HW 4

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February 21, 2025

1. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions:

   (a) $a_n = 4a_{n-1} + 1, \quad a_0 = 1;$
   $a_1 = 5, a_2 = 21, a_3 = 85, a_4 = 341, a_5 = 1365$

   (b) $a_n = a_{n-1}^2 - 2, \quad a_0 = 2;$
   $a_{1 \to 5} = 2$

   (c) $a_n = a_{n-1}^2 - 1, \quad a_1 = 2;$
   $a_2 = 3, a_3 = 8, a_4 = 63, a_5 = 3968$

   (d) $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1, \quad a_1 = 2;$
   $a_2 = 5, a_3 = 11, a_4 = 26, a_5 = 59$

   (e) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = a_1 = 1.$
   $a_2 = 6, a_3 = 27, a_4 = 204, a_5 = 1695$

2. Let $x > 0$ be a positive real number. What are the possible values of $\lfloor 3x \rfloor$ in terms of $\lfloor x \rfloor$? Under what conditions will $\lfloor 3x \rfloor$ take each of these values? Give one example for each case.

   $\lfloor x \rfloor$ would be rounded down to the closest integer. For example: $\frac{3}{2} = 1.5, \lfloor 1.5 \rfloor = 1$. Multiplying the X value by 3 and then getting the floor of it would still be rounded down to the closest integer. Following the previous example: $\lfloor 3(1.5) \rfloor = \lfloor 4.5 \rfloor = 4$

3. The formula
   $$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$
   holds for every integer $n \geq 0$ and every real number $r \neq 1$. Use this formula to find the sum of each of the following sequences:

   (a) $1 + 2 + 2^2 + \cdots + 2^{n-1}$
   $S_n = 2^n - 1$

   (b) $3^{n-1} + 3^{n-2} + \cdots + 3 + 1$
   $S_n = \frac{3^n - 1}{2}$

(c) $2^n + 3 \cdot 2^{n-2} + \cdots + 3 \cdot 2 + 3$

$3 \cdot (2^n + 1)$

(d) $2^n - 2^{n-1} + 2^{n-2} - \cdots + (-1)^{n-1} \cdot 2 + (-1)^n$

$S_n \frac{2^{n+1}+(-1)^n}{3}$

4. Calculate the number of trailing zeros of 5678!.

we need to count how many factors of 5 there are in 5678!

- $5678/5 = 1135$
- $5678/25 = 227$
- $5678/125 = 45$
- $5678/625 = 9$
- $5678/3125 = 1$
- total $= 1417$ zeros

5. Deduce the formula for the summation of cubes:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2 .$$

Starting from the RHS (right hand side) $(\sum_{i=1}^{n} i)^2$
$(1 + 2 + 3 + \cdots + n) \cdot (1 + 2 + 3 + \cdots + n)$
$1(1)+2(1+2+1)3(1+2+3+2+1)+\cdots n[1+2+\cdots+n+(n-1)+\cdots+1]$
factoring out
$1(1) + 2(2 + 2) + 3(3 + 3 + 3) + \cdots + n(n + n \cdots + n) =$ adding up
$1(1) + 2(2 \cdot 2) + 3(3 \cdot 3) + \cdots + n(n \cdot n) =$ re-writing expression
$= 1^3 + 2^3 + 3^3 + \cdots + n^3 =$ LHS (left hand side)

6. Evaluate the following products:

(a) $\prod_{i=0}^{50} i^2$

$0^2 \cdot 1^2 \cdots 50^2 = 0$ (since first term is 0)

(b) $\prod_{i=5}^{8} i$

$5 \times 6 \times 7 \times 8 = 1,680$

(c) $\prod_{i=1}^{10} 2$

$2^10 = 1024$ (since 2 is the constant, we count count up to 10)

(d) $\prod_{i=1}^{100} (-1)^i$

$-1^{100} = 1$ (since 100 is even, there are an even number of -1 factors)

7. Evaluate the following sum and product:

(a) $\prod_{i=1}^{n} \frac{i}{i+1}$

$1/2 \cdot 2/3 \cdot 3/4 \cdot 4/5 \cdots \frac{i}{i+1}$ it is a telescoping pattern... $= \frac{1}{n+1}$

(b) $\sum_{i=1}^{n} \frac{1}{i(i+1)}$
$\frac{1}{i} - \frac{1}{i+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{i(i+1)}$

$(1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) \cdots + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1})$
becomes a telescoping pattern, leaving us with:
$1 - \frac{1}{n+1}$ Example 5.1.10 in the book

8. Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then either $a = b$ or $a = -b$.

given $a|b$, $b = ka$ for some integer k
given $b|a$, $a = bm$ for some integer m
substituting $b = ka$ into $a = bm$:
$a = m(ka) = (mk)a = a$ since a $\neq 0$ we can divide a out:
$mk = 1$
since m and k are integers, the only solutions are:
$m = k = 1$ or $m = k = -1$
Thus $b = a$ or $b = -a$

9. Prove that if n is an integer not divisible by 3, then $n^2 - 1$ is divisible by 3.

10. Prove that if n is a positive integer, then 4 does not divide $n^2 + 2$. (Hint: Consider cases when n is even and odd separately.)

case 1: n is even, so $n = 2k$, substituting that:
$(2k)^2 + 2 = 4k^2 + 2$
Since $4k^2$ can be divided by 4, we analyze the term: $4k^2 + 2 \equiv 2 \pmod 4$
Since $2 \neq 0$ mod 4, we conclude that 4 does not divide n^2+2 when n is even

case 2: when n is odd, so $n = 2k + 1$ for some integer k:
$(2k + 1)^2 + 2 = 4k^2 + 4k + 1 + 2 = 4k^2 + 4k + 3$
Since $4k^2$ is divisible by 4, we look at the remainder:
$4k^2 + 4k + 3 \equiv 3 \pmod 4$
since $3 \neq 0 \pmod 4$, we can conclude that 4 does not divide $n^2 + 2$ when n is odd