1.2 The Set of Real Numbers

Last time, $\forall x \in F$ $0 \cdot x = 0$. By commutativity $x \cdot 0 = 0$

We talked about ordered sets and fields last week.

Def A field is an <u>ordered field</u> if it also an ordered set (trichotomy/transitivity) such that

(i) $\forall x, y, z \in F$ if $x < y$, then $x + z < y + z$

(ii) $\forall x, y \in F$ if $x > 0$ and $y > 0$, then $xy > 0$.

Tell students to read Prop 1.1.8 (along w/ proofs) (If $\exists$ Questions, see me in the office)

$(-x)y = -(xy)$ pf use uniqueness of $-(xy)$: $xy + (-x)y = yx + y(-x) = y(x - x) = y \cdot 0 = 0$

Corollary $(-1)y = -(1 \cdot y) = -y$

$-(-y) = y$ pf By def. $-(-y)$ satisfies $(-y) + -(-y) = 0$ but $-y$, by def. is the additive inverse of y so $y + -y = 0$. This also says (after commutativity) that y is the additive inverse of $-y$. By uniqueness of additive inverse, $y = -(-y)$.

Theorem $1 > 0$.

By def (see M4) $1 \neq 0$. It suffices to prove $\forall x \in F$ with $x \neq 0$, $x^2 > 0$. Observe that $1 = 1^2$.

Assume $x > 0$. then prop. 2 of the def. of ordered field, using $y = x$, says $x^2 > 0 \cdot 0$. Last time we proved $0 \cdot x = 0$, so $x^2 > 0$.

not enough time

Now assume $x < 0$. Then $0 = x - x < 0 - x = -x$. i.e. $-x > 0$.

By the def. of ordered field $(-x)(-x) > 0 \cdot 0 = 0$.

By the results above $(-x)(\underbrace{-x}_{y}) = -(x(\underbrace{-x}_{y})) = -((-x)x) = -(-(x \cdot x)) = x \cdot x$

Thus, $x^2 > 0$. Consequently, $x^2 > 0$ if $x \neq 0$.

Streamline: By def. $1 \neq 0$. If $1 < 0$, then $0 = 1 - 1 < 0 - 1 = -1$. By part 2 of the ordered field def, $(-1)(-1) > 0$. From above, $(-1)(-1) = -(1 \cdot (-1)) = -(-1) = 1$. Thus, $1 > 0$ contradicting the assumption $1 < 0$.

Summary: If $x \neq 0$, then $x^2 > 0$. $\therefore$ $1 > 0$. Also, $\mathbb{C}$ is not an ordered field.

Thm $\exists$ unique* ordered field $\mathbb{R}$ with the least upper bd property such that $\mathbb{Q} \subseteq \mathbb{R}$
(Every nonempty subset bd. above has a l.u.b.)

$1 > 0$ If $n > 0$, then $n + 1 > n + 0 = n > 0$ so $n + 1 > 0$. By induction $n > 0$ $\forall n \in \mathbb{N}$. $\frac{1}{n} > 0$ as well $0 \neq 1 = n(\frac{1}{n}) \Rightarrow \frac{1}{n} \neq 0$; $\frac{1}{n} < 0 \Rightarrow n(\frac{1}{n}) < n \cdot 0 \ \ast$ $\therefore \frac{1}{n} > 0$. Adding 1 to $1 > 0$ implies $2 > 1$. Since $\frac{1}{2} > 0$, then $1 > \frac{1}{2}$.

Prop If $x \in \mathbb{R}$ s.t. $x \geq 0$ and $x \leq \varepsilon$ $\forall \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, then $x = 0$. Assume $x > 0$.

pf $\frac{1}{2} > 0$ (see above), so $x/2 > 0$. Since $1 > \frac{1}{2}$ and $x > 0$, then $x > \frac{x}{2}$. Thus $0 < x/2 < x$.

Let $\varepsilon = x/2$. Then $0 < \varepsilon < x$. $\ast$ $\therefore$ $x = 0$.

$a + a = a \cdot 1 + a \cdot 1 = a(1 + 1) = (1+1)a$

If $a < b$, then $a < \frac{a+b}{2} < b$. Idea Show $2a < a + b < 2b$; $a + a < a + b$; $a + b < b + b$. Slick pf: Adding a to both sides of $a < b$ yields $2a = a + a < a + b$. Since $\frac{1}{2} > 0$, $a = \frac{1}{2}(2a) < \frac{1}{2}(a + b)$

Tell students to Read the Example 1.2.3 $\exists!$ $r > 0$ s.t. $r^2 = 2$. Denote r by $\sqrt{2}$.

pf $r := \sup \{x \in \mathbb{R} \mid x^2 < 2\}$ Lebl shows how to employ the l.u.b. property to show $r^2 = 2$ ($r^2 \leq 2$ and $r^2 \geq 2$)

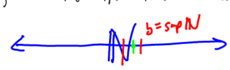
In foundations, we prove such number(s) $\notin \mathbb{Q}$. Thus, we now know irrationals exist.

Archimedean Property (i) (Archimedean property) If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ s.t. $nx > y$.] equivalent

(ii) ($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x, y \in \mathbb{R}$ and $x < y$, then $\exists r \in \mathbb{Q}$ s.t. $x < r < y$.

pf Notice that (i) implies $\exists n \in \mathbb{N}$ s.t. $n > \frac{y}{x}$ given $x, y \in \mathbb{R}$, $x > 0$. i.e. $\mathbb{N}$ not bounded above. (equivalent characterization) Every $r = \frac{r}{1}$.

pf by contradiction: Assume $\mathbb{N}$ is bdd. above. Then we may let $b := \sup \mathbb{N}$.



$b - 1 < b$ so $b - 1$ is not an upper bd. for $\mathbb{N}$.

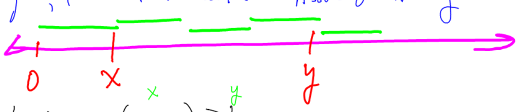
b-1 not u.b. $\exists n > b-1$ so $n+1 > b = \sup \mathbb{N}$ $\ast$

Then $\exists n \in \mathbb{N}$ s.t. $b - 1 < n$. Then $b < n + 1$. $n + 1 \in \mathbb{N}$ $\ast$.

Therefore $\forall r \in \mathbb{R}$ $\exists n$ s.t. $n > r$. (r is not an upper bd)

In particular, if $r = \frac{y}{x}$ with $x > 0$, $\exists n \in \mathbb{N}$ s.t. $n > \frac{y}{x}$. then $nx > y$.

If $x > y$, then $n = 1$ works. Assume $0 < x < y$



(ii) $\boxed{\text{Assume } x > 0}$. Since $y - x > 0$, then by (i) $\exists n \in \mathbb{N}$ s.t. $n(y - x) > 1$

Also, by (i) the set $A = \{k \in \mathbb{N} \mid k > nx\} \neq \emptyset$. "x" = 1 "y" = nx

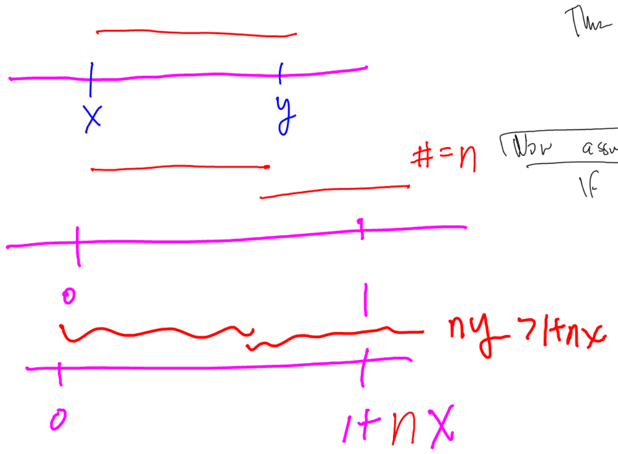
$ny - nx > 1$

If $m = 1$, then $1 > nx$

By the well-ordering property (Every nonempty subset of $\mathbb{N}$ has a least elt.) (equiv. to induction) If $m \geq 1$

A has a least element m. Since $m \in A$, then $m > nx$. By minimality, $m - 1 \notin A$. and $m - 1 \in \mathbb{N}$.

So $m-1 \le nx$. If $m=1$, then $m-1=0 \le nx$ still holds.
Thus $m-1 \le nx < m$ Divide by n to get $\frac{m-1}{n} \le x < \frac{m}{n}$.
Using $n(y-x) > 1$, $ny > 1+nx \geq 1+(m-1)=m$ so $y > \frac{m}{n}$.
i.e. $x < \frac{m}{n} < y$.



$\boxed{\text{Now assume } x<0}$. If $y>0$, then $r=0$ works for $x<0<y$.
If $y<0$, then $-x>0$ and $-y>0$ and $0<-y<-x$.
from above, $\exists\, q \in \mathbb{Q}$ s.t. $-y < q < -x$.
or $x < -q < y$ and $r=-q \in \mathbb{Q}$ as well

Corollary $\inf \{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$. pf $A \neq \emptyset$. $\frac{1}{n} > 0$ so A is bdd. below. so $\inf A$ exists. and $\inf A \geq 0$
If $a>0$ arbitrary, then by (i) $\exists\, n \in \mathbb{N}$ s.t. $na > 1$.
i.e. $\frac{1}{n} < a$. Thus, $a \neq \inf A$ so $\inf A = 0$.

For any real $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ s.t. $n\varepsilon > 1$. seems like a special case $(y=1, x=\varepsilon)$ but actually is equivalent.

Using Supremum and Infimum

If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, $x+A := \{x+y \mid y \in A\}$
$xA := \{xy \mid y \in A\}$

Prop 1.2.6 Let $A \subset \mathbb{R}$ be bdd. and nonempty.

i) If $x \in \mathbb{R}$, then $\sup(x+A) = x + \sup A$
ii) If $x \in \mathbb{R}$, then $\inf(x+A) = x + \inf A$
iii) If $x>0$, then $\sup(xA) = x \sup A$
iv) If $x>0$, then $\inf(xA) = x \inf(A)$
v) If $x<0$, then $\sup(xA) = x \inf A$
vi) If $x<0$, then $\inf(xA) = x \sup A$

pfs: (i) in book

(ii) $\inf(x+A) \le x + \inf A$. $\Leftrightarrow$ $\inf(x+A) - x \le \inf A$ show the Left side is a lower bound for A. show $\inf(x+A) - x \le y\ \forall y \in A$.
$\inf(x+A)$ is a lower bound for $x+A$ so $\inf(x+A) \le x+y\ \forall y \in A$,
Then $\inf(x+A) - x \le y\ \forall y \in A$, which says $\inf(x+A)-x$ is a lower bd. for A.
By maximality, $\inf(x+A) - x \le \inf A$. i.e.
$\inf(x+A) \le x + \inf A$.

show $\inf(x+A) \geq x + \inf A$ show $x+\inf A$ is a lower bd for $x+A$. $x+\inf A \le x+y\ \forall y \in A$,
$\inf A$ is a lower bd. for A so $\inf A \le y\ \forall y \in A$.
Thus $x + \inf A \le x+y\ \forall y \in A$,
This says $x+\inf A$ is a lower bound for $x+A$ so
$x+\inf A \le \inf(x+A)$ by maximality,
Therefore, $\inf(x+A) = x + \inf A$

(iii) Similar to (i)
Prove $\sup(xA) \geq x\sup(A)$ by showing $\frac{1}{x}\sup(xA)$ is an upper bd. for A,
Prove $\sup(xA) \le x\sup(A)$ by showing $x\sup(A)$ is an upper bd. for xA.

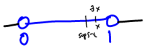
(iv) Similar to (iii)

(v) Assume $x<0$. $\sup(xA) \le x \inf A$ u.b. By def. $\inf A \le y\ \forall y \in A$. Since $x<0$, $x\inf A \geq xy\ \forall y \in A$ so $x\inf$ is an upper bd. for xA.
By minimality of sup, $\sup(xA) \le x\inf A$.
$\sup(xA) \geq x\inf A$ $\frac{1}{x}\sup(xA) \le \inf A$ l.b. By def. of sup, $\sup(xA) \geq xy\ \forall y \in A$.
Since $x<0$, $\frac{1}{x}\sup(xA) \le y\ \forall y \in A$ so
$\frac{1}{x}\sup(xA)$ is a lower bd. for A. By maximality of inf,
$\frac{1}{x}\sup(xA) \le \inf A$ Thus, $\sup(xA) \geq x\inf A$.

(vi) Similar to (v).

Prop 1.2.8 If $S \subset \mathbb{R}$ is a nonempty bdd. set, then
$\forall \varepsilon>0$, $\exists\, x \in S$ such that $\sup S - \varepsilon < x \le \sup S$

pf $\sup S$ is the least upper bound for S, so $\sup S - \varepsilon$ is not an u.b.
Then $\exists\, x \in S$ with $\sup S - \varepsilon < x$. Since $\sup S$ is an upper bd. for S, then
$x \le \sup S$. Consequently, $\sup S - \varepsilon < x \le \sup S$.



$\sup(0,1)=1$
In general if $\sup S - \varepsilon < x \le \sup S$, we do not know if $x \in S$ or $x \notin S$

* Have class guess $\exists\, x \in S$ with $\inf S \le x < (\inf S) + \varepsilon$

* Skip extended reals
* If time, discuss how or have student groups finish Prop 1.2.6

In example 1.2.3 Lebl proves that

if $A = \{x \in \mathbb{R} \mid x^2 < 2\}$, then $r = \sup A$ is the unique number satisfying $r > 0$ and $r^2 = 2$.

* 2 is an u.b. $\left(x \geq 2 \overset{\text{Prop 1.1.8}}{\Rightarrow} x^2 \geq 4\right)$

and $1 \in A$ so $r = \sup A$ exists.

(If

To show $s < r$ (r an u.b.) and not just $s \leq r$, we need $s < s+h \leq r$ with $s+h \in A$ as well

* $r^2 \geq 2$ by showing $s < r$ whenever $s > 0$ and $s^2 < 2$.

want $(s+h)^2 < 2$ $\quad s^2 + 2sh + h^2 < 2$ $\quad s^2 \quad (s+h)^2 \quad 2$

$s^2 + 2sh + h^2 < 2$

$2sh + h^2 < 2 - s^2$

$h(2s+h) < 2 - s^2$

Require $h(2s+h) < h(2s+1) < 2 - s^2$

$h < 1$ Then $0 < h < \frac{2-s^2}{2s+1}$

Now check that if $h < 1$ and $0 < h < \frac{2-s^2}{2s+1}$ implies $s^2 < (s+h)^2 < 2$.

and * $r^2 \leq 2$ by showing $s > r$ whenever $s > 0$ and $s^2 > 2$.