2.5 Series

Def 2.5.1 Given a sequence $\{x_n\}$, the partial sums are
$s_n = \sum_{k=1}^{n} s_k = x_1 + x_2 + \cdots + x_{n-1} + x_n \quad (n \in \mathbb{N})$.
If $\lim_{n\to\infty} s_n$ exists, we denote it by $\sum_{k=1}^{\infty} x_n$. Note: We use this notation $\sum x_n$ even when s_n diverges, although it is then merely a formal object

Prop 2.5.5 Let $\sum x_n$ be a series. Let $M \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} x_n$ converges iff $\sum_{n=M}^{\infty} x_n$ converges.
pf: $\sum_{n=1}^{k} x_n = \underbrace{\left(\sum_{n=1}^{M-1} x_n\right)}_{\text{fixed number } T} + \sum_{n=M}^{k} x_n$ for $k \geq M$.
If $\sum_{n=1}^{\infty} x_n$ converges, then $\sum_{n=M}^{k} x_n = \sum_{n=1}^{k} x_n - T$ converges.
If $\sum_{n=M}^{k} x_n$ converges, then $\sum_{n=1}^{k} x_n = T + \sum_{n=M}^{k} x_n$ converges.

Def 2.5.6 A series $\sum x_n$ is Cauchy (Cauchy series) if $\{s_n\}$ is a Cauchy seq.

Note $\sum x_n$ converges iff $\sum x_n$ is a Cauchy series.

$\{s_n\}$ is Cauchy if $\forall \varepsilon > 0$ $\exists M \in \mathbb{N}$ s.t. $n, k \geq M$ implies $|s_k - s_n| < \varepsilon$
Assume WLOG $n < k$: $\left|\sum_{j=1}^{k} x_j - \sum_{j=1}^{n} x_j\right| = \left|\sum_{j=n+1}^{k} x_j\right| < \varepsilon$.

Prop 2.5.7 $\sum x_n$ is Cauchy if $\forall \varepsilon > 0$ $\exists M \in \mathbb{N}$ st. $\forall n \geq M$ and $\forall k > n$, $\left|\sum_{j=n+1}^{k} x_j\right| < \varepsilon$

Prop 2.5.8 If $\sum x_n$ converges, then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = 0$.
Let $\varepsilon > 0$. Since $\sum x_n$ converges, $\sum x_n$ is Cauchy. $\exists M$ st. $n \geq M$ (use $k = n+1$) $\left|\sum_{j=n+1}^{n+1} x_j\right| < \varepsilon$ so $|x_{n+1}| < \varepsilon$.
So for all $n \geq M+1$ $|x_n| < \varepsilon$. Thus, $\lim_{n\to\infty} x_n = 0$.

Example 2.5.9 The converse of Prop 2.5.8 is false
Nicole Oresme (1323 – 1382) $x_n = \frac{1}{n}$
Argument: produce an unbounded ($\therefore$ divergent) subseq. of $\{s_n\}$. s_{2^k}:

$s_1 = 1$
$s_2 = 1 + \frac{1}{2}$
$s_4 = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right)$
$s_8 = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right)$
$s_{2^k} = 1 + \sum_{j=1}^{k} \left(\sum_{m=2^{k-1}+1}^{2^k} \frac{1}{m}\right) \geq 1 + \sum_{j=1}^{k} \frac{1}{2} = 1 + \frac{k}{2}$.

Prop 2.5.10 (Linearity)