THE KELLER–SEGEL EQUATION COMPACT SURFACES

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ABSTRACT. We study the parabolic–elliptic Keller–Segel equations on compact surfaces.

[rest of to be completed...]

1. INTRODUCTION

This is the part that we will write last.

Organization of the paper. To be completed...

Acknowledgment. *To be completed...*

2. REFORMULATION OF THE EQUATION

Let Σ be a smooth, closed surface with a Riemannian metric g and area form ω. Let G be the Green operator of the Laplacian, Δ, on $\mathrm{L}^2(\Sigma, g)$ and let us fix $\varrho_0 \in \mathrm{L}^2(\Sigma, g)$. Note that $\mathrm{L}^2 \hookrightarrow \mathrm{L}^1$ on domains of finite measure. We say that a positive function, $\varrho \in \mathrm{C}^1\big((0, \mathrm{T}); \mathrm{L}^2(\Sigma, g)\big)$ is said to satisfy the *parabolic–elliptic Keller–Segel equations* on (Σ, g) with initial value ϱ_0 if it is a solution to the following system:

$$\partial_t \varrho = -\Delta \varrho + \mathrm{d}^*\big(\varrho\, \mathrm{d}\mathrm{G}(\varrho)\big), \tag{2.1a}$$

$$\lim_{t \to 0^+} \varrho_t = \varrho_0. \tag{2.1b}$$

where for all $t \in (0, \mathrm{T})$

$$\varrho_t := \varrho|_{\{t\} \times \Sigma},$$

regarded as a function in $\mathrm{L}^2(\Sigma, g)$.

Now let

$$m(t) := \int_\Sigma \varrho_t\, \omega.$$

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Then m is constant, because

$$\dot{m}(t) = \int_\Sigma \left(-\Delta\varrho + \mathrm{d}^*\big(\varrho\,\mathrm{dG}(\varrho)\big)\right)\omega = \int_\Sigma \mathrm{d}^*\big(-\mathrm{d}\varrho + \varrho\,\mathrm{dG}(\varrho)\big)\,\omega = 0.$$

Thus we drop the t-dependence of m from its notation.

For the rest of the paper, let $\mathrm{A}_\Sigma := \mathrm{Area}\big(\Sigma, g\big)$. The following lemma recasts equations (2.1a) and (2.1b) in a simpler form.

Lemma 2.1. *Let* $\chi_0 := \varrho_0 - \frac{m}{\mathrm{A}_\Sigma}$ *and* $\chi := \varrho - \frac{m}{\mathrm{A}_\Sigma}$. *Then equations* (2.1a) *and* (2.1b) *is equivalent to*

$$\partial_t\chi = \left(\tfrac{m}{\mathrm{A}_\Sigma} - \Delta\right)\chi + \mathrm{d}^*\big(\chi\,\mathrm{dG}(\chi)\big), \tag{2.2a}$$

$$\chi|_{\{0\}\times\Sigma} = \chi_0. \tag{2.2b}$$

Proof. To be completed... □

3. The iteration

Let now $(\Psi_a)_{a\in\mathbb{N}}$ be a orthonormal eigenbasis of Δ and

$$\Delta\Psi_a = \lambda_a\Psi_a.$$

Let us order this basis so that

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \lambda_a \leqslant \lambda_{a+1} \leqslant \ldots$$

In particular, $\Psi_0 = \frac{1}{\sqrt{\mathrm{A}_\Sigma}}$.

Assume that χ is a solution to equations (2.2a) and (2.2b) and write

$$\mathrm{R}_a(t) := \big\langle \Psi_a \big| \chi|_{\{t\}\times\Sigma} \big\rangle_{\mathrm{L}^2(\Sigma,g)}.$$

Then for each $a \in \mathbb{N}$, we have that $\mathrm{R}_a \in \mathrm{C}^1((0,\mathrm{T});\mathbb{R})$. Note that $\mathrm{R}_0 \equiv \frac{m}{\mathrm{A}_\Sigma}$. Finally let

$$\forall a,b,c \in \mathbb{N}: \quad \varphi_{abc} := \int_\Sigma \Psi_a\Psi_b\Psi_c\,\omega.$$

Theorem 3.1. *The function* ϱ *is a solution to equations* (2.1a) *and* (2.1b) *exactly when the functions* $(\mathrm{R}_a)_{a\in\mathbb{N}}$ *satisfy*

$$\forall t \in (0,\mathrm{T}): \quad (\mathrm{R}_a(t))_{a\in\mathbb{N}} \in l^2(\mathbb{N}),$$

$$\varrho_0 = \sum_{a\in\mathbb{N}} \left(\lim_{t\to 0^+} \mathrm{R}_a(t)\right)\Psi_a,$$

$$\forall a \in \mathbb{N} - \{0\}: \quad \dot{\mathrm{R}}_a = \left(\tfrac{m}{\mathrm{A}_\Sigma} - \lambda_a\right)\mathrm{R}_a + \sum_{b,c\in\mathbb{N}} \frac{\lambda_a - \lambda_b + \lambda_c}{\lambda_c}\varphi_{abc}\mathrm{R}_b\mathrm{R}_c.$$

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