Certified Higher-Order Recursive Path Ordering

... a short story of a never ending formalization

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Abstract

The paper reports on a formalization of a proof of well-foundedness of the higher-order recursive path ordering (HORPO) in the proof checker Coq. The development is axiom-free and fully constructive. Three substantive parts that could be used also in other developments are the formalizations of the simply-typed lambda calculus, of finite multisets and of the multiset ordering. The Coq code consists of more than 1000 lemmas and 300 definitions.

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1. Background
   - Termination of rewriting
   - Recursive path ordering (RPO)
   - Higher-order rewriting
   - Higher-order recursive path ordering
2. Formalization
   - History
   - Motivation & Goals
   - Overview
3. Conclusions

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Termination of rewriting

- Termination is an important concept in term rewriting
- ... but in general it is undecidable
- ... but there are many techniques for proving termination.
- The simplest approach is embedding into a well-founded partial order.

$$t \to_R u \implies \phi(t) > \phi(u) \qquad (> \ \mathrm{well-founded})$$

- The simplest class of techniques are direct techniques using reduction orderings.
- Those are basic techniques but important nevertheless as they are heavily used in more complex transformational techniques.

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Reduction orderings

Definition

A strict order $>$ on $\mathcal{T}(\Sigma, \mathcal{V})$ is called a reduction ordering iff it is:

- monotonic

$$t > u \implies f(\ldots, t, \ldots) > f(\ldots, u, \ldots)$$

- stable

$$t > u \implies t\sigma > u\sigma$$

- well-founded

Theorem

A term rewriting system R is terminating iff there exist a reduction ordering $>$ that satisfied $\ell > r$ for all $\ell \to r \in R$.

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Recursive path ordering (RPO)

Definition (RPO, Dershowtiz 1982)

Given order on function symbols $\triangleright$ called precedence and a status we define the RPO ordering $\succ_{rpo}$ as follows:

$s = f(s_1, \ldots, s_n) \succ_{rpo} g(t_1, \ldots, t_m) = t \iff$

1. $s_i \succeq_{rpo} t$ for some $1 \leq i \leq n$.
2. $f \triangleright g$ and $s \succ_{rpo} t_i$ for all $1 \leq i \leq m$
3. $f = g$ and $(s_1, \ldots, s_n) \succ_{rpo}^{\tau(f)} (t_1, \ldots, t_m)$

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RPO is a reduction ordering.

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Higher-order rewriting

There are three variants of higher-order rewriting:

HRS Higher-order rewriting systems (Nipkow)
- $\lambda^{\to}$ terms
- Rules restricted to patterns
- Rewriting modulo $\beta\eta$

AFS Algebraic functional systems (Jouannaud and Okada)
- Algebraic terms with arity
- Plain pattern matching

CRS Combinatory reduction systems (Klop)
- Can be encoded via the other two

In this talk we concentrate on AFSs.

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Higher-order terms

Definition (Simple types, $\mathcal{T}_\mathcal{S}$)

Given a set of sorts $\mathcal{S}$ we inductively define a set of simple types as follows:

$$\mathcal{T}_\mathcal{S} \ ::= \ \mathcal{S} \mid \mathcal{T}_\mathcal{S} \to \mathcal{T}_\mathcal{S}$$

Definition (Signature, $\mathcal{F}$)

A signature is a set of function declarations of the shape:

$$f : \alpha_1 \times \ldots \times \alpha_n \to \beta$$

Definition (Environment, $\mathcal{E}$)

Environment is a finite set of distinct variable declarations:

$$\mathcal{E} \subset \mathcal{V} \times \mathcal{T}_\mathcal{S}$$

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Higher-order terms cont.

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A set of algebraic terms is defined by the following grammar:

$$\mathcal{P}t := \mathcal{V} \mid @(\mathcal{P}t, \mathcal{P}t) \mid \lambda\mathcal{V}{:}\mathcal{T}_{\mathcal{S}}.\mathcal{P}t \mid \mathcal{F}(\mathcal{P}t, \ldots, \mathcal{P}t)$$

Definition (Typing rules)

$$\frac{x : \alpha \in \Gamma}{\Gamma \vdash x : \alpha} \qquad \frac{f : \alpha_1 \times \ldots \times \alpha_n \to \beta \in \Sigma \quad \Gamma \vdash t_1 : \alpha_1, \ldots, \Gamma \vdash t_n : \alpha_n}{\Gamma \vdash f(t_1, \ldots, t_n) : \beta}$$

$$\frac{\Gamma \vdash t : \alpha \to \beta \qquad \Gamma \vdash u : \alpha}{\Gamma \vdash @(t, u) : \beta} \qquad \frac{\Gamma \cup \{x : \alpha\} \vdash t : \beta}{\Gamma \vdash \lambda x : \alpha . t : \alpha \to \beta}$$

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Examples of higher-order rewriting

Example (AFS for map)

$$\begin{array}{rcl} \mathcal{S} & = & \{\mathsf{List}, \mathsf{Nat}\} \\ \mathsf{nil} & : & \mathsf{List} \\ \mathsf{cons} & : & \mathsf{Nat} \times \mathsf{List} \to \mathsf{List} \\ \mathsf{map} & : & \mathsf{List} \times (\mathsf{Nat} \to \mathsf{Nat}) \to \mathsf{List} \end{array}$$

$$\begin{array}{rcl} \mathsf{map}(\mathsf{nil}, F) & \to & \mathsf{nil} \\ \mathsf{map}(\mathsf{cons}(x, l), F) & \to & \mathsf{cons}(@(F, x), \mathsf{map}(l, F)) \end{array}$$

Example (AFS for summation)

Function $\Sigma(n, F)$ computes $\Sigma_{0 \leq i \leq n} F(i)$.

$$\begin{array}{rcl} \Sigma(0, F) & \to & @(F, 0) \\ \Sigma(s(n), F) & \to & +(\Sigma(n, F), @(F, s(n))) \end{array}$$

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- monotonic $t > u \implies s[t]_p > s[u]_p$
- stable $t > u \implies t\sigma > u\sigma$
- coherent

$$\left.\begin{array}{c} \Gamma \vdash s : \delta > \Gamma \vdash t : \delta \\ \Delta \leftrightsquigarrow \Gamma \\ \Delta \vdash s : \delta \\ \Delta \vdash t : \delta \end{array}\right\} \implies \Delta \vdash s : \delta > \Delta \vdash t : \delta$$

- functional $t \rightarrow_\beta u \implies t > u$
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- coherent $\left.\begin{array}{c} \Gamma \vdash s : \delta > \Gamma \vdash t : \delta \\ \Delta \leftrightsquigarrow \Gamma \\ \Delta \vdash s : \delta \\ \Delta \vdash t : \delta \end{array}\right\} \implies \Delta \vdash s : \delta > \Delta \vdash t : \delta$
- functional $t \rightarrow_\beta u \implies t > u$
- well-founded

Higher-order reduction orderings

Definition

A strict order $>$ on $\mathcal{T}(\Sigma, \mathcal{V})$ is called a higher-order reduction ordering iff it is:

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Reduction orderings

First-order

- monotonic
  $t > u \implies f(\ldots, t, \ldots) > f(\ldots, u, \ldots)$
- stable
  $t > u \implies t\sigma > u\sigma$
- well-founded

Higher-order

- monotonic
  $t > u \implies s[t]_p > s[u]_p$
- stable
  $t > u \implies t\sigma > u\sigma$
- coherent
  $\Gamma \vdash s : \delta > \Gamma \vdash t : \delta \ \wedge \ \ldots$
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Formalization

- monotonic
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  $\Gamma \vdash s : \delta > \Gamma \vdash t : \delta \ \wedge \ \ldots$
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- $> \cup \rightarrow_\beta$ well-founded

Outline

1. Background
   - Termination of rewriting
   - Recursive path ordering (RPO)
   - Higher-order rewriting
   - Higher-order recursive path ordering
2. Formalization
3. Conclusions

Higher-order recursive path ordering

Definition (HORPO, Jouannaud and Rubio)

$\Gamma \vdash t : \delta \succ \Gamma \vdash u : \delta$ iff one of the following holds:

1. $t = f(t_1, \ldots, t_n), \exists i \in \{1, \ldots, n\} \; . \; t_i \succeq u$
2. $t = f(t_1, \ldots, t_n), u = g(u_1, \ldots, u_k), f \rhd g, t \ggcurly \{u_1, \ldots u_k\}$
3. $t = f(t_1, \ldots, t_n), u = f(u_1, \ldots, u_k),$ $\{\{t_1, \ldots t_n\}\} \succ_{mul} \{\{u_1, \ldots, u_k\}\}$
4. $@(u_1, \ldots, u_k)$ is a partial flattening of u, $t \ggcurly \{u_1, \ldots u_k\}$
5. $t = @(t_l, u_r), u = @(t_l, u_r), \{\{t_l, t_r\}\} \succ_{mul} \{\{u_l, u_r\}\}$
6. $t = \lambda x : \alpha . t', u = \lambda x : \alpha . u', t' \succ u'$

where $\ggcurly$ is defined as:

$t = f(t_1, \ldots, t_k) \ggcurly \{u_1, \ldots, u_n\}$ iff
$\forall i \in \{1, \ldots, n\} \; . \; t \succ u_i \vee (\exists j \; . \; t_j \succeq u_i)$.

HORPO in the formalization

Differences between HORPO as defined in [1] ($\succ_{JR}$) and the formalized variant ($\succ$).

- $\succ$ can only compare terms of equal types whereas in $\succ_{JR}$ terms of equivalent type can be compared
- $\succ_{JR}$ uses statuses (lexicographic / multiset) whereas in $\succ$ arguments can be compared only as multisets.
- $\succ$ uses a different variant of a definition of the multiset extension of a relation.
- $\succ$ assumes ground output type for functions.

J.-P. Jouannaud and A. Rubio.
The higher-order recursive path ordering.
In *LICS '99*, pages 402–411, Trento, Italy, July 1999.

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2. Formalization
   - History
   - Motivation & Goals
   - Overview
3. Conclusions

Timeline of the project

Two stages of the project:

- Jan 2004 - Jul 2004
  - Author's Master Thesis at the Vrije Universiteit supervised by dr Femke van Raamsdonk.
  - Proof completed but computability properties as axioms.
- Nov 2004 - Apr 2006
  - Development continued at the Technical University Eindhoven.
  - Complete, axiom free proof.
  - Many improvements on the way.
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Motivation & Goals

Motivation: Why making such formalization?

- Verification of the theory (especially for complicated, not very well-known proofs).
- Stimulus for improvement of formal methods, in general, and theorem proving, in particular.
- CoLoR: Coq library on rewriting and termination, `http://color.loria.fr`.

Goal: formalization that is:

- complete (axiom-free),
- fully constructive,
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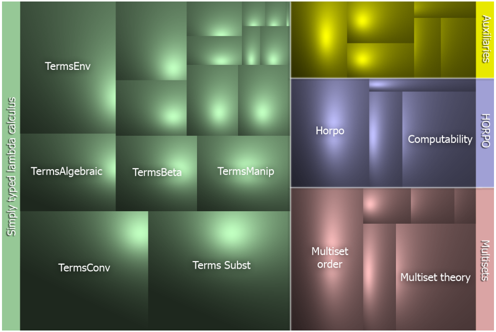
- Terms with typing a'la Church.
- Typing properties: uniqueness of types, decidability of typing.
- Many variable, typed substitution.
- Convertibility relation extending α-convertibility to free variables.
- $\longrightarrow_\beta$ and its properties.



- Computability predicate.
- Computability properties.
- Definition of the HORPO.
- Properties of HORPO (inc. decidability).
- Well-foundedness of $\succ \cup \longrightarrow_\beta$.

- Multisets as an abstract data-type.
- Concrete implementation using lists.
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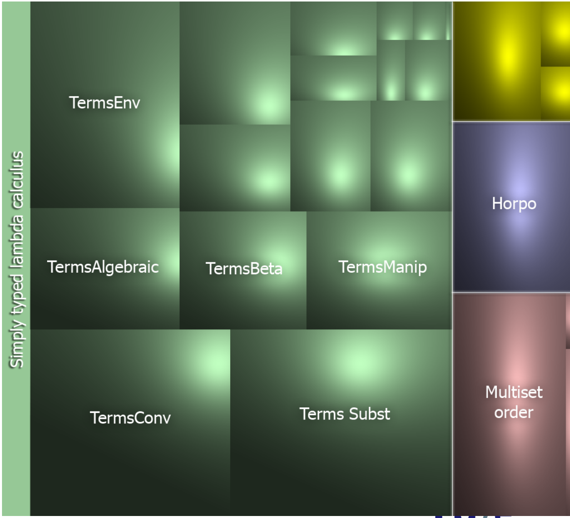

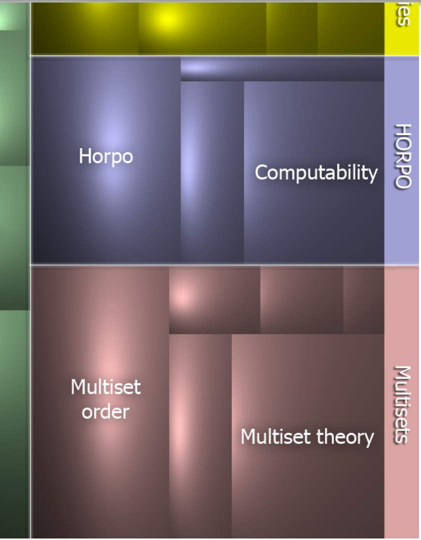

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- 33 files.
- $> 24,000$ lines.
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Evaluation of Coq

- Big developments in Coq are possible...
- ... but are still rather time consuming.
- Often simple things turn out not to be that simple (intuition).
- Dependent types are powerful and expressive...
- ... but reasoning about them is difficult.
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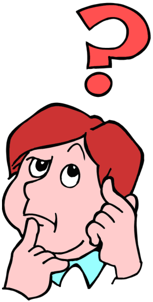
So now we can be certain that HORPO is correct.



Thank you for your attention.