THE KELLER–SEGEL EQUATION COMPACT SURFACES

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Abstract. We study the (parabolic-elliptic) Keller–Segel equations on compact surfaces.

[rest of to be completed...]

1. Introduction

This is the part that we will write last.

Organization of the paper. To be completed...

Acknowledgment. *To be completed...*

2. Reformulation of the equation

Let Σ be a smooth, closed surface with a Riemannian metric g and area form ω. Let G be the Green operator of the Laplacian, Δ, on $L^2(\Sigma, g)$ and let us fix $\rho_0 \in L^2(\Sigma, g)$. Note that $L^2 \hookrightarrow L^1$ on domains of finite measure. We say that a positive function, $\rho \in C^1\big((0,T); L^2(\Sigma, g)\big)$ is said to satisfy the *parabolic–elliptic Keller–Segel equations* on (Σ, g) with initial value ρ_0 if it is a solution to the following system:

$$\partial_t \rho = -\Delta \rho + \mathrm{d}^*(\rho \, \mathrm{d}G(\rho)), \tag{2.1a}$$

$$\lim_{t \to 0^+} \rho_t = \rho_0. \tag{2.1b}$$

where for all $t \in (0, T)$

$$\rho_t := \rho|_{\{t\} \times \Sigma},$$

regarded as a function in $L^2(\Sigma, g)$.

Now let

$$m(t) := \int_\Sigma \rho_t \, \mathrm{d}A.$$

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Then m is constant, because

$$\dot{m}(t) = \int_\Sigma (-\Delta\rho + \mathrm{d}^*(\rho\, \mathrm{d}G(\rho)))\, \mathrm{d}A = \int_\Sigma \mathrm{d}^*(-\mathrm{d}\rho + \rho\, \mathrm{d}G(\rho))\, \mathrm{d}A = 0.$$

Thus we drop the t-dependence of m from its notation.

For the rest of the paper, let $A_\Sigma := \mathrm{Area}(\Sigma, g)$. The following lemma recasts equations (2.1a) and (2.1b) in a simpler form.

Lemma 2.1. *Let $\chi_0 := \rho_0 - \frac{m}{A_\Sigma}$ and $\chi := \rho - \frac{m}{A_\Sigma}$. Then equations* (2.1a) *and* (2.1b) *is equivalent to*

$$\partial_t \chi = \left(\tfrac{m}{A_\Sigma} - \Delta\right)\chi + \mathrm{d}^*(\chi\, \mathrm{d}G(\chi)), \tag{2.2a}$$

$$\chi|_{\{0\}\times\Sigma} = \chi_0. \tag{2.2b}$$

Proof. The equivalency of equation (2.1b) and equation (2.2b) is obvious.

Note that G annihilates constants and since χ is orthogonal to constants, we have that $\Delta(G(\chi)) = \chi$. Since $\chi = \rho - \frac{m}{A_\Sigma}$, we get, using equation (2.1a), that

$$\begin{aligned}
\partial_t \chi &= \partial_t\left(\rho - \tfrac{m}{A_\Sigma}\right) \\
&= \partial_t \rho - 0 \\
&= -\Delta\rho + \mathrm{d}^*(\rho\, \mathrm{d}G(\rho)) \\
&= -\Delta\left(\tfrac{m}{A_\Sigma} + \chi\right) + \mathrm{d}^*\left(\left(\tfrac{m}{A_\Sigma} + \chi\right)\mathrm{d}G\left(\tfrac{m}{A_\Sigma} + \chi\right)\right) \\
&= -\Delta\chi + \mathrm{d}^*\left(\left(\tfrac{m}{A_\Sigma} + \chi\right)\mathrm{d}G(\chi)\right) \\
&= -\Delta\chi + \tfrac{m}{A_\Sigma}\mathrm{d}^*\, \mathrm{d}G(\chi) + \mathrm{d}^*(\chi\, \mathrm{d}G(\chi)) \\
&= \left(\tfrac{m}{A_\Sigma} - \Delta\right)\chi + \mathrm{d}^*(\chi\, \mathrm{d}G(\chi)),
\end{aligned}$$

which completes the proof. □

Remark 2.2. *Let λ_1 be the smallest nonzero eigenvalue of Δ and note that quantity $M_\Sigma = \lambda_1 A_\Sigma$ only depends on the geometry of (Σ, g). When $m < M_\Sigma$, then the linear term in equation* (2.2a) *is strictly negative definite.*

3. The generalized Fourier transform

Let now $\left(\Psi_a \in L^2(\Sigma, g)\right)_{a\in\mathbb{N}}$ be an orthonormal eigenbasis of Δ and

$$\Delta\Psi_a = \lambda_a \Psi_a.$$

Let us order this basis so that

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \lambda_a \leqslant \lambda_{a+1} \leqslant \dots$$

In particular, $\Psi_0 = \frac{1}{\sqrt{A_\Sigma}}$.

Assume that χ is a solution to equations (2.2a) and (2.2b) and write

$$R_a(t) := \left\langle \Psi_a \middle| \chi|_{\{t\}\times\Sigma} \right\rangle_{L^2(\Sigma,g)}.$$

Then for each $a \in \mathbb{N}$, we have that $R_a \in C^1((0,T);\mathbb{R})$. Note that $R_0 \equiv \frac{m}{A_\Sigma}$. Finally let

$$\forall a,b,c \in \mathbb{N}: \quad \varphi_{abc} := \int_\Sigma \Psi_a \Psi_b \Psi_c \, \mathrm{d}A.$$

Note that φ_{abc} is a completely symmetric 3-tensor.

Theorem 3.1. *The function ρ is a solution to equation* (2.1a) *exactly when*

$$\forall t \in (0,T): \quad (R_a(t))_{a\in\mathbb{N}} \in l^2(\mathbb{N}), \tag{3.1a}$$

$$\forall a \in \mathbb{N}_+: \quad \dot{R}_a = \left(\tfrac{m}{A_\Sigma} - \lambda_a\right) R_a + \sum_{b,c\in\mathbb{N}} \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c} \varphi_{abc} R_b R_c. \tag{3.1b}$$

Furthermore, if equation (2.1b) *is also satisfied, then for all $a \in \mathbb{N}$, $\lim_{t\to 0^+} R_a(t)$ exists and*

$$\rho_0 = \sum_{a\in\mathbb{N}} \left(\lim_{t\to 0^+} R_a(t) \right) \Psi_a. \tag{3.2}$$

Proof. Since $(\Psi_a)_{a\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Sigma,g)$ and for all $t \in (0,T)$, ρ_t is in $L^2(\Sigma,g)$, we get condition (3.1a).

Fix $a \in \mathbb{N}_+$. Then

$$G(\Psi_a) = \lambda_a^{-1} \Psi_a.$$

Using the above equation, the self-adjointness of Δ, and equation (2.2a), we get

$$\begin{aligned}
\dot{R}_a &= \langle \Psi_a | \partial_t \chi \rangle_{L^2(\Sigma,g)} \\
&= \left\langle \Psi_a \middle| \left(\tfrac{m}{A_\Sigma} - \Delta\right)\chi + \mathrm{d}^*(\chi \, \mathrm{d}G(\chi)) \right\rangle_{L^2(\Sigma,g)} \\
&= \left\langle \left(\tfrac{m}{A_\Sigma} - \Delta\right)\Psi_a \middle| \chi \right\rangle_{L^2(\Sigma,g)} + \langle \Psi_a | \mathrm{d}^*(\chi \, \mathrm{d}G(\chi)) \rangle_{L^2(\Sigma,g)} \\
&= \left(\tfrac{m}{A_\Sigma} - \lambda_a\right) \langle \Psi_a | \chi \rangle_{L^2(\Sigma,g)} + \sum_{b,c\in\mathbb{N}_+} \langle \Psi_a | \mathrm{d}^*(\Psi_b \, \mathrm{d}G(\Psi_c)) \rangle_{L^2(\Sigma,g)} R_b R_c \\
&= \left(\tfrac{m}{A_\Sigma} - \lambda_a\right) R_a + \sum_{b,c\in\mathbb{N}_+} \langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} \lambda_c^{-1} R_b R_c.
\end{aligned} \tag{3.3}$$

Note that

$$\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} = \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{d}A, \tag{3.4}$$

and

$$\Delta(\Psi_a\Psi_c) = (\Delta\Psi_a)\Psi_c + \Psi_a(\Delta\Psi_c) - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = (\lambda_a + \lambda_c)\Psi_a\Psi_c - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c).$$

Thus

$$g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = \frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c).$$

Plugging the above equation into equation (3.4) we get

$$\begin{aligned}
\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} &= \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{d}A \\
&= \int_\Sigma \Psi_b\Big(\frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c)\Big) \, \mathrm{d}A \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{abc} - \frac{1}{2}\int_\Sigma (\Delta\Psi_b)\Psi_a\Psi_c \, \mathrm{d}A \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{abc} - \frac{\lambda_b}{2}\int_\Sigma \Psi_b\Psi_a\Psi_c \, \mathrm{d}A \\
&= \frac{\lambda_a - \lambda_b + \lambda_c}{2}\varphi_{abc}.
\end{aligned}$$

Inserting this into equation (3.3) yields equation (3.1b).

The equivalency of equation (2.1b) and equation (3.2) is straightforward. □

Remark 3.2. *The moral of Theorem 3.1 is that the Keller–Segel equations, which is a (hard) elliptic-parabolic system of partial differential equations, can be transformed (on closed surfaces) into a infinite system of ordinary differential equations, which is potentially easier to handle.*

In the rest of the paper we show that this system can be further simplified under certain extra hypotheses.

3.1. **Analytic solutions.** In order to further simplify equation (3.1b), we search for analytic solutions, that is

$$\forall a \in \mathbb{N} : \forall t \in (0,T) : \quad R_a(t) = \sum_{n\in\mathbb{N}} R_{a,n} t^n,$$

and the right-hand side is assumed to be absolute convergent in $l^2(\mathbb{N})$.

The next lemma rewrites equation (3.1b) in terms of the coefficients $(R_{a,n})_{(a,n)\in\mathbb{N}\times\mathbb{N}}$.

Lemma 3.3. *Under the above assumption, the function ρ is a t-analytic solution to equation* (2.1a) *with mass m exactly when $R_{0,0} = \frac{m}{A_\Sigma}$, for all $n \in \mathbb{N}_+$, $R_{0,n} = 0$, and*

$$\forall a, n \in \mathbb{N}: \quad R_{a,n+1} = \frac{1}{n+1}\left(\left(\tfrac{m}{A_\Sigma} - \lambda_a\right)R_{a,n} + \sum_{b,c\in\mathbb{N}}\sum_{m=0}^{n}\frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c}\varphi_{abc}R_{b,m}R_{c,n-m}\right),$$

$$\forall n \in \mathbb{N}: \quad \mathcal{R}_n := (R_{a,n})_{a\in\mathbb{N}} \in l^2(\mathbb{N}),$$

$$\limsup_{n\to\infty} \|\mathcal{R}_n\|_{l^2(\mathbb{N})}^{\frac{1}{n}} = \limsup_{n\to\infty}\left(\sum_{a\in\mathbb{N}} R_{a,n}^2\right)^{\frac{1}{n}} < \frac{1}{T}.$$

Proof. To be completed... □

4. Round spheres

To be completed...

5. Flat tori

To be completed...

6. Solutions on $\mathbb{R}^2/\mathbb{Z}^2$

If we figure this out.

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