Math 387: Constructing the Real Numbers

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January 26, 2016

Constructions of the various number structures $\mathbb{N}, \mathbb{N} \cup \{0\}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ vary and multiple definitions have been proposed. Fortunately, they (the successful ones) can be shown to be equivalent, so we are free to choose our favorite. Below, I briefly summarize one such construction. The brevity is intentional, but comes at a heavy cost. There are many unjustified claims about extending additiion/multiplication/order operations and axioms from sets to equivalence classes of ordered pairs. You can think of this summary as an outline of a more rigorous proof.

Brief History

Archaelogists teach that "anatomically modern humans" or the subspecies Homo sapiens sapiens, the subspecies of Homo sapiens that includes all modern humans including you and me, evolved about 200,000 years ago [1]. For a bit more perspective, the Homo genus which includes our extinct relatives Homo habilis and Homo neanderthalensis is about 2.8 million year old. [2]

Prehistoric humans cut notches into bones that have been dated to be between 35,000 and 25,000 years old. [3] One wonders how many millenia of counting fingers and toes predated this revolution of tallying with objects other than body parts.

Over time we developed our tally marks technology into abstract numeral systems. Early systems had no concept of place value which made the representation of large numbers very difficult. [4] The earliest discovered place value systems come from Mesopotamia (ca. 3400 BC) and Egypt (ca 3200 BC). Various incarnations of a zero place holder have been discovered from Egyptians (ca. 1740 BC) and Babylonians, but the status of zero as a number, as something, was discomforting, especially to the Greeks who started to philosophize about such things. [4]

An abstract concept of negative numbers was used in ancient China (ca. 100 BC - 50 BC) and in India (ca. 600), but were resisted by Europeans until the 1600s or even as aas the 1800s. Philosophically, the idea of something than less than nothing was problematic as an object in itself, although very useful in finance to represent debts.

A very different account of numbers can be given that is mathematical rather than historical. Such attempts were made in the 19th century by mathematical philosophers and grew out of philosophical discussions about the true nature of numbers. Naturalists believed the natural numbers were a consequence of the human psyche. Constructivists viewed this as insufficient foundation and sought rigorous definition. Set-theoretical definitions were started by Frege (1848-1925), although they were problematic and motivated Bertrand Russell's famous paradox (1872-1970). These attempts were refined by Guiseppe Peano (1858 - 1932) [5].

Constructing $\mathbb{N} \cup \{0\}$

For convenience, denote the "whole" numbers by $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. These numbers can be constructed via set theory by declaring:

$$0 := \emptyset, \text{ the empty set.}$$

$$1 := 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}.$$

Some set-theoretic simplification is included above for convenience.

$$2 := 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{0, 1\}$$

In, general once a number x has been defined, we define its successor as $x \cup \{x\}$. As a consquence, the formaly defined number n has (informally understood) n many elements. Frege wanted to define 2 as a concept that stood for all sets have two elements and not just the particular set $\{0, 1\}$ as above. Frege's attempt motivated Russel's paradox:

> Consider the set $B = \{\text{sets } x | x \notin x\}$, the set of all sets that do not contain themselves as members. Question: Is $B \in B$? If yes, then by definition of B, it must be the case that $B \notin B$. This is a contradiction. Thus, it must be the case that $B \notin B$. By definition of B,...

Put simply, Peano's refinement was to circumvent Russell's objections to Frege by defining numbers by particular sets having a fixed cardinality, rather than as some all-consuming conglomeration. This construction is something to bear in mind when you read the cardinality discussion at the end of section 0.3 in your textbook.

Constructing $\mathbb{Z}$

The goal is to use set-theoretic concepts of cartesian products and ordered pairs and equivalence relations/classes to extent $\mathbb{N}_0$ to $\mathbb{Z}$ by defining negative numbers.

On the Cartesian product $\mathbb{N}_0 \times \mathbb{N}_0$ define the relation

$$(a, b) \sim (x, y) \text{ if and only if } a + y = b + x.$$

For example $(3, 5)$ is related to $(2, 4)$ because $3 + 4 = 5 + 2$. You can prove that this is an equivalence relation. The set of integers $\mathbb{Z}$ is defined to be the collection of equivalence classes. The symbol -2 is defined to be the equivalence class containing $(0, n)$. This also includes the pairs above $(3, 5)$ and $(2, 4)$. Observe (informally), that this relation is motivated by the fact that $-2 = 3 - 5 = 2 - 4$. The whole numbers are embedded into $\mathbb{Z}$ by associating each whole number n with the equivalence class of $(n, 0)$. This mapping can be proven to be injective. Furthermore, the familiar operations of addition and multiplication can be extended in a well-defined manner from N_0 to $\mathbb{Z}$, a collection of equivalence classes of ordered pairs of whole numbers, so that this embedding is structure-preserving and all the familiar axioms and rules of $\mathbb{Z}$ are valid. In particular, for each whole number n, the "new" number $-n$ satisfies its defining property as the additive inverse: $n + (-n) = 0$. Observe that this follows from the fact that n is really recycled notation for an equivalence class $[(n, 0)]$ and $-n$ stands for $[(0, n)]$. Since addition of equivalence classes is well-defined by addition of (any choice of) representatives.

$$n+(-n) = [(n, 0)]+[(0, n)] = [(n, 0)+(0, n)] = [(n+0, 0+n)] = [(n, n)] = [(0, 0)] = 0.$$

The equation above abuses notation a bit with how n is used as both an equivalence class and as a number within the ordered pairs, but the context should be clear. This computation gives you a flavor of how other addition/multiplication/order axioms of $\mathbb{Z}$ would be proven. For those that have taken abstract algebra, the additional structure of additive inverses makes $\mathbb{Z}$ an *abelian group* with respect to addition. Taking into account both addition and multiplication, $\mathbb{Z}$ is a *ring.*

Constructing $\mathbb{Q}$

If you think of the main theme of the previous construction as "cross addition", then the main idea to constructing $\mathbb{Q}$ would be "cross multiplication." Division by zero is problematic, so we denote the set of nonzero integers by $\mathbb{Z}^*$. We define a relation on the cartesian product $\mathbb{Z} \times \times Z*$ by declaring

$$(a, b) \sim (x, y) \text{ if and only if } ay = bx.$$

This relation turns out to be an equivalence relation and we define the rational number $\mathbb{Q}$ to be the collection of all equivalence classes. We can embed $\mathbb{Z}$ injectively into $\mathbb{Q}$ by mapping each integer z to $[(z, 1)]$. We also "recycle," a euphemism for "abuse," notation by writing z in place of $[(z, 1)]$. For nonzero z, the symbols $1/z$ or z^{-1} are used for the class $[(1, z)]$ and it can easily be proven that this is the multiplicative inverse of z. As before, the familiar operations of addition, multiplication, and order are preserved from $\mathbb{Z}$. Furthermore, the embedding of $\mathbb{Z}$ into $\mathbb{Q}$ is structure-preserving. For those that have taken linear algebra or abstract algebra, the additional structure of multiplicative inverses makes $\mathbb{Q}$ a *field.* Moreover, it is an *ordered field*, although it is not *complete*, as you will read about in Chapter 1.

Constructing $\mathbb{R}$

The details of this construction will not be readily understood until we have learned about Cauchy sequences in class. Until then, here is a brief summary. A sequence of numbers $\{x_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N > 0$ such that $\|x_n - x_m\| < \epsilon$ whenever $n, m \geq N$. A major difference between $\mathbb{Q}$ and $\mathbb{R}$ is that there are Cauchy sequences in $\mathbb{Q}$ that do not converge to a limit that is also in $\mathbb{Q}$. For example, consider decimal approximations of $\sqrt{2}$ as a sequence of rational numbers. This sequence does not have a rational limit (and no limit whatsoever if you live in a universe consisting of only rationals.) However, On the other hand, every Cauchy sequence in $\mathbb{R}$ has a limit also in $\mathbb{R}$. In fact, the point of this constructivist note is that $\mathbb{R}$ is defined precisely so that this is the case.

Let C be the set of all Cauchy sequences in $\mathbb{Q}$. Again, these are sequences $\{q_n\}$ of rational numbers satisfying the Cauchy sequence definition above. Some have rational limits; some have no limit currently. Sequences that are obviously convergent and Cauchy are constant sequences $\{q, q, q, \dots\}$. They converge to q and they are Cauchy because the expressions $\|q_n - q_m\| = 0 < \epsilon$ no matter what n, m, or $\epsilon > 0$ happen to be. We define a relation on $C \times C$ by asserting

$$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

This turns out to be an equivalence relation and we define $\mathbb{R}$ to be the collection of all equivalence classes. We can embed $\mathbb{Q}$ injectively into $\mathbb{R}$ by associating $q \in \mathbb{Q}$ with the constant sequence $\{q, q, q, , \dots\}$. This embedding is structure-preserving, the additive/multiplicative/order structures of $\mathbb{Q}$ can be extended to $\mathbb{R}$ by using equivalence class representatives. Afterwards, all the familiar addition/multiplication/order axioms of $\mathbb{R}$ are true, making $\mathbb{R}$ a *complete ordered field.*

Final note

Interestingly, all of the equivalence-relation constructions used to define $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are generalizable to more abstract algebraic structures. Moreover, these generalizations are actually important and not just done for the sake of generalizing stuff. For examples, see the Grothendiek group [6], localization of a ring, [7] and Cauchy completion in metric spaces [8]. We'll see the last one at the end of the semester, but the first two examples are too advanced for this course, not to mention undergraduate mathematics.

References

[1] https://en.wikipedia.org/wiki/Anatomically_modern_human

[2] https://en.wikipedia.org/wiki/Homo

[3] https://en.wikipedia.org/wiki/Tally_marks#Early_history

[4] https://en.wikipedia.org/wiki/Number#First_use_of_numbers

[5] https://en.wikipedia.org/wiki/Natural_number#Modern_definitions

[6] https://en.wikipedia.org/wiki/Grothendieck_group

[7] https://en.wikipedia.org/wiki/Localization_of_a_ring

[8] https://en.wikipedia.org/wiki/Complete_metric_space#Completion