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New numerical method and application to Keller-Segel model with fractional order derivative

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ABSTRACT

Using the fundamental theorem of fractional calculus together with the well-known Lagrange polynomial interpolation, we constructed a new numerical scheme. The new numerical scheme is suggested to solve non-linear and linear partial differential equation with fractional order derivative. The method was used to solve numerically the time fractional Keller-Segel model. The existence and uniqueness solution of the model with fractional Mittag-Leffler kernel derivative are presented in detail. Some simulations are performed to access the efficiency of the newly proposed method.

1. Introduction

In the last past years, several lives have been taken away due to cancer diseases. No wonder therefore why scientists in several fields are devoted to find a suitable to stop such fatal disease amount mankind. An effort is being doing by chemists to understanding the chemical reaction of this disease while mathematicians are interested in understand the dynamical system underpinning the spread of this disease. One-way to understanding this, mathematicians rely on ordinary differential and partial differential equations to build a mathematical able to replicate the observed facts. One of the first pioneers of this model is perhaps the Lee Aaron Segel who suggested the model of chemo taxis. For those do not know the scientific mean of cancer, we recall that, it is a group of diseases connecting nonconventional cell growth with a possibility of invading or spreading to other parts of the body for clear understanding of this process, we request readers to read through the following references [21–25]. Also we recall that, chemo taxis is a scientific word that is an addition of two others including chemo and taxis, meaning the diffusion of an organism in response to chemical stimulus more details please visit the following work [26–29]. Biologically speaking, somatic cells bacteria and other single/multicellular beings undeviating their diffusion accordingly to some chemical within their environment. It has been reported that, the mechanisms that tolerate chemo taxis in animals can also be destabilised during cancer metastasis. We note that, positive chemo taxis is observed if the diffusion is toward a higher concentration of the mentioned chemical, while on the other hand negative chemo taxis is identified when the movement is in the opposite direction. Using a capillary tube assay one can model the process of chemo taxis. With their biological instinct, motile prokaryotes are able to sense chemicals in their vicinities then, alter their motility consequently. The randomness is observed when no chemicals are present and the repellent chemical is present, the motility also change, runs become longer and tumbles because less frequent so that diffusion in the direction or in opposite direction to the chemical can be reached. Thus, the net diffusion can be considered. Thus, the diffusion can be considered in the beaker, where bacteria amass around the attractant and away from the repellent.

The concept of anomalous diffusion cannot be captured with classical differential operator as have been indicated in several already published studies. This is due to the fact that, the process of the spread does not follow a conventional pattern, and such pattern cannot be capturing using the concept of rate of change. It was suggested that only nonconventional or non-local differential operators are better candidates to capture such processes. However, due to the change in states during the diffusion of cancer within a human body, even those non-local differential operators based on power-law decay kernel cannot as they do not have ability if crossover either in waiting time distribution, mean square displacement and also density probability. Thus one will therefore rely on those differential operators with crossover behaviour and the better candidates are nothing more than the Atangana-

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Baleanu fractional differential operators [27,28]. Noting that, the mathematical model able to replicate this spread as suggested by Segel is highly non-linear, the model can only be solved using either numerical methods or iterative methods. Within the theoretical framework of the applications of these differential operators to modeling real world problem, some interesting results have been obtained and these results are very important when dealing with applications, we can quote the following works [30–42].

Within the passed years, many iterative methods have been introduced in order to take care of nonlinear partial differential equation with integer and non-integer order derivatives [1–6]. However, the problems faced by these methods are perhaps, the stability, the convergency and the ability of these methods to handle strong non-linearities. Methods like homotopy perturbation method, homotopy decomposition method, variational iterative method; Laplace/Sumudu perturbation method and Adomian decomposition method have been used and applied extensively within the passed decade [7–10]. Nevertheless, all these methods are not satisfactory due to their inabilities to secure the stability and the convergency for non-linear equations. Their limitations can be visible in the field of chaos as small change in the initial input can affect greatly the outputs. On the other hand a very powerful mathematical tools known as Adams-Bashforth method for solving nonlinear ordinary differential equations has been introduced, this method is very efficient and converges toward exact solution must of the times [11–15]. But this method has some limitations it is only application for ordinary differential equations and in addition, one may one to add the predictor-corrector to insure accuracy. However, the method was recognized as powerful numerical method for solving chaotic's, epidemiologic, and nonlinear models that arise in biology, chemistry, mechanic, technology and engineering. Thus there is a need of such numerical scheme for solving non-linear partial differential equations with no requirement corrector-predictor method. The new method will therefore be used to solve numerically the Keller-Segel model with Atangana-Baleanu fractional differentiation in Caputo sense.

2. Existence and uniqueness of Keller-Segel model with Atangana-Baleanu derivative

In this section the Keller-Segel model describing the aggregation advancement of cellular slime mold by chemical attraction. The mathematical model has been studied by several authors within the scope of classical differentiation and also fractional differentiation but with Riemann-Liouville and Caputo derivative. However, recently due the limitations of the power law kernel used in Riemann-Liouville derivative, Caputo and Fabrizio [16–20] suggested a different kernel, exponential decay law, this kernel although not non-local but has the property that it portrays fading memory which is observed in several physical problems in nature. To solve the problem on non-locality, Atangana and Baleanu suggested [19] new fractional derivatives with Mittag-leffler kernel. In this section we consider the following fractional non-linear model.

$$\begin{cases} {}_0^{ABC}D_t^{\alpha}u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,t))}{\partial x}\right), \\ {}_0^{ABC}D_t^{\alpha}\rho(x,t) = b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t), \\ u(x,0) = f(x), \ \ \rho(x,0) = g(x), \\ {}_a^{ABC}D_t^{\alpha}u(x,t) = \frac{AB(\alpha)}{1-\alpha}\int\limits_0^t \frac{\partial u(x,l)}{\partial l}E_\alpha\left[-\alpha\frac{(t-l)^\alpha}{1-\alpha}\right]dl. \end{cases} \tag{1}$$

We shall present the detailed proof of existence and uniqueness of the exact solution using the fixed-point method.

2.1. Existence and uniqueness

To achieve the existence and uniqueness of exacts solution of Eq. (1), we apply on both sides the fractional integral, thus by the fundamental theorem of fractional calculus, we obtain:

$$\begin{cases} u(x,t) - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\left(a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial\eta(u(x,t))}{\partial x}\right)\right) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t\left(a\frac{\partial^2 u(x,l)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial\eta(u(x,l))}{\partial x}\right)\right)(t-l)^{\alpha-1}dl, \\ \rho(x,t) - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\int\limits_0^t b\frac{\partial^2\rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t b\frac{\partial^2\rho(x,l)}{\partial x^2} + cu(x,l) - d\rho(x,l)(t-l)^{\alpha-1}dl. \end{cases} \tag{2}$$

For the sake of simplicity, we put

$$\begin{cases} \pi_1(x,t,u) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial\eta(u(x,t))}{\partial x}\right) \\ \pi_2(x,t,\rho) = b\frac{\partial^2\rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t) \end{cases} \tag{3}$$

Thus Eq. (2) become

$$\begin{cases} u(x,t) - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,u) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \\ \qquad \int_0^t \pi_1(x,l,u)(t-l)^{\alpha-1}dl, \\ \rho(x,t) - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\rho) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \\ \qquad \int_0^t \pi_2(x,l,\rho)(t-l)^{\alpha-1}dl. \end{cases} \tag{4}$$

To start we consider the infinite norm defines and also

$$\|\psi(t)\|_\infty = \sup_{t\in I}|\psi(t)| \tag{5}$$

$$M_1 = \max\left\{\pi_1(x,t,u)_{(x,t)\in I\times J}\right\}$$

$$M_2 = \max\left\{\pi_2(x,t,\rho)_{(x,t)\in I\times J}\right\}$$

We now consider the following fractional operators

$$\begin{cases} N\Phi - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,\Phi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl, \\ \Delta\Xi - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\Xi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl. \end{cases} \tag{6}$$

We show that the above fractional operators are well defined; to achieve this we apply the infinite norm

$$\begin{cases} \|N\Phi - u(x,0)\|_\infty = \left\|\begin{matrix}\frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,\Phi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl\end{matrix}\right\|_\infty, \\ \|\Delta\Xi - \rho(x,0)\|_\infty = \left\|\begin{matrix}\frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\Xi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl\end{matrix}\right\|_\infty. \end{cases}$$

$$\begin{cases} \|N\Phi - u(x,0)\|_\infty \le \frac{1-\alpha}{AB(\alpha)}\|\pi_1(x,t,\Phi)\|_\infty \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\left\|\int\limits_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl\right\|_\infty, \\ \|\Delta\Xi - \rho(x,0)\|_\infty \le \frac{1-\alpha}{AB(\alpha)}\|\pi_2(x,t,\Xi)\|_\infty \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\left\|\int\limits_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl\right\|_\infty. \end{cases} \tag{7}$$

Thus using the max of the defined kernel, we obtained

$$\begin{cases} \|N\Phi - u(x,0)\|_\infty \le \frac{1-\alpha}{AB(\alpha)}M_1 + \frac{M_1T_{\max}^\beta}{AB(\alpha)\Gamma(\alpha)}, \\ \|\Delta\Xi - \rho(x,0)\|_\infty \le \frac{1-\alpha}{AB(\alpha)}M_2 + \frac{M_2T_{\max}^\beta}{AB(\alpha)\Gamma(\alpha)} + . \end{cases} \tag{8}$$

We define b the upper bound of I and $T_{\max}$ to be the upper bound of J, thus the functions are well defined if the following inequalities are reached.

$$T_{\max} < \min_{\alpha\in(0,1)}\left\{\left(\frac{b-\frac{1-\alpha}{AB(\alpha)}}{\frac{M_1}{AB(\alpha)\Gamma(\alpha)}}\right)^{\frac{1}{\alpha}},\left(\frac{b-\frac{1-\alpha}{AB(\alpha)}}{\frac{M_2}{AB(\alpha)\Gamma(\alpha)}}\right)^{\frac{1}{\alpha}}\right\}. \tag{9}$$

We next prove that the function N, Ξ are Lipchitz for the variables u and ρ respectively and also find the conditions of contraction.

Theorem 1. *Let us suppose that the function is bounded with infinite norm such that, one can find a positive constant verifying:*

$$\left\|u\frac{\partial\eta(u)}{\partial x}-v\frac{\partial\eta(u)}{\partial x}\right\|_\infty < \chi\|u-v\|_\infty \tag{10}$$

Thus the function $N(x, t, u)$ has Lipchitz property a.

Proof. Let u, v be two different functions from $C_{T_{\max,b}} = I\times J$ then,

$$\begin{gathered}\|N(x,t,u)-N(x,t,v)\|_\infty \\ =\left\|\begin{array}{l}\frac{1-\alpha}{AB(\alpha)}(\pi_1(x,t,u)-\pi_1(x,t,v))+\\ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t(\pi_1(x,t,u)-\pi_1(x,t,v))(t-l)^{\alpha-1}dl\end{array}\right\|_\infty, \\ \|N(x,t,u)-N(x,t,v)\|_\infty \\ =\left\|\begin{array}{l}\frac{1-\alpha}{AB(\alpha)}\left(a\frac{\partial^2(u-v)}{\partial x^2}-\frac{\partial}{\partial x}\left(u\frac{\partial\eta u}{\partial x}-v\frac{\partial\eta v}{\partial x}\right)\right)+\\ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t\left(a\frac{\partial^2(u-v)}{\partial x^2}-\frac{\partial}{\partial x}\left(u\frac{\partial\eta u}{\partial x}-v\frac{\partial\eta v}{\partial x}\right)\right)(t-l)^{\alpha-1}dl\end{array}\right\|_\infty.\end{gathered} \tag{11}$$

Using the triangular inequality and also the theorem hypothesis, the above is converted to

$$\begin{gathered}\|N(x,t,u)-N(x,t,v)\|_\infty \\ <\left[\frac{1-\alpha}{AB(\alpha)}\left(O^2a+\chi\right)+\frac{\chi\left(O^2a+\chi\right)T_{\max}^\alpha}{AB(\alpha)\Gamma(\alpha)}\right]\|u-v\|_\infty \\ \|N(x,t,u)-N(x,t,v)\|_\infty < K\|u-v\|_\infty\end{gathered}$$

And the function N will be a contraction if

$$T_{\max} < \left[\frac{AB(\alpha)\Gamma(\alpha)}{\chi}\left(\frac{1}{\left(O^2a+\chi\right)}+\frac{\alpha-1}{AB(\alpha)}\right)\right]^{\frac{1}{\alpha}}. \tag{12}$$

Similarly, the function Ξ is a contraction if

$$T_{\max} < \left[AB(\alpha)\Gamma(\alpha)\left(\frac{1}{\left(O_1^2b+d\right)}+\frac{\alpha-1}{AB(\alpha)}\right)\right]^{\frac{1}{\alpha}}. \tag{13}$$

System of equation has a unique solution if and only if the following inequality is satisfied

$$T_{\max} < \min_{\alpha\in(0,1)}\left\{\begin{array}{c}\left(\frac{b-\frac{1-\alpha}{AB(\alpha)}}{\frac{M_1}{AB(\alpha)\Gamma(\alpha)}}\right)^{\frac{1}{\alpha}},\left(\frac{b-\frac{1-\alpha}{AB(\alpha)}}{\frac{M_2}{AB(\alpha)\Gamma(\alpha)}}\right)^{\frac{1}{\alpha}}\\ \left[\frac{AB(\alpha)\Gamma(\alpha)}{\chi}\left(\frac{1}{(O^2a+\chi)}+\frac{\alpha-1}{AB(\alpha)}\right)\right]^{\frac{1}{\alpha}},\left[AB(\alpha)\Gamma(\alpha)\left(\frac{1}{(O_1^2b+d)}+\frac{\alpha-1}{AB(\alpha)}\right)\right]^{\frac{1}{\alpha}}\end{array}\right\}.$$

□

3. Derivation of new numerical scheme

Recently the literature has received an impressive numbers of research papers in which the well-known Adams-Basforth numerical method was used to solve linear, non-linear, and ordinary equation with local and non-local derivatives. The method is proven to be a powerful mathematical tool to solve non-linear ordinary differential equation. Up to 2017 March, the method was not applicable for partial differential equation; recently, Atangana and Batogna suggested an extended version of Adams-Basforth to partial differential using Laplace transform, Lagrance polynomial interpolation and the forward/backward scheme. In this paper we will propose an alternative numerical scheme, easier to be implemented.

Let us consider a general partial differential equation with Atangana-Baleanu fractional differentiation as follows

$${}_0^{ABC}D_t^\alpha u(x,t) = Lu(x,t)+Nu(x,t). \tag{14}$$

Where L and N are linear and non-linear operators respectively. The derivation of the method consists of applying the Laplace transform/ Sumudu transform on both sides of equation in x-direction to obtain:

$$\begin{aligned} l\left({}_0^{ABC}D_t^\alpha u(x,t)\right)(p) &= l(Lu(x,t)+Nu(x,t))(p),\\ {}_0^{ABC}D_t^\alpha U(p,t) &= l(Lu(x,t)+Nu(x,t))(p) = F(t,p,U),\\ {}_0^{ABC}D_t^\alpha U(p,t) &= F(t,p,U).\end{aligned} \tag{15}$$

We then apply the fundamental theorem of fractional calculus on the last equation of the above to obtain

$$\begin{aligned}U(t)-U(0) &= \frac{(1-\alpha)}{AB(\alpha)}F(t,U)\\ &\quad+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t F(U,y)(t-y)^{-\alpha}dy.\end{aligned} \tag{16}$$

Thus at t_n we have the following

$$\begin{aligned}U(t_n)-U(0) &= \frac{(1-\alpha)}{AB(\alpha)}F(t_{n-1},U)\\ &\quad+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{t_n}F(U,y)(t_n-y)^{-\alpha}dy,\\ &= \frac{(1-\alpha)}{AB(\alpha)}F(t_{n-1},U)\\ &\quad+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\left(\int\limits_{t_j}^{t_{j+1}}F_j(U,y)(t_n-y)^{-\alpha}dy\right)\end{aligned} \tag{17}$$

At the right hand side, we have for the first function as it was suggested by in the method, this is suggested to avoid having implicit scheme. Using the Lagrange interpolation for F, we obtain

$$F_j(u,t)\approx P_j(t) = \frac{t-t_{j-1}}{t_j-t_{j-1}}F(t_j,U_j)+\frac{t-t_j}{t_j-t_{j-1}}F(t_{j-1},U_{j-1}). \tag{18}$$

Replacing the above in Eq. (17), we obtain

$$\begin{aligned}U_n-U_0 &= \frac{(1-\alpha)}{AB(\alpha)}F(t_{n-1},U)\\ &\quad+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left[\begin{array}{l}\frac{t-t_{j-1}}{t_j-t_{j-1}}F(t_j,U_j)\\ +\frac{t-t_j}{t_j-t_{j-1}}F(t_{j-1},U_{j-1})\end{array}\right](t_n-y)^{-\alpha}dy,\end{aligned}$$

We note that the Lagrange interpolation is done within the following interval $[t_j, t_{j+1}]$, thus integrating we obtain the following

$$\begin{aligned}U_n &= U_0+\frac{(1-\alpha)}{AB(\alpha)}F(t_{n-1},U)\\ &\quad+\frac{\alpha}{AB(\alpha)}\sum_{j=1}^{n}\left[\begin{array}{l}\frac{h^\alpha}{\Gamma(\alpha+2)}F(t_{j-1},U_{j-1})\left\{\begin{array}{l}(n-k+1)^\alpha(n-k+2+\alpha)\\-(n-k)^\alpha(n-k+2+2\beta)\end{array}\right\}\\ -\frac{h^\alpha}{\Gamma(\alpha+2)}F(t_{j-1},U_{j-1})\left\{\begin{array}{l}(n-k+1)^{\alpha+1}\\-(n-k)^\alpha(n-k+\alpha+1)\end{array}\right\}\end{array}\right]\end{aligned} \tag{19}$$

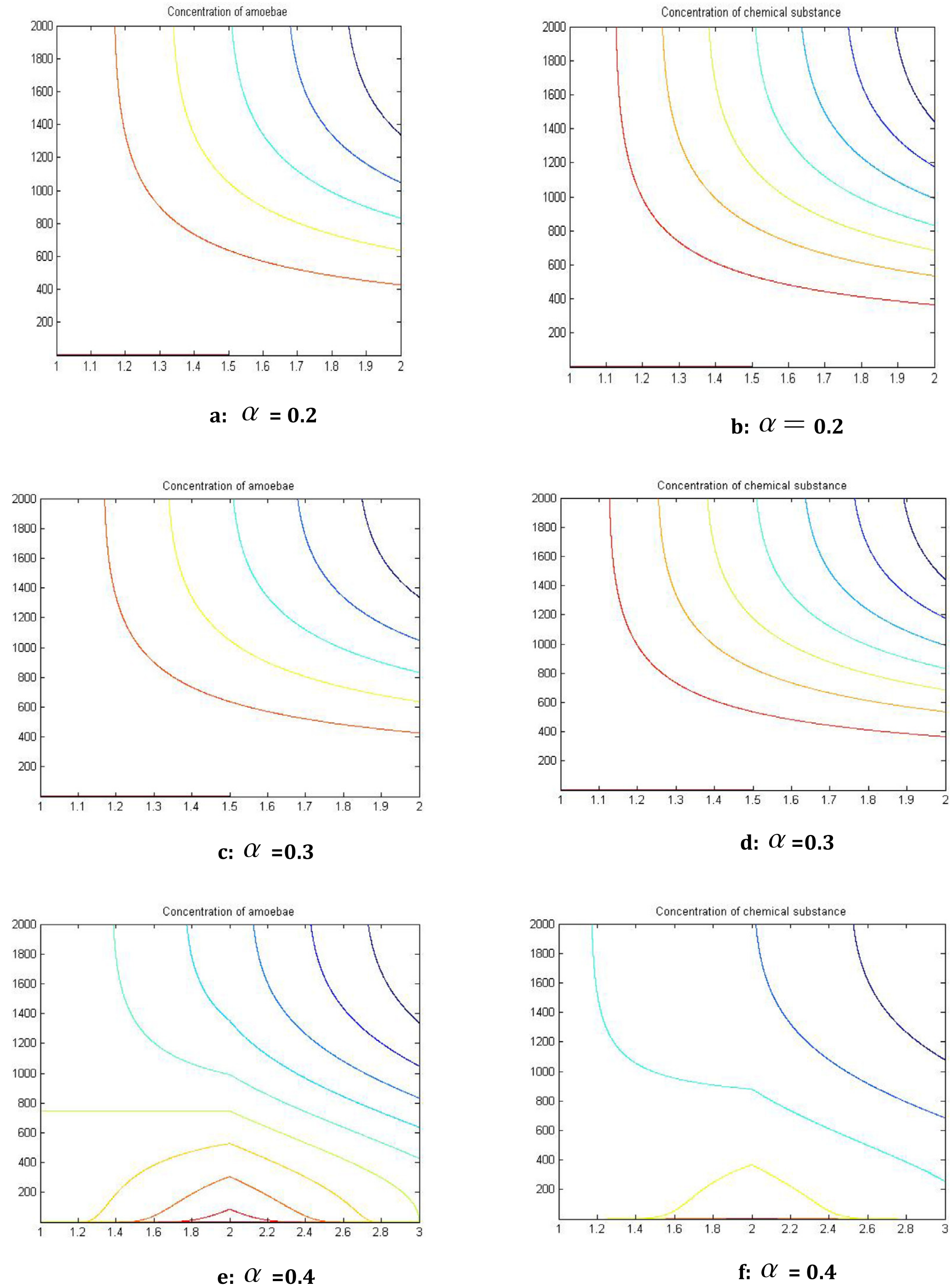

Fig. 1. Numerical simulations for fractional order from 0.2 to 0.4.

Thus to finish, we applied the inverse Laplace transform on both sides to obtain

$$u_n(x,t) = u_0(x,t) + \frac{(1-\alpha)}{AB(\alpha)} F(t_{n-1}, u(x,t)) + \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{j-1}, u_{j-1}(x,t)) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{j-1}, u_{j-1}(x,t)) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \tag{20}$$

The above equation can be later discritised in space using forward Euler or backward Euler. Thus for forward we have:

$$u_n^i = u_0^i + \frac{(1-\alpha)}{AB(\alpha)} F^i(t_{n-1}, u_{n-1}) + \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \tag{21}$$

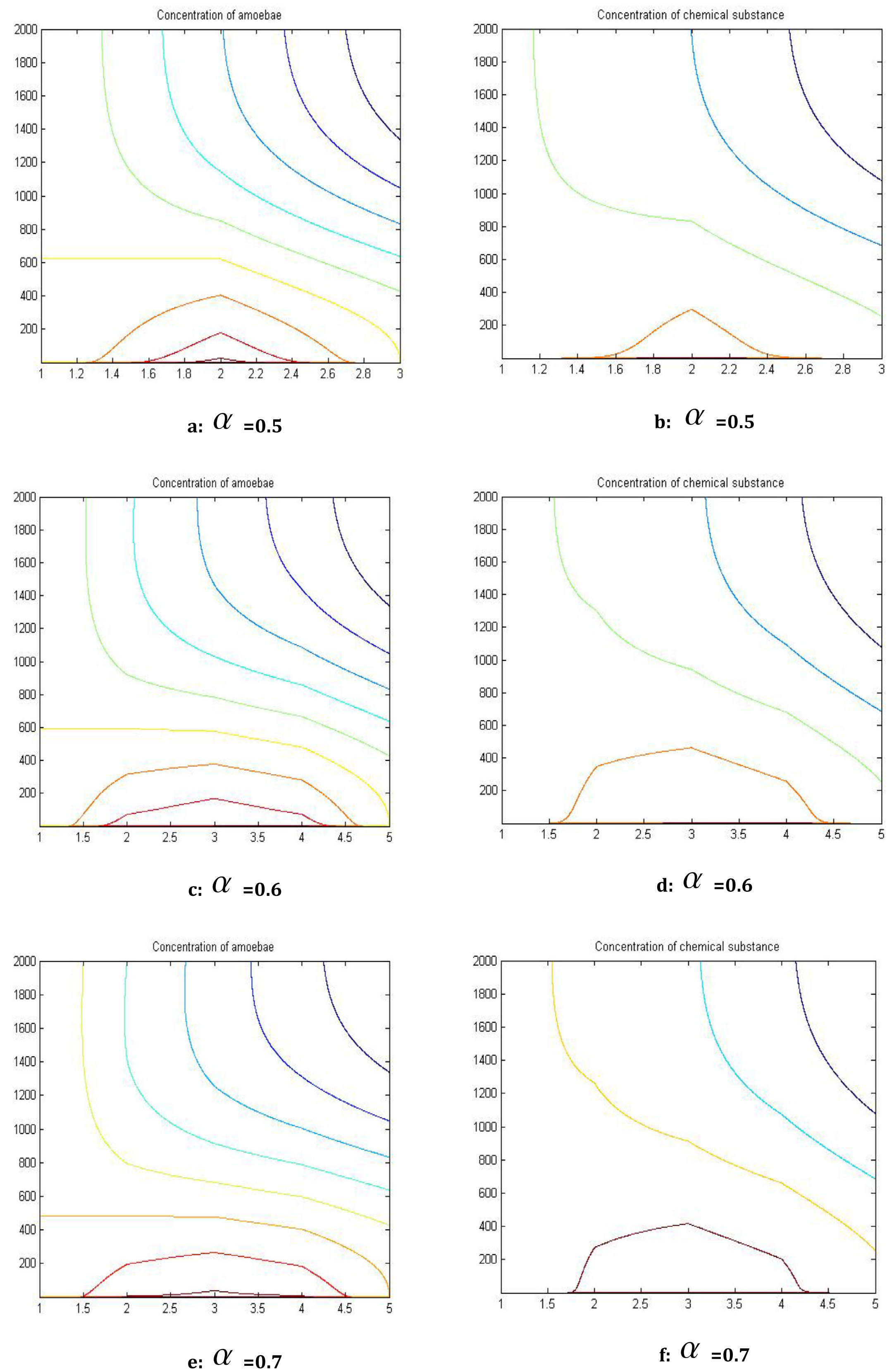

Fig. 2. Numerical simulation for alpha from 0.4 to 0.7.

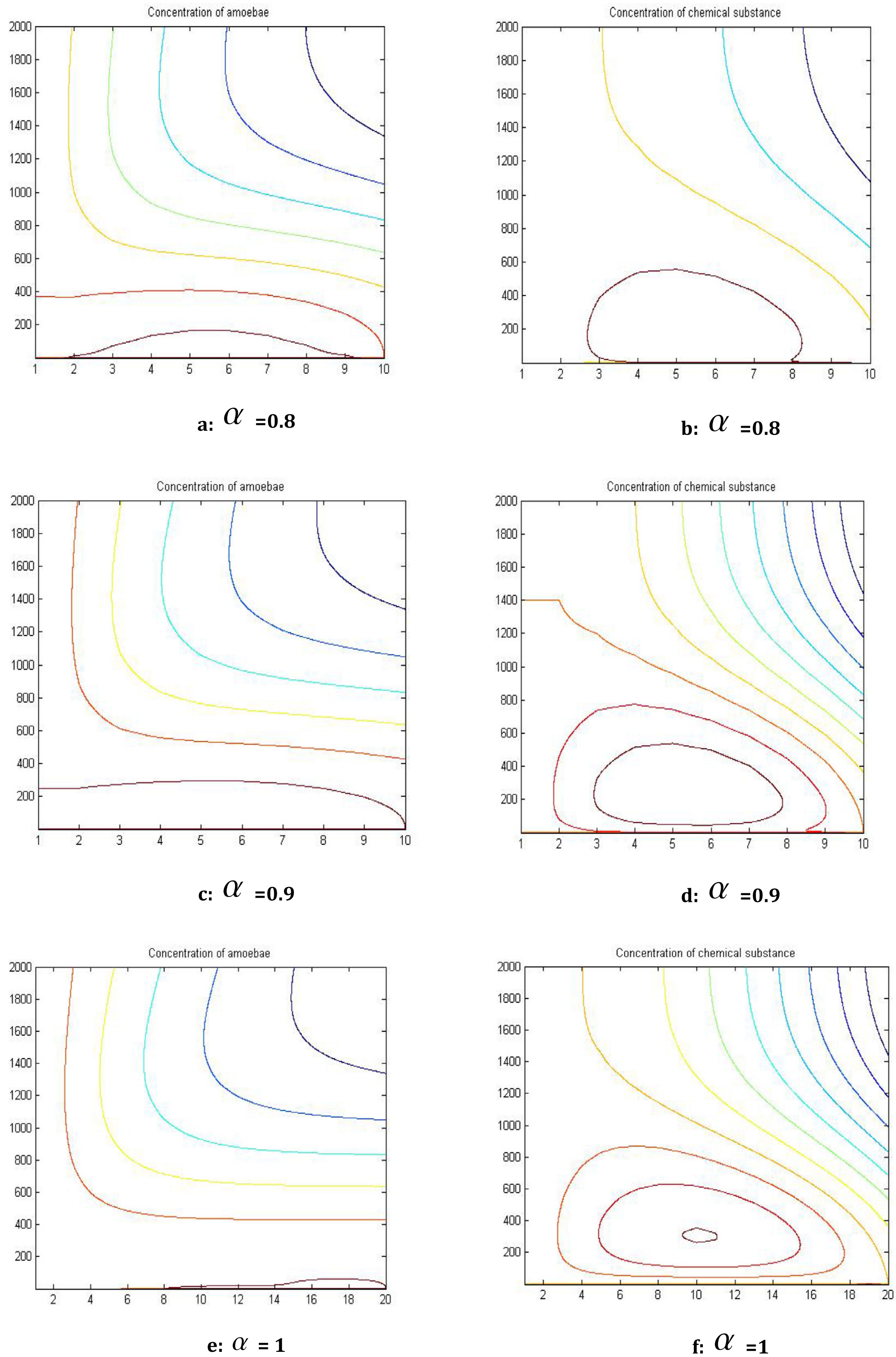

Fig. 3. Numerical simulation for alpha 0.8 to 1.

For backward we have

$$u_n(x,t) = u_0(x,t) + \frac{(1-\alpha)}{AB(\alpha)} F^{i+1}(t_{n-1}, u_{n-1}(x,t))$$
$$+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^{\alpha}}{\Gamma(\alpha+2)} F^{i+1}(t_{j-1}, u_{j-1}(x,t)) \left\{ \begin{array}{l} (n-k+1)^{\alpha}(n-k+2+\alpha) \\ -(n-k)^{\alpha}(n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^{\alpha}}{\Gamma(\alpha+2)} F^{i+1}(t_{j-1}, u_{j-1}(x,t)) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^{\alpha}(n-k+\alpha+1) \end{array} \right\} \end{array} \right] \tag{22}$$

The following theorem provides the error or the boundedness of the remainder

Theorem 2. *Let u(x, t) be n−times differentiable respect to time, thus a nonlinear partial differential equation with Atangana-Baleanu fractional derivative can be discritised as following:*

$$\begin{aligned} u_n^i &= u_0^i + \frac{(1-\alpha)}{AB(\alpha)} F^i(t_{n-1}, u_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \\ &+ R_n^{\alpha,k} + O(l^n + h^\alpha), \\ \left\| R_n^{\alpha,k} \right\|_\infty &< M < \infty. \end{aligned} \tag{23}$$

For backward we have

$$\begin{aligned} u_n^{i+1} &= u_0^{i+1} + \frac{(1-\alpha)}{AB(\alpha)} F^{i+1}(t_{n-1}, u_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F^{i+1}(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F^{i+1}(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \\ &+ R_n^{\alpha,k} + O(l^n + h^\alpha), \\ \left\| R_n^{\alpha,k} \right\|_\infty &< M < \infty. \end{aligned} \tag{24}$$

Proof. Without lost of generality, we consider forward case, thus using the suggested methodology; Eq. (1) can be discritized as:

$$\begin{aligned} u_n &= u_0 + \frac{(1-\alpha)}{AB(\alpha)} F(t_{n-1}, u_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(y-t_k)(y-t_{k+1})}{2!} \left[\frac{\partial^2 f(x,y,u)}{\partial y^2} \right]_{y=g} (t_n - y)^{\alpha-1} dy \end{aligned} \tag{25}$$

Next applying the forward scheme in x-direction, we obtained

$$\begin{aligned} u_n^i &= u_0^i + \frac{(1-\alpha)}{AB(\alpha)} F^i(t_{n-1}, u_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} F^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] \\ &+ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(y-t_k)(y-t_{k+1})}{2!} \left[\frac{\partial^2 f(x,y,u)}{\partial y^2} \right]_{y=g} (t_n - y)^{\alpha-1} dy \\ &+ O(l^n + h^\alpha) \end{aligned} \tag{26}$$

Take

$$R_n^{\alpha,k} = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(y-t_k)(y-t_{k+1})}{2!} \left[\frac{\partial^2 f(x,y,u)}{\partial y^2} \right]_{y=g} \times (t_n - y)^{\alpha-1} dy$$

Thus applying norm on both sides results

$$\left\| R_n^{\alpha,k} \right\| = \left\| \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(y-t_k)(y-t_{k+1})}{2!} \left[\frac{\partial^2 f(x,y,u(x,y))}{\partial y^2} \right]_{y=g} (t_n - y)^{\alpha-1} dy \right\|$$

$$< \frac{h^{\alpha+2}}{2AB(\alpha)\Gamma(\alpha+1)} \max_{[0,t_n]} \left\| \frac{\partial^2 f(x,t,u)}{\partial t^2} \right\| \sum_{k=0}^{n-1} \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n+1-k)^\alpha (n-k+2+\alpha) \end{array}$$

$$< \frac{h^{\alpha+2}}{2AB(\alpha)\Gamma(\alpha+1)} \max_{[0,t_n]} \left\| \frac{\partial^2 f(x,t,u)}{\partial t^2} \right\| \frac{n(n+4+2\alpha)}{2}$$

Thus the requested result is obtained. □

4. Application to Keller-Segel model

In this section, the novel numerical scheme suggested in this paper will be used to solve the nonlinear system of partial fractional differential equation representing the model of interaction via the process of chemotaxis. To situate the readers, we shall note that, in biology several organisms for instance, bacteria, amoebea, cells and others, or social insects including ants and swarms interact via the process of chemotaxis. More importantly, the word chemotaxis refers to a long-range correspondence that considers the orientation of individuals besides chemical signals that reproduce themselves. The model under investigation here consists of two coupled fractional differential equations with the Atangana-Baleanu fractional differentiation and govern the progress of the density of cells represented by and the progress of the secreted chemical. Therefore to apply the new method to solve this system, we consider

$$\begin{aligned} U(t) - U(0) &= F(t,u) = L\left[a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,t))}{\partial x} \right)(p) \right] \\ P(t) - P(0) &= S(t,\rho,u) = L\left[b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t) \right] \end{aligned} \tag{27}$$

Thus applying the new numerical scheme on the above we get:

$$\begin{aligned} u_n^i &= u_0^i + \frac{(1-\alpha)}{AB(\alpha)} f^i(t_{n-1}, u_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} f^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} f^i(t_{j-1}, u_{j-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] + \\ \rho_n^i &= \rho_0^i + \frac{(1-\alpha)}{AB(\alpha)} s^i(t_{n-1}, u_{n-1}, \rho_{n-1}) \\ &+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} \frac{h^\alpha}{\Gamma(\alpha+2)} g^i(t_{j-1}, u_{j-1}, \rho_{n-1}) \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \\ -\frac{h^\alpha}{\Gamma(\alpha+2)} g^i(t_{j-1}, u_{j-1}, \rho_{n-1}) \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\} \end{array} \right] + \end{aligned} \tag{28}$$

For simplicty, we put

$$a_1 = \frac{h^\alpha}{\Gamma(\alpha+2)} \left\{ \begin{array}{l} (n-k+1)^\alpha (n-k+2+\alpha) \\ -(n-k)^\alpha (n-k+2+2\beta) \end{array} \right\} \tag{29}$$

$$a_2 = \frac{h^\alpha}{\Gamma(\alpha+2)} \left\{ \begin{array}{l} (n-k+1)^{\alpha+1} \\ -(n-k)^\alpha (n-k+\alpha+1) \end{array} \right\}$$

Thus Eq. (28) can be converted to

$$u_n^i = u_0^i + \frac{(1-\alpha)}{AB(\alpha)} f^i(t_{n-1}, u_{n-1}) \tag{30}$$

$$+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} a_1 f^i(t_{j-1}, u_{j-1}) - \\ a_2 f^i(t_{j-1}, u_{j-1}) \end{array} \right]$$

$$\rho_n^i = \rho_0^i + \frac{(1-\alpha)}{AB(\alpha)} s^i(t_{n-1}, u_{n-1}, \rho_{n-1}) +$$

$$+ \frac{\alpha}{AB(\alpha)} \sum_{j=1}^{n} \left[\begin{array}{l} a_1 g^i(t_{j-1}, u_{j-1}, \rho_{n-1}) - \\ a_2 g^i(t_{j-1}, u_{j-1}, \rho_{n-1}) \end{array} \right]$$

The above numerical scheme can now be used to show the numerical solution of the studied system for different values of alpha and for given set of parameters inputs. This will be done in the next section.

5. Numerical simulations

In this section, we present the numerical solutions of the Keller-Segel model with non-local fading memory induced by the Atangana-Baleanu fractional differentiation. The numerical simulations are depicted in the following figures for different values of fractional order.

The numerical solutions are thus represented in Figs. (1,2,3) a, (1,2,3) b, (1,2,3) c and (1,2,3) d different values of fractional orders alpha. The numerical simulations are providing the spread within 2 different layers, when the fractional order is closer to 0.9, one can see that there are two different spreads that occur these is nothing more than the expression of crossover behaviors.

6. Conclusion

A reliable, efficient and friendly user numerical method for solving fractional partial differential equations with non-local fading memory was derived in this paper. The scheme consists of applying any integral transform on a given partial differential equation with fading memory to reduce the equation to ordinary differential equation. The second step if to use the Lagrange polynomial and then take the inverse integral operator and apply the finite difference method. The Keller Segel model with non-local fading memory was considered in this paper. The existence and uniqueness were studied in detail. The new numerical scheme was applied to the non-local fading memory system and some numerical simulations for different alphas were presented.

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References

[1] He J-H. Homotopy perturbation technique. Comput Methods Appl Mech Eng 1999;178(3–4):257–62.
[2] Matinfar M, Ghanbari M. The application of the modified variational iteration method on the generalized Fisher's equation. J Appl Math Comput 2008;31(1–2):165–75.
[3] Kurulay M, Secer A, Akinlar MA. A new approximate analytical solution of kuramoto-Sivashinsky equation using homotopy analysis method. Appl Math Inform Sci 2013;7(1):267–71.
[4] Yıldırım A. An algorithm for solving the fractional nonlinear schrödinger equation by means of the homotopy perturbation method. Intern J Nonlin SciNumer Simul 2009;10(4):445–50.
[5] Ganji DD, Rafei M. Solitary wave solutions for a generalized hirota-Satsuma coupled KdV equation by homotopy perturbation method. Phys Lett A 2006;356(2):131–7.
[6] Zeng Q, Qin YM. The laplace-Adomian-Pade technique for the seepage flows with the riemann-Liouville derivatives. Comm Fract Calculus 2012;3:26–9.
[7] Duan S, Rach R, Bulean D, Wazwaz AM. A review of the adomian decomposition method and its applications to fractional differential equations. Comm Fract Calculus 2012;3:73–99.
[8] Watugala GK. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. Int J Math Educ Sci Technol 1993;24(1):35–43.
[9] Weerakoon S. Application of sumudu transform to partial differential equations. Int J Math Educ Sci Technol 1994;25(2):277–83.
[10] Atangana A, Alabaraoye E. Solving system of fractional partial differential equations arisen in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equation. Adv Dif Eq 2013;2013.
[11] Majid ZA, See PP, Suleiman M. Solving directly two point non linear boundary value problems using direct adams moulton method. J Math Stat 2011;7(2):124–8.
[12] Ehigie J.O., Okunuga S.A., Sofoluwe A.B. 3-Point block methods for direct integration of general second-order ordinary differential equations. Adv Num Anal 2011. Article ID 51314814.
[13] Martín A, García O. Variable multistep methods for higher-order delay differential equations. Math Comput Model 2002;36(7–8):805–20.
[14] Ishak F, Suleiman MB, Omar Z. Two-point predictor-corrector block method for solving delay differential equations. Matematika 2008;24(2):131–40.
[15] El-Safty A. Approximate solution of the delay differential equation y ″ =f(x,y(x),y(α(x))) with cubic spline functions. Bullet Faculty Sci Assiut Uni 1993;22:67–73.
[16] Hristov J. Transient heat diffusion with a non-singular fading memory: from the cattaneo constitutive equation with Jeffrey's kernel to the caputo-fabrizio time-fractional derivative. Therm Sci 2016;20(2):757–62.
[17] Ahmood WA, Kiliçman A. On some applications of the multi-dimensional new fractional calculus for the riemann-Liouville with atangana-Baleanu. Intern J Pure ApplMath 2016;111(3):373–80.
[18] Atangana A, Koca I. Chaos in a simple nonlinear system with atangana–Baleanu derivatives with fractional order. Chaos, Solitons and Fractals 2016;89:447–54.
[19] Atangana A, Dumitru B. New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. Therm Sci 2016;20(2):763–9.
[20] Alkahtani BST. Chua'S circuit model with atangana–Baleanu derivative with fractional order. Chaos, Solitons & Fractals 2016;89:547–51.
[21] Alkahtani BST. Chua'S circuit model with atangana–Baleanu derivative with fractional order. Chaos, Solitons & Fractals 2016;89:547–51.
[22] Keller EF, Segel LA. Initiation of slime mold aggregation viewed as an instability. J Theor Biol. March. 1970;26(3):399–415.
[23] Keller EF, Segel LA. Model for chemotaxis. J Theor Biol 1971;30:225–34.
[24] Hillen T, Painter KJ. A user's guide to PDE models for chemotaxis. J Math Biol. Jan. 2009;58(1=2):183–217.
[25] Segel LA, Jackson JL. Dissipative structure: an explanation and an ecological example. J Theor Biol 1972;37:545–59.
[26] Segal LA. Distant sidewalls cause slow amplitude modulation of cellular convection. J Fluid Mech 1969;38:203.
[27] Snyderman R, Gewurz H, Mergenhagen SE. Interactions of the complement system with endotoxic lipopolysaccharide. generation of a factor chemotactic for polymorphonuclear leukocytes. J Exp Med 1968;128(2):259–75.
[28] Ervin Lenzi K, Tateishi Angel A, Haroldo Ribeiro V. The role of fractional time-Derivative operators on anomalous diffusion. Front Phys 2017;5:1–9.
[29] Abdon A. Non validity of index law in fractional calculus: a fractional differential operator with markovian and non-Markovian properties. Physica A 2018;505:688–706.
[30] Atangana A, Aguila JFG. Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena. Eur Phys J Plus 2018;133:166.
[31] Abdeljawad T, Baleanu D. Integration by parts and its applications of a new nonlocal fractional derivative with mittag-Leffler nonsingular kernel. J Nonlinear Sci Appl 2017;10(3):1098–107.
[32] Abdeljawad T, Al-Mdallal QM. Discrete mittag-Leffler kernel type fractional difference initial value problems and gronwalls inequality. J Comput Appl Math. Special Issue: SI. 2018;339:218–30. doi:10.1016/j.cam.2017.10.021.
[33] Abdeljawad T, Baleanu D. Monotonicity analysis of a nabla discrete fractional operator with discrete mittag-Leffler kernel. Chaos, Solitons and Fractals. Published: SEP. 2017;102:106–10.
[34] Abdeljawad T, Al-Mdallal QM. Discrete mittag-Leffler kernel type fractional difference initial value problems and gronwalls inequality. J Comput Appl Math. Special Issue: SI. 2018;339:218–30. doi:10.1016/j.cam.2017.10.021.
[35] Abdeljawad T. A lyapunov type inequality for fractional operators with nonsingular mittagleffler kernel. J Inequal Appl. 2017:130. 2017. doi:10.1186/s13660-017-1400-5.
[36] Abdeljawad T, Madjidi F. Lyapunov type inequalities for fractional difference operators with discrete mittag-Leffler kernels of order $2 < \alpha < 5/2$. Eur Phys J Special Topics. December. 2017;226(1618):33553368.
[37] Fernandez A, Baleanu D. The mean value theorem and taylors theorem for fractional derivatives with mittag-Leffler kernel. Adv Diff Eq 2018. 2018:86.
[38] Abdeljawad T, Baleanu D. Discrete fractional differences with non-singular discrete mittagleffler kernels. Adv Diff Eq. 2016:232. 2016. doi:10.1186/s13662-016-0949-5.
[39] Bozkurt F, Abdeljawad T, Hajji MA. Stability analysis of a fractional order differential equation model of a brain tumor growth depending on the density. Appl Comput Math 2015;14(1):50–62.
[40] Al-Mdallal QM, Syam MI. An efficient method for solving non-linear singularly perturbed two points boundary-value problems of fractional order. Commun Nonlinear Sci Numer Simul 2012;17(6):2299–308.
[41] Al-Mdallal QM, Hajji MA. A convergent algorithm for solving higher-order nonlinear fractional boundary value problems. Fract Calculus Appl Anal 2015;18(6):1423–40.
[42] Al-Mdallal QM, Omer ASA. Fractional-order legendre-collocation method for solving fractional initial value problems. Appl Math Comput 2018;321:74–84.