3.3 Min-Max and Intermediate Value Theorems

Lemma 3.3.1 Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is bounded.

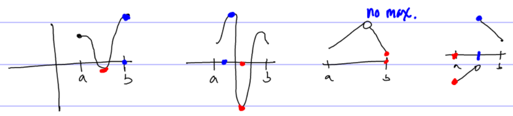
pf Suppose f is not bouded. Then $\forall n\in\mathbb{N}$ $\exists$ $x_n\in[a,b]$ s.t. $|f(x_n)|\geq n$.
$\{x_n\}$ is bounded becaus $a\leq x_n\leq b$ $\forall n$. By the Bolzano-Weierstrass Thm, $\exists$ convergent subseq. $\{x_{n_k}\}$.
Let $x=\lim\limits_{k\to\infty} x_{n_k}$. $a\leq x_{n_k}\leq b$ $\forall k$ implies $a\leq x\leq b$. $\lim\limits_{k\to\infty} f(x_{n_k})$ DNE since the seq. $\{f(x_{n_k})\}$ is unbounded
$|f(x_{n_k})|\geq n_k\geq k$ However, $f(x)=f(\lim\limits_{k\to\infty} x_{n_k})$ but $\lim\limits_{k\to\infty} f(x_{n_k})$ DNE
Thus, f is not continuous. □

Thm 3.3.2 (Minimum-maximum Thm) Let $f:[a,b]\to\mathbb{R}$ be cont. Then f achieves both an absolute max. and abs. min. on $[a,b]$.

pf From the lemma, $f([a,b])$ is bounded so has a sup and inf. $\exists$ seqs. $\{f(x_n)\}$ and $\{f(y_n)\}$ approaching them. $\lim\limits_{n\to\infty} f(x_n)=\inf f([a,b])$ and $\lim f(y_n)=\sup f([a,b])$
However x_n and y_n need not converge. Each seq. is bdd (in $[a,b]$), so apply B-W to get convergent subseqs.
$\lim\limits_k x_{n_k}=x$ and $\lim\limits_k y_{n_k}=y$. Also, $x,y\in[a,b]$. By continuity, $\inf f([a,b])=\lim\limits_n f(x_n)=\lim\limits_k f(x_{n_k})=f(\lim\limits_k x_{n_k})=f(x)$.
Similarly, $\sup f([a,b])=f(y)$.

"neripole construction"

Examples



Lemma 3.3.7 Let $f:[a,b]\to\mathbb{R}$ be cont, with $f(a)<0$ and $f(b)>0$. $\exists$ $c\in(a,b)$ with $f(c)=0$.

Define two seqs. 1 $a_1=a$ $b_1=b$ 2 If $f(\frac{a_n+b_n}{2})\geq 0$, then $a_{n+1}=a_n$ and $b_{n+1}=\frac{a_n+b_n}{2}$. 3 If $f(\frac{a_n+b_n}{2})<0$, let $a_{n+1}=\frac{a_n+b_n}{2}$ and $b_{n+1}=b_n$.
If $a_n<b_n$, then $a_{n+1}<b_{n+1}$. By induction $a_n<b_n$ $\forall n$.
$a_n\leq a_{n+1}$ and $b_n\geq b_{n+1}$
$b_{n+1}-a_{n+1}=\frac{b_n-a_n}{2}$. By induction, $b_n-a_n=\frac{b_1-a_1}{2^{n-1}}=2^{1-n}(b-a)$
The a_n and b_n seq are bdd monotone so $c=\lim a_n$ and $d=\lim b_n$.
$a_n<b_n \Rightarrow c\leq d$. and $c=\sup a_n$ and $d=\inf b_n$. So $d-c\leq b_n-a_n$
$|d-c|=d-c\leq b_n-a_n=2^{1-n}(b-a)$ $\forall n$.
$2^{1-n}(b-a)\to 0$ so $c=d$. $f(c)=f(\lim a_n)\leq 0$ since $f(a_n)\leq 0$
$f(d)=f(\lim b_n)\geq 0$ because $f(b_n)\geq 0$.
$c=d$ implies $f(c)=0$ as desired (note $a<c<b$)

Theorem 3.3.8 (Bolzano's Intermediate Value Theorem) Let $f:[a,b]\to\mathbb{R}$ be continuous
Suppose $\exists$ y s.t. $f(a)<y<f(b)$ or $f(a)>y>f(b)$. Then $\exists c\in(a,b)$ with $f(c)=y$.
pf If $f(a)<y<f(b)$, then apply the lemma to the continuous function $g(x)=f(x)-y$

Note that it is often useful to restrict $f:S\to\mathbb{R}$ to $f:[a,b]\to\mathbb{R}$ (which preserves continuity)