Problem Sheet 7

1. (2+2 points) Fix a prime number p. Let $(K, |\cdot|)$ be a complete non-archimedean field. We call K a *perfectoid field* if there exists an element $\varpi \in K$ such that $|p| \leq |\varpi^p| < 1$ and the Frobenius $\varphi\colon \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ is surjective.

   (a) Show that if K is a perfectoid, then the value group $|K^*| \subseteq \mathbb{R}_{>0}$ is p-divisible. In particular, the absolute value is not discrete.

   (b) Define $\mathbb{Q}_p^{\mathrm{cycl}}$ to be the completion of $\bigcup\limits_{n\geq 1} \mathbb{Q}_p(\zeta_{p^n})$. Prove that $\mathbb{Q}_p^{\mathrm{cycl}}$ is perfectoid. Is the p-power map $\mathbb{Q}_p^{\mathrm{cycl}} \to \mathbb{Q}_p^{\mathrm{cycl}}$ surjective?

2. (4 points) Recall from Sheet 6, Problem 3 (b), that there is exactly one Galois extension $K/\mathbb{Q}_2$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$. Compute its upper and lower ramification groups.

3. (4 points) Compute the upper and lower ramification groups of $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$.

4. (2+2 points) Consider the formal group laws $\hat{\mathbb{G}}_a, \hat{\mathbb{G}}_m$ defined by the power series

$$F_{\hat{\mathbb{G}}_a}(X,Y) = X + Y \in R[[X,Y]],$$
$$F_{\hat{\mathbb{G}}_m}(X,Y) = X + Y + XY \in R[[X,Y]].$$

   (a) Prove that over $R = \mathbb{Q}$ all formal group laws are isomorphic. Write down an explicit isomorphism $\hat{\mathbb{G}}_a \simeq \hat{\mathbb{G}}_m$.

   (b) Converseley, if $\hat{\mathbb{G}}_a \simeq \hat{\mathbb{G}}_m$, then prove that R contains $\mathbb{Q}$ as a subring.

   (Hint: For (b), consider first fields of positive characteristic.)

Please hand in your solutions in the lecture on Tuesday, 4th of December. You may work in groups of at most three students.