Dimensionality Reduction and (Bucket) Ranking: A Mass Transportation Approach

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ALT 2019, Chicago

Outline

1. Introduction
2. Dimensionality Reduction on $\mathfrak{S}_n$
3. Theoretical results
4. Numerical Experiments on Real-world Preference Data

Outline

Introduction - Ranking Data

Consider a set of items $[\![n]\!] := \{1, \ldots, n\}$.

A ranking is an **ordered list** (of any size) **of items** of $[\![n]\!]$

 $\rightarrow$  $\prec$  $\prec$  $\prec$ 

Example: *travel* $\prec$ *sports* $\prec$ *finance* $\prec$ *clothing*

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Many applications involve rankings/comparisons:

- Modelling human preferences (elections, surveys, online implicit feedback)
- Computer systems (search engines, recommendation systems)
- Other (competitions, biological data...)

Ranking data - Permutations

A full ranking can be seen as the permutation σ that maps an item to its rank:

$$a_1 \prec a_2 \prec \cdots \prec a_n \qquad \Leftrightarrow \qquad \sigma \in \mathfrak{S}_n \text{ such that } \sigma(a_i) = i$$

$$2 \prec 1 \prec 3 \prec 4 \qquad \Leftrightarrow \qquad \sigma = 2134 \ \ (\sigma(2) = 1, \sigma(1) = 2, \dots)$$

Let $\mathfrak{S}_n$ be set of permutations of $[\![n]\!]$, the symmetric group.
Ex: $\mathfrak{S}_4 = 1234, 1324, 1423, \dots, 4321$

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How to summarize P ?

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Our proposal

Summarize P on $\mathfrak{S}_n$ by:

- a bucket ordering (**a partial order**) $\mathcal{C}$
- a **sparse** ranking distribution $P_{\mathcal{C}}$

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Background on Consensus Ranking

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Find the dirac distribution closest to P:

$$\delta_{\sigma^*} = \min_{\sigma \in \mathfrak{S}_n} W_{d,q}\left(P, \delta_\sigma\right)$$

where $W_{d,q}\left(P, P'\right) = \inf_{\Sigma \sim P,\ \Sigma' \sim P'} \mathbb{E}\left[d^q(\Sigma, \Sigma')\right]$ is the Wassertein distance.

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$\Longrightarrow W_{d,q}\left(P, \delta_\sigma\right) = \mathbb{E}_{\Sigma \sim P}[d(\Sigma, \sigma)]$.
$\Longrightarrow$ **ranking aggregation/consensus ranking** as a radical dimensionality reduction procedure

We choose the Kendall's τ distance:

- $d_\tau(\sigma, \sigma') = \sum_{i<j} \mathbb{I}\{(\sigma(i) - \sigma(j))(\sigma'(i) - \sigma'(j)) < 0\}$

Kemeny medians are solutions of:

$$\sigma^*_P = \min_{\sigma \in \mathfrak{S}_n} \sum_{1 \leq i < j \leq n} p_{i,j} \mathbb{I}\{\sigma(i) > \sigma(j)\} + (1 - p_{i,j}) \mathbb{I}\{\sigma(i) < \sigma(j)\} \quad (1)$$

where $p_{i,j} = \mathbb{P}[\Sigma(i) < \Sigma(j)]$ when $\Sigma \sim P$ (prob. that item i is preferred to j).

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[Korba et al., 2017] $\Rightarrow$ (1) is given by Copeland ranking

$$\sigma_P^*(i) = 1 + \sum_{j \neq i} \mathbb{I}\{p_{i,j} < 1/2\}.$$

The rank of item i in σ_P^* is its number of pairwise defeats against other items

if P **strictly stochastically transitive**:

- $p_{i,j} \neq 1/2$ for all $i < j$
- $p_{i,j} \geq 1/2$ and $p_{j,k} \geq 1/2 \ \Rightarrow \ p_{i,k} \geq 1/2$

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- $\mathcal{C}_k$'s disjoint non empty subsets of $[\![n]\!]$
- $\cup_{k=1}^{K} \mathcal{C}_k = [\![n]\!]$

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Find the distribution $P_{\mathcal{C}}$ closest to P:

$$\Lambda_P(\mathcal{C}) = \min_{P' \in \mathbf{P}_{\mathcal{C}}} W_{d_\tau,1}(P, P')$$

where $\mathbf{P}_{\mathcal{C}}$ set of distributions associated to $\mathcal{C}$.

Sparsity and Bucket orders

Sparse distributions

$\mathbf{P}_{\mathcal{C}}$: set of all bucket distributions P' associated to $\mathcal{C}$

- P' distribution on $\mathfrak{S}_n$
- if $i \prec_{\mathcal{C}} j$ (i.e. $\exists k < l$, s.t. $(i,j) \in (\mathcal{C}_k, \mathcal{C}_l)$), then $p'_{j,i} = \mathbb{P}_{\Sigma' \sim P'}\{\Sigma'(j) < \Sigma'(i)\} = 0$

i.e. the order of two items in two $\neq$ buckets is deterministic

$\Longrightarrow P' \in \mathbf{P}_{\mathcal{C}}$ described by $d_{\mathcal{C}} = \prod_{1 \leq k \leq K} \#\mathcal{C}_k! - 1 \leq n! - 1$ parameters

Dimensionality reduction with optimal coupling

Proposition (Optimal Coupling)

$$\Lambda_P(\mathcal{C}) = \min_{P' \in \mathbf{P}_{\mathcal{C}}} W_{d_\tau,1}(P, P') = W_{d_\tau,1}(P, P_{\mathcal{C}}) = \sum_{i \prec_{\mathcal{C}} j} p_{j,i}$$

optimal when $P' = P_{\mathcal{C}}$ the distribution of $\Sigma_{\mathcal{C}}$:

$\forall k \in \{1, \ldots, K\}, \forall i \in \mathcal{C}_k, \;\; \Sigma_{\mathcal{C}}(i) = 1 + \sum_{l<k} \#\mathcal{C}_l + \sum_{j \in \mathcal{C}_k} \mathbb{I}\{\Sigma(j) < \Sigma(i)\},$

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Dimensionality Reduction

Let $K \leq n$ and $\mathbf{C}_{K,\lambda}$ the set of all bucket orders of size K and shape λ. A natural dimensionality reduction approach consists in finding a solution $C^{*(K)}$ of:

$$\min_{\mathcal{C} \in \mathbf{C}_{K,\lambda}} \Lambda_P(\mathcal{C})$$

as well as a solution $P_{C^{*(K)}}$ of $\Lambda_P(C^{*(K)})$ and a coupling $(\Sigma, \Sigma_{C^{*(K)}})$ s.t. $\mathbb{E}\left[d_\tau(\Sigma, \Sigma_{C^{*(K)}})\right]$.

Outline

Optimality

Assume that P is strongly (and strictly*) stochastically transitive i.e.:

$$p_{i,j} \geq 1/2 \text{ and } p_{j,k} \geq 1/2 \;\Rightarrow\; p_{i,k} \geq max(p_{i,j}, p_{j,k}).$$

*: $p_{i,j} \neq 1/2$.

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Theorem

(i) $\Lambda_P(\cdot)$ *has a unique minimizer* $\mathcal{C}^{*(K,\lambda)}$ *over* $\mathbf{C}_{K,\lambda}$.

(ii) $\mathcal{C}^{*(K,\lambda)}$ *is the unique bucket order in* $\mathbf{C}_{K,\lambda}$ *agreeing with the Kemeny median* σ^*_P*:* $\mathcal{C}^{*(K,\lambda)} = (\mathcal{C}_1^{*(K,\lambda)}, \ldots, \mathcal{C}_K^{*(K,\lambda)})$*, where*

$$\mathcal{C}_k^{*(K,\lambda)} = \left\{ i \in [\![n]\!] : \sum_{l<k} \lambda_l < \sigma^*_P(i) \leq \sum_{l \leq k} \lambda_l \right\} \text{ for } k \in \{1, \ldots, K\}.$$

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$\Longrightarrow$ this result will lead to our practical method

Empirical setting

How to recover optimal buckets from a training sample $\Sigma_1, \dots, \Sigma_N \sim P$?

- Empirical pairwise probabilities:

$$\widehat{p}_{i,j} = \frac{1}{N} \sum_{s=1}^{N} \mathbb{I}\{\Sigma_s(i) < \Sigma_s(j)\}.$$

- Empirical distortion of any bucket order $\mathcal{C}$:

$$\widehat{\Lambda}_N(\mathcal{C}) = \Lambda_{\widehat{P}_N}(\mathcal{C}) = \sum_{1 \le k < l \le K} \sum_{(i,j) \in \mathcal{C}_k \times \mathcal{C}_l} \widehat{p}_{j,i}.$$

Rate bound

Empirical distortion minimizer $\widehat{C}_{K,\lambda}$ is solution of:

$$\min_{\mathcal{C} \in \mathbf{C}_{K,\lambda}} \widehat{\Lambda}_N(\mathcal{C}),$$

where $\mathbf{C}_{K,\lambda}$ set of bucket orders $\mathcal{C}$ of size K and shape $\lambda = (\lambda_1, \ldots, \lambda_K)$ (i.e. $\#\mathcal{C}_k = \lambda_k$ for all $1 \leq k \leq K$).

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For all $\delta \in (0,1)$, we have with probability at least $1-\delta$:

$$\Lambda_P(\widehat{C}_{K,\lambda}) - \inf_{\mathcal{C} \in \mathbf{C}_{K,\lambda}} \Lambda_P(\mathcal{C}) \leq \beta(n,\lambda) \times \sqrt{\frac{\log(\frac{1}{\delta})}{N}}.$$

Outline

Experiments

Sushi dataset (Kamishima, 2003):

- $n = 10$ sushi dishes
- $N = 5000$ full rankings.

Cars dataset

- $n = 10$ cars
- $N = 2500$ pairwise comparisons.

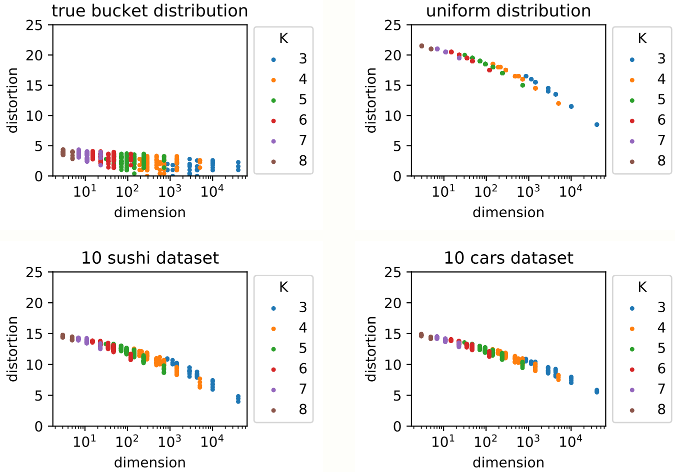
Method

1. Compute empirical pairwise probabilities $\widehat{p}_{i,j}$
2. Compute $\sigma_{\widehat{P}_N}$ with Copeland method

$$\sigma^*_{\widehat{P}_N}(i) = 1 + \sum_{j \neq i} \mathbb{I}\{\widehat{p}_{i,j} < 1/2\}.$$

3. Choose a size K, shape λ and segment $\sigma_{\widehat{P}_N}$ according to λ

Dimension-Distortion plot - $n = 10$ items



On top: true bucket distribution and uniform distribution.
Below: real preference data.

Conclusion

This paper introduces:

- theoretical concepts to represent in a sparse manner ranking distributions (bucket distributions)
- a distortion measure based on a mass transportation metric (Wassertein), to evaluate the accuracy of these representations

Future work: investigate how to exploit such representations in some tasks (e.g. clustering, ranking prediction)

Thank you!

Korba, A., Clémençon, S., and Sibony, E. (2017).
A learning theory of ranking aggregation.
In *Artificial Intelligence and Statistics*, pages 1001–1010.