

Using semantic labelling with natural numbers for  
proving termination automatically.

Adam Koprowski

29th June 2005

# Outline

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<http://www.win.tue.nl/tpa>

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- ... more to come?

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Contributions are very welcome!

<http://color.loria.fr/>

## SLnat - general idea

Is the following system terminating?

- (1)  $f(x, c) \rightarrow c$
- (2)  $f(c, y) \rightarrow c$
- (3)  $f(p(x), p(y)) \rightarrow p(f(x, y))$
- (5)  $g(x, c) \rightarrow x$
- (4)  $g(c, y) \rightarrow y$
- (6)  $g(p(x), p(y)) \rightarrow p(g(x, y))$
- (7)  $w(x, c) \rightarrow x$
- (8)  $w(p(x), p(y)) \rightarrow w(x, y)$
- (9)  $h(p(x), p(y)) \rightarrow h(w(p(g(x, y)), p(f(x, y))), p(f(x, y)))$

## SLnat - general idea (continued)

And how about this one?

- (1)  $\min(x, 0) \rightarrow 0$
- (2)  $\min(0, y) \rightarrow 0$
- (3)  $\min(s(x), s(y)) \rightarrow s(\min(x, y))$
- (5)  $\max(x, 0) \rightarrow x$
- (4)  $\max(0, y) \rightarrow y$
- (6)  $\max(s(x), s(y)) \rightarrow s(\max(x, y))$
- (7)  $x - 0 \rightarrow x$
- (8)  $s(x) - s(y) \rightarrow x - y$
- (9)  $\gcd(s(x), s(y)) \rightarrow \gcd(s(\max(x, y)) - s(\min(x, y)), s(\min(x, y)))$

Natural interpretation for function symbols:

$[0]$	$=$	$0$	$[\min(x, y)]$	$=$	$\min(x, y)$
$[s(x)]$	$=$	$x + 1$	$[\max(x, y)]$	$=$	$\max(x, y)$
$[x - y]$	$=$	$x - y$	$[\gcd(x, y)]$	$=$	$\gcd(x, y)$

Natural interpretation for function symbols:

$$\begin{array}{ll}
 [0] & = 0 \\
 [s(x)] & = x + 1 \\
 [x - y] & = x - y
 \end{array}
 \qquad
 \begin{array}{ll}
 [\min(x, y)] & = \min(x, y) \\
 [\max(x, y)] & = \max(x, y) \\
 [\gcd(x, y)] & = \gcd(x, y)
 \end{array}$$

$$\begin{array}{ll}
 \min_{i,0}(x, 0) & \rightarrow 0 \\
 \min_{0,j}(0, y) & \rightarrow 0 \\
 \min_{i+1,j+1}(s_i(x), s_j(y)) & \rightarrow s_j(\min_{i,j}(x, y)) & \text{if } i \geq j \\
 \min_{i+1,j+1}(s_i(x), s_j(y)) & \rightarrow s_i(\min_{i,j}(x, y)) & \text{if } i < j \\
 \max_{i,0}(x, 0) & \rightarrow x \\
 \max_{0,j}(0, y) & \rightarrow y \\
 \max_{i+1,j+1}(s_i(x), s_j(y)) & \rightarrow s_i(\max_{i,j}(x, y)) & \text{if } i \geq j \\
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 x -_{i,0} 0 & \rightarrow x \\
 s_i(x) -_{i+1,j+1} s_j(y) & \rightarrow x -_{i,j} y \\
 \gcd_{i+1,j+1}(s_i(x), s_j(y)) & \rightarrow \gcd_{i-j,j+1}(s_i(\max_{i,j}(x, y)) -_{i+1,j+1} s_j(\min_{i,j}(x, y)), \min_{i,j}(x, y)) & \text{if } i \geq j \\
 \gcd_{i+1,j+1}(s_i(x), s_j(y)) & \rightarrow \gcd_{j-i,i+1}(s_j(\max_{i,j}(x, y)) -_{j+1,i+1} s_i(\min_{i,j}(x, y)), \min_{i,j}(x, y)) & \text{if } i < j
 \end{array}$$

If we label only gcd symbol with  $\pi_{\text{gcd}}(x, y) = x + y$  then we get the following system:

- (1)  $\min(x, 0) \rightarrow 0$
- (2)  $\min(0, y) \rightarrow 0$
- (3)  $\min(s(x), s(y)) \rightarrow s(\min(x, y))$
- (5)  $\max(x, 0) \rightarrow x$
- (4)  $\max(0, y) \rightarrow y$
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which can easily be proved to be terminating by RPO with the following precedence:

$\min$	$>$	$s$	$\text{gcd}_i$	$>$	$\min$
$\max$	$>$	$s$	$\text{gcd}_i$	$>$	$\max$
$\text{gcd}_i$	$>$	$-$	$\text{gcd}_i$	$>$	$s$
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TPA does the following transformation to reduce any TRS to a TRS containing only constants, unary and binary symbols.

$$f(x_1, \dots, x_n) \equiv f'(x_1, f''(x_2, \dots))$$

# Proving termination using semantic labelling.

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Combinations of those functions are tried in the search for a (quasi-)model. The search space is finite.

For every obtained (quasi-)model some techniques are applied to the labelled system in order to try to prove its termination.



## SLnat and polynomial interpretations

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$$\begin{aligned} [c] &\in I_c \\ [s_m(x)] &\in \{i_u, i_u + m \mid i_u \in I_u\} \\ [f_{m,n}(x, y)] &\in \{i_b, i_b + m, i_b + n, i_b + m + n \mid i_b \in I_b\} \end{aligned}$$

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Precedence is well-founded if the corresponding directed graph is acyclic.

(1)	$0 + x$	$\rightarrow$	$x$
(2)	$s(x) + y$	$\rightarrow$	$s(x + y)$
(3)	$0 * x$	$\rightarrow$	$0$
(4)	$s(x) * y$	$\rightarrow$	$y + (x * y)$

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$$* \longrightarrow + \longrightarrow s$$

$$\text{compare} : \text{funS} \times \text{funS} \rightarrow \{<, =, >, ?\}$$

## RPO with $\mathbf{SL}_{\text{nat}}$

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 $\text{cmpRes} : \{<, >, ?\}$   
 $\text{argsF} : \{\leftarrow, \rightarrow, \leftrightarrow\}$

if  $\text{compare}(f, g) = (\Gamma_f, \Gamma_g, \Delta_<, \Delta_>, \Delta_?)$

$\text{args}_{\leftarrow}(l, r) = l$

$\text{args}_{\rightarrow}(l, r) = r$

$\text{args}_{\leftrightarrow}(l, r) = l + r$

$$f_{i,j} \otimes g_{k,l} \quad \text{if} \quad \text{args}_{\Gamma_f}(i, j) \begin{matrix} < \\ = \\ > \end{matrix} \text{args}_{\Gamma_g}(k, l) \wedge \begin{matrix} \Delta_< \\ \Delta_> \\ \Delta_? \end{matrix} = \otimes$$



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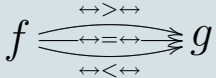
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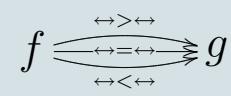
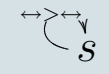
Where condition is of the shape:

$$\begin{aligned} X \otimes Y \\ X, Y \in \{\leftarrow, \rightarrow, \leftrightarrow\} \\ \otimes \in \{<, =, >\} \end{aligned}$$

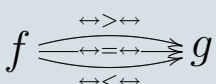

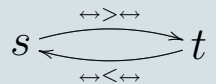
# RPO with SLnat - examples of precedence

Expected result	Precedence function	Edges in a multigraph
$f_{i,j} > g_{k,l}$ for all $i, j, k, l$	$\Omega(f, g) = (\leftrightarrow, \leftrightarrow, >, >, >)$	

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$s_1 < s_2 < \dots$	$\Omega(s, s) = (\leftrightarrow, \leftrightarrow, <, =, >)$	

# RPO with SLnat - examples of precedence

Expected result	Precedence function	Edges in a multigraph
$f_{i,j} > g_{k,l}$ for all $i, j, k, l$	$\Omega(f, g) = (\leftrightarrow, \leftrightarrow, >, >, >)$	
$s_1 < s_2 < \dots$	$\Omega(s, s) = (\leftrightarrow, \leftrightarrow, <, =, >)$	
$s_1 < t_1 < s_2 < t_2 < \dots$	$\Omega(s, t) = (\leftrightarrow, \leftrightarrow, <, <, >)$	

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- for every two consecutive edges  $\circ \xrightarrow{W \otimes_1 X} \circ \xrightarrow{Y \otimes_2 Z} \circ, X = Y$ .

Those three conditions assure that some weight-function on indexes decrease along the cycle.

$$\dots \xrightarrow{X \geq Y} f \xrightarrow{Y \geq Z} \dots \quad X \geq Y = Y \geq Z$$

Conjecture:

Relation is well-founded, and hence can be extended to precedence, if every cycle in its multigraph is safe.

## Motivating example

Consider the following TRS (SUBST) which describes the process of substitution in combinatory categorical logic.

$$\begin{array}{lll} (1) & \lambda(x) \circ y & \rightarrow \lambda(x \circ (1 \cdot (y \circ \uparrow))) \\ (2) & (x \cdot y) \circ z & \rightarrow (x \circ z) \cdot (y \circ z) \\ (3) & (x \circ y) \circ z & \rightarrow x \circ (y \circ z) \\ (4) & \text{id} \circ x & \rightarrow x \\ (5) & 1 \circ \text{id} & \rightarrow 1 \\ (6) & \uparrow \circ \text{id} & \rightarrow \uparrow \\ (7) & 1 \circ (x \cdot y) & \rightarrow x \\ (8) & \uparrow \circ (x \cdot y) & \rightarrow y \end{array}$$

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- Using technique of SLnat a very concise proof can be given.
- Now it can be given by TPA.

## Termination proof of SUBST as given by TPA

TPA v.1.0b

Result: TRS is terminating

Default interpretations for symbols are not printed. For polynomial interpretations and semantic labelling over  $N \setminus \{0,1\}$  defaults are 2 for constants, identity for unary symbols and  $x+y-2$  for binary symbols. For semantic labelling over  $\{0,1\}$  (booleans) defaults are 0 for constants, identity for unary symbols and disjunction for binary symbols.

[1] TRS as loaded from the input file:

- (1)  $o(\lambda(x), y) \rightarrow \lambda(o(x, d(1, o(y, p))))$
- (2)  $o(d(x, y), z) \rightarrow d(o(x, z), o(y, z))$
- (3)  $o(o(x, y), z) \rightarrow o(x, o(y, z))$
- (4)  $\lambda(x) \rightarrow x$
- (5)  $o(x, y) \rightarrow x$
- (6)  $o(x, y) \rightarrow y$
- (7)  $d(x, y) \rightarrow x$
- (8)  $d(x, y) \rightarrow y$

[2] Label this TRS using following interpretation over  $N \setminus \{0,1\}$ :

$[\lambda(x)] = x + 1$   
 $[d(x, y)] = \max(x, y)$   
rest default

## Termination proof of SUBST as given by TPA (cont.)

This interpretation is a quasi-model and yields following TRS:

```
(D1) lambda{i + 1}(x) ->= lambda{i}(x)
(D2) o{i + 1, j}(x, y) ->= o{i, j}(x, y)
(D3) o{i, j + 1}(x, y) ->= o{i, j}(x, y)
(D4) d{i + 1, j}(x, y) ->= d{i, j}(x, y)
(D5) d{i, j + 1}(x, y) ->= d{i, j}(x, y)
(1) o{i + 1, j}(lambda{i}(x), y) -> lambda{j + i - 2}(o{i, j}(x, d{2, j}(1, o{j, 2}(y, p))))
(2<) o{i, k}(d{i, j}(x, y), z) -> d{k + i - 2, k + j - 2}(o{i, k}(x, z), o{j, k}(y, z)) for i >= j
(2>) o{j, k}(d{i, j}(x, y), z) -> d{k + i - 2, k + j - 2}(o{i, k}(x, z), o{j, k}(y, z)) for j >= i
(3) o{j + i - 2, k}(o{i, j}(x, y), z) -> o{i, k + j - 2}(x, o{j, k}(y, z))
(4) lambda{i}(x) -> x
(5) o{i, j}(x, y) -> x
(6) o{i, j}(x, y) -> y
(7<) d{i, j}(x, y) -> x for i >= j
(7>) d{i, j}(x, y) -> x for j >= i
(8>) d{i, j}(x, y) -> y for j >= i
(8<) d{i, j}(x, y) -> y for i >= j
```

[3] All the rules of this TRS can be oriented with RPO with the following precedence:

Status: o: Lex-LR,

Precedence:

```
o{i, j} > o{k, l} for i+j > k+l
o{i, j} > lambda{k} for all i, j, k
o{i, j} > d{k, l} for all i, j, k, l
o{i, j} > 1 for all i, j
o{i, j} > p for all i, j
lambda{i} > lambda{k} for i > k
d{i, j} > d{k, l} for i+j > k+l
```



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Questions?