5.1 The Riemann Integral

Def 5.1.1 A partition P of the interval $[a,b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. We write $\Delta x_i = x_i - x_{i-1}$ for $1 \le i \le n$.

Let $f : [a,b] \to \mathbb{R}$ be bounded and P a partition of $[a,b]$.
We define $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ $\quad M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$

$L(P,f) = \sum_{i=1}^n m_i \Delta x_i$ (lower Darboux sum)

$U(P,f) = \sum_{i=1}^n M_i \Delta x_i$ (upper Darboux sum)

Prop 5.1.2 Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose $m \le f(x) \le M$ for all $x \in [a,b]$. Then $\forall$ partitions P,
$m(b-a) \le L(P,f) \le U(P,f) \le M(P,f)$.

pf For each $x \in [x_{i-1}, x_i]$, $m \le m_i \le M_i \le M$. Also, $\sum_{i=1}^n \Delta x_i = (\cancel{x_1} - x_0) + (x_2 - \cancel{x_1}) + \cdots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})$
$= x_n - x_0 = b - a$

For each $i = 1, \ldots, n$, $m \Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le M \Delta x_i$. Summing from $i=1$ to n,
$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.

This proposition says $\{L(P,f) : P \text{ is a partition}\}$ and $\{U(P,f) : P \text{ is a partition}\}$ are bounded sets, justifying the next definition.

Def 5.1.3 $\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$ (lower Darboux integral)

and $\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \text{ is a partition of } [a,b]\}$ (upper Darboux integral)

Def 5.1.9 A bounded function $f : [a,b] \to \mathbb{R}$ is **Riemann Integrable** if $\underline{\int_a^b} f = \overline{\int_a^b} f$. In this case, we simply write $\int_a^b f(x)\,dx$ or $\int_a^b f(x)$ or $\int_a^b f$

Def 5.1.6 For partitions $P = \{x_0, x_1, \ldots, x_n\}$ and $\tilde{P} = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_m\}$, we say $\tilde{P}$ is a refinement of P if $P \subseteq \tilde{P}$
e.g. $\{1,2,3,4\}$ is a refinement of $\{1,3,4\}$. **Intuitively,** refinement yield better approximations.

Prop 5.1.7 Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let P be a partition of $[a,b]$.
Let $\tilde{P}$ be a refinement of P. Then $L(P,f) \le P(\tilde{P},f)$ and $U(\tilde{P},f) \le U(P,f)$.

Prop 5.1.8 If $f : [a,b] \to \mathbb{R}$ is bounded with $m \le f(x) \le M$, then $m(b-a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b-a)$.

pf We know $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ for arbitrary partitions. Since $\underline{\int_a^b} f$ is an upper bound of all $L(P,f)$, then $m(b-a) \le L(P,f) \le \underline{\int_a^b} f$. Similarly, $\overline{\int_a^b} f \le U(P,f) \le M(b-a)$. The middle inequality is the result of taking refinements and using Prop 5.1.7.
If P_1 and P_2 are arbitrary partitions, then $\tilde{P} = P_1 \cup P_2$ is a refinement for both P_1 and P_2.
$L(P_1, f) \le L(\tilde{P}, f) \le U(\tilde{P}, f) \le U(P_2, f)$. Fix P_1. Then $L(P_1, f) \le U(P_2, f)$ for arbitrary P_2.
Thus, $L(P_1, f)$ is a lower bound for $\{U(P_2, f) : P_2 \text{ partition}\}$ so $L(P_1, f) \le \overline{\int_a^b} f$ (g.l.b)
However P_1 was arbitrarily fixed, so $\overline{\int_a^b} f$ is an u.b. for $\{L(P_1, f)\}$ so (l.u.b.) $\underline{\int_a^b} f \le \overline{\int_a^b} f$.

Prop 5.1.9 Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable and $m \le f(x) \le M$ $\forall x \in [a,b]$. Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Prop 5.1.13 Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann Integrable if $\forall \varepsilon > 0$ $\exists$ partition P of $[a,b]$ so that $U(P,f) - L(P,f) < \varepsilon$.

pf hint: $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P,f) - L(P,f)$. $(\underline{\int_a^b} f) - \varepsilon$ is not an u.b. $(\overline{\int_a^b} f) + \varepsilon$ is not a l.b. This gives P_1 and P_2. Consider $P_1 \cup P_2$.

Note: In class I proved if $f : [a,b] \to \mathbb{R}$ is bounded, then $\exists$ partition P s.t. $(\overline{\int_a^b} f - \underline{\int_a^b} f) \le U(P,f) - L(P,f) < (\overline{\int_a^b} f - \underline{\int_a^b} f) + \varepsilon$. This estimate yields a converse to the above prop.

Example $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$ is integrable. f is increasing so for any partition P, $f(x_{i-1}) \le f(x) \le f(x_i)$ for $x \in [x_{i-1}, x_i]$.
So $m_i = x_{i-1}^2$ and $M_i = x_i^2$. observe that $0^2 \le f(x) \le 1^2$ so $\int_0^1 f$, if it exists, is in $[0(1-0), 1(1-0)] = [0,1]$.

Key properties of f
monotonic, bdd. derivative

$M_i - m_i = x_i^2 - x_{i-1}^2$ so $U(P,f) - L(P,f) = \sum_{i=1}^n \boxed{(x_i^2 - x_{i-1}^2)} \Delta x_i$.

$\exists$ Easier argument using continuity in next section and not differentiability.

$f(x) = x^2$ is diff. on $[0,1]$ so also on $[x_{i-1}, x_i]$.
By the MVT $f(x_i) - f(x_{i-1}) = f'(\tilde{x}_i)(x_i - x_{i-1})$ for some $\tilde{x}_i \in (x_{i-1}, x_i)$
Actually $\frac{y^2 - x^2}{y - x} = \frac{\cancel{(y-x)}(y+x)}{\cancel{y-x}} = y + x = f'(c) = 2c$ says $c = \frac{x+y}{2}$ works.
So $\boxed{f(x_i) - f(x_{i-1})} = f'(\tilde{x}_i)\Delta x_i = f'\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$ $\quad f'(x) = 2x$ is bounded on $[0,1]$ by 2.

So $U(P,f) - L(P,f) = \sum_{i=1}^n \underbrace{\boxed{(x_{i-1} + x_i)\Delta x_i}}_{f'(\tilde{x}_i)\Delta x_i} \Delta x_i$

Goal $f'(x) \le K$. Given $\varepsilon > 0$ Pick any partition P with $\max\{\Delta x_i\} < \frac{\varepsilon}{K(b-a)} = \frac{\varepsilon}{2}$
E.g. regular partition $\Delta x = \frac{b-a}{N}$ using $N > \frac{K(b-a)^2}{\varepsilon} = \frac{2}{\varepsilon}$
$\frac{b-a}{N} < \frac{\varepsilon}{K(b-a)}$, $N >$

$\le \sum 2\left(\frac{\varepsilon}{2}\right)\Delta x_i = \varepsilon \sum \Delta x_i = \varepsilon$.