1. GENERAL DISCUSSION

Let Σ be any smooth, compact surface (potentially with boundary), and let Δ be its Laplace operators. Let $\left(f_a \in \mathrm{L}^2(\Sigma)\right)_{a\in \mathrm{I}}$ be a complete, orthonormal set of eigenfunctions (thus I is countable infinite), that is $\Delta f_a = \lambda_a f_a$. Without any loss of generality, assume that $0 \in \mathrm{I}$ and f_0 is a constant. Then write

$$\varrho_t = \sum_{a\in \mathrm{I}} \sum_{n\in\mathbb{N}} \mathrm{R}_{n,a} t^n f_a.$$

Let

$$\varphi_{a,b,c} := \int_\Sigma f_a f_b f_c \mathrm{dA}.$$

If ϱ_t solves the Keller–Segel equation, then and $a \neq 0$, then

$$\mathrm{R}_{n+1,a} = \frac{\mathrm{R}_{0,0} - \lambda_a}{n+1} \mathrm{R}_{n,a} + \sum_{b,c\in \mathrm{I}-\{0\}} \sum_{m=0}^{n} \frac{\lambda_a - \lambda_b + \lambda_c}{(n+1)\lambda_c} \varphi_{a,b,c} \mathrm{R}_{b,m} \mathrm{R}_{c,n-m}.$$

Remark 1.1. *For all $\Lambda \in \mathbb{R}_+$, let*

$$n(\Lambda) := |\{\ a \in \mathrm{I} \mid \lambda_a \leqslant \Lambda\ \}|.$$

Weyl's law, that states that

$$\lim_{\Lambda\to\infty} \frac{\Lambda}{n(\Lambda)} = \frac{2}{\mathrm{Area}\left(\Sigma, g\right)}.$$

2. CONVERGENCE OF THE ITERATIVE METHOD ON THE STANDARD TORUS

Let us write

$$\varrho(x,t) = \sum_{\underline{k}\in\mathbb{Z}^2} \sum_{n\in\mathbb{N}} \mathrm{R}_{n,\underline{k}} t^n \exp\left(2\pi i\, \underline{k}\cdot\underline{x}\right).$$

If ϱ solves the Keller–Segel equation, then and $\underline{k} \neq \underline{0}$, then

$$\mathrm{R}_{n+1,\underline{k}} = \tfrac{\mathrm{R}_{0,\underline{0}} - 4\pi^2 |\underline{k}|^2}{n+1} \mathrm{R}_{n,\underline{k}} + \sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0},\underline{k}\}} \tfrac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} \sum_{m=0}^{n} \tfrac{1}{n+1} \mathrm{R}_{m,\underline{l}} \mathrm{R}_{n-m,\underline{k}-\underline{l}}. \tag{2.1}$$

Finally let

$$\forall \underline{k} \in \mathbb{Z}^2: \quad \mathrm{S}_{\underline{k}}(t) := \mathrm{R}_{\underline{k}}(t) \exp\left(4\pi^2 \left|\underline{k}\right|^2 t\right),$$

and write

$$\mathrm{S}_{\underline{k}}(t) = \sum_{n\in\mathbb{N}} \mathrm{S}_{n,\underline{k}} t^n.$$

Then, in terms of the $\mathrm{S}_{n,\underline{k}}$ coefficients, equation (2.1) becomes

$$\mathrm{S}_{n+1,\underline{k}} = \frac{1}{n+1} \sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0},\underline{k}\}} \tfrac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} \sum_{a=0}^{n} \sum_{b=0}^{n-a} \mathrm{S}_{a,\underline{l}} \mathrm{S}_{b,\underline{k}-\underline{l}} \frac{\left(8\pi^2 \left(\underline{k}-\underline{l}\right)\cdot\underline{l}\right)^{n-a-b}}{(n-a-b)!}. \tag{2.2}$$

If that the coefficients in equation (2.1) exist after the n^{th} step, then let

$$\mathrm{Z}_n := \left\{\ \underline{k} \in \mathbb{Z}^2\ \middle|\ \mathrm{S}_{n,\underline{k}} \neq 0\ \right\} \subseteq \mathbb{Z}^2.$$

Lemma 2.1. *Let us assume that Z_0 is finite and symmetric, that is $-\mathrm{Z}_0 = \mathrm{Z}_0$, and let*

$$d := \max\left(\left\{\ \left|\underline{k}\right|\ \middle|\ \underline{k} \in \mathrm{Z}_0\ \right\}\right) > 0.$$

Then for all $n \in \mathbb{N}_+$, the coefficients in equation (2.1) *exist and Z_n is finite and symmetric, moreover*

$$\mathrm{Z}_n \subseteq \mathrm{Z}_{n-1} + \mathrm{Z}_{n-1},$$

and

$$\max\left(\left\{\ \left|\underline{k}\right|\ \middle|\ \underline{k} \in \mathrm{Z}_n\ \right\}\right) = \max\left(\left\{\ \left|\underline{k}\right|\ \middle|\ \mathrm{S}_{n,\underline{k}} \neq 0\ \right\}\right) \leqslant 2^n d.$$

Proof. *Insert Adam's proof here.* ☐

For all $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$ let

$$\mathrm{N}(\underline{k}) := \min(\{ n \in \mathbb{N} \mid \underline{k} \in \mathrm{Z}_n \}).$$

Lemma 2.2. *Under the assumptions above, we have that*

$$\forall \underline{k} \in \mathbb{Z}^2 - \{\underline{0}\} : \forall n \in \mathbb{N} : \quad \left|\mathrm{S}_{n,\underline{k}}\right| < \epsilon^n. \tag{2.3}$$

Proof. By Lemma 2.1, we can rewrite equation (2.2) as

$$\mathrm{S}_{n+1,\underline{k}} = \frac{1}{n+1} \sum_{x=0}^{n} \sum_{\underline{l}:\mathrm{N}(\underline{k})=x} \frac{\underline{k}\cdot\underline{l}}{\left|\underline{l}\right|^2} \sum_{a=x}^{n} \sum_{b=0}^{n-a} \mathrm{S}_{a,\underline{l}} \mathrm{S}_{b,\underline{k}-\underline{l}} \frac{\left(8\pi^2 (\underline{k}-\underline{l})\cdot\underline{l}\right)^{n-a-b}}{(n-a-b)!}. \tag{2.4}$$

The inequality (2.3) is true for $n = 0$, so we can argue by induction.

Let us assume now that we have proved inequality (2.3) for every nonnegative integer up to $n \in \mathbb{N}$. Without any loss of generality, we can assume that $n > \mathrm{N}(\underline{k})$, then inequality (2.3) holds. Now using equation (2.4) we have:

$$\begin{aligned}
\left|\mathrm{S}_{n+1,\underline{k}}\right| &\leqslant \frac{1}{n+1} \sum_{x=0}^{n} \sum_{\underline{l}:\mathrm{N}(\underline{l})=x} \sum_{a=x}^{n} \sum_{b=0}^{n-a} \epsilon^{a+b} \frac{\left|8\pi^2 (\underline{k}-\underline{l})\cdot\underline{l}\right|^{n-a-b}}{(n-a-b)!} \frac{|\underline{k}\cdot\underline{l}|}{\left|\underline{l}\right|^2} \\
&\leqslant \frac{\epsilon^n}{n+1} \sum_{x=0}^{n} \sum_{\underline{l}:\mathrm{N}(\underline{l})=x} \sum_{a=0}^{n-x} a \frac{\left|8\pi^2 (\underline{k}-\underline{l})\cdot\underline{l}/\epsilon\right|^{n-x-a}}{(n-x-a)!} \frac{|\underline{k}\cdot\underline{l}|}{\left|\underline{l}\right|^2} \\
&\leqslant \frac{\epsilon^n}{n+1} \sum_{x=0}^{n} \sum_{\underline{l}:\mathrm{N}(\underline{l})=x} \sum_{a=0}^{n-x} (n-x-a) \frac{\left|8\pi^2 (\underline{k}-\underline{l})\cdot\underline{l}/\epsilon\right|^{a}}{a!} \frac{|\underline{k}\cdot\underline{l}|}{\left|\underline{l}\right|^2} \\
&\leqslant \frac{\epsilon^n}{n+1} \sum_{x=0}^{n} \sum_{a=0}^{n-x} (n-x-a) \sum_{\underline{l}:\mathrm{N}(\underline{l})=x} \frac{\left|8\pi^2 (\underline{k}-\underline{l})\cdot\underline{l}/\epsilon\right|^{a}}{a!} \frac{|\underline{k}\cdot\underline{l}|}{\left|\underline{l}\right|^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\left| \sum_{m=0}^{n} \sum_{\substack{\underline{l}\in\mathrm{Z}_m \\ \underline{k}-\underline{l}\in\mathrm{Z}_{n-m}}} \tfrac{1}{n+1} \mathrm{R}_{m,\underline{l}} \mathrm{R}_{n-m,\underline{k}-\underline{l}} \right| &\leqslant \sum_{m=0}^{n} \frac{\mathrm{C}^{n+2}}{(n+1)!} \exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right) \binom{n}{m} 2\pi 4^m 4^{n\ln(n)} \\
&\leqslant \frac{\mathrm{C}^{n+2}}{(n+1)!} \exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right) 4^{n\ln(n)} \sum_{m=0}^{n} \binom{n}{m} 2\pi 4^m \\
&= \frac{\mathrm{C}^{n+2}}{(n+1)!} \exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right) 4^{n\ln(n)} 2\pi \frac{5^{n+1}-1}{5-1}.
\end{aligned}$$

Inserting this to the outer sum, we get

$$\left| \sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0},\underline{k}\}} \tfrac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} \sum_{m=0}^{n} \tfrac{1}{n+1} \mathrm{R}_{m,\underline{l}} \mathrm{R}_{n-m,\underline{k}-\underline{l}} \right| \leqslant \frac{\mathrm{C}^{n+2}}{(n+1)!} \exp\left(-\tfrac{|\underline{k}|^2}{\mathrm{C}}\right) \sum_{\underline{l}\in\mathbb{Z}^2-\{0\}} \tfrac{|\underline{k}\cdot\underline{l}|}{|\underline{l}|^2} \left(\left|\underline{l}\right|^2 + \left|\underline{k}-\underline{l}\right|^2\right)^n \exp\left(\tfrac{2(\underline{k}-\underline{l})\cdot\underline{l}}{\mathrm{C}}\right)$$

$\square$