Problem Sheet 3

1. (4 points) Consider the power series in $\mathbb{Q}_p[[T]]$, $(p \neq 2)$
$$f(T) = \sum_{n=0}^{\infty} \binom{1/2}{n} T^n,$$
where $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$. Compute its radius of convergence. What is $f\left(\frac{7}{9}\right)$ in $\mathbb{Q}_7$?

2. (4 points) Let K be a field, and let $|\cdot|_1, \dots, |\cdot|_n$ be absolute values on K such that no two of them are equivalent. Assume we are given $x_1, \dots, x_n \in K$. Prove that for every every $\varepsilon > 0$ there exists an x such that $|x_i - x| < \varepsilon$. For $K = \mathbb{Q}$, compare this to the Chinese Remainder Theorem.

3. (4 points) *(Product formula)* Let K be a number field, and Σ_K the set of places. Normalize the valuations by
$$\begin{cases} |\pi_v|_v = q_v^{-1} & \text{if } v \text{ is non-archimedean, with } \pi_v \in \mathcal{O}_v \text{ a uniformizer, } q_v = |\mathcal{O}_v/\pi_v|, \\ \text{usual absolute value} & \text{if } v \text{ is archimedean.} \end{cases}$$
Set $\varepsilon_v = 2$ for all complex places v, and $\varepsilon_v = 1$ for all other places. Show that for every $x \in K$,
$$\prod_{v \in \Sigma_K} |x|_v^{\varepsilon_v} = 1.$$
(Hint: Reduce to the case $K = \mathbb{Q}$.)

4. (4 points) Let $(K, |\cdot|)$ be a non-archimedean valued field, with $\mathcal{O} = \{x \colon |x| \leq 1\}$ its ring of integers, $\mathfrak{m} = \{x \colon |x| < 1\}$ its maximal ideal, and $k = \mathcal{O}/\mathfrak{m}$ its residue field. Assume that $|\cdot|$ extends to a separable closure K^{sep}. Show that the following are equivalent.

   i) The local ring $\mathcal{O}$ is Henselian, i.e. given a monic polynomial $f \in \mathcal{O}[X]$, such that its reduction $\bar{f} \in k[X]$ splits as the product of two coprime polynomials $\bar{g}, \bar{h} \in k[X]$, there exist lifts $g, h \in \mathcal{O}[X]$ such that $f = gh$.

   ii) Consider a separable polynomial $f \in K[x]$ with roots $\alpha_1, \dots, \alpha_n \in K^{sep}$, and let $\beta \in K^{sep}$ such that
$$|\beta - \alpha_1| < |\alpha_i - \alpha_1|$$
for $i = 2, \dots, n$. Then $K(\alpha_1) \subseteq K(\beta)$.

Please hand in your solutions in the lecture on Tuesday, 6th of November. You may work in groups of at most three students.