Predictive Labeling with Dependency Pairs using SAT

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CADE
Bremen

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Termination of Rewriting

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \gcd(\max(x,y) - \min(x,y),\ s(\min(x,y))) \end{aligned}$$

$\gcd(6,4)$

Termination of Rewriting

Example

$$\mathsf{min}(x,0) \to 0 \qquad \mathsf{max}(x,0) \to x$$
$$\mathsf{min}(0,y) \to 0 \qquad \mathsf{max}(0,y) \to y$$
$$\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{min}(x,y)) \qquad \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\mathsf{max}(x,y))$$
$$\mathsf{gcd}(0,\mathsf{s}(x)) \to \mathsf{s}(x) \qquad x - 0 \to x$$
$$\mathsf{gcd}(\mathsf{s}(x),0) \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$$\mathsf{gcd}(6,4) \to \mathsf{gcd}(\mathsf{max}(5,3) - \mathsf{min}(5,3), \mathsf{s}(\mathsf{min}(5,3)))$$

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \color{red}{\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y))} &\color{red}{\to \mathsf{s}(\mathsf{max}(x, y))} \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$\color{red}{\mathsf{gcd}(6, 4)} \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{max}(4, 2)) - \mathsf{min}(5, 3), \mathsf{s}(\mathsf{min}(5, 3)))$

Termination of Rewriting

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\min(0,y) &\to 0 & \max(0,y) &\to y\\
\min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y))\\
\gcd(0,s(x)) &\to s(x) & x-0 &\to x\\
\gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y
\end{aligned}$$

$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\gcd(6,4) \to^{+} \gcd(s(s(\max(3,1))) - \min(5,3), s(\min(5,3)))$

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{max}(2, 0)))) - \mathsf{min}(5, 3), \mathsf{s}(\mathsf{min}(5, 3)))$

Termination of Rewriting

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \end{aligned}$$

$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

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\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y)))
\end{aligned}
$$

$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(5 - 3, 4)$

Termination of Rewriting

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$\mathrm{gcd}(6, 4) \to^{+} \mathrm{gcd}(2, 4)$

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
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$$\min(s(x),s(y)) \to s(\min(x,y)) \qquad \max(s(x),s(y)) \to s(\max(x,y))$$
$$\gcd(0,s(x)) \to s(x) \qquad x - 0 \to x$$
$$\gcd(s(x),0) \to s(x) \qquad s(x) - s(y) \to x - y$$
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$\gcd(6,4) \to^{+} \gcd(2,2)$

Termination of Rewriting

Example

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$\mathsf{gcd}(6,4) \to^{+} \mathsf{gcd}(0,2)$

Termination of Rewriting

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$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
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$\gcd(6,4) \to^{+} 2$

Definition

A TRS is terminating if it does not admit an infinite rewrite sequence.

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Termination methods

Knuth-Bendix order, polynomial interpretations, lexicographic path order, multiset order, multiset path order, recursive path order, semantic path order, recursive decomposition order, transformation order, elementary interpretations, well-founded monotone algebra, general path order, semantic labeling, type introduction, freezing, top-down labeling, dependency pair method, matchbounds, size-change principle, predictive labeling, uncurrying, matrix interpretations, quasi-periodic interpretations, bounded increase . . .

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Outline

1. Theory
   - Semantic Labeling (SL)
   - Predictive Labeling (PL)
   - Dependency Pairs (DP)
2. Practice
   - SAT encoding
   - Experimental results
3. Conclusions

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Semantic Labeling (SL)

Example

Is this TRS terminating?

$$
\begin{aligned}
\mathsf{f}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{f}(x,y))\\
\mathsf{f}(x,\mathsf{c}) &\to \mathsf{c}\\
\mathsf{f}(\mathsf{c},y) &\to \mathsf{c}\\
\mathsf{g}(x,\mathsf{c}) &\to x\\
\mathsf{g}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{g}(x,y))\\
\mathsf{g}(\mathsf{c},y) &\to y\\
\mathsf{h}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{h}(x,y)\\
\mathsf{h}(x,\mathsf{c}) &\to x\\
\mathsf{l}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{l}(\mathsf{h}(\mathsf{g}(x,y),\mathsf{f}(x,y)),\mathsf{s}(\mathsf{f}(x,y)))
\end{aligned}
$$

Semantic Labeling (SL)

Example

How about this one?

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}$$

Semantic Labeling (SL)

Example

$$
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\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}
$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

Semantic Labeling (SL)

Example

$$
\begin{array}{rcll}
\mathsf{min}(x, 0) & \to & 0 & x \geq 0 \\
\mathsf{min}(0, y) & \to & 0 & 0 \geq 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{min}(x, y)) & 2x + 1 \geq 2x + 1 \\
\mathsf{max}(x, 0) & \to & x & x \geq x \\
\mathsf{max}(0, y) & \to & y & y \geq y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{max}(x, y)) & 2x + 2y + 2 > 2x + 2y + 1 \\
\mathsf{s}(x) - \mathsf{s}(y) & \to & x - y & 2x + 1 > x \\
x - 0 & \to & x & x \geq x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) & 0 \geq 0
\end{array}
$$

$$
0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y
$$

$$
-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0
$$

Semantic Labeling (SL)

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\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \qquad i, j \in \mathbb{N} \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y) \qquad i > j
\end{aligned}
$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathsf{gcd}}(x, y) = 2x + y$$

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\mathsf{min}(x, 0) &\to 0 \\
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\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}
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$\Rightarrow$ LPO applicable

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

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\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
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\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}
$$

$\Rightarrow$ LPO applicable

$\ldots > \mathsf{gcd}_1 > \mathsf{gcd}_0 > \{\mathsf{min}, \mathsf{max}, -\} > \mathsf{s}$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathsf{gcd}}(x, y) = 2x + y$$

Semantic Labeling (SL)

Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$

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Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_\mathcal{A}, \gtrsim_\mathcal{A})$
- Labeling:
  - for every $f \in \mathcal{F}$ a set of labels $L_f \subseteq A$
  - labeling functions $\ell_f : A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$

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- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$
- Labeling:
  - for every $f \in \mathcal{F}$ a set of labels $L_f \subseteq A$
  - labeling functions $\ell_f : A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$
- Labeling of terms (for a variable assignment $\alpha : \mathcal{V} \to A$).

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$

Semantic Labeling (SL)

Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$
- Labeling:
  - for every $f \in \mathcal{F}$ a set of labels $L_f \subseteq A$
  - labeling functions $\ell_f : A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$
- Labeling of terms (for a variable assignment $\alpha : \mathcal{V} \to A$).

$$\mathrm{lab}_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathrm{lab}_\alpha(t_1), \ldots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathrm{lab}_\alpha(t_1), \ldots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

  with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$
- Labeled TRS:
  - $\mathcal{R}_{\mathrm{lab}} = \{\mathrm{lab}_\alpha(l) \to \mathrm{lab}_\alpha(r) \mid l \to r \in \mathcal{R}, \alpha : \mathcal{V} \to A\}$
  - $\mathrm{Decr} = \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid f \in \mathcal{F}, L_f \neq \varnothing; a, b \in L_f, a > b\}$

Semantic Labeling (SL)

Theorem (Zantema, 1995)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a* weakly-monotone *$\mathcal{F}$-algebra and*
- *a* weakly-monotone *labeling ℓ*

such that:

- $\mathcal{R} \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Predictive Labeling (PL)

Example

$$\min(x, 0) \to 0 \qquad \max(x, 0) \to x$$

$$\min(0, y) \to 0 \qquad \max(0, y) \to y$$

$$\min(s(x), s(y)) \to s(\min(x, y)) \qquad \max(s(x), s(y)) \to s(\max(x, y))$$

$$\gcd(0, s(x)) \to s(x) \qquad x - 0 \to x$$

$$\gcd(s(x), 0) \to s(x) \qquad s(x) - s(y) \to x - y$$

$$\gcd(s(x), s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\min(x,0) &\to 0 & \max(x,0) &\to x \\
\min(0,y) &\to 0 & \max(0,y) &\to y \\
\min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\
\gcd(0,s(x)) &\to s(x) & x - 0 &\to x \\
\gcd(s(x),0) &\to s(x) & s(x) - s(y) &\to x - y \\
\gcd(s(x),s(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}
$$

$$
0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = 2x+1 \quad \min_{\mathbb{N}}(x,y) = x \quad \max_{\mathbb{N}}(x,y) = x+y
$$
$$
-_{\mathbb{N}}(x,y) = x \quad \gcd_{\mathbb{N}}(x,y) = 0
$$

Problem: how to obtain a quasi-model?

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}
$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x,y) = x \quad \mathsf{max}_{\mathbb{N}}(x,y) = x + y$$
$$-_{\mathbb{N}}(x,y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x,y) = 0$$

Problem: how to obtain a quasi-model?

$\Rightarrow$ use predictive labeling (quasi-model constraints only for usable rules)

Predictive Labeling (PL)

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$

$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$

$$\min(s(x),s(y)) \to s(\min(x,y)) \qquad \max(s(x),s(y)) \to s(\max(x,y))$$

$$\mathsf{gcd}(0,s(x)) \to s(x) \qquad x - 0 \to x$$

$$\mathsf{gcd}(s(x),0) \to s(x) \qquad s(x) - s(y) \to x - y$$

$$\mathsf{gcd}(s(x),s(y)) \to \mathsf{gcd}(\max(x,y) - \min(x,y),\ s(\min(x,y)))$$

Computation of usable rules:

1. Look at symbols that will get labels.

Predictive Labeling (PL)

Example

$$\mathsf{min}(x, 0) \to 0 \qquad \mathsf{max}(x, 0) \to x$$
$$\mathsf{min}(0, y) \to 0 \qquad \mathsf{max}(0, y) \to y$$
$$\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\mathsf{min}(x, y)) \qquad \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\mathsf{max}(x, y))$$
$$\mathsf{gcd}(0, \mathsf{s}(x)) \to \mathsf{s}(x) \qquad x - 0 \to x$$
$$\mathsf{gcd}(\mathsf{s}(x), 0) \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{gcd}(\mathsf{max(x,y)} - \mathsf{min(x,y)}, \mathsf{s}(\mathsf{min(x,y)}))$$

Computation of usable rules:

1. Look at symbols that will get labels.
2. Look at their subterms.

Predictive Labeling (PL)

Example

$$
\begin{aligned}
\min(x,0) &\to 0 & \max(x,0) &\to x \\
\min(0,y) &\to 0 & \max(0,y) &\to y \\
\min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\
\gcd(0,s(x)) &\to s(x) & x-0 &\to x \\
\gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y
\end{aligned}
$$

$$\gcd(s(x),s(y)) \to \gcd(\max(x,y) - \min(x,y), s(\min(x,y)))$$

Computation of usable rules:

2. Look at their subterms.
3. All function symbols occurring there are usable.

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \end{aligned}$$

$$\gcd(s(x),s(y)) \to \gcd(\max(x,y) - \min(x,y), s(\min(x,y)))$$

Computation of usable rules:

3. All function symbols occurring there are usable.
4. Also the symbols that depend on them.

Predictive Labeling (PL)

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ \mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))} \end{aligned}$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

Computation of usable rules:

4. Also the symbols that depend on them.
5. Semantics needed only for usable symbols.

Predictive Labeling (PL)

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x-0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) &\to x-y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Computation of usable rules:

5. Semantics needed only for usable symbols.
6. Quasi-model constraints only required for usable rules.

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x - 0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x) - s(y) &\to x - y \end{aligned}$$

$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Computation of usable rules:

1. Look at symbols that will get labels.
2. Look at their subterms.
3. All function symbols occurring there are usable.
4. Also the symbols that depend on them.
5. Semantics needed only for usable symbols.
6. Quasi-model constraints only required for usable rules.

Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \unlhd r$

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- $F^* \equiv \{g \mid f \rhd_{\mathrm{d}}^* g$ for some $f \in \mathcal{F}\}$.

Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \unlhd r$
- $F^* \equiv \{g \mid f \rhd_{\mathrm{d}}^* g$ for some $f \in \mathcal{F}\}$.
- $\mathcal{G}_\ell(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \mathcal{F}(t_1)^* \cup \cdots \cup \mathcal{F}(t_n)^* & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing, \\ \mathcal{G}_\ell(t_1) \cup \cdots \cup \mathcal{G}_\ell(t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \end{cases}$

Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \trianglelefteq r$
- $F^* \equiv \{g \mid f \rhd_{\mathrm{d}}^* g \text{ for some } f \in \mathcal{F}\}$.
- $\mathcal{G}_\ell(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \mathcal{F}(t_1)^* \cup \cdots \cup \mathcal{F}(t_n)^* & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing, \\ \mathcal{G}_\ell(t_1) \cup \cdots \cup \mathcal{G}_\ell(t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \end{cases}$
- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$

Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \trianglelefteq r$
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- $\mathcal{G}_\ell(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \mathcal{F}(t_1)^* \cup \cdots \cup \mathcal{F}(t_n)^* & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing, \\ \mathcal{G}_\ell(t_1) \cup \cdots \cup \mathcal{G}_\ell(t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \end{cases}$
- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$
- $\mathcal{U}(\mathcal{R}, \ell) = \{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{G}_\ell(\mathcal{R})\}$

Predictive Labeling (PL)

Theorem (Hirokawa, Middeldorp, 2006)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a weakly-monotone $\sqcup$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Dependency Pairs (DP)

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
$$\min(s(x),s(y)) \to s(\min(x,y)) \qquad \max(s(x),s(y)) \to s(\max(x,y))$$
$$\gcd(0,s(x)) \to s(x) \qquad x - 0 \to x$$
$$\gcd(s(x),0) \to s(x) \qquad s(x) - s(y) \to x - y$$
$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Dependency Pairs (DP)

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ \mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))} \end{aligned}$$

$\Rightarrow$ apply dependency pairs

Dependency Pairs (DP)

Example

$$
\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y))) & &
\end{aligned}
$$

Dependency Pairs (DP)

Example

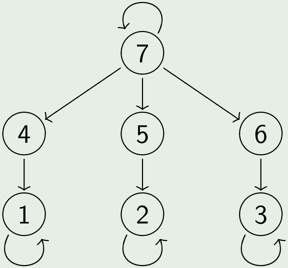
$$\begin{aligned}
\mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x\\
\mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y\\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y))\\
\mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x-0 &\to x\\
\mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x)-\mathsf{s}(y) &\to x-y\\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$$\begin{array}{rlrl}
(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)\\
(2) & \mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) & (6) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y)\\
(3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) & & \\
(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) & & \\
(7) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) & &
\end{array}$$

Dependency Pairs (DP)

Example

(1)	$\mathsf{minus}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{minus}^\sharp(x, y)$	(5)	$\mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{min}^\sharp(x, y)$
(2)	$\mathsf{min}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{min}^\sharp(x, y)$	(6)	$\mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{max}^\sharp(x, y)$
(3)	$\mathsf{max}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{max}^\sharp(x, y)$		
(4)	$\mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x, y), \mathsf{min}(x, y))$		
(7)	$\mathsf{gcd}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))$		



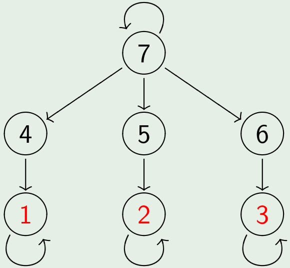
Dependency Pairs (DP)

Example

(1)	$\mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y)$	(5)	$\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)$
(2)	$\mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y)$	(6)	$\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y)$
(3)	$\mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y)$		

(4) $\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y))$

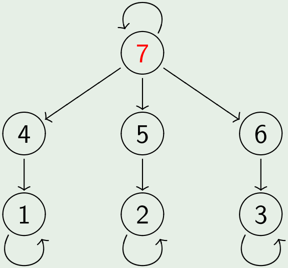
(7) $\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y) - \mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y)))$



Dependency Pairs (DP)

Example

$$
\begin{array}{llll}
(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) \\
(2) & \mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) & (6) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) \\
(3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) & & \\
(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) & & \\
(7) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) & &
\end{array}
$$



Proving termination with PL

(1) $\mathsf{SN}(\mathcal{R})$

$\Big\downarrow$ DP transformation

(2) $\mathsf{SN}(\mathsf{DP}(\mathcal{R}), \mathcal{R})$

$\Big\downarrow$ DP graph analysis

(3) $\mathsf{SN}(\mathcal{C}, \mathcal{R})$

$\Big\downarrow$ Predictive labeling

(4) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

$\Big\downarrow$ Reduction pair processor modulo π

(5) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}} \setminus >^{\pi}_{\mathsf{LPO}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

$\Big\downarrow$ Unlabeling

(6) $\mathsf{SN}(\mathcal{C}', \mathcal{R})$

Proving termination with PL

(1) $\mathsf{SN}(\mathcal{R})$

$\downarrow$ DP transformation

(2) $\mathsf{SN}(\mathsf{DP}(\mathcal{R}), \mathcal{R})$

$\downarrow$ DP graph analysis

(3) $\mathsf{SN}(\mathcal{C}, \mathcal{R})$

$\downarrow$ Predictive labeling

(4) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

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(5) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}} \setminus >^{\pi}_{\mathsf{LPO}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

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(6) $\mathsf{SN}(\mathcal{C}', \mathcal{R})$

Problem: $\mathcal{C}_{\mathsf{lab}}$ and $\mathcal{R}_{\mathsf{lab}}$ are infinite!

Proving termination with PL

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PL in the DP setting

Theorem

A DP problem $(\mathcal{P}, \mathcal{R})$ *is finite if there exists:*

- *a weakly-monotone* $\sqcup$*-algebra and*
- *a weakly-monotone labeling* ℓ

such that:

- $\mathcal{P} \subseteq \mathsf{DP}(\mathcal{R})$
- $\mathcal{R}$ *is finitely branching,*
- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $(\mathcal{P}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr})$ *is finite.*

Outline

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$\Rightarrow$ Idea: use SAT

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Quasi-model constraints

- Symbol dependencies:

$$\Delta_f(\mathcal{P},\mathcal{R}) = \bigcup_{l \to r \in \mathcal{P} \cup \mathcal{R}} \{g \in \mathcal{F}(t) \mid \mathrm{root}(t) = f \text{ and } t \trianglelefteq l \text{ or } t \trianglelefteq r\}.$$

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Encoding LPO

- LPO lifts a well-founded (quasi)-ordering of function symbols to a well-founded (quasi)-ordering of terms.

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$$\ulcorner f_i \succ g_j \urcorner = \ulcorner f_{\mathrm{L}} >_{\mathbb{N}} g_{\mathrm{L}} \urcorner \vee \Big(\ulcorner f_{\mathrm{L}} =_{\mathbb{N}} g_{\mathrm{L}} \urcorner \wedge \mathsf{L}_f \wedge \mathsf{L}_g \wedge \big(\ulcorner i >_{\mathcal{A}} j \urcorner \vee (\ulcorner i \gtrsim_{\mathcal{A}} j \urcorner \wedge \ulcorner f_{\mathrm{SL}} >_{\mathbb{N}} g_{\mathrm{SL}} \urcorner) \big) \Big)$$

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Experimental results

Table: Experiments with TPA on 864 TRSs from the TPDB version 3.2.

		60 seconds timeout			10 minutes timeout		
	technique	yes	time	timeout	yes	time	timeout
1×1	SL	440	1178	2	440	1351	0
	PL	456	1193	2	456	1316	0
	PL′	426	752	1	426	893	0

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2×2	SL	503	6905	51	506	24577	30
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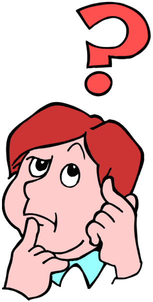
Innermost Termination of Rewrite Systems by Labeling.
WRS 2007.

The end



Thank you for your attention.