Problem Sheet 5

1. (2+1+1 points) Let $K = \mathbb{Q}_p(\zeta_{p^n})$ for $n \geq 1$.
   (a) Prove that $K/\mathbb{Q}_p$ is Galois and totally ramified. Compute its Galois group. *(Hint: Prove that $\zeta_{p^n} - 1$ is a uniformizer.)*
   (b) What is $\mathcal{O}_K$?
   (c) Find an example of a totally ramified extension of the form $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$ with $m \neq p^n$.

2. (3+1+0 points) *(Tamely ramified extensions)*
   (a) Assume that L/K is a totally ramified extension of non-archimedean local fields of degree d, with d coprime to the residue characteristic p. Prove that L is generated by a root, i.e. there exists an $a \in K$ such that
   $$L = K\left(\sqrt[d]{a}\,\right).$$
   (b) Suppose we are in the situation of (a), and assume further that
   $$|k_K| \equiv 1 \mod d.$$
   Prove that L/K is Galois and cyclic.
   (c) Philosophical question: How does this relate to class field theory?

3. (2+2 points) (a) Let K be a local field of characteristic 0, and fix a separable closure K^{sep}. Show that for every $n > 0$ there are only finitely many intermediate extensions $K \subseteq L \subseteq K^{sep}$ of degree $[L : K] = n$
   (b) What happens if instead we assume that K has positive characteristic?

4. (2+2 points) For a subset $A \subset \mathbb{R}^2$ its *lower convex hull* is the highest convex polygonal line such that all points in A lie on or above it.

   For a power-series $f(T) = a_0 + a_1T + \cdots \in \mathbb{C}_p[[T]]$ its *Newton polygon* $N(f)$ is the lower convex hull of
   $$\{(i, v_p(a_i)) \colon i \in \mathbb{N}, a_i \neq 0\} \subseteq \mathbb{R}^2\,.$$
   Here, $v_p = -\log_p |\cdot|_p$, i.e. it is normalized so that $v_p(p) = 1$.
   (a) Draw the Newton polygons for $1 + T + pT^4 + p^2T^6$ and $\prod_{i=1}^{p^2}(1 - pT)$.
   (b) Assume that f is a polynomial with $f(0) = 1$, and that it splits as
   $$f(T) = \prod_{i=1}^{n} \left(1 - \frac{T}{\alpha_i}\right).$$
   Prove: If λ is a slope of $N(f)$ with multiplicity d, then there are exactly d roots α_i with $v_p(1/\alpha_i) = \lambda$.

Please hand in your solutions in the lecture on Tuesday, 20th of November. You may work in groups of at most three students.