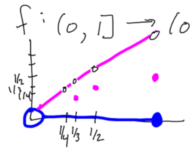
1.4 Intervals and Size of $\mathbb{R}$

(a,b), $[a,b)$, $(a,b]$, $[a,b]$, $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$, and $(-\infty,\infty)$ all have the same cardinality.

$\tan : (-\pi,\pi) \longrightarrow \mathbb{R}$ is a bijection

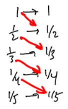
$f : (0,1] \to (0,1)$



$f(\frac{1}{n}) = \frac{1}{n+1}$ for $n = 1,2,3,\dots$
$f(x) = x$ for all other x.

$\frac{1}{\frac{1}{x}+1}$ $\quad \frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x}$

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{if } \frac{1}{x} \in \mathbb{N} \\ x & \text{if } \frac{1}{x} \notin \mathbb{N} \end{cases}$$



Card $[a,b]$ = Card $\mathbb{R}$

(There is a HW problem in which figure out another piece of the puzzle)

Theorem (Cantor) 1874 $\mathbb{R}$ is uncountable

pf Lebl proves that if $X \subset \mathbb{R}$ has the following property, then $X \subsetneq \mathbb{R}$.

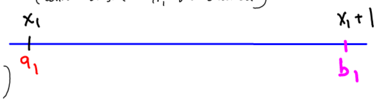
Assume X is countably infinite and $\forall\, a,b \in \mathbb{R}$, $\exists\, x \in X$ with $a < x < b$.
$(a,b) \cap X \neq \emptyset$.

$\exists$ bijection $\mathbb{N} \to X$. i.e., we may enumerate

$x_1, x_2, x_3, \dots$ (cannot assume x_i are ordered)

Define sequences $\{a_k\}$, $\{b_k\}$ inductively:

$a_1 := x_1$ $\quad b_1 = x_1 + 1$



Then $a_1 < b_1$ and $x_1 \notin (a_1, b_1)$

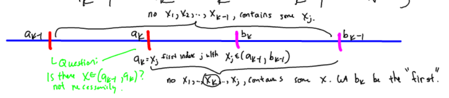
For $k > 1$, suppose a_{k-1} and b_{k-1} have been defined so that

$(a_{k-1}, b_{k-1}) \cap \{x_1, x_2, \dots x_{k-1}\} = \emptyset$. i.e. $x_j \notin (a_{k-1}, a_{k-1})$ for $j = 1,2,\dots,k-1$

Define $a_k := x_j$ where j is the smallest $j \in \mathbb{N}$ such that $x_j \in (a_{k-1}, b_{k-1})$ ← (Such an x_j exists by assumption on X and well-ordering of $\mathbb{N}$. Also, $j \geq k$ x_j can't be x_{k-1}. It could be x_k or some later x_j.)

$b_k := x_j$ where j is the smallest $j \in \mathbb{N}$ such that $x_j \in (a_k, b_{k-1})$.

Then $a_k < b_k$ and $a_{k-1} < a_k < b_k < b_{k-1}$. Also, (a_k, b_k) does not contain x_k (or $x_{k-1}, x_{k-2}, \dots x_2, x_1$)



Claim: $a_j < b_k$ $\forall\, j,k \in \mathbb{N}$. (case $j=k$) $a_k < b_k$

(Case: $j<k$) $a_j < a_{j+1} < \dots < a_{k-1} < a_k < b_k$

(case $j>k$) $a_j < b_j < b_{j-1} < \dots < b_{k+1} < b_k$

Let $A = \{a_j : j \in \mathbb{N}\}$ and $B = \{b_j \mid j \in \mathbb{N}\}$

Then $\sup A \leq \inf B$

Define $y = \sup A$

Show $y \notin X$. $y \notin A$. otherwise, $\exists\, j \in \mathbb{N}$ s.t. $y = a_j < a_{j+1}$ contradicting y being an u.b. for A.

$y \notin B$. " " " $y = b_j > b_{j+1}$ " " " a l.b. for B

$a_j < y < b_j$ $\forall\, j \in \mathbb{N}$. $\quad y \in (a_j, b_j)$ $\forall\, j$.

By the construction of A and B, $x_j \notin (a_j, b_j)$ $\forall\, j$, so $y \neq x_j$ $\forall\, j$. i.e. $y \notin X$.

$\therefore$ X is a proper subset of $\mathbb{R}$.

$\mathbb{R}$ satisfying property ③ $(\forall\, a,b \in \mathbb{R}\ \exists\, x \in X$ with $a < x < b)$.

Thus, $\mathbb{R}$ is uncountable. □

If there is time, review $*\ |x+y| \leq |x| + |y|$ and pf of $|x-y| \geq \big||x| - |y|\big|$

Def 1.3.6 We say $f : D \to \mathbb{R}$ is bounded if $\exists\, M \in \mathbb{R}$ such that $|f(x)| \leq M$ $\forall\, x \in D$

Notation: $\sup\limits_{x \in D} f(x) = \sup f(D)$ $\quad \inf\limits_{x \in D} f(x) = \inf f(D)$

Prop 1.3.7 If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ $(D \neq \emptyset)$ are bounded and $f(x) \leq g(x)$ $\forall\, x \in D$, then

$\sup\limits_{x \in D} f(x) \leq \sup\limits_{x \in D} g(x)$ and $\inf\limits_{x \in D} f(x) \leq \inf\limits_{x \in D} g(x)$

(Be careful about the variables)

We can't conclude $\sup\limits_{x \in D} f(x) \leq \inf\limits_{x \in D} g(x)$

We would need the stronger hypothesis: $f(x) \leq g(y)$ $\forall\, x, y \in D$.