MULTI-CONTROLLED NOT GATE DECOMPOSITION

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This document explains the contents of the accompanying Jupyter notebook, MCX.IPYNB. We construct a multi-controlled NOT gate following the work of da Silva et al. in [1]. We do not reproduce the proof of their results here, those are readily available in the published paper, only the construction.

1. The multi-controlled NOT gate

Let $\mathrm{MCX}^{[n]}$ be a quantum gate on $n+1$ qubits, that is controlled by the first n qubits and if all of those are in state $|1\rangle$, then it acts as a NOT, or X, gate on the last. In other words, its matrix in the computational basis (still referred to as MCX^{n}) has the form

$$\mathrm{MCX}^{[n]}_{ij} = \begin{cases} 1, & \text{if } i = j \,\&\, i < 2^{n+1}-2, \text{or} \big(i,j\big) \in \big\{\big(2^{n+1}-1, 2^{n+1}-2\big), \big(2^{n+1}-2, 2^{n+1}-1\big)\big\}, \\ 0, & \text{otherwise.} \end{cases}$$

for example, when $n = 2$, we get

$$\mathrm{MCX}^{[2]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 1.1. *Note that* QISKIT *uses "endian" labeling for qubits, thus one gets the above matrix if the target qubit is the* first *one, not the last. Hence, in my code, I do that same get the matrix above. The target qubit can be easily changed from the first to the* k^{th} *by replacing the line*

```
QControl, QTarget = QRegs[1:], QRegs[0]
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to

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QControl, QTarget = QRegs[:k - 1] + QRegs[k:], QRegs[k - 1]
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where $1 \leqslant k \leqslant n+1$.

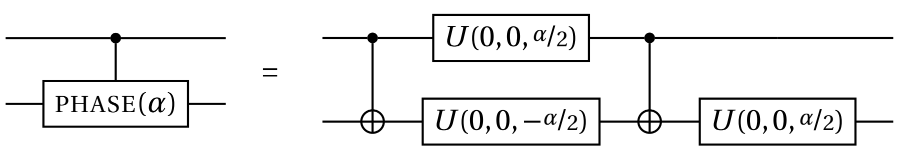
Date: October 15, 2023.

In order to simplify our construction, we note that the Hadamard gates, H, satisfies $H^2 = \mathbb{1}_2$, $H = U\left(\frac{\pi}{2}, 0, \pi\right)$, and $X = HZH$, all of which is easily verifiable, thus it is enough to construct a multi-controlled Z gate, call $\mathrm{MCZ}^{[n]}$, (for which any qubit can be regarded as the target) and then $\mathrm{MCX}^{[n]} = U^{(t)}\left(\frac{\pi}{2}, 0, \pi\right)\mathrm{MCZ}^{[n]}U^{(t)}\left(\frac{\pi}{2}, 0, \pi\right)$, where the superscript $^{(t)}$ means that the gate is applied to the target qubit. Hence, we outline the construction of $\mathrm{MCZ}^{[n]}$ below only.

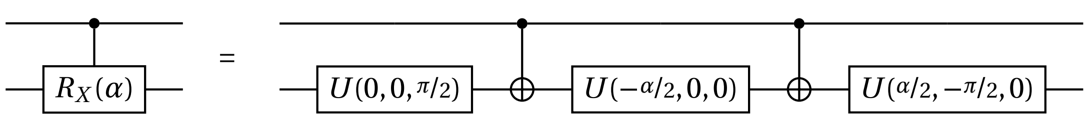
Using [1, Equation 3] or [1, Figure 1], with $U = Z$ (in their notation), we can break down the construction of $\mathrm{MCZ}^{[n]}$ to the following steps:

1. For each $i \in \{1, \ldots, n-1\}$, in the usual order, add a $\sqrt[i+1]{Z}$ to the target qubit, controlled by the $(n+1-i)^{\text{th}}$ control qubit and then for each $j \in \{n-i, \ldots, n-1\}$, in the usual order, add an $R_X\left(\frac{\pi}{2^i}\right)$ gate to the j^{th} control qubit, controlled by the $(n+1-i)^{\text{th}}$ control qubit.
2. apply (1)for $i = n$ also.
3. Apply the inverse of (1).
4. Apply (1) again, but without the controlled $\sqrt[i+1]{Z}$ gates.
5. Apply (4) for $i = n$ also.
6. Apply the inverse of (4).

Finally, note that we use controlled gates that are not, yet, decomposed into U and CNOT gates, but that can always be done; see [2, Chapter 4.3]. We only need two types, controlled R_X and controlled phase gates, which can be given (for any $\alpha \in [0, 2\pi)$ and $x, y \in \{0, 1\}$) as follows:



and



For a concrete implementation of the code (and some further comments), please see the accompanying code.

References

[1] Adenilton J. Da Silva and Daniel K. Park, *Linear-depth quantum circuits for multiqubit controlled gates*, Physical Review A **106** (2022), no. 4, 042602. ↑1, 2

[2] Michael A Nielsen and Isaac L Chuang, *Quantum computation and quantum information*, Phys. Today **54** (2001), no. 2, 60. ↑2

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