

Problem Sheet 8

1. (1+3 points) Let K/\mathbb{Q}_p be a finite extension and \check{K} be the completion of the maximal unramified extension of K .

(a) Prove that there exists a unique automorphism φ of \check{K} fixing K such that φ reduces to the Frobenius on $\mathcal{O}_{\check{K}}/\mathfrak{m}$.

(b) Prove that there exist short exact sequences

$$\begin{aligned} 1 &\rightarrow \mathcal{O}_K^\times \rightarrow \mathcal{O}_{\check{K}}^\times \xrightarrow{x \mapsto \phi(x)/x} \mathcal{O}_{\check{K}}^\times \rightarrow 1, \\ 0 &\rightarrow \mathcal{O}_K \rightarrow \mathcal{O}_{\check{K}} \xrightarrow{x \mapsto \phi(x) - x} \mathcal{O}_{\check{K}} \rightarrow 0. \end{aligned}$$

(Hint: The sequences reduce to well-known sequence in Galois theory. Prove that they are exact by approximation.)

2. (4 points) Let K/\mathbb{Q}_p be a finite extension. Show that any two Lubin–Tate formal group laws F_f and $F_{f'}$ become isomorphic over $\mathcal{O}_{\check{K}}$.

3. (2+2 points) (*Weierstrass preparation theorem*) Let K be a non-archimedean local field and $\pi \in K$ be a uniformizer. Consider a power-series $f(T) = a_0 + a_1T + a_2T^2 + \dots \in \mathcal{O}_K[[T]]$.

(a) Assume that $a_i \in \mathfrak{m}_K$ for $i < n$, but $a_n \notin \mathfrak{m}_K$. Prove that for every $g \in \mathcal{O}_K[[T]]$ there exists a unique factorization $g = qf + r$, with $q \in \mathcal{O}_K[[T]]$ and $r \in \mathcal{O}_K[T]$ a polynomial of degree at most $n - 1$.

(b) Show that f admits a unique presentation $f = \pi^m gh$, with $m \geq 0$, g a polynomial satisfying $g \equiv X^n \pmod{\mathfrak{m}_K}$ and $h \in (\mathcal{O}_K[[T]])^\times$ is a unit.

4. (4 points) Let k be a field of characteristic $p > 0$ and $F(X, Y) \in k[[X, Y]]$ a formal group law. The *height* of F is defined to be the h such that

$$[p]_F = a_h X^h + a_{h+1} X^{h+1} + \dots$$

and $a_h \neq 0$.

Let now K/\mathbb{Q}_p be a finite extension and $F(X, Y) \in \mathcal{O}_K[[X, Y]]$ a formal group law. Let h be the height of its reduction modulo \mathfrak{m}_K . Prove that there exists a finite extension K'/K such that the p -torsion of the group $F(\mathfrak{m}_{K'})$ is $(\mathbb{Z}/p\mathbb{Z})^h$.

(Hint: Use exercise 3.)

Please hand in your solutions in the lecture on Tuesday, 11th of December. You may work in groups of at most three students.