# Topics in Termination of Term Rewriting

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Prose (Process Seminar)

# Outline

1. Project overview
   - Proving termination
   - Automation of termination proving
   - Certification of termination proving
   - Application of termination proving
   - Project roadmap
2. Recursive path ordering for infinite labelled systems
   - Semantic labelling
   - Recursive path ordering (RPO)
   - RPO for infinite systems
   - Automation of RPO for infinite systems

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# Introduction to term rewriting

Let's define plus in Peano arithmetic.

**Example (Plus)**

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\begin{aligned}
0 + y &= y \\
s(x) + y &= s(x + y)
\end{aligned}
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**Example (Computing with plus)**

Now let us do some some maths... how about $2 + 2$?

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A TRS is terminating iff it does not admin infinite reductions.

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# More examples

## Example (Minus)

$$\begin{aligned} x - 0 &\rightarrow x \\ s(x) - s(y) &\rightarrow x - y \end{aligned}$$

## Example (Min)

$$\begin{aligned} min(x, 0) &\rightarrow 0 \\ min(0, x) &\rightarrow 0 \\ min(s(x), s(y)) &\rightarrow s(min(x, y)) \end{aligned}$$

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# Relative termination

**Example**

$$\begin{array}{rcl} T(I(x), y) & \rightarrow & T(x, y) \\ T(x, y) & \rightarrow & T(x, I(y)) \end{array}$$

This system is not terminating.

**Example**

$$\begin{array}{rcl} T(I(x), y) & \rightarrow & T(x, y) \\ T(x, y) & \xrightarrow{=} & T(x, I(y)) \end{array}$$

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# How about commutativity and associativity?

## Example (Plus with commutativity rule)

$$\begin{array}{rcl} 0 + y & \rightarrow & y \\ s(x) + y & \rightarrow & s(x + y) \\ x + y & \rightarrow & y + x \end{array}$$

This system is not terminating.

This is addressed by rewriting modulo AC, which is a special case of rewriting modulo equations, which is generalized by relative termination.

## Example (Plus with commutativity rule cont.)

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# Automation of termination proving

- Termination is undecidable.
- But there are many techniques for proving termination.
- Recently focus in the area is on automation of termination proving process.
- An annual competition of termination tools is being organized.
- There is a number of tools from different authors: AProVE, TTT, TORPA, TPA, Jambox, C*i*ME, Matchbox, TEPARLA, Cariboo, Termptation.

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# TPA: a tool for proving termination

## TPA

TPA: Termination Proved Automatically

`http://www.win.tue.nl/tpa`

Why developing yet another tool? What makes TPA different?

- Support for relative termination.
- Usage of semantic labelling with natural numbers.
- Aiming at certified proofs.

TPA facts:

- Written in Ocaml (around 10,000 lines of code).
- 3rd place in the last edition of the termination competition (Nara 2005).

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It is not uncommon that termination tools contain bugs (just as any other piece of software does).

**CoLoR**

CoLoR: a Coq library on rewriting and termination

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- Objective: formalization of theory of term rewriting in the theorem prover Coq.
- Ultimate objective: certification of termination proofs produced by tools for proving termination of rewriting.

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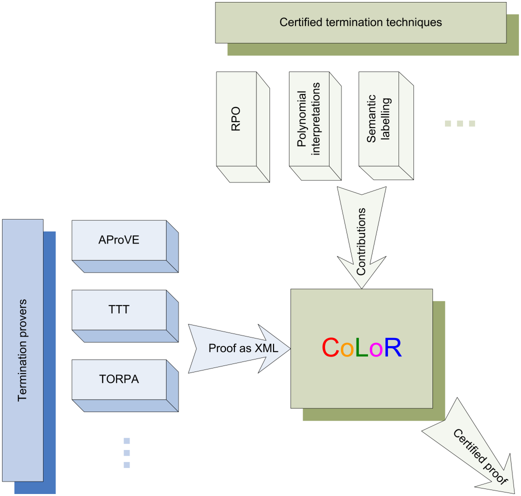
Certified termination techniques
RPO
Polynomial interpretations
Semantic labelling
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Certified proof

# CoLoR contributions

To date contributions to CoLoR project:

- Polynomial interpretations [Hinderer]
- Dependancy pairs [Blanqui]
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- There is also a tool support for termination.
- Idea: let's use that for verification.
- Concentrate on liveness properties.
- Concentrate on termination in rewriting.
- A restricted set of liveness properties.

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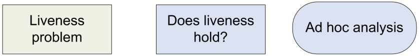
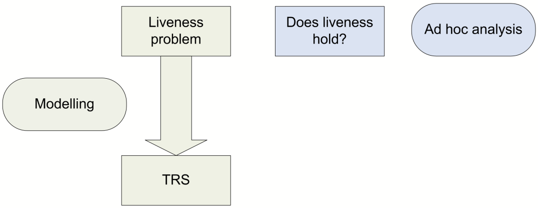
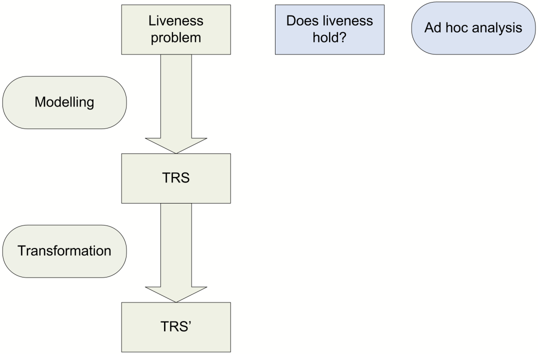
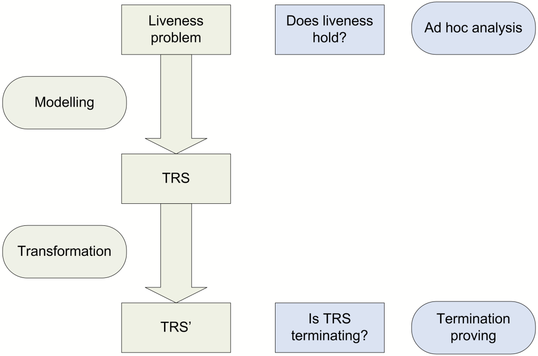

Liveness problem
Does liveness hold?
Ad hoc analysis


Liveness problem
Does liveness hold?
Ad hoc analysis
Modelling
TRS


Liveness problem
Does liveness hold?
Ad hoc analysis
Modelling
TRS
Transformation
TRS'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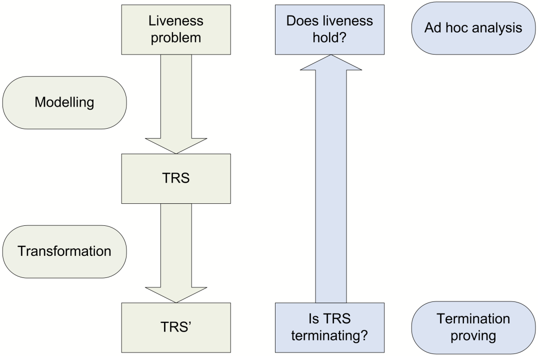
Termination proving


Liveness problem
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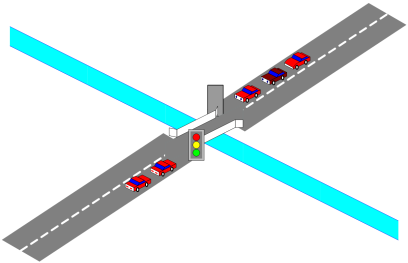
# Liveness example: cars over a bridge

Liveness: no car will wait forever.



Remarks:

- We need an assumption about the lights: Fairness
- State space is infinite!

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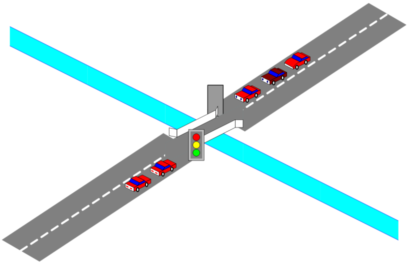

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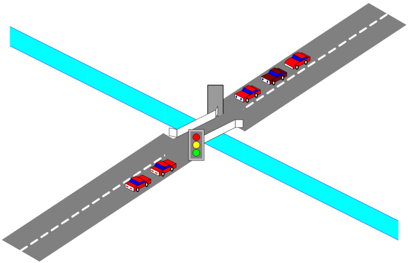

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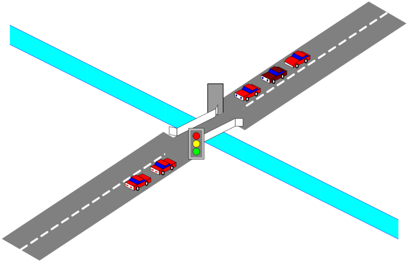

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# Cars over a bridge: model (1/2)

**System configuration**

top
left/right
left lane    right lane

**Possible system configuration**

top
left
car    bot
new    car
old    bot

**Lane configuration**

car
old/new    car
old/new
bot

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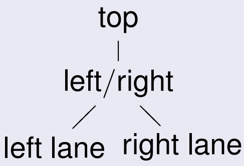
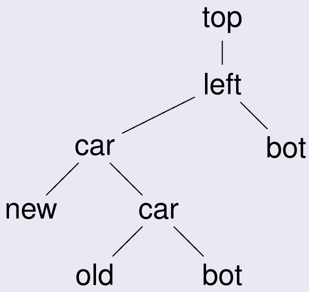
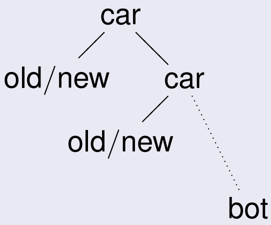



## Possible system configuration




## Lane configuration



# Cars over a bridge: model (2/2)

## Example

$$
\begin{array}{lrcl}
(1) & \mathrm{top}(\mathrm{left}(\mathrm{car}(x), y)) & \to & \mathrm{top}(\mathrm{right}(x, y)) \\
(2) & \mathrm{top}(\mathrm{right}(x, \mathrm{car}(y))) & \to & \mathrm{top}(\mathrm{left}(x, y)) \\
(3) & \mathrm{top}(\mathrm{left}(\mathrm{bot}, y)) & \to & \mathrm{top}(\mathrm{right}(\mathrm{bot}, y)) \\
(4) & \mathrm{top}(\mathrm{right}(x, \mathrm{bot})) & \to & \mathrm{top}(\mathrm{left}(x, \mathrm{bot})) \\
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# Liveness transformation

J. Giesl and H. Zantema in 2003 proposed two transformations:

**Transformation L**

- Sound and complete.
- Complicated.

**Transformation LS**

- Only sound.
- Significantly simpler.

A. Koprowski and H. Zantema in 2005 presented another transformation:

**Transformation LT**

- Sound.
- Complete (with some additional assumptions).
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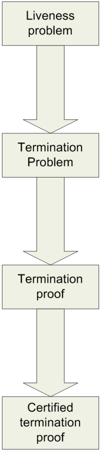
# Outline

# Project roadmap



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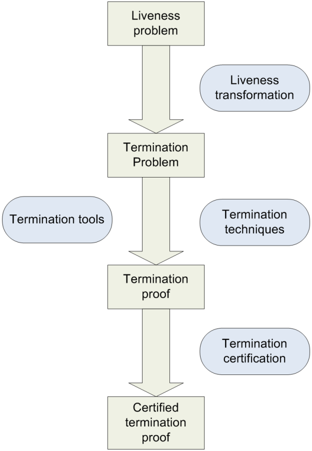

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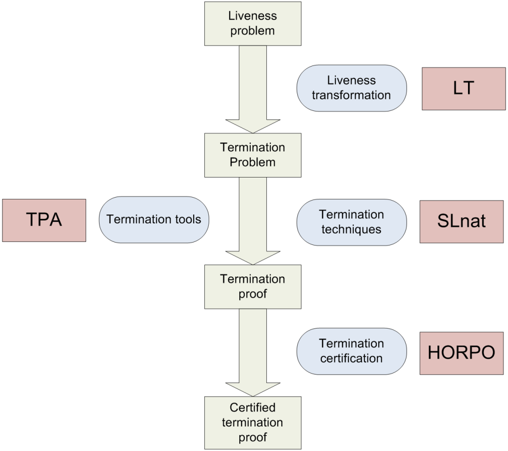

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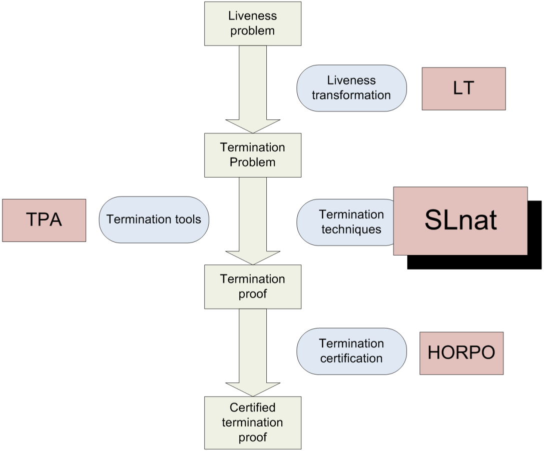

# Outline

1. Project overview

2. Recursive path ordering for infinite labelled systems
   - Semantic labelling
   - Recursive path ordering (RPO)
   - RPO for infinite systems
   - Automation of RPO for infinite systems

# Semantic labelling - introduction

Is the following system terminating?

## Example

$$
\begin{array}{lrcl}
(1) & \mathsf{f}(x, \mathsf{c}) & \to & \mathsf{c} \\
(2) & \mathsf{f}(\mathsf{c}, y) & \to & \mathsf{c} \\
(3) & \mathsf{f}(\mathsf{p}(x), \mathsf{p}(y)) & \to & \mathsf{p}(\mathsf{f}(x, y)) \\
(5) & \mathsf{g}(x, \mathsf{c}) & \to & x \\
(4) & \mathsf{g}(\mathsf{c}, y) & \to & y \\
(6) & \mathsf{g}(\mathsf{p}(x), \mathsf{p}(y)) & \to & \mathsf{p}(\mathsf{g}(x, y)) \\
(7) & \mathsf{w}(x, \mathsf{c}) & \to & x \\
(8) & \mathsf{w}(\mathsf{p}(x), \mathsf{p}(y) & \to & \mathsf{w}(x, y) \\
(9) & \mathsf{h}(\mathsf{p}(x), \mathsf{p}(y)) & \to & \mathsf{h}(\mathsf{w}(\mathsf{p}(\mathsf{g}(x, y)), \mathsf{p}(\mathsf{f}(x, y))), \mathsf{p}(\mathsf{f}(x, y)))
\end{array}
$$

# Semantic labelling - introduction

And how about this one?

## Example (GCD)

$$
\begin{array}{lrcl}
(1) & \mathrm{min}(x, 0) & \to & 0 \\
(2) & \mathrm{min}(0, y) & \to & 0 \\
(3) & \mathrm{min}(\mathrm{s}(x), \mathrm{s}(y)) & \to & \mathrm{s}(\mathrm{min}(x, y)) \\
(4) & \mathrm{max}(x, 0) & \to & x \\
(5) & \mathrm{max}(0, y) & \to & y \\
(6) & \mathrm{max}(\mathrm{s}(x), \mathrm{s}(y)) & \to & \mathrm{s}(\mathrm{max}(x, y)) \\
(7) & x - 0 & \to & x \\
(8) & \mathrm{s}(x) - \mathrm{s}(y) & \to & x - y \\
(9) & \mathrm{gcd}(\mathrm{s}(x), \mathrm{s}(y)) & \to & \mathrm{gcd}(\mathrm{s}(\mathrm{max}(x, y)) - \mathrm{s}(\mathrm{min}(x, y)), \\
 & & & \mathrm{s}(\mathrm{min}(x, y)))
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$$

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(4) & \max(x, 0) & \to & x \\
(5) & \max(0, y) & \to & y \\
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# GCD: interpretation

**Example (GCD)**

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\end{array}
$$

Natural interpretation for function symbols:

**Example (Interpretation for GCD)**

$$
\begin{array}{rcl}
[0] & = & 0 \\
[\mathsf{s}(x)] & = & x + 1 \\
[x - y] & = & x - y
\end{array}
\qquad
\begin{array}{rcl}
[\mathsf{min}(x, y)] & = & \mathsf{min}(x, y) \\
[\mathsf{max}(x, y)] & = & \mathsf{max}(x, y) \\
[\mathsf{gcd}(x, y)] & = & \mathsf{gcd}(x, y)
\end{array}
$$

# GCD: labelled system

## Example (Labelled GCD)

$$
\begin{array}{rcll}
\mathrm{min}_{i,0}(x,0) & \to & 0 & \\
\mathrm{min}_{0,j}(0,y) & \to & 0 & \\
\mathrm{min}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_j(\mathrm{min}_{i,j}(x,y)) & \text{if } i >= j \\
\mathrm{min}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_i(\mathrm{min}_{i,j}(x,y)) & \text{if } i < j \\
\mathrm{max}_{i,0}(x,0) & \to & x & \\
\mathrm{max}_{0,j}(0,y) & \to & y & \\
\mathrm{max}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_i(\mathrm{max}_{i,j}(x,y)) & \text{if } i >= j \\
\mathrm{max}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_j(\mathrm{max}_{i,j}(x,y)) & \text{if } i < j \\
x -_{i,0} 0 & \to & x & \\
\mathrm{s}_i(x) -_{i+1,j+1} \mathrm{s}_j(y)) & \to & x -_{i,j} y & \\
\mathrm{gcd}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{gcd}_{i-j,j+1}(\mathrm{s}_i(\mathrm{max}_{i,j}(x,y)) -_{i+1,j+1} & \text{if } i >= j \\
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- Fix a poset $(A, \geq)$.
- Fix weakly monotonic interpretations for all function symbols: $[f] : A^n \to A$
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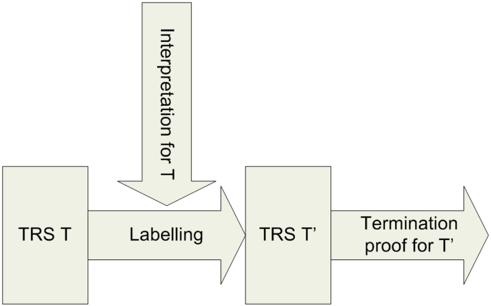


**Theorem**

*Given interpretation that is a model R is terminating iff $\overline{R}$ is terminating.*

# Semantic labelling with natural numbers - challenges

So far semantic labelling was used with $A = \{0, 1\}$.
What if we want to use it with $A = \mathbb{N}$?

- Labelled system is infinite (infinitely many rules, infinite signature)
- We need to be able to represent and manipulate it.
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**Example (SUBST)**

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\begin{array}{clcl}
(1) & \lambda(x) \circ y & \rightarrow & \lambda(x \circ (1 \cdot (y \circ \uparrow))) \\
(2) & (x \cdot y) \circ z & \rightarrow & (x \circ z) \cdot (y \circ z) \\
(3) & (x \circ y) \circ z & \rightarrow & x \circ (y \circ z) \\
(4) & \mathrm{id} \circ x & \rightarrow & x \\
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(7) & 1 \circ (x \cdot y) & \rightarrow & x \\
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- This system describes the process of substitution in combinatory categorical logic.
- Showing its termination is difficult...
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1. Project overview
2. Recursive path ordering for infinite labelled systems
   - Semantic labelling
   - Recursive path ordering (RPO)
   - RPO for infinite systems
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# RPO - definition

## Definition

Given well-founded ordering on function symbols $\succ$, called a precedence and a status function $\tau$ we define RPO as:

$$s = f(s_1, \ldots, s_n) \succ_{RPO} t \iff$$

(1) $\exists_{1 \leq i \leq n} \,.\, s_i \succeq_{RPO} t$, or

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- Orient every rule of a given TRS with RPO (choices).
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We need to restrict to precedences such that:

- they can be captured in finite description,
- are "powerful",
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**Definition (Precedence description)**

A *precedence description* consists of:

- for every $f \in \Sigma$ of arity $n$, $\phi_f : \mathbb{N}^n \to \mathbb{N}$ and
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This gives rise to a precedence $\succ$: $f_{k_1,\ldots,k_n} \succ g_{l_1,\ldots,l_m}$ iff:

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$$\begin{aligned} &\mathrm{pd}(f,g) = \top \lor \\ &(\mathrm{pd}(f,g) = \geq \ \land \phi_f(k_1,\ldots,k_n) \geq \phi_g(l_1,\ldots,l_m)) \lor \\ &(\mathrm{pd}(f,g) = > \ \land \phi_f(k_1,\ldots,k_n) > \phi_g(l_1,\ldots,l_m)). \end{aligned}$$

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We need to restrict to precedences such that:

- they can be captured in finite description,
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**Definition (Precedence description)**

A precedence description consists of:

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## Example

Let us consider $\phi$ functions to be:

- Always identity (id) for unary symbols,
- For binary symbols one of the following: summation ($+$), left projection ($\pi_1$) or right projection ($\pi_2$).

Then to have $f_{i+1,j+1} \succ g_{i+2,j}$ one of the following precedence descriptions is possible:

- $\mathrm{pd}(f, g) = \top$,
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- $\phi(f) = +,\ \phi(g) = \pi_2,\ \mathrm{pd}(f, g) = > / \geq$,
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## Definition (Precedence graph)

pd gives rise to a precedence graph having $\Sigma$ as its node set, and having connections between all $f, g \in \Sigma$, $\text{pd}(f, g) \neq \bot$:

| $\text{pd}(f, g)$ | Edge type | Notation |
|---|---|---|
| $\top$ | Unconditional | $\Longrightarrow$ |
| $\geq$ | Non-strict | $\dashrightarrow$ |
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## Theorem

*A precedence description* pd *gives rise to a well-founded precedence* $\succ$ *if every cycle in the corresponding precedence graph:*

(1) *contains no unconditional edge, and*
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# Outline

1. Project overview
2. Recursive path ordering for infinite labelled systems
   - Semantic labelling
   - Recursive path ordering (RPO)
   - RPO for infinite systems
   - Automation of RPO for infinite systems

# Implementing RPO for infinite systems

- Approach: Fix some finite domains for $\phi$ functions.
- Problem: we do not know in advance what $\phi$ function are appropriate for given symbols.
- Solution: let us localize this choice and make it part of the search procedure.

**Definition (Precedence description scheme)**

Precedence description schema is a function:

$$\text{pds}(f, g) \subseteq \{\bot\} \cup (\{>, \geq\} \times \mathbb{N}^{\mathbb{N}^n} \times \mathbb{N}^{\mathbb{N}^m}) \cup \{\top\}$$

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## Definition (Compatibility with precedence description)

A precedence description $(\{\phi_f\}_{f \in \Sigma}, \mathrm{pd})$ is compatible with a precedence description schema pds if:

$$\forall f, g \in \Sigma \,.\, \begin{cases} \mathrm{pd}(f, g) = \top \implies \top \in \mathrm{pds}(f, g) \\ \mathrm{pd}(f, g) = \geq \implies (\geq, \phi_f, \phi_g) \in \mathrm{pds}(f, g) \\ \mathrm{pd}(f, g) = > \implies (>, \phi_f, \phi_g) \in \mathrm{pds}(f, g) \end{cases}$$

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## Theorem

*Suppose at least three different $\phi$ functions are allowed. Then given a precedence description scheme* pds*, the problem of finding a precedence description compatible with it is NP-complete.*

## Definition (Algorithm)

(1) Compute SCCs in the precedence graph. Refine pds.

$$\mathrm{pds}'(f, g) := \begin{cases} \{\top\} & \text{if } SCC(f) \neq SCC(g) \\ \{s \in \mathrm{pds}(f, g) \mid s \neq \top\} & \text{otherwise} \end{cases}$$

(2) If for any pair $f$ and $g$, $\mathrm{pds}(f, g) = \emptyset$ answer NO and stop.

(3) For every function symbol $f$ compute $\mathrm{IN}_f$, $\mathrm{OUT}_f$:

(4) For every SCC:

- (4a) Compute possible label synthesis functions:
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- (4e) Consider all possible precedences compatible with pds and check whether they comply with condition (2). If none does, answer NO and stop. Otherwise continue with step (4) with the next SCC or with step (5) if there are no more SCCs.

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$$\begin{aligned} \text{IN}_f &= \{g \mid \text{pds}(g, f) \neq \{\bot\},\ SCC(f) = SCC(g)\} \\ \text{OUT}_f &= \{g \mid \text{pds}(f, g) \neq \{\bot\},\ SCC(f) = SCC(g)\} \end{aligned}$$

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$$\mathrm{PSF}_f = \bigcap_{g \in \mathrm{OUT}_f} \{\phi_f \mid (\geq / >, \phi_f, \phi_g) \in \mathrm{pds}(f, g)\} \cap \bigcap_{g \in \mathrm{IN}_f} \{\phi_f \mid (\geq / >, \phi_g, \phi_f) \in \mathrm{pds}(g, f)\}$$

(4b) Refine pds:

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- (4d) If $\mathrm{pds}' \neq \mathrm{pds}$ set $\mathrm{pds} := \mathrm{pds}'$ and go to (4a), otherwise continue with (4e).
- (4e) Consider all possible precedences compatible with pds and check whether they comply with condition (2). If none does, answer NO and stop. Otherwise continue with step (4) with the next SCC or with step (5) if there are no more SCCs.

(5) Combine solutions for all SCCs to get a precedence description compatible with pds giving rise to a well-founded precedence.

## Definition (Algorithm)

(1) Compute SCCs in the precedence graph. Refine pds.

(2) If for any pair $f$ and $g$, $\text{pds}(f, g) = \emptyset$ answer NO and stop.

(3) For every function symbol $f$ compute $\text{IN}_f$, $\text{OUT}_f$:

(4) For every SCC:

- (4a) Compute possible label synthesis functions:
- (4b) Refine pds:
- (4c) If for any $f$ and $g$, $\text{pds}'(f, g) = \emptyset$ answer NO and stop.
- (4d) If $\text{pds}' \neq \text{pds}$ set $\text{pds} := \text{pds}'$ and go to (4a), otherwise continue with (4e).
- (4e) Consider all possible precedences compatible with pds and check whether they comply with condition (2). If none does, answer NO and stop. Otherwise continue with step (4) with the next SCC or with step (5) if there are no more SCCs.

(5) Combine solutions for all SCCs to get a precedence description compatible with pds giving rise to a well-founded precedence.

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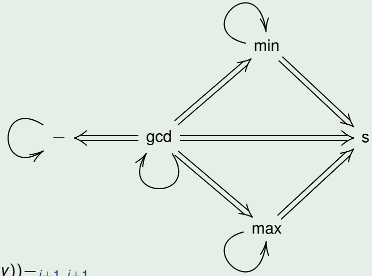
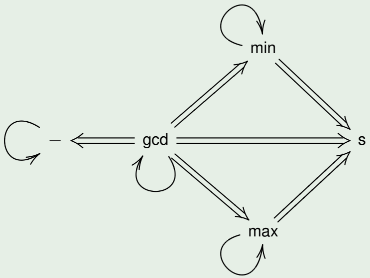
## Example (Precedence description for GCD)

$$
\begin{array}{rcl}
\mathsf{min}_{i,0}(x,0) & \to & 0 \\
\mathsf{min}_{0,j}(0,y) & \to & 0 \\
\mathsf{min}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{s}_j(\mathsf{min}_{i,j}(x,y)) \\
\mathsf{min}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{s}_i(\mathsf{min}_{i,j}(x,y)) \\
\mathsf{max}_{i,0}(x,0) & \to & x \\
\mathsf{max}_{0,j}(0,y) & \to & y \\
\mathsf{max}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{s}_i(\mathsf{max}_{i,j}(x,y)) \\
\mathsf{max}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{s}_j(\mathsf{max}_{i,j}(x,y)) \\
x -_{i,0} 0 & \to & x \\
\mathsf{s}_i(x) -_{i+1,j+1} \mathsf{s}_j(y)) & \to & x -_{i,j} y \\
\mathsf{gcd}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{gcd}_{i-j,j+1}(\mathsf{s}_i(\mathsf{max}_{i,j}(x,y)) -_{i+1,j+1} \\
 & & \mathsf{s}_j(\mathsf{min}_{i,j}(x,y)), \mathsf{min}_{i,j}(x,y))) \\
\mathsf{gcd}_{i+1,j+1}(\mathsf{s}_i(x),\mathsf{s}_j(y)) & \to & \mathsf{gcd}_{j-i,i+1}(\mathsf{s}_j(\mathsf{max}_{i,j}(x,y)) -_{j+1,i+1} \\
 & & \mathsf{s}_i(\mathsf{min}_{i,j}(x,y)), \mathsf{min}_{i,j}(x,y)))
\end{array}
$$




$$
\begin{aligned}
\phi_{min} &= \pi_1 \\
\phi_{max} &= \pi_1 \\
\phi_{s} &= \mathrm{id} \\
\phi_{-} &= \mathrm{id} \\
\phi_{gcd} &= +
\end{aligned}
$$

## Example

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\begin{array}{rcll}
\mathrm{min}_{i,j} & > & \mathrm{min}_{k,l} & \text{if } i > k \\
\mathrm{gcd}_{i,j} & > & \mathrm{gcd}_{k,l} & \text{if } i + j > k + l \\
\mathrm{min}_{i,j} & > & \mathrm{s}_{k} & \forall i, j, k \\
\mathrm{gcd}_{i,j} & > & \mathrm{min}_{k,l} & \forall i, j, k, l \\
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-_{i,j} & > & -_{k,l} & \text{if } i > k \\
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\end{array}
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## Example (Precedence for GCD)

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\begin{array}{rcl}
\mathrm{min}_{i,0}(x,0) & \to & 0 \\
\mathrm{min}_{0,j}(0,y) & \to & 0 \\
\mathrm{min}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_j(\mathrm{min}_{i,j}(x,y)) \\
\mathrm{min}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_i(\mathrm{min}_{i,j}(x,y)) \\
\mathrm{max}_{i,0}(x,0) & \to & x \\
\mathrm{max}_{0,j}(0,y) & \to & y \\
\mathrm{max}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_i(\mathrm{max}_{i,j}(x,y)) \\
\mathrm{max}_{i+1,j+1}(\mathrm{s}_i(x),\mathrm{s}_j(y)) & \to & \mathrm{s}_j(\mathrm{max}_{i,j}(x,y)) \\
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# Conclusions

Conclusions:

- Semantic labelling with natural numbers is an interesting and powerful technique for proving termination.
- It can be implemented (and has been implemented in TPA).
- For some systems (SUBST, GCD) it is the only successful technique at the moment.

Further research:

- Choice of interpretation functions.
- Choice of label synthesis functions.
- Approach for case analysis,
- Comparison of polynomials in presence of side conditions.
- Extension of other techniques to infinite signatures (KBO?).

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Thank you for your attention.