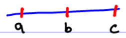
5.2 Properties of the Integral

Additivity



Lemma 5.2.1 Suppose $a<b<c$ and $f:[a,c]\to\mathbb{R}$ is bounded. Then
$\underline{\int}_a^c f = \underline{\int}_a^b f + \underline{\int}_b^c f$ and $\overline{\int}_a^c f = \overline{\int}_a^b f + \overline{\int}_b^c f$

key idea.
pf: Take arbitrary partitions P_1 and P_2 of $[a,b]$ and $[b,c]$. Consider $P=P_1\cup P_2$ for $[a,c]$.
Note $P=P_1\cup P_2$ is not the most general partition of $[a,c]$ but is for those with $b\in P$. However, a refinement $P=\tilde{P}\cup\{b\}$ makes $P=P_1\cup P_2$ sufficient for purposes of sup / inf.

Theorem 5.2.2 Let $a<b<c$. A function $f:[a,c]\to\mathbb{R}$ is Riemann Integrable if and only if f is Riemann integrable on $[a,b]$ and $[c,d]$. If f is Riemann integrable, then $\int_a^c f = \int_a^b f + \int_b^c f$

pf The hardest part (and only nontrivial part) is showing f is integrable on $[a,b]$ and $[b,c]$ if it is on $[a,c]$.
Using the lemma above if f is integrable on $[a,c]$, then

$$\underline{\int}_a^b f + \underline{\int}_b^c f \underset{\text{lemma}}{=} \underline{\int}_a^c f \underset{\text{hypothesis}}{=} \overline{\int}_a^c f \underset{\text{lemma}}{=} \overline{\int}_a^b f + \overline{\int}_b^c f$$

Then

$$\begin{aligned}
\underline{\int}_a^b f &= \underline{\int}_a^b f + \underbrace{\underline{\int}_b^c f - \underline{\int}_b^c f}_{0} \\
&= \left(\overline{\int}_a^b f + \overline{\int}_b^c f\right) - \underline{\int}_b^c f \quad \text{(hypothesis and lemma)} \\
&= \overline{\int}_a^b f + \underbrace{\left(\overline{\int}_b^c f - \underline{\int}_b^c f\right)}_{\geq 0} \quad \underline{\int}_b^c f \leq \overline{\int}_b^c f \\
&\geq \overline{\int}_a^b f.
\end{aligned}$$

Since we always have $\underline{\int}_a^b f \leq \overline{\int}_a^b f$, then we conclude $\underline{\int}_a^b f = \overline{\int}_a^b f$.

Corollary 5.2.3 If $f\in\mathcal{R}[a,b]$ and $[c,d]\subset[a,b]$, then $f|_{[c,d]}\in\mathcal{R}[c,d]$.

Linearity and Monotonicity

Proposition 5.2.4 (Linearity) If $f,g\in\mathcal{R}[a,b]$ and $\alpha\in\mathbb{R}$, then
(i) $\alpha f\in\mathcal{R}[a,b]$ and $\int_a^b \alpha f = \alpha\int_a^b f$
(ii) $f+g\in\mathcal{R}[a,b]$ and $\int_a^b f+g = \int_a^b f + \int_a^b g$

Proposition 5.2.5 (Montonicity) If $f,g\in\mathcal{R}[a,b]$ and $f(x)\leq g(x)$ for all $x\in[a,b]$, then $\int_a^b f \leq \int_a^b g$.
Key idea
pf Show $L(P,f)\leq L(P,g)$.