

## 1. GENERAL DISCUSSION

Let  $\Sigma$  be any smooth, compact surface (potentially with boundary), and let  $\Delta$  be its Laplace operators. Let  $(f_a \in L^2(\Sigma))_{a \in I}$  be a complete, orthonormal set of eigenfunctions (thus  $I$  is countable infinite), that is  $\Delta f_a = \lambda_a f_a$ . Without any loss of generality, assume that  $0 \in I$  and  $f_0$  is a constant. Then write

$$\varrho_t = \sum_{a \in I} \sum_{n \in \mathbb{N}} R_{n,a} t^n f_a.$$

Let

$$\varphi_{a,b,c} := \int_{\Sigma} f_a f_b f_c dA.$$

If  $\varrho_t$  solves the Keller–Segel equation, then and  $a \neq 0$ , then

$$R_{n+1,a} = \frac{R_{0,0} - \lambda_a}{n+1} R_{n,a} + \sum_{b,c \in I - \{0\}} \sum_{m=0}^n \frac{\lambda_a - \lambda_b + \lambda_c}{(n+1)\lambda_c} \varphi_{a,b,c} R_{b,m} R_{c,n-m}.$$

**Remark 1.1.** For all  $\Lambda \in \mathbb{R}_+$ , let

$$n(\Lambda) := |\{a \in I \mid \lambda_a \leq \Lambda\}|.$$

*Weyl's law*, that states that

$$\lim_{\Lambda \rightarrow \infty} \frac{\Lambda}{n(\Lambda)} = \frac{2}{\text{Area}(\Sigma, g)}.$$

## 2. CONVERGENCE OF THE ITERATIVE METHOD ON THE STANDARD TORUS

Let us write

$$\varrho(x, t) = \sum_{\underline{k} \in \mathbb{Z}^2} \sum_{n \in \mathbb{N}} R_{n,\underline{k}} t^n \exp(2\pi i \underline{k} \cdot \underline{x}).$$

If  $\varrho$  solves the Keller–Segel equation, then and  $\underline{k} \neq \underline{0}$ , then

$$R_{n+1,\underline{k}} = \frac{R_{0,0} - 4\pi^2 |\underline{k}|^2}{n+1} R_{n,\underline{k}} + \sum_{\underline{l} \in \mathbb{Z}^2 - \{\underline{0}, \underline{k}\}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} \sum_{m=0}^n \frac{1}{n+1} R_{m,\underline{l}} R_{n-m,\underline{k}-\underline{l}}. \quad (2.1)$$

Finally let

$$\forall \underline{k} \in \mathbb{Z}^2 : S_{\underline{k}}(t) := R_{\underline{k}}(t) \exp\left(4\pi^2 |\underline{k}|^2 t\right),$$

and write

$$S_{\underline{k}}(t) = \sum_{n \in \mathbb{N}} S_{n,\underline{k}} t^n.$$

Then, in terms of the  $S_{n,\underline{k}}$  coefficients, equation (2.1) becomes

$$S_{n+1,\underline{k}} = \frac{1}{n+1} \sum_{\underline{l} \in \mathbb{Z}^2 - \{\underline{0}, \underline{k}\}} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} \sum_{a=0}^n \sum_{b=0}^{n-a} S_{a,\underline{l}} S_{b,\underline{k}-\underline{l}} \frac{(8\pi^2 (\underline{k} \cdot \underline{l}) \cdot |\underline{l}|)^{n-a-b}}{(n-a-b)!}. \quad (2.2)$$

If that the coefficients in equation (2.1) exist after the  $n^{\text{th}}$  step, then let

$$Z_n := \{ \underline{k} \in \mathbb{Z}^2 \mid S_{n,\underline{k}} \neq 0 \} \subseteq \mathbb{Z}^2.$$

**Lemma 2.1.** Let us assume that  $Z_0$  is finite and symmetric, that is  $-Z_0 = Z_0$ , and let

$$d := \max(\{|\underline{k}| \mid \underline{k} \in Z_0\}) > 0.$$

Then for all  $n \in \mathbb{N}_+$ , the coefficients in equation (2.1) exist and  $Z_n$  is finite and symmetric, moreover

$$Z_n \subseteq Z_{n-1} + Z_{n-1},$$

and

$$\max(\{|\underline{k}| \mid \underline{k} \in Z_n\}) = \max(\{|\underline{k}| \mid S_{n,\underline{k}} \neq 0\}) \leq 2^n d.$$

*Proof.* \*Insert Adam's proof here.\*

□

For all  $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$  let

$$N(\underline{k}) := \min(\{ n \in \mathbb{N} \mid \underline{k} \in Z_n \}).$$

**Lemma 2.2.** *Under the assumptions above, we have that*

$$\forall \underline{k} \in \mathbb{Z}^2 - \{\underline{0}\} : \forall n \in \mathbb{N} : |S_{n,\underline{k}}| < \epsilon^n. \quad (2.3)$$

*Proof.* By Lemma 2.1, we can rewrite equation (2.2) as

$$S_{n+1,\underline{k}} = \frac{1}{n+1} \sum_{x=0}^n \sum_{\underline{l}: N(\underline{l})=x} \frac{\underline{k} \cdot \underline{l}}{|\underline{l}|^2} \sum_{a=x}^n \sum_{b=0}^{n-a} S_{a,\underline{l}} S_{b,\underline{k}-\underline{l}} \frac{(8\pi^2 (\underline{k} \cdot \underline{l}))^{n-a-b}}{(n-a-b)!}. \quad (2.4)$$

The inequality (2.3) is true for  $n = 0$ , so we can argue by induction.

Let us assume now that we have proved inequality (2.3) for every nonnegative integer up to  $n \in \mathbb{N}$ . Without any loss of generality, we can assume that  $n > N(\underline{k})$ , then inequality (2.3) holds. Now using equation (2.4) we have:

$$|S_{n+1,\underline{k}}| \leq \frac{1}{n+1} \sum_{x=0}^n \sum_{\underline{l}: N(\underline{l})=x} \sum_{a=x}^n \sum_{b=0}^{n-a} \epsilon^{a+b} \frac{|8\pi^2 (\underline{k} \cdot \underline{l}) \cdot \underline{l}|^{n-a-b}}{(n-a-b)!} \frac{|\underline{k} \cdot \underline{l}|}{|\underline{l}|^2}$$

□