THE KELLER–SEGEL EQUATION COMPACT SURFACES

1. The Keller–Segel equation on the torus

Consider a distribution of germs $\rho(t,x_1,x_2) = \rho(t,\underline{x})$ and food $c(t,x_1,x_2) = c(t,\underline{x})$. We impose, as a model of nature,

$$\partial_t\rho = \partial_a^2\rho - \partial_a(\rho\partial_a c), \tag{1.1a}$$

$$\partial_a^2 c = -\rho. \tag{1.1b}$$

When a appears as an index, summation over $a \in \{1,2\}$ is implied. We take the Fourier transform of ρ: For all $\underline{k} \in \mathbb{Z}^2$, let $f_{\underline{k}}(\underline{x}) = e^{2\pi i \underline{k}\cdot\underline{x}}$. Note that $f_{\underline{k}}$ is an eigenfunction of the Laplacian; i.e. $\partial_a^2 f_{\underline{k}} = -4\pi^2|k|^2 f_{\underline{k}}$. Also note that $\partial_a f_{\underline{k}} = 2\pi i k_a f_{\underline{k}}$.

Let us write

$$\rho(t,\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}). \tag{1.2}$$

Now any solution, c, to equation (1.1b) has the form

$$c(t,\underline{x}) = c_0 + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{1}{4\pi^2|\underline{l}|^2} R_{\underline{l}}(t) f_{\underline{l}}(\underline{x}),$$

where $c_0 \in \mathbb{C}$ can be chosen arbitrarily.

Using equations (1.1a) and (1.2) we get that

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2|\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -\partial_a\left(\sum_{\underline{m}\in\mathbb{Z}^2} R_{\underline{m}} f_{\underline{m}} \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{R_{\underline{l}}}{4\pi^2|\underline{l}|^2}\partial_a f_{\underline{l}}\right)\\ &= -\sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{2\pi i l_a}{4\pi^2|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} \partial_a\Big(f_{\underline{l}} f_{\underline{m}}\Big).\end{aligned}$$

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Using that $f_{\underline{m}} f_{\underline{l}} = f_{\underline{m}+\underline{l}}$ and substituting $\underline{k} = \underline{l} + \underline{m}$ on the right-hand side, we get

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -i^2 \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{l_a(l_a+m_a)}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}} \\ &= \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot(\underline{l}+\underline{m})}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}} \\ &= \sum_{\substack{\underline{l},\underline{k}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}} f_{\underline{k}}.\end{aligned}$$

After pairing with $f_{\underline{k}}$ for any $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$ and separating out $R_{\underline{k}}$ terms, we get

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right) R_{\underline{k}} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}}.$$

Let us consider solutions with analytic $R_{\underline{k}}$. Taking $R_{\underline{k}}(t) = \sum_{i=0}^{\infty} R_{\underline{k},i} t^i$, we have

$$\sum_{i=0}^{\infty} (i+1) R_{\underline{k},i+1} t^i = \left(R_0 - 4\pi^2|\underline{k}|^2\right) \sum_{i=0}^{\infty} R_{\underline{k},i} t^i + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} R_{\underline{l},i} R_{\underline{k}-\underline{l},j} t^{i+j}.$$

Equating coefficients of powers of t, we obtain

$$R_{\underline{k},i+1} = \frac{1}{i+1}\left(\left(R_0 - 4\pi^2|\underline{k}|^2\right) R_{\underline{k},i} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} \sum_{j=0}^{i} R_{\underline{l},j} R_{\underline{k}-\underline{l},i-j}\right)$$

which is an explicit recursive formula for the R_k's given intitial coefficients. This formula may be used for numerical computation; we might have polynomial initial conditions and produce simple series solutions for the R_k's. Along this line of inquiry, we prove the following theorem about the number of terms in successive truncations: it may at most double.

Theorem 1.1. *Suppose for fixed i and D, $R_{\underline{k},I} = 0$ for $I \leq i$ and $|\underline{k}|_\infty > D$. Then $R_{\underline{k},i+1} = 0$ for $|\underline{k}|_\infty > 2D$.*

Proof. We have

$$
\begin{aligned}
(i+1)R_{\underline{k},i+1} &= \Big(R_0 - 4\pi^2|\underline{k}|^2\Big)R_{\underline{k},i} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq 0,\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} \sum_{j=0}^{i} R_{\underline{l},j} R_{\underline{k}-\underline{l},i-j} \\
&= \Big(R_0 - 4\pi^2|\underline{k}|^2\Big)R_{\underline{k},i} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ |\underline{l}|_\infty \le D \\ \underline{l}\neq 0,\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} \sum_{j=0}^{i} R_{\underline{l},j} R_{\underline{k}-\underline{l},i-j}.
\end{aligned}
$$

Take $|\underline{k}|_\infty > 2D$. Since $|\underline{l}|_\infty \le D \le |\underline{k}|_\infty$, the reverse triangle inequality yields

$$
|\underline{k}-\underline{l}|_\infty \ge |\underline{k}|_\infty - |\underline{l}|_\infty > 2D - |\underline{l}|_\infty = D + (D - |\underline{l}|_\infty) \ge D + 0 = D
$$

where the rightmost inequality follows from $|l|_\infty \le D$. Then $R_{\underline{k}-\underline{l},i-j} = 0$ for all $i-j$. Of course $R_{\underline{k},i} = 0$ by assumption, so $(i+1)R_{\underline{k},i+1} = 0$.

2. The general case

Let Σ now some compact surface, with (positive definite) Laplace operator Δ (we can discuss what that means at some point), and assume that $f_0, f_1, f_2, \ldots, f_n, \ldots$ are an orthonormal basis of eigenvectors for $L^2(\Sigma)$, that is there are numbers $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ so that for all $n, o \in N$ we have

$$
\Delta f_n = \lambda_n f_n, \quad \& \quad \langle f_n | f_o \rangle_{L^2(\Sigma)} = \delta_{n,o}.
$$

For all $n, o, p \in \mathbb{N}$, let

$$
\varphi_{n,o,p} := \int_\Sigma f_n f_o f_p \,\mathrm{d}A.
$$

With that in mind, at some point prove the following: If $\rho \in C^1\Big([0,T]; L^2(\Sigma)\Big)$ solve the Keller–Segel equations on Σ and $R_n(t) := \langle f_n | \rho(t,\cdot)\rangle_{L^2(\Sigma)}$, then R_0 is constant and

$$
\forall n \in \mathbb{N} - \{0\}: \quad \dot{R}_n = (R_0 - \lambda_n)R_n + \sum_{o,p\in\mathbb{N}-\{0\}} \frac{\lambda_m - \lambda_o + \lambda_p}{\lambda_p} \varphi_{n,o,p} R_o R_p.
$$

Remark 2.1. *Note how the sign of the first term changes depending on whether $R_0 = \int_\Sigma \rho \,\mathrm{d}A$ is small or greater than λ_n!*

3. Banach Business

Consider the map $\mathcal{K} : \{R_{\underline{k}}\}_{\underline{k}\in\mathbb{Z}^2} \to C$ (C is something, perhaps $\{R_{\underline{k}}\}_{\underline{k}\in\mathbb{Z}^2}$) defined by

$$\mathcal{K}(R_{\underline{k}}) = R_{\underline{k}}(0) + \int_0^t (R_0 - 4\pi^2|\underline{k}|^2)R_{\underline{k}}(\tau) + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{\underline{k}-\underline{l}}(\tau)R_{\underline{l}}(\tau)\,\mathrm{d}\tau$$

Collections of fourier coefficients with $\mathcal{K}(R) = R$ satisfy equation (1.1a). We seek, therefore, conditions (on $(R_{\underline{k}})_{\underline{k}\in\mathbb{Z}^2}$ and t) which yield fixed points of $\mathcal{K}$. It would be sufficient to bound

$$\|\mathcal{K}(R) - \mathcal{K}(S)\| \le \theta\|R - S\|$$

where $\|\cdot\|$ is perhaps

$$\sup_{t\in[0,t]} \sum_{\underline{k}\in\mathbb{Z}^2} |R_k(t)|^2.$$

and $0 \le \theta < 1$ is some constant. We might begin to consider $\|\mathcal{K}(R) - \mathcal{K}(S)\| =$

$$\left\| \int_0^t (R_0 - 4\pi^2|\underline{k}|^2)\Big(R_{\underline{k}}(\tau) - S_{\underline{k}}(\tau)\Big) + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2}\Big(R_{\underline{k}-\underline{l}}(\tau)R_{\underline{l}}(\tau) - S_{\underline{k}-\underline{l}}(\tau)S_{\underline{l}}(\tau)\Big)\,\mathrm{d}\tau \right\|.$$

It may be fruitful to bound the summands of $\|\cdot\|$ corresponding to values of $\underline{k}$, sharply enough that the infinite sum converges.

4. Bounding Blunders

Circuitously, we will show $\sum_{\underline{k}\in[1,2,\ldots]^2} \frac{1}{|\underline{k}|^2} = \infty$. See that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+m^2} = \frac{\coth(\pi m)\pi m - 1}{2m^2}.$$

Now $\coth(x) > \frac{1}{\pi x} + \frac{\pi x}{3} - \frac{\pi^3 x^3}{45}$ so that

$$\begin{aligned}\sum_{n,m=1}^{\infty} \frac{1}{n^2+m^2} = \sum_{m=1}^{\infty} \frac{\coth(\pi m)\pi m - 1}{2m^2} &> \sum_{m=1}^{\infty} \frac{\left(\frac{1}{\pi^2 m} + \frac{\pi^2 m}{3} - \frac{\pi^6 m^3}{45}\right)\pi m - 1}{2m^2} \\ &= \sum_{m=1}^{\infty} \frac{1 + \frac{\pi^4 m^2}{3} - \frac{\pi^8 m^4}{45} - \pi}{2\pi m^2} = \infty.\end{aligned}$$

Details may be filled in later. In the mean time, I will think of other sums.

We have

$$S(p)=\sum_{\substack{n_0=1\\n_1=1}}^{\infty}\frac{1}{n_0^{2p}+n_1^{2p}}=\frac{-p+n_1\pi\sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}}\cot\left((-1)^{\frac{2i-1}{2p}}n_1\pi\right)}{2pn_1^{2p}}$$

for $p>0$ an integer. Consistent with ζ, for odd powers of n_0 the sum is more difficult to evaluate. We expand the cotangents into infinite sums and combine:

$$\begin{aligned}S(p)&=\frac{-p+n_1\pi\sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}}\sum_{j=0}^{\infty}\frac{(-1)^j2^{2j}B_{2j}}{(2j)!}\left((-1)^{\frac{2i-1}{2p}}n_1\pi\right)^{2j-1}}{2pn_1^{2p}}\\&=\frac{-p+n_1\pi\sum_{j=0}^{\infty}(n_1\pi)^{2j-1}\frac{(-1)^j2^{2j}B_{2j}}{(2j)!}\sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}}(-1)^{\frac{(2i-1)(2j-1)}{2p}}}{2pn_1^{2p}}\\&=\frac{-p+n_1\pi\sum_{j=0}^{\infty}(n_1\pi)^{2j-1}\frac{(-1)^j2^{2j}B_{2j}}{(2j)!}\sum_{i=1}^{p}(-1)^{\frac{(2i-1)j}{p}}}{2pn_1^{2p}}.\end{aligned}$$

This particular sum of roots of unity is a number theoretic one. It evaluates to $(-1)^{\frac{j}{p}}p$ if p divides j, and 0 if p does not divide j. That is,

$$\sum_{i=1}^{p}(-1)^{\frac{(2i-1)j}{p}}=\begin{cases}(-1)^{\frac{j}{p}}p & \text{if } p|j\\ 0 & \text{if } p\nmid j.\end{cases}$$

Then

$$\begin{aligned}S(p)&=\frac{-p+n_1\pi\sum_{k=0}^{\infty}(n_1\pi)^{2pk-1}\frac{(-1)^{pk}2^{2pk}B_{2pk}}{(2pk)!}(-1)^kp}{2pn_1^{2p}}\\&=\frac{-1+\sum_{k=0}^{\infty}(n_1\pi)^{2pk}\frac{(-1)^{(p+1)k}2^{2pk}B_{2pk}}{(2pk)!}}{2n_1^{2p}}\end{aligned}$$

5. Symmetric in R summands

We have

$$\dot{R}_{\underline{k}}=\left(R_0-4\pi^2|\underline{k}|^2\right)R_{\underline{k}}+\sum_{\substack{\underline{l}\in\mathbb{Z}^2\\\underline{l}\neq\underline{0},\underline{k}}}\frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2}R_{\underline{l}}R_{\underline{k}-\underline{l}}.$$

Substitute $\underline{j}=\underline{l}-\frac{\underline{k}}{2}$. Then

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{\underline{j}\in\mathbb{Z}^2+\frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j}\neq\pm\frac{\underline{k}}{2}}} \frac{(\underline{j}+\frac{\underline{k}}{2})\cdot\underline{k}}{|\underline{j}+\frac{\underline{k}}{2}|^2} R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}$$

where $\mathrm{mod}(n)$ is zero for even n and 1 for odd n (that is, $\underline{j}$ and $\frac{\underline{k}}{2}$ lie on common integer grids with offsets of 0 or $\frac{1}{2}$) (and it is evaluated elementwise). Let us make each summand have unique R products: we only sum over j with $0 \leq \arctan \underline{j} < \pi$:

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{0\leq\arctan\underline{j}<\pi \\ \underline{j}\in\mathbb{Z}^2+\frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j}\neq\pm\frac{\underline{k}}{2}}} \left(\frac{(\underline{j}+\frac{\underline{k}}{2})\cdot\underline{k}}{|\underline{j}+\frac{\underline{k}}{2}|^2} + \frac{(-\underline{j}+\frac{\underline{k}}{2})\cdot\underline{k}}{|-\underline{j}+\frac{\underline{k}}{2}|^2} \right) R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}.$$

We write the coefficients more explicitly:

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{0\leq\arctan\underline{j}<\pi \\ \underline{j}\in\mathbb{Z}^2+\frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j}\neq\pm\frac{\underline{k}}{2}}} \frac{|\underline{k}|^2\left(|\underline{j}|^2 + \frac{|\underline{k}|^2}{4}\right) - 2(\underline{j}\cdot\underline{k})^2}{\left(|\underline{j}|^2 - \underline{j}\cdot\underline{k} + \frac{|\underline{k}|^2}{4}\right)\left(|\underline{j}|^2 + \underline{j}\cdot\underline{k} + \frac{|\underline{k}|^2}{4}\right)} R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}.$$