Problem Sheet 2

1. (1+3 points) (a) How many digits 0 does the factorial 2018! have at the end of its decimal expansion?

   (b) Prove that for $n \in \mathbb{Z}$
   $$v_p(n!) = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor .$$
   Can you read this off from the p-adic expansion of n?

2. (1+1+2 points) (a) Let K be a non-archimedean complete field. Consider a power series $f(T) = \sum_{n \geq 0} a_n T^n \in K[[T]]$. Prove that its radius of convergence is
   $$R = \frac{1}{\limsup |a_n|^{1/n}},$$
   i.e. $f(x)$ converges for $|x| < R$ and diverges for $|x| > R$, for x in any finite field extension of K. *(The proof is much simpler than in the archimedean case.)*

   (b) Define the power series
   $$\exp(T) = \sum_{n \geq 0} \frac{T^n}{n!} \text{ and } \log(1+T) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} T^n$$
   Compute their radii of convergence in $\mathbb{Q}_p[[T]]$.

   (c) Use exp and log to describe $\mathbb{Q}_p^*$ as an abelian group, i.e. write $\mathbb{Q}_p^*$ as a product of cyclic groups and $\mathbb{Z}_p$'s.

   (d) (2 bonus points) The same as (c), but for a finite extension $K/\mathbb{Q}_p$.

3. (4 points) Consider a continuous representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ of a profinite group G. Prove that $\ker(\rho)$ is open in G. Deduce that ρ factors through a finite quotient. What does this imply about complex representations of $\mathbb{Z}_p, \mathbb{Z}_p^*, \mathbb{Q}_p$, or $\mathbb{Q}_p^*$?

   (Hint: Prove first that $1 \in \mathrm{GL}_n(\mathbb{C})$ has a neighborhood that doesn't contain any non-trivial subgroup.)

4. (4 points) *(Geometric Ostrowski)* Describe all equivalence classes of valuations on $\mathbb{F}_q(T)$.

Please hand in your solutions in the lecture on Tuesday, 30th of October. You may work in groups of at most three students.