Certification of Termination

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TeReSe

Outline

1. CoLoR
   - Motivation
   - CoLoR architecture
   - History
   - Overview
   - Related work
   - Certified competition
2. Formalization of matrix interpretations
   - Introduction to matrix interpretations
   - Monotone algebras
   - Matrices
   - Matrix interpretations
   - Practicalities

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   - Practicalities

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- Certification of results of termination provers.
- Common proof format for termination provers:
  - common tools (proof presentation, manipulation, dots),
  - control language for provers (integration of tools)
- Extension of proof assistance kernels.

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How to certify termination results?

- Possibility: certification of tools source code.
  $\Rightarrow$ difficult, tool dependent, extra work with every change, ...
- CoLoR approach:
  - TPG: common format for termination proofs.
  - Tools output proofs in TPG format.
  - CoLoR: a Coq library of results on termination.
  - Rainbow: a tool for translation from proofs in TPG format to Coq proofs, using results from CoLoR.

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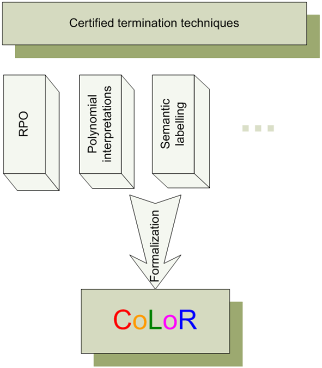

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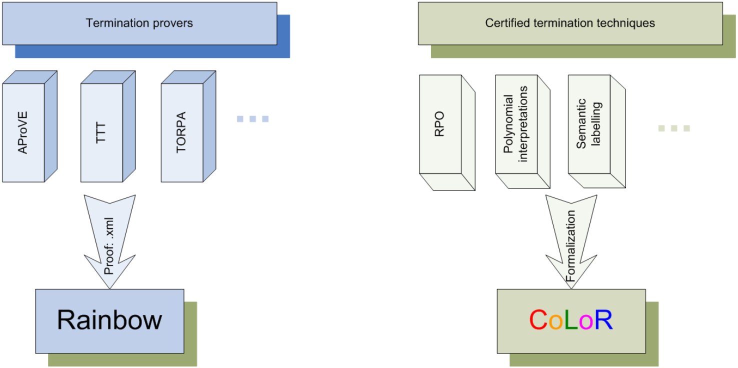

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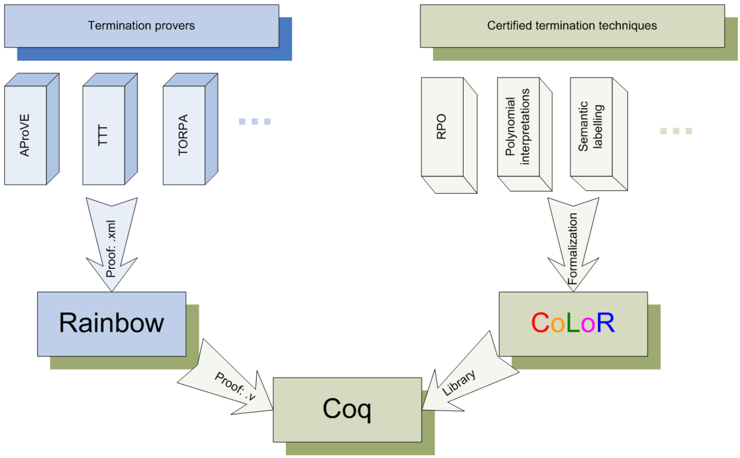

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- First release March 2005
- First certified proofs July 2006
- First certification workshop May 2007
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  - dependency graph cycles [Blanqui]
  - higher-order recursive path ordering [Koprowski]
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Size of CoLoR

- 42.000 lines of code.
- half of the size of Coq standard library.
- 5% of Coq contribs.

Structure:

Terms	44%
Data structures	29%
Termination criteria	17%
Mathematical structures	10%

Coq constructs:

Inductive definitions	38
Recursive functions	116
Non-recursive definitions	560
Lemmas and theorems	2170

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Certified competition

- In the termination competition this year a new “certified” category introduced.
- Participants:
  - CiME + A3PAT
  - TPA + CoLoR
  - $\mathsf{T_TT_2}$ + CoLoR
  - AProVE + A3PAT (?)
- Many questions remain, like
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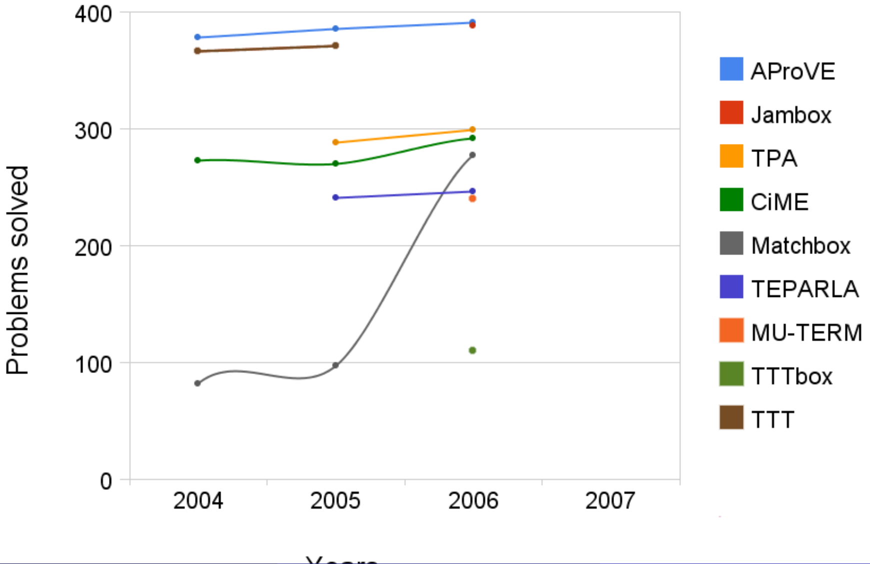

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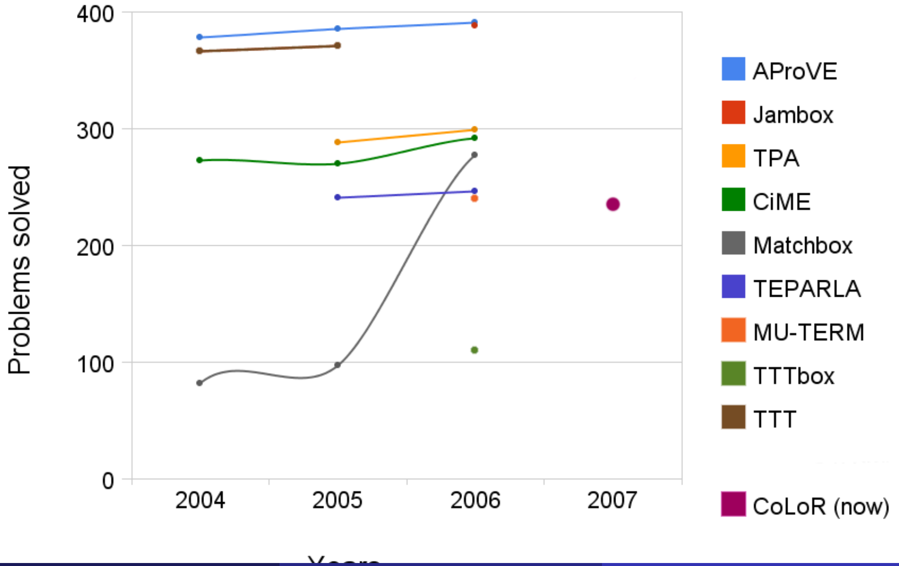

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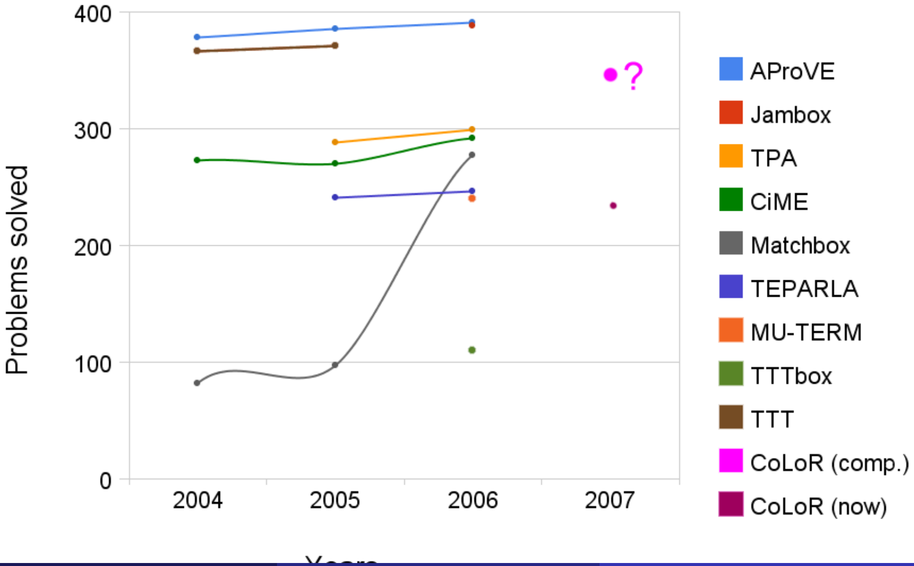

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Example

z086.trs

$a(a(x)) \to c(b(x)), \quad b(b(x)) \to c(a(x)), \quad c(c(x)) \to b(a(x))$

Matrix interpretation for z086.trs

$$a(x) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$b(x) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c(x) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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Example ctd.

Termination proof for z086.trs

$$a(a(x)) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$c(b(x)) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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$$
\begin{aligned}
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&= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \\
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\end{aligned}
$$

Monotone algebras

Definition (Monotonicity)

An operation $[f] : A \times \cdots \times A \to A$ is *monotone* with respect to a binary relation $\rhd$ on A if
$a_i \rhd a'_i \implies [f](a_1, \ldots, a_i, \ldots a_n) \rhd [f](a_1, \ldots, a'_i, \ldots, a_n)$.

Definition

Given a relation $\rhd$ on A we define its extension to a relation on terms as:

$$s \rhd_{\mathcal{T}} t \equiv \forall \alpha : \mathcal{X} \to A, [s, \alpha] \rhd [t, \alpha]$$

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Monotone algebras

Definition (A weakly monotone Σ-algebra)

A *weakly monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a Σ-algebra $(A, [\cdot])$ equipped with two binary relations $>$, $\gtrsim$ on A such that

- $>$ is well-founded;
- $> \cdot \gtrsim \subseteq >$;
- for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $\gtrsim$.

Definition (An *extended monotone Σ-algebra*)

An *extended monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a weakly monotone Σ-algebra $(A, [\cdot], >, \gtrsim)$ in which moreover for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $>$.

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Monotone algebras

Theorem

Let R, R', S, S' be TRSs over a signature Σ, $(A, [\cdot], >, \gtrsim)$ be an extended monotone Σ-algebra such that:

- $\ell \gtrsim_{\mathcal{I}} r$ *for every rule* $\ell \to r$ *in* $R \cup S$ *and*
- $\ell >_{\mathcal{I}} r$ *for every rule* $\ell \to r$ *in* $R' \cup S'$

Then $\mathrm{SN}(R/S)$ *implies* $\mathrm{SN}(R \cup R' \;/\; S \cup S')$.

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Formalization of monotone algebras

- Monotone algebras are formalized as a functor.
- Apart for the aforementioned requirements there is one additional required to deal with concrete examples: $>_{\mathcal{T}}$ and $\gtrsim_{\mathcal{T}}$ must be decidable.
- More precisely the requirement is to provide a relation $\gg$, such that
  - $\gg \subseteq >_{\mathcal{T}}$ and
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  - similarly for $\gtrsim$.
- The structure returned by the functor contains all the machinery required to prove (relative)-(top)-termination in Coq.

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Formalization of matrices

- Matrices are formalized as a functor taking as an argument the semi-ring of coefficients $\mathcal{R}$ and providing a structure of matrices of arbitrary sizes with coefficients in $\mathcal{R}$ and
- a number of basic operations over matrices such as:

$$[\cdot], \quad M_{i,j}, \quad M + N, \quad M * N, \quad M^T, \ \ldots$$

- and a number of basic properties such as:
  - $M + N = N + M$,
  - $M * (N * P) = (M * N) * P$
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Polynomial interpretations in the setting of monotone algebras

- $A = \mathbb{Z}$,
- $> \; = \; >_{\mathbb{Z}}$, $\gtrsim = \geq_{\mathbb{Z}}$,
- interpretations represented by polynomials $[f(x_1, \ldots, x_n)] = P_{\mathbb{Z}}(x_1, \ldots, x_n)$,
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Matrix interpretations in the setting of monotone algebras

- fix a dimension d,
- $A = \mathbb{N}^d$,
- $(u_1, \ldots, u_d) \gtrsim (v_1, \ldots, v_d)$ iff $\forall i, u_i \geq_{\mathbb{N}} v_i$,
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- interpretations represented as:
  $[f(x_1, \ldots, x_n)] = M_1 x_1 + \ldots + M_n x_n + v$
  where $M_i \in \mathbb{N}^{d \times d}$, $v \in \mathbb{N}^d$,
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Practicalities

Formalization size (LOC):

• Monotone algebras:	351
• Matrices:	642
• Matrix interpretations:	673
• Polynomial interpretations in MA setting:	116

Evaluation of Rainbow

Evaluation of TPA + Rainbow on TPDB 3.2 (864 TRSs):

- polynomial interpretations: 167
- matrix interpretations: 237
- polynomial and matrix interpretations: 275
  - Verification time: AVG: 5sec. MAX: 75sec.
  - Certificate size: AVG: 25kB. MAX: 437kB
  - Proof steps: AVG: 5 MAX: 29

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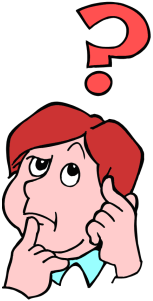
Figure: Before

Figure: Now

The end

http://color.loria.fr



Thank you for your attention.