

Problem Sheet 4

1. (2+2 points) Let $(K, |\cdot|)$ be a non-archimedean valued field.

(a) (*Continuity of roots*) Define the norm of polynomials by

$$\|a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0\| = \max_{i=0, \dots, n} |a_i|.$$

Let $f \in K[X]$ be a monic polynomial, and $\alpha \in K$ a root. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every monic $g \in K[X]$ satisfying

1. $\deg g = \deg f$,
2. g splits completely in $K[X]$,
3. $\|g - f\| < \delta$,

there is a root $\beta \in K$ of g such that $|\alpha - \beta| < \varepsilon$.

(b) Assume that K is algebraically closed. Prove that its completion \hat{K} is algebraically closed, too.

2. (4 points) Let $(K, |\cdot|)$ be a complete non-archimedean valued field with its ring of integers \mathcal{O} . Let $f \in \mathcal{O}[X]$ be a monic polynomial. Suppose there exists an $\alpha \in \mathcal{O}$ such that

$$|f(\alpha)| < |f'(\alpha)|^2.$$

Prove that there exists a $\beta \in \mathcal{O}$ such that $f(\beta) = 0$ and $|f(\beta)| < |f'(\alpha)|$.

3. (2+2 points) (a) Let $p > 2$. Consider a finite extension K/\mathbb{Q}_p with ramification degree e and residue class degree f . Find a formula in terms of e and f for the number of quadratic extensions of K .
- (b) Prove that \mathbb{Q}_2 has exactly one Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$.
4. (4 points) (a) Let $f(T) = 1 + a_1 T + \cdots \in \mathbb{Q}_p[T]$. Show that all $a_i \in \mathbb{Z}_p$ if and only if

$$\frac{f(T^p)}{f(T)^p} \in 1 + pT \mathbb{Z}_p[T].$$

(b) Define the *Artin-Hasse exponential* by

$$E_p(T) = \exp \left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots \right).$$

Use (a) to prove that its coefficients lie in \mathbb{Z}_p . What elementary number theoretic fact corresponds to the fact that the coefficient of T^p lies in \mathbb{Z}_p ?

Please hand in your solutions in the lecture on Tuesday, 13th of November. You may work in groups of at most three students.