4.2 The Mean Value Theorem

Def 4.2.1 Let $S \subset \mathbb{R}$ and $f: S \to \mathbb{R}$ be a fcn. f has a relative maximum (resp. minimum) at $c \in S$ if $\exists \delta > 0$ so that $f(x) \le f(c)$ (resp. $f(x) \ge f(c)$) for all $|x - c| < \delta$.

Theorem 4.2.2 If $f: [a,b] \to \mathbb{R}$ is diff. at $c \in (a,b)$ with relative extremum at c, then $f'(c) = 0$.

pf Suppose f has a relative max. $f(x) - f(c) \le 0$. If $x > c$, $\frac{f(x)-f(c)}{x-c} \le 0$ and if $x < c$ $\frac{f(x)-f(c)}{x-c} \ge 0$.

Since $f'(c)$ exists, $\exists$ seqs. $\{x_n\} \subseteq (c, c+\delta)$ and $\{y_n\} \subseteq (c-\delta, c)$ both converging to c.

$0 \le \lim \frac{f(y_n) - f(c)}{y_n - c} = f'(c) = \lim \frac{f(x_n) - f(c)}{x_n - c} \le 0$

Theorem 4.2.3 (Rolle) If $f: [a,b] \to \mathbb{R}$ is cont. and diff. on (a,b) with $f(a) = f(b) = 0$, then $\exists c \in (a,b)$ with $f'(c) = 0$

pf By the Extreme Value Theorem, f attains absolute max. and min on $[a,b]$. If either abs. max or min are attained at $c \in (a,b)$, then $f'(c) = 0$. Otherwise both abs. extrema are attained at a or b. Since $f(a) = f(b)$, the max. and min coincide, so f is constant. $f'(x) = 0$ $\forall x \in [a,b]$ and at $c = \frac{a+b}{2}$ in particular.

Theorem 4.2.4 (Mean Value Theorem) If $f: [a,b] \to \mathbb{R}$ is cont. on $[a,b]$ and diff. on (a,b), then $\exists c \in (a,b)$ with

$$f(b) - f(a) = f'(c)(b-a)$$

pf Apply Rolle's theorem to $g: [a,b] \to \mathbb{R}$ defined as $g(x) = \underbrace{f(x)}_{f(x)} - \underbrace{\left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]}_{\text{secant line btw } (a, f(a)) \text{ and } (b, f(b))}$

Applications:

Prop 4.2.5 Let I be an interval and $f: I \to \mathbb{R}$ diff. with $f'(x) = 0$ $\forall x \in I$. Then f is constant.

pf Pick any $x, y \in I$ with $x < y$. Apply MVT to $f: [x,y] \to \mathbb{R}$ to get $c \in (x,y)$ with

$f(y) - f(x) = f'(c)(y-x) = 0$

Prop 4.2.6 Let $f: I \to \mathbb{R}$ be differentiable.

(i) f is increasing iff $f'(x) \ge 0$ $\forall x \in I$.

(ii) f is decreasing iff $f'(x) \le 0$ $\forall x \in I$.

Prop 4.2.8 Let $f: (a,b) \to \mathbb{R}$ be continuous. Let $c \in (a,b)$ and suppose f diff. on (a,c) and (c,b).

(i) If $f'(x) \le 0$ on (a,c) and $f'(x) \ge 0$ on (c,b), then f has an absolute minimum at c.

(ii) If $f'(x) \ge 0$ on (a,c) and $f'(x) \le 0$ on (c,b), then f " " " maximum " c.

Theorem 4.2.9 (Darboux) Let $f: [a,b] \to \mathbb{R}$ be differentiable. Suppose $\exists y \in \mathbb{R}$ so that $f'(a) < y < f'(b)$ or $f'(a) > y > f'(b)$.

Then $\exists c \in (a,b)$ such that $f'(c) = y$.

This result is interesting because $f'(x)$ does not need to be continuous