Arctic Termination ... Below Zero

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Abstract. We introduce the arctic matrix method for automatically proving termination of term rewriting. We use vectors and matrices over the arctic semi-ring: natural numbers extended with $-\infty$, with the operations "max" and "plus". This extends the matrix method for term rewriting and the arctic matrix method for string rewriting. In combination with the Dependency Pairs transformation, this allows for some conceptually simple termination proofs in cases where only much more involved proofs were known before. We further generalize to arctic numbers "below zero": integers extended with $-\infty$. This allows to treat some termination problems with symbols that require a predecessor semantics. The contents of the paper has been formally verified in the Coq proof assistant and the formalization has been contributed to the CoLoR library of certified termination techniques. This allows formal verification of termination proofs using the arctic matrix method. We also report on experiments with an implementation of this method which, compared to results from 2007, outperforms TPA (winner of the certified termination competition for term rewriting), and in the string rewriting category is as powerful as Matchbox was but now all of the proofs are certified.

1 Introduction

One method of proving termination is interpretation into a well-founded algebra. Polynomial interpretations (over the naturals) are a well-known example of this approach. Another example is the recent development of the matrix method [17, 7] that uses linear interpretations over vectors of naturals, or equivalently, $\mathbb{N}$-weighted automata. In [23, 22] one of the authors extended this method (for string rewriting) to arctic automata, i.e. on the max/plus semi-ring on $\{-\infty\} \cup \mathbb{N}$. Its implementation in the termination prover Matchbox [21] contributed to this prover winning the string rewriting division of the 2007 termination competition [26].

The first contribution of the present work is a *generalization of arctic termination to term rewriting*. We use interpretations given by functions of the form $(x_1, \ldots, x_n) \mapsto M_0 + M_1 \cdot x_1 + \ldots + M_n \cdot x_n$. Here, x_i are (column) vector variables, M_0 is a vector and $M_1, \ldots, M_n$ are square matrices, where all entries are arctic numbers, and operations are understood in the arctic semi-ring.

Since the max operation is not strictly monotone in single arguments, we obtain monotone interpretations only for the case when all function symbols are at most unary, i.e. string rewriting. For symbols of higher arity, arctic interpretations are weakly monotone. These cannot prove termination, but only top termination, where rewriting steps are only applied at the root of terms. This is a restriction but it fits with the framework of the dependency pairs method [2] that transforms a termination problem to a top termination problem.

The second contribution is a *generalization from arctic naturals to arctic integers*, i.e. $\{-\infty\} \cup \mathbb{Z}$. Arctic integers allow e.g. to interpret function symbols by the predecessor function and this matches the "intrinsic" semantics of some termination problems. There is previous work on polynomial interpretations with negative coefficients [14], where the interpretation for predecessor is also expressible using ad-hoc max operations. Using arctic integers, we obtain verified termination proofs for 10 of the 24 rewrite systems Beerendonk/* from TPDB, simulating imperative computations. Previously, they could only be handled by the method of Bounded Increase [12].

The third contribution is that definitions, theorems and proofs (excluding Section 5 with results on full termination) have been *formalized with the proof assistant Coq* [25]. This extends previous work [19] and will become part of the CoLoR project [4] that gathers formalizations of termination techniques and employs them to certify termination proofs found automatically. In 2007, the certified category of the termination competition was won by the termination prover TPA [18] that uses CoLoR.

A method to search for arctic interpretations is implemented for the termination prover Matchbox. It works by transformation to a boolean satisfiability problem and application of a state-of-the-art SAT solver (in this case, Minisat). For several termination problems that could not be solved in last year's certified termination competition it finds proofs via arctic interpretations and the new CoLoR version certifies them.

The paper is organized as follows. We present notation and basic facts on rewriting and the arctic semi-ring in Section 2. Then in Section 3 we describe what kind of functions we use for interpretation and in Section 4 we discuss the appropriate ordering relations. We present arctic interpretations for termination in Section 5, for top termination in Section 6 and the generalization to arctic integers in Section 7. We report on the formal verification in Section 8 and on performance of our implementation in Section 9. We present some discussion of the method, its limitations and related work in Section 10 and we conclude in Section 11.

2 Notation and Preliminaries

We follow the notation of [3] for term rewriting. The top one-step derivation relation of a rewriting system $\mathcal{R}$ is denoted by $\xrightarrow{\text{top}}_{\mathcal{R}}$ and the full one-step derivation relation is $\rightarrow_{\mathcal{R}}$. We often abbreviate these by $\mathcal{R}_{\text{top}}$ and $\mathcal{R}$, respectively. A relation $\rightarrow$ is terminating if it does not admit infinite descending chains $t_0 \rightarrow t_1 \rightarrow \ldots$, denoted as $\mathrm{SN}(\to)$. For relations $\to_1, \to_2$, we define $\to_1 / \to_2$ by $(\to_1) \circ (\to_2)^*$. If $\mathrm{SN}(\mathcal{R}/\mathcal{S})$, we say that $\mathcal{R}$ is terminating relative to $\mathcal{S}$.

We cite notation for monotone algebras [7]. A k-ary operation $[f]$ is *monotone* with respect to a relation $\to$, if it is monotone in each argument individually: $x_i \to x_i'$ implies $[f](x_1, \dots, x_i, \dots, x_k) \to [f](x_1, \dots, x_i', \dots, x_k)$. A *weakly monotone algebra* for a signature Σ is a Σ-algebra $(A, [\cdot])$ with two relations $>, \gtrsim$ such that $>$ is well-founded, $> \cdot \gtrsim \subseteq >$ and for every $f \in \Sigma$, the operation $[f]$ is monotone with respect to $\gtrsim$. Such an algebra is called *extended monotone* if additionally each $[f]$ is monotone with respect to $>$. For terms ℓ, r with variables from a set $\mathcal{X}$, we write $[\ell] >_\alpha [r]$ to abbreviate $[\ell, \alpha] > [r, \alpha]$ for every $\alpha : \mathcal{X} \to A$. Now we present a slight variant of the main theorem from [7], for proving relative (top)-termination with monotone algebras:

Theorem 1. *Let $\mathcal{R}, \mathcal{R}', \mathcal{S}, \mathcal{S}'$ be TRSs over a signature Σ.*

1. *Let $(A, [\cdot], >, \gtrsim)$ be an extended monotone algebra such that: $[\ell] \gtrsim_\alpha [r]$ for every rule $\ell \to r \in \mathcal{R} \cup \mathcal{S}$ and $[\ell] >_\alpha [r]$ for every rule $\ell \to r \in \mathcal{R}' \cup \mathcal{S}'$. Then* $\mathrm{SN}(\mathcal{R}/\mathcal{S})$ *implies* $\mathrm{SN}(\mathcal{R} \cup \mathcal{R}'/\mathcal{S} \cup \mathcal{S}')$.
2. *Let $(A, [\cdot], >, \gtrsim)$ be a weakly monotone algebra such that: $[\ell] \gtrsim_\alpha [r]$ for every rule $\ell \to r \in \mathcal{R} \cup \mathcal{S}$ and $[\ell] >_\alpha [r]$ for every rule $\ell \to r \in \mathcal{R}'$. Then* $\mathrm{SN}(\mathcal{R}_{\mathrm{top}}/\mathcal{S})$ *implies* $\mathrm{SN}(\mathcal{R}_{\mathrm{top}} \cup \mathcal{R}'_{\mathrm{top}}/\mathcal{S})$. □

A *commutative semi-ring* [13] consists of a carrier D, two designated elements $d_0, d_1 \in D$ and two binary operations $\oplus, \otimes$ on D, such that both $(D, d_0, \oplus)$ and $(D, d_1, \otimes)$ are commutative monoids and multiplication distributes over addition: $\forall x, y, z \in D : x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

One example of semi-rings are the natural numbers with the standard operations. We will need the *arctic semi-ring* (also called the *max/plus algebra*) [9] with carrier $\mathbb{A}_\mathbb{N} \equiv \{-\infty\} \cup \mathbb{N}$, where semi-ring addition is the max operation with neutral element $-\infty$ and semi-ring multiplication is the standard plus operation with neutral element 0 ($x \otimes y = -\infty$ if either $x = -\infty$ or $y = -\infty$). We also consider these operations for arctic numbers *below zero* (ie. *arctic integers*), that is, on the carrier $\mathbb{A}_\mathbb{Z} \equiv \{-\infty\} \cup \mathbb{Z}$.

For any semi-ring D, we can consider the space of linear functions (square matrices) on n-dimensional vectors over D. These functions (matrices) again form a semi-ring (though a non-commutative one), and indeed we write $\oplus$ and $\otimes$ for its operations as well.

A semi-ring is *ordered* [8] by $\geq$ if $\geq$ is a partial order compatible with the operations: $\forall x \geq y, z : x \oplus z \geq y \oplus z$ and $\forall x \geq y, z : x \otimes z \geq y \otimes z$.

The standard semi-ring of natural numbers is ordered by the standard $\geq$ relation. The semi-ring of arctic naturals and arctic integers is ordered by $\geq$, being the reflexive closure of $>$ defined as $\ldots > 1 > 0 > -1 > \ldots > -\infty$. Note that standard integers with standard operations form a semi-ring but it is not ordered in this sense, as we have for instance $1 \geq 0$ but $1 * (-1) = -1 \not\geq 0 = 0 * (-1)$.

3 Max/Plus Linear Algebra

We consider vectors of arctic numbers. They form a monoid under component-wise arctic addition. For arctic matrices we define arctic addition and multiplication as usual. Square matrices form a non-commutative semi-ring with these operations. E.g. the 3×3 identity matrix is

$$\begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$

A square matrix M then maps a (column) vector x to a (column) vector $M \otimes x$ and this mapping is linear: $M \otimes (x \oplus y) = M \otimes x \oplus M \otimes y$. We use the following shape of vector-valued functions of several vector arguments:

$$f(x_1, \ldots, x_n) = f_0 \oplus f_1 \otimes x_1 \oplus \ldots \oplus f_n \otimes x_n.$$

Here, x_i are column vectors, f_0 is a column vector and $f_1, \ldots, f_n$ are square matrices. We call this an *arctic linear function* (with linear factors $f_1, \ldots, f_n$ and absolute part f_0).

Note that for brevity in all the examples we use the following notation for such linear functions:

$$f(x_1, \ldots, x_n) = f_0 \oplus f_1 x_1 \oplus \ldots \oplus f_n x_n.$$

Definition 2.
- *A number $a \in \mathbb{A}$ is called* finite *if $a > -\infty$.*
- *A number $a \in \mathbb{A}$ is called* positive *if $a \geq 0$.*
- *A vector $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$ is called finite if x_1 is finite and it is called positive if x_1 is positive.*
- *A matrix $M \in \mathbb{A}^{m \times n}$ is called finite if $M_{1,1}$ is finite.*
- *A linear function f is called* somewhere finite *if $\exists 0 \leq i \leq n : \text{finite}(f_i)$.*
- *A linear function f is called* absolute positive *if* positive(f_0)*.* $\diamond$

Example 3. Consider a linear function:

$$f(x, y) = \begin{pmatrix} 1 & -\infty \\ 0 & -\infty \end{pmatrix} x \oplus \begin{pmatrix} -\infty & -\infty \\ 0 & 1 \end{pmatrix} y \oplus \begin{pmatrix} -\infty \\ 0 \end{pmatrix}$$

which is somewhere finite, as the upper-leftmost element of the matrix coefficient of x is 1, which is finite. It is not absolute positive, as the constant vector has $-\infty$ on its first position.

Evaluation of this function on some exemplary arguments yields:

$$f(\begin{pmatrix} -\infty \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\infty \end{pmatrix}) = \begin{pmatrix} 1 & -\infty \\ 0 & -\infty \end{pmatrix} \begin{pmatrix} -\infty \\ 0 \end{pmatrix} \oplus \begin{pmatrix} -\infty & -\infty \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\infty \end{pmatrix} \oplus \begin{pmatrix} -\infty \\ 0 \end{pmatrix} = \begin{pmatrix} -\infty \\ 1 \end{pmatrix}.$$

$\lhd$

Lemma 4. *For numbers, vectors, matrices:*

– *if a is finite and b arbitrary, then $a \oplus b$ is finite.*
– *if a is positive and b arbitrary, then $a \oplus b$ is positive.*
– *if a and b are finite, then $a \otimes b$ is finite.* □

Lemma 5. *For a linear function f:*

1. *if f is somewhere finite, then* $\forall x_1, \ldots, x_n : (\forall i : \text{finite}(x_i)) \Rightarrow \text{finite}(f(x_1, \ldots, x_n))$.
2. *if f is absolute positive, then* $\forall x_1, \ldots, x_n : \text{positive}(f(x_1, \ldots, x_n))$. □

4 Orders on Max/Plus

Arctic addition (i.e., the max operation) is not strictly monotone in single arguments: we have e.g. $5 > 3$ but $5 \oplus 6 = 6 \not> 6 = 3 \oplus 6$. It is, however, "half strict" in the following sense: a strict increase in both arguments simultaneously gives a strict increase in the result, e.g. $5 > 3$ and $6 > 4$ implies $5 \oplus 6 > 3 \oplus 4$. Compared to the standard matrix method, this special property of arctic addition requires a somewhat different treatment of monotonicity. In several places where the standard matrix method needs just one strict inequality (among several non-strict ones), the arctic matrix method needs all inequalities to be strict. There is one exception: arctic addition is obviously strict if one argument is arctic zero, i.e., $-\infty$. This explains the definition of $\gg$ below. In this section, we consider arctic integers.

Definition 6. – *We write $\geq$ for reflexive closure of the standard ordering $\ldots > 1 > 0 > -1 > \ldots > -\infty$ and extend this notation component-wise to vectors, matrices and linear functions.*
– *We write $a \gg b$ if $(a > b) \vee (a = b = -\infty)$, and we extend this notation component-wise to vectors, matrices and linear functions.* ◇

Note that $\gg \cdot \geq \subseteq \gg$, which is required to apply the monotone algebra theorem.

Lemma 7. *For arctic integers a, a_1, a_2, b_1, b_2,*

– *if $a_1 \geq a_2 \wedge b_1 \geq b_2$, then $a_1 \oplus b_1 \geq a_2 \oplus b_2$ and $a_1 \otimes b_1 \geq a_2 \otimes b_2$.*
– *if $a_1 \gg a_2 \wedge b_1 \gg b_2$, then $a_1 \oplus b_1 \gg a_2 \oplus b_2$.*
– *if $b_1 \gg b_2$, then $a \otimes b_1 \gg a \otimes b_2$.* □

The following lemma allows to establish order on results of two functions by comparison of their coefficients. It is the arctic counter-part of the absolute-positiveness criterion used for polynomial interpretations.

Lemma 8. *For linear functions f, g with $f \geq g$ (resp. $f \gg g$), and for each tuple of vectors $x_1, \ldots, x_n$: $f(x_1, \ldots, x_n) \geq g(x_1, \ldots, x_n)$ (resp. $f(x_1, \ldots, x_n) \gg g(x_1, \ldots, x_n)$).* □

Lemma 9. *Every linear function f is monotone with respect to $\geq$.*

Proof. For $x_i \geq x'_i$ we have:

$$f_0 \oplus f_1 \otimes x_1 \oplus \ldots f_i \otimes x_i \ldots \oplus f_n \otimes x_n \geq f_0 \oplus f_1 \otimes x_1 \oplus \ldots f_i \otimes x'_i \ldots \oplus f_n \otimes x_n$$

using Lemma 7 lifted to vectors. □

5 Full Arctic Termination

In this section we present a method of using arctic matrices to prove full termination (as opposed to top termination, see Section 6). For some fixed dimension d we choose the algebra over the domain, $\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$, that is over vectors of arctic naturals where the first position of the vector is finite. The algebra is ordered with $\gg$ and the ordering is well-founded due to restriction to finite elements on first vector positions. Function symbols are interpreted by linear arctic functions.

The following theorem provides a termination criterion with such monotone interpretations. A linear Σ-interpretation is an interpretation that associates an arctic linear function $[f]$ with every $f \in \Sigma$. As noted at the beginning of Section 4, "$\oplus$" is not strictly monotone. Therefore, a function of the shape $f_0 \oplus f_1 \otimes x_1 \oplus \ldots \oplus f_n \otimes x_n$ is monotone only if the $\oplus$ operation is essentially redundant. This happens in the following cases.

Theorem 10. *Let $\mathcal{R}, \mathcal{R}', \mathcal{S}, \mathcal{S}'$ be TRSs over a signature Σ and $[\cdot]$ be a linear Σ-interpretation with coefficients in $\mathbb{A}_{\mathbb{N}}$. If:*

- *every function symbol has arity at most 1,*
- *for every constant $f \in \Sigma$, $[f]_0$ is finite,*
- *for every unary symbol $f \in \Sigma$, $[f]_0$ is the arctic zero vector and $[f]_1$ is finite,*
- $[\ell] \geq [r]$ *for every rule* $\ell \to r \in \mathcal{R} \cup \mathcal{S}$,
- $[\ell] \gg [r]$ *for every rule* $\ell \to r \in \mathcal{R}' \cup \mathcal{S}'$ *and*
- $\mathrm{SN}(\mathcal{R}/\mathcal{S})$.

Then $\mathrm{SN}(\mathcal{R} \cup \mathcal{R}'/\mathcal{S} \cup \mathcal{S}')$.

Proof. By Theorem 1.1. Note that, by Lemma 8, $[\ell] \geq [r]$ (resp. $[\ell] \gg [r]$) implies $[\ell] \geq_\alpha [r]$ (resp. $[\ell] \gg_\alpha [r]$). So we only need to show that $(\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}, [\cdot], \gg, \geq)$ is an extended monotone algebra. The order $\gg$ is well-founded on this domain as with every decrease we get a decrease in the first component of the vector, which differs from $-\infty$. It is an easy observation that, due to the first three premises of this theorem, such interpretations are monotone. Finally evaluation of interpretations stays within the domain by Lemma 5.1 as every $[f]$ is somewhere finite by assumption. □

For symbols of arity $n > 1$ there is no arctic linear function that is monotone, hence the arctic matrix method for full termination is only applicable for string rewriting (plus constants). As such, it had been described in [22] and had been applied by Matchbox in the 2007 termination competition. The following example illustrates the method.

Example 11. The relative termination problem `SRS/Waldmann/r2` is

$$\{\mathrm{c\,a\,c} \to \epsilon,\ \mathrm{a\,c\,a} \to \mathrm{a}^4\ \ /\ \epsilon \to \mathrm{c}^4\}.$$

In the 2007 termination competition, it had been solved by Jambox [6] via "self labeling" and by Matchbox via essentially the following arctic proof.

We use the following arctic interpretation

$$[\mathrm{a}](x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & -\infty \\ 1 & 1 & 0 \end{pmatrix} x \oplus \begin{pmatrix} -\infty \\ -\infty \\ -\infty \end{pmatrix} \qquad [\mathrm{c}](x) = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \\ -\infty & 0 & -\infty \end{pmatrix} x \oplus \begin{pmatrix} -\infty \\ -\infty \\ -\infty \end{pmatrix}$$

It is immediate that [c] is a permutation (it swaps the second and third component of its argument vector), so $[\mathrm{c}]^2 = [\mathrm{c}]^4$ is the identity and we have $[\epsilon] = [\mathrm{c}]^4$. A short calculation shows that [a] is idempotent, so $[\mathrm{a}] = [\mathrm{a}^4]$. We compute

$$[\mathrm{c\,a\,c}](x) = \begin{pmatrix} 0 & -\infty & 0 \\ 1 & 0 & 1 \\ 0 & -\infty & 0 \end{pmatrix} x \quad [\mathrm{a\,c\,a}](x) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} x \quad [\mathrm{a}^4](x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & -\infty \\ 1 & 1 & 0 \end{pmatrix} x$$

and therefore $[\mathrm{c\,a\,c}] \geq [\epsilon]$ and $[\mathrm{a\,c\,a}] \gg [\mathrm{a}^4]$. Note that indeed we have point-wise $\gg$ and the top left entries of matrices are finite. This allows to remove one strict rule. The remaining strict rule can be removed by counting letters `a`. $\lhd$

6 Arctic Top Termination

As explained earlier, there are no monotone linear arctic functions of more than one argument. We therefore change our attention from proving full termination to proving top termination. This fits with the Dependency Pairs method that replaces a full termination problem with an equivalent top termination problem.

The domain, as in Section 5, is $\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$ for some fixed dimension d and we use the ordering relations $\gg$ (strict) and $\geq$ (weak). The following theorem allows us to prove top termination in this setting:

Theorem 12. *Let $\mathcal{R}, \mathcal{R}', \mathcal{S}$ be TRSs over a signature Σ and $[\cdot]$ be a linear Σ-interpretation with coefficients in $\mathbb{A}_{\mathbb{N}}$. If:*

- *for each $f \in \Sigma$, $[f]$ is somewhere finite,*
- *$[\ell] \geq [r]$ for every rule $\ell \to r \in \mathcal{R} \cup \mathcal{S}$,*
- *$[\ell] \gg [r]$ for every rule $\ell \to r \in \mathcal{R}'$ and*
- *$\mathrm{SN}(\mathcal{R}_{\mathrm{top}}/\mathcal{S})$.*

Then $\mathrm{SN}(\mathcal{R}_{\mathrm{top}} \cup \mathcal{R}'_{\mathrm{top}}/\mathcal{S})$.

Proof. By Theorem 1.2; we need to show that $(\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}, [\cdot], \gg, \geq)$ is a weakly monotone algebra. The proof is essentially the same as the proof of Theorem 10. Note that now we only need a weakly monotone algebra and indeed by allowing function symbols of arity > 1, we lose the strict monotonicity property. $\square$

Example 13. Consider the rewriting system `secret05/tpa2`:

$$\begin{aligned} \mathrm{f}(\mathrm{s}(x), y) &\to \mathrm{f}(\mathrm{p}(\mathrm{s}(x) - y), \mathrm{p}(y - \mathrm{s}(x))), & \mathrm{p}(\mathrm{s}(x)) &\to x, \\ \mathrm{f}(x, \mathrm{s}(y)) &\to \mathrm{f}(\mathrm{p}(x - \mathrm{s}(y)), \mathrm{p}(\mathrm{s}(y) - x)), & x - 0 &\to x, \\ & & \mathrm{s}(x) - \mathrm{s}(y) &\to x - y. \end{aligned}$$

It was solved in the 2007 competition by AProVE [11] using narrowing followed by polynomial interpretations and by $\mathsf{T_TT2}$ [15] using polynomial interpretations with negative constants.

After the DP transformation 9 dependency pairs can be removed using polynomial interpretations leaving the essential two dependency pairs:

$$\mathrm{f}^\sharp(\mathrm{s}(x), y) \to \mathrm{f}^\sharp(\mathrm{p}(\mathrm{s}(x) - y), \mathrm{p}(y - \mathrm{s}(x)))$$
$$\mathrm{f}^\sharp(x, \mathrm{s}(y)) \to \mathrm{f}^\sharp(\mathrm{p}(x - \mathrm{s}(y)), \mathrm{p}(\mathrm{s}(y) - x))$$

Now the arctic interpretation

$$\begin{aligned}
[\mathrm{f}^\sharp(x,y)] &= \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix} x \oplus \begin{pmatrix} 0 & 0 \\ -\infty & -\infty \end{pmatrix} y \oplus \begin{pmatrix} 0 \\ -\infty \end{pmatrix} & [0] &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
[x - y] &= \begin{pmatrix} 0 & -\infty \\ 0 & 0 \end{pmatrix} x \oplus \begin{pmatrix} -\infty & -\infty \\ 0 & 0 \end{pmatrix} y \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} & [\mathrm{p}(x)] &= \begin{pmatrix} 0 & -\infty \\ 0 & -\infty \end{pmatrix} x \oplus \begin{pmatrix} -\infty \\ -\infty \end{pmatrix} \\
[\mathrm{f}(x,y)] &= \begin{pmatrix} 0 & 0 \\ 0 & -\infty \end{pmatrix} x \oplus \begin{pmatrix} 2 & 0 \\ 0 & -\infty \end{pmatrix} y \oplus \begin{pmatrix} 0 \\ -\infty \end{pmatrix} & [\mathrm{s}(x)] &= \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} x \oplus \begin{pmatrix} 0 \\ 2 \end{pmatrix}
\end{aligned}$$

removes the second dependency pair as we have:

$$\begin{aligned}
[\mathrm{f}^\sharp(x, \mathrm{s}(y))] &= \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix} x \oplus \begin{pmatrix} 2 & 1 \\ -\infty & -\infty \end{pmatrix} y \oplus \begin{pmatrix} 2 \\ -\infty \end{pmatrix} \\
[\mathrm{f}^\sharp(\mathrm{p}(x - \mathrm{s}(y)), \mathrm{p}(\mathrm{s}(y) - x))] &= \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix} x \oplus \begin{pmatrix} 0 & 0 \\ -\infty & -\infty \end{pmatrix} y \oplus \begin{pmatrix} 0 \\ -\infty \end{pmatrix}
\end{aligned}$$

and it is weakly compatible with all the rules. The remaining dependency pair can be removed by a standard matrix interpretation of dimension two. $\lhd$

7 ... Below Zero

We extend the domain of matrix and vector coefficients from $\mathbb{A}_\mathbb{N}$ (arctic naturals) to $\mathbb{A}_\mathbb{Z}$ (arctic integers). This allows to interpret some function symbols by the "predecessor" function $x \mapsto x - 1$, and so represents their "intrinsic" semantics. This is the same motivation as the one for allowing polynomial interpretations with negative coefficients [14].

We need to be careful though, as the relation $\gg$ on vectors of arctic integers is not well-founded.

Theorem 14. *Let $\mathcal{R}, \mathcal{R}', \mathcal{S}$ be TRSs over a signature Σ and $[\cdot]$ be a linear Σ-interpretation with coefficients in $\mathbb{A}_\mathbb{Z}$. If:*

- *for each $f \in \Sigma$, $[f]$ is absolute positive,*
- *$[\ell] \geq [r]$ for every rule $\ell \to r \in \mathcal{R} \cup \mathcal{S}$,*
- *$[\ell] \gg [r]$ for every rule $\ell \to r \in \mathcal{R}'$ and*
- *$\mathrm{SN}(\mathcal{R}_{\mathrm{top}}/\mathcal{S})$.*

Then $\mathrm{SN}(\mathcal{R}_{\mathrm{top}} \cup \mathcal{R}'_{\mathrm{top}}/\mathcal{S})$.

Proof. The proof goes along the same lines as the proof of Theorem 12. Note however that as we are working with integers now, to ensure that we stay within the domain, we need a stronger assumption on interpretations; we get that property by Lemma 5.2. □

Example 15. Let us consider the Beerendonk/2.trs TRS from the TPDB [27], consisting of the following six rules:

$$\begin{aligned} \mathrm{cond}(\mathrm{true}, x, y) &\to \mathrm{cond}(\mathrm{gr}(x, y), \mathrm{p}(x), \mathrm{s}(y)), & \mathrm{gr}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{gr}(x, y), \\ \mathrm{gr}(0, x) &\to \mathrm{false}, & \mathrm{gr}(\mathrm{s}(x), 0) &\to \mathrm{true}, \\ \mathrm{p}(0) &\to 0, & \mathrm{p}(\mathrm{s}(x)) &\to x \end{aligned}$$

This is a straightforward encoding of the following imperative program

```
while x > y do (x, y) := (x-1, y+1);
```

which is obviously terminating. However this TRS posed a serious challenge for the tools in the termination competition. Only AProVE could deal with this system (as well as a number of others coming from such transformations from imperative programs) using a specialized bounded increase method [12]. We will now show a termination proof for this system using the arctic below zero interpretations.

We begin by applying the dependency pair method and obtaining four dependency pairs, three of which can be easily removed (for instance using standard matrix or polynomial interpretations) leaving the following single dependency pair:

$$\mathrm{cond}^\sharp(\mathrm{true}, x, y) \to \mathrm{cond}^\sharp(\mathrm{gr}(x, y), \mathrm{p}(x), \mathrm{s}(y))$$

Now, consider the following arctic matrix interpretation:

$$\begin{aligned} [\mathrm{cond}^\sharp(x, y, z)] &= (0)x \oplus (0)y \oplus (-\infty)z \oplus (0), & [0] &= (0), \\ [\mathrm{cond}(x, y, z)] &= (0)x \oplus (2)y \oplus (-\infty)z \oplus (0), & [\mathrm{false}] &= (0), \\ [\mathrm{gr}(x, y)] &= (-1)x \oplus (-\infty)y \oplus (0), & [\mathrm{true}] &= (2), \\ [\mathrm{p}(x)] &= (-1)x \oplus (0), & [\mathrm{s}(x)] &= (2)x \oplus (3). \end{aligned}$$

With this interpretation we get a decrease for the dependency pair:

$$\begin{aligned} [\mathrm{cond}^\sharp(\mathrm{true}, x, y)] &= (\ \ 0)x \oplus (-\infty)y \oplus (2) \\ [\mathrm{cond}^\sharp(\mathrm{gr}(x, y), \mathrm{p}(x), \mathrm{s}(y))] &= (-1)x \oplus (-\infty)y \oplus (0) \end{aligned}$$

and all the original rules are oriented weakly. ◁

Remark 16. We discuss a variant that looks more liberal, but turns out to be equivalent to the one given here. We cannot allow $\mathbb{Z} \times \mathbb{A}_{\mathbb{Z}}^{d-1}$ for the domain, because it is not well-founded for $\gg$. So we can restrict the admissible range of negative values by some bound $c > -\infty$, and use the domain $\mathbb{A}_{\mathbb{Z}\geq c} \times \mathbb{A}_{\mathbb{Z}}^{d-1}$ where $\mathbb{A}_{\mathbb{Z}\geq c} := \{b \in \mathbb{A}_{\mathbb{Z}} \mid b \geq c\}$. Now to ensure that we stay within this domain we would demand that the first position of the constant vector of every interpretation is greater or equal than c.

Note however that this c can be fixed to 0 without any loss of generality as every interpretation using lower values in those positions can be "shifted" upwards. For any interpretation $[\cdot]$ and arctic number d construct an interpretation $[\cdot]'$ by $[t]' := [t] \otimes d$. This is obtained by going from $[f] = f_0 \oplus f_1 x_1 \oplus \ldots f_k x_k$ to $[f]' = f_0 \otimes d \oplus f_1 x_1 \oplus \ldots f_k x_k$. (A linear function with absolute part can be scaled by scaling the absolute part.) □

8 Certification

The certification has been carried out within the CoLoR library [4]: a library of termination techniques formalized in Coq. This library is then used by a tool Rainbow to transform termination proofs in the common termination proof format, designed within the CoLoR project, to actual Coq proofs certifying termination.

The basis of this work was the certification of the matrix interpretations method [19], which consists of formalizations of:

- a semi-ring structure,
- vectors and matrices over arbitrary semi-rings of coefficients,
- the monotone algebras framework and
- the matrix interpretation method.

The framework of monotone algebras was used without any changes at all. Vectors and matrices were formalized for arbitrary semi-rings, however all the results involving orders were developed for the usual orders on natural numbers, as used in the matrix interpretations method. So the first step in the certification process was to generalize the semi-ring structure to a semi-ring equipped with two orders $(>, \geq)$ and to adequately generalize results on vectors and matrices. Then the arctic semi-ring was developed in this setting.

As for the technique itself it has a lot in common with the technique of matrix interpretations. Therefore the common parts were extracted to a module `MatrixBasedInt` which was then specialized to the matrix interpretation method (`MatrixInt`) and to a basis for arctic based methods (`ArcticBasedInt`), which was narrowed down to the methods of arctic interpretations (`ArcticInt`) and arctic below-zero interpretations (`ArcticBZInt`). This hierarchy is depicted in Figure 1.

Considering the extension of the proof format in Rainbow it was minimal. The format for the matrix interpretation proofs was already developed in [19] and it essentially requires to provide matrix interpretations for all the function symbols in the signature. The format for arctic interpretations is the same except that:

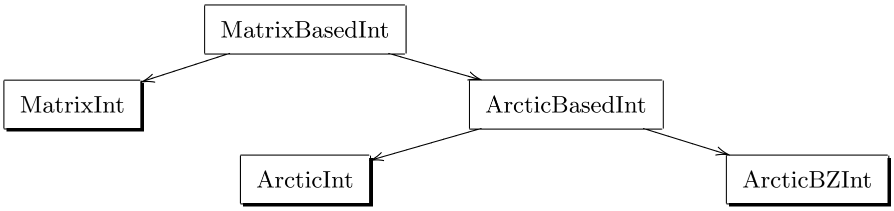

Fig. 1. Hierarchy of different matrix-based methods in CoLoR.

- it indicates which matrix-based method is to be used, indicated by different XML tags (as the common proof format of CoLoR is specified using XML syntax),
- the entries of vectors and matrices are from a different domain.

The experimental data concerning certification results is presented in the following section.

9 Implementation

The implementation in Matchbox follows the scheme described in [7]. The constraint problem for the arctic interpretation is translated to a constraint problem for matrices, for arctic numbers and, finally, for Boolean variables. This is then solved by Minisat [5].

An arctic number is represented by a pair $a = (b; v_0, v_1, \ldots, v_n)$ where b is a Boolean value and $v_0, \ldots, v_n$ is sequence of Booleans (all numbers have fixed bit-width). If b is 1, then a represents $-\infty$, if b is 0, then a represents the binary value of $v_0, \ldots, v_n$.

To represent integers, we use two's complement representation, i.e., the most significant bit is the "sign bit".

Note that implementation of max/plus operation is less expensive than standard plus/times: with a binary representation both max and plus can be computed (encoded) with a linear size formula (whereas a naive implementation of the standard multiplication requires quadratic size and asymptotically better schemes do not pay off for small bit widths).

It is useful to require the following, for each arctic number $a = (b, v)$: if the infinity bit b is set, then $v = 0$. Then $(b, v) \oplus (b', v') = (b \wedge b', \max(v, v'))$. For $(b, v) \otimes (b', v')$ we compute $c = b \vee b'$, $u = (u_0, \ldots, u_n) = v + v'$ and the result is $(c; \neg c \wedge u_0, \ldots, \neg c \wedge u_n)$.

To represent arctic integers, we use a similar convention: if the infinity flag b is set, we require that the number v represents the lowest value of its range.

The following table lists the numbers of certified proofs that we obtain with DP transformation (without SCC decomposition, see below) and these matrix methods: (s)tandard, (a)rctic, below (z)ero. For comparison, we give the corresponding numbers for last year's winner of the (certified, where applicable) termination competition.

problem set	time	s	sa	sz	saz	2007 winner
975 TRS	1 min	361	376	388	389	TPA: 354
	10 min	365	381	393	394	
517 SRS	1 min	178	312	298	320	Matchbox: 337
	10 min	185	349	323	354	

Runs were executed on a single core of an Intel X5365 processor running at 3GHz. All proofs will be made available for inspection at the Matchbox web page [21]. In all cases we used standard matrices of dimension 1 and 2 to remove rules before the DP transformation, and then matrix dimensions d from 1 up; with numbers of bit width $\max(1, 4 - \lfloor d/2 \rfloor)$, and a timeout of $5 + 2^d$ seconds for each individual attempt.

It should be noted that TPA 2007 additionally used (non-linear) polynomial interpretations, and that Matchbox 2007 also used additional methods (e.g. RFC match-bounds) and was running uncertified.

Here, we count only verified proofs, so we are missing about 3 to 5 proofs where Coq does not finish in reasonable time. (This happened—for exactly the same problems—also in 2007.)

To certify termination of string rewriting, we use the standard transformation to a term rewriting system with all symbols unary. We do this for the original system $\mathcal{R}$ as well as for the system $\text{reverse}(\mathcal{R}) = \{\text{reverse}(l) \to \text{reverse}(r) \mid (l \to r) \in \mathcal{R}$. It is obvious (though presently not included in CoLoR) that this transformation preserves termination both ways. Half of the allotted time is spent for each of $\mathcal{R}$ and $\text{reverse}(\mathcal{R})$. This increases the score considerably (by about one third).

The dependency pairs transformation is often combined with a decomposition of the resulting top termination problem into independent subproblems; analyzing strongly connected components of the estimated dependency graph [10]. Currently, CoLoR provides only a simple graph approximation by top symbols of dependency pairs, but at the moment it is not efficient. Our current implementation therefore does not do decomposition. However, with only this simple graph approximation, this does not decrease power: note that an interpretation that removes rules from a maximal component in the DP graph (with no incoming arrows) can be extended to the complete graph by assigning constant zero to all top symbols not occurring in this component.

10 Discussion

Arctic naturals form a sub-semi-ring of arctic integers. So the question comes up whether Theorem 14 subsumes Theorem 12. Note that the prerequisites for both theorems are incomparable. Still there might be a method to construct from a somewhere-finite interpretation (above zero) an equivalent absolute-positive interpretation (below zero). We are not aware of any. Experience with implementation shows that it is useful to have both methods, especially for string rewriting. Naturals are easier to handle than integers because they do not require signed arithmetics. So typically we can increase the bit width or the matrix dimension for naturals. Our implementation finds several proofs according to Theorem 12 where it fails to find a proof according to Theorem 14 and vice versa.

It is interesting to ask whether the preconditions of Theorems 10,12,14 can be weakened. We discussed one variant in Remark 16. In general, a linear interpretation $[\cdot]$ with coefficients in $\mathbb{A}_\mathbb{N}$ ($\mathbb{A}_\mathbb{Z}$ respectively) is admissible for a termination proof if for each ground term t, the value $[t]$ is finite (positive, respectively). This is in fact a reachability problem for weighted (tree) automata. It is decidable for interpretations on arctic naturals, but it is undecidable for arctic integers (follows from a result of Krob [20] on tropical word automata). In our setting, we do not guess an interpretation and then decide whether it is admissible. Rather, we have to formulate the decision algorithm as part of the constraint system for the interpretation. Therefore we chose sharper conditions on interpretations that imply finiteness (positiveness, respectively) and have an easy constraint encoding.

Another question is the relation of the standard matrix method with the arctic matrix method(s). Performance of our implementation suggests that neither method subsumes the other, but this may well be a problem of computing resources, as we hardly reach matrix dimension 5 and bit width 3.

As for the relation to other termination methods (e.g. path orderings), the only information we have is that arctic (and other) matrix methods can do non-simple termination, while path orders and polynomial interpretations cannot; and on the other hand, the arctic matrix method implies a linear bound on derivational complexity (see below), which is easily surpassed by path orders and other interpretations.

The full arctic termination method bounds lengths of derivations:

Lemma 17. *For a rewriting system $\mathcal{R}$ that fulfils the requirements of Theorem 10 for $\mathcal{S} = \emptyset$, the derivational complexity of $\mathcal{R}$ is linear.*

Proof. For a finite arctic vector $x = (x_1, \ldots, x_k)$, define $|x| = \max(x_1, \ldots, x_k)$.

Then $|x \oplus y| \leq \max(|x|, |y|)$ and $|x \otimes y^T| \leq |x| + |y|$.

For a finite arctic matrix A of dimension $k \times k$, define $|A| = \max\{A_{i,j} \mid 1 \leq i, j \leq k\}$. Then $|A \otimes x| \leq |A| + |x|$ and $|A \otimes B| \leq |A| + |B|$.

For an interpretation $[\cdot]$ of some signature Σ, and any word $w \in \Sigma^*$, this implies that $|[w]| \leq c \cdot |w|$ where $c = \max\{|[f]| : f \in \Sigma\}$.

Now we remark that $u \to_\mathcal{R} v$ implies $[u] \gg [v]$, and $x \gg y$ implies $|x| > |y|$. Thus the derivational complexity of $\mathcal{R}$ is linear: any derivation starting from u has at most $c \cdot |u|$ steps. □

This means that rewriting systems with higher derivational complexity (e.g. quadratic: $\{ab \to ba\}$, or exponential $\{ab \to b^2a\}$) do not admit an arctic termination proof. Note that both these systems admit a standard matrix proof.

It seems very difficult to combine this argument with the dependency pairs transformation, as it can drastically alter (i.e., reduce) derivational complexity.

Example 18. The following rewriting system [16] has a derivational complexity that is not primitive recursive:

$$\{\mathrm{s}(x) + (y + z) \to x + (\mathrm{s}(\mathrm{s}(y)) + z), \mathrm{s}(x) + (y + (z + w)) \to x + (z + (y + w))\}$$

and still it has, after DP transformation, an easy termination proof by "counting symbols" [7]. Note however that arctic interpretations cannot count globally: to compute the interpretation $[f(t_1, t_2)]$, it is impossible to add values from subtrees $[t_1], [t_2]$, as we can only take the maximum of $[t_1], [t_2]$. Yet we find an arctic proof, as follows. The given system is in fact an encoding of a length-preserving string rewriting system on the infinite alphabet $\mathbb{N}$. Both rules keep the right spine of terms (corresponding to the length of the simulated string) intact, so we can remove dependency pairs that shrink it, using the interpretation $[+](x, y) = y \otimes 1$. We are left with two dependency pairs (that directly correspond to the original rules). They can be handled by $[+](x, y) = x$ and $[\mathrm{s}](x) = x \otimes 1$. So instead of numbers of symbols, we were just using path lengths. $\triangleleft$

Arctic interpretations subsume quasi-periodic interpretations [24]. This has been remarked in [22] for string rewriting and it easily extends to term rewriting.

Max/Plus polynomials have been used by Amadio [1] as quasi-interpretations (i.e. functions are weakly monotone), to bound the space complexity of derivations. Proving termination directly was not intended.

11 Conclusions

We presented the arctic interpretations method for proving termination of term rewriting. It is based on the matrix interpretation method [7] where the usual plus/times operations on $\mathbb{N}$ are generalized to an arbitrary semi-ring, in this case instantiated by the arctic semi-ring (max/plus algebra) on $\{-\infty\} \cup \mathbb{N}$.

We also generalized this to arctic integers. This generalization allowed us to solve 10 of Beerendonk/* examples that are difficult to prove terminating and thus far could only be solved by AProVE with the Bounded Increase [12] technique, dedicated to such class of problems coming from transformations from imperative programs.

Our presentation of the theory is accompanied by a formalization in the Coq proof assistant. By becoming part of the CoLoR project this formalization allows us to formally verify termination proofs involving the arctic matrix method. With this contribution CoLoR can now certify more than half of the systems that could be proven terminating in the 2007 competition in term rewriting and essentially all (and some more) systems in the string rewriting category.

We want to remark here that all performance data and all examples presented in this paper were collected from problems of TPDB 2007, and we did not "cook up" any special examples to show off the arctic method. The emphasis of these examples (in fact, of the whole paper) is not to provide termination proofs where none were known before, but rather to provide certified (and often conceptually simpler) termination proofs where only uncertified proofs were available up to now.

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