7.3 Sequences and Convergence

Def A <u>sequence</u> in a metric space (X,d) is a function $x: \mathbb{N} \to X$ denoted $\{x_n\}_{n=1}^{\infty}$

A sequence is <u>bounded</u> if $\exists p \in X$ and $B \in \mathbb{R}$ with $d(p, x_n) \leq B \ \forall n \in \mathbb{N}$

Def A sequence $\{x_n\}$ in (X,d) <u>Converges to $p \in X$</u> if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ with $d(x_n, p) < \varepsilon$ for all $n \geq N$.
$(\lim_{n\to\infty} x_n = p)$

Proposition A convergent sequence in a metric space has a unique limit.

Prop A convergent sequence is bounded.

Prop A sequence $\{x_n\}$ converges to $p \in X$ iff $\exists$ sequence of real numbers $\{a_n\}$ such that $d(x_n, p) \leq a_n \ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$.

Prop Let $\{x_n\}$ be a sequence
(i) If $\{x_n\}$ converges to $p \in X$, then every subsequence $\{x_{n_k}\}$ converges to p.
(ii) If for some $K \in \mathbb{N}$ the K-tail $\{x_n\}_{n=K+1}^{\infty}$ converges to p, then $\{x_n\}$ converges to p.

Prop 7.3.8 $\{x_n\}$ converges to $x \in X$ iff for every open neighborhood U of x, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n \in U$ $\boxed{\text{pf}}$ If $\lim_{n\to\infty} x_n = x$ an $x \in U$ for U open, then $\exists$ ball $x \in B(x,\varepsilon) \subseteq U$ for some $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, then $\exists M$ with $d(x, x_n) < \varepsilon \ \forall n \geq M$. I.e., $x_n \in B(x,\varepsilon) \ \forall n \geq M$. So $x_n \in U \ \forall n \geq M$. The converse follows by letting $U = B(x,\varepsilon)$ given $\varepsilon > 0$. ∎

Prop If $E \subset X$ is closed and $\{x_n\}$ is a sequence in E converging to $x \in X$, then $x \in E$.
$\boxed{\text{pf}}$ If $x \notin E$, then since E^c is open, then E^c is an open neighborhood of x. Therefore, $\exists M$ with $x_n \in E^c \ \forall n \geq M$.
Therefore $\{x_n\}$ is not a sequence in E. ∎