

THE KELLER–SEGEL EQUATION COMPACT SURFACES

1. THE KELLER–SEGEL EQUATION ON THE TORUS

Consider a distribution of germs $\rho(t, x_1, x_2) = \rho(t, \underline{x})$ and food $c(t, x_1, x_2) = c(t, \underline{x})$. We impose, as a model of nature,

$$\partial_t \rho = \partial_a^2 \rho - \partial_a(\rho \partial_a c), \quad (1.1a)$$

$$\partial_a^2 c = -\rho. \quad (1.1b)$$

When a appears as an index, summation over $a \in \{1, 2\}$ is implied. We take the Fourier transform of ρ : For all $\underline{k} \in \mathbb{Z}^2$, let $f_{\underline{k}}(\underline{x}) = e^{2\pi i \underline{k} \cdot \underline{x}}$. Note that $f_{\underline{k}}$ is an eigenfunction of the Laplacian; i.e. $\partial_a^2 f_{\underline{k}} = -4\pi^2 |\underline{k}|^2 f_{\underline{k}}$. Also note that $\partial_a f_{\underline{k}} = 2\pi i k_a f_{\underline{k}}$.

Let us write

$$\rho(t, \underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}). \quad (1.2)$$

Now any solution, c , to equation (1.1b) has the form

$$c(t, \underline{x}) = c_0 + \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq 0}} \frac{1}{4\pi^2 |\underline{l}|^2} R_{\underline{l}}(t) f_{\underline{l}}(\underline{x}),$$

where $c_0 \in \mathbb{C}$ can be chosen arbitrarily.

Using equations (1.1a) and (1.2) we get that

$$\begin{aligned} \sum_{\underline{k} \in \mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k} \in \mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -\partial_a \left(\sum_{\underline{m} \in \mathbb{Z}^2} R_{\underline{m}} f_{\underline{m}} \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq 0}} \frac{R_{\underline{l}}}{4\pi^2 |\underline{l}|^2} \partial_a f_{\underline{l}} \right) \\ &= - \sum_{\substack{\underline{l}, \underline{m} \in \mathbb{Z}^2 \\ \underline{l} \neq 0}} \frac{2\pi i l_a}{4\pi^2 |\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} \partial_a (f_{\underline{l}} f_{\underline{m}}). \end{aligned}$$

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Using that $f_{\underline{m}}f_{\underline{l}} = f_{\underline{m}+\underline{l}}$ and substituting $\underline{k} = \underline{l} + \underline{m}$ on the right-hand side, we get

$$\begin{aligned} \sum_{\underline{k} \in \mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k} \in \mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -i^2 \sum_{\substack{\underline{l}, \underline{m} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{l_a(l_a + m_a)}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}} \\ &= \sum_{\substack{\underline{l}, \underline{m} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{\underline{l} \cdot (\underline{l} + \underline{m})}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}} \\ &= \sum_{\substack{\underline{l}, \underline{k} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{\underline{l} \cdot \underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}} f_{\underline{k}}. \end{aligned}$$

After pairing with $f_{\underline{k}}$ for any $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$ and separating out $R_{\underline{k}}$ terms, we get

$$\dot{R}_{\underline{k}} = (R_0 - 4\pi^2 |\underline{k}|^2) R_{\underline{k}} + \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}, \underline{k}}} \frac{\underline{l} \cdot \underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}}.$$

It remains to be shown that there are such $R_{\underline{k}}$ which satisfy the equations.

2. THE GENERAL CASE

Let Σ now some compact surface, with (positive definite) Laplace operator Δ (we can discuss what that means at some point), and assume that $f_0, f_1, f_2, \dots, f_n, \dots$ are an orthonormal basis of eigenvectors for $L^2(\Sigma)$, that is there are numbers $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ so that for all $n, o \in \mathbb{N}$ we have

$$\Delta f_n = \lambda_n f_n, \quad \& \quad \langle f_n | f_o \rangle_{L^2(\Sigma)} = \delta_{n,o}.$$

For all $n, o, p \in \mathbb{N}$, let

$$\varphi_{n,o,p} := \int_{\Sigma} f_n f_o f_p \, dA.$$

With that in mind, at some point prove the following: If $\rho \in C^1([0, T]; L^2(\Sigma))$ solve the Keller–Segel equations on Σ and $R_n(t) := \langle f_n | \rho(t, \cdot) \rangle_{L^2(\Sigma)}$, then R_0 is constant and

$$\forall n \in \mathbb{N} - \{0\} : \quad \dot{R}_n = (R_0 - \lambda_n) R_n + \sum_{o,p \in \mathbb{N} - \{0\}} \frac{\lambda_m - \lambda_o + \lambda_p}{\lambda_p} \varphi_{n,o,p} R_o R_p.$$

Remark 2.1. Note how the sign of the first term changes depending on whether $R_0 = \int_{\Sigma} \rho \, dA$ is small or greater than λ_n !

3. BANACH BUSINESS

Consider the map $\mathcal{K} : \{R_{\underline{k}}\}_{\underline{k} \in \mathbb{Z}^2} \rightarrow C$ (C is something, perhaps $\{R_{\underline{k}}\}_{\underline{k} \in \mathbb{Z}^2}$) defined by

$$\mathcal{K}(R_{\underline{k}}) = R_{\underline{k}}(0) + \int_0^t (R_0 - 4\pi^2 |\underline{k}|^2) R_{\underline{k}}(\tau) + \sum_{\substack{l \in \mathbb{Z}^2 \\ l \neq 0}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} R_{\underline{k}-\underline{l}}(\tau) R_{\underline{l}}(\tau) d\tau$$

Collections of fourier coefficients with $\mathcal{K}(R) = R$ satisfy equation (1.1a). We seek, therefore, conditions (on $(R_{\underline{k}})_{\underline{k} \in \mathbb{Z}^2}$ and t) which yield fixed points of \mathcal{K} . It would be sufficient to bound

$$\|\mathcal{K}(R) - \mathcal{K}(S)\| \leq \theta \|R - S\|$$

where $\|\cdot\|$ is perhaps

$$\sup_{t \in [0, t]} \sum_{\underline{k} \in \mathbb{Z}^2} |R_{\underline{k}}(t)|^2.$$

and $0 \leq \theta < 1$ is some constant. We might begin to consider $\|\mathcal{K}(R) - \mathcal{K}(S)\| =$

$$\left\| \int_0^t (R_0 - 4\pi^2 |\underline{k}|^2) (R_{\underline{k}}(\tau) - S_{\underline{k}}(\tau)) + \sum_{\substack{l \in \mathbb{Z}^2 \\ l \neq 0}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} (R_{\underline{k}-\underline{l}}(\tau) R_{\underline{l}}(\tau) - S_{\underline{k}-\underline{l}}(\tau) S_{\underline{l}}(\tau)) d\tau \right\|.$$

It may be fruitful to bound the summands of $\|\cdot\|$ corresponding to values of \underline{k} , sharply enough that the infinite sum converges.

4. BOUNDING BLUNDERS

Circuitously, we will show $\sum_{\underline{k} \in [1, 2, \dots]^2} \frac{1}{|\underline{k}|^2} = \infty$. See that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + m^2} = \frac{\coth(\pi m) \pi m - 1}{2m^2}.$$

Now $\coth(x) > \frac{1}{\pi x} + \frac{\pi x}{3} - \frac{\pi^3 x^3}{45}$ so that

$$\begin{aligned} \sum_{n,m=1}^{\infty} \frac{1}{n^2 + m^2} &= \sum_{m=1}^{\infty} \frac{\coth(\pi m) \pi m - 1}{2m^2} > \sum_{m=1}^{\infty} \frac{\left(\frac{1}{\pi^2 m} + \frac{\pi^2 m}{3} - \frac{\pi^6 m^3}{45} \right) \pi m - 1}{2m^2} \\ &= \sum_{m=1}^{\infty} \frac{1 + \frac{\pi^4 m^2}{3} - \frac{\pi^8 m^4}{45} - \pi}{2\pi m^2} = \infty. \end{aligned}$$

Details may be filled in later. In the mean time, I will think of other sums.