Eur. Phys. J. Plus (2018) **133**: 200
DOI 10.1140/epjp/i2018-12038-6

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Analytical solutions of the Keller-Segel chemotaxis model involving fractional operators without singular kernel

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Received: 1 February 2018
Published online: 29 May 2018

Abstract. This paper discusses the application of analytical techniques, namely the Laplace homotopy perturbation method and the modified homotopy analysis transform method, for solving a coupled one-dimensional time-fractional Keller-Segel chemotaxis model. The first method is based on a combination of the Laplace transform and homotopy methods, while the second method is an analytical technique based on the homotopy polynomial. Fractional derivatives with exponential and Mittag-Leffler laws in Liouville-Caputo sense are considered. The effectiveness of both methods is demonstrated by finding the exact solutions of the Keller-Segel chemotaxis model. Some examples have been presented in order to compare the results obtained with both fractional-order derivatives.

1 Introduction

Fractional calculus (FC) has become an alternative mathematical method to describe models with nonlocal behavior. In the recent decades several physical problems have been represented mathematically by fractional derivatives; these representations have offered great results in the modelling of real-world problems. Some fundamental definitions of fractional operators were given by Coimbra, Riesz, Riemann-Liouville, Hadamard, Weyl, Grünwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio, Atangana-Baleanu, among others [1–10]. Many powerful methods for finding exact solutions have been developed to study the solutions of nonlinear FPDEs, for instance, the Hermite colocation method [11], the invariant subspace method [12], the optimal homotopy asymptotic method [13], the q-homotopy analysis transform technique [5], the Adomian decomposition method [14], the homotopy analysis Sumudu transform method [15], the homotopy perturbation transform method [16, 17], the Sumudu transform series expansion method [18], the Padé-approximation and homotopy-Padé technique [19]. The homotopy analysis method transforms a problem into an infinite number of linear problems without using the perturbation techniques; this method employs the concept of the homotopy from topology to generate a convergent series solution [20, 21]. The Laplace homotopy perturbation method (LHPM) is a combination of the homotopy analysis method proposed by Liao [22] and of the Laplace transform [23, 24].

Chemotaxis is the directed motion of cells in response to the gradient of some chemical substance. It plays a key role in development biology, and more generally in the self-organization of cell populations. The first mathematical model of chemotaxis was proposed in 1970 for Evelyn Keller and Lee Segel. Their presented parabolic systems for describing the aggregation process of a cellular slime mold based on chemical attraction [25]. In this paper we consider

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the coupled time-fractional Keller-Segel chemotaxis model (K-S) of the form

$$\frac{\partial}{\partial t}u(x,t) = a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial\chi(\rho)}{\partial x}\right), \tag{1}$$

$$\frac{\partial}{\partial t}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{2}$$

subject to the boundaries conditions

$$\frac{\partial}{\partial t}u(\alpha,t) = \frac{\partial}{\partial t}u(\beta,t) = \frac{\partial}{\partial t}\rho(\alpha,t) = \frac{\partial}{\partial t}\rho(\beta,t) = 0, \tag{3}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad \rho(x,0) = \rho_x, \quad x \in I, \tag{4}$$

the unknown function $u(x,t)$ describes the concentration of amoebae and $\rho(x,t)$ describes the concentration of the chemical substance; $\frac{\partial}{\partial x}(u(x,t)\frac{\partial\chi(\rho)}{\partial x})$ denotes the chemotactic term and shows that the cells are sensitive to the chemicals and are attracted by them; $\chi(\rho)$ is the sensitivity function while a, b, c, and d are positive constants; $0 < \alpha \leq 1$ is the parameter representing the order of the fractional derivative. Recently, the K-S model has been analyzed extensively. For instance, Atangana in [26,27] and [28] obtained the solution to the K-S model by using modified homotopy perturbation, Laplace transform and the homotopy decomposition method, respectively. Zayernouri in [29] developed a fractional class of explicit Adams-Bashforth and implicit Adams-Moulton methods. The Liouville-Caputo fractional derivative was considered. The modified homotopy analysis transform method (MHATM) was proposed in [30]; this method is an analytical technique based on the combination of the homotopy analysis method and Laplace transform with homotopy polynomial. In [30], considering the Liouville-Caputo fractional derivative, the authors developed the MHATM method with homotopy polynomial for solving time-fractional K-S equation. A convergence analysis of MHATM was obtained by the proposed method and verified through different graphical representations.

In this paper, we use the Laplace homotopy perturbation method and the modified homotopy analysis transform method to obtain the solution to the K-S model using the Caputo-Fabrizio and Atangana-Baleanu fractional operators in Liouville-Caputo sense. This work is organized as follows: we give some definitions of the fractional calculus in sect. 2. In sects. 3 and 4, we apply the LHPM and MHATM, respectively, for approximate solution of time-fractional coupled K-S equations, respectively. Implementation and numerical results of the LHPM and MHATM method for approximate solution of time-fractional coupled K-S model are shown in sect. 5. Finally, sect. 6 ends with some conclusions and perspectives.

2 Basic definitions

In this section, some basic definitions of fractional calculus are presented.

Definition 1. The Liouville-Caputo operator (C) is defined as follows [7]:

$${}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\theta)^{n-\alpha-1}u^{(n)}(x,\theta)\mathrm{d}\theta, \quad n-1<\alpha<n, \tag{5}$$

where $u^{(n)}(x,\theta)$ is the derivative of integer n-th order of $u(x,t)$, $n = 1,2,\ldots \in N$ and $n-1<\alpha\leq n$.

If $0<\alpha\leq 1$, then we define the Laplace transform for the Liouville-Caputo fractional derivative as follows:

$$\mathcal{L}\left[{}_0^C\mathcal{D}_t^\alpha u(x,t)\right](s) = s^\alpha\mathcal{L}[u(x,t)](s) - s^{\alpha-1}[u(x,0)]. \tag{6}$$

Definition 2. The Caputo-Fabrizio-Caputo operator (CFC) is expressed as follows [9]:

$${}_0^{CFC}\mathcal{D}_t^\alpha u(x,t) = \frac{(2-\alpha)M(\alpha)}{2(n-\alpha)}\int_0^t \exp\left(-\alpha\frac{(t-\theta)}{n-\alpha}\right)u^{(n)}(x,\theta)\mathrm{d}\theta, \tag{7}$$

where $M(\alpha)$ is a normalization function, $M(0) = M(1) = 1$. This fractional operator uses the exponential law as nonsingular kernel.

If $0<\alpha\leq 1$, then we define the Laplace transform for the Caputo-Fabrizio-Caputo fractional derivative as follows:

$$\mathcal{L}\left[{}_0^{CFC}\mathcal{D}_t^\alpha u(x,t)\right](s) = \left(\frac{s\mathcal{L}[u(x,t)](s) - u(x,0)}{s+\alpha(1-s)}\right). \tag{8}$$

Definition 3. The fractional derivative with generalized Mittag-Leffler law in Liouville-Caputo sense (ABC) is defined as follows [10]:

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha}u(x,t)=\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{n-\alpha}\right]u^{(n)}(x,\theta)\mathrm{d}\theta, \tag{9} $$

where $B(\alpha)$ is a normalization function, $B(0)=B(1)=1$. This fractional operator uses the Mittag-Leffler law as nonsingular and nonlocal kernel.

If $0<\alpha\leq 1$, then we define the Laplace transform for the Atangana-Baleanu fractional derivative as follows:

$$ \mathcal{L}\left[{}_0^{ABC}D_t^{\alpha}u(x,t)\right](s)=\left(\frac{s^\alpha\mathcal{L}[u(x,t)](s)-s^{\alpha-1}[u(x,0)]}{s^\alpha(1-\alpha)+\alpha}\right). \tag{10} $$

In this article we consider the Caputo-Fabrizio-Caputo fractional-order derivative (7) and the fractional derivative with generalized Mittag-Leffler law (9). In the next section we apply the Laplace homotopy perturbation method and modified homotopy analysis transform method for solving coupled one-dimensional time-fractional Keller-Segel chemotaxis model.

3 Implementation of the LHPM for approximate solution of time-fractional coupled K-S chemotaxis model via CFC and ABC fractional-order derivatives

The LHPM is a combination of the homotopy analysis method proposed by Liao and the Laplace transform. Following the methodology described in [31] we solved the time-fractional Keller-Segel chemotaxis model via Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo fractional-order derivatives.

The main steps of this method are described as follows:

Step 1. Let us consider the following equation:

$$ \frac{\partial^p}{\partial t^p}F(x,t)=\Lambda(F(x,t))+\Xi(F(x,t))+u(x,t),\quad p=1,2,3,\ldots, \tag{11} $$

subject to the initial condition

$$ \frac{\partial^i}{\partial t^i}F(x,0)=z_i(x),\quad \frac{\partial^{p-1}}{\partial t^{p-1}}F(x,0)=0,\quad i=0,1,2,3,\ldots,p-2, \tag{12} $$

where p is the order of the derivative, $u(x,t)$ is a known function, Ξ is the general nonlinear differential operator, and Λ represents a linear differential operator.

Step 2. Applying the Laplace transform operator to both sides of eq. (11) we obtain

$$ \mathcal{L}[f(x,t)]=s^p\mathscr{L}[\Lambda(F(x,t))]+s^p\mathscr{L}[\Xi(F(x,t))]+s^p\mathscr{L}[u(x,t)], \tag{13} $$

where the Laplace transform can be applied in the Caputo-Fabrizio (8) and Atangana-Baleanu (10) sense.

Step 3. Applying the inverse Laplace transform operator on both sides of eq. (13) we obtain

$$ F(x,t)=H(x,t)+\mathscr{L}^{-1}[s^p\mathscr{L}[\Lambda(F(x,t))]]+s^p\mathscr{L}[\Xi(F(x,t))]. \tag{14} $$

Step 4. Using the homotopy scheme, the solution of the above equation is given in series form as

$$ \begin{aligned} F(x,t,k)&=\sum_{n=0}^{\infty}k^nF_n(x,t),\\ F(x,t)&=\lim_{k\to1}F(x,t,k), \end{aligned} \tag{15} $$

and the nonlinear term can be decomposed as

$$ \Xi\,F(r,t)=\sum_{n=1}^{\infty}k^n\mathfrak{H}_n(F(x,t)), $$

where $k \in (0,1]$ is an embedding parameter, $\mathfrak{H}_n(F(x,t))$ are the He polynomials, which can be generated by

$$\mathfrak{H}_n(F_0,\ldots,F_n) = \frac{1}{n!}\frac{\partial^n}{\partial k^n}\left[\Xi\left(\sum_{j=0}^{n} k^j F_j(x,t)\right)\right], \quad \text{for } n = 0,1,2,\ldots. \tag{16}$$

Finally, the LHPM is obtained by coupling the decomposition method with the Abel integral, which is given by

$$\sum_{j=0}^{\infty} k^n F_n(x,t) = T(x,t) + k\frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1}\left[u(x,\tau) + \Lambda\left(\sum_{n=0}^{\infty} k^n F_n(x,\tau)\right) + \sum_{n=0}^{\infty} k^n \mathfrak{H}_n(F(x,t))\right]\mathrm{d}\tau, \tag{17}$$

where $T(x,t) = H(x,t)$.

Comparing the terms with the same powers of k, produces solutions of various orders. The initial estimated of the approximation is $T(x,t)$, which is actually the Taylor series for the exact solution of order p.

Applying the aforesaid technique we have used the LHPM for solving the Keller-Segel model in the Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo sense.

Example 1. Let us consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho$ in the CFC sense:

$$\frac{{}^{CFC}\partial^\alpha}{\partial t^\alpha}u(x,t) = a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x), \tag{18}$$

$$\frac{{}^{CFC}\partial^\alpha}{\partial t^\alpha}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{19}$$

with initial conditions

$$u(x,0) = m\,e^{-x}, \qquad \rho(x,0) = n\,e^{-x}. \tag{20}$$

Solution. Applying the Laplace transform (8) to eq. (18), we obtain

$$\frac{s\mathcal{L}[u(x,t)] - u(x,0)}{s+\alpha(1-s)} = \mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \tag{21}$$

Taking initial conditions (20) and simplifying the above equation, we get

$$\mathcal{L}[u(x,t)] = \frac{u(x,0)}{s} + \frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \tag{22}$$

Applying the inverse Laplace transform to eq. (22), we obtain

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}\right\}. \tag{23}$$

Applying the LHPM to eq. (23) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - \sum_{n=0}^{\infty}\frac{\partial}{\partial x}u_n(x,t)\,\frac{\partial}{\partial x}\rho_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\,\frac{\partial^2}{\partial x^2}\rho_n(x)\right\}\right\} \tag{24}$$

and, for eq. (19), we obtain

$$\sum_{n=0}^{\infty}\rho_n(x,t) = \rho(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right) + c\sum_{n=0}^{\infty} p^n u_n(x,t) - d\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right\}\right\}. \tag{25}$$

Comparing terms with the same power in p we obtain

$$\begin{aligned}
p^0 &: u_0(x,t) = u(x,0) = m\,e^{-x},\\
p^0 &: \rho_0(x,t) = \rho(x,0) = n\,e^{-x},\\
p^1 &: u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\left(u_0(x,t)\right) - \frac{\partial}{\partial x}u_0(x,t)\,\frac{\partial}{\partial x}\rho_0(x,t) + u_0(x,t)\,\frac{\partial^2}{\partial x^2}\rho_0(x)\right\}\right\}\\
&\qquad = am\,e^{-x}\left[\alpha t + (1-\alpha)\right],\\
p^1 &: \rho_1(x,t) = \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\rho_0(x,t)\right) + c\,u_0(x,t) - d\,\rho_0(x,t)\right\}\right\},\\
&\qquad = e^{-x}(cm + (b-d)n)\left[\alpha t + (1-\alpha)\right],
\end{aligned}$$

$$
\begin{aligned}
p^2 : u_2(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_1(x,t)) - \frac{\partial}{\partial x}u_1(x,t)\,\frac{\partial}{\partial x}\rho_1(x,t) + u_1(x,t)\,\frac{\partial^2}{\partial x^2}\rho_1(x)\right\}\right\}\\
&= a^2 e^{-x} m\left[(1-\alpha)^2 + 2\alpha(1-\alpha)\,t + \frac{\alpha^2 t^2}{2}\right],\\
p^2 : \rho_2(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_1(x,t)) + c\,u_1(x,t) - d\,\rho_1(x,t)\right\}\right\},\\
&= e^{-x}\Big(acm + (b-d)\,(cm + (b-d)n\Big)\left[(1-\alpha)^2 + 2\alpha(1-\alpha)\,t + \frac{\alpha^2 t^2}{2}\right],\\
&\ \vdots\\
p^n : u_n(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_{n-1}) - \sum_{k=0}^{n-1}\frac{\partial}{\partial x}u_k\,\frac{\partial}{\partial x}\rho_{n-k-1} + \sum_{k=0}^{n-1}u_j\,\frac{\partial^2}{\partial x^2}\rho_{n-j-1}(x)\right\}\right\}, \qquad (26)\\
p^n : \rho_n(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_{n-1}(x,t)) + c\,u_{n-1}(x,t) - d\,\rho_{n-1}(x,t)\right\}\right\}. \qquad (27)
\end{aligned}
$$

We only calculate some terms of the solution, the other terms can be found using the iterative formula. Thus, the asymptotic solutions of eqs. (18), (19) are given as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \qquad (28)$$
$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots. \qquad (29)$$

Example 2. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho$ in the ABC sense:

$$\frac{{}^{ABC}\partial^\alpha}{\partial t^\alpha}u(x,t) = a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x), \qquad (30)$$
$$\frac{{}^{ABC}\partial^\alpha}{\partial t^\alpha}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \qquad (31)$$

with initial conditions

$$u(x,0) = m\,e^{-x}, \qquad \rho(x,0) = n\,e^{-x}. \qquad (32)$$

Solution. Applying the Laplace transform (10) to eq. (30), we obtain

$$\frac{B(\alpha)}{1-\alpha}\,\frac{s^\alpha\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)}{s^\alpha + \frac{\alpha}{1-\alpha}} = \mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \qquad (33)$$

Taking initial conditions (32) and simplifying the above equation, we get

$$\mathcal{L}[u(x,t)] = \frac{u(x,0)}{s} + \frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \qquad (34)$$

Applying the inverse Laplace transform to eq. (34), we obtain

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}\right\}. \qquad (35)$$

Applying the LHPM to eq. (35) we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}u_n(x,t) = u(x,0) + p\,\mathcal{L}^{-1}\Bigg\{&\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\Bigg\{a\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}p^n\,u_n(x,t)\right)\\
&- \sum_{n=0}^{\infty}\frac{\partial}{\partial x}u_n(x,t)\,\frac{\partial}{\partial x}\rho_n(x,t) + \sum_{n=0}^{\infty}u_n(x,t)\,\frac{\partial^2}{\partial x^2}\rho_n(x)\Bigg\}\Bigg\} \qquad (36)
\end{aligned}
$$

and, for eq. (31), we obtain

$$\sum_{n=0}^{\infty} \rho_n(x,t) = \rho(x,0)$$
$$+\, p\,\mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right) + c\sum_{n=0}^{\infty} p^n u_n(x,t) - d\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right\}\right\}. \quad (37)$$

Comparing terms with the same power in p we obtain

$$p^0 : u_0(x,t) = u(x,0) = m\,e^{-x},$$

$$p^0 : \rho_0(x,t) = \rho(x,0) = n\,e^{-x},$$

$$p^1 : u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_0(x,t)) - \frac{\partial}{\partial x}u_0(x,t)\,\frac{\partial}{\partial x}\rho_0(x,t) + u_0(x,t)\,\frac{\partial^2}{\partial x^2}\rho_0(x)\right\}\right\}$$
$$= \frac{1}{B(\alpha)}\,am\,e^{-x}\left[\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + (1-\alpha)\right],$$

$$p^1 : \rho_1(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_0(x,t)) + c\,u_0(x,t) - d\,\rho_0(x,t)\right\}\right\},$$
$$= \frac{1}{B(\alpha)}\,e^{-x}(cm + (b-d)n)\left[\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + (1-\alpha)\right],$$

$$p^2 : u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_1(x,t)) - \frac{\partial}{\partial x}u_1(x,t)\,\frac{\partial}{\partial x}\rho_1(x,t) + u_1(x,t)\,\frac{\partial^2}{\partial x^2}\rho_1(x)\right\}\right\}$$
$$= \frac{1}{B^2(\alpha)}a^2 e^{-x} m\left[(1-\alpha)^2 + \frac{2\alpha(1-\alpha)\,t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$p^2 : \rho_2(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_1(x,t)) + c\,u_1(x,t) - d\,\rho_1(x,t)\right\}\right\},$$
$$= e^{-x}\Big(acm + (b-d)\,(cm + (b-d)n\Big)\left[(1-\alpha)^2 + \frac{2\alpha(1-\alpha)\,t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$\vdots$$

$$p^n : u_n(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_{n-1}) - \sum_{k=0}^{n-1}\frac{\partial}{\partial x}u_k\,\frac{\partial}{\partial x}\rho_{n-k-1} + \sum_{k=0}^{n-1} u_j\,\frac{\partial^2}{\partial x^2}\rho_{n-j-1}(x)\right\}\right\}, \quad (38)$$

$$p^n : \rho_n(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^{\alpha}+\alpha)}{B(\alpha)\,s^{\alpha}}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_{n-1}(x,t)) + c\,u_{n-1}(x,t) - d\,\rho_{n-1}(x,t)\right\}\right\}. \quad (39)$$

We only calculate some terms of the solution, the other terms can be found using the iterative formula. Thus, the asymptotic solutions of eqs. (30), (31) are given as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \quad (40)$$

$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots. \quad (41)$$

Example 3. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho^2$ in the CFC sense:

$$\frac{{}^{CFC}\partial^{\alpha}}{\partial t^{\alpha}}u(x,t) = a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x), \quad (42)$$

$$\frac{{}^{CFC}\partial^{\alpha}}{\partial t^{\alpha}}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \quad (43)$$

with initial conditions

$$u(x,0) = m\sin(x),$$
$$\rho(x,0) = n\sin(x). \tag{44}$$

Solution. Applying the Laplace transform (8) to eq. (42), we obtain

$$\frac{s\mathcal{L}[u(x,s)] - u(x,0)}{s+\alpha(1-s)} = \mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}. \tag{45}$$

Taking initial conditions (44) and simplifying the above equation, we get

$$\mathcal{L}[u(x,s)] = \frac{u(x,0)}{s} + \frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}. \tag{46}$$

Applying the inverse Laplace transform to eq. (46), we obtain

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}\right\}. \tag{47}$$

Applying the LHPM to eq. (47) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n\,u_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n(x,t) + \sum_{n=0}^{\infty} K_n(x,t)\right\}\right\} \tag{48}$$

and, for eq. (43), we have

$$\sum_{n=0}^{\infty}\rho_n(x,t) = \rho(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n\rho_n(x,t)\right) + c\sum_{n=0}^{\infty} p^n u_n(x,t) - d\sum_{n=0}^{\infty} p^n\rho_n(x,t)\right\}\right\}. \tag{49}$$

Comparing terms with the same power in p, we obtain

$$p^0 : u_0(x,t) = u(x,0) = m\sin(x),$$
$$p^0 : \rho_0(x,t) = \rho(x,0) = n\sin(x),$$
$$\begin{aligned}
p^1 : u_1(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2 u_0(x,t)}{\partial x^2} - 2\frac{\partial u_0}{\partial x}\,\frac{\partial \rho_0}{\partial x}\rho_0 + 2u_0\,\rho_0\,\frac{\partial^2\rho_0}{\partial x^2} + 2u_0\left(\frac{\partial\rho_0}{\partial x}\right)^2\right\}\right\}\\
&= -m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)[\alpha t + (1-\alpha)],
\end{aligned}$$
$$\begin{aligned}
p^1 : \rho_1(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_0(x,t)) + c\,u_0(x,t) - d\,\rho_0(x,t)\right\}\right\},\\
&= \sin(x)\,(cm - (b+d)n)\,[\alpha t + (1-\alpha)],
\end{aligned}$$
$$\begin{aligned}
p^2 : u_2(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{a\,\frac{\partial^2 u_1(x,t)}{\partial x^2} - 2\frac{\partial u_1}{\partial x}\,\frac{\partial \rho_1}{\partial x}\rho_1 + 2u_1\,\rho_1\,\frac{\partial^2\rho_1}{\partial x^2} + 2u_1\left(\frac{\partial\rho_1}{\partial x}\right)^2\right\}\right\}\\
&= -m\sin(x)\left[a + n(\alpha t + (1-\alpha))\,(2cm - an - 2bn - 2dn + n^3 + n^3\cos(2x))\sin^2(x)\right][\alpha t + (1-\alpha)],
\end{aligned}$$
$$\begin{aligned}
p^2 : \rho_2(x,t) &= \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_1(x,t)) + c\,u_1(x,t) - d\,\rho_1(x,t)\right\}\right\},\\
&= -\sin(x)\,\left(acm + bcm + cdm - b^2n - 2dbn - d^2n + cnm^2 + cmn^2\cos(2x)\right)\left[(1-\alpha)^2 + 2\alpha(1-\alpha)\,t + \frac{\alpha^2t^2}{2}\right],
\end{aligned}$$
$$\vdots$$
$$\begin{aligned}
p^n : u_n(x,t) = \mathcal{L}^{-1}\Bigg\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\Bigg\{&a\,\frac{\partial^2}{\partial x^2}\,(u_{n-1}) - 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}\rho_k\,\frac{\partial}{\partial x}u_{j-k}\,\frac{\partial}{\partial x}\rho_{n-j-1}\\
&+ 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}u_k\,\rho_{j-k}\,\frac{\partial^2}{\partial x^2}\rho_{n-j-1} + 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}u_k\,\frac{\partial}{\partial x}\rho_{j-k}\,\frac{\partial}{\partial x}\rho_{n-j-1}\Bigg\}\Bigg\},
\end{aligned} \tag{50}$$
$$p^n : \rho_n(x,t) = \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_{n-1}(x,t)) + c\,u_{n-1}(x,t) - d\,\rho_{n-1}(x,t)\right\}\right\}. \tag{51}$$

We only calculate some terms of the solution, the other terms can be found using the iterative formula. Thus, the asymptotic solutions of eqs. (42), (43) are given as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \tag{52}$$
$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots. \tag{53}$$

Example 4. Considering the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho^2$ in the ABC sense.

$$\frac{{}^{ABC}\partial^\alpha}{\partial t^\alpha} u(x,t) = a\,\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial x} u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x), \tag{54}$$
$$\frac{{}^{ABC}\partial^\alpha}{\partial t^\alpha} \rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{55}$$

with initial conditions

$$u(x,0) = m\sin(x), \qquad \rho(x,0) = n\sin(x). \tag{56}$$

Solution. Applying the Laplace transform (10) to eq. (54), we obtain

$$\frac{B(\alpha)}{1-\alpha}\,\frac{s^\alpha \mathcal{L}[u(x,s)] - s^{\alpha-1}u(x,0)}{s^\alpha + \frac{\alpha}{1-\alpha}} = \mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}. \tag{57}$$

Taking initial conditions (56) and simplifying the above equation, we get

$$\mathcal{L}[u(x,s)] = \frac{u(x,0)}{s} + \frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}. \tag{58}$$

Applying the inverse Laplace transform to eq. (58), we obtain

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x)\right\}\right\}. \tag{59}$$

Applying the LHPM to eq. (59) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n\,u_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n(x,t) + \sum_{n=0}^{\infty} K_n(x,t)\right\}\right\}, \tag{60}$$

and, for eq. (55), we obtain

$$\sum_{n=0}^{\infty} \rho_n(x,t) = \rho(x,0) + p\,\mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right) + c\sum_{n=0}^{\infty} p^n u_n(x,t) - d\sum_{n=0}^{\infty} p^n \rho_n(x,t)\right\}\right\}. \tag{61}$$

Comparing terms with the same power in p, we obtain

$$\begin{aligned}
p^0 &: u_0(x,t) = u(x,0) = m\sin(x),\\
p^0 &: \rho_0(x,t) = \rho(x,0) = n\sin(x),\\
p^1 &: u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha + \alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2 u_0(x,t)}{\partial x^2} - 2\frac{\partial u_0}{\partial x}\,\frac{\partial \rho_0}{\partial x}\rho_0 + 2u_0\,\rho_0\,\frac{\partial^2 \rho_0}{\partial x^2} + 2u_0\left(\frac{\partial \rho_0}{\partial x}\right)^2\right\}\right\}\\
&\qquad = -\frac{1}{B(\alpha)}\,m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right],
\end{aligned}$$

$$p^1 : \rho_1(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha+\alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_0(x,t)) + c\,u_0(x,t) - d\,\rho_0(x,t)\right\}\right\},$$
$$= \frac{1}{B(\alpha)}\sin(x)\,(cm-(b+d)n)\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right],$$
$$p^2 : u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha+\alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2 u_1(x,t)}{\partial x^2} - 2\frac{\partial u_1}{\partial x}\,\frac{\partial \rho_1}{\partial x}\rho_1 + 2u_1\,\rho_1\,\frac{\partial^2\rho_1}{\partial x^2} + 2u_1\,\left(\frac{\partial \rho_1}{\partial x}\right)^2\right\}\right\}$$
$$= -m\sin(x)\left[a + n\left(\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right)(2cm - an - 2bn - 2dn + n^3\right.$$
$$\left. + n^3\cos(2x))\sin^2(x)\right]\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right],$$
$$p^2 : \rho_2(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha+\alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_1(x,t)) + c\,u_1(x,t) - d\,\rho_1(x,t)\right\}\right\},$$
$$= -\sin(x)\,\left(acm + bcm + cdm - b^2n - 2dbn - d^2n + cnm^2 + cmn^2\cos(2x)\right)$$
$$\cdot\left[(1-\alpha)^2 + \frac{2\alpha(1-\alpha)\,t^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$
$$\vdots$$
$$p^n : u_n(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha+\alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}\,(u_{n-1}) - 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}\rho_k\,\frac{\partial}{\partial x}u_{j-k}\,\frac{\partial}{\partial x}\rho_{n-j-1}\right.\right.$$
$$\left.\left. + 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}u_k\,\rho_{j-k}\,\frac{\partial^2}{\partial x^2}\rho_{n-j-1} + 2\sum_{j=0}^{n-1}\sum_{k=0}^{j}u_k\,\frac{\partial}{\partial x}\rho_{j-k}\,\frac{\partial}{\partial x}\rho_{n-j-1}\right\}\right\}, \tag{62}$$
$$p^n : \rho_n(x,t) = \mathcal{L}^{-1}\left\{\frac{((1-\alpha)s^\alpha+\alpha)}{B(\alpha)\,s^\alpha}\,\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\,(\rho_{n-1}(x,t)) + c\,u_{n-1}(x,t) - d\,\rho_{n-1}(x,t)\right\}\right\}. \tag{63}$$

The remaining terms can be obtained using the iterative formula. However, we only consider a few terms of the series of solutions and the asymptotic solutions (54), (55) are given as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \tag{64}$$
$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots. \tag{65}$$

4 Implementation of the MHATM for approximate solution of time-fractional coupled K-S chemotaxis model via CFC and ABC fractional-order derivatives

The MHATM is a combination of the homotopy analysis method and Laplace transform with homotopy polynomial. Following the methodology described in [30] we solved the time-fractional Keller-Segel chemotaxis model via Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo fractional-order derivatives.

The main steps of this method are described as follows.

Step 1. Let us consider the following equation:

$$\mathcal{D}_t^\alpha\{f(x,t)\} + \Xi[x]f(x,t) + \Lambda[x]f(x,t) = \Psi(x,t), \quad t>0,\ x\in\Re,\ 0<\alpha\le 1, \tag{66}$$

where $\Xi[x]$ is a bounded linear operator in x. While the non-linear operator $\Lambda[x]$ in x is Lipschitz continuos and satisfying $|\Lambda(f) - \Lambda(\phi)| \le \vartheta|f-\phi|$, where $\vartheta > 0$ and $\Psi(x,t)$ is a continuos function. The boundary and initial conditions can be treated in a similar way.

Step 2. Applying the methodology proposed in [32,33] we get the following m-th–order deformation equation:

$$f_m(x,t) = (\chi_m + \hbar) f_{m-1} - \hbar(1-\chi_m)\sum_{i=0}^{j-1} t^i f^{(i-1)}(0) + \hbar\mathcal{L}^{-1}\left(\mathcal{L}\left(\Xi_{m-1}[x] f_{m-1}(x) + \sum_{k=0}^{m-1} P_k(f_0, f_1, \dots, f_m) - \Psi(x,t)\right)\right), \tag{67}$$

where the Laplace transform is applied in the Caputo-Fabrizio and Atangana-Baleanu sense and P_k is the homotopy polynomial defined by Odibat in [34].

Step 3. The nonlinear term $\Lambda[x] f(x,t)$ is expanded in terms of homotopy polynomials as

$$\Lambda[f(x,t)] = \Lambda\left(\sum_{k=0}^{m-1} f_m(x,t)\right) = \sum_{m=0}^{\infty} P_m\, f^m. \tag{68}$$

Step 4. Expanding the nonlinear term in (67) as a series of homotopy polynomials, we can calculate the various $f_m(x,t)$ for $m \geq 1$ and the solutions of eq. (66) is considered as the summation of an infinite series which usually converges rapidly to the exact solutions

$$f(x,t) = \sum_{m=0}^{\infty} f_m(x,t). \tag{69}$$

Therefore, the series solution of eq. (66) can be obtained as

$$F(x,t) = \sum_{m=0}^{\infty} F_m(x,t). \tag{70}$$

Convergence analysis

Theorem 1. *The series solution* $\sum_{m=0}^{\infty} F_m(x,t)$ *converges if for a pre-assigned positive* ξ *there exist a natural number* υ*, such that* $|F_{m+p}| < \xi$*,* $\forall_m \geq \upsilon$ *and* $p = 1, 2, 3, \dots$ *.*

Proof. We define the sequence of function $\{y_m\}_{m=0}^{\infty}$ as follows:

$$\begin{aligned} y_0 &= F_0, \\ y_1 &= F_0 + F_1, \\ y_2 &= F_0 + F_1 + F_2, \\ &\dots \\ y_m &= F_0 + F_1 + F_2 + \dots + F_m. \end{aligned} \tag{71}$$

We have shown that $\{y_m\}_{m=0}^{\infty}$ converges uniformly on $\mathbb{R}$. From the assumption of theorem 1, we have

$$|y_{m+p} - y_m| = |F_{m+p}| < \xi \quad \forall\ m \geq \upsilon \text{ and } p = 1, 2, 3, \dots . \tag{72}$$

Therefore, by the Cauchy criterion, we have the sequence $\{y_m\}_{m=0}^{\infty}$ converges uniformly on $\mathbb{R}$, and hence series solution $\sum_{m=0}^{\infty} F_m(x,t)$ converges. □

Applying the aforesaid technique we have used the MHATM for solving the Keller-Segel model in the Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo sense.

Example 1. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho$ in the CFC sense:

$$\frac{{}^{CFC}\partial^\alpha}{\partial t} u(x,t) = a\,\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial x} u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x), \tag{73}$$

$$\frac{{}^{CFC}\partial^\alpha}{\partial t}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{74}$$

with initial conditions

$$u(x,0) = m\,e^{-x^2}, \qquad \rho(x,0) = n\,e^{-x^2}. \tag{75}$$

Solution. Applying the Laplace transform (8) to eq. (73), we obtain

$$\frac{s\mathcal{L}[u(x,t)] - u(x,0)}{s+\alpha(1-s)} = \mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \tag{76}$$

Taking initial conditions (75) and simplifying the above equation, we get

$$\mathcal{L}[u(x,t)] = \frac{u(x,0)}{s} + \frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}. \tag{77}$$

Applying the inverse Laplace transform to eq. (77), we obtain

$$u(x,t) = m\,e^{-x^2} + \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x)\right\}\right\}$$

and, for eq. (74), we obtain

$$\rho(x,t) = n\,e^{-x^2} + \mathcal{L}^{-1}\left\{\frac{(s+\alpha(1-s))}{s}\mathcal{L}\left\{b\,\frac{\partial^2}{\partial x^2}\left(\rho(x,t)\right) + c\,u(x,t) - d\,\rho_(x,t)\right\}\right\}. \tag{78}$$

We choose the linear operator as

$$\mathfrak{F}[\phi_j(x,t;q)] = \mathcal{L}[\phi_j(x,t;q)], \quad j=1,2, \tag{79}$$

with property $\mathfrak{F}(c)=0$, where c is constant. Next, we define the system of the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;q)] &= \mathcal{L}[\phi(x,t;q)] - m\,e^{-x^2} - \frac{(s+\alpha(1-s))}{s}\,\mathcal{L}[a\,\phi_{xx} - \phi_x\,\Phi_x + \phi\,\Phi_{xx}],\\ N[\Phi(x,t;q)] &= \mathcal{L}[\Phi(x,t;q)] - n\,e^{-x^2} - \frac{(s+\alpha(1-s))}{s}\,\mathcal{L}[b\,\Phi_{xx} + c\,\phi - d\,\Phi]. \end{aligned} \tag{80}$$

Now we construct the so-called zeroth-order deformation equation of the following manner:

$$(1-q)\,\mathfrak{F}[\phi_j(x,t;q) - u_0(x,t)] = q\,\hbar\,N[\phi_j(x,t;q)], \quad j=1,2. \tag{81}$$

When $q=0$ and $q=1$, we have

$$\phi_j(x,t;0) = u_0(x,t), \qquad \phi_j(x,t;1) = u(x,t), \quad j=1,2. \tag{82}$$

Thus, we obtain the m-th–order deformation equations as

$$\begin{aligned} \mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= \hbar\,R_m(\overrightarrow{u_{m-1}},x,t),\\ \mathcal{L}[\rho_m(x,t) - \chi_m \rho_{m-1}(x,t)] &= \hbar\,R_m(\overrightarrow{\rho_{m-1}},x,t). \end{aligned} \tag{83}$$

Applying the inverse Laplace transform to eq. (83) we get

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) + \hbar\,R_m(\overrightarrow{u_{m-1}},x,t),\\ \rho_m(x,t) &= \chi_m \rho_{m-1}(x,t) + \hbar\,R_m(\overrightarrow{\rho_{m-1}},x,t), \end{aligned} \tag{84}$$

where

$$\begin{aligned} R_m(\overrightarrow{u_{m-1}},x,t) &= \mathcal{L}[u_{m-1}(x,t)] - (1-\chi_m)m\,e^{-x^2} - \frac{s+\alpha(1-s)}{s}\Big[a(u_{m-1})_{xx} - (u_{m-1})_x\,(\rho_{m-1})_x + u_{m-1}\,(\rho_{m-1})_{xx}\Big],\\ R_m(\overrightarrow{\rho_{m-1}},x,t) &= \mathcal{L}[\rho_{m-1}(x,t)] - (1-\chi_m)m\,e^{-x^2} - \frac{s+\alpha(1-s)}{s}\Big[b(\rho_{m-1})_{xx} + c\,u_{m-1} - d\,\rho_{m-1}\Big]. \end{aligned} \tag{85}$$

Now the solutions of the m-th–order deformation equations (83) are given as

$$\begin{aligned} u_m(x,t) &= (\chi_m+\hbar)u_{m-1} - \hbar(1-\chi_m)m\,e^{-x^2} - \hbar\,\mathcal{L}^{-1}\left\{\frac{s+\alpha(1-s)}{s}\,\mathcal{L}[a(u_{m-1})_{xx} + H_m + K_m]\right\},\\ \rho_m(x,t) &= (\chi_m+\hbar)\rho_{m-1} - \hbar(1-\chi_m)n\,e^{-x^2} - \hbar\,\mathcal{L}^{-1}\left\{\frac{s+\alpha(1-s)}{s}\,\mathcal{L}[a(\rho_{m-1})_{xx} + c\,u_{m-1} - d\,\rho_{m-1}]\right\}, \end{aligned} \tag{86}$$

where

$$H_m = \frac{1}{\Gamma(m+1)} \left[\frac{\partial^m}{\partial q^m} N[(q\,\phi(x,t;q))_x\,(q\,\Phi(x,t;q))_x] \right]_{q=0},$$

$$K_m = \frac{1}{\Gamma(m+1)} \left[\frac{\partial^m}{\partial q^m} N[(q\,\phi(x,t;q))\,(q\,\Phi(x,t;q))_{xx}] \right]_{q=0}. \tag{87}$$

Taking the initial conditions and the iterative scheme (86), we obtain the following iterations:

$$u_1 = 2ame^{-x^2}\,\hbar\,[\alpha\,t + (1-\alpha)] + 2mne^{-2x^2}\hbar\,[\alpha\,t + (1-\alpha)] - 4ame^{-x^2}\,\hbar\,[\alpha\,t + (1-\alpha)],$$

$$\rho_1 = 4ak^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\,[\alpha\,t + (1-\alpha)] + 4b^2k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\,[\alpha\,t + (1-\alpha)]$$
$$+ 4b\sqrt{a+b^2}k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\,[\alpha\,t + (1-\alpha)],$$

$$u_2 = 2ame^{-x^2}\hbar(1+\hbar)\,[\alpha\,t + (1-\alpha)] + 2mne^{-2x^2}\hbar(1+\hbar)\,[\alpha\,t + (1-\alpha)]$$
$$- 4amx^2e^{-x^2}\hbar(1+\hbar)\,[\alpha\,t + (1-\alpha)] + 12a^2me^{-x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right],$$
$$- 2cm^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right] + 12amne^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right]$$
$$+ 12bmne^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right] + 2dmne^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right]$$
$$+ mn^2e^{-3x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right] - 48a^2mx^2e^{-x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right]$$
$$- 24amnx^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right] - 24bmnx^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right]$$
$$+ 8mn^2x^2e^{-3x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right] + 16a^2mx^4e^{-x^2}\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right],$$

$$\rho_2 = 4ak^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\,[\alpha\,t+(1-\alpha)]4b^2k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\,[\alpha\,t+(1-\alpha)]$$
$$+ 4b\sqrt{a+b^2}\,k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\,[\alpha\,t + (1-\alpha)]$$
$$-4ak^4\lambda^2\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2+2\alpha(1-\alpha)t+\frac{\alpha^2t^2}{2}\right] - 4b^2k^4\lambda^2\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2+2\alpha(1-\alpha)t+\frac{\alpha^2t^2}{2}\right]$$
$$+ 4b\sqrt{a+b^2}\,k^4\lambda^2\cosh[2k(c+x)]\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)t + \frac{\alpha^2t^2}{2}\right]. \tag{88}$$

Finally, the solutions to eqs. (73), (74) are given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots = \sum_{m=0}^{\infty} u_m(x,t), \tag{89}$$

$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots = \sum_{m=0}^{\infty} \rho_m(x,t). \tag{90}$$

Example 2. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho$ in the ABC sense:

$$\frac{{}^{ABC}\partial^\alpha}{\partial t}u(x,t) = a\,\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}u(x,t)\,\frac{\partial}{\partial x}\rho(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho(x), \tag{91}$$

$$\frac{{}^{ABC}\partial^\alpha}{\partial t}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{92}$$

with initial conditions

$$u(x,0) = m\,e^{-x^2}, \qquad \rho(x,0) = n\,e^{-x^2}. \tag{93}$$

Solution. Applying the aforesaid technique as in example 3, we obtain the following iterations:

$$u_1 = 2ame^{-x^2}\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]+2mne^{-2x^2}\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]-4ame^{-x^2}\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right],$$

$$\begin{aligned}\rho_1 = {} & 4ak^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & +4b^2k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & +4b\sqrt{a+b^2}k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right],\end{aligned}$$

$$\begin{aligned}u_2 = {} & 2ame^{-x^2}\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]+2mne^{-2x^2}\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & -4amx^2e^{-x^2}\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]+12a^2me^{-x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right],\\ & -2cm^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+12amne^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & +12bmne^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+2dmne^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & +mn^2e^{-3x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]-48a^2mx^2e^{-x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & -24amnx^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]-24bmnx^2e^{-2x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & +8mn^2x^2e^{-3x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+16a^2mx^4e^{-x^2}\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right],\end{aligned}$$

$$\begin{aligned}\rho_2 = {} & 4ak^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & \times 4b^2k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & +4b\sqrt{a+b^2}\,k^3\lambda\coth[k(c+x)]\sec^2[k(c+x)]\hbar(1+\hbar)\left[\frac{\alpha\, t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\\ & -4ak^4\lambda^2\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & -4b^2k^4\lambda^2\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ & \times 4b\sqrt{a+b^2}\,k^4\lambda^2\cosh[2k(c+x)]\sec^4[k(c+x)]\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\, t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right].\end{aligned} \tag{94}$$

Finally, the solutions to eqs. (91), (92) are given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots = \sum_{m=0}^{\infty} u_m(x,t), \tag{95}$$

$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots = \sum_{m=0}^{\infty} \rho_m(x,t). \tag{96}$$

Example 3. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho^2$ in the CFC sense:

$$\frac{{}^{CFC}\partial^\alpha}{\partial t} u(x,t) = a\,\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial x} u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x), \tag{97}$$

$$\frac{{}^{CFC}\partial^\alpha}{\partial t}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{98}$$

with initial conditions

$$u(x,0) = m\sin(x), \qquad \rho(x,0) = n\sin(x). \tag{99}$$

Solution. Applying the aforesaid technique, we obtain the following iterations:

$$\begin{aligned}
u_0(x,t) &= m\sin(x),\\
\rho_0(x,t) &= n\sin(x),\\
u_1(x,t) &= m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)\,\hbar[\alpha t + (1-\alpha)],\\
\rho_1(x,t) &= -\sin(x)\,(cm - (b+d)n)\,\hbar[\alpha t + (1-\alpha)],\\
u_2(x,t) &= m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)\,\hbar(1+\hbar)[\alpha t + (1-\alpha)]\\
&\quad - m\sin(x)\,[a + n\,\hbar(\alpha t + (1-\alpha))\,(2cm - an - 2bn - 2dn + n^3 + n^3\cos(2x))\sin^2(x)]\,\hbar[\alpha t + (1-\alpha)],\\
\rho_2(x,t) &= -\sin(x)\,(cm - (b+d)n)\,\hbar(1+\hbar)\,[\alpha t + (1-\alpha)]\\
&\quad -\sin(x)\,\left(acm + bcm + cdm - b^2 n - 2dbn - d^2 n + cnm^2 + cmn^2\cos(2x)\right)\hbar^2\left[(1-\alpha)^2 + 2\alpha(1-\alpha)\,t + \frac{\alpha^2 t^2}{2}\right].
\end{aligned} \tag{100}$$

Finally, the solutions to eqs. (97), (98) are given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots = \sum_{m=0}^{\infty} u_m(x,t), \tag{101}$$

$$\rho(x,t) = \rho_0(x,t) + \rho_1(x,t) + \rho_2(x,t) + \cdots = \sum_{m=0}^{\infty} \rho_m(x,t). \tag{102}$$

Example 4. We consider the K-S chemotaxis model with the sensitivity function $\chi(\rho) = \rho^2$ in the ABC sense:

$$\frac{{}^{ABC}\partial^\alpha}{\partial t} u(x,t) = a\,\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial x} u(x,t)\,\frac{\partial}{\partial x}\rho^2(x,t) + u(x,t)\,\frac{\partial^2}{\partial x^2}\rho^2(x), \tag{103}$$

$$\frac{{}^{ABC}\partial^\alpha}{\partial t}\rho(x,t) = b\,\frac{\partial^2}{\partial x^2}\rho(x,t) + c\,u(x,t) - d\,\rho(x,t), \tag{104}$$

with initial conditions

$$u(x,0) = m\sin(x), \qquad \rho(x,0) = n\sin(x). \tag{105}$$

Solution. Applying the aforesaid technique, we obtain the following iterations:

$$\begin{aligned}
u_0(x,t) &= m\sin(x),\\
\rho_0(x,t) &= n\sin(x),\\
u_1(x,t) &= \frac{1}{B(\alpha)}\,m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right]\hbar,\\
\rho_1(x,t) &= -\frac{1}{B(\alpha)}\sin(x)\,(cm - (b+d)n)\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right]\hbar,\\
u_2(x,t) &= \frac{1}{B(\alpha)}\,m\sin(x)\,\left(a + 2n^2\cos^2(x)\right)\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right]\hbar(1+\hbar)\\
&\quad - \frac{m}{B(\alpha)^2}\sin(x)\left[a + n\,\hbar\left(\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right)\,(2cm - an - 2bn - 2dn + n^3 + n^3\cos(2x))\sin^2(x)\right]\\
&\quad \times\left[\frac{\alpha t^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\right]\hbar,
\end{aligned}$$

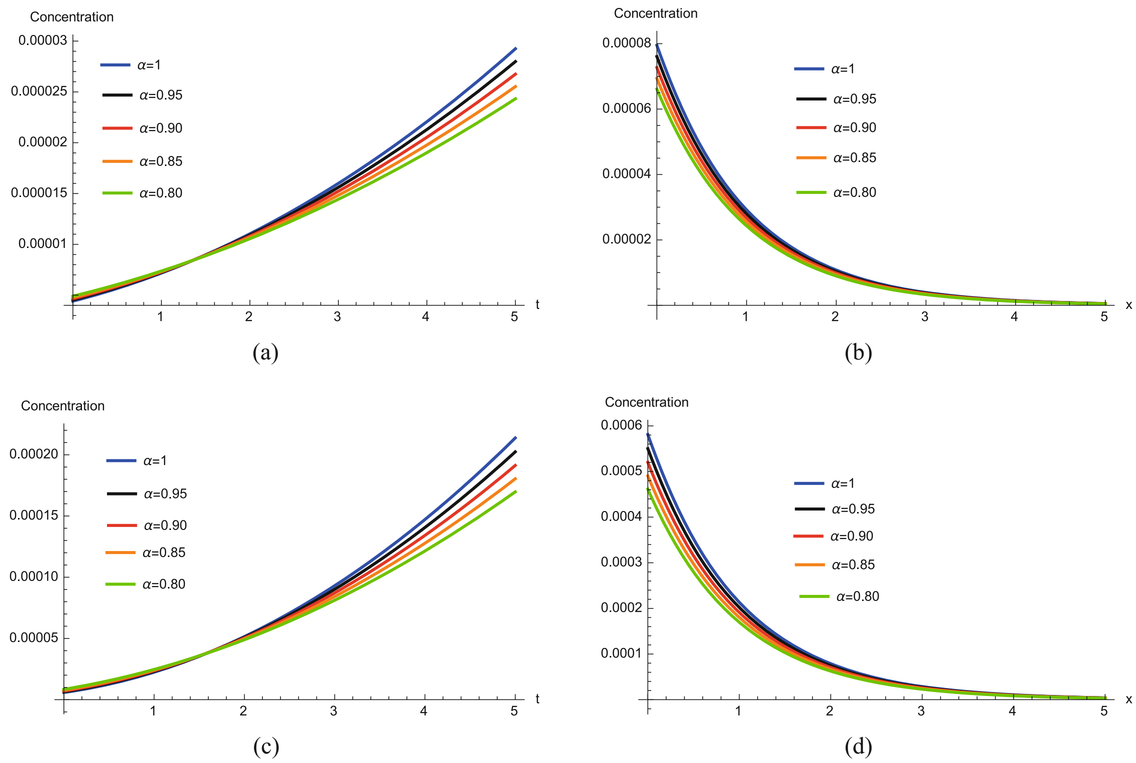

Fig. 1. Numerical simulation for the approximate solutions given by eqs. (26) and (27) for different values of α. In (a) plot of concentrations of the chemical substance in the human body as a function of space, when $x = 1$. In (b) plot of concentrations of the chemical substance in the human body as a function of time, when $t = 5$. In (c) plot of concentrations of amoebae in the human body as a function of space, when $x = 1$. In (b) plot of concentrations of amoebae in the human body as a function of time, when $t = 5$.

$$\begin{aligned}\rho_2(x,t) = &-\frac{1}{B(\alpha)}\sin(x)\,(cm-(b+d)n)\left[\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\right]\hbar(1+\hbar)-\frac{\sin(x)}{B(\alpha)^2}\,(acm+bcm+cdm-b^2n\\ &-2dbn-d^2n+cnm^2+cmn^2\cos(2x))\cdot\hbar^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right].\end{aligned} \tag{106}$$

Finally, the solutions to eqs. (103), (104) are given by

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots=\sum_{m=0}^{\infty}u_m(x,t), \tag{107}$$

$$\rho(x,t) = \rho_0(x,t)+\rho_1(x,t)+\rho_2(x,t)+\cdots=\sum_{m=0}^{\infty}\rho_m(x,t). \tag{108}$$

5 Numerical results and discussions

In this section, we show the numerical solution of the chemotaxis model obtained by LHPM and MHATM through a graphical representation. The following numerical solutions were obtained by LHPM. Consider the following values $a = 0.5$, $b = 3$, $c = 1$, $d = 2$, $m = 0.000012$ and $n = 0.000016$, arbitrarily chosen. Figures 1(a)–(d) show numerical simulations for eqs. (26) and (27) for different values of α. Figures 1(a) and (b) show the biological behavior of these solutions describing the concentrations of the chemical substance in the human body as a function of space, when $x = 1$, and the biological behavior of the concentrations of the chemical substance in the human body as a function of time, when $t = 5$. Additionally, plots (c) and (d) show the concentrations of amoebae in the human body as a function of space, when $x = 1$, and the concentrations of amoebae in the human body as a function of time, when $t = 5$.

Figures 2(a)–(d) show numerical simulations for eqs. (38), (39) for different values of α.

The following numerical solutions were obtained by MHATM through a graphical representation. Consider the following values $a = 0.5$, $b = 3$, $c = 1$, $d = 2$, $h = 0.8$, $m = 0.000012$ and $n = 0.000016$, arbitrarily chosen. Figures 3(a)–(d) shows numerical simulations for eqs. (101), (102) for different values of α.

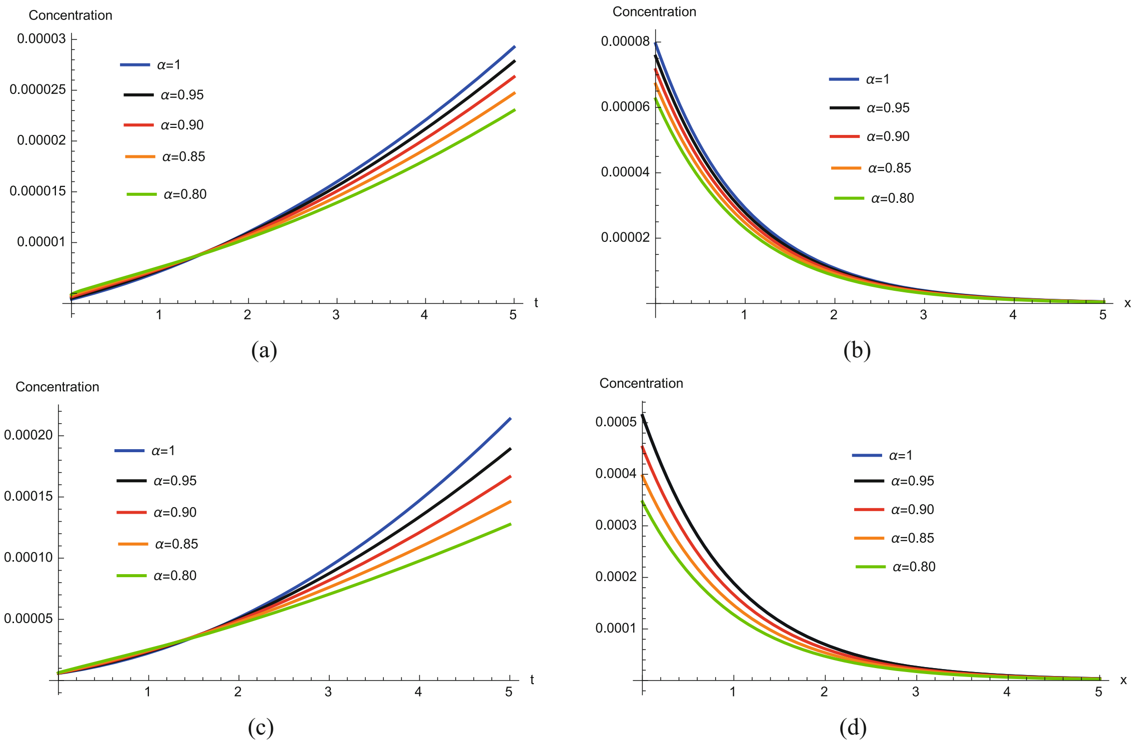

Fig. 2. Numerical simulation for the approximate solutions given by eqs. (38) and (39) for different values of α. In (a) plot of concentrations of the chemical substance in the human body as a function of space, when $x = 1$. In (b) plot of concentrations of the chemical substance in the human body as a function of time, when $t = 5$. In (c) plot of concentrations of amoebae in the human body as a function of space, when $x = 1$. In (b) plot of concentrations of amoebae in the human body as a function of time, when $t = 5$.

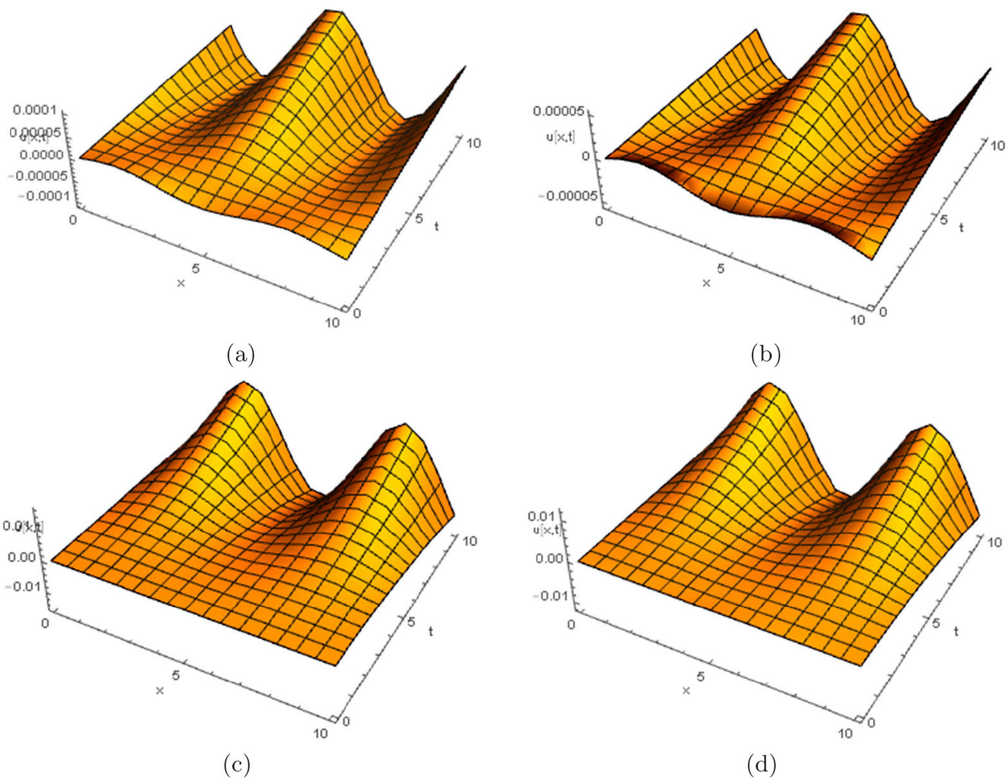

Fig. 3. The surface graph for the approximate solutions given by eqs. (101) and (102) for different values of α. In (a) concentration of the chemical substance in the human body for $\alpha = 1$. In (b) concentration of the chemical substance in the human body for $\alpha = 0.8$. In (c) concentration of amoebae in the human body for $\alpha = 1$. In (b) concentration of amoebae in the human body for $\alpha = 0.8$.

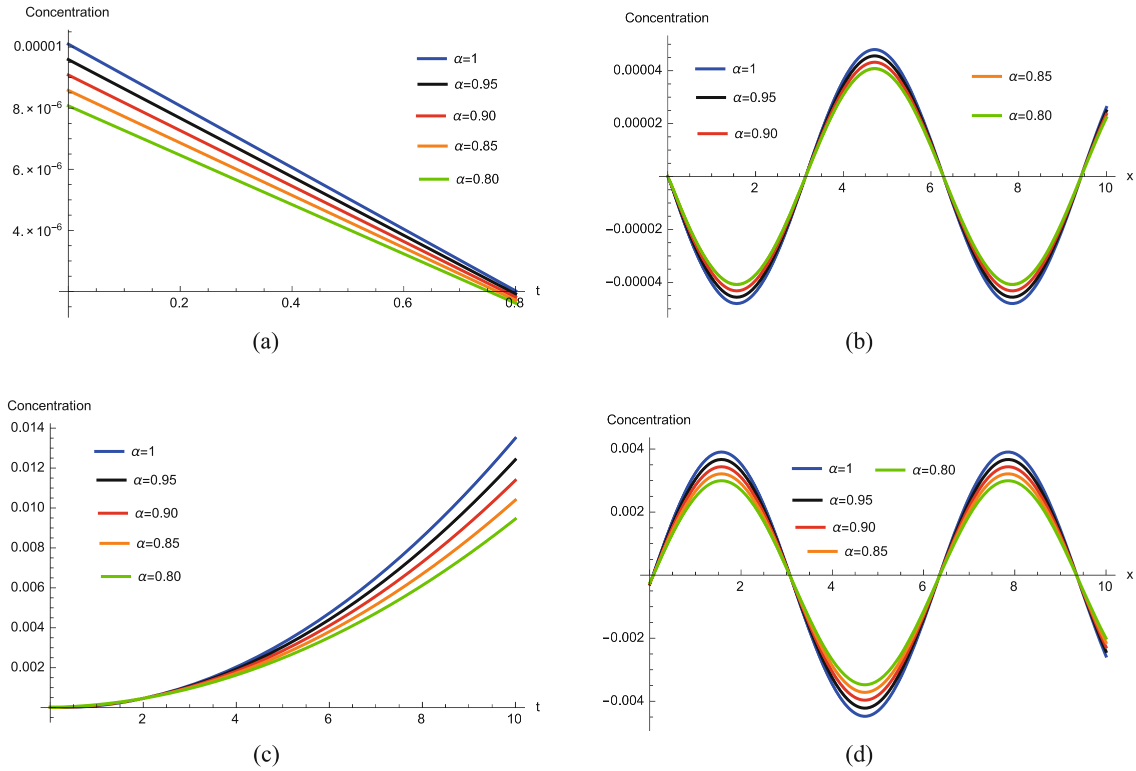

Fig. 4. Numerical simulation for the approximate solutions given by eqs. (101) and (102) for different values of α. In (a) plot of concentration of the chemical substance in the human body as a function of space when $x = 1$. In (b) plot of concentration of the chemical substance in the human body as a function of time when $t = 5$. In (c) plot of concentration of amoebae in the human body as a function of space when $x = 1$. In (b) plot of concentration of amoebae in the human body as a function of time when $t = 5$.

Figures 4(a)–(d) show the two-dimensional physical behavior of the K-S chemotaxis model (eqs. (101), (102)) as a function of space for a fixed time $t = 5$ and as a function of time for a fixed distance $x = 1$, respectively, for different values of α.

Figures 5(a)–(d) show the three-dimensional behavior of the approximate solution obtained by the proposed method for eqs. (107), (108) for different values of α.

Figures 6(a)–(d) show numerical simulations for eqs. (107) and (108) for different values of α.

6 Conclusions

In this work, the LHPM and the MHATM were considered to obtain analytical approximated solution for the K-S chemotaxis model considering the fractional derivatives of Caputo-Fabrizio-Caputo and fractional derivatives with generalized Mittag-Leffler law in Liouville-Caputo sense. Approximate analytical solutions were obtained for different sensitivity functions. Also numerical simulations for the K-S chemotaxis model were obtained with the HPTM and MHATM for these two different fractional derivatives. Both presented methods have proved to be important mathematical tools for scientists working in various areas of natural sciences. The two definitions of fractional operators considered in this work must be applied conveniently depending on the nature and the phenomenological behavior of the system under consideration.

The polynomial expansion considered in the HTPM method permits to obtain an infinite series solution for the K-S chemotaxis model. Based on this method, a general scheme was developed to obtain approximate solutions of fractional equations and the solutions are given in a series form, which converges rapidly. The novelty of MHATM scheme, however, lies in the combination of the homotopy analysis method and Laplace transform with homotopy polynomial for solving nonlinear fractional partial differential equations. This work shows that the LHPM and the MHATM are efficient tools for solving nonlinear FODEs considering fractional operators of Caputo-Fabrizio-Caputo and the fractional derivative with generalized Mittag-Leffler law of Liouville-Caputo type.

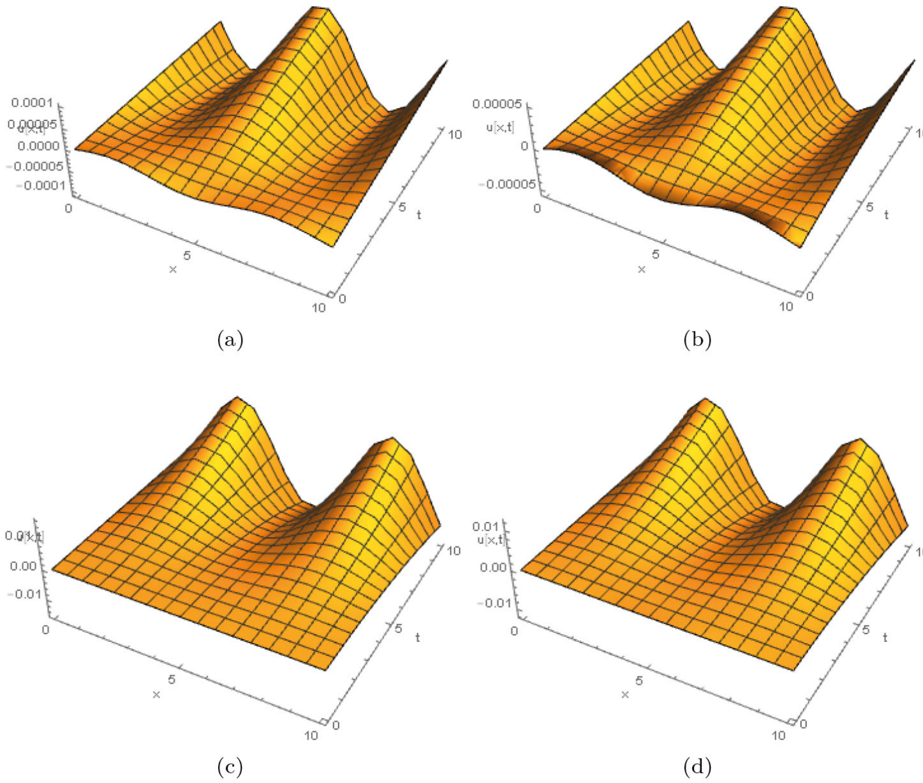

Fig. 5. The surface graph for the approximate solutions given by eqs. (107) and (108) for different values of α. In (a) concentration of the chemical substance in the human body for $\alpha = 1$. In (b) concentration of the chemical substance in the human body for $\alpha = 0.8$. In (c) concentration of amoebae in the human body for $\alpha = 1$. In (b) concentration of amoebae in the human body for $\alpha = 0.8$.

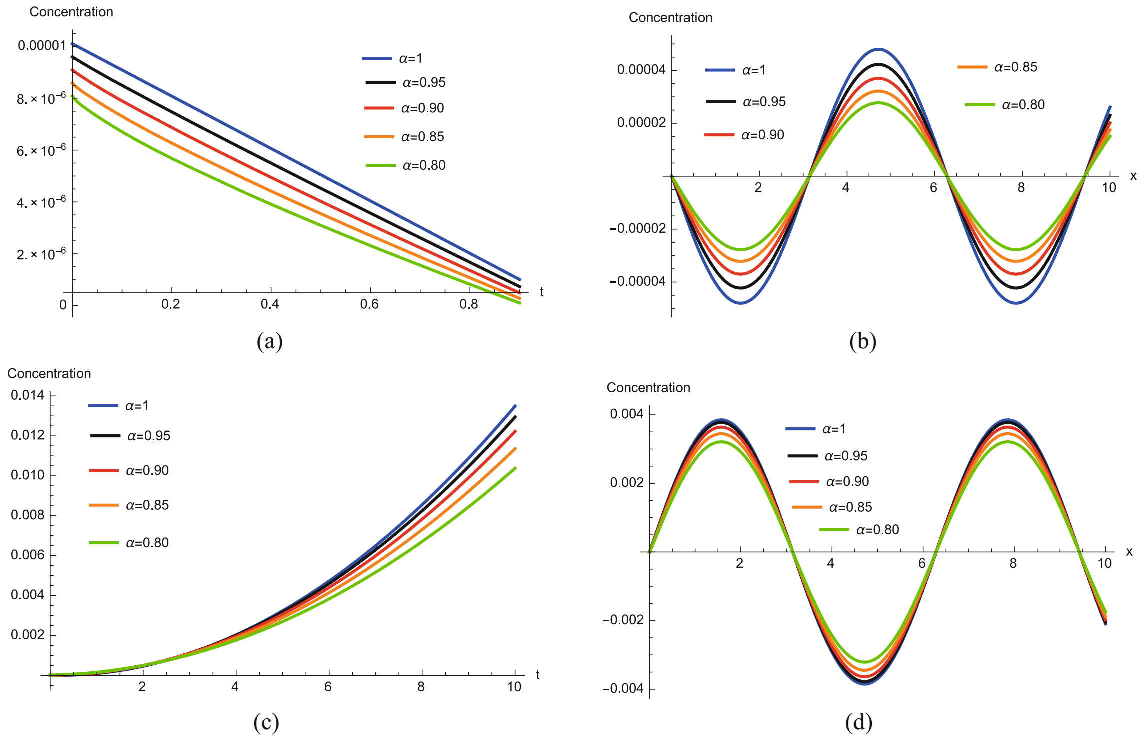

Fig. 6. Numerical simulation for the approximate solutions given by eqs. (107) and (108) for different values of α. In (a) plot of concentration of the chemical substance in the human body as a function of space when $x = 1$. In (b) plot of concentration of the chemical substance in the human body as a function of time when $t = 5$. In (c) plot of concentration of amoebae in the human body as a function of space when $x = 1$. In (b) plot of concentration of amoebae in the human body as a function of time when $t = 5$.

JFGA acknowledges the support provided by CONACyT: Cátedras CONACyT para jóvenes investigadores 2014. JFGA and MATH acknowledge the support provided by SNI-CONACyT.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Author contribution statement

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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