THE KELLER–SEGEL EQUATION COMPACT SURFACES

1. The Keller–Segel equation on the torus

Consider a distribution of germs $\rho(t, x_1, x_2) = \rho(t, \underline{x})$ and food $c(t, x_1, x_2) = c(t, \underline{x})$. We impose, as a model of nature,

$$\partial_t \rho = \partial_a^2 \rho - \partial_a(\rho \partial_a c), \tag{1.1a}$$

$$\partial_a^2 c = -\rho. \tag{1.1b}$$

When a appears as an index, summation over $a \in \{1, 2\}$ is implied. We take the Fourier transform of ρ: For all $\underline{k} \in \mathbb{Z}^2$, let $f_{\underline{k}}(\underline{x}) = e^{2\pi i \underline{k} \cdot \underline{x}}$. Note that $f_{\underline{k}}$ is an eigenfunction of the Laplacian; i.e. $\partial_a^2 f_{\underline{k}} = -4\pi^2 |k|^2 f_{\underline{k}}$. Also note that $\partial_a f_{\underline{k}} = 2\pi i k_a f_{\underline{k}}$.

Let us write

$$\rho(t, \underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}). \tag{1.2}$$

Now any solution, c, to equation (1.1b) has the form

$$c(t, \underline{x}) = c_0 + \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{1}{4\pi^2 |\underline{l}|^2} R_{\underline{l}}(t) f_{\underline{l}}(\underline{x}),$$

where $c_0 \in \mathbb{C}$ can be chosen arbitrarily.

Using equations (1.1a) and (1.2) we get that

$$\sum_{\underline{k} \in \mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k} \in \mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} = -\partial_a \left(\sum_{\underline{m} \in \mathbb{Z}^2} R_{\underline{m}} f_{\underline{m}} \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{R_{\underline{l}}}{4\pi^2 |\underline{l}|^2} \partial_a f_{\underline{l}} \right)$$

$$= -\sum_{\substack{\underline{l}, \underline{m} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{2\pi i l_a}{4\pi^2 |\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} \partial_a \left(f_{\underline{l}} f_{\underline{m}} \right).$$

Using that $f_{\underline{m}}f_{\underline{l}} = f_{\underline{m}+\underline{l}}$ and substituting $\underline{k} = \underline{l} + \underline{m}$ on the right-hand side, we get

$$\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}}f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2|\underline{k}|^2 R_{\underline{k}}f_{\underline{k}} = -i^2 \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{l_a(l_a + m_a)}{|\underline{l}|^2} R_{\underline{l}}R_{\underline{m}}f_{\underline{l}+\underline{m}}$$

$$= \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot(\underline{l}+\underline{m})}{|\underline{l}|^2} R_{\underline{l}}R_{\underline{m}}f_{\underline{l}+\underline{m}}$$

$$= \sum_{\substack{\underline{l},\underline{k}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}}R_{\underline{k}-\underline{l}}f_{\underline{k}}.$$

After pairing with $f_{\underline{k}}$ for any $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$ and separating out $R_{\underline{k}}$ terms, we get

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}}R_{\underline{k}-\underline{l}}.$$

It remains to be shown that there are such $R_{\underline{k}}$ which satisfy the equations.

2. The general case

Let Σ now some compact surface, with (positive definite) Laplace operator Δ (we can discuss what that means at some point), and assume that $f_0, f_1, f_2, \ldots, f_n, \ldots$ are an orthonormal basis of eigenvectors for $L^2(\Sigma)$, that is there are numbers $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ so that for all $n, o \in N$ we have

$$\Delta f_n = \lambda_n f_n, \quad \& \quad \langle f_n | f_o \rangle_{L^2(\Sigma)} = \delta_{n,o}.$$

For all $n, o, p \in \mathbb{N}$, let

$$\varphi_{n,o,p} := \int_\Sigma f_n f_o f_p \, dA.$$

With that in mind, at some point prove the following: If $\rho \in C^1\big([0,T]; L^2(\Sigma)\big)$ solve the Keller–Segel equations on Σ and $R_n(t) := \langle f_n | \rho(t,\cdot)\rangle_{L^2(\Sigma)}$, then R_0 is constant and

$$\forall n \in \mathbb{N} - \{0\}: \quad \dot{R}_n = (R_0 - \lambda_n)R_n + \sum_{o,p\in\mathbb{N}-\{0\}} \frac{\lambda_m - \lambda_o + \lambda_p}{\lambda_p} \varphi_{n,o,p} R_o R_p.$$

Remark 2.1. Note how the sign of the first term changes depending on whether $R_0 = \int_\Sigma \rho \, dA$ is small or greater than λ_n!

3. Banach Business

Consider the map $\mathcal{K} : \{R_{\underline{k}}\}_{\underline{k} \in \mathbb{Z}^2} \to C$ (C is something, perhaps $\{R_{\underline{k}}\}_{\underline{k} \in \mathbb{Z}^2}$) defined by

$$\mathcal{K}(R_{\underline{k}}) = R_{\underline{k}}(0) + \int_0^t (R_0 - 4\pi^2 |\underline{k}|^2) R_{\underline{k}}(\tau) + \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} R_{\underline{k}-\underline{l}}(\tau) R_{\underline{l}}(\tau) \, d\tau$$

Collections of fourier coefficients with $\mathcal{K}(R) = R$ satisfy equation (1.1a). We seek, therefore, conditions (on $(R_{\underline{k}})_{\underline{k} \in \mathbb{Z}^2}$ and t) which yield fixed points of $\mathcal{K}$. It would be sufficient to bound

$$\|\mathcal{K}(R) - \mathcal{K}(S)\| \leq \theta \|R - S\|$$

where $\|\cdot\|$ is perhaps

$$\sup_{t \in [0,t]} \sum_{\underline{k} \in \mathbb{Z}^2} |R_k(t)|^2.$$

and $0 \leq \theta < 1$ is some constant. We might begin to consider $\|\mathcal{K}(R) - \mathcal{K}(S)\| =$

$$\left\| \int_0^t (R_0 - 4\pi^2 |\underline{k}|^2) \Big(R_{\underline{k}}(\tau) - S_{\underline{k}}(\tau) \Big) + \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} \Big(R_{\underline{k}-\underline{l}}(\tau) R_{\underline{l}}(\tau) - S_{\underline{k}-\underline{l}}(\tau) S_{\underline{l}}(\tau) \Big) d\tau \right\|.$$

It may be fruitful to bound the summands of $\|\cdot\|$ corresponding to values of $\underline{k}$, sharply enough that the infinite sum converges.

4. Bounding Blunders

Circuitously, we will show $\sum_{\underline{k} \in [1,2,\dots]^2} \frac{1}{|\underline{k}|^2} = \infty$. See that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + m^2} = \frac{\coth(\pi m)\pi m - 1}{2m^2}.$$

Now $\coth(x) > \frac{1}{\pi x} + \frac{\pi x}{3} - \frac{\pi^3 x^3}{45}$ so that

$$\sum_{n,m=1}^{\infty} \frac{1}{n^2 + m^2} = \sum_{m=1}^{\infty} \frac{\coth(\pi m)\pi m - 1}{2m^2} > \sum_{m=1}^{\infty} \frac{\left(\frac{1}{\pi^2 m} + \frac{\pi^2 m}{3} - \frac{\pi^6 m^3}{45} \right) \pi m - 1}{2m^2}$$

$$= \sum_{m=1}^{\infty} \frac{1 + \frac{\pi^4 m^2}{3} - \frac{\pi^8 m^4}{45} - \pi}{2\pi m^2} = \infty.$$

Details may be filled in later. In the mean time, I will think of other sums.

We have

$$S(p) = \sum_{\substack{n_0=1 \\ n_1=1}}^{\infty} \frac{1}{n_0^{2p} + n_1^{2p}} = \frac{-p + n_1\pi \sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}} \cot\left((-1)^{\frac{2i-1}{2p}} n_1\pi\right)}{2pn_1^{2p}}$$

for $p > 0$ an integer. Consistent with ζ, for odd powers of n_0 the sum is more difficult to evaluate. We expand the cotangents into infinite sums and combine:

$$S(p) = \frac{-p + n_1\pi \sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}} \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!}\left((-1)^{\frac{2i-1}{2p}} n_1\pi\right)^{2j-1}}{2pn_1^{2p}}$$

$$= \frac{-p + n_1\pi \sum_{j=0}^{\infty}(n_1\pi)^{2j-1}\frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} \sum_{i=1}^{p}(-1)^{\frac{2i-1}{2p}}(-1)^{\frac{(2i-1)(2j-1)}{2p}}}{2pn_1^{2p}}$$

$$= \frac{-p + n_1\pi \sum_{j=0}^{\infty}(n_1\pi)^{2j-1}\frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} \sum_{i=1}^{p}(-1)^{\frac{(2i-1)j}{p}}}{2pn_1^{2p}}.$$

This particular sum of roots of unity is a number theoretic one. It evaluates to $(-1)^{\frac{j}{p}}p$ if p divides j, and 0 if p does not divide j. That is,

$$\sum_{i=1}^{p}(-1)^{\frac{(2i-1)j}{p}} = \begin{cases} (-1)^{\frac{j}{p}}p & \text{if } p|j \\ 0 & \text{if } p \nmid j. \end{cases}$$

Then

$$S(p) = \frac{-p + n_1\pi \sum_{k=0}^{\infty}(n_1\pi)^{2pk-1}\frac{(-1)^{pk} 2^{2pk} B_{2pk}}{(2pk)!}(-1)^k p}{2pn_1^{2p}}$$

$$= \frac{-1 + \sum_{k=0}^{\infty}(n_1\pi)^{2pk}\frac{(-1)^{(p+1)k} 2^{2pk} B_{2pk}}{(2pk)!}}{2n_1^{2p}}$$

5. Symmetric in R summands

We have

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2}R_{\underline{l}}R_{\underline{k}-\underline{l}}.$$

Substitute $\underline{j} = \underline{l} - \frac{k}{2}$. Then

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$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{\underline{j} \in \mathbb{Z}^2 + \frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j} \neq \pm\frac{\underline{k}}{2}}} \frac{(\underline{j} + \frac{\underline{k}}{2}) \cdot \underline{k}}{|\underline{j} + \frac{\underline{k}}{2}|^2} R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}$$

where $\mathrm{mod}(n)$ is zero for even n and 1 for odd n (that is, $\underline{j}$ and $\frac{\underline{k}}{2}$ lie on common integer grids with offsets of 0 or $\frac{1}{2}$) (and it is evaluated elementwise). Let us make each summand have unique R products: we only sum over $\underline{j}$ with $0 \leq \arctan \underline{j} < \pi$:

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{0 \leq \arctan \underline{j} < \pi \\ \underline{j} \in \mathbb{Z}^2 + \frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j} \neq \pm\frac{\underline{k}}{2}}} \left(\frac{(\underline{j} + \frac{\underline{k}}{2}) \cdot \underline{k}}{|\underline{j} + \frac{\underline{k}}{2}|^2} + \frac{(-\underline{j} + \frac{\underline{k}}{2}) \cdot \underline{k}}{|-\underline{j} + \frac{\underline{k}}{2}|^2} \right) R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}.$$

We write the coefficients more explicitly:

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right)R_{\underline{k}} + \sum_{\substack{0 \leq \arctan \underline{j} < \pi \\ \underline{j} \in \mathbb{Z}^2 + \frac{\mathrm{mod}(\underline{k},2)}{2} \\ \underline{j} \neq \pm\frac{\underline{k}}{2}}} \frac{|\underline{k}|^2\left(|\underline{j}|^2 + \frac{|\underline{k}|^2}{4}\right) - 2(\underline{j} \cdot \underline{k})^2}{\left(|\underline{j}|^2 - \underline{j} \cdot \underline{k} + \frac{|\underline{k}|^2}{4}\right)\left(|\underline{j}|^2 + \underline{j} \cdot \underline{k} + \frac{|\underline{k}|^2}{4}\right)} R_{\frac{\underline{k}}{2}+\underline{j}} R_{\frac{\underline{k}}{2}-\underline{j}}.$$