7.1 Metric Spaces

Def 7.1.1 Let X be a set and let $d: X \times X \to \mathbb{R}$ be a function such that

(i) $d(x,y) \geq 0$ for all $x, y \in X$

(ii) $d(x,x) = 0$ iff $x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

Then the pair (X, d) is called a metric space.

d is called a metric or distance function.

Examples

* $(\mathbb{R}^n, d)$ where d is the Euclidean metric
  $d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$

* $(C([a,b]), d)$ where $C([a,b]) = \{ f: [a,b] \to \mathbb{R} : f \text{ is continuous} \}$ and
  $d(f,g) = \sup \{ |f(x) - g(x)| : x \in [a,b] \}$

Definition A subset $S \subset X$ of a metric space (X, d) is bounded if there exists $p \in X$ and $M \in \mathbb{R}$ so that
$d(p, x) \leq M$ for all $x \in S$.

Intuitively, S is contained in a closed ball centered at p with radius M