2.2 Facts about Limits of Sequences

Lemma 2.2.1 (Squeeze Lemma) Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be seq. s.t. $a_n \le x_n \le b_n$ $\forall n \in \mathbb{N}$.
Suppose $\{a_n\}$ and $\{b_n\}$ converge and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$. Then x_n converges to x.

pf Let $\varepsilon > 0$. Observe that $a_n \le x_n \le b_n$ implies $|x_n - a_n| = x_n - a_n \le b_n - a_n$

Since $\lim_{n\to\infty} a_n = x$, there exists $N_1 \in \mathbb{N}$ s.t. $n \ge N_1 \Rightarrow |a_n - x| < \frac{\varepsilon}{3}$.

Likewise $\exists$ N_2 s.t. $n \ge N_2 \Rightarrow |b_n - x| < \frac{\varepsilon}{3}$

Take $N = \max\{N_1, N_2\}$. Then if $n \ge N$,

$$
\begin{aligned}
|x_n - x| = |x_n - a_n + a_n - x| &\le |x_n - a_n| + |a_n - x| \le (b_n - a_n) + |a_n - x| \\
&= |b_n - a_n| + |a_n - x| \\
&\le |b_n - x| + |x - a_n| + |a_n - x| \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \\
&= \varepsilon \quad \square
\end{aligned}
$$

Lemma 2.2.3 If $\{x_n\}$ and $\{y_n\}$ are convergent seqs. and $x_n \le y_n$ $\forall n \in \mathbb{N}$, then $x = \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n = y$.

pf Let $\varepsilon > 0$ be given. $\exists N_1$ s.t. $|x_n - x| < \varepsilon/2$ for $n \ge N$ and $\exists N_2$ s.t. $|y_n - y| < \varepsilon/2$ for $n \ge N$.

Take $N = \max\{N_1, N_2\}$ and $n \ge N$. Then $y_n - x_n + x - y = y_n - y + x - x_n \le |y_n - y| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

so $y_n - x_n < y - x + \varepsilon$. Also, $0 \le y_n - x_n < y - x + \varepsilon$ so $x - y < \varepsilon$

If $x \ge y$, then $0 \le x - y < \varepsilon$ $\forall \varepsilon > 0$ implies $x = y$.

Thus, $x \le y$.

Corollary Suppose $\{x_n\}$ converges to x

(i) $x_n \ge 0$ $\forall n$ implies $x \ge 0$.

(ii) If $a, b \in \mathbb{R}$ with $a \le x_n \le b$ $\forall n \in \mathbb{N}$, then $a \le x \le b$.

Prop 2.2.5 Continuity of Algebraic Operations

Suppose $\{x_n\}$ and $\{y_n\}$ converge. Then $\lim_{n\to\infty} x_n + y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.
$\lim_{n\to\infty} x_n - y_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$, $\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right)$, and if $\lim y_n \ne 0$ and $y_n \ne 0$ $\forall n \in \mathbb{N}$, then
$\lim_{n\to\infty} \frac{x_n}{y_n} = \left(\lim_{n\to\infty} x_n\right) / \left(\lim_{n\to\infty} y_n\right)$ $\quad |(x_n + y_n) - (x+y)| = |x_n - x + y_n - y| \le |x_n - x| + |y_n - y|$

Prop 2.2.6 If $\{x_n\}$ is a convergent seq. with $x_n \ge 0$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$

Prop 2.2.7. If $\{x_n\}$ is convergent, then $\{|x_n|\}$ is as well and $\lim_{n\to\infty} |x_n| = |\lim_{n\to\infty} x_n|$ reverse triangle inequality.

Recursively-Defined Sequences

Ex $x_1 = 1/3$ $\quad x_{n+1} = \frac{1}{2} x_n + x_n^2$

$L = \frac{1}{2} L + L^2 \quad 0 = L^2 - \frac{1}{2} L \Rightarrow L = 0$ or $L = \frac{1}{2}$

Claim $x_n \in [0, \frac{1}{2}]$ for all n. Then $0 \le \frac{1}{3} \le \frac{1}{2}$. Assume $0 \le x_k \le \frac{1}{2}$.

Then $0 \le \frac{1}{2} x_k \le \frac{1}{4}$ and $0 \le x_k^2 \le \frac{1}{4}$

Then $0 \le \frac{1}{2} x_k + x_k^2 \le \frac{1}{4} + \frac{1}{4}$ i.e., $0 \le x_{k+1} \le \frac{1}{2}$.

Claim: $x_{k+1} \le x_k$ for all $k \in \mathbb{N}$.

$x_2 = \frac{1}{2}\frac{1}{3} + \left(\frac{1}{3}\right)^2 = \frac{1}{6} + \frac{1}{9} = \frac{2+3}{18} \le \frac{6}{18} = x_1$

Assume $x_{k+1} \le x_k$ show $x_{k+2} \le x_{k+1}$.

Then $\frac{1}{2} x_{k+1} \le \frac{1}{2} x_k$ and, as $x_{k+1} \ge 0$ and $x_k \ge 0$, $x_{k+1}^2 \le x_k^2$

Then $x_{k+2} = \frac{1}{2} x_{k+1} + x_{k+1}^2 \le \frac{1}{2} x_k + x_k^2 = x_{k+1}$.

Then, $\lim_{n\to\infty} x_n = L$ exists. $\quad x_{n+1} = \frac{1}{2} x_n + x_n^2$

k-tail result $\quad L = \frac{1}{2} L + L^2 \quad L^2 - \frac{1}{2} L = 0 \quad L = 0$ or $L = 1/2$. Can't argue $x_n < \frac{1}{2}$ $\forall n$

Since the sequence is monotone $x_k \le x_1 = \frac{1}{3}$ for all k, so $\lim_{n\to\infty} x_n \le \frac{1}{3} < \frac{1}{2}$. Then, $L = 0$. so $L < \frac{1}{2}$.

Convergence Results

Prop 2.2.10 Let $\{x_n\}$ be a seq. If $x \in \mathbb{R}$ and $\exists$ seq. $\{a_n\}$ s.t. $\lim_{n\to\infty} a_n = 0$ and $|x_n - x| \le a_n$ $\forall n$, then $\lim_{n\to\infty} x_n = x$.

Prop 2.2.11 For $c > 0$, $c < 1$ implies $\lim_{n\to\infty} c^n = 0$ and $c > 1$ implies $\{c^n\}$ is unbounded. $\quad c^n = (1+r)^n \ge 1 + nr$ or $c^n = \frac{1}{(1+r)^n} \le \frac{1}{1+nr} \le \frac{1}{r}\frac{1}{n}$

Lemma 2.2.12 (Ratio Test) Let $\{x_n\}$ be a seq. s.t. $x_n \ne 0$ $\forall n$ and $L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$ exists.

(i) If $L < 1$, then $\{x_n\}$ converges to 0 $\quad$ Pick $L < r < 1$ $\left|\frac{|x_{n+1}|}{|x_n|} - L\right| < r - L$ eventually $\frac{|x_{n+1}|}{|x_n|} < r$. For $n \ge M$ $|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \cdot \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} \le (|x_M| r^{-M}) r^n \to 0$ so M-tail of $x_n \to 0$ so $x_n \to 0$.

(ii) If $L > 1$, then $\{x_n\}$ is unbounded (and thus diverges)