of a risk-free investment, a whole new set of portfolio combinations becomes possible.
- We can estimate the return on a portfolio made up of RF asset (r_f) and a risky asset A letting the weight (*w)* invested in the risky asset and the weight invested in RF as (1-*w*)
- Our portfolio return: $R_{p\,=}\,(1-w)(r_f)+\,wR_A$

The Risk-Free Asset

- Note that by definition
  - $E[r_f] = r_f$
  - $Var[r_f] = 0$
  - $Cov(A,r_f) = 0$ for any other portfolio A

The New Efficient Frontier

Risk-Free Investing

- Expected return and risk on a two asset portfolio made up of risky asset *A* and *RF*:

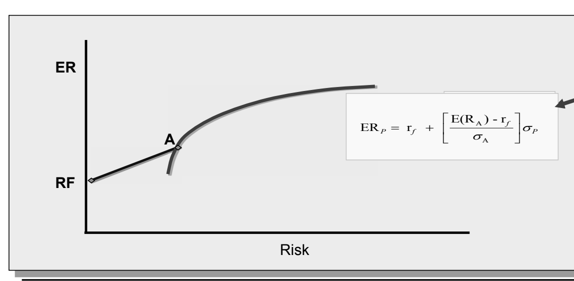
$$E[R_p] = (1\text{-w})(r_f) + wE[R_A]$$
$$\sigma_P^2 = w^2\ \sigma_A^2$$
$$\sigma_P = w\sigma_A$$

Equation of a line

- From
$$E[R_p] = (1\text{-w})(r_f) + wE[R_A]$$
$$\sigma_P = w\sigma_A$$
- We obtain
$$E[R_p] = r_f + w(E[R_A]\text{-}r_f)$$
$$= r_f + \sigma_p\left(\frac{E[R_A]\text{-}r_f}{\sigma_A}\right)$$

The New Efficient Frontier

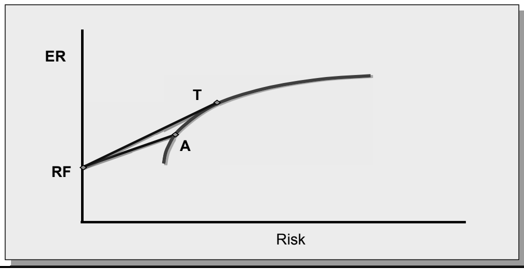
Attainable Portfolios Using *RF* and *A*



This means you can achieve any portfolio combination along the blue coloured line simply by changing the relative weight of *RF* and *A* in the two asset portfolio.

The New Efficient Frontier

Attainable Portfolios using the *RF* and *A*, and *RF* and *T*



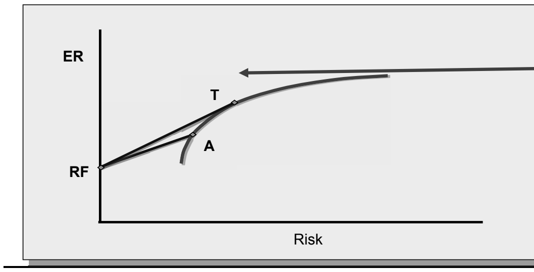
Which risky portfolio would a rational risk-averse investor choose in the presence of a *RF* investment?

Portfolio *A*?

Tangent Portfolio *T*?

The New Efficient Frontier

Efficient Portfolios using the Tangent Portfolio *T*

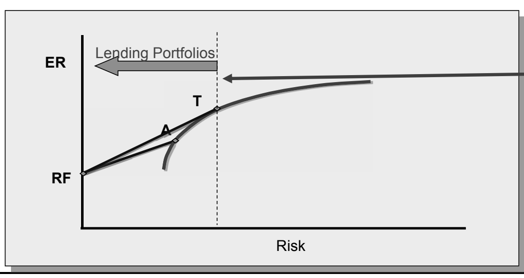


Clearly *RF* with *T* (the tangent portfolio) offers a series of portfolio combinations that dominate those produced by *RF* and *A*.

Further, they dominate all but one portfolio on the efficient frontier!

The New Efficient Frontier

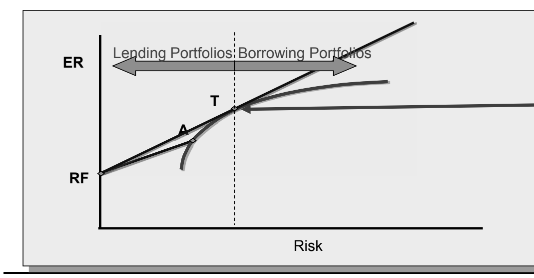
Lending Portfolios



Portfolios between *RF* and *T* are 'lending' portfolios, because they are achieved by investing in the Tangent Portfolio and lending funds to the government (purchasing a T-bill, the *RF*).

The New Efficient Frontier

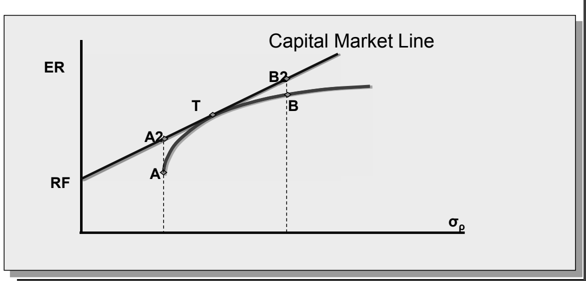
Borrowing Portfolios



The line can be extended to risk levels beyond 'T' by borrowing at RF and investing it in T. This is a levered investment that increases both risk and expected return of the portfolio.

The New Efficient Frontier

The New (Super) Efficient Frontier



This is now called the new (or super) efficient frontier of risky portfolios.

Investors can achieve any one of these portfolio combinations by borrowing or investing in RF in combination with the market portfolio.

The Market Portfolio

- Portfolio T, called the tangent portfolio, is also called the Market Portfolio
- The Capital Market Line is tangent to the efficient frontier, so every portfolio on the efficient frontier is below this line.
- ALL the best investment portfolios are on this line.

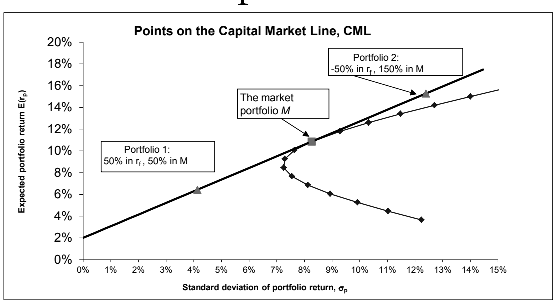
The Capital Market Line

- Is the set of optimal portfolio investments
- Each point on the line is
  - A combination of some percentage invested in the risk free asset
  - Another percentage invested in the market portfolio M.

About the CML

- All the portfolios on the CML incorporate this choice: Each CML portfolio is a combination of an investment in the risk-free asset Rf and the market portfolio M.
- Any portfolio on the CML is optimal in the sense that it could possibly be a rational investor's choice of his best investment portfolio.

Numerical Example



Examples of CML Portfolios

$E(r_M) = 10.85\%$, $r_f = 2\%$, and $\sigma_M = 8.26\%$.

Portfolio Proportions and Investment Returns on the Capital Market Line (CML)

Percentage invested in market portfolio *M*	$E(r_p) = \%\ in\ risk\text{-}free * r_f + \%\ in\ market * E(r_M)$	$\sigma_p = \%\ in\ market * \sigma_M$
0% (invest all your wealth in risk-free asset r_f)	$E(r_p) = 100\% * r_f = 2\%$	$\sigma_p = 0\% * \sigma_M = 0$
50% (invest 50% of your wealth in market portfolio *M* and 50% in risk-free asset)	$E(r_p) = 50\% * r_f + 50\% * E(r_M)$ $= 50\% * 2\% + 50\% * 10.85\%$ $= 6.43\%$	$\sigma_p = 50\% * \sigma_M$ $= 50\% * 8.26\% = 4.13\%$
100% (invest all your wealth in market portfolio *M*)	$E(r_p) = 0\% * r_f + 100\% * E(r_M)$ $= 100\% * 10.85\%$ $= 10.85\%$	$\sigma_p = 100\% * \sigma_M$ $= 100\% * 8.26\% = 8.26\%$
125% (borrow 25% of your wealth to increase investment in risky assets *M*)	$E(r_p) = -25\% * r_f + 125\% * E(r_M)$ $= -25\% * 2\% + 125\% * 10.85\%$ $= -0.5\% + 13.57\% = 13.06\%$	$\sigma_p = 125\% * \sigma_M$ $= 125\% * 8.26\% = 10.33\%$
150% (borrow 50% of your wealth to increase investment in risky assets *M*)	$E(r_p) = -50\% * r_f + 150\% * E(r_M)$ $= -50\% * 1\% + 150\% * 10.85\%$ $= -1\% + 16.28\% = 15.28\%$	$\sigma_p = 150\% * \sigma_M$ $= 150\% * 8.26\% = 12.39\%$
200% (borrow 100% of your wealth to increase investment in risky assets *M*)	$E(r_p) = -100\% * r_f + 200\% * E(r_M)$ $= -100\% * 2\% + 200\% * 10.85\%$ $= -2\% + 21.70\% = 19.70\%$	$\sigma_p = 200\% * \sigma_M$ $= 200\% * 8.26\% = 16.52\%$

The Separation Property: task 1

- Introduced by James Tobin, the 1983 Nobel Laureate for Economics.
- It implies that portfolio choice can be separated into two independent tasks.
- **Task 1**: determining the optimal risky portfolio M (the tangent portfolio)
- Given the particular input data, the best risky portfolio is the same for all clients regardless of risk aversion.

The Separation Property: task 2

- The second task, construction of the complete portfolio from a risk free asset (tbills, say) and portfolio M, however, depends on personal preferences.
- Here the client is the decision maker.
- If the optimal portfolio is the same for all clients, management is more efficient and less costly-the real competition among money managers is their choice of securities.

Practical Implications (summary)

- The analyst or planner should identify what they believe will be the best performing well diversified portfolio, call it P.
- P may include funds, stocks, bonds, international and other alternative investments.
- This portfolio will serve as the starting point for all their clients.
- The planner will then change the asset allocation between the risky portfolio and “near cash” investments according to risk tolerance of client.
- The risky portfolio P may have to be adjusted for individual clients for tax and liquidity concerns if relevant and for the client’s opinions.