2.4 Cauchy Sequences

Def A sequence $\{x_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists M \in \mathbb{N}$ so that $\forall n, m \geq M$ $|x_n - x_k| < \varepsilon$

Example $\{\frac{1}{n^2}\}$ Given $\varepsilon > 0$. find M s.t. $M > \frac{\sqrt{2}}{\sqrt{\varepsilon}}$. Suppose $n, k > M$. Then $n > \frac{\sqrt{2}}{\sqrt{\varepsilon}}$ or $n^2 > \frac{2}{\varepsilon}$ so $\frac{1}{n^2} < \frac{\varepsilon}{2}$ Similarly $\frac{1}{k^2} < \frac{\varepsilon}{2}$.

$\left|\frac{1}{n^2} - \frac{1}{k^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{k^2}\right| = \frac{1}{n^2} + \frac{1}{k^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Prop 2.4.4 A Cauch seq. is bdd.

pf $\exists M$ s.t. $n, m \geq M$ implie $|x_n - x_m| < 1$.

Then $\forall n \geq M$, $|x_n| - |x_M| \leq |x_n - x_M| < 1$ so $|x_n| < 1 + |x_M|$.

Let $B = \max\{|x_1|, |x_2|, \dots, |x_{M-1}|, 1 + |x_M|\}$. Then $|x_n| \leq B$ $\forall n \in \mathbb{N}$.

Thm 2.4.5 $\{x_n\}$ is Cauchy iff convergent.

pf Suppose $\lim_{n\to\infty} x_n = x$. Given $\varepsilon > 0$ chose M s.t. $n \geq M$ implies $|x_n - x| < \frac{\varepsilon}{2}$.

If $n, m \geq M$, then $|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Suppose $\{x_n\}$ is Cauchy. We know $\{x_n\}$ is bounded. We show $\overset{a}{\limsup_{n\to\infty} x_n} = \overset{b}{\liminf_{n\to\infty} x_n}$.

Use Thm 2.3.7 to get subseqs. $\lim_{i\to\infty} x_{n_i} = a$ and $\lim_{j\to\infty} x_{m_j} = b$. $\exists M_1$ s.t. $i \geq M_1 \Rightarrow |x_{n_i} - a| < \frac{\varepsilon}{3}$ and $\exists M_2$ s.t. $i \geq M_2 \Rightarrow |x_{m_i} - b| < \frac{\varepsilon}{3}$. Since $\{x_n\}$ is cauchy, $\exists M_3$ s.t. $n, m \geq M_3$ implies $|x_n - x_m| < \frac{\varepsilon}{3}$. Take $M = \max\{M_1, M_2, M_3\}$

If $i \geq M$, then $n_i \geq M$ and $m_i \geq M$. $|a - b| \leq |a - x_{n_i}| + |x_{n_i} - x_{m_i}| + |x_{m_i} - b| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

As $\varepsilon > 0$ was arbitrary, $a = b$ so $\{x_n\}$ converges