

## Problem Sheet 3

1. (4 points) Consider the power series in  $\mathbb{Q}_p[[T]]$ , ( $p \neq 2$ )

$$f(T) = \sum_{n=0}^{\infty} \binom{1/2}{n} T^n,$$

where  $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$ . Compute its radius of convergence. What is  $f\left(\frac{7}{9}\right)$  in  $\mathbb{Q}_7$ ?

2. (4 points) Let  $K$  be a field, and let  $|\cdot|_1, \dots, |\cdot|_n$  be valuations on  $K$  such that no two are equivalent. Assume we are given  $x_1, \dots, x_n \in K$ . Prove that for every  $\varepsilon > 0$  there exists an  $x$  such that  $|x_i - x| < \varepsilon$ . For  $K = \mathbb{Q}$ , compare this to the Chinese Remainder Theorem.
3. (4 points) (*Product formula*) Let  $K$  be a number field, and  $\Sigma_K$  the set of places. Normalize the valuations by

$$\begin{cases} |\pi_v|_v = q_v^{-1} & \text{if } v \text{ is non-archimedean, with } \pi_v \in \mathcal{O}_v \text{ a uniformizer, } q_v = |\mathcal{O}_v/\pi_v|, \\ \text{usual absolute value} & \text{if } v \text{ is archimedean.} \end{cases}$$

Set  $\varepsilon_v = 2$  for all complex places  $v$ , and  $\varepsilon_v = 1$  for all other places. Show that for every  $x \in K$ ,

$$\prod_{v \in \Sigma_K} |x|_v^{\varepsilon_v} = 1.$$

(*Hint: Reduce to the case  $K = \mathbb{Q}$ .*)

4. (4 points) Let  $(K, |\cdot|)$  be a valued field, with  $\mathcal{O} = \{x: |x| \leq 1\}$  its ring of integers,  $\mathfrak{m} = \{x: |x| < 1\}$  its maximal ideal, and  $k = \mathcal{O}/\mathfrak{m}$  its residue field. Assume that  $|\cdot|$  extends to a separable closure  $K^{sep}$ . Show that the following are equivalent.
- i) The local ring  $\mathcal{O}$  is Henselian, i.e. given a monic polynomial  $f \in \mathcal{O}[X]$ , such that its reduction  $\bar{f} \in k[X]$  splits as the product of two coprime polynomials  $\bar{g}, \bar{h} \in k[X]$ , there exist lifts  $g, h \in \mathcal{O}[X]$  such that  $f = gh$ .
- ii) Consider a separable polynomial  $f \in K[x]$  with roots  $\alpha_1, \dots, \alpha_n \in K^{sep}$ , and let  $\beta \in K^{sep}$  such that

$$|\beta - \alpha_1| < |\alpha_i - \alpha_1|$$

for  $i = 2, \dots, n$ . Then  $K(\alpha_1) \subseteq K(\beta)$ .

Please hand in your solutions in the lecture on Tuesday, 6th of November. You may work in groups of at most three students.