

Problem Sheet 2

1. (1+3 points) (a) How many digits 0 does the factorial $2018!$ have at the end of its decimal expansion?
(b) Prove that for $n \in \mathbb{Z}$

$$v_p(n!) = \sum_{k \geq 0} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Can you read this off from the p -adic expansion of n ?

2. (1+1+2 points) (a) Let K be a non-archimedean complete field. Consider a power series $f(T) = \sum_{n \geq 0} a_n T^n \in K[[T]]$. Prove that its radius of convergence is

$$R = \frac{1}{\limsup |a_n|^{1/n}},$$

i.e. $f(x)$ converges for $|x| < R$ and diverges for $|x| > R$, for x in any finite field extension of K . (*The proof is much simpler than in the archimedean case.*)

- (b) Define the power series

$$\exp(T) = \sum_{n \geq 0} \frac{T^n}{n!} \text{ and } \log(1+T) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} T^n$$

Compute their radii of convergence in $\mathbb{Q}_p[[T]]$.

- (c) Use \exp and \log to describe \mathbb{Q}_p^* as an abelian group, i.e. write \mathbb{Q}_p^* as a product of cyclic groups and \mathbb{Z}_p 's.
(d) (2 bonus points) The same as (c), but for a finite extension K/\mathbb{Q}_p .
3. (4 points) Consider a continuous representation $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{C})$ of a profinite group G . Prove that $\ker(\rho)$ is open in G . Deduce that ρ factors through a finite quotient. What does this imply about complex representations of $\mathbb{Z}_p, \mathbb{Z}_p^*, \mathbb{Q}_p$, or \mathbb{Q}_p^* ?
(*Hint: Prove first that $1 \in \mathrm{GL}_n(\mathbb{C})$ has a neighborhood that doesn't contain any non-trivial subgroup.*)
4. (4 points) (*Geometric Ostrowski*) Describe all equivalence classes of valuations on $\mathbb{F}_q(T)$.

Please hand in your solutions in the lecture on Tuesday, 30th of October. You may work in groups of at most three students.