4.1 The Derivative

Def 4.1.1 Let I be an interval, let $f: I \to \mathbb{R}$ be a function, and let $c \in I$. If $L := \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, then we say f is **differentiable at c**, that L is the derivative of f at c, and write $f'(c) = L$.

Note: I is allowed to be closed and c may be an endpt.
By def. of limit: $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in I \setminus \{c\})\left(|x-c| < \delta \Rightarrow \left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon\right)$

Examples
* $f: \mathbb{R}\to\mathbb{R}$ $f(x) = a$ (constant) $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{a-a}{x-c} = \lim_{x\to c} 0 = 0$. So $f'(c) = 0$ $\forall x \in I$
* $g: \mathbb{R}\to\mathbb{R}$ $g(x) = x^3$ $\lim_{x\to c} \frac{x^3-c^3}{x-c} = \lim_{x\to c} \frac{(x-c)(x^2+cx+c^2)}{x-c} = \lim_{x\to c} x^2+cx+c^2 = c^2+c^2+c^2 = 3c^2$ $g'(c) = 3c^2$
* $h: \mathbb{R}\to\mathbb{R}$ $h(x) = x$ $\lim_{x\to c} \frac{x-c}{x-c} = \lim_{x\to c} 1 = 1$ $h'(c) = 1$
* $k: (0,\infty) \to \mathbb{R}$ $k(x) = \frac{1}{x}$ $\lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} = \lim_{x\to c} \frac{\frac{c-x}{xc}}{x-c} = \lim_{x\to c} \frac{-1}{xc} = \frac{-1}{c}\lim_{x\to c}\frac{1}{x} = \frac{-1}{c}\cdot\frac{1}{c} = \frac{-1}{c^2}$

Prop 4.1.4 Diff. implies cont. pf hint: $f(x)-f(c) = \frac{f(x)-f(c)}{x-c}(x-c) \to f'(c)\cdot 0 = 0$ as $x \to c$.

Prop 4.1.5 Suppose $f: I\to\mathbb{R}$ and $g: I\to\mathbb{R}$ are differentiable at $c\in I$. The αf and $f+g$ are as well. $(\alpha f)'(c) = \alpha f'(c)$ and $(f+g)'(c) = f'(c) + g'(c)$

Prop 4.1.6 With the same hypotheses above, $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
pf hint: $f(x)g(x) - f(c)g(c) = f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)$

Prop 4.1.7 (as above and $g(c)\neq 0$) $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$
pf hint:
If $g(c)\neq 0$, then $\exists \delta_1 > 0$ so that $g(x)\neq 0$ $\forall x \in I \cap (c-\delta_1, c+\delta_1)$ by continuity.
$\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)} = \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)} = \frac{1}{g(x)g(c)}\left[f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)\right]$

Prop 4.1.8 Let $g: I_1 \to I_2$ be diff. at c and $f: I_2 \to \mathbb{R}$ be diff. at $g(c)$.
Then $(f\circ g)'(c) = f'(g(c))\, g'(c)$

pf Trap: $\frac{f(g(x)) - f(g(c))}{x-c} = \underbrace{\frac{f(g(x)) - f(g(c))}{g(x) - g(c)}}_{f'(g(c))?} \cdot \underbrace{\frac{g(x)-g(c)}{x-c}}_{g'(c)}$ but what if $g(x) = g(c)$ for infinitely many x close to c.

pf Define auxillary functions $u: I_2 \to \mathbb{R}$ $v: I_1 \to \mathbb{R}$ by
$$u(y) = \begin{cases} \frac{f(y)-f(g(c))}{y-g(c)} & \text{if } y \neq g(c) \\ f'(g(c)) & \text{if } y = g(c) \end{cases}$$
$$v(x) = \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x\neq c \\ g'(c) & \text{if } x = c. \end{cases}$$

Then $f(y) - f(g(c)) = u(y)(y - g(c))$ and $g(x) - g(c) = v(x)(x-c)$
$f(g(x)) - f(g(c)) = u(g(x))(g(x)-g(c)) = u(g(x))\,v(x)(x-c)$
so $(f(g(x)) - f(g(c)))/(x-c) = u(g(x))\,v(x)$
$\lim_{y\to g(c)} u(y) = f'(g(c))$ and $\lim_{x\to c} v(x) = g'(c)$. u is cont. @ $g(c)$. v is cont. @ c and g is cont. @ c. $\lim_{x\to c} u(g(x))\,v(x) = f'(g(c))\,g'(c)$ so $f\circ g$ is diff. at $x=c$. ∎