Problem Sheet 10

1. (4 points) Let A be a local Noetherian algebra with perfect residue field k of prime characteristic p. Let F, F' be formal group laws such that the height of $F \otimes_A k$ is finite.

   Prove that the natural map
   $$\operatorname{Hom}_A(F, F') \to \operatorname{Hom}_k(F \otimes_A k, F' \otimes_A k)$$
   is injective.

   (Notation: $F \otimes_A k$ denotes the base change *of the formal group law to k, i.e.you reduce all coefficients of the power series. Hint: It is enough to prove that if ϕ is a homomorphism that reduces to zero, then it's zero modulo $\mathfrak{m}_A^n$ for all n. (Why?) So we can replace A by $A/\mathfrak{m}_A^n$. This is a finite dimensional k-vector space, so an element is zero if and only if it is killed by all functionals $A/\mathfrak{m}_A^n \to k$.)*

2. (1+3 points) Let k be either a finite field of characteristic p or an algebraic closure $\bar{\mathbb{F}}_p$. Let $\mathcal{C}$ be the category of local Artinian rings with residue field k.

   (a) Prove that $\mathcal{C}$ has all finite products.

   (b) Fix a formal group law F_0 over k. Consider the functor
   $$\mathcal{F} \colon \mathcal{C} \to \mathcal{S}ets \colon A \mapsto \{F \text{ formal group law over } A \colon F \otimes_A k = F_0\} / \cong .$$
   Prove that if $A \to C$ and $B \to C$ are morphisms in C, with at least one of the two being surjective, then the natural map
   $$\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$$
   is bijective.

3. (4 points) Let H be a normal subgroup of a group G and let A be a G-module. Prove that the sequence below is exact
   $$0 \to H^1(G/H, A^H) \xrightarrow{Inf} H^1(G, A) \xrightarrow{Res} H^1(H, A)$$

4. (4 points) (Continuation of exercise 3) Assume further that $H^i(H, A) = 0$ for $1 \leq i \leq q-1$. Prove that the sequence below is exact
   $$0 \to H^q(G/H, A^H) \xrightarrow{Inf} H^q(G, A) \xrightarrow{Res} H^q(H, A).$$
   (Hint: Induction on q, exercise 3 being the base case.) Both results could be obtained directly from the Hochschild–Serre spectral sequence, but your job is to give a direct proof.

Please hand in your solutions in the lecture on Tuesday, 15th of January. You may work in groups of at most three students.