THE KELLER–SEGEL EQUATION COMPACT SURFACES

1. LaTeXcomments

A few "best practices" that might be worth adopting (in no particular order):

- Cleaner code: Try adding tabs and empty line (between paragraphs) for better readability. Nothing I ever write is good on the first try, so being able to conveniently go back and re-read the whole thing is a life-saver. :)
  I also like adding some amount of spaces in the the math environment, so when I have to change something later in a huge formula it's easier.
- Don't use \[...\] (or $$ for that matter), but the equation environment. Better formatting, easier to add labels later, etc. You can make a macro for it if you think it's too long to type it out.
- Instead of arrays with an equation, use the align environment.
- Less important: integrals and sums look nicer in equation mode (but not in text) if you use \limits.
- To make what I mean clearer, I've implemented the changes in your code. Let me know what you think!

2. The Keller–Segel equation on the torus

I've found a mistake in the equation that probably I've made too in the meeting. Fixed it below, please double-check!

Consider a distribution of germs $\rho(t, x_1, x_2) = \rho(t, \underline{x})$ and food $c(t, x_1, x_2) = c(t, \underline{x})$. We impose, as a model of nature,

$$\partial_t \rho = \partial_a^2 \rho - \partial_a(\rho \partial_a c), \tag{2.1a}$$

$$\partial_a^2 c = -\rho. \tag{2.1b}$$

When a appears as an index, summation over $a \in \{1, 2\}$ is implied. We take the Fourier transform of ρ: For all $\underline{k} \in \mathbb{Z}^2$, let $f_{\underline{k}}(\underline{x}) = e^{2\pi i \underline{k} \cdot \underline{x}}$. Note that $f_{\underline{k}}$ is an eigenfunction of the Laplacian; i. e. $\partial_a^2 f_{\underline{k}} = 4\pi^2 |k^2| f_{\underline{k}}$. Also note that $\partial_a f_{\underline{k}} = 2\pi i k_a f_{\underline{k}}$.

Date: June 30, 2021.
Key words and phrases. Keller–Segel equations.

Now let us write

$$\rho(t,\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}). \tag{2.2}$$

Now any solution, c, to equation (2.1b) has the form

$$c(t,\underline{x}) = c_0 + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{1}{4\pi^2|\underline{l}|^2} R_{\underline{l}}(t) f_{\underline{l}}(\underline{x}),$$

where $c_0 \in \mathbb{C}$ can be chosen arbitrarily.

Using equations (2.1a) and (2.2) we get that

$$\begin{aligned}
\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2|\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -\partial_a\left(\sum_{\underline{m}\in\mathbb{Z}^2} R_{\underline{m}} f_{\underline{m}} \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{R_{\underline{l}}}{4\pi^2|\underline{l}|^2}\partial_a f_{\underline{l}}\right)\\
&= -\sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{2\pi i l_a}{4\pi^2|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} \partial_a\Big(f_{\underline{l}} f_{\underline{m}}\Big).
\end{aligned}$$

Using that $f_{\underline{m}} f_{\underline{l}} = f_{\underline{m}+\underline{l}}$ and substituting $\underline{k} = \underline{l}+\underline{m}$ on the right-hand side, we get

$$\begin{aligned}
\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2|\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -i^2 \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{l_a(l_a+m_a)}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}}\\
&= \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot(\underline{l}+\underline{m})}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}}\\
&= \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}} f_{\underline{k}}.
\end{aligned}$$

After pairing with $f_{\underline{k}}$ for any $\underline{k}\in\mathbb{Z}^2-\{\underline{0}\}$ and separating out $R_{\underline{k}}$ terms, we get

$$\dot{R}_{\underline{k}} = \Big(R_0 - 4\pi^2|\underline{k}|^2\Big)R_{\underline{k}} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{\underline{k}-\underline{l}} R_{\underline{l}}.$$

It remains to be shown that there are such $R_{\underline{k}}$ which satisfy the equations.

3. The general case

Let Σ now some compact surface, with (positive definite) Laplace operator Δ (we can discuss what that means at some point), and assume that $f_0, f_1, f_2, \ldots, f_n, \ldots$ are an orthonormal basis of eigenvectors for $L^2(\Sigma)$, that is there are numbers $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ so that for all $n, o \in N$ we have

$$\Delta f_n = \lambda_n f_n, \quad \& \quad \langle f_n | f_o \rangle_{L^2(\Sigma)} = \delta_{n,o}.$$

For all $n, o, p \in \mathbb{N}$, let

$$\varphi_{n,o,p} := \int\limits_{\Sigma} f_n f_o f_p \,\mathrm{d}A.$$

With that in mind, at some point prove the following: If $\rho \in C^1\big([0,T]; L^2(\Sigma)\big)$ solve the Keller–Segel equations on Σ and $R_n(t) := \langle f_n | \rho(t, \cdot) \rangle_{L^2(\Sigma)}$, then R_0 is constant and

$$\forall n \in \mathbb{N} - \{0\} : \quad \dot{R}_n = (R_0 - \lambda_n) R_n + \sum_{o,p \in \mathbb{N} - \{0\}} \frac{\lambda_m - \lambda_o + \lambda_p}{\lambda_p} \varphi_{n,o,p} R_o R_p.$$

Remark 3.1. Note how the sign of the first term changes depending on whether $R_0 = \int_\Sigma \rho \,\mathrm{d}A$ is small or greater than λ_n!