Table 4.1 Penetration depth, $\lambda(0)$, and coherence length, $\xi(0)$, at zero temperature for some important superconductors. Data values are taken from Poole (2000).

	T_c (K)	$\lambda(0)$ (nm)	$\xi(0)$ (nm)	κ
Al	1.18	1550	45	0.03
Sn	3.72	180	42	0.23
Pb	7.20	87	39	0.48
Nb	9.25	39	52	1.3
Nb_3Ge	23.2	3	90	30
YNi_2B_2C	15	8.1	103	12.7
K_3C_{60}	19.4	2.8	240	95
$YBa_2Cu_3O_{7-\delta}$	91	1.65	156	95

In terms of the original free energy GL parameters, $\dot{a}$ and b, the superfluid density, n_s is given by

$$n_s = 2|\psi^2| = 2\frac{\dot{a}(T_c - T)}{b}. \tag{4.65}$$

Therefore the London penetration depth, $\lambda(T)$ is given by

$$\lambda(T) = \left(\frac{m_e b}{2\mu_0 e^2 \dot{a}(T_c - T)}\right)^{1/2}. \tag{4.66}$$

Clearly this will diverge at the critical temperature, T_c, since it is proportional to $(T_c - T)^{-1/2}$. We saw earlier that the GL coherence length, $\xi(T)$, also diverges with the same power of $(T_c - T)$, and so the dimensionless ratio,

$$\kappa = \frac{\lambda(T)}{\xi(T)}, \tag{4.67}$$

is independent of temperature within the GL theory. Table 4.1 summarizes the measured values of penetration depth and coherence length at zero temperature, $\lambda(0)$, $\xi(0)$, for a selection of superconductors.

4.8 Flux quantization

Let us now apply the GL theory to the case of a superconducting ring, as shown in Fig. 3.4. Describing the system using cylindrical polar coordinates, $\mathbf{r} = (r, \phi, z)$, with the z-axis perpendicular to the plane of the ring, we see that the order parameter $\psi(\mathbf{r})$ must be periodic in the angle ϕ,

$$\psi(r, \phi, z) = \psi(r, \phi + 2\pi, z). \tag{4.68}$$

We assume that the variations of $\psi(\mathbf{r})$ across the cross section of the ring are unimportant, and so we can neglect r and z dependence. Therefore the possible order parameters inside the superconductor are of the form

$$\psi(\phi) = \psi_0 e^{in\phi}, \tag{4.69}$$

where n is an integer and ψ_0 is a constant. We can interpret n as a **winding number** of the macroscopic wave function, exactly as for the case of superfluid helium in Fig. 2.9.

However, unlike the case of superfluid helium, a circulating current in a superconductor will induce magnetic fields. Assuming that there is a magnetic flux Φ through the ring, then the vector potential can be chosen to be in the tangential direction, $\mathbf{e}_\phi$ and is given by

$$A_\phi = \frac{\Phi}{2\pi R}, \tag{4.70}$$

where R is the radius of the area enclosed by the ring. This follows from

$$\Phi \equiv \int \mathbf{B}\cdot d\mathbf{S} = \int (\nabla \times \mathbf{A})\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{r} = 2\pi R A_\phi. \tag{4.71}$$

The free energy corresponding to this wave function and vector potential is

$$\begin{aligned} F_s(T) &= F_n(T) + \int d^3r \left(\frac{\hbar^2}{2m^*}\left|\left(\nabla + \frac{2ei}{\hbar}\mathbf{A}\right)\psi\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4\right) + E_B \\ &= F_s^0(T) + V\left(\frac{\hbar^2}{2m^*}\left|\frac{in}{R} - \frac{2ei\Phi}{2\pi\hbar R}\right|^2 |\psi|^2\right) + \frac{1}{2\mu_0}\int B^2 d^3r \end{aligned} \tag{4.72}$$

where we have used the expression to gradient in cylindrical polar coordinates

$$\nabla X = \frac{\partial X}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial X}{\partial \phi}\mathbf{e}_\phi + \frac{\partial X}{\partial z}\mathbf{e}_z \tag{4.73}$$

(Boas 1983), V is the total volume of the superconducting ring, and $F_s^0(T)$ is the ground state free energy of the ring in the absence of any currents and magnetic fluxes. The vacuum magnetic field energy $E_B = (1/2\mu_0)\int B^2 d^3r$ can be expressed in terms of the inductance, L of the ring and the current I,

$$E_B = \frac{1}{2}LI^2. \tag{4.74}$$

Clearly, it will be proportional to the square of the total flux, Φ through the ring

$$E_B \propto \Phi^2.$$

On the other hand, the energy of the superconductor contains a term depending on both the flux Φ and the winding number, n. This term can be expressed as,

$$V\frac{\hbar^2}{2m^*R^2}|\psi|^2(\Phi - n\Phi_0)^2,$$

where the **flux quantum** is $\Phi_0 = h/2e = 2.07 \times 10^{-15}$ Wb.

We therefore see that the free energy is equal to the bulk free energy plus two additional terms depending only on the winding number n and the flux Φ. The energy of the superconducting ring is therefore of the general form,

$$F_s(T) = F_s^{\text{bulk}}(T) + \text{const.}(\Phi - n\Phi_0)^2 + \text{const.}\Phi^2. \tag{4.75}$$

This energy is sketched in Fig. 4.6. We can see from the figure that the free energy is a minimum whenever the flux through the loop obeys $\Phi = n\Phi_0$. This is the phenomenon of **flux quantization** in superconductors.

Taking a ring in its normal state above T_c and cooling it to below T_c will result in the system adopting one of the metastable minima in Fig. 4.6, depending on the applied field. It will then be trapped in the minimum, and a persistent current will flow around the ring to maintain a constant flux $\Phi = n\Phi_0$. Even if any external magnetic fields are turned off, the current in the ring must maintain

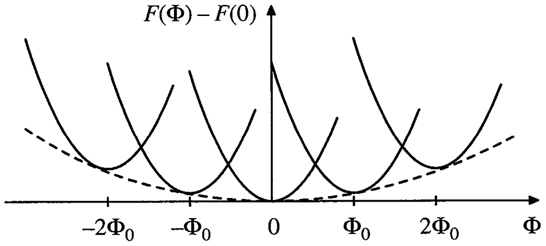

Fig. 4.6 Flux quantization in a superconducting ring. Metastable energy minima exist when the flux is an integer multiple of the flux quantum $\Phi_0 = h/2e$. There is an overall background increase with Φ corresponding to the self-inductance of the ring, making the zero flux state $\Phi = 0$ the global energy minimum. Thermal fluctuations and quantum tunnelling allow transitions between neighboring metastable energy minima.

a constant flux Φ in the ring. It is possible to directly measure the magnetic flux directly in such rings, and hence confirm that it is indeed quantized in units of Φ_0, or multiples of 2×10^{-15} Wb. Incidentally, the fact that flux quantization is observed in units of $\Phi_0 = h/2e$ and not units of h/e is clear experimental proof that the relevant charge is $2e$ not e, hence implying the existence of Cooper pairs.

Given that a system is prepared in one of the metastable minima, it can, in principle, escape over the energy barriers to move into a neighboring lower energy minimum. This would be a mechanism for the persistent current to decay, and hence for dissipation. Such an event corresponds to a change in the winding number, n, and is called a **phase-slip**. However, the rate for thermally hopping over these barriers is exponentially small, of order

$$\frac{1}{\tau} \sim e^{-E_0/k_B T} \tag{4.76}$$

where E_0 is the barrier height between minima in Fig. 4.6. Clearly this thermal hopping rate can be made negligibly small. For example, E_0 is formally proportional to the ring volume, V, and so can be made arbitrarily large in a macroscopic system. In practice persistent currents have been observed to flow for years, with essentially no decay!

Another interesting possible mechanism for a phase-slip would be a quantum tunnelling from one minimum to another. This would be possible at any temperature. But again the rate is impractically small in macroscopic systems. However, a very interesting recent development has been the direct observation of these tunnelling events in small **mesoscopic** superconducting rings. These experiments have demonstrated **macroscopic quantum coherence** and are discussed briefly in the next chapter.

4.9 The Abrikosov flux lattice

The great beauty of the GL theory is that it allows one to solve many difficult problems in superconductvity, without any reference to the underlying microscopic BCS theory. In some sense one could argue that it is more general, for example, it would almost certainly apply to exotic superconductors, such as the high T_c cuprates, even though the original BCS theory does not seem to explain these systems. The other great advantage of the GL theory is that it is considerably easier to work with than the BCS theory, especially in cases