Problem Sheet 4

1. (2+2 points) Let $(K, |\cdot|)$ be a non-archimedean valued field.
   (a) *(Continuity of roots)* Define the norm of polynomials by

   $$\|a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0\| = \max_{i=0,\dots,n} |a_i|.$$

   Let $f \in K[X]$ be a monic polynomial, and $\alpha \in K$ a root. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every monic $g \in K[X]$ satisfying
   1. $\deg g = \deg f$,
   2. g splits completely in $K[X]$,
   3. $\|g - f\| < \delta$,

   there is a root $\beta \in K$ of g such that $|\alpha - \beta| < \varepsilon$.

   (b) Assume that K is algebraically closed. Prove that its completion $\hat{K}$ is algebraically closed, too.

2. (4 points) Let $(K, |\cdot|)$ be a complete non-archimedean valued field with its ring of integers $\mathcal{O}$. Let $f \in \mathcal{O}[X]$ be a monic polynomial. Suppose there exists an $\alpha \in \mathcal{O}$ such that

   $$|f(\alpha)| < |f'(\alpha)|^2.$$

   Prove that there exists a $\beta \in \mathcal{O}$ such that $f(\beta) = 0$ and $|f(\beta)| < |f'(\alpha)|$.

3. (2+2 points) (a) Let $p > 2$. Consider a finite extension $K/\mathbb{Q}_p$ with ramification degree e and residue class degree f. Find a formula in terms of e and f for the number of quadratic extensions of K.

   (b) Prove that $\mathbb{Q}_2$ has exactly one Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$.

4. (4 points) (a) Let $f(T) = 1 + a_1 T + \cdots \in \mathbb{Q}_p[[T]]$. Show that all $a_i \in \mathbb{Z}_p$ if and only if

   $$\frac{f(T^p)}{f(T)^p} \in 1 + pT\,\mathbb{Z}_p[[T]].$$

   (b) Define the *Artin–Hasse exponential* by

   $$E_p(T) = \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \dots\right).$$

   Use (a) to prove that its coefficients lie in $\mathbb{Z}_p$. What elementary number theoretic fact corresponds to the fact that the coefficient of T^p lies in $\mathbb{Z}_p$?

Please hand in your solutions in the lecture on Tuesday, 13th of November. You may work in groups of at most three students.