Note: You can turn in the 7.1 Problems this week or with the next homework.
Unassigned, but suggested: Problems 4,5,10,11 in Section 6.1
Unassigned, but suggested: Problems 3,5 in Section 6.2
Unassigned, but suggested: Problems 6,7,8 in Section 7.1

12.1 Problem 6.1.1

Let f and g be bounded functions on $[a, b]$. Prove

$$\|f + g\|_u \leq \|f\|_u + \|g\|_u .$$

Solution. □

12.2 Problem 6.1.2

a) Find the pointwise limit $\dfrac{e^{x/n}}{n}$ for $x \in \mathbb{R}$.
b) Is the limit uniform on $\mathbb{R}$?
c) Is the limit uniform on $[0, 1]$?

Solution. □

12.3 Problem 6.1.7

Suppose there exists a sequence of functions $\{g_n\}$ uniformly converging to 0 on A. Now suppose we have a sequence of functions $\{f_n\}$ and a function f on A such that

$$|f_n(x) - f(x)| \leq g_n(x)$$

for all $x \in A$. Show that $\{f_n\}$ converges uniformly to f on A.

Solution. □

12.4 Problem 6.2.1

While uniform convergence preserves continuity, it does not preserve differentiability. Find an explicit example of a sequence of differentiable functions on $[-1, 1]$ that converge uniformly to a function f such that f is not differentiable. Hint: Consider $|x|^{1+1/n}$, show that these functions are differentiable, converge uniformly, and then show that the limit is not differentiable.

Solution. □

12.5 Problem 6.2.2

Let $f_n(x) = \frac{x^n}{n}$. Show that $\{f_n\}$ converges uniformly to a differentiable function f on $[0, 1]$ (find f). However, show that $f'(1) \neq \lim\limits_{n\to\infty} f_n'(1)$.

Solution. □

12.6 Problem 7.1.1

Show that for any set X, the discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$) does give a metric space (X, d).

Solution. □

12.7 Problem 7.1.3

Let $X := \{a, b\}$ be a set. Can you make it into two distinct metric spaces? (define two distinct metrics on it)

Solution. □

12.8 Problem 7.1.7

Let X be the set of continuous functions on $[0,1]$. Let $\varphi\colon [0,1] \to (0,\infty)$ be continuous. Define

$$d(f,g) := \int_0^1 |f(x) - g(x)| \, \varphi(x) \, dx.$$

Show that (X,d) is a metric space.

Solution. □