1.1 Basic Properties

$\mathbb{Q}$ is an ordered field
$\mathbb{R}$ is a complete ordered field

Def An ordered set is a set A together with a relation $<$ such that

(i) $\forall\, x, y \in A$, exactly one of $x<y$, $x=y$ or $y<x$ holds.
(ii) If $x<y$ and $y<z$, then $x<z$

Examples: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$. $x<y$ iff $y-x$ is positive.

Notation
$x \le y$ means $x<y$ or $x=y$
$x > y$ means $y<x$
$x \ge y$ means $y \le x$.

Def Let $E \subset A$ where A is an ordered set

(i) If $\exists\, b \in A$ such that $x \le b\ \forall x \in E$, we say E is bounded above and b is an upper bound of E.
(ii) If $\exists\, b \in A$ such that $x \ge b\ \forall x \in E$, " " E is bounded below and b is a lower bound of E
(iii) If $\exists$ upper bound b_0 of E such that whenever b is an upper bound for E, we have $b_0 \le b$, then b_0 is called the least upper bound (or the supremum of E)
Write $b_0 = \sup E$
(iv) Similarly, if there exists a lower bound b_0 of E such that whenever b is any lower bound for E, we have $b_0 \ge b$, then b_0 is called the greatest lower bound or infimum of E
$b_0 = \inf E$
(v) When E is bounded above and below, we say E is bounded.

Examples
$\{x \in \mathbb{Q} \mid x < 2\}$ $\quad \mathbb{N} \subseteq \mathbb{R}$
$\{x \in \mathbb{Q} \mid x^2 < 2\}$ $\quad \{x \in \mathbb{Z} \mid 2|x\} \cap \{x \in \mathbb{Z} \mid 5|x\} \subseteq \mathbb{Z}$
$\{1853\} \subseteq \mathbb{R}$ $\quad \mathbb{N}$ " $\mathbb{N} \subseteq \mathbb{N}$

Questions: bounded above, bdd. below, bdd? Compute sup, inf, exist? in E?

* Pick 3 students S_1, S_2, S_3 and order by height
$h: \{S_1, S_2, S_3\} \to \mathbb{R}$ injective
order by "amount of physical money on you (or in bag nearby)"

* $S = \{1,2,3,4\}$ Consider $\mathcal{P}(S)$ as an ordered set?
$A < B$ if $A \subsetneq B$. (Not everything is comparable) partially ordered set
$S = \{1\} \Rightarrow \mathcal{P}(S) = \{\phi, S\}$ ordered set?

Def An ordered set A has the least-upper-bound property if every nonempty subset $E \subset A$ that is bounded above has a least upper bound (in A).
i.e. $\phi \ne E \subseteq A$ is bdd above $\Rightarrow \sup E$ exists.

Counterexample $\mathbb{Q}$ $\quad \sup\{x \in \mathbb{Q} \mid x^2 < 2\}$ DNE.

$E = \{x \in \mathbb{Q} \mid x^2 < 2\}$

(Lots of details missing)

$x \in E \Rightarrow x^2 < 2 < 4 \Rightarrow x < 2 \Rightarrow 2$ is an upper bound

If $\underbrace{y = \sup E \in \mathbb{Q}, \text{ then } y^2}_{\text{see Ex. 1.2.3}} = 2$. $\therefore y$ is irrational (proved in 243)

Fields A set F is a *field* if it has two binary operations addition ($x+y$) and multiplication (xy) satisfying the following

A1 (closure) If $x,y \in F$, then $x+y \in F$. ($+ : F \times F \to F$ is a function)

A2 (commutativity of addition) If $x,y \in F$, then $x+y = y+x$ (Show students Lebl's writing error to explain style)

A3 (associativity of addition) If $x,y,z \in F$, then $(x+y)+z = x+(y+z)$

A4 (additive identity) There exists an element $0 \in F$ s.t. $0 + x = x \ \forall x \in F$. (Discuss notation, uniqueness, commutativity)

A5 (additive inverse) $\forall x \in F$ there exists an element $-x \in F$ such that $x + (-x) = 0$

M1 (multiplicative closure) If $x,y \in F$, then $xy \in F$.

M2 (comm. of mult.) If $x,y \in F$, then $xy = yx$

M3 (ass. of mult.) If $x,y,z \in F$, then $(xy)z = x(yz)$

M4 (mult. identity) $\exists\, 1 \in F$ (and $1 \neq 0$) s.t. $1x = x \ \forall x \in F$

M5 (mult. inverse) $\forall x \in F$ s.t. $x \neq 0$ $\exists\, \frac{1}{x} \in F$ s.t. $x(\frac{1}{x}) = 1$.

D (distributive law) $x(y+z) = xy + xz \quad \forall x,y,z \in F$.

$(\mathbb{Q}, +, \times)$ is a field, but $(\mathbb{Z}, +, \times)$ are not $2x = 1$ has no sol'n. (Is 1 even or odd?)

finite examples? $\mathbb{Z}_p$ where p is a prime $\{0,1,2,3,4\}$ what is 3^{-1}? $3\cdot 0 \equiv 0$, $3 \cdot 1 \equiv 3$ $3\cdot 2 = 6 \equiv 1$ $3\cdot 3 = 9 \equiv 4$ $3 \cdot 4 = 12 \equiv 2$

Def A field is an *ordered field* if F is also an ordered set s.t.

(O1) (i) $\forall x,y,z \in F$, $x < y \Rightarrow x + z < y + z$

(O2) (ii) $\forall x,y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$.

If $x > 0$, we say x is positive If $x < 0$, we say x is negative

$x \geq 0$ nonnegative $x \leq 0$ nonpositive.

Now the fun begins.

Claim $0x = 0$. pf $xx \overset{A4}{=} x(0+x) \overset{D}{=} x \cdot 0 + xx \overset{M2}{=} 0 \cdot x + xx$ (want to cancel xx using A5) missing associativity of add. Doing things to both sides is valid, but subtle.

$$0 \overset{A5}{=} xx + -(xx) = (0\cdot x + xx) + -(xx) \overset{A3}{=} 0 \cdot x + (xx + -(xx)) \overset{A5}{=} 0 \cdot x + 0 \overset{A2}{=} 0 + 0 \cdot x \overset{A4}{=} 0 \cdot x$$

↑ not $(-x)x$ or $(-1)xx$ unless proven.

If uniqueness of 0 is known, then that says $0 \cdot x = 0$? not quite unless xx represents all elts. in F.

Claim $0, 1, -x$, and $\frac{1}{x}$ are all unique so their notation is justified

Suppose A satisfies $A + x = x \ \forall x \in F$.

$A = 0 + A$ 0 is an additive identity.

$= A + 0$ commutativity

$= 0$ A is an additive identity. uniqueness of 1 is similar.

Suppose $\exists\, y \in F$ st. $x + y = 0$.

$y = 0 + y$ A4

$= (x + -x) + y$ A5

$= (-x + x) + y$ A2

$= -x + (x + y)$ A3

$= -x + 0$ assumption of y Similar proof for unique inv of $\frac{1}{x}$.

$= 0 + -x$ A2

$= -x$ A4

Claim $(-x)y = -(xy)$ $-x$ satisfies $x + -x = 0$

$-(xy)$ satisfies $xy + -xy = 0$.

Use the uniqueness of $-(xy)$ and show $xy + (-x)y = 0$.

$xy + (-x)y = yx + y(-x)$ M2

$= y(x + -x)$ D

$= y \cdot 0$ A5

$= 0 \cdot y$ M2

$= 0$ above.

Prop 1.1.8 Let $\mathbb{F}$ be an ordered field and $x, y, z \in \mathbb{F}$. Then

(i) $x > 0$ iff $-x < 0$

(ii) If $x > 0$ and $y < z$, then $xy < xz$

(exercise) (iii) If $x < 0$ and $y < z$, then $xy > xz$

(iv) If $x \neq 0$, then $x^2 > 0$. (explain $1 > 0$)

(v) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Pf (i) Assume $x > 0$, then O1 gives $x + -x > 0 + -x$

A5 on the left side: $0 > 0 + -x$

A4 $0 > -x$

(ii) Assume $x > 0$ and $y < z$ show $z - y > 0$.

By O1 $y + -y < z + -y$

By A5 $0 < z - y$.

By O2, $x(z - y) > 0$.

By D, $xz + x(-y) > 0$.