THE KELLER–SEGEL EQUATION CLOSED SURFACES

ADAM MENDENHALL AND ÁKOS NAGY

ABSTRACT. We study the (parabolic-elliptic) Keller–Segel equations on closed surfaces.
To be completed.

1. INTRODUCTION

To be completed. This is the part that we will write last.

Organization of the paper. **To be completed.**

Acknowledgment. ***To be completed.***

2. REFORMULATION OF THE EQUATION

Let Σ be a smooth, closed surface with a Riemannian metric g and area form ω. Let G be the Green operator of the Laplacian, Δ. For each function, $\varrho \in \mathrm{C}^1\big((0,\mathrm{T});\mathrm{L}^1(\Sigma)\big)$, let

$$\forall t \in (0,\mathrm{T}): \quad \varrho_t \mathrel{\dot{=}} \varrho|_{\{t\}\times\Sigma} \in \mathrm{L}^1(\Sigma).$$

Let us fix $\varrho_0 \in \mathrm{L}^1(\Sigma, g)$. A function, $\varrho \in \mathrm{C}^1\big((0,\mathrm{T});\mathrm{L}^1(\Sigma)\big)$, satisfies the *(parabolic-elliptic) Keller–Segel equations* with initial value ϱ_0 if for all $t \in (0,\mathrm{T})$, $\varrho_t \in \mathrm{L}^2_{1,\mathrm{loc}}$ and ϱ is (weak) solution to the following system:

$$\partial_t \varrho = -\Delta\varrho + \mathrm{d}^*\big(\varrho\, \mathrm{dG}(\varrho)\big), \tag{2.1a}$$

$$\lim_{t\to 0^+} \varrho_t = \varrho_0. \tag{2.1b}$$

The convergence in equation (2.1b) is in $\mathrm{L}^1(\Sigma, g)$.

The mass of ϱ_t is

$$\mathrm{M}(t) \mathrel{\dot{=}} \int_\Sigma \varrho_t \, \mathrm{dA}.$$

Date: August 23, 2021.
2020 *Mathematics Subject Classification.* 35J15, 35Q92, 92C17.
Key words and phrases. chemotaxis, Keller–Segel equations.

Then

$$\dot{\mathrm{M}}(t) = \int_\Sigma \left(-\Delta\varrho + \mathrm{d}^*\left(\varrho\,\mathrm{dG}(\varrho)\right)\right)\,\mathrm{dA} = \int_\Sigma \mathrm{d}^*\left(-\mathrm{d}\varrho + \varrho\,\mathrm{dG}(\varrho)\right)\,\mathrm{dA} = 0,$$

Thus M constant and hence we drop the t-dependence from its notation.

For the rest of the paper, let $\mathrm{A}_\Sigma \doteq \mathrm{Area}(\Sigma, g)$. The following lemma recasts equations (2.1a) and (2.1b) in a simpler form.

Lemma 2.1. *Let* $\chi_0 \doteq \varrho_0 - \frac{\mathrm{M}}{\mathrm{A}_\Sigma} \in \mathrm{L}^1(\Sigma)$ *and* $\chi \doteq \varrho - \frac{\mathrm{M}}{\mathrm{A}_\Sigma} \in \mathrm{C}^1\left((0,\mathrm{T});\mathrm{L}^1(\Sigma)\right)$. *Then equations* (2.1a) *and* (2.1b) *are equivalent to*

$$\partial_t\chi = \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\chi - \Delta\chi + \mathrm{d}^*\left(\chi\,\mathrm{dG}(\chi)\right), \tag{2.2a}$$

$$\chi|_{\{0\}\times\Sigma} = \chi_0. \tag{2.2b}$$

Proof. The equivalency of equation (2.1b) and equation (2.2b) is obvious.

Note that G annihilates constants and since χ is orthogonal to constants, we have that $\Delta(\mathrm{G}(\chi)) = \chi$. Since $\chi = \varrho - \frac{\mathrm{M}}{\mathrm{A}_\Sigma}$, we get, using equation (2.1a), that

$$\begin{aligned}
\partial_t\chi &= \partial_t\left(\varrho - \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\right)\\
&= \partial_t\varrho - 0\\
&= -\Delta\varrho + \mathrm{d}^*\left(\varrho\,\mathrm{dG}(\varrho)\right)\\
&= -\Delta\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \chi\right) + \mathrm{d}^*\left(\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \chi\right)\mathrm{dG}\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \chi\right)\right)\\
&= -\Delta\chi + \mathrm{d}^*\left(\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \chi\right)\mathrm{dG}(\chi)\right)\\
&= -\Delta\chi + \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\mathrm{d}^*\,\mathrm{dG}(\chi) + \mathrm{d}^*\left(\chi\,\mathrm{dG}(\chi)\right)\\
&= \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\chi - \Delta\chi + \mathrm{d}^*\left(\chi\,\mathrm{dG}(\chi)\right),
\end{aligned}$$

which completes the proof. □

Remark 2.2. *Let* λ_1 *be the smallest nonzero eigenvalue of* Δ *and note that quantity* $\mathrm{M}_\Sigma = \lambda_1\mathrm{A}_\Sigma$ *only depends on the geometry of* (Σ, g). *When* $\mathrm{M} < \mathrm{M}_\Sigma$, *then the linear term in equation* (2.2a) *is strictly negative definite.*

3. The generalized Fourier transform

Since Σ is compact, for all $p \in (1,\infty]$, we have that $\mathrm{L}^p(\Sigma) \hookrightarrow \mathrm{L}^1(\Sigma, g)$. Let us assume now that $\varrho \in \mathrm{C}^1\left((0,\mathrm{T});\mathrm{L}^2(\Sigma)\right) \hookrightarrow \mathrm{C}^1\left((0,\mathrm{T});\mathrm{L}^1(\Sigma)\right)$ is a solution to the Keller–Segel equation (2.1a).

Let now $\left(\Psi_a \in \mathrm{L}^2(\Sigma, g)\right)_{a\in\mathbb{N}}$ be an orthonormal eigenbasis of Δ and

$$\Delta\Psi_a = \lambda_a \Psi_a.$$

Let us order this basis so that

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \lambda_a \leqslant \lambda_{a+1} \leqslant \ldots$$

In particular, $\Psi_0 = \frac{1}{\sqrt{\mathrm{A}_\Sigma}}$.

Assume that χ is a solution to equation (2.2a) and write

$$\forall a \in \mathbb{N}: \ \forall t \in (0,\mathrm{T}): \quad \mathrm{R}_a(t) \mathrel{\dot{=}} \langle \Psi_a | \chi|_{\{t\}\times\Sigma} \rangle_{\mathrm{L}^2(\Sigma,g)}.$$

Then for each $a \in \mathbb{N}$, we have that $\mathrm{R}_a \in \mathrm{C}^1((0,\mathrm{T});\mathbb{R})$. Note that $\mathrm{R}_0 \equiv 0$. Finally let

$$\forall a,b,c \in \mathbb{N}: \quad \varphi_{a,b,c} \mathrel{\dot{=}} \int_\Sigma \Psi_a \Psi_b \Psi_c \,\mathrm{dA}.$$

Then φ is a completely symmetric 3-tensor, and

$$\forall b,c \in \mathbb{N}: \quad \varphi_{0,b,c} = \frac{1}{\sqrt{\mathrm{A}_\Sigma}} \delta_{b,c}.$$

Theorem 3.1. *Under the above assumptions, the function χ is a solution to equation* (2.2a) *exactly when*

$$\forall t \in (0,\mathrm{T}): \quad (\mathrm{R}_a(t))_{a\in\mathbb{N}_+} \in l^2(\mathbb{N}_+), \tag{3.1a}$$

$$\forall a \in \mathbb{N}_+: \quad \dot{\mathrm{R}}_a = \left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - \lambda_a\right)\mathrm{R}_a + \sum_{b,c\in\mathbb{N}_+} \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c} \varphi_{a,b,c}\mathrm{R}_b\mathrm{R}_c. \tag{3.1b}$$

Proof. Since $(\Psi_a)_{a\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{L}^2(\Sigma,g)$ and for all $t \in (0,\mathrm{T})$, ϱ_t is in $\mathrm{L}^2(\Sigma,g)$, we get condition (3.1a).

For all $a \in \mathbb{N}_+$, we have $\mathrm{G}(\Psi_a) = \lambda_a^{-1}\Psi_a$. Using this, the self-adjointness of Δ, and equation (2.2a), we get

$$\begin{aligned}
\dot{\mathrm{R}}_a &= \langle \Psi_a | \partial_t \chi \rangle_{\mathrm{L}^2(\Sigma,g)} \\
&= \left\langle \Psi_a \middle| \frac{\mathrm{M}}{\mathrm{A}_\Sigma}\chi - \Delta\chi + \mathrm{d}^*\left(\chi\,\mathrm{dG}(\chi)\right) \right\rangle_{\mathrm{L}^2(\Sigma,g)} \\
&= \left\langle \frac{\mathrm{M}}{\mathrm{A}_\Sigma}\Psi_a - \Delta\Psi_a \middle| \chi \right\rangle_{\mathrm{L}^2(\Sigma,g)} + \langle \mathrm{d}\Psi_a | \chi\,\mathrm{dG}(\chi) \rangle_{\mathrm{L}^2(\Sigma,g)} \\
&= \left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - \lambda_a\right) \langle \Psi_a | \chi \rangle_{\mathrm{L}^2(\Sigma,g)} + \sum_{b,c\in\mathbb{N}_+} \langle \mathrm{d}\Psi_a | \Psi_b\,\mathrm{dG}(\Psi_c) \rangle_{\mathrm{L}^2(\Sigma,g)} \mathrm{R}_b\mathrm{R}_c \\
&= \left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - \lambda_a\right)\mathrm{R}_a + \sum_{b,c\in\mathbb{N}_+} \langle \mathrm{d}\Psi_a | \Psi_b\,\mathrm{d}\Psi_c \rangle_{\mathrm{L}^2(\Sigma,g)} \lambda_c^{-1} \mathrm{R}_b\mathrm{R}_c.
\end{aligned} \tag{3.2}$$

Note that

$$\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{\mathrm{L}^2(\Sigma,g)} = \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{dA}, \tag{3.3}$$

and

$$\Delta(\Psi_a\Psi_c) = (\Delta\Psi_a)\Psi_c + \Psi_a(\Delta\Psi_c) - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = (\lambda_a + \lambda_c)\Psi_a\Psi_c - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c).$$

Thus

$$g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = \frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c).$$

Plugging the above equation into equation (3.3) we get

$$\begin{aligned}
\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{\mathrm{L}^2(\Sigma,g)} &= \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{dA} \\
&= \int_\Sigma \Psi_b \left(\frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c) \right) \mathrm{dA} \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{a,b,c} - \frac{1}{2}\int_\Sigma (\Delta\Psi_b)\Psi_a\Psi_c \, \mathrm{dA} \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{a,b,c} - \frac{\lambda_b}{2}\int_\Sigma \Psi_b\Psi_a\Psi_c \, \mathrm{dA} \\
&= \frac{\lambda_a - \lambda_b + \lambda_c}{2}\varphi_{a,b,c}.
\end{aligned}$$

Inserting this into equation (3.2) yields equation (3.1b). □

Remark 3.2. *The moral of Theorem 3.1 is that the Keller–Segel equations, which is a (hard) parabolic-elliptic system of partial differential equations, can be transformed (on closed surfaces) into a infinite system of ordinary differential equations, which is potentially easier to handle.*

In the rest of the paper we show that this system can be further simplified under certain extra hypotheses.

3.1. **Integrating factors.** Using the notation and assumptions of the previous section, let us define (for all $a \in \mathbb{N}_+$ and $n \in \mathbb{N}$)

$$\mathrm{S}_a(t) \vdots= \exp\left(\left(\lambda_a - \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\right)t\right)\mathrm{R}_a(t).$$

These are the "Fourier" coefficient of $\exp\left(\Delta - \frac{\mathrm{M}}{\mathrm{A}_\Sigma}\mathbb{1}\right)\chi$. Using equation (3.1b), we get that

$$\dot{\mathrm{S}}_a(t) \vdots= \sum_{b,c\in\mathbb{N}_+} \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c}\varphi_{a,b,c} \exp\left(\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \lambda_a - \lambda_b - \lambda_c\right)t\right)\mathrm{S}_b(t)\mathrm{R}_c(t). \tag{3.4}$$

3.2. **Analytic solutions.** In order to further simplify equations (3.1b) and (3.4), we search for analytic solutions, that is

$$\forall a \in \mathbb{N}_+ : \forall t \in (0,\mathrm{T}) : \quad \mathrm{R}_a(t) = \sum_{n\in\mathbb{N}} \mathrm{R}_{n,a} t^n, \tag{3.5}$$

and the sums

$$\forall t \in (0,\mathrm{T}) : \sum_{n\in\mathbb{N}} \left(\mathrm{R}_{n,a}\right)_{a\in\mathbb{N}_+} t^n,$$

are assumed to be absolute convergent in $l^2(\mathbb{N}_+)$. Similarly we define

$$\forall a \in \mathbb{N}_+ : \forall t \in (0,\mathrm{T}) : \quad \mathrm{S}_a(t) = \sum_{n\in\mathbb{N}} \mathrm{S}_{n,a} t^n,$$

The next lemma rewrites equations (3.1b) and (3.4) in terms of the coefficients $\left(\mathrm{R}_{n,a}\right)_{(n,a)\in\mathbb{N}\times\mathbb{N}_+}$ and $\left(\mathrm{S}_{n,a}\right)_{(n,a)\in\mathbb{N}\times\mathbb{N}_+}$.

Lemma 3.3. *Under the above assumptions, the function* χ *is a* t*-analytic solution to equation* (2.2a) *exactly when (for all relevant* a *and* n*) we have*

$$\mathrm{R}_{n+1,a} = \frac{1}{n+1}\left(\left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - \lambda_a\right)\mathrm{R}_{n,a} + \sum_{b,c\in\mathbb{N}_+} \sum_{m=0}^{n} \frac{\lambda_a-\lambda_b+\lambda_c}{2\lambda_c} \varphi_{a,b,c} \mathrm{R}_{m,b}\mathrm{R}_{n-m,c}\right), \tag{3.6a}$$

$$\left(\mathrm{R}_{n,a}\right)_{a\in\mathbb{N}} \in l^2(\mathbb{N}), \tag{3.6b}$$

$$\limsup_{n\to\infty}\left(\sum_{a\in\mathbb{N}} \mathrm{R}_{n,a}^2\right)^{\frac{1}{n}} \leqslant \frac{1}{\mathrm{T}}. \tag{3.6c}$$

If $\exp\left(\Delta - \frac{\mathrm{M}}{\mathrm{A}_\Sigma}\mathbb{1}\right)\chi$ *is also defined for all* $t \in (0,\mathrm{T})$*, then*

$$\mathrm{S}_{n+1,a} = \frac{1}{n+1} \sum_{b,c\in\mathbb{N}_+} \sum_{x+y+z=n} \frac{\lambda_a-\lambda_b+\lambda_c}{2\lambda_c} \varphi_{a,b,c} \mathrm{S}_{x,b} \mathrm{S}_{y,c} \frac{\left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma}+\lambda_a-\lambda_b-\lambda_c\right)^z}{z!}. \tag{3.7}$$

Proof. Inserting equation (3.5) into equation (3.1b) yields equation (3.6a). The equations (3.6b) and (3.6c) are necessary (and, in fact, sufficient) to have that the convergence radius of the Taylor series of ϱ in the L^2 topology is at least T. The claim about equation (3.7) is straightforward to check. □

In the following two sections we investigate two special cases when iteration in equation (3.6a) exists for all a and n.

4. Round spheres

Let Σ be the 2-sphere and g be the round metric of radius r. Then we have that $(\Psi_a)_{a\in\mathbb{N}}$ are the spherical harmonics. Then $\mathrm{A}_\Sigma = 4\pi r^2$. In fact, after relabeling them, we can write the eigenvalue has the form $\lambda_{l,m} \doteq \frac{l(l+1)}{r^2}$, where $l \in \mathbb{N}$ and M is any integer satisfying $|m| \leqslant l$. Let us now write

$$\Psi_l^m \doteq \Psi_{(l,m)}, \quad \& \quad \mathrm{R}_{l,a}^m \doteq \mathrm{R}_{(l,m),a}, \quad \& \quad \varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} \doteq \varphi_{(l_1,m_1),(l_2,m_2),(l_3,m_3)}.$$

Using this new set of indices and notation, we can rewrite equation (3.6a) as (for all $l \in \mathbb{N}_+$)

$$\begin{aligned} \mathrm{R}_{l,n+1}^m = &\frac{1}{n+1}\left(\frac{\mathrm{M}}{4\pi r^2} - \frac{l(l+1)}{r^2}\right)\mathrm{R}_{(l,m),n} \\ &+ \sum_{l_1,l_2 \in \mathbb{N}_+} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \sum_{x=0}^{n} \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)} \varphi_{l,l_1,l_2}^{m,m_1,m_2} \mathrm{R}_{l_1,x}^{m_1} \mathrm{R}_{l_2,n-x}^{m_2}. \end{aligned} \tag{4.1}$$

Using the Clebsch–Gordan Theorem, we have that if $l_1 \geqslant l_2 + l_3$ or $l_1 \leqslant |l_2 - l_3|$, then for all m_1, m_2, and m_3, we have $\varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} = 0$. Thus equation (4.1) becomes

$$\begin{aligned} \mathrm{R}_{l,n+1}^m = &\frac{1}{n+1}\left(\frac{\mathrm{M}}{4\pi r^2} - \frac{l(l+1)}{r^2}\right)\mathrm{R}_{l,n}^m \\ &+ \sum_{l_1 \in \mathbb{N}_+} \sum_{l_2=\max(\{1,|l-l_1|\})}^{l+l_1} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \sum_{x=0}^{n} \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)} \varphi_{l,l_1,l_2}^{m,m_1,m_2} \mathrm{R}_{l_1,x}^{m_1} \mathrm{R}_{l_2,n-x}^{m_2}. \end{aligned} \tag{4.2}$$

Before prove the main result of this section, let us make the following definition:

$$\forall n \in \mathbb{N}: \quad \mathrm{Z}_n \doteq \left\{ (l,m) \in \mathbb{N} \times \mathbb{Z} \;\middle|\; \mathrm{R}_{l,n}^m \neq 0 \right\}.$$

Theorem 4.1. *Assume that* Z_0 *is finite. Then for all* $(l,m) \in \mathbb{N} \times \mathbb{Z}$ *and* $n \in \mathbb{N}$, $\mathrm{R}_{l,n}^m$ *exists. Furthermore,* Z_n *is also finite, and equation* (4.2) *becomes*

$$\begin{aligned} \mathrm{R}_{l,n+1}^m = &\frac{1}{n+1}\left(\frac{\mathrm{M}}{4\pi r^2} - \frac{l(l+1)}{r^2}\right)\mathrm{R}_{l,n}^m \\ &+ \sum_{x=0}^{n} \sum_{\substack{(l_1,m_1) \in \mathrm{Z}_x \\ (l_2,m_2) \in \mathrm{Z}_{n-x}}} \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)} \varphi_{l,l_1,l_2}^{m,m_1,m_2} \mathrm{R}_{l_1,x}^{m_1} \mathrm{R}_{l_2,n-x}^{m_2}. \end{aligned} \tag{4.3}$$

Proof. Let us prove by induction.

Since the claim for $n = 0$ is the hypothesis of the theorem, we only need to assume that we have already proven the claim for all nonnegative integers up to and including $n \in \mathbb{N}_+$.

The right-hand side of equation (4.2) contains coefficients of the form $\mathrm{R}_{l_1,x}^{m_1}$ and $\mathrm{R}_{l_2,n-x}^{m_2}$, with $0 \leqslant x \leqslant n$, we have that $\mathrm{R}_{l_1,x}^{m_1}$ unless $(l_1, m_1) \in \mathrm{Z}_x$ and $\mathrm{R}_{l_2,n-x}^{m_2}$ unless $(l_2, m_2) \in \mathrm{Z}_{n-x}$. Thus for every x we have the contribution of a finite sum, and we only consider finitely many x's, this proves equation (4.3).

Since now $\mathrm{R}_{l,n+1}^m$ is expressed as a finite sum, it exists, which concludes the proof. $\square$

Remark 4.2. *Similar results can be proven about the corresponding* $\mathrm{S}_{n,a}$ *coefficients, which we omit here.*

5. Flat tori

Let now Σ be a flat torus. Thus, without any loss of generality, we can assume that there are vectors

$$\underline{e}_1 = \begin{pmatrix} \mathrm{L}_1 \\ 0 \end{pmatrix}, \quad \& \quad \underline{e}_2 = \begin{pmatrix} \mathrm{L}_2 \cos(\theta) \\ \mathrm{L}_2 \sin(\theta) \end{pmatrix}.$$

such that, if we define the *lattice* $\Lambda \stackrel{\cdot}{=} \mathbb{Z}\underline{e}_1 \oplus \mathbb{Z}\underline{e}_2$, then

$$\Sigma = \mathbb{R}^2/\Lambda.$$

Note that $\mathrm{A}_\Sigma = \mathrm{L}_1\mathrm{L}_2 \sin(\theta)$. Let the *dual* lattice be

$$\Lambda^* \stackrel{\cdot}{=} \left\{ \underline{k} \in \mathbb{R}^2 \;\middle|\; \forall \underline{x} \in \Lambda : \; \underline{k} \cdot \underline{x} \in \mathbb{Z} \right\}.$$

It is easy to see that if

$$\underline{f}_1 \stackrel{\cdot}{=} \begin{pmatrix} \frac{1}{\mathrm{L}_1} \\ -\frac{\cot(\theta)}{\mathrm{L}_1} \end{pmatrix}, \quad \& \quad \underline{f}_2 = \begin{pmatrix} 0 \\ \frac{1}{\mathrm{L}_2 \sin(\theta)} \end{pmatrix}.$$

then $\underline{e}_i \cdot \underline{f}_j = \delta_{i,j}$ and thus

$$\Lambda^* = \mathbb{Z}\underline{f}_1 \oplus \mathbb{Z}\underline{f}_2.$$

Now let

$$\forall \underline{x} \in \Sigma : \forall \underline{k} \in \Lambda^* : \quad \Psi_{\underline{k}}\left(\underline{x}\right) \stackrel{\cdot}{=} \frac{1}{\sqrt{\mathrm{A}_\Sigma}} e^{2\pi i \underline{k}\cdot\underline{x}}.$$

Then $\left(\Psi_{\underline{k}}\right)_{\underline{k}\in\Lambda^*}$ is an orthonormal basis for the *complex* Hilbert space $\mathrm{L}^2_{\mathbb{C}}\left(\Sigma, g\right)$. Furthermore, note that $\Psi_{\underline{k}} = \overline{\Psi_{-\underline{k}}}$. Finally, note that

$$\Delta\Psi_{\underline{k}} = 4\pi^2 \left|\underline{k}\right|^2 \Psi_{\underline{k}},$$

thus $\left(\Psi_{\underline{k}}\right)_{\underline{k}\in\Lambda^*}$ is an eigenbasis for the Laplacian, albeit a complex one. The corresponding spectrum is $\left(4\pi^2\left|\underline{k}\right|^2\right)_{\underline{k}\in\Lambda^*}$.

Function on Σ can be viewed as Λ-periodic functions on $\mathbb{R}^2$, thus if χ is an ($\mathrm{L}^2_{\mathbb{C}}$) function on Σ, then we use Fourier decomposition to get:

$$\forall \underline{k} \in \Lambda^* : \quad \mathrm{R}_{\underline{k}} \stackrel{\cdot}{=} \frac{1}{\sqrt{\mathrm{A}_\Sigma}} \int_\Sigma e^{-2\pi i \underline{k}\cdot\underline{x}} \chi\left(\underline{x}\right) \,\mathrm{dA}\left(\underline{x}\right), \quad \Leftrightarrow \quad \chi = \sum_{\underline{k}\in\Lambda^*} \mathrm{R}_{\underline{k}} \Psi_{\underline{k}}.$$

Note again that $\mathrm{R}_{\underline{0}} \equiv 0$. If χ is real, then $\mathrm{R}_{\underline{k}} = \overline{\mathrm{R}_{-\underline{k}}}$. In this section we slightly deviate from our previous method and use the above complex basis and coefficients.

The ideas and proofs of the previous sections still apply, and if $\chi \in \mathrm{C}^1\left((0,\mathrm{T}), \mathrm{L}^2_{\mathbb{C}}\left(\Sigma, g\right)\right)$, then we can define the coefficients functions $\mathrm{R}_{\underline{k}}, \mathrm{S}_{\underline{k}} \in \mathrm{C}^1_{\mathbb{C}}(\Sigma)$, so that

$$\chi\left(t, \underline{x}\right) = \sum_{n\in\mathbb{N}} \sum_{\underline{k}\in\Lambda^* - \{\underline{0}\}} \mathrm{R}_{\underline{k}}(t) \Psi_{\underline{k}}\left(\underline{x}\right),$$

$$\exp\left(\left(\Delta - \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma}\right)t\right)\chi\left(t,\underline{x}\right) = \sum_{n\in\mathbb{N}}\sum_{\underline{k}\in\Lambda^*-\{\underline{0}\}} \mathrm{S}_{\underline{k}}(t)\Psi_{\underline{k}}\left(\underline{x}\right).$$

If, furthermore, χ is a solution to the Keller–Segel equation (2.2a), then we get (after a straightforward computation) that for all $\underline{k}\in\Lambda^*-\{\underline{0}\}$

$$\dot{\mathrm{R}}_{\underline{k}} = \left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\left|\underline{k}\right|^2\right)\mathrm{R}_{\underline{k}} + \sum_{\underline{l}\in\Lambda^*-\{\underline{0},\underline{k}\}} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2\sqrt{\mathrm{A}_\Sigma}}\mathrm{R}_{\underline{l}}\mathrm{R}_{\underline{k}-\underline{l}},$$

$$\dot{\mathrm{S}}_{\underline{k}} = \sum_{\underline{l}\in\Lambda^*-\{\underline{0},\underline{k}\}} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2\sqrt{\mathrm{A}_\Sigma}}\exp\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + 8\pi^2\left(\underline{k}-\underline{l}\right)\cdot\underline{l}\right)\mathrm{S}_{\underline{l}}\mathrm{S}_{\underline{k}-\underline{l}}$$

Finally, if χ is analytic in t, and we define $\mathrm{R}_{n,\underline{k}},\mathrm{S}_{n,\underline{k}}\in\mathbb{C}$ through

$$\mathrm{R}_{\underline{k}}(t) = \sum_{n\in\mathbb{N}}\mathrm{R}_{n,\underline{k}}t^n,$$
$$\mathrm{S}_{\underline{k}}(t) = \sum_{n\in\mathbb{N}}\mathrm{S}_{n,\underline{k}}t^n,$$

then we get the corresponding iteration is (for all $n\in\mathbb{N}$ and $\underline{k}\in\Lambda^*-\{\underline{0}\}$)

$$\mathrm{R}_{n+1,\underline{k}} = \frac{1}{n+1}\left(\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\left|\underline{k}\right|^2\right)\mathrm{R}_{n,\underline{k}} + \sum_{\underline{l}\in\Lambda^*-\{\underline{0},\underline{k}\}}\sum_{m=0}^{n} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2\sqrt{\mathrm{A}_\Sigma}}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right), \tag{5.1a}$$

$$\mathrm{S}_{n+1,\underline{k}} = \frac{1}{n+1}\sum_{\underline{l}\in\Lambda^*-\{\underline{0},\underline{k}\}}\sum_{a+b+c=n} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2\sqrt{\mathrm{A}_\Sigma}}\mathrm{S}_{a,\underline{l}}\mathrm{S}_{b,\underline{k}-\underline{l}}\frac{\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + 8\pi^2\left(\underline{k}-\underline{l}\right)\cdot\underline{l}\right)^c}{c!}, \tag{5.1b}$$

As in Section 4, let us make the following definition: For all $n\in\mathbb{N}$, let

$$\mathrm{Z}_n \mathrel{\vdots}= \left\{\ \underline{k}\in\Lambda^*-\{\underline{0}\}\ \middle|\ \mathrm{R}_{n,\underline{k}}\neq 0\ \right\}\cup\{\underline{0}\},$$
$$\hat{\mathrm{Z}}_n \mathrel{\vdots}= \left\{\ \underline{k}\in\Lambda^*-\{\underline{0}\}\ \middle|\ \mathrm{S}_{n,\underline{k}}\neq 0\ \right\}\cup\{\underline{0}\}.$$

Furthermore, let

$$d_n \mathrel{\vdots}= \max\left(\left\{\ \left|\underline{k}\right|\ \middle|\ \underline{k}\in\mathrm{Z}_n\ \right\}\right),$$
$$\hat{d}_n \mathrel{\vdots}= \max\left(\left\{\ \left|\underline{k}\right|\ \middle|\ \underline{k}\in\hat{\mathrm{Z}}_n\ \right\}\right).$$

Note that $\mathrm{Z}_0 = \hat{\mathrm{Z}}_0$, and thus $d_0 = \hat{d}_0$.

Theorem 5.1. *Assume that* Z_0 *is finite. Then for all* $\underline{k}\in\Lambda^*-\{\underline{0}\}$ *and* $n\in\mathbb{N}$, $\mathrm{R}_{n,\underline{k}}$ *exists. Moreover,* Z_n *is also finite and equations* (5.1a) *and* (5.1b) *become, for all* $\underline{k}\in\Lambda^*-\{\underline{0}\}$

$$\mathrm{R}_{n+1,\underline{k}} = \frac{1}{n+1}\left(\left(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\left|\underline{k}\right|^2\right)\mathrm{R}_{n,\underline{k}} + \sum_{m=0}^{n}\sum_{\substack{\underline{l}\in\mathrm{Z}_m\\ \underline{k}-\underline{l}\in\mathrm{Z}_{n-m}}} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2\sqrt{\mathrm{A}_\Sigma}}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right), \tag{5.2}$$

$$\mathrm{S}_{n+1,\underline{k}} = \frac{1}{n+1} \sum_{x+y+z=n} \sum_{\substack{\underline{l}\in\hat{\mathrm{Z}}_a \\ \underline{k}-\underline{l}\in\hat{\mathrm{Z}}_b}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2\sqrt{\mathrm{A}_\Sigma}} \mathrm{S}_{a,\underline{l}} \mathrm{S}_{b,\underline{k}-\underline{l}} \frac{\left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} + 8\pi^2 \left(\underline{k}-\underline{l}\right)\cdot\underline{l}\right)^z}{z!}.$$

Moreover

$$d_n, \hat{d}_n \leqslant (n+1)d_0,$$

Proof. We only prove the claim for the $\mathrm{R}_{n,\underline{k}}$ coefficients. The proof is analogous for the $\mathrm{S}_{n,\underline{k}}$ coefficients.

Let us prove by induction. The claim for $n = 0$ is the hypothesis of the theorem.

Let us now assume that all nonnegative integers, m, up to but not including $n \in \mathbb{N}_+$, we have that for all $\underline{k} \in \Lambda^* - \{\underline{0}\}$, $\mathrm{R}_{m,\underline{k}}$ exists, Z_m is finite, equation (5.2) holds (with n replaced by m), and $d_m \leqslant (m+1)d_0$.

Since equation (5.2) holds when n is replaced by $m = n-1$, we have that

$$\forall \underline{k} \in \Lambda^* - \{\underline{0}\} : \mathrm{R}_{n,\underline{k}} = \frac{1}{n}\left(\left(\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\left|\underline{k}\right|^2\right)\mathrm{R}_{n-1,\underline{k}} + \sum_{m=0}^{n-1} \sum_{\substack{\underline{l}\in\mathrm{Z}_m \\ \underline{k}-\underline{l}\in\mathrm{Z}_{n-1-m}}} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2} \mathrm{R}_{m,\underline{l}} \mathrm{R}_{n-1-m,\underline{k}-\underline{l}}\right). \tag{5.3}$$

Let

$$\mathrm{Z}'_n \doteq \bigcup_{m=0}^{n-1} (\mathrm{Z}_m + \mathrm{Z}_{n-1-m}),$$

By assumption, Z'_n is a finite union of finite sets, so it is also finite. Pick any $\underline{k} \in \mathrm{Z}'_n$. Since $\mathrm{Z}_{n-1} \subseteq \mathrm{Z}'_n$, we have that the first term in equation (5.3) vanishes. By construction, there cannot exist $m \in [0, n-1] \cap \mathbb{N}$ and $\underline{l} \in \mathrm{Z}_m$, such that $\underline{k} - \underline{l} \in \mathrm{Z}_{n-1-m}$, so all summands of the second term of equation (5.3) also vanish, hence $\mathrm{Z}_n \subseteq \mathrm{Z}'_n$, thus Z_n is finite.

Using the finiteness of Z_m, now for all $m \in [0, n] \cap \mathbb{N}$, and equation (5.1a) implies equation (5.2).

Since now $\mathrm{R}_{n,\underline{k}}$ is expressed as a finite sum, it exists.

Finally, if $\underline{k} \in \mathrm{Z}_n - \mathrm{Z}_{n-1}$, then there exist $m \in [0, n-1] \cap \mathbb{N}$, $\underline{l}_1 \in \mathrm{Z}_m$, and $\underline{l}_2 \in \mathrm{Z}_{n-m}$, such that $\underline{k} = \underline{l}_1 + \underline{l}_2$. Thus

$$\left|\underline{k}\right| = \left|\underline{l}_1 + \underline{l}_2\right| \leqslant \left|\underline{l}_1\right| + \left|\underline{l}_2\right| \leqslant d_m + d_{n-m} \leqslant (m+1)d_0 + (n-m+1)d_0 = ((n+1)+1)d_0,$$

which concludes the proof. $\square$

Corollary 5.2. *Under the assumptions of Theorem 5.1, let* $d \doteq d_0$. *Then we have*

$$|\mathrm{Z}_n| \leqslant 2\pi n^2 d^2 \mathrm{A}_\Sigma.$$

We are now ready to prove our main theorem.

Theorem 5.3. *Let* $\varrho_0 \in \mathrm{L}^1\big(\Sigma, g\big)$ *be such that it has only finitely many nonzero Fourier coefficients, that is,* Z_0 *is finite. Then there exists* $\mathrm{T} > 0$*, and* $\varrho \in \mathrm{C}^\omega\big((0,\mathrm{T}), \mathrm{L}^2\big(\Sigma, g\big)\big)$*, such that it solves equations* (2.1a) *and* (2.1b)*.*

Proof. By Theorem 5.1, for all $n \in \mathbb{N}$ and $\underline{k} \in \Lambda^* - \{\underline{0}\}$ the $\mathrm{R}_{n,\underline{k}}$ coefficients exist. We first prove that there exists C, such that for all $n \in \mathbb{N}_+$ and $\underline{k} \in \Lambda^* - \{\underline{0}\}$

$$\left|\mathrm{R}_{n,\underline{k}}\right| \leqslant \mathrm{C}^{n+1} \tfrac{n^n}{(n+1)!} \exp\Big(\tfrac{|\underline{k}|}{\mathrm{C}}\Big), \tag{5.4}$$

and thus

$$\limsup_{n\in\mathbb{N}} \sqrt[n]{\|\mathrm{R}_n\|_{l^1(\Lambda^*)}} = \lim_{n\to\infty} \sqrt[n]{\|\mathrm{R}_n\|_{l^1(\Lambda^*)}} \leqslant \frac{\mathrm{C}}{e} =: \frac{1}{\mathrm{T}}.$$

One can pick $\mathrm{C} > 0$, so that inequality (5.4) holds for $n = 0$. Let us assume that we have already proven **??** for all nonnegative integers up to and including $n \in \mathbb{N}_+$. Then, by Theorem 5.1, for $n+1$, we only need to consider $\underline{k} \in \mathrm{Z}_{n+1}$, in particular, $\left|\underline{k}\right| \leqslant (n+2)d_0$

$$\left|\frac{\frac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2|\underline{k}|^2}{n+1} \mathrm{R}_{n,\underline{k}}\right| \leqslant \tfrac{\mathrm{C}|\underline{k}|^2}{2} \mathrm{C}^{n+1} \tfrac{n^n}{(n+1)!} \exp\Big(\tfrac{|\underline{k}|}{\mathrm{C}}\Big)$$

, using equation (5.2), we have

—

*This part is **To be completed.***

—

Now for all $t \in (0,\mathrm{T})$ and $\underline{x} \in \Sigma$, let

$$\varrho\big(t, \underline{x}\big) \vcentcolon= \tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} + \sum_{\underline{k}\in\Lambda^*-\{\underline{0}\}} \mathrm{S}_{\underline{k}}(t) \exp\Big(\Big(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\big|\underline{k}\big|^2\Big)t\Big) \tfrac{e^{2\pi i \underline{k}\cdot\underline{x}}}{\sqrt{\mathrm{A}_\Sigma}}. \tag{5.5}$$

Now note that if $\underline{k} \in \hat{\mathrm{Z}}_n - \hat{\mathrm{Z}}_{n-1}$, then

$$\left|\mathrm{S}_{\underline{k}}(t) - \mathrm{S}_{\underline{k}}(0)\right| \leqslant \mathrm{C}\frac{t^n}{\mathrm{T} - |t|}.$$

In particular, if $n > 0$, then

$$\left|\mathrm{S}_{\underline{k}}(t)\right| \leqslant \mathrm{C}\frac{t^n}{\mathrm{T} - |t|}.$$

Thus

$$\begin{aligned}
\sum_{\underline{k}\in\Lambda^*-\{\underline{0}\}} \left|\mathrm{S}_{\underline{k}}(t) \exp\Big(\Big(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\big|\underline{k}\big|^2\Big)t\Big) \tfrac{e^{2\pi i \underline{k}\cdot\underline{x}}}{\sqrt{\mathrm{A}_\Sigma}}\right| &\leqslant \sum_{\underline{k}\in\mathrm{Z}_0} \left|\mathrm{S}_{\underline{k}}(t)\right| \exp\Big(\Big(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\big|\underline{k}\big|^2\Big)t\Big) \tfrac{1}{\sqrt{\mathrm{A}_\Sigma}} \\
&\quad + \sum_{n\in\mathbb{N}_+} \sum_{\underline{k}\in\hat{\mathrm{Z}}_n-\hat{\mathrm{Z}}_{n-1}} \left|\mathrm{S}_{\underline{k}}(t)\right| \exp\Big(\Big(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} - 4\pi^2\big|\underline{k}\big|^2\Big)t\Big) \tfrac{1}{\sqrt{\mathrm{A}_\Sigma}} \\
&\leqslant \frac{\mathrm{C}}{(\mathrm{T} - |t|)\sqrt{\mathrm{A}_\Sigma}} \exp\Big(\tfrac{\mathrm{M}}{\mathrm{A}_\Sigma} t\Big)\Big(\|\varrho_0\|_{\mathrm{L}^1(\Sigma,g)} + \sum_{n\in\mathbb{N}} |\hat{\mathrm{Z}}_n| t^n\Big)
\end{aligned}$$

$$\leqslant \frac{C}{(T-|t|)\sqrt{A_\Sigma}} \exp\left(\tfrac{M}{A_\Sigma} T\right) \left(\|\varrho_0\|_{L^1(\Sigma,g)} + C \frac{T^2}{(T-|t|)^3} \right).$$

Thus right-hand side of equation (5.5) is absolute convergent for all $t \in (-T, T)$. In particular

$$\begin{aligned}
\lim_{t\to 0^+} \varrho\big(t, \underline{x}\big) &= \varrho\big(0, \underline{x}\big) \\
&= \tfrac{M}{A_\Sigma} + \sum_{\underline{k}\in\Lambda^* - \{\underline{0}\}} S_{\underline{k}}(0) \exp\left(\left(\tfrac{M}{A_\Sigma} - 4\pi^2 \big|\underline{k}\big|^2\right) 0\right) \tfrac{e^{2\pi i \underline{k}\cdot\underline{x}}}{\sqrt{A_\Sigma}} \\
&= \tfrac{M}{A_\Sigma} + \sum_{\underline{k}\in\Lambda^* - \{\underline{0}\}} R_{\underline{k}}(0) \tfrac{e^{2\pi i \underline{k}\cdot\underline{x}}}{\sqrt{A_\Sigma}} \\
&= \varrho_0\big(\underline{x}\big),
\end{aligned}$$

which concludes the proof. □

6. Exact solutions

(Adam Mendenhall) University of California, Santa Barbara
Email address: amendenhall@ucsb.edu

(Ákos Nagy) University of California, Santa Barbara
Email address: contact@akosnagy.com
URL: akosnagy.com