Using Supremum and Infimum (Cont. from last time 1.2)

write this but only prove parts below

If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, $x + A := \{x + y \mid y \in A\}$
$xA := \{xy \mid y \in A\}$

Prop 1.2.6 Let $A \subset \mathbb{R}$ be bdd. and nonempty.

(i) If $x \in \mathbb{R}$, then $\sup(x + A) = x + \sup A$
(ii) If $x \in \mathbb{R}$, then $\inf(x + A) = x + \inf A$ *
(iii) If $x > 0$, then $\sup(xA) = x \sup A$
(iv) If $x > 0$, then $\inf(xA) = x \inf(A)$
(v) If $x < 0$, then $\sup(xA) = x \inf A$ *
(vi) If $x < 0$, then $\inf(xA) = x \sup A$

pfs: (i) in book

(ii) $\inf(x + A) \le x + \inf A \iff \inf(x + A) - x \le \inf A$ — Show the Left side is a lower bound for A. Show $\inf(x+A) - x \le y \;\forall y \in A$.

$\inf(x + A)$ is a lower bound for $x + A$ so $\inf(x + A) \le x + y \;\forall y \in A$.
Then $\inf(x + A) - x \le y \;\forall y \in A$, which says $\inf(x + A) - x$ is a lower bd. for A.
By maximality, ~~inf(~~ $\inf(x + A) - x \le \inf A$. i.e.
$\inf(x + A) \le x + \inf A$.

Show $\inf(x + A) \ge x + \inf A$ — Show $x + \inf A$ is a lower bd for $x + A$. $x + \inf A \le x + y \;\forall y \in A$.

$\inf A$ is a lower bd. for A so $\inf A \le y \;\forall y \in A$.
Thus $x + \inf A \le x + y \;\forall y \in A$,
This says $x + \inf A$ is a lower bound for $x + A$ so
$x + \inf A \le \inf(x + A)$ by maximality.
Therefore, $\inf(x + A) = x + \inf A$

make shorter: $\forall y \in A$ $\inf A \le y$ so $\forall y \in A$ $x + \inf A \le x + y$. Therefore $x + \inf A$ is a l.b. for $x + A$.
and $\boxed{x + \inf A \le \inf(x + A)}$. $\inf(x + A) \le x + y \;\forall y \in A$, so
$\inf(x + A) - x \le y \;\forall y \in A$. This says $\inf(x + A) - x$ is
a l.b. for A so $\inf(x + A) - x \le \inf A$. Therefore,
$\boxed{x + \inf A \ge \inf(x + A)}$.

(iii) Similar to (i)
Prove $\sup(xA) \ge x \sup(A)$ by showing $\frac{1}{x}\sup(xA)$ is an upper bd. for A,
Prove $\sup(xA) \le x \sup(A)$ by showing $x \sup(A)$ is an upper bd. for xA.

(iv) Similar to (iii)

(v) Assume $x < 0$. $\sup(xA) \le \underbrace{x \inf A}_{\text{u.b}}$

By def. $\inf A \le y \;\forall y \in A$. Since $x < 0$, $x \inf A \ge xy \;\forall y \in A$ so $x \inf$ is an upper bd. for xA.
By minimality of sup, $\sup(xA) \le x \inf A$.

$\sup(xA) \ge x \inf A$ $\quad \underbrace{\tfrac{1}{x}\sup(xA)}_{\text{l.b.}} \le \inf A$

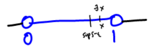
By def. of sup, $\sup(xA) \ge xy \;\forall y \in A$.
Since $x < 0$, $\frac{1}{x}\sup(xA) \le y \;\forall y \in A$ so
$\frac{1}{x}\sup(xA)$ is a lower bd. for A. By maximality of inf,
$\frac{1}{x}\sup(xA) \le \inf A$. Thus, $\sup(xA) \ge x \inf A$.

Just sketch:
Show $x \inf A$ is an u.b. for xA to get $\sup(xA) \le x \inf A$.
Show $\frac{1}{x}\sup(xA)$ is a l.b. for A to get $\frac{1}{x}\sup(xA) \le \inf A$. $\sup(xA) \ge x \inf A$

(vi) Similar to (v).

* **Prop 1.2.8** If $S \subset \mathbb{R}$ is a nonempty bdd. set, then $\forall \varepsilon > 0$, $\exists x \in S$ such that $\sup S - \varepsilon < x \le \sup S$

pf $\sup S$ is the least upper bound for S, so $\sup S - \varepsilon$ is not an u.b.
Then $\exists x \in S$ with $\sup S - \varepsilon < x$. Since $\sup S$ is an upper bd. for S, then
$x \le \sup S$. Consequently, $\sup S - \varepsilon < x \le \sup S$.



$\sup(0,1) = 1$
in general if $\sup S - \varepsilon < x \le \sup S$, we do not know if $x \in S$ or $x \notin S$

* Have class guess $\exists x \in S$ with $\inf S \le x < (\inf S) + \varepsilon$

* Skip extended reals

* If time, discuss hw or have student groups finish Prop 1.2.6

1.3 Absolute Value

Def $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$ $\quad |\cdot| : \mathbb{R} \to \mathbb{R}$

Prop 1.3.1 $\forall x,y \in \mathbb{R}$ (i) $|x| \ge 0$ and $|x| = 0$ iff $x = 0$ (ii) $|-x| = |x|$ (iii) $|xy| = |x||y|$ (iv) $|x|^2 = x^2$ (v) $|x| \le y$ iff $-y \le x \le y$ (vi) $-|x| \le x \le |x|$

Prop 1.3.2 (Triangle Inequality) $|x + y| \le |x| + |y| \quad \forall x,y \in \mathbb{R}$

By Proposition 1.3.1 part (v), it suffices to prove $-(|x| + |y|) \le x + y \le |x| + |y|$
Part (vi) tells us $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$.
$-(|x| + |y|) = -|x| - |y| \le x + y \le |x| + |y|$ □

Corollary 1.3.3 Let $x, y \in \mathbb{R}$
(i) (Reverse triangle inequality) $\big||x| - |y|\big| \le |x - y|$

$|x| = |(x - y) + y| \le |x - y| + |y|$. Thus, $|x| - |y| \le |x - y|$ ①
$|y| = |(y - x) + x| \le |y - x| + |x|$ $\quad$ $|y| - |x| \le |y - x| = |x - y|$ ②
so $-|x - y| \overset{②}{\le} |x| - |y| \overset{①}{\le} |x - y|$ so $\big||x| - |y|\big| \le |x - y|$

(ii) $|x - y| \le |x| + |y|$ $\quad |x - y| \le |x| + |-y| = |x| + |y|$.

Def 1.3.6 We say $f : D \to \mathbb{R}$ is bounded if $\exists M \in \mathbb{R}$ such that $|f(x)| \le M \;\forall x \in D$

Notation: $\sup_{x \in D} f(x) = \sup f(D)$ $\quad \inf_{x \in D} f(x) = \inf f(D)$

Prop 1.3.7 If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ $(D \ne \emptyset)$ are bounded and $f(x) \le g(x) \;\forall x \in D$, then
$\sup_{x \in D} f(x) \le \sup_{x \in D} g(x)$ and $\inf_{x \in D} f(x) \le \inf_{x \in D} g(x)$

(Be careful about the variables)

We can't conclude $\sup_{x \in D} f(x) \le \inf_{x \in D} g(x)$
We would need the stronger hypothesis: $f(x) \le g(y) \;\forall x, y \in D$.