6.1 Pointwise and Uniform Convergence

Def 6.1.1 Let $f_n : S \to \mathbb{R}$ be a function for each $n \in \mathbb{N}$. We say $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f : S \to \mathbb{R}$ if for every $x \in S$, $f(x) = \lim_{n\to\infty} f_n(x)$.

Example $f_n(x) = x^n$ on $[0,1]$. $\{f_n\}$ converge pointwise to $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ Observe each f_n is continuous on $[a,b]$ but f is not.

Proposition 6.1.5 $\{f_n\}$ converges pointwise to f iff $\forall x \in S$ and $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. Note: N may depend on ε and x.

Definition Let $f_n : S \to \mathbb{R}$ be functions. We say $\{f_n\}$ converges uniformly to $f : S \to \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ so that $\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$.

Key idea: N depends only on $\varepsilon > 0$, not x.

Proposition If $\{f_n\}$ is a sequence of continuous function converging uniformly to $f : S \to \mathbb{R}$, then f is continuous.

pf Let $\varepsilon > 0$ and fix $x \in S$. By uniform convergence, there exists $N \in \mathbb{N}$ so that $n \geq N$ implies $|f_n(y) - f(y)| < \varepsilon/3$ for all $y \in S$.
Since $f_N : S \to \mathbb{R}$ is continuous at x, there exists $\delta > 0$ so that $|y - x| < \delta$ and $y \in S$ imply $|f_N(y) - f_N(x)| < \varepsilon/3$.
For such y, we then have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \underbrace{\tfrac{\varepsilon}{3}}_{\text{① uniform convergence } f_n \to f \text{ and } N \geq N} + \underbrace{\tfrac{\varepsilon}{3}}_{\text{② continuity of } f_N \text{ at } x \text{ and } |y-x|<\delta} + \underbrace{\tfrac{\varepsilon}{3}}_{\text{③ uniform convergence (again) and } N \geq N} = \varepsilon \end{aligned}$$

We have shown $\exists \delta > 0$ so that $|f(y) - f(x)| < \varepsilon$ whenever $y \in S$ with $|y - x| < \delta$.
Therefore, f is continuous at $x \in S$. As x was arbitrary, $f : S \to \mathbb{R}$ is continuous.

Note: This result tells us $\lim_{y\to x}\left(\lim_{n\to\infty} f_n(y)\right) = \lim_{n\to\infty}\left(\lim_{y\to x} f_n(y)\right)$
Uniform convergence $f_n \to f$ makes this "interchange of limits" possible.
It need not be true if we only have $f_n(x) \to f(x)$ pointwise.