2.3 Limit Superior, Limit Inferior, Bolzano-Weierstrass

Suppose $\{x_n\}$ is bdd. Then each n-tail $\{x_n, x_{n+1}, \dots\}$ is bdd as well

Def Let $\{x_n\}$ be bdd. Let $a_n = \sup\{x_k : k \geq n\}$ and $b_n = \inf\{x_k : k \geq n\}$

Then $\{a_n\}$ is bdd. monotone decreasing and $\{b_n\}$ is bdd. monotone increasing

We define limsup and liminf as

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

Prop 2.3.2 Let $\{x_n\}$ be bdd. Define a_n and b_n as above.

(i) $\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\}$ and $\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\}$ pf a_n and b_n are monotone

(ii) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ pf $\inf X_n \leq \sup X_n$ so $b_n \leq a_n$ $\forall n$. Lemma 2.2.3 say's $\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$.

Note on bdd. monotonicity of a_n & b_n: If $A \subseteq B$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.

Set $T_n = \{a_k : k \geq n\}$ and $T = \{a_k : k \in \mathbb{N}\}$ Then for all n, $T_{n+1} \subseteq T_n \subseteq T$. Since T is bdd, T_n is bdd. $\inf T \leq \inf T_n = b_n$; $\inf T \leq x_k \leq \sup T$ $\forall k \geq n$. $a_n = \sup T_n \leq \sup T$

The first inclusion shows $\sup T_{n+1} \leq \sup T_n$ i.e., $a_{n+1} \leq a_n$. The seq $\{a_n\}$ is bdd. and monotone dec.

Example

$$x_n = \begin{cases} \frac{n}{n^2+1} & n \equiv 0 \bmod 3 \\ -1 + \frac{1}{n} & n \equiv 1 \bmod 3 \text{ (* start *)} \\ 1 + \frac{1}{n} & n \equiv 2 \bmod 3 \end{cases}$$

$a_n = \sup\{x_k : k \geq n\}$

$a_1 = 1$ $b_1 = -1$

$a_2 = 1$ $b_2 = -1$

$a_n = 1$ $b_n = -1$

$\limsup x_n = 1$ $\liminf x_n = -1$

(See Sage cell)

n	1	2	3	4	5	6	7	8	
a_n	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\to \limsup x_n = 1 = \inf\{a_n\}$
b_n	-1	-1	-1	-1	-1	-1	-1	-1	$\nearrow \liminf x_n = -1 = \sup\{b_n\}$

Thm 2.3.4 If $\{x_n\}$ is a bounded seq., then $\exists$ subseq. $\{x_{n_k}\}$ s.t. $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$. Similarly, $\exists$ (perhaps different) subseq. $\{x_{m_k}\}$ s.t. $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$

pf Define $a_n = \sup\{x_k : k \geq n\}$ and write $x = \limsup x_n = \lim a_n$. Define the subseq. as

Pick $n_1 = 1$ (so $x_{n_1} = x_1$) Suppose $x_{n_1}, x_{n_2}, \dots, x_{n_k}$ have been def. $\Big(\underbrace{\text{so that } a_{(n_{k-1}+1)} - \tfrac{1}{k} < x_{n_k}}_{\text{not stated in book}}\Big)$

Pick some $m > n_k$ so that $a_{(n_k+1)} - x_m < \frac{1}{k+1}$

Why? $a_{(n_k+1)} = \sup\{x_k : k \geq n_k+1\} - \frac{1}{k+1}$ is not an u.b. for T_{n_k+1}

Set $n_{k+1} = m$ i.e. x_{n_k}, x_m. Claim $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k}$

$n_k > n_{k-1}$ so $n_k \geq n_{k-1}+1$ If $m \geq n_k$, then $m \geq n_{k-1}+1$ so $T_{n_k} \subseteq T_{n_{k-1}+1}$ $\Rightarrow$ $\boxed{a_{n_k} \leq a_{n_{k-1}+1}}$ $x_{n_k} \in T_{n_k}$ so $a_{n_k} \geq x_{n_k}$.

For $k > 1$, $|a_{n_k} - x_{n_k}| = a_{n_k} - x_{n_k} \leq a_{(n_{k-1}+1)} - x_{n_k} < \frac{1}{k}$

Show x_{n_k} converges. Let $\varepsilon > 0$ $a_n \to x$ implies $a_{n_k} \xrightarrow{k\to\infty} x$. $\exists M_1 \in \mathbb{N}$ s.t.

$\forall k \geq M_1$ $|a_{n_k} - x| < \frac{\varepsilon}{2}$. Choose $M_2 \in \mathbb{N}$ s.t. $\frac{1}{M_2} \leq \frac{\varepsilon}{2}$ and take

$M = \max\{M_1, M_2\}$. For all $k \geq M$,

$$|x - x_{n_k}| = |a_{n_k} - x_{n_k} + x - a_{n_k}| \leq |a_{n_k} - x_{n_k}| + |x - a_{n_k}| < \frac{1}{k} + \frac{\varepsilon}{2} \leq \frac{1}{M_2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thm 2.3.5 Let $\{x_n\}$ be bounded. $\{x_n\}$ converges iff $\liminf x_n = \limsup x_n$.

If $\{x_n\}$ converges, $\lim x_n = \limsup x_n = \liminf x_n$.

pf With $a_n = \sup T_n$ and $b_n = \inf T_n$, $b_n \leq x_n \leq a_n$. If $\liminf x_n = \limsup x_n$, apply squeeze lemma

If $\lim x_n = x$, then by Thm 2.3.4 $\exists$ subseq. $x_{n_k} \to \limsup x_n$.

So $\lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = \limsup x_n$. (Similarly, deal w/ liminf.)

Prop 2.3.6 Suppose $\{x_n\}$ is bdd. and $\{x_{n_k}\}$ is a subseq., then $\underset{\text{(smallest subseq. limit)}}{\liminf_{n\to\infty} x_n} \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \overset{\text{(largest subseq. limit)}}{\limsup_{n\to\infty} x_n}$.

Thm 2.3.7 A bounded sequence $\{x_n\}$ is convergent and converges to x iff every convergent subseq. converges to x.

Thm 2.3.8 (Bolzano-Weierstrass) Every bounded sequence has a convergent subsequence.

pf Proved in 2.3.4. See book for more general pf that works in $\mathbb{R}^n$.