3.1 Limits of Functions (continued)

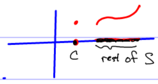
Lemma 3.1.7 Let $S \subset \mathbb{R}$ and c be a cluster point of S. Let $f: S \to \mathbb{R}$ be a function. Then $f(x) \to L$ as $x \to c$, if and only if for every sequence $\{x_n\}$ of numbers such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim_{n\to\infty} x_n = c$, we have that the sequence $\{f(x_n)\}$ converges to L.

3.2 Continuous Functions

Def 3.2.1 Let $S \subset \mathbb{R}$, $c \in S$ (not necessarily a cluster point), and let $f: S \to \mathbb{R}$ be a function. We say f is continuous at c if $\forall \varepsilon > 0\ \exists \delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Prop 3.2.2 Suppose $f: S \to \mathbb{R}$ and $c \in S$

i) If c is not a cluster point of S, then f is continuous at c. E.g.



pf $\exists \delta > 0$ s.t. $(S \setminus \{c\}) \cap (c - \delta, c + \delta)$ is empty.

For every $\varepsilon > 0$, use this δ. For every $x \in S \cap (c-\delta, c+\delta)$, we have $|f(x) - f(c)| < \varepsilon$ because $x = c$ is the ONLY element in $S \cap (c - \delta, c+\delta)$. Trivially, $|f(c) - f(c)| = 0 < \varepsilon$.

In Calc I, domains were typically intervals so phenomena such as this involving "non cluster points" was never seen. Beware: your intuition was imperfectly trained.

ii) If c is a cluster point of S, then f is cont. at c iff the limit of $f(x)$ as $x \to c$ exists and $\lim_{x \to c} f(x) = f(c)$.

pf Assume c is a cluster point.

"$\Rightarrow$" Suppose f cont. at c: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in S$ $(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$

If we restrict x to $S \setminus \{c\} \subsetneq S$: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in S \setminus \{c\}$ $(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$

This says $\lim_{x \to c} f(x) = f(c)$, by def. of limit of $f(x)$ as $x \to c$.

"$\Leftarrow$" Suppose $\lim_{x \to c} f(x) = f(c)$: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\boxed{\forall x \in S \setminus \{c\}}$ $(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$.

We need to also consider $x = c$. $|f(c) - f(c)| = 0 < \varepsilon$ trivially, so

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\boxed{\forall x \in S}$ $(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$

This is the def. of f is continuous at c.

iii) f is cont. at c iff $\forall$ seq. $\{x_n\} \subseteq S$ with $\lim x_n = c$, the seq. $\{f(x_n)\}$ converges to $f(c)$.

pf: "$\Rightarrow$" Suppose f is cont. Let $\{x_n\} \subseteq S$ be a seq. with $\lim x_n = c$.

Let $\varepsilon > 0$. Since f is cont. at c, $\exists \delta > 0$ s.t. $\forall x \in S$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

We need $x_n \in (c - \delta, c + \delta)$ "eventually". Fortunately, $\lim x_n = c$, so $\exists N \in \mathbb{N}$ with $|x_n - c| < \delta$ for all $n \geq N$.

For $n \geq N$, $|x_n - c| < \delta$ and $x_n \in S$, so $|f(x_n) - f(c)| < \varepsilon$. In other words, $\exists N \in \mathbb{N}$ with $|f(x_n) - f(c)| < \varepsilon$ for all $n \geq N$ so the seq. $\{f(x_n)\}$ converges to $f(c)$. Summary: For $\varepsilon > 0$, continuity of f at c gives us δ. $\lim x_n = c$ gives N

"$\Leftarrow$" Prove the contrapositive: If f is not cont. @ $x = c$, then $\{f(x_n)\}$ does not converge to $f(c)$

Note $\{f(x_n)\}$ may diverge or converge to $L \neq f(c)$.

Negate the def. of continuity: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in S$ $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

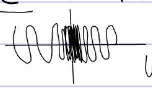
f is not cont. at c: $\boxed{\exists \varepsilon > 0}\boxed{\forall \delta > 0}\boxed{\exists x \in S}$ s.t. $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.

Take the $\varepsilon > 0$ that exists above. For each n, $\delta = \frac{1}{n}$ to get some $x_n \in S$ with $|x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(c)| \geq \varepsilon$

In this way, we get a sequence $\{x_n\}$ that converges to c (since $|x_n - c| < \frac{1}{n}\ \forall n \in \mathbb{N}$)

but $\{f(x_n)\}$ does not converge to $f(c)$ ($|f(x_n) - f(c)| \geq \varepsilon$ for some particular ε given above. i.e. $|f(x_n) - f(c)|$ cannot be made arbitrarily small)

Example Use the previous result to show we cannot make $\begin{cases} \sin(1/x) & x \neq 0 \\ L & x = 0 \end{cases}$ cont. @ $x=0$ by choosing L.



idea: Find two sequences $x_n \to 0$ and $y_n \to 0$ with $\lim \sin\left(\frac{1}{x_n}\right) \neq \lim \sin\left(\frac{1}{y_n}\right)$.

$\sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$ and $\sin\left(-\frac{\pi}{2} + 2n\pi\right) = -1$ for all $n \in \mathbb{N}$.

Let $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} = \frac{2}{(4n+1)\pi}$ and $y_n = \frac{1}{-\frac{\pi}{2} + 2n\pi} = \frac{2}{(4n-1)\pi}$

This is a lot easier than showing $\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in S\ |x - 0| < \delta$ but $|f(x) - L| \geq \varepsilon$.

Prop 3.2.5 Let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be continuous at $c \in S$.

(i) $h(x) := f(x) + g(x)$ is continuous at c.

pf Let $\varepsilon > 0$. $\exists \delta_1 > 0$ s.t. $\forall x \in S\ |x - c| < \delta_1 \Rightarrow |f(x) - f(c)| < \frac{\varepsilon}{2}$.

$\exists \delta_2 > 0$ s.t. $\forall x \in S\ |x - c| < \delta_2 \Rightarrow |g(x) - g(c)| < \frac{\varepsilon}{2}$.

Then let $\delta = \min\{\delta_1, \delta_2\}$. and $x \in S$ with $|x - c| < \delta$.

$|(f(x) + g(x)) - (f(c) + g(c))| \leq |f(x) - f(c)| + |g(x) - g(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

ii) $f - g$ cont. @ c Hint $|f(x) - g(x) - (f(c) - g(c))| \leq |f(x) - f(c)| + |g(c) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

iii) fg cont. @ c Hint $|f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)|$

iv) If $g(x) \neq 0$ for all $x \in S$, $\frac{f}{g}$ cont. @ c. $\left|\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right| = \left|\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right|$

Prop 3.2.7 Let $A, B \subset \mathbb{R}$, $f: B \to \mathbb{R}$, $g: A \to B$. If g is cont. at $c \in A$ and f is cont. at $g(c)$, then $f \circ g$ is cont. at c.

pf (1) Suppose $\{x_n\} \subseteq A$ and $\lim x_n = c$. Since g is cont.

$\lim g(x_n) = g(c)$. Since $g(x_n) \subseteq B$ and f is cont. @ $g(c)$,

then $\lim f(g(x_n)) = f(g(c))$. As $\{x_n\}$ was arbitrary,

$f \circ g$ is cont. @ c.

(2) Alternative pf Let $\varepsilon > 0$. $\exists \delta_1$ s.t.

$|w - g(c)| < \delta_1$ implies $|f(w) - f(g(c))| < \varepsilon$.

Since g is cont. @ c $\exists \delta_2 > 0$ with

$x \in A$ and $|x - c| < \delta_2$ implies $|g(x) - g(c)| < \delta_1$.

As $|g(x) - g(c)| < \delta_1$, then $|f(g(x)) - f(g(c))| < \varepsilon$.