Semantic labeling for proving termination of term rewriting

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Termination of Rewriting

Example

$$\begin{aligned} \min(x, 0) &\to 0 & \max(x, 0) &\to x \\ \min(0, y) &\to 0 & \max(0, y) &\to y \\ \min(s(x), s(y)) &\to s(\min(x, y)) & \max(s(x), s(y)) &\to s(\max(x, y)) \\ \gcd(0, s(x)) &\to s(x) & x - 0 &\to x \\ \gcd(s(x), 0) &\to s(x) & s(x) - s(y) &\to x - y \end{aligned}$$

$$\gcd(s(x), s(y)) \to \gcd(\max(x,y) - \min(x,y), s(\min(x,y)))$$

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to$ gcd(max(x,y) - min(x,y), s(min(x,y)))

$\mathsf{gcd}(6, 4)$

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ \mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y))) \end{aligned}$$

$$\mathsf{gcd}(6, 4) \to \mathsf{gcd}(\mathsf{max}(5, 3) - \mathsf{min}(5, 3), \mathsf{s}(\mathsf{min}(5, 3)))$$

Example

$$\min(x, 0) \to 0 \qquad \max(x, 0) \to x$$
$$\min(0, y) \to 0 \qquad \max(0, y) \to y$$
$$\min(s(x), s(y)) \to s(\min(x, y)) \qquad \max(s(x), s(y)) \to s(\max(x, y))$$
$$\gcd(0, s(x)) \to s(x) \qquad x - 0 \to x$$
$$\gcd(s(x), 0) \to s(x) \qquad s(x) - s(y) \to x - y$$
$$\gcd(s(x), s(y)) \to \gcd(\max(x,y) - \min(x,y), s(\min(x,y)))$$

$$\gcd(6, 4) \to^{+} \gcd(s(\max(4, 2)) - \min(5, 3), s(\min(5, 3)))$$

Termination of Rewriting

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}$$

$$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{max}(3, 1))) - \mathsf{min}(5, 3), \mathsf{s}(\mathsf{min}(5, 3)))$$

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{max}(2, 0)))) - \mathsf{min}(5, 3), \mathsf{s}(\mathsf{min}(5, 3)))$$

Termination of Rewriting

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
$$\min(s(x),s(y)) \to s(\min(x,y)) \qquad \max(s(x),s(y)) \to s(\max(x,y))$$
$$\gcd(0,s(x)) \to s(x) \qquad x - 0 \to x$$
$$\gcd(s(x),0) \to s(x) \qquad s(x) - s(y) \to x - y$$
$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\gcd(6,4) \to^{+} \gcd(s(s(s(2))) - \min(5,3), s(\min(5,3)))$

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \gcd(\max(x,y) - \min(x,y), s(\min(x,y))) \end{aligned}$$

$\gcd(6,4) \to^{+} \gcd(5-3,4)$

Termination of Rewriting

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
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$$\gcd(0,s(x)) \to s(x) \qquad x - 0 \to x$$
$$\gcd(s(x),0) \to s(x) \qquad s(x) - s(y) \to x - y$$
$$\gcd(s(x),s(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\gcd(6,4) \to^{+} \gcd(2,4)$

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(2, 2)$

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$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$\mathsf{gcd}(6, 4) \to^{+} \mathsf{gcd}(0, 2)$

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \qquad \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \qquad \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \qquad \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & \qquad x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \qquad \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}
\end{aligned}
$$

$\mathsf{gcd}(6, 4) \to^{+} 2$

Definition

A TRS is terminating if it does not admit an infinite rewrite sequence.

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Termination methods

Knuth-Bendix order, polynomial interpretations, lexicographic path order, multiset order, multiset path order, recursive path order, semantic path order, recursive decomposition order, transformation order, elementary interpretations, well-founded monotone algebra, general path order, semantic labeling, type introduction, freezing, top-down labeling, dependency pair method, matchbounds, size-change principle, predictive labeling, uncurrying, matrix interpretations, quasi-periodic interpretations, bounded increase . . .

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Termination tools

AProVE, Cariboo, Cime, JamBox, MatchBox, MultumNonMulta, MuTerm, Teparla, Torpa, TPA, TTT, TTTbox, ...

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Outline

technische universiteit eindhoven

Outline

Semantic Labeling (SL)

Example

Is this TRS terminating?

$$
\begin{aligned}
\mathsf{f}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{f}(x, y)) \\
\mathsf{f}(x, \mathsf{c}) &\to \mathsf{c} \\
\mathsf{f}(\mathsf{c}, y) &\to \mathsf{c} \\
\mathsf{g}(x, \mathsf{c}) &\to x \\
\mathsf{g}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{g}(x, y)) \\
\mathsf{g}(\mathsf{c}, y) &\to y \\
\mathsf{h}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{h}(x, y) \\
\mathsf{h}(x, \mathsf{c}) &\to x \\
\mathsf{l}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{l}(\mathsf{h}(\mathsf{g}(x, y), \mathsf{f}(x, y)), \mathsf{s}(\mathsf{f}(x, y)))
\end{aligned}
$$

Semantic Labeling (SL)

Example

How about this one?

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}
$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y)))
\end{aligned}
$$

$$
0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y
$$

$$
-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0
$$

Semantic Labeling (SL)

Example

$$
\begin{array}{rcll}
\mathsf{min}(x,0) & \to & 0 & x \geq 0 \\
\mathsf{min}(0,y) & \to & 0 & 0 \geq 0 \\
\mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{min}(x,y)) & 2x+1 \geq 2x+1 \\
\mathsf{max}(x,0) & \to & x & x \geq x \\
\mathsf{max}(0,y) & \to & y & y \geq y \\
\mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{max}(x,y)) & 2x+2y+2 > 2x+2y+1 \\
\mathsf{s}(x) - \mathsf{s}(y) & \to & x - y & 2x+1 > x \\
x - 0 & \to & x & x \geq x \\
\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y))) & 0 \geq 0
\end{array}
$$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x+1 \quad \mathsf{min}_{\mathbb{N}}(x,y) = x \quad \mathsf{max}_{\mathbb{N}}(x,y) = x+y$$
$$-_{\mathbb{N}}(x,y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x,y) = 0$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\min(x, 0) &\to 0 \\
\min(0, y) &\to 0 \\
\min(s(x), s(y)) &\to s(\min(x, y)) \\
\max(x, 0) &\to x \\
\max(0, y) &\to y \\
\max(s(x), s(y)) &\to s(\max(x, y)) \\
s(x) - s(y) &\to x - y \\
x - 0 &\to x \qquad i, j \in \mathbb{N} \\
\mathrm{gcd}_{4i+2j+3}(s(x), s(y)) &\to \mathrm{gcd}_{4i+2j+1}(\max(x, y) - \min(x, y), s(\min(x, y))) \\
\mathrm{gcd}_i(x, y) &\to \mathrm{gcd}_j(x, y) \qquad i > j
\end{aligned}
$$

$$0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = 2x + 1 \quad \min_{\mathbb{N}}(x, y) = x \quad \max_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathrm{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathrm{gcd}}(x, y) = 2x + y$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\mathsf{min}(x, 0) &\to 0 \\
\mathsf{min}(0, y) &\to 0 \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) \\
\mathsf{max}(x, 0) &\to x \\
\mathsf{max}(0, y) &\to y \\
\mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\mathsf{max}(x, y) - \mathsf{min}(x, y), \mathsf{s}(\mathsf{min}(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}
$$

$\Rightarrow$ LPO applicable

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \mathsf{min}_{\mathbb{N}}(x, y) = x \quad \mathsf{max}_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

$$\ell_{\mathsf{gcd}}(x, y) = 2x + y$$

Semantic Labeling (SL)

Example

$$
\begin{aligned}
\min(x, 0) &\to 0 \\
\min(0, y) &\to 0 \\
\min(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\min(x, y)) \\
\max(x, 0) &\to x \\
\max(0, y) &\to y \\
\max(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\max(x, y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
x - 0 &\to x \\
\mathsf{gcd}_{4i+2j+3}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}_{4i+2j+1}(\max(x, y) - \min(x, y), \mathsf{s}(\min(x, y))) \\
\mathsf{gcd}_i(x, y) &\to \mathsf{gcd}_j(x, y)
\end{aligned}
$$

$\Rightarrow$ LPO applicable

$\ldots > \mathsf{gcd}_1 > \mathsf{gcd}_0 > \{\min, \max, -\} > \mathsf{s}$

$$0_{\mathbb{N}} = 0 \quad \mathsf{s}_{\mathbb{N}}(x) = 2x + 1 \quad \min_{\mathbb{N}}(x, y) = x \quad \max_{\mathbb{N}}(x, y) = x + y$$

$$-_{\mathbb{N}}(x, y) = x \quad \mathsf{gcd}_{\mathbb{N}}(x, y) = 0$$

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Semantic Labeling (SL)

Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$

Semantic Labeling (SL)

Definition (Semantic labeling)

- Semantics:
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}}, >_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$
- Labeling:
  - for every $f \in \mathcal{F}$ a set of labels $L_f \subseteq A$
  - labeling functions $\ell_f : A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$

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- Labeling of terms (for a variable assignment $\alpha : \mathcal{V} \to A$).

$$\mathrm{lab}_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathrm{lab}_\alpha(t_1), \dots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathrm{lab}_\alpha(t_1), \dots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_n))$

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  with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$
- Labeled TRS:
  - $\mathcal{R}_{\mathrm{lab}} = \{\mathrm{lab}_\alpha(l) \to \mathrm{lab}_\alpha(r) \mid l \to r \in \mathcal{R}, \alpha : \mathcal{V} \to A\}$
  - $\mathrm{Decr} = \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid f \in \mathcal{F}, L_f \neq \varnothing; a, b \in L_f, a > b\}$

Semantic Labeling (SL)

Theorem (Zantema, 1995)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a weakly-monotone $\mathcal{F}$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{R} \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Semantic Labeling (SL) - automation

Finite domain ($\{a, b\}$ or $\{a, b, c\}$):

Infinite domain ($\mathbb{N}$):

Semantic Labeling (SL) - automation

Finite domain ($\{a, b\}$ or $\{a, b, c\}$):

- Functions: (a subset of) all possible functions.

Infinite domain ($\mathbb{N}$):

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Finite domain ({a, b} or {a, b, c}):

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- Model search: exhaustive or random search.

Infinite domain ($\mathbb{N}$):

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- Functions: (a subset of) all possible functions.
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Infinite domain ($\mathbb{N}$):

- Functions: a set of "natural" functions ($\lambda x.x + 1,\ \lambda xy.x + y,\ \lambda xy.min(x, y) \ldots$).

Semantic Labeling (SL) - automation

Finite domain ({a, b} or {a, b, c}):

- Functions: (a subset of) all possible functions.
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- Tool support: AProVE, Jambox, TORPA, TPA, TEPARLA.

Infinite domain ($\mathbb{N}$):

- Functions: a set of "natural" functions ($\lambda x.x + 1,\ \lambda xy.x + y,\ \lambda xy.min(x, y) \ldots$).
- Search: exhaustive search.

Semantic Labeling (SL) - automation

Finite domain ({a, b} or {a, b, c}):

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Semantic Labeling (SL) - example

Explicit substitution in combinatory categorical logic:

Example

$$
\begin{aligned}
\lambda(x) \circ y &\to \lambda(x \circ (1 \cdot (y \circ \uparrow))) \\
(x \cdot y) \circ z &\to (x \circ z) \cdot (y \circ z) \\
(x \circ y) \circ z &\to x \circ (y \circ z) \\
\mathrm{id} \circ x &\to x \\
1 \circ \mathrm{id} &\to 1 \\
\uparrow \circ \ \mathrm{id} &\to \uparrow \\
1 \circ (x \cdot y) &\to x \\
\uparrow \circ (x \cdot y) &\to y
\end{aligned}
$$

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- Two publications devoted to proving termination of this system (Hardin and Lavills [1986], Curien et al. [1992]).

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\end{aligned}
$$

- Two publications devoted to proving termination of this system (Hardin and Lavills [1986], Curien et al. [1992]).
- Elementary termination proof with semantic labeling.

Outline

Predictive Labeling (PL)

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
$$\min(s(x),s(y)) \to s(\min(x,y)) \qquad \max(s(x),s(y)) \to s(\max(x,y))$$
$$\color{red}{gcd(0,s(x)) \to s(x)} \qquad x - 0 \to x$$
$$\color{red}{gcd(s(x),0) \to s(x)} \qquad s(x) - s(y) \to x - y$$
$$gcd(s(x),s(y)) \to gcd(max(x,y) - min(x,y), s(min(x,y)))$$

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \gcd(\max(x,y) - \min(x,y), s(\min(x,y))) \end{aligned}$$

$$0_{\mathbb{N}} = 0 \quad s_{\mathbb{N}}(x) = 2x+1 \quad \min_{\mathbb{N}}(x,y) = x \quad \max_{\mathbb{N}}(x,y) = x+y$$
$$-_{\mathbb{N}}(x,y) = x \quad \gcd_{\mathbb{N}}(x,y) = 0$$

Problem: how to obtain a quasi-model?

Predictive Labeling (PL)

Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))} \end{aligned}$$

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Problem: how to obtain a quasi-model?

$\Rightarrow$ use predictive labeling (quasi-model constraints only for usable rules)

Predictive Labeling (PL)

Example

$$\begin{aligned}
\min(x,0) &\to 0 & \max(x,0) &\to x\\
\min(0,y) &\to 0 & \max(0,y) &\to y\\
\min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y))\\
\textcolor{red}{\mathrm{gcd}}(0,s(x)) &\to s(x) & x-0 &\to x\\
\textcolor{red}{\mathrm{gcd}}(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y\\
\textcolor{red}{\mathrm{gcd}}(s(x),s(y)) &\to \textcolor{red}{\mathrm{gcd}}(\max(x,y) - \min(x,y),\ s(\min(x,y)))
\end{aligned}$$

Computation of usable rules:

1. Look at symbols that will get labels.

Predictive Labeling (PL)

Example

$$\min(x,0) \to 0 \qquad \max(x,0) \to x$$
$$\min(0,y) \to 0 \qquad \max(0,y) \to y$$
$$\min(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\min(x,y)) \qquad \max(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{s}(\max(x,y))$$
$$\gcd(0,\mathsf{s}(x)) \to \mathsf{s}(x) \qquad x - 0 \to x$$
$$\gcd(\mathsf{s}(x),0) \to \mathsf{s}(x) \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$\gcd(\mathsf{s}(x),\mathsf{s}(y)) \to \gcd(\max(x,y) - \min(x,y), \mathsf{s}(\min(x,y)))$$

Computation of usable rules:

1. Look at symbols that will get labels.
2. Look at their subterms.

Predictive Labeling (PL)

Example

$$\min(x, 0) \to 0 \qquad \max(x, 0) \to x$$

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$$\min(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\min(x, y)) \qquad \max(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{s}(\max(x, y))$$

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3. All function symbols occurring there are usable.

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Example

$$\begin{aligned} \min(x,0) &\to 0 & \max(x,0) &\to x \\ \min(0,y) &\to 0 & \max(0,y) &\to y \\ \min(s(x),s(y)) &\to s(\min(x,y)) & \max(s(x),s(y)) &\to s(\max(x,y)) \\ \gcd(0,s(x)) &\to s(x) & x-0 &\to x \\ \gcd(s(x),0) &\to s(x) & s(x)-s(y) &\to x-y \\ \gcd(s(x),s(y)) &\to \gcd(\max(x,y) - \min(x,y), s(\min(x,y))) \end{aligned}$$

Computation of usable rules:

3. All function symbols occurring there are usable.
4. Also the symbols that depend on them.

Predictive Labeling (PL)

Example

$$
\begin{array}{rclrcl}
\min(x,0) & \to & 0 & \max(x,0) & \to & x \\
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\min(s(x),s(y)) & \to & s(\min(x,y)) & \max(s(x),s(y)) & \to & s(\max(x,y)) \\
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Computation of usable rules:

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Predictive Labeling (PL)

Example

$$\mathsf{min}(x, 0) \to 0 \qquad \mathsf{max}(x, 0) \to x$$
$$\mathsf{min}(0, y) \to 0 \qquad \mathsf{max}(0, y) \to y$$
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1. Look at symbols that will get labels.
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Predictive Labeling (PL)

Definition (Usable rules)

- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \unlhd r$

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- $f \rhd_{\mathrm{d}} g \equiv \exists f(\ldots) \to r \in \mathcal{R}, g(\ldots) \trianglelefteq r$
- $F^* \equiv \{g \mid f \rhd_{\mathrm{d}}^* g$ for some $f \in \mathcal{F}\}$.

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- $\mathcal{G}_\ell(t) = \begin{cases} \varnothing & \text{if } t \text{ is a variable,} \\ \mathcal{F}(t_1)^* \cup \cdots \cup \mathcal{F}(t_n)^* & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing, \\ \mathcal{G}_\ell(t_1) \cup \cdots \cup \mathcal{G}_\ell(t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \end{cases}$

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- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$

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- $\mathcal{G}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{G}_\ell(l) \cup \mathcal{G}_\ell(r)$
- $\mathcal{U}(\mathcal{R}, \ell) = \{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{G}_\ell(\mathcal{R})\}$

Predictive Labeling (PL)

Theorem (Hirokawa, Middeldorp, 2006)

A TRS $\mathcal{R}$ is terminating if there exists:

- *a weakly-monotone $\sqcup$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $\mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}$ *is terminating*

Predictive Labeling (SL) - automation

- Larger search space: which symbols to label?

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- Larger search space: which symbols to label?
- Functions: (linear) polynomials with bounded coefficients.
- Search: encoding to SAT.
- min & max: problematic, not supported at the moment.
- Tool support: TPA

Outline

Dependency Pairs (DP)

Example

$$\begin{aligned} \mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\ \mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\ \mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\ \mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

$$\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

Dependency Pairs (DP)

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$\Rightarrow$ apply dependency pairs

Dependency Pairs (DP)

Example

$$\begin{aligned}
\mathsf{min}(x, 0) &\to 0 & \mathsf{max}(x, 0) &\to x \\
\mathsf{min}(0, y) &\to 0 & \mathsf{max}(0, y) &\to y \\
\mathsf{min}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x, y)) & \mathsf{max}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x, y)) \\
\mathsf{gcd}(0, \mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\
\mathsf{gcd}(\mathsf{s}(x), 0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(\mathsf{max}(x,y) - \mathsf{min}(x,y), \mathsf{s}(\mathsf{min}(x,y)))
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Dependency Pairs (DP)

Example

$$\begin{aligned} \mathsf{min}(x,0) &\to 0 & \mathsf{max}(x,0) &\to x \\ \mathsf{min}(0,y) &\to 0 & \mathsf{max}(0,y) &\to y \\ \mathsf{min}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{min}(x,y)) & \mathsf{max}(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{s}(\mathsf{max}(x,y)) \\ \mathsf{gcd}(0,\mathsf{s}(x)) &\to \mathsf{s}(x) & x - 0 &\to x \\ \mathsf{gcd}(\mathsf{s}(x),0) &\to \mathsf{s}(x) & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

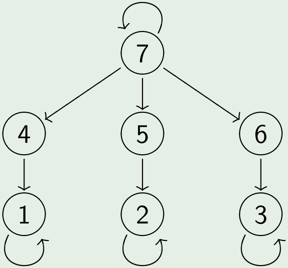
$$\mathsf{gcd}(\mathsf{s}(x),\mathsf{s}(y)) \to \text{gcd(max(x,y) - min(x,y), s(min(x,y)))}$$

$$\begin{aligned} &(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}^\sharp(x,y) & &(5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{min}^\sharp(x,y) \\ &(2) & \mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{min}^\sharp(x,y) & &(6) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{max}^\sharp(x,y) \\ &(3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{max}^\sharp(x,y) \\ &(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) \\ &(7) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) &\to \mathsf{gcd}^\sharp(\mathsf{max}(x,y) - \mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) \end{aligned}$$

Dependency Pairs (DP)

Example

$$
\begin{array}{llll}
(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) \\
(2) & \mathsf{min}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) & (6) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) \\
(3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) & & \\
(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) & & \\
(7) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) & &
\end{array}
$$



Dependency Pairs (DP)

Example

$$
\begin{array}{llll}
(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) \\
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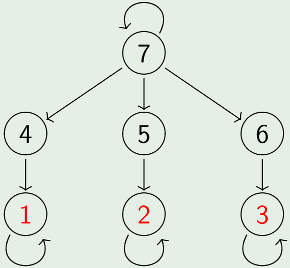
\end{array}
$$



Dependency Pairs (DP)

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(1) & \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(x,y) & (5) \quad \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{min}^\sharp(x,y) \\
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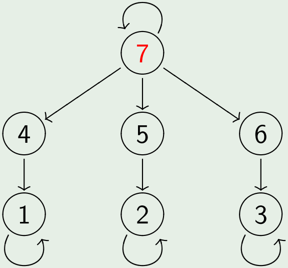
(3) & \mathsf{max}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{max}^\sharp(x,y) & \\
(4) & \mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{minus}^\sharp(\mathsf{max}(x,y),\mathsf{min}(x,y)) & \\
\color{red}(7) & \color{red}\mathsf{gcd}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{gcd}^\sharp(\mathsf{max}(x,y)-\mathsf{min}(x,y),\mathsf{s}(\mathsf{min}(x,y))) &
\end{array}
$$



Proving termination with PL

(1) $\mathsf{SN}(\mathcal{R})$

DP transformation $\downarrow$

(2) $\mathsf{SN}(\mathsf{DP}(\mathcal{R}), \mathcal{R})$

DP graph analysis $\downarrow$

(3) $\mathsf{SN}(\mathcal{C}, \mathcal{R})$

Predictive labeling $\downarrow$

(4) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

Reduction pair processor modulo π $\downarrow$

(5) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}} \setminus >^{\pi}_{\mathsf{LPO}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

Unlabeling $\downarrow$

(6) $\mathsf{SN}(\mathcal{C}', \mathcal{R})$

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\quad\big\downarrow \text{Unlabeling} \\
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\end{array}$$

Problem: $\mathcal{C}_{\mathsf{lab}}$ and $\mathcal{R}_{\mathsf{lab}}$ are infinite!

Proving termination with PL

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$\downarrow$ Reduction pair processor modulo π

Correct if $\mathcal{R}_{\mathsf{lab}}$ is finitely branching

(5) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}} \setminus >^{\pi}_{\mathsf{LPO}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

$\downarrow$ Unlabeling

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Proving termination with PL

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↓ DP transformation

(2) $\mathsf{SN}(\mathsf{DP}(\mathcal{R}), \mathcal{R})$

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(3) $\mathsf{SN}(\mathcal{C}, \mathcal{R})$

↓ Predictive labeling — Problem: is this correct?

(4) $\mathsf{SN}(\mathcal{C}_{\mathsf{lab}}, \mathcal{R}_{\mathsf{lab}} \cup \mathsf{Decr}_{\mathcal{F}^\sharp})$

↓ Reduction pair processor modulo π

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PL in the DP setting

Theorem

A DP problem $(\mathcal{P}, \mathcal{R})$ is finite if there exists:

- *a weakly-monotone $\sqcup$-algebra and*
- *a weakly-monotone labeling ℓ*

such that:

- $\mathcal{P} \subseteq \mathrm{DP}(\mathcal{R})$
- *$\mathcal{R}$ is finitely branching,*
- $\mathcal{U}(\mathcal{R}, \ell) \subseteq \gtrsim_{\mathcal{A}}$ *and*
- $(\mathcal{P}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{Decr})$ *is finite.*

Outline

Experimental results

Table: Experiments with TPA on 864 TRSs from the TPDB version 3.2.

	technique	60 seconds timeout: yes	60 seconds timeout: time	60 seconds timeout: timeout	10 minutes timeout: yes	10 minutes timeout: time	10 minutes timeout: timeout
1×1	SL	440	1178	2	440	1351	0
	PL	456	1193	2	456	1316	0
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	PL	456	1193	2	456	1316	0
	PL′	426	752	1	426	893	0
2×2	SL	503	6905	51	506	24577	30
	PL	527	6906	53	532	25582	32
	PL′	522	5211	33	524	11328	8

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Conclusions

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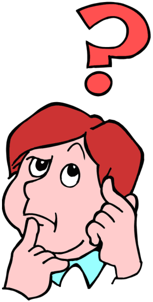
Improvements and innermost termination:

R. Thiemann, A .Middeldorp.
Innermost Termination of Rewrite Systems by Labeling.
WRS 2007.

The end



Thank you for your attention.