3.4 Uniform Continuity

Def 3.4.1 Let $S \subset \mathbb{R}$ and let $f: S \to \mathbb{R}$ be a function. Suppose $\forall \varepsilon > 0$ $\exists \delta > 0$ so that whenever $x, c \in S$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Then we say f is **uniformly continuous**.

Example $f: [a,b] \to \mathbb{R}$ $f(x) = x^2$ is uniformly continuous. Given $\varepsilon > 0$, take $\boxed{\delta = \frac{\varepsilon}{2\max\{|a|,|b|\}}}$ (Does not depend on c). Suppose $x, c \in [a,b]$ and $|x-c| < \delta$.

$$|x^2 - c^2| = |(x-c)(x+c)| = |x-c| \cdot |x+c| \le |x-c|\left(|x| + |c|\right) \le |x-c|\left(2\max\{|a|,|b|\}\right) < \frac{\varepsilon}{2\max\{|a|,|b|\}} \cdot 2\max\{|a|,|b|\} = \varepsilon.$$

Example $f: (0,3) \to \mathbb{R}$ $f(x) = \frac{1}{2x}$ is **not** uniformly continuous.

We require $\left|\frac{1}{2x} - \frac{1}{2c}\right| = \left|\frac{c-x}{2xc}\right| < \varepsilon$. for some $\delta > 0$ and $|x-c| < \delta$.

We need $|c-x| < 2xc\varepsilon$. We would require some $\delta \le 2xc\varepsilon$

Since the domain is $(0,3)$, such δ will need to depend on c. Observe that taking $c_n = \frac{1}{n}$ would result in $\delta \le 2x\frac{\varepsilon}{n}$. Since $0 < x < 3$, then $\delta \le \frac{6\varepsilon}{n}$ Taking $n \to \infty$ shows $\delta = 0$ but δ is supposed to be positive.

Theorem Let $f: [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

pf Suppose f is not uniformly continuous.

Then $\neg\left(\forall \varepsilon > 0\ \exists \delta > 0\ \forall c \in S\ \forall x \in S\ \left(|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon\right)\right)$

$\equiv$ $\exists \varepsilon > 0\ \forall \delta > 0\ \exists c \in S\ \exists x \in S$ $|x-c| < \delta$ but $|f(x) - f(c)| \ge \varepsilon$.

(Understand that this is not say f is discontinuous. It says that (for some $\varepsilon > 0$) every δ_ε fails for some $c \in S$. Such c might not cause a different δ_ε to "fail". With this $\varepsilon > 0$ fixed,

Now, use $\delta = \frac{1}{n}$ to construct sequences: $x_n, c_n \in S$ with $|x_n - c_n| < \frac{1}{n}$ and $|f(x_n) - f(c_n)| \ge \varepsilon$

$\{x_n\}$ might diverge, but Bolzano-Weierstrass yields convergent $\{x_{n_k}\}$ set $w = \lim x_{n_k}$. $a \le x_{n_k} \le b$ implies $a \le w \le b$.

$$|w - c_{n_k}| = |w - x_{n_k} + x_{n_k} - c_{n_k}| \le |w - x_{n_k}| + |x_{n_k} - c_{n_k}| \le |w - x_{n_k}| + \frac{1}{n_k}$$

$|w - x_{n_k}|$ and $\frac{1}{n_k}$ both tend to 0 as $k \to \infty$ so

$\{c_{n_k}\}$ also convergences to w.

$||A| - |B|| \le |A-B|$

$|A| - |B| \le |A-B|$

$|B| - |A| \le |A-B|$

$$\begin{aligned}|f(w) - f(x_{n_k})| &= |f(w) - f(c_{n_k}) + f(c_{n_k}) - f(x_{n_k})| \\ &= |f(w) - f(c_{n_k}) - (f(x_{n_k}) - f(c_{n_k}))| \\ &\ge |f(x_{n_k}) - f(c_{n_k})| - |f(w) - f(c_{n_k})| \\ &\ge \varepsilon - |f(w) - f(c_{n_k})|\end{aligned}$$

$$|f(w) - f(x_{n_k})| + |f(w) - f(c_{n_k})| \ge \varepsilon$$

Either $\{f(x_{n_k})\}$ or $\{f(c_{n_k})\}$ does not converge to $f(w)$. f is not continuous.

Def 3.4.7 Let $f: S \to \mathbb{R}$ be a function so that $\exists K \in \mathbb{R}$ so that $\forall x, y \in S$ $|f(x) - f(y)| \le K|x-y|$

Then we say f is **Lipschitz continuous**.

Note: $K \le 0$ implies f is constant

Theorem Lipschitz continuous functions are uniformly continuous. pf Uniform continuity follows trivially for const. functions Suppose $K > 0$. and let $\varepsilon > 0$

Take $\delta = \frac{\varepsilon}{K}$ and suppose $x, y \in S$ with $|x-y| < \delta$. Then $|f(x) - f(y)| \le K|x-y| < K\left(\frac{\varepsilon}{K}\right) = \varepsilon$.

Example $f(x): [1,\infty) \to \mathbb{R}$ $f(x) = \sqrt{x}$ is Lipschitz cont., but $\sqrt{x}: [0,\infty) \to \mathbb{R}$ is not. (However, it is uniform cont.)

Lipschitz cont. $\Rightarrow$ uniformly cont. $\Rightarrow$ cont. $\Rightarrow$ cont. for some c