1. CONVERGENCE OF THE ITERATIVE METHOD

Let us write

$$\varrho(x,t) = \sum_{\underline{k}\in\mathbb{Z}^2}\sum_{n\in\mathbb{N}} \mathrm{R}_{n,\underline{k}}\, t^n \exp\bigl(2\pi i\, \underline{k}\cdot\underline{x}\bigr).$$

If ϱ solves the Keller–Segel equation, then and $\underline{k}\neq\underline{0}$, then

$$\mathrm{R}_{n+1,\underline{k}} = \tfrac{\mathrm{R}_{0,\underline{0}}-4\pi^2|\underline{k}|^2}{n+1}\mathrm{R}_{n,\underline{k}} + \sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0},\underline{k}\}} \tfrac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2}\sum_{m=0}^{n}\tfrac{1}{n+1}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}. \tag{1.1}$$

If that the coefficients in equation (1.1) exist after the n^{th} step, then let

$$\mathrm{Z}_n := \left\{\, \underline{k}\in\mathbb{Z}^2 \;\middle|\; \mathrm{R}_{n,\underline{k}}\neq 0 \,\right\} \subseteq \mathbb{Z}^2.$$

Lemma 1.1. *Let us assume that* Z_0 *is finite and symmetric, that is* $-\mathrm{Z}_0=\mathrm{Z}_0$*, and let*

$$d := \max\bigl(\{\, |\underline{k}| \mid \underline{k}\in\mathrm{Z}_0 \,\}\bigr) > 0.$$

Then for all $n\in\mathbb{N}_+$*, the coefficients in equation* (1.1) *exist and* Z_n *is finite and symmetric, moreover*

$$\mathrm{Z}_n \subseteq \mathrm{Z}_{n-1}+\mathrm{Z}_{n-1},$$

and

$$\max\bigl(\{\, |\underline{k}| \mid \underline{k}\in\mathrm{Z}_n \,\}\bigr) = \max\bigl(\{\, |\underline{k}| \mid \mathrm{R}_{n,\underline{k}}\neq 0 \,\}\bigr) \leqslant 2^n d.$$

Proof. *Insert Adam's proof here.* □

For all $\underline{k}\in\mathbb{Z}^2-\{\underline{0}\}$ let

$$\mathrm{N}(\underline{k}) := \min\bigl(\{\, n\in\mathbb{N} \mid \underline{k}\in\mathrm{Z}_n \,\}\bigr).$$

For the rest of the section, let us assume that Z_0 is finite, and let d be as in Lemma 1.1. Then let

$$\mathrm{C} := \max\bigl(\{\, 2\pi^2, 2\mathrm{R}_{0,\underline{0}}, \|\mathrm{R}_0\|_{\mathrm{L}^\infty} \,\}\bigr).$$

Lemma 1.2. *Let* C *be as above. Then*

$$\forall \underline{k}\in\mathbb{Z}^2-\{\underline{0}\} : \forall n\in\mathbb{N} : \quad \bigl|\mathrm{R}_{n,\underline{k}}\bigr| \leqslant \frac{\mathrm{C}^{n+1}4^{n\ln(n)}}{n!}\exp\Bigl(-\frac{|\underline{k}|}{\mathrm{C}}\Bigr). \tag{1.2}$$

Proof. By Lemma 1.1, inequality (1.2) is true for $n=0$, so we can argue by induction.

Let us assume now that we have proved inequality (1.2) for every nonnegative integer up to $n\in\mathbb{N}$. If $n<\mathrm{N}(\underline{k})$, then inequality (1.2) holds. When $n\geqslant\mathrm{N}(\underline{k})$, then using that $|\underline{k}|^2\leqslant 4^{\mathrm{N}(\underline{k})}d$ and that $|\underline{k}|\geqslant 1$, we then have that

$$\begin{aligned}
\Bigl|\tfrac{\mathrm{R}_{0,\underline{0}}-4\pi^2|\underline{k}|^2}{n+1}\mathrm{R}_{n,\underline{k}}\Bigr| &\leqslant \frac{\bigl|\mathrm{R}_{0,\underline{0}}-4\pi^2|\underline{k}|^2\bigr|}{n+1}\,\frac{\mathrm{C}^{n+1}4^{n\ln(n)}}{n!}\exp\Bigl(-\frac{|\underline{k}|}{\mathrm{C}}\Bigr)\\
&\leqslant \frac{\max\Bigl(\Bigl\{\mathrm{R}_{0,\underline{0}}, 4\pi^2|\underline{k}|^2\Bigr\}\Bigr)\mathrm{C}^{n+1}4^{n\ln(n)}}{(n+1)!}\exp\Bigl(-\frac{|\underline{k}|}{\mathrm{C}}\Bigr)\\
&\leqslant \frac{\mathrm{C}^{n+2}4^{(n+1)\ln(n+1)}}{2(n+1)!}\exp\Bigl(-\tfrac{|\underline{k}|}{\mathrm{C}}\Bigr).
\end{aligned}$$

Now let us analyze the inner summand in the quadratic term in equation (1.1):

$$
\begin{aligned}
\left|\sum_{\substack{\underline{l}\in Z_m\\ \underline{k}-\underline{l}\in Z_{n-m}}} \frac{1}{n+1}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right| &\leqslant \frac{1}{n+1}\sum_{\substack{\underline{l}\in Z_m\\ \underline{k}-\underline{l}\in Z_{n-m}}} \left|\mathrm{R}_{m,\underline{l}}\right|\left|\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right| \\
&\leqslant \frac{1}{n+1}\sum_{\substack{\underline{l}\in Z_m\\ \underline{k}-\underline{l}\in Z_{n-m}}} \frac{\mathrm{C}^{m+1}4^{m\ln(m)}}{m!}\exp\left(-\frac{|\underline{l}|}{\mathrm{C}}\right)\frac{\mathrm{C}^{n-m+1}4^{(n-m)\ln(n-m)}}{(n-m)!}\exp\left(-\frac{|\underline{k}-\underline{l}|}{\mathrm{C}}\right) \\
&= \frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{l}|+|\underline{k}-\underline{l}|}{\mathrm{C}}\right)\binom{n}{m}\sum_{\substack{\underline{l}\in Z_m\\ \underline{k}-\underline{l}\in Z_{n-m}}} 4^{m\ln(m)+(n-m)\ln(n-m)} \\
&= \frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right)\binom{n}{m}2\pi 4^{m}4^{n\ln(n)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\sum_{m=0}^{n}\sum_{\substack{\underline{l}\in Z_m\\ \underline{k}-\underline{l}\in Z_{n-m}}} \tfrac{1}{n+1}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right| &\leqslant \sum_{m=0}^{n}\frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right)\binom{n}{m}2\pi 4^{m}4^{n\ln(n)} \\
&\leqslant \frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right)4^{n\ln(n)}\sum_{m=0}^{n}\binom{n}{m}2\pi 4^{m} \\
&= \frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{k}|}{\mathrm{C}}\right)4^{n\ln(n)}2\pi\frac{5^{n+1}-1}{5-1}.
\end{aligned}
$$

Inserting this to the outer sum, we get

$$
\left|\sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0},\underline{k}\}}\frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2}\sum_{m=0}^{n}\tfrac{1}{n+1}\mathrm{R}_{m,\underline{l}}\mathrm{R}_{n-m,\underline{k}-\underline{l}}\right| \leqslant \frac{\mathrm{C}^{n+2}}{(n+1)!}\exp\left(-\tfrac{|\underline{k}|^2}{\mathrm{C}}\right)\sum_{\underline{l}\in\mathbb{Z}^2-\{\underline{0}\}}\frac{|\underline{k}\cdot\underline{l}|}{|\underline{l}|^2}\left(|\underline{l}|^2+|\underline{k}-\underline{l}|^2\right)^n\exp\left(\tfrac{2(\underline{k}-\underline{l})\cdot\underline{l}}{\mathrm{C}}\right)
$$

□