3.1 Limits of Functions

Def 3.1.1 Let $S \subset \mathbb{R}$ be a set. A number $x \in \mathbb{R}$ is called a **cluster point** of S if for every $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\}$ is not empty.
(Equiv.: $\forall \varepsilon > 0$, $\exists y \in S$ st. $\underbrace{y \neq x \text{ and } |x-y| < \varepsilon}_{0 < |x-y| < \varepsilon}$

E.g. $(S, \text{cluster pts of } S)$: $((a,b), [a,b])$ $(\mathbb{Q}, \mathbb{R})$ $(\{1,2\}, \emptyset)$ $(\{2 - \frac{1}{n} \mid n \in \mathbb{N}\}, \{2\})$, $(\mathbb{N}, \emptyset)$

Prop 3.1.2 Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster pt. of S iff $\exists$ convergent seq. $\{x_n\}$ with $x_n \neq x$, $x_n \in S$ and $\lim x_n = x$.

pf: If x is a cluster pt. Use $\varepsilon = \frac{1}{n}$ to get a sequence $x_n \in S$: $0 < |x - x_n| < \frac{1}{n}$. $\lim x_n = x$.
Conversely, if $\{x_n\} \subseteq S$, $x_n \neq x$ and $\lim x_n = x$, then $\forall \varepsilon > 0$ $\exists M$ st. $n \geq M \Rightarrow |x_n - x| < \varepsilon$.
so $x_M \neq x$, $x_M \in S$ and $0 < |x_M - x| < \varepsilon$.

Def 3.1.3 Let $f: S \to \mathbb{R}$ be a function and c a cluster pt. of S. Suppose there exists an $L \in \mathbb{R}$ and $\forall \varepsilon > 0$ $\exists \delta > 0$ so that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.
"$f(x)$ converges to L as x goes to c."
$\lim_{x \to c} f(x) = L$.
$f(x) \to L$ as $x \to c$.
If no such L exists, then we say the limit DNE or "f diverges at c"

Prop 3.1.4 Let c be a cluster point of $S \subset \mathbb{R}$ and let $f: S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ a x goes to c is unique.

pf: Suppose L_1 and L_2 are both limits. Take an $\varepsilon > 0$. Then $\exists \delta_1 > 0$ s.t. $|f(x) - L_1| < \frac{\varepsilon}{2}$ $\forall x \in S \setminus \{c\}$ with $|x - c| < \delta_1$. Similarly $\exists \delta_2 > 0$ s.t. $|f(x) - L_2| < \frac{\varepsilon}{2}$ $\forall x \in S \setminus \{c\}$ with $|x - c| < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\}$. If $|x - c| < \delta$ and $x \neq c$,
Then $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
As $|L_1 - L_2| < \varepsilon$ for arbitrary $\varepsilon > 0$, $L_1 = L_2$.

Examples Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x + 5$. Then $\lim_{x \to c} f(x) = 3(c) + 5$.

Work: Want $|3x + 5 - (3c + 5)| < \varepsilon$. $|3(x - c)| < \varepsilon$ Try $|x - c| < \frac{\varepsilon}{3}$.

pf: Let $c \in \mathbb{R}$ be fixed and $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{3}$ and Take $x \neq c$ with $|x - c| < \delta$. So $|x - c| < \frac{\varepsilon}{3}$. Therefore,

$$
\begin{aligned}
|f(x) - (3c+5)| &= |3x + 5 - 3c - 5| \\
&= |3x - 3c| \\
&= 3|x - c| \\
&< 3\left(\tfrac{\varepsilon}{3}\right) = \varepsilon
\end{aligned}
$$

let $g(x): \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^3 + 2x^2$ show $\lim_{x \to c} c^3 + 2c^2$

work: want $|g(x) - (c^3 + 2c^2)| < \varepsilon$ $|x^3 + 2x^2 - (c^3 + 2c^2)| = |x^3 - c^3 + 2(x^2 - c^2)| \leq |x^3 - c^3| + 2|x^2 - c^2| = |(x - c)(x^2 + xc + c^2)| \leq |x - c| \cdot |x^2 + xc + c^2|$ $+ 2|x - c| \cdot |x + c|$

$|x + c| \leq |x| + |c|$ $|x + c| = |x - c + 2c| \leq \underbrace{|x - c| + 2|c|}_{\text{want } |x - c| \leq 1} \leq 1 + 2|c|$

$|x^2 + xc + c^2| \leq |x| \cdot |x + c| + |c|^2 \leq (1 + |c|)(1 + 2|c|) + |c|^2$ and $|x - c| < \frac{\varepsilon/2}{2(1 + 2|c|)}$

$\underbrace{|x| - |c| \leq |x - c| \leq 1}_{\text{rev. } \Delta} \Rightarrow |x| \leq (1 + |c|)$ want $|x - c| \leq \frac{\varepsilon/2}{(1 + |c|)((1 + 2|c|) + |c|^2}$

pf Let $\varepsilon > 0$, c be fixed. Choose $\delta = \min\left\{1, \frac{\varepsilon}{4(1 + 2|c|)}, \frac{\varepsilon}{2(1 + |c|)(1 + 2|c|) + |c|^2}\right\}$
Suppose $x \neq c$ with $|x - c| < \delta$. In particular, $|x - c| < 1$ and by the reverse triangle inequality,
$|x| - |c| \leq |x - c| < 1$ so $|x| < 1 + |c|$. Thus, $|x + c| \leq |x| + |c| < 1 + 2|c|$

$$
\begin{aligned}
|g(x) - (c^3 + 2c^2)| &= |x^3 + 2x^2 - c^3 - 2c^2| \\
&= |(x^3 - c^3) + 2(x^2 - c^2)| \\
&\leq |x^3 - c^3| + 2|x^2 - c^2| \\
&= |x - c| \cdot |x^2 + xc + c^2| + 2|x - c||x + c| \\
&\leq |x - c|\left(|x||x + c| + |c|^2\right) + 2|x - c| \cdot |x + c| \\
&< |x - c|\left[(1 + |c|)(1 + 2|c|) + |c|^2\right] + 2|x - c|(1 + 2|c|) \\
&< \frac{\frac{\varepsilon}{2}}{(1 + |c|)(1 + 2|c|) + |c|^2} + 2\,\frac{\frac{\varepsilon}{2}}{2(1 + 2|c|)}(1 + 2|c|) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square.
\end{aligned}
$$