

THE KELLER–SEGEL EQUATION COMPACT SURFACES

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ABSTRACT. We study the (parabolic-elliptic) Keller–Segel equations on compact surfaces.

To be completed.

1. INTRODUCTION

To be completed. This is the part that we will write last.

Organization of the paper. To be completed.

Acknowledgment. To be completed.

2. REFORMULATION OF THE EQUATION

Let Σ be a smooth, closed surface with a Riemannian metric g and area form ω . Let G be the Green operator of the Laplacian, Δ , on $L^2(\Sigma, g)$ and let us fix $\rho_0 \in L^2(\Sigma, g)$. Note that $L^2 \hookrightarrow L^1$ on domains of finite measure. We say that a positive function, $\rho \in C^1((0, T); L^2(\Sigma, g))$ is said to satisfy the *parabolic-elliptic Keller–Segel equations* on (Σ, g) with initial value ρ_0 if it is a solution to the following system:

$$\partial_t \rho = -\Delta \rho + d^*(\rho dG(\rho)), \quad (2.1a)$$

$$\lim_{t \rightarrow 0^+} \rho_t = \rho_0. \quad (2.1b)$$

where for all $t \in (0, T)$

$$\rho_t := \rho|_{\{t\} \times \Sigma},$$

regarded as a function in $L^2(\Sigma, g)$.

Now let

$$M(t) := \int_{\Sigma} \rho_t \, dA.$$

Date: August 18, 2021.

2020 Mathematics Subject Classification. 35J15, 35Q92, 92C17.

Key words and phrases. chemotaxis, Keller–Segel equations.

Then M is constant, because

$$\dot{M}(t) = \int_{\Sigma} (-\Delta\rho + d^*(\rho dG(\rho))) dA = \int_{\Sigma} d^*(-d\rho + \rho dG(\rho)) dA = 0.$$

Thus we drop the t -dependence of M from its notation.

For the rest of the paper, let $A_{\Sigma} := \text{Area}(\Sigma, g)$. The following lemma recasts equations (2.1a) and (2.1b) in a simpler form.

Lemma 2.1. *Let $\chi_0 := \rho_0 - \frac{M}{A_{\Sigma}}$ and $\chi := \rho - \frac{M}{A_{\Sigma}}$. Then equations (2.1a) and (2.1b) is equivalent to*

$$\partial_t \chi = \frac{M}{A_{\Sigma}} \chi - \Delta \chi + d^*(\chi dG(\chi)), \quad (2.2a)$$

$$\chi|_{\{0\} \times \Sigma} = \chi_0. \quad (2.2b)$$

Proof. The equivalency of equation (2.1b) and equation (2.2b) is obvious.

Note that G annihilates constants and since χ is orthogonal to constants, we have that $\Delta(G(\chi)) = \chi$. Since $\chi = \rho - \frac{M}{A_{\Sigma}}$, we get, using equation (2.1a), that

$$\begin{aligned} \partial_t \chi &= \partial_t \left(\rho - \frac{M}{A_{\Sigma}} \right) \\ &= \partial_t \rho - 0 \\ &= -\Delta \rho + d^*(\rho dG(\rho)) \\ &= -\Delta \left(\frac{M}{A_{\Sigma}} + \chi \right) + d^* \left(\left(\frac{M}{A_{\Sigma}} + \chi \right) dG \left(\frac{M}{A_{\Sigma}} + \chi \right) \right) \\ &= -\Delta \chi + d^* \left(\left(\frac{M}{A_{\Sigma}} + \chi \right) dG(\chi) \right) \\ &= -\Delta \chi + \frac{M}{A_{\Sigma}} d^* dG(\chi) + d^*(\chi dG(\chi)) \\ &= \frac{M}{A_{\Sigma}} \chi - \Delta \chi + d^*(\chi dG(\chi)), \end{aligned}$$

which completes the proof. \square

Remark 2.2. *Let λ_1 be the smallest nonzero eigenvalue of Δ and note that quantity $M_{\Sigma} = \lambda_1 A_{\Sigma}$ only depends on the geometry of (Σ, g) . When $M < M_{\Sigma}$, then the linear term in equation (2.2a) is strictly negative definite.*

3. THE GENERALIZED FOURIER TRANSFORM

Let now $(\Psi_a \in L^2(\Sigma, g))_{a \in \mathbb{N}}$ be an orthonormal eigenbasis of Δ and

$$\Delta \Psi_a = \lambda_a \Psi_a.$$

Let us order this basis so that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_a \leq \lambda_{a+1} \leq \dots$$

In particular, $\Psi_0 = \frac{1}{\sqrt{A_\Sigma}}$.

Assume that χ is a solution to equations (2.2a) and (2.2b) and write

$$R_a(t) := \left\langle \Psi_a | \chi |_{\{t\} \times \Sigma} \right\rangle_{L^2(\Sigma, g)}.$$

Then for each $a \in \mathbb{N}$, we have that $R_a \in C^1((0, T); \mathbb{R})$. Note that $R_0 \equiv \frac{M}{A_\Sigma}$. Finally let

$$\forall a, b, c \in \mathbb{N} : \quad \varphi_{abc} := \int_{\Sigma} \Psi_a \Psi_b \Psi_c \, dA.$$

Note that φ_{abc} is a completely symmetric 3-tensor.

Theorem 3.1. *The function ρ is a solution to equation (2.1a) exactly when*

$$\forall t \in (0, T) : \quad (R_a(t))_{a \in \mathbb{N}} \in l^2(\mathbb{N}), \quad (3.1a)$$

$$\forall a \in \mathbb{N}_+ : \quad \dot{R}_a = \left(\frac{M}{A_\Sigma} - \lambda_a \right) R_a + \sum_{b, c \in \mathbb{N}} \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c} \varphi_{abc} R_b R_c. \quad (3.1b)$$

Furthermore, if equation (2.1b) is also satisfied, then for all $a \in \mathbb{N}$, $\lim_{t \rightarrow 0^+} R_a(t)$ exists and

$$\rho_0 = \sum_{a \in \mathbb{N}} \left(\lim_{t \rightarrow 0^+} R_a(t) \right) \Psi_a. \quad (3.2)$$

Proof. Since $(\Psi_a)_{a \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Sigma, g)$ and for all $t \in (0, T)$, ρ_t is in $L^2(\Sigma, g)$, we get condition (3.1a).

Fix $a \in \mathbb{N}_+$. Then

$$G(\Psi_a) = \lambda_a^{-1} \Psi_a.$$

Using the above equation, the self-adjointness of Δ , and equation (2.2a), we get

$$\begin{aligned} \dot{R}_a &= \langle \Psi_a | \partial_t \chi \rangle_{L^2(\Sigma, g)} \\ &= \left\langle \Psi_a \left| \frac{M}{A_\Sigma} \chi - \Delta \chi + d^*(\chi dG(\chi)) \right. \right\rangle_{L^2(\Sigma, g)} \\ &= \left\langle \frac{M}{A_\Sigma} \Psi_a - \Delta \Psi_a \left| \chi \right. \right\rangle_{L^2(\Sigma, g)} + \langle d\Psi_a | \chi dG(\chi) \rangle_{L^2(\Sigma, g)} \\ &= \left(\frac{M}{A_\Sigma} - \lambda_a \right) \langle \Psi_a | \chi \rangle_{L^2(\Sigma, g)} + \sum_{b, c \in \mathbb{N}_+} \langle d\Psi_a | \Psi_b dG(\Psi_c) \rangle_{L^2(\Sigma, g)} R_b R_c \\ &= \left(\frac{M}{A_\Sigma} - \lambda_a \right) R_a + \sum_{b, c \in \mathbb{N}_+} \langle d\Psi_a | \Psi_b d\Psi_c \rangle_{L^2(\Sigma, g)} \lambda_c^{-1} R_b R_c. \end{aligned} \quad (3.3)$$

Note that

$$\langle d\Psi_a | \Psi_b d\Psi_c \rangle_{L^2(\Sigma, g)} = \int_{\Sigma} \Psi_b g(d\Psi_a, d\Psi_c) dA, \quad (3.4)$$

and

$$\Delta(\Psi_a \Psi_c) = (\Delta \Psi_a) \Psi_c + \Psi_a (\Delta \Psi_c) - 2g(d\Psi_a, d\Psi_c) = (\lambda_a + \lambda_c) \Psi_a \Psi_c - 2g(d\Psi_a, d\Psi_c).$$

Thus

$$g(d\Psi_a, d\Psi_c) = \frac{\lambda_a + \lambda_c}{2} \Psi_a \Psi_c - \frac{1}{2} \Delta(\Psi_a \Psi_c).$$

Plugging the above equation into equation (3.4) we get

$$\begin{aligned} \langle d\Psi_a | \Psi_b d\Psi_c \rangle_{L^2(\Sigma, g)} &= \int_{\Sigma} \Psi_b g(d\Psi_a, d\Psi_c) dA \\ &= \int_{\Sigma} \Psi_b \left(\frac{\lambda_a + \lambda_c}{2} \Psi_a \Psi_c - \frac{1}{2} \Delta(\Psi_a \Psi_c) \right) dA \\ &= \frac{\lambda_a + \lambda_c}{2} \varphi_{abc} - \frac{1}{2} \int_{\Sigma} (\Delta \Psi_b) \Psi_a \Psi_c dA \\ &= \frac{\lambda_a + \lambda_c}{2} \varphi_{abc} - \frac{\lambda_b}{2} \int_{\Sigma} \Psi_b \Psi_a \Psi_c dA \\ &= \frac{\lambda_a - \lambda_b + \lambda_c}{2} \varphi_{abc}. \end{aligned}$$

Inserting this into equation (3.3) yields equation (3.1b).

The equivalency of equation (2.1b) and equation (3.2) is straightforward. \square

Remark 3.2. *The moral of Theorem 3.1 is that the Keller–Segel equations, which is a (hard) elliptic-parabolic system of partial differential equations, can be transformed (on closed surfaces) into a infinite system of ordinary differential equations, which is potentially easier to handle.*

In the rest of the paper we show that this system can be further simplified under certain extra hypotheses.

3.1. Analytic solutions. In order to further simplify equation (3.1b), we search for analytic solutions, that is

$$\forall a \in \mathbb{N} : \forall t \in (0, T) : \quad R_a(t) = \sum_{n \in \mathbb{N}} R_{a,n} t^n, \quad (3.5)$$

and the right-hand side is assumed to be absolute convergent in $l^2(\mathbb{N})$.

The next lemma rewrites equation (3.1b) in terms of the coefficients $(R_{a,n})_{(a,n) \in \mathbb{N} \times \mathbb{N}}$.

Lemma 3.3. *Under the above assumption, the function ρ is a t -analytic solution to equation (2.1a) with mass M exactly when $R_{0,0} = \frac{M}{A_\Sigma}$, for all $n \in \mathbb{N}_+$, $R_{0,n} = 0$, and*

$$\forall a, n \in \mathbb{N} : R_{a,n+1} = \frac{1}{n+1} \left(\left(\frac{M}{A_\Sigma} - \lambda_a \right) R_{a,n} + \sum_{b,c \in \mathbb{N}_+} \sum_{m=0}^n \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c} \varphi_{abc} R_{b,m} R_{c,n-m} \right), \quad (3.6a)$$

$$\forall n \in \mathbb{N} : \mathcal{R}_n := (R_{a,n})_{a \in \mathbb{N}} \in l^2(\mathbb{N}), \quad (3.6b)$$

$$\limsup_{n \rightarrow \infty} \|\mathcal{R}_n\|_{l^2(\mathbb{N})}^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} \left(\sum_{a \in \mathbb{N}} R_{a,n}^2 \right)^{\frac{1}{n}} < \frac{1}{T}. \quad (3.6c)$$

Proof. Inserting equation (3.5) into equation (3.1b) yields equation (3.6a). The equations (3.6b) and (3.6c) are necessary (and, in fact, sufficient) to have that the convergence radius of the Taylor series of ρ in the L^2 topology is at least T . \square

In the following two sections we investigate two special cases when iteration in equation (3.6a) exists for all a and n .

4. ROUND SPHERES

Let Σ be the 2-sphere and g be the round metric of radius r . Then we have that $(\Psi_a)_{a \in \mathbb{N}}$ are the spherical harmonics. Then $A_\Sigma = 4\pi r^2$. In fact, after relabeling them, we can write the eigenvalue has the form $\lambda_{l,m} := \frac{l(l+1)}{r^2}$, where $l \in \mathbb{N}$ and M is any integer satisfying $|m| \leq l$. Let us now write

$$\begin{aligned} \Psi_{(l,m)} &:= \Psi_l^m, \\ R_{l,a}^m &:= R_{(l,m),a}, \\ \varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} &:= \varphi_{(l_1,m_1),(l_2,m_2),(l_3,m_3)}. \end{aligned}$$

Using this new set of indices and notation, we can rewrite equation (3.6a) as

$$\begin{aligned} \forall l \in \mathbb{N}_+ : R_{l,n+1}^m &= \frac{1}{n+1} \left(\frac{M}{4\pi r^2} - \frac{l(l+1)}{r^2} \right) R_{(l,m),n} \\ &+ \sum_{l_1, l_2 \in \mathbb{N}_+} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \sum_{x=0}^n \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_1l_2} \varphi_{l_1,l_2}^{m,m_1,m_2} R_{l_1,x}^{m_1} R_{l_2,n-x}^{m_2}. \end{aligned} \quad (4.1)$$

Using the Clebsch–Gordan Theorem, we have that if $l_1 \geq l_2 + l_3$ or $l_1 \leq |l_2 - l_3|$, then for all m_1, m_2 , and m_3 , we have $\varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} = 0$. Thus equation (4.1) becomes

$$R_{l,n+1}^m = \frac{1}{n+1} \left(\frac{M}{4\pi r^2} - \frac{l(l+1)}{r^2} \right) R_{l,n}^m$$

$$+ \sum_{l_1 \in \mathbb{N}_+} \sum_{l_2 = \max(\{1, |l-l_1|\})}^{l+l_1} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \sum_{x=0}^n \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)} \varphi_{l,l_1,l_2}^{m,m_1,m_2} R_{l_1,x}^{m_1} R_{l_2,n-x}^{m_2}. \quad (4.2)$$

Before prove the main result of this section, let us make the following definition:

$$\forall n \in \mathbb{N} : Z_n := \{ (l, m) \in \mathbb{N} \times \mathbb{Z} \mid R_{l,n}^m \neq 0 \}.$$

Proposition 4.1. *Assume that Z_0 is finite. Then for all $(l, m) \in \mathbb{N} \times \mathbb{Z}$ and $n \in \mathbb{N}$, $R_{l,n}^m$ exists. Furthermore, Z_n is also finite, and equation (4.2) becomes*

$$R_{l,n+1}^m = \frac{1}{n+1} \left(\frac{M}{4\pi r^2} - \frac{l(l+1)}{r^2} \right) R_{l,n}^m + \sum_{x=0}^n \sum_{\substack{(l_1, m_1) \in Z_x \\ (l_2, m_2) \in Z_{n-x}}} \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)} \varphi_{l,l_1,l_2}^{m,m_1,m_2} R_{l_1,x}^{m_1} R_{l_2,n-x}^{m_2}.$$

Proof. To be completed. □

5. FLAT TORI

To be completed.

6. SOLUTIONS ON $\mathbb{R}^2/\mathbb{Z}^2$

If we figure this out.

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