Certification of Arctic Termination

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(joint work with Frédéric Blanqui and Johannes Waldmann)

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TCS seminar, VU

Outline

1. CoLoR
   - Background: termination of rewriting
   - Why?... motivation
   - How?... CoLoR's approach to certification
   - When?... history of the project
   - What?... overview of the content
   - Related work
   - Certified competition
2. Arctic Termination
   - Monotone Algebras
   - Polynomial Interpretations
   - Matrix Interpretations
   - Arctic Interpretations
   - Arctic Below Zero Interpretations
   - Performance & Summary

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Example (Plus)

Let's define plus in Peano arithmetic.

$$
\begin{aligned}
0 + y &= y \\
s(x) + y &= s(x + y)
\end{aligned}
$$

Example (Computing with plus)

Now let us do some some maths... how about $2 + 2$?

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\begin{aligned}
&s(s(0)) + s(s(0)) \to s(s(0) + s(s(0))) \to \\
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Definition

A TRS is terminating iff it does not admin infinite reductions.

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Termination of rewriting:

- Is undecidable.
- Is an important topic in term rewriting.
- Many methods exist and new ones are constantly being developed.
- Recently the emphasis is on automation.
- There exists a number of tools for proving termination.
- Stimulated by an annual termination competition.
- Tools (and proofs that they produce) are getting more and more complex, so reliability is an issue.

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- Certification of results of termination provers.
- Common proof format for termination provers:
  - common tools (proof presentation, manipulation, . . .),
  - control language for provers (integration of tools)
- Extension of proof assistance kernels.

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CoLoR's approach to certification

CoLoR

http://color.loria.fr

CoLoR: Coq Library on Rewriting and Termination.
Goal: certification of termination proofs produced by various termination provers.

How to certify termination results?

- Possibility: certification of tools source code.
  $\Rightarrow$ difficult, tool dependent, extra work with every change, ...
- CoLoR's approach:
  - TPG: common format for termination proofs.
  - Tools output proofs in TPG format.
  - CoLoR: a Coq library of results on termination.
  - Rainbow: a tool for translation from proofs in TPG format to Coq proofs, using results from CoLoR.

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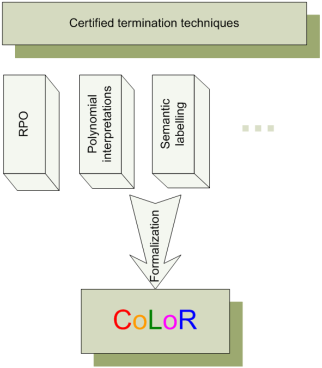
CoLoR's architecture overview



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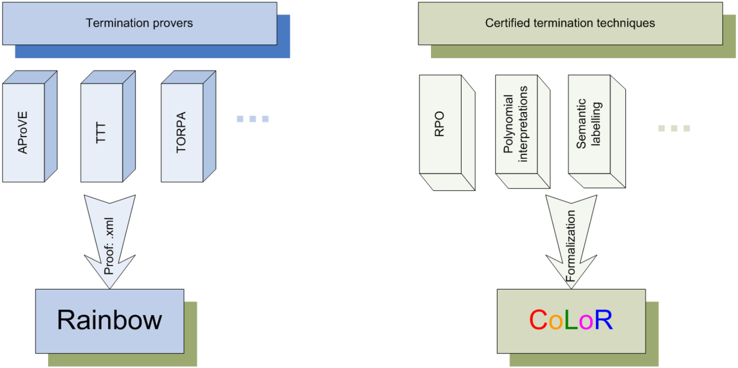

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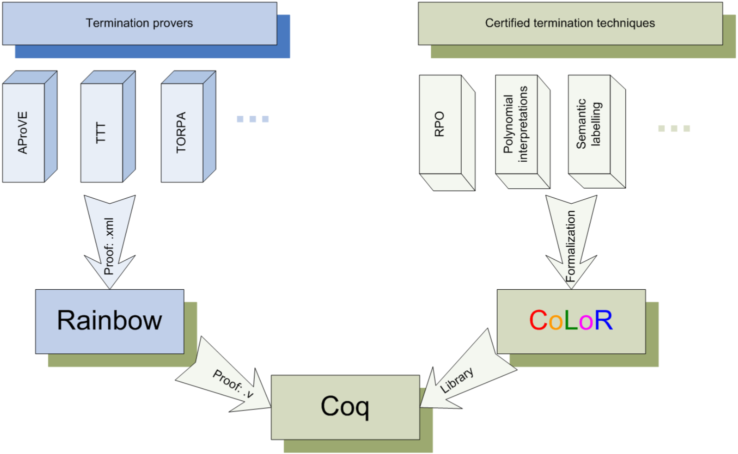

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- First release March 2005
- First certified proofs July 2006
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  - integer polynomials with multiple variables [Hinderer]
  - lists, vectors, relations, etc.
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Related work

- CoLoR project
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  Tool: T-TPA, . . .
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- A3PAT project
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- Participants:
  - CiME + A3PAT
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TOP


TERMINATION
CERTIFIED
BY COQ
STOP

Outline

Monotone algebras

Definition (Monotonicity)

An operation $[f] : A \times \cdots \times A \to A$ is *monotone* with respect to a binary relation $\rhd$ on A if
$a_i \rhd a'_i \implies [f](a_1, \ldots, a_i, \ldots a_n) \rhd [f](a_1, \ldots, a'_i, \ldots, a_n)$.

Definition (Monotone Σ-algebras)

A *weakly monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a Σ-algebra $(A, [\cdot])$ equipped with two binary relations $>, \gtrsim$ on A such that

- $>$ is well-founded;
- $> \cdot \gtrsim \subseteq >$;
- for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $\gtrsim$.

An *extended monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a weakly monotone Σ-algebra $(A, [\cdot], >, \gtrsim)$ in which moreover for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $>$.

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Monotone algebras

Theorem

Let R, R', S, S' be TRSs over a signature Σ, $(A, [\cdot], >, \gtrsim)$ be an extended monotone Σ-algebra such that:

- $\ell \gtrsim_{\mathcal{I}} r$ *for every rule* $\ell \to r$ *in* $R \cup S$ *and*
- $\ell >_{\mathcal{I}} r$ *for every rule* $\ell \to r$ *in* $R' \cup S'$

Then $\mathrm{SN}(R/S)$ *implies* $\mathrm{SN}(R \cup R' \;/\; S \cup S')$.

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Given TRSs $\mathcal{R}$ and $\mathcal{S}$ define:

- *top rewrite relation:* $t \xrightarrow{\text{top}}_{\mathcal{R}} u$ if and only if there is a rewrite rule $\ell \to r \in \mathcal{R}$ and a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\Sigma, \mathcal{V})$ such that $t = \ell\sigma$ and $u = r\sigma$.
- *rewrite relation:* $\to_{\mathcal{R}}$ is the smallest relation such that $\xrightarrow{\text{top}}_{\mathcal{R}} \subseteq \to_{\mathcal{R}}$ and $\to_{\mathcal{R}}$ is context-closed.
- *relation modulo:* $\to_1 / \to_2 \equiv \to_2^* \cdot \to_1$.
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Polynomial interpretations

- Interpretation domain: $\mathbb{N}$.
- Semi-ring computation structure: $\langle \mathbb{N}, +, * \rangle$.

Example

$$x * (y + z) \to x * y + x * z$$

$$[x + y] = x + y + 2, \qquad [x * y] = 2x + 2y + 2xy + 1$$

$$[x * (y + z)] = 2x + 2(y + z + 2) + 2x(y + z + 2) + 1$$
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- To obtain strict monotonicity we require that for every interpretation $[f(x_1, \ldots, x_n)]$, $\forall i, \exists c > 0, c * x_i \in [f(x_1, \ldots, x_n)]$.

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Matrix interpretations

- Interpretation domain: $\mathbb{N}^d$, for some fixed d.
- Semi-ring computation structure: $\langle \mathbb{N}, +, * \rangle$.
- $\vec{u} \geq \vec{v}$ iff $\forall i, \vec{u}_i \geq \vec{v}_i$.
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Example

$$a(a(x)) \to a(b(a(x))).$$

$$[a(x)] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad [b(x)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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- Now we need to restrict to linear interpretations.
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- $\vec{u} \geq \vec{v}$ iff $\forall i, \vec{u}_i \geq \vec{v}_i$.
- $\vec{u} > \vec{v}$ iff $\forall i, \vec{u}_i \gg \vec{v}_i$.

- Problem: arctic addition is not strictly monotonic in single arguments, ie. $5 > 3$ but $5 \oplus 6 = 6 \not> 6 = 3 \oplus 6$. We cannot get strict monotonicity for symbols of arity > 1.
- $\Rightarrow$ full termination only for SRSs.
- $\Rightarrow$ for TRSs we can "only" prove top-termination.
- Problem: well-foundedness of $>$ (as $-\infty \gg -\infty$)
- Problem: we need to ensure that we stay within the domain.
- $\Rightarrow$ for every interpretation $[f(x_1, \ldots, x_n)] = F_1 x_1 + \ldots F_n x_n + \vec{f}$ we require $\exists i, \text{finite}((F_i)_{1,1})$ or $\text{finite}(\vec{f}_1)$.

Arctic interpretations

- $\mathbb{A}_{\mathbb{N}} \equiv \{-\infty\} \cup \mathbb{N}$.
- We say that $a \in \mathbb{A}_{\mathbb{N}}$ is *finite* iff $a \neq -\infty$.
- Interpretation domain: $\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$, for some fixed d.
- Semi-ring computation structure: $\langle \mathbb{A}_{\mathbb{N}}, max, + \rangle$.
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Example arctic proof

Example

$$\{\mathsf{c\,a\,c} \to \epsilon,\ \mathsf{a\,c\,a} \to \mathsf{a}^4 \ /\ \epsilon \to \mathsf{c}^4\}.$$

$$[\mathsf{a}](x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & -\infty \\ 1 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} -\infty \\ -\infty \\ -\infty \end{pmatrix} \qquad [\mathsf{c}](x) = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \\ -\infty & 0 & -\infty \end{pmatrix} x + \begin{pmatrix} -\infty \\ -\infty \\ -\infty \end{pmatrix}$$

- $[\mathsf{c}]$ is a permutation (it swaps the second and third component), so $[\mathsf{c}]^2 = [\mathsf{c}]^4 = [\epsilon]$.
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Arctic below zero interpretations

- $\mathbb{A}_{\mathbb{Z}} \equiv \{-\infty\} \cup \mathbb{Z}$.
- Interpretation domain: $\mathbb{N} \times \mathbb{A}_{\mathbb{Z}}^{d-1}$, for some fixed d.
- $\Rightarrow$ we restrict first component to $\mathbb{N}$ to get well-foundedness.
- Semi-ring structure: $\langle \mathbb{A}_{\mathbb{Z}}, max, + \rangle$.
- $\geq, >$ as before.

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Example arctic below zero proof

Example

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while x > y do x := x - 1;
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$$\mathrm{cond}(\mathrm{true}, x, y) \to \mathrm{cond}(\mathrm{gr}(x, y), \mathrm{p}(x), y), \qquad \mathrm{gr}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{gr}(x, y),$$
$$\mathrm{gr}(0, x) \to \mathrm{false}, \qquad \mathrm{gr}(\mathrm{s}(x), 0) \to \mathrm{true},$$
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$$\mathrm{cond}^\sharp(\mathrm{true}, x, y) \to \mathrm{cond}^\sharp(\mathrm{gr}(x, y), \mathrm{p}(x), y)$$

$$[\mathrm{cond}^\sharp(x, y, z)] = (0)x + (0)y + (-\infty)z + (0), \qquad [0] = (0),$$
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Performance & Summary

- We presented an extension of matrix interpretations method by replacing the usual semi-ring structure with the arctic semi-ring.
- The method can prove full termination for SRSs and (relative) top termination for TRSs.
- We extended this from naturals to integers, resulting in arctic below zero interpretations.
- The whole method has been formalized in Coq within the CoLoR project.
- It has also been implemented in Matchbox, by transforming the constraints to propositional satisfiability problem and running Minisat.

problem set	time	s	sa	sz	saz	2007 winner
975 TRS	1 min	361	376	388	389	TPA: 354
	10 min	365	381	393	394	
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The end

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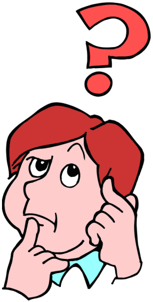

Thank you for your attention.