Homework 8: Graph Theory

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1. Given a graph G with n vertices and m edges, and each vertex has degree between d_{min} and d_{max}, show that:

$$\frac{1}{2}d_{\text{min}}n \leq m \leq \frac{1}{2}d_{\text{max}}n$$

- The sum of the degrees is $\sum \deg(v) = 2m$. Since each vertex has degree between d_{min} and d_{max}, we have:

$$d_{\text{min}}n \leq 2m \leq d_{\text{max}}n \Rightarrow \frac{1}{2}d_{\text{min}}n \leq m \leq \frac{1}{2}d_{\text{max}}n$$

2. (a) How many edges does the complete graph K_n have?

- $\binom{n}{2} = \frac{n(n-1)}{2}$.

(b) How many edges does the complete bipartite graph $K_{p,q}$ have?

- pq.

- A path: $A - B - D$
- A trail: $A - B - D - B - C$ (no repeated edge)
- A walk: $A - B - D - B - C - D$ (allows repeated edges and vertices)

4. (a) Prove that if a walk contains a repeated edge, then it contains a repeated vertex. (b) Since repeating an edge requires revisiting at least one of its endpoints (a repeated vertex), then if no vertices are repeated, no edges can be repeated either.

- An edge connects two vertices. Repeating an edge requires visiting at least one of its vertices again, so a repeated edge implies a repeated vertex.

(b) Use (a) to explain why a walk with no repeated vertices has no repeated edges.

- Contrapositive of (a): no repeated vertex $\Rightarrow$ no repeated edge.

5. (a) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \\ -2 & 1 & 2 \end{bmatrix}$$

Compute AB and BA.

- $$AB = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -6 & -9 \\ -2 & 1 & 2 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 6 & 3 \\ 2 & -6 & -3 \\ -4 & 3 & 2 \end{bmatrix}$$

6. Given an adjacency matrix, draw the corresponding graph.
7. (a) Show that if there is a walk from u to v, then there is one of length $\leq n-1$.

- Any walk can be shortened by removing cycles. A simple path between u and v uses at most $n-1$ edges.

(b) Prove: G is connected iff all entries of $A + A^2 + \cdots + A^{n-1}$ are positive.

- Entry (i, j) in A^k counts walks of length k from i to j. If all such entries are > 0, there exists a path between all vertex pairs $\Rightarrow$ graph is connected.

8. Draw all nonisomorphic trees with 5 vertices.

- There are 3 nonisomorphic trees:

  a) Path: $v_1 - v_2 - v_3 - v_4 - v_5$
  b) Star: one central vertex connected to 4 leaves
  c) Fork: central node with two branches of lengths 2 and 1

9. (a) What is the minimum height of a binary tree with 27 leaves?

- $\lceil \log_2 27 \rceil = 5$

(b) 48 leaves?

- $\lceil \log_2 48 \rceil = 6$

(c) 64 leaves?

- $\lceil \log_2 64 \rceil = 6$

10. Suppose each of n people is acquainted with at least one other. Show two have same number of acquaintances.

- By the pigeonhole principle, two people must have the same number of acquaintances in a group of n people.

11. Refer to Figure 2 (path graph: $1 - 2 - 3 - 4$) (a) Adjacency Matrix A:

- $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Compute A^2:

- $$A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Symmetric: Yes
- Nonzero entries: 8
- $A^2_{1,1} = 1$ (one 2-step walk from 1 to 1)

(c) Compute A^3:

- $$A^3 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

- $A^3_{1,3} = 0$ (no 3-step walk from 1 to 3)

(d) Distance from 2 to 4:

- 2 (since $A^2_{2,4} = 1$)

12. let weighted graph G = (V,E,w) as shown in Fig 3.

- what is the shortest path distance between V3 and v4?

  3,1,2,4 = 2+1+1 = 4

- find the MST for G. Explicitly list the edges in your tree and calculate the total weight of your MST.

- $$\begin{aligned} e_1 &= (1,2), \quad w = 1 \\ e_3 &= (2,3), \quad w = 4 \\ e_4 &= (2,4), \quad w = 1 \\ e_5 &= (3,4), \quad w = 6 \end{aligned}$$

- Sort edges by increasing weight:

$$(1,2), w = 1; \quad (2,4), w = 1; \quad (2,3), w = 4; \quad (3,4), w = 6$$

- Add $(1, 2)$: no cycle
- Add $(2, 4)$: no cycle
- Add $(2, 3)$: no cycle

- Final MST edges:

$$\{(1, 2),\ (2, 4),\ (2, 3)\}$$

- Total weight of the MST:

$$1 + 1 + 4 = \boxed{6}$$