5.3 Fundamental Theorem of Calculus

Thm 5.3.1 (Part I of the FTC)

Let $F:[a,b]\to\mathbb{R}$ be continuous and suppose F is differentiable on (a,b). Let $f\in\mathcal{R}[a,b]$ with $f(x)=F'(x)$ for $x\in(a,b)$.

Then $\int_a^b f = F(b)-F(a)$.

pf Let P be an arbitrary partition of $[a,b]$. Apply the MVT to F over each $[x_{k-1},x_k]$ to get

$c_k\in[x_{k-1},x_k]$ with $F'(c_k)(x_k-x_{k-1}) = F(x_k)-F(x_{k-1})$.

Then m_k and M_k (recall $m_k=\inf\{f(x): x\in[x_{k-1},x_k]\}$ and $M_k=\sup\{f(x): x\in[x_{k-1},x_k]\}$) satisfy

$m_k \le f(c_k)\le M_k$. Therefore, $m_k\Delta x_k \le f(c_k)\Delta x_k \le M_k\Delta x_k$. By summing, $L(P,f)\le \sum_{k=1}^n f(c_k)\Delta x_k \le U(P,f)$.

Since $f(x)=F'(x)$, then the middle expression equals $\sum_{k=1}^n F'(c_k)\Delta x_k = \sum_{k=1}^n F(x_k)-F(x_{k-1}) \overset{\text{telescoping}}{=} F(b)-F(a)$.

Therefore $F(b)-F(a)\le U(P,f)$ for arbitrary P so we have

$F(b)-F(a)\le \overline{\int_a^b} f$. Similarly, $L(P,f)\le F(b)-F(a)$ for arbitrary P shows $\underline{\int_a^b} f \le F(b)-F(a)$.

Since $\underline{\int_a^b} f = \overline{\int_a^b} f$, then $\int_a^b f = F(b)-F(a)$ as desired. □

Thm 5.3.2 (Part II of FTC)

If $f:[a,b]\to\mathbb{R}$ is Riemann integrable, then $F(x):=\int_a^x f$ is continuous.

If f is continuous at $c\in[a,b]$, then F is differentiable at c and $F'(c)=f(c)$.

pf Since f is bounded, $|f(x)|\le M$ for some M and for all $x\in[a,b]$.

Suppose $x,y\in[a,b]$ and WLOG assume $x<y$.

Then $|F(y)-F(x)| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_a^y f + \int_x^a f\right| = \left|\int_x^y f\right| \le M|y-x|$. Since F is Lipschitz continuous on $[a,b]$, F is continuous over $[a,b]$.

Assume f is cont. at c and let $\varepsilon>0$. Then $\exists\,\delta>0$ so that $|f(x)-f(c)|<\frac{\varepsilon}{2}$ whenever $x\in[a,b]$ and $|x-c|<\delta$.

Then $f(c)-\frac{\varepsilon}{2} < f(x) < f(c)+\frac{\varepsilon}{2}$ for $x\in[a,b]$ and $|x-c|<\delta$.

If $x>c$, then $\int_c^x f(c)-\varepsilon/2 \le \int_c^x f \le \int_c^x f(c)+\varepsilon/2$ so $(f(c)-\varepsilon)(x-c)\le\int_c^x f\le (f(c)+\varepsilon)(x-c)$

Thus, $f(c)-\varepsilon/2 \le \frac{\int_c^x f}{x-c}\le f(c)+\varepsilon/2$. We obtain a similar inequality when $x<c$.

$\int_x^c f(c)-\varepsilon/2 \le \int_x^c f\le \int_x^c f(c)+\varepsilon/2$ so $(f(c)-\frac{\varepsilon}{2})(c-x)\le \int_x^c f \le (f(c)+\frac{\varepsilon}{2})(c-x)$.

Dividing thru by $c-x>0$ and using $\int_x^c f = -\int_c^x f$, we obtain $f(c)-\frac{\varepsilon}{2}\le \frac{-\int_c^x f}{c-x}\le f(c)+\frac{\varepsilon}{2}$.

or $f(c)-\frac{\varepsilon}{2} \le \frac{\int_c^x f}{x-c}\le f(c)+\frac{\varepsilon}{2}$.

Observe that the middle numerator equals $\int_c^a f + \int_a^x f = -\int_a^c f + \int_a^x f = -F(c)+F(x)$ so

the middle expression is really $\frac{F(x)-F(c)}{x-c}$. Thus, we have $\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| < \frac{\varepsilon}{2}$ for $|x-c|<\delta$.

So $f(c) = \lim_{x\to c}\frac{F(x)-F(c)}{x-c} = F'(c)$. □

Example: To compute $\frac{d}{dx}\int_{x^3}^{x^5} e^{-t^2}\,dt$ we use the fact that $\frac{d}{dx}\int_a^x e^{-t^2}\,dt = e^{-x^2}$. $\frac{d}{dx}F(x)=e^{-x^2}$

$\int_{x^3}^{x^5} e^{-t^2}dt = \int_{x^3}^0 e^{-t^2}dt + \int_0^{x^5} e^{-t^2}dt = -\int_0^{x^3} e^{-t^2}dt + \int_0^{x^5}e^{-t^2}dt = F(x^5)-F(x^3)$.

By the chain-rule, $\frac{d}{dx}F(x^5)-F(x^3) = F'(x^5)5x^4 - F'(x^3)3x^2 = e^{-(x^5)^2}5x^4 - e^{-(x^3)^2}3x^2 = 5x^4e^{-x^{10}} - 3x^2e^{-x^6}$