(Continuation from last time - 1.4)

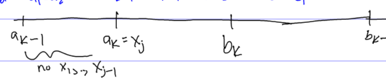
$X = \{x_1, x_2, x_3, \dots\}$ and $\forall a,b \in \mathbb{R}$ with $a<b$, $(a,b) \cap X \neq \emptyset$.

$a_1 := x_1$ and $b_1 := x_1 + 1$. Then $a_1 < b_1$ and $x_1 \notin (a_1, b_1)$.

Suppose $a_1, \dots, a_{k-1}$ and $b_1, \dots, b_{k-1}$ are already defined with

$a_{k-1} < b_{k-1}$ and $\{x_1, x_2, \dots, x_{k-1}\} \cap (a_{k-1}, b_{k-1}) = \emptyset$

need $a_1 < a_2 < \dots < a_{k-1} < b_{k-1} < b_{k-2} < \dots < b_2 < b_1$



(a_{k-1}, b_{k-1}) contains some x_j but none of $x_1, x_2, \dots, x_{k-1}$.
Let $a_k = x_j$ where $j = \min\{n \in \mathbb{N} \mid x_n \in (a_{k-1}, b_{k-1})\}$ (well-ordering)
observe that $j \geq k$ and $a_{k-1} < a_k$.

Similarly (a_k, b_{k-1}) contains some x_t but none of $x_1, \dots, x_k, \dots, x_j$ (maybe equal)
Let $b_k = x_t$ where $t = \min\{n \in \mathbb{N} \mid x_n \in (a_k, b_{k-1})\}$

Then $a_{k-1} < a_k < b_k < b_{k-1}$.

Consequences (Don't prove)
* $a_j < b_k \ \forall j,k \in \mathbb{N}$ (all the a's are smaller than all the b's)
* $\sup_{j \in \mathbb{N}} \{a_j\} \leq \inf_{k \in \mathbb{N}} \{b_k\}$

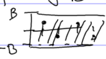
* Let $y = \sup_{j \in \mathbb{N}} \{a_j\}$
* $y \notin A$ $(a_{k-1} < a_k)$ and $y \notin B$ $(b_k < b_{k-1})$
* so $a_j < y < b_j \ \forall j$ but $x_j \notin (a_j, b_j) \ \forall j$.
* $y \notin X$ so $X \subsetneq \mathbb{R}$. □

2 Sequences & Series

2.1 Sequences & limits

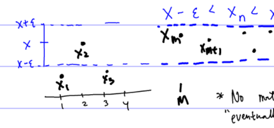
Def * A sequence is a function $x : \mathbb{N} \to \mathbb{R}$ but denoted $\{x_n\}$

* $\{x_n\}$ is bounded if $\exists B \in \mathbb{R}$ with $|x_n| \leq B \ \forall n \in \mathbb{N}$.



* A sequence $\{x_n\}$ converges to $x \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists M \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n \geq M$

$x - \varepsilon < x_n < x + \varepsilon$



* No matter how small $\varepsilon > 0$ is, "eventually all" the x_n's are in $(x - \varepsilon, x + \varepsilon)$.

Prop 2.1.6 Limits are unique.

pf Suppose $\{x_n\}$ converges to x and y.
Let $\varepsilon > 0$. By def. of convergence, there exists $M_1 \in \mathbb{N}$ with $|x_n - x| < \varepsilon/2$ for all $n \geq M_1$.
Similarly $\exists M_2 \in \mathbb{N}$, $|x_n - y| < \frac{\varepsilon}{2}$ for $n \geq M_2$. Let $M = M_1 + M_2$ (or $\max\{M_1, M_2\}$)
Then $|x - y| = |x - x_M + x_M - y| \leq |x - x_M| + |x_M - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
As $0 \leq |x - y| < \varepsilon \ \forall \varepsilon > 0$, $|x - y| = 0$ so $x = y$. □

This justifies the notation $\lim_{n \to \infty} x_n = x$.

Ex Let $c \in \mathbb{R}$ and consider the constant sequence $c, c, c, \dots$

Show $\lim_{n \to \infty} c = c$.

work backwards: want $|x_n - c| < \varepsilon$ for $n \geq$ some M.
$|c - c| = 0 < \varepsilon$ automatically
so we just use $M = 1$

pf Let $\varepsilon > 0$. Pick $M = 1$.
Then whenever $n \geq 1$, $|x_n - c| = |c - c| = 0 < \varepsilon$ □.

Ex Show $\lim_{n \to \infty} \frac{1}{n} = 0$. Want $\left|\frac{1}{n} - 0\right| < \varepsilon$ $\quad \frac{1}{n} < \varepsilon \quad n > \frac{1}{\varepsilon}$.
Use the Archimedean property.

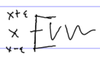
pf Let $\varepsilon > 0$. By the Archimedean property, $\exists M \in \mathbb{N}$ with $M > \frac{1}{\varepsilon}$. Then, $\frac{1}{M} < \varepsilon$. If $n \geq M$, $\frac{1}{n} \leq \frac{1}{M}$. Therefore, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{M} < \varepsilon$ whenever $n \geq M$.

Ex $1, 2, 3, \dots$ diverges: $\forall x \in \mathbb{R}$, $\{n\}$ does not converge to x

$\exists \varepsilon > 0$ such that $\forall M \in \mathbb{N} \ \exists n \geq M$ so that $|x_n - x| \geq \varepsilon$.

Consider $\varepsilon = 1$. Given any $M \in \mathbb{N}$, there exists natural $n > M + (x+1)$ (archimedes). Then $x_n = n \geq x + 1$ and $n \geq M$ so we have found $n \geq M$ with $|x_n - x| \geq \varepsilon$.

want $n \geq M$ and $x_n \leq x - \varepsilon$ or $x_n \geq x + \varepsilon$
want $n \geq x+1$ and $n \geq M$



no matter how far out you go ($\forall M$) there exists a further x_n ($\exists n$) outside the ε-tube

Example $\lim_{n \to \infty} \frac{3n-1}{n+1} = 3$.

not part of proof:
want $\left|\frac{3n-1}{n+1} - 3\right| < \varepsilon$ $\quad \left|\frac{3n-1-3n-3}{n+1}\right| < \varepsilon$ $\quad \frac{4}{n+1} < \varepsilon$
$\frac{n+1}{4} > \frac{1}{\varepsilon}$ $\quad n+1 > \frac{4}{\varepsilon}$ $\to$ $n > \frac{4}{\varepsilon} - 1$ magic sauce hiding in fridge.

By Archimedean Prop. $\exists M \in \mathbb{N}$ $M > \frac{4}{\varepsilon} - 1$.
Suppose $n \geq M$. Now show $|x_n - 3| < \varepsilon$.

Prop 2.1.7 A convergent sequence is bounded. pf Suppose x_n converges to x.

pf $\exists M \in \mathbb{N}$ with $|x_n - x| < 1853$ whenever $n \geq M$
Then $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1853 + |x|$ for $n \geq M$
Let $B_1 = 1853 + |x|$. (fixed bound).
Set $B_2 = \max\{|x_1|, |x_2|, \dots, |x_{M-1}|\}$ and $B = \max\{B_1, B_2\}$
Then for all $n \in \mathbb{N}$ $|x_n| \leq B$.

This result explains $\{n\}$ diverges. However, the converse is not true. $x_n = (-1)^n$ is bounded but divergent.

Monotone Sequences ← START HERE next lecture

Def A sequence $\{x_n\}$ is monotone increasing if $x_n \leq x_{n+1} \ \forall n \in \mathbb{N}$.
monotone decreasing if $x_n \geq x_{n+1} \ \forall n \in \mathbb{N}$.
monotone (or monotonic) if it is either monotone inc. or dec.

Thm 2.1.10 (MCT) A monotone sequence $\{x_n\}$ is bounded iff it converges. Furthermore, if $\{x_n\}$ is increasing and bdd, then $\lim_{n \to \infty} x_n = \sup\{x_n\}$. If decreasing, then $\lim_{n \to \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

pf Suppose x_n increases and is bdd. $\exists B$ with $x_n \leq B$. By the l.u.b. property, $x := \sup\{x_n : n \in \mathbb{N}\}$ exists.
Let $\varepsilon > 0$. By supremum properties $\exists M \in \mathbb{N}$ with $x_M > x - \varepsilon$. Since $\{x_n\}$ is monotone increasing, $x_n \geq x_M > x - \varepsilon$ (induction). Thus, $|x_n - x| = x - x_n \leq x - x_M < \varepsilon$ for all $n \geq M$.

(Students: Read about tails / subsequences and the next section)

Lecture 2: Sequences & Limits

Example Show $\lim_{n\to\infty} \frac{2n-3}{7n-10} = \frac{2}{7}$

Find N so that $n \geq N$ implies $\varepsilon > \left|\frac{2n-3}{7n-10} - \frac{2}{7}\right| = \left|\frac{(2n-3)7 - 2(7n-10)}{(7n-10)7}\right| = \left|\frac{14n-21-(14n+20)}{7(7n-10)}\right| = \frac{1}{7}\left|\frac{-1}{7n-10}\right| = \frac{1}{7}\frac{1}{|7n-10|} < \varepsilon$

$|7n-10| > \frac{1}{7\varepsilon}$

Given $\varepsilon>0$, only true finitely many n: $7n < 10 - \frac{1}{7\varepsilon}$ or much better: $7n > 10 + \frac{1}{7\varepsilon}$, $n > \frac{10}{7} + \frac{1}{49\varepsilon}$

pf Let $\varepsilon > 0$. Choose $N > \frac{10}{7} + \frac{1}{49\varepsilon}$. If $n \geq N$, then $\left|x_n - \frac{2}{7}\right| = \left|\frac{2n-3}{7n-10} - \frac{2}{7}\right| = \left|\frac{(2n-3)7 - 2(7n-10)}{7(7n-10)}\right| = \left|\frac{-1}{7(7n-10)}\right| = \frac{1}{7}\frac{1}{|7n-10|} \underset{n \geq \frac{10}{7}}{=} \frac{1}{7(7n-10)} < \varepsilon$

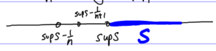
$n \geq N > \frac{10}{7} + \frac{1}{49\varepsilon}$ $\quad n - \frac{10}{7} > \frac{1}{49\varepsilon}$ $\quad \frac{7n-10}{7} > \frac{1}{49\varepsilon}$ $\quad 7(7n-10) > \frac{1}{\varepsilon}$ $\quad \frac{1}{7(7n-10)} < \varepsilon$

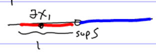
$\frac{1}{n - \frac{10}{7}} < 49\varepsilon$

$\frac{1}{49(n-\frac{10}{7})} < \frac{1}{49} 49\varepsilon = \varepsilon$

Prop 2.1.13 Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then $\exists$ monotone sequences $\{x_n\}$ and $\{y_n\}$ in S s.t. $\sup S = \lim_{n\to\infty} x_n$ and $\inf S = \lim_{n\to\infty} y_n$.

($\forall n \in \mathbb{N}$ $\exists$ $x_n \in S$ with $\sup S - \frac{1}{n} < x_n \leq \sup S$ not necessarily monotone)





$\exists$ $x_1 \in S$ s.t. $\sup S - 1 < x_1 \leq \sup S$ If $x_1 = \sup S$

For the next ε, use $\sup S - x_1$. Unfortunately, ε might be 0. ($\sup S = x_1 \in S$)

Case 1 If $\sup S \in S$, then the constant sequence $x_k = \sup S$ $\forall k \in \mathbb{N}$ works.

Otherwise $\exists$ $x_1 \in S$ s.t. $\sup S - 1 < x_1 < \sup S$.

$\exists$ $x_2 \in S$ s.t. $\sup S - \min\{\frac{1}{2}, \sup S - x_1\} < x_2 < \sup S$.

Then $|\sup S - x_2| = \sup S - x_2 < \frac{1}{2}$ and $\sup S - x_2 < \min\{\frac{1}{2}, \sup S - x_1\}$ $(x_1 < x_2)$

Suppose $x_1, \ldots, x_k$ have been defined so that

$x_1, \ldots, x_k \in S$ is monotone and $\sup S - x_k < \frac{1}{k}$.

Then $\exists$ $x_{k+1} \in S$ s.t. $\sup S - \min\{\frac{1}{k+1}, \sup S - x_k\} < x_{k+1} < \sup S$.

Then $\sup S - x_{k+1} < \frac{1}{k+1}$ and $x_k < x_{k+1}$.

Therefore, $\forall \varepsilon > 0$, $\exists N$ s.t. $\frac{1}{N} < \varepsilon$.

For $n \geq N$, $|\sup S - x_n| = \sup S - x_n < \min\{\frac{1}{n}, \sup S - x_k\} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

$\lim_{n\to\infty} x_n = \sup S$.

2.1.2 Tail of a sequence

Def 2.1.14 The K-tail (or tail) of a seq. $\{x_k\}$ is the seq. starting at $k+1$ $\{x_{n+k}\}_{n=1}^{\infty}$ or $\{x_n\}_{n=k+1}^{\infty}$

Prop 2.1.15 For any $K \in \mathbb{N}$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges iff the K-tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges

Further, if the limit exists, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}$.

pf Hint. Let $\varepsilon > 0$. "$\Rightarrow$" $\exists N$... consider N. "$\Leftarrow$" $\exists N$... consider $N+K$.

2.1.3 Subsequences

Def 2.1.16 Let $\{x_n\}$ be a seq. Let $\{n_i\}$ be a strictly inc. seq. in $\mathbb{N}$

$\{x_{n_i}\}_{i=1}^{\infty}$ is called a subseq.

$x_1, x_2, x_3, x_4, x_5, \ldots$

$x_2, x_5, x_{17}, x_{317}, \ldots$

$-1, 1, -1, 1, -1$ diverges

$-1 \quad -1 \quad -1$ converges

Prop 2.1.17 If $\{x_n\}$ is a convergent seq., then any subseq. $\{x_{n_i}\}$ is convergent and has the same limit. $\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}$

Another (ε-N) example: $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0$

Work: $\left|\frac{1}{\sqrt{n^2+n}} - 0\right| < \varepsilon$ $\quad \frac{1}{n^2+n} < \varepsilon^2$ $\quad n^2+n > \frac{1}{\varepsilon^2}$ $\quad n^2+n > n^2 > \frac{1}{\varepsilon^2}$ Choose $N > \frac{1}{\varepsilon}$.

Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon}$ in $\mathbb{N}$. Suppose $n \geq N$. Then $n^2 + n > n^2 \geq N^2 > \frac{1}{\varepsilon^2}$ so $\frac{1}{n^2+n} < \varepsilon^2$

Taking sq. roots, $\frac{1}{\sqrt{n^2+n}} < \varepsilon$ so $|x_n - 0| < \varepsilon$.