HW 7

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1. (a) There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

   P(6,6) = 6! = 720

   (b) In how many different orders can five runners finish a race if no ties are allowed?

   P (5,5) = 5! = 120

2. Six women (6W) and eight men (8M) are on the faculty of the math department in a college.

   (a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

   Total committees = $\binom{14}{5} = \frac{14!}{5!(14!-5!)} = 2002$

   committees only consisting of men: $\binom{8}{5} = \frac{8!}{5!(8!-5!)} = 56$

   Therefore, the number of ways to form a committee with at least 1 woman is: $2002 - 56 = 1946$

   (b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

   $\sum_{k=1}^{5} \binom{6}{k}\binom{8}{5-k}$:

   $$= \binom{6}{1}\binom{8}{4} + \binom{6}{2}\binom{8}{3} + \binom{6}{3}\binom{8}{2} + \binom{6}{4}\binom{8}{1} + \binom{6}{5}\binom{8}{0}$$

   $$= (6 \times 70) + (15 \times 56) + (20 \times 28) + (15 \times 8) + (6 \times 1)$$

   $$= 420 + 840 + 560 + 120 + 6 = 1,946$$

   Answer: $1,946$ ways.

3. How many ways are there for three (3) (distinct) penguins and six (6) (distinct) puffins to stand in a line so that:

(a) all puffins stand together?

all 6 puffins + 1 penguin block = 7! = 5040

(b) all penguins stand together?

3! = 6

4. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women must alternate?

n!×n!

5. (a) What is the coefficient of x^9 in $(2-x)^{19}$?

$= \sum_{k=0}^{19}(2^{19-k})(-x)^k$

x^9 is where $k = 9$: $\binom{19}{9}2^1 0(-1)^{-9})$

$\binom{19}{9} \times 1024 \times -1 = \frac{19!}{9!(10!)} \times 1024 \times -1 = -94,594,048$

(b) What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$? What about the coefficient of x^8y^8?

$\binom{17}{8,9}(3x)^8(2y)^9$

$\binom{17}{8}3^8 2^9$

$\binom{17}{8} = \frac{17!}{8!(9!)} = 24310$

$3^8 = 6561, 2^9 = 512$

$24310 \times 6561 \times 512 = 8,187,962,880$

6. Prove that a nonempty set has the same number of subsets with an odd number of elements as it has of subsets with an even number of elements.

using binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

if x=1:

$2^n = \sum_{k=0}^{n} \binom{n}{k}$

if x=-1:

$0^n = \sum_{k=0}^{n} \binom{n}{k} - 1^k$

$\sum_{\text{even k}} \binom{n}{k} - \sum_{\text{odd k}} \binom{n}{k} = 0$

even - odd = 0, even = odd

7. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 18$, where each x_i, $1 \leq i \leq 4$ is a non-negative integer such that

(a) $x_1 \geq 1$?

$x`_1 = x_1 - 1$, so $x`_1 \leq 0$ transforming the equation into:

$x`_1 + x_2 + x_3 + x_4 = 17$

$\binom{17+3}{3} = \binom{20}{3} = 1140$

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4$?

let $x`_i = x_i - 2$ making the equation into:

$x`_1 + x`_2 + x`_3 + x`_4 = 18 - 8 = 10$

$\binom{10+3}{3} = \binom{13}{3} = 286$

(c) $0 \leq x_1 \leq 9$?

8. There are 12 questions on a Discrete Structures final exam. How many ways are there to assign scores to all the questions if the sum of these scores is 100 and each question is worth at least 5 points?

Let $y_i = x_i - 5 \geq 0$:

$y_1 + y_2 \cdots y_1 2 = 100 - 12(5) = 40$

using stars and bars:

$\binom{40+11}{11} = \binom{51}{11} = 3,586,825,500$

9. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

there are 51 houses. 1000 and 1099 (100 possible numbers)

Since $51 > 50$ (pairs), pigeonhole principle guarantees that at one pair of houses must have consecutive numbers.

10. Ten students $A, B, \ldots, J$ are in a class. A committee of three is chosen at random to represent the class. Find the probability that:

(a) A belongs to the committee;

(b) Both A and B belong to the committee;

(c) B belongs to the committee;

(d) A or B (but not both) belongs to the committee.