7.2 Open and Closed Sets

Def * For $\delta > 0$ and $x \in X$ the open ball centered at x with radius δ is
$$B(x,\delta) := \{y \in X : d(x,y) < \delta\}.$$
* The closed ball is $C(x,\delta) := \{y \in X : d(x,y) \le \delta\}$
* A subset $V \subset X$ is an open set if for all $x \in V$, there exists some $\delta > 0$ with $B(x,\delta) \subset V$.
* A subset $E \subset X$ is a closed set if $X \setminus E$ (the complement of E) is an open set.

Proposition Let (X,d) be a metric space

(i) $\emptyset$ and X are open

(ii) Let $n \in \mathbb{N}$. If $U_1, U_2, \ldots, U_n$ are open, then $\bigcap_{i=1}^{n} U_i$ is open (finite intersection)

Hint: Let $x \in \bigcap_{i=1}^{n} U_i$. For each $i=1,\ldots,n$ $\exists\, \delta_i > 0$ so that $B(x,\delta_i) \subset U_i$. Then $B(x, \min\{\delta_1, \delta_2, \ldots, \delta_n\}) \subset \bigcap_{i=1}^{n} U_i$

(iii) Let Λ be an arbitrary indexing set. If U_λ is open $\forall \lambda \in \Lambda$, then $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open (arbitrary unions)

Let $x \in \bigcup_{\lambda\in\Lambda} U_\lambda$. Then $\exists\, \tilde{\lambda} \in \Lambda$ so that $x \in U_{\tilde{\lambda}} \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$.

Since $U_{\tilde{\lambda}}$ is open, $\exists\, \delta > 0$ so that $x \in B(x,\delta) \subseteq U_{\tilde{\lambda}} \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$.

There is a similar (but different) proposition for closed sets (assigned for HW)

Remarks * The proposition above is usually false if finite intersections are replaced by arbitrary intersections. E.g in $(\mathbb{R}, |\cdot|)$ $\bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open although each $(-\frac{1}{n}, \frac{1}{n})$ is open.
* $[0,2)$ is neither open nor closed.
* In $\mathbb{R}$ the only sets that are "clopen" (both closed and open) are $\emptyset$, $\mathbb{R}$. In more general spaces, there may be additional clopen sets (disconnected)

Def Given $A \subset X$ with (X,d) a metric space, the closure of A is
$$\bar{A} := \bigcap \{E \subset X \mid E \text{ is closed and } A \subset E\}$$
(intersection of all closed sets containing A)

Prop The closure $\bar{A}$ is closed. If A is closed, then $\bar{A} = A$.

Prop $x \in \bar{A}$ iff $\forall \delta > 0$ $B(x,\delta) \cap A \neq \emptyset$.
(Every ball centered at x intersects A)

Def If $A \subset X$, the interior of A is the set
$$A^\circ := \bigcup \{V \subset X \mid V \text{ is open and } V \subset A\}$$

Prop A° is open. If A is open, then $A = A^\circ$.

Prop $x \in A^\circ$ iff $\exists\, \delta > 0$ so that $B(x,\delta) \subset A$.

Def The boundary of A is the set $\partial A = \bar{A} \setminus A^\circ$

Prop ∂A is closed   pf $\partial A = \bar{A} \cap (A^\circ)^c$

Prop $x \in \partial A$ iff $\forall \delta > 0$, both $B(x,\delta) \cap A$ and $B(x,\delta) \cap A^c$ are both nonempty.