1.1 Basic Properties

Definition 1.1.1. An *ordered set* is a set A, together with a relation $<$ such that

1. For any $x, y \in A$, exactly one of $x < y$, $x = y$, or $y < x$ holds.

2. If $x < y$ and $y < z$, then $x < z$.

Definition 1.1.2. Let $E \subset A$, where A is an ordered set.

(i) If there exists a $b \in A$ such that $x \leq b$ for all $x \in E$, then we say E is *bounded above* and b is an *upper bound* of E.

(ii) If there exists a $b \in A$ such that $x \geq b$ for all $x \in E$, then we say E is *bounded below* and b is a *lower bound* of E.

(iii) If there exists an upper bound b_0 of E such that whenever b is any upper bound for E we have $b_0 \leq b$, then b_0 is called the *least upper bound* or the *supremum* of E. We write

$$\sup E := b_0.$$

(iv) Similarly, if there exists a lower bound b_0 of E such that whenever b is any lower bound for E we have $b_0 \geq b$, then b_0 is called the *greatest lower bound* or the *infimum* of E. We write

$$\inf E := b_0.$$

Definition 1.1.3. An ordered set A has the *least-upper-bound property* if every nonempty subset $E \subset A$ that is bounded above has a least upper bound, that is $\sup E$ exists in A.

Definition 1.1.5. A set F is called a *field* if it has two operations defined on it, addition $x + y$ and multiplication xy, and if it satisfies the following axioms.

(A1) If $x \in F$ and $y \in F$, then $x + y \in F$.

(A2) *(commutativity of addition)* If $x + y = y + x$ for all $x, y \in F$.

(A3) *(associativity of addition)* If $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) There exists an element $0 \in F$ such that $0 + x = x$ for all $x \in F$.

(A5) For every element $x \in F$ there exists an element $-x \in F$ such that $x + (-x) = 0$.

(M1) If $x \in F$ and $y \in F$, then $xy \in F$.

(M2) *(commutativity of multiplication)* If $xy = yx$ for all $x, y \in F$.

(M3) *(associativity of multiplication)* If $(xy)z = x(yz)$ for all $x, y, z \in F$.

(M4) There exists an element 1 (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.

(M5) For every $x \in F$ such that $x \neq 0$ there exists an element $1/x \in F$ such that $x(1/x) = 1$.

(D) *(distributive law)* $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Definition 1.1.7. A field F is said to be an *ordered field* if F is also an ordered set such that:

(i) For $x, y, z \in F$, $x < y$ implies $x + z < y + z$.

(ii) For $x, y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$.

If $x > 0$, we say x is *positive*. If $x < 0$, we say x is *negative*. We also say x is *nonnegative* if $x \geq 0$, and x is *nonpositive* if $x \leq 0$.

Proposition 1.1.8. *Let F be an ordered field and $x, y, z \in F$. Then:*

(i) If $x > 0$, then $-x < 0$ (and vice-versa).

(ii) If $x > 0$ and $y < z$, then $xy < xz$.

(iii) If $x < 0$ and $y < z$, then $xy > xz$.

(iv) If $x \neq 0$, then $x^2 > 0$.

(v) If $0 < x < y$, then $0 < 1/y < 1/x$.

Proposition 1.1.9. *Let $x, y \in F$ where F is an ordered field. Suppose $xy > 0$. Then either both x and y are positive, or both are negative.*