# THE KELLER–SEGEL EQUATION COMPACT SURFACES

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Abstract. We study the (parabolic-elliptic) Keller–Segel equations on compact surfaces. **To be completed.**

## 1. Introduction

**To be completed.**This is the part that we will write last.

**Organization of the paper.** **To be completed.**

**Acknowledgment.** ***To be completed.***

## 2. Reformulation of the equation

Let $\Sigma$ be a smooth, closed surface with a Riemannian metric $g$ and area form $\omega$. Let $G$ be the Green operator of the Laplacian, $\Delta$, on $L^2(\Sigma, g)$ and let us fix $\rho_0 \in L^2(\Sigma, g)$. Note that $L^2 \hookrightarrow L^1$ on domains of finite measure. We say that a positive function, $\rho \in C^1\big((0,T); L^2(\Sigma, g)\big)$ is said to satisfy the *parabolic–elliptic Keller–Segel equations* on $(\Sigma, g)$ with initial value $\rho_0$ if it is a solution to the following system:

$$\partial_t \rho = -\Delta\rho + \mathrm{d}^*(\rho\, \mathrm{d}G(\rho)), \tag{2.1a}$$

$$\lim_{t \to 0^+} \rho_t = \rho_0. \tag{2.1b}$$

where for all $t \in (0, T)$

$$\rho_t := \rho|_{\{t\} \times \Sigma},$$

regarded as a function in $L^2(\Sigma, g)$.

Now let

$$M(t) := \int\limits_{\Sigma} \rho_t \, \mathrm{d}A.$$

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Then $M$ is constant, because

$$\dot{M}(t) = \int_\Sigma (-\Delta\rho + \mathrm{d}^*(\rho\,\mathrm{d}G(\rho)))\,\mathrm{d}A = \int_\Sigma \mathrm{d}^*(-\mathrm{d}\rho + \rho\,\mathrm{d}G(\rho))\,\mathrm{d}A = 0.$$

Thus we drop the $t$-dependence of $M$ from its notation.

For the rest of the paper, let $A_\Sigma := \mathrm{Area}(\Sigma, g)$. The following lemma recasts equations (2.1a) and (2.1b) in a simpler form.

**Lemma 2.1.** *Let $\chi_0 := \rho_0 - \frac{M}{A_\Sigma}$ and $\chi := \rho - \frac{M}{A_\Sigma}$. Then equations* (2.1a) *and* (2.1b) *is equivalent to*

$$\partial_t \chi = \tfrac{M}{A_\Sigma}\chi - \Delta\chi + \mathrm{d}^*(\chi\,\mathrm{d}G(\chi)), \tag{2.2a}$$

$$\chi|_{\{0\}\times\Sigma} = \chi_0. \tag{2.2b}$$

*Proof.* The equivalency of equation (2.1b) and equation (2.2b) is obvious.

Note that $G$ annihilates constants and since $\chi$ is orthogonal to constants, we have that $\Delta(G(\chi)) = \chi$. Since $\chi = \rho - \frac{M}{A_\Sigma}$, we get, using equation (2.1a), that

$$\begin{aligned}
\partial_t\chi &= \partial_t\Big(\rho - \tfrac{M}{A_\Sigma}\Big)\\
&= \partial_t\rho - 0\\
&= -\Delta\rho + \mathrm{d}^*(\rho\,\mathrm{d}G(\rho))\\
&= -\Delta\Big(\tfrac{M}{A_\Sigma} + \chi\Big) + \mathrm{d}^*\Big(\Big(\tfrac{M}{A_\Sigma} + \chi\Big)\mathrm{d}G\Big(\tfrac{M}{A_\Sigma} + \chi\Big)\Big)\\
&= -\Delta\chi + \mathrm{d}^*\Big(\Big(\tfrac{M}{A_\Sigma} + \chi\Big)\mathrm{d}G(\chi)\Big)\\
&= -\Delta\chi + \tfrac{M}{A_\Sigma}\,\mathrm{d}^*\,\mathrm{d}G(\chi) + \mathrm{d}^*(\chi\,\mathrm{d}G(\chi))\\
&= \tfrac{M}{A_\Sigma}\chi - \Delta\chi + \mathrm{d}^*(\chi\,\mathrm{d}G(\chi)),
\end{aligned}$$

which completes the proof. □

**Remark 2.2.** *Let $\lambda_1$ be the smallest nonzero eigenvalue of $\Delta$ and note that quantity $M_\Sigma = \lambda_1 A_\Sigma$ only depends on the geometry of $(\Sigma, g)$. When $M < M_\Sigma$, then the linear term in equation* (2.2a) *is strictly negative definite.*

## 3. The generalized Fourier transform

Let now $\big(\Psi_a \in L^2(\Sigma, g)\big)_{a\in\mathbb{N}}$ be an orthonormal eigenbasis of $\Delta$ and

$$\Delta\Psi_a = \lambda_a\Psi_a.$$

Let us order this basis so that

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \lambda_a \leqslant \lambda_{a+1} \leqslant \dots$$

In particular, $\Psi_0 = \frac{1}{\sqrt{A_\Sigma}}$.

Assume that $\chi$ is a solution to equations (2.2a) and (2.2b) and write

$$R_a(t) := \left\langle \Psi_a \middle| \chi|_{\{t\}\times\Sigma} \right\rangle_{L^2(\Sigma,g)}.$$

Then for each $a \in \mathbb{N}$, we have that $R_a \in C^1((0,T);\mathbb{R})$. Note that $R_0 \equiv \frac{M}{A_\Sigma}$. Finally let

$$\forall a,b,c \in \mathbb{N}: \quad \varphi_{abc} := \int_\Sigma \Psi_a \Psi_b \Psi_c \, \mathrm{d}A.$$

Note that $\varphi_{abc}$ is a completely symmetric 3-tensor.

**Theorem 3.1.** *The function $\rho$ is a solution to equation* (2.1a) *exactly when*

$$\forall t \in (0,T): \quad (R_a(t))_{a\in\mathbb{N}} \in l^2(\mathbb{N}), \tag{3.1a}$$

$$\forall a \in \mathbb{N}_+: \quad \dot{R}_a = \left(\tfrac{M}{A_\Sigma} - \lambda_a\right) R_a + \sum_{b,c\in\mathbb{N}} \frac{\lambda_a - \lambda_b + \lambda_c}{2\lambda_c} \varphi_{abc} R_b R_c. \tag{3.1b}$$

*Furthermore, if equation* (2.1b) *is also satisfied, then for all $a \in \mathbb{N}$, $\lim_{t\to 0^+} R_a(t)$ exists and*

$$\rho_0 = \sum_{a\in\mathbb{N}} \left( \lim_{t\to 0^+} R_a(t) \right) \Psi_a. \tag{3.2}$$

*Proof.* Since $(\Psi_a)_{a\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Sigma,g)$ and for all $t \in (0,T)$, $\rho_t$ is in $L^2(\Sigma,g)$, we get condition (3.1a).

Fix $a \in \mathbb{N}_+$. Then

$$G(\Psi_a) = \lambda_a^{-1} \Psi_a.$$

Using the above equation, the self-adjointness of $\Delta$, and equation (2.2a), we get

$$\begin{aligned}
\dot{R}_a &= \langle \Psi_a | \partial_t \chi \rangle_{L^2(\Sigma,g)} \\
&= \left\langle \Psi_a \middle| \tfrac{M}{A_\Sigma}\chi - \Delta\chi + \mathrm{d}^*(\chi \, \mathrm{d}G(\chi)) \right\rangle_{L^2(\Sigma,g)} \\
&= \left\langle \tfrac{M}{A_\Sigma}\Psi_a - \Delta\Psi_a \middle| \chi \right\rangle_{L^2(\Sigma,g)} + \langle \mathrm{d}\Psi_a | \chi \, \mathrm{d}G(\chi) \rangle_{L^2(\Sigma,g)} \\
&= \left(\tfrac{M}{A_\Sigma} - \lambda_a\right) \langle \Psi_a | \chi \rangle_{L^2(\Sigma,g)} + \sum_{b,c\in\mathbb{N}_+} \langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}G(\Psi_c) \rangle_{L^2(\Sigma,g)} R_b R_c \\
&= \left(\tfrac{M}{A_\Sigma} - \lambda_a\right) R_a + \sum_{b,c\in\mathbb{N}_+} \langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} \lambda_c^{-1} R_b R_c.
\end{aligned} \tag{3.3}$$

Note that

$$\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} = \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{d}A, \tag{3.4}$$

and

$$\Delta(\Psi_a \Psi_c) = (\Delta\Psi_a)\Psi_c + \Psi_a(\Delta\Psi_c) - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = (\lambda_a + \lambda_c)\Psi_a\Psi_c - 2g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c).$$

Thus

$$g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) = \frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c).$$

Plugging the above equation into equation (3.4) we get

$$\begin{aligned}
\langle \mathrm{d}\Psi_a | \Psi_b \, \mathrm{d}\Psi_c \rangle_{L^2(\Sigma,g)} &= \int_\Sigma \Psi_b g(\mathrm{d}\Psi_a, \mathrm{d}\Psi_c) \, \mathrm{d}A \\
&= \int_\Sigma \Psi_b \Big( \frac{\lambda_a + \lambda_c}{2}\Psi_a\Psi_c - \frac{1}{2}\Delta(\Psi_a\Psi_c) \Big) \, \mathrm{d}A \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{abc} - \frac{1}{2}\int_\Sigma (\Delta\Psi_b)\Psi_a\Psi_c \, \mathrm{d}A \\
&= \frac{\lambda_a + \lambda_c}{2}\varphi_{abc} - \frac{\lambda_b}{2}\int_\Sigma \Psi_b\Psi_a\Psi_c \, \mathrm{d}A \\
&= \frac{\lambda_a - \lambda_b + \lambda_c}{2}\varphi_{abc}.
\end{aligned}$$

Inserting this into equation (3.3) yields equation (3.1b).

The equivalency of equation (2.1b) and equation (3.2) is straightforward. □

**Remark 3.2.** *The moral of Theorem 3.1 is that the Keller–Segel equations, which is a (hard) elliptic-parabolic system of partial differential equations, can be transformed (on closed surfaces) into a infinite system of ordinary differential equations, which is potentially easier to handle.*

*In the rest of the paper we show that this system can be further simplified under certain extra hypotheses.*

3.1. **Analytic solutions.** In order to further simplify equation (3.1b), we search for analytic solutions, that is

$$\forall a \in \mathbb{N} : \forall t \in (0,T) : \quad R_a(t) = \sum_{n\in\mathbb{N}} R_{a,n} t^n, \tag{3.5}$$

and the right-hand side is assumed to be absolute convergent in $l^2(\mathbb{N})$.

The next lemma rewrites equation (3.1b) in terms of the coefficients $(R_{a,n})_{(a,n)\in\mathbb{N}\times\mathbb{N}}$.

**Lemma 3.3.** *Under the above assumption, the function $\rho$ is a $t$-analytic solution to equation* (2.1a) *with mass $M$ exactly when $R_{0,0} = \frac{M}{A_\Sigma}$, for all $n \in \mathbb{N}_+$, $R_{0,n} = 0$, and*

$$\forall a, n \in \mathbb{N}: \quad R_{a,n+1} = \frac{1}{n+1}\left(\left(\tfrac{M}{A_\Sigma} - \lambda_a\right)R_{a,n} + \sum_{b,c\in\mathbb{N}_+}\sum_{m=0}^{n} \tfrac{\lambda_a-\lambda_b+\lambda_c}{2\lambda_c}\varphi_{abc}R_{b,m}R_{c,n-m}\right), \tag{3.6a}$$

$$\forall n \in \mathbb{N}: \quad \mathcal{R}_n := (R_{a,n})_{a\in\mathbb{N}} \in l^2(\mathbb{N}), \tag{3.6b}$$

$$\limsup_{n\to\infty} \|\mathcal{R}_n\|_{l^2(\mathbb{N})}^{\frac{1}{n}} = \limsup_{n\to\infty}\left(\sum_{a\in\mathbb{N}} R_{a,n}^2\right)^{\frac{1}{n}} < \frac{1}{T}. \tag{3.6c}$$

*Proof.* Inserting equation (3.5) into equation (3.1b) yields equation (3.6a). The equations (3.6b) and (3.6c) are necessary (and, in fact, sufficient) to have that the convergence radius of the Taylor series of $\rho$ in the $L^2$ topology is at least $T$. □

In the following two sections we investigate two special cases when iteration in equation (3.6a) exists for all $a$ and $n$.

## 4. Round spheres

Let $\Sigma$ be the 2-sphere and $g$ be the round metric of radius $r$. Then we have that $(\Psi_a)_{a\in\mathbb{N}}$ are the spherical harmonics. Then $A_\Sigma = 4\pi r^2$. In fact, after relabeling them, we can write the eigenvalue has the form $\lambda_{l,m} := \frac{l(l+1)}{r^2}$, where $l \in \mathbb{N}$ and $M$ is any integer satisfying $|m| \leqslant l$. Let us now write

$$\Psi_l^m := \Psi_{(l,m)}, \quad \& \quad R_{l,a}^m := R_{(l,m),a}, \quad \& \quad \varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} := \varphi_{(l_1,m_1),(l_2,m_2),(l_3,m_3)}.$$

Using this new set of indices and notation, we can rewrite equation (3.6a) as

$$\begin{aligned} R_{l,n+1}^m = \frac{1}{n+1}&\left(\tfrac{M}{4\pi r^2} - \tfrac{l(l+1)}{r^2}\right)R_{(l,m),n} \\ &+ \sum_{l_1,l_2\in\mathbb{N}_+}\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\sum_{x=0}^{n} \frac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)}\varphi_{l,l_1,l_2}^{m,m_1,m_2}R_{l_1,x}^{m_1}R_{l_2,n-x}^{m_2}. \end{aligned} \tag{4.1}$$

Using the Clebsch–Gordan Theorem, we have that if $l_1 \geqslant l_2 + l_3$ or $l_1 \leqslant |l_2 - l_3|$, then for all $m_1, m_2$, and $m_3$, we have $\varphi_{l_1,l_2,l_3}^{m_1,m_2,m_3} = 0$. Thus equation (4.1) becomes

$$R_{l,n+1}^m = \frac{1}{n+1}\left(\tfrac{M}{4\pi r^2} - \tfrac{l(l+1)}{r^2}\right)R_{l,n}^m$$

$$+\sum_{l_1\in\mathbb{N}_+}\sum_{l_2=\max(\{1,|l-l_1|\})}^{l+l_1}\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\sum_{x=0}^{n}\tfrac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)}\varphi_{l,l_1,l_2}^{m,m_1,m_2}R_{l_1,x}^{m_1}R_{l_2,n-x}^{m_2}. \quad (4.2)$$

Before prove the main result of this section, let us make the following definition:

$$\forall n\in\mathbb{N}: \quad Z_n := \left\{ (l,m)\in\mathbb{N}\times\mathbb{Z} \;\middle|\; R_{l,n}^m \neq 0 \right\}.$$

**Proposition 4.1.** *Assume that $Z_0$ is finite. Then for all $(l,m)\in\mathbb{N}\times\mathbb{Z}$ and $n\in\mathbb{N}$, $R_{l,n}^m$ exists. Furthermore, $Z_n$ is also finite, and equation* (4.2) *becomes*

$$\begin{aligned} R_{l,n+1}^m &= \frac{1}{n+1}\left(\tfrac{M}{4\pi r^2}-\tfrac{l(l+1)}{r^2}\right)R_{l,n}^m \\ &+\sum_{x=0}^{n}\sum_{\substack{(l_1,m_1)\in Z_x\\(l_2,m_2)\in Z_{n-x}}}\tfrac{l(l+1)-l_1(l_1+1)+l_2(l_2+1)}{2(n+1)l_2(l_2+1)}\varphi_{l,l_1,l_2}^{m,m_1,m_2}R_{l_1,x}^{m_1}R_{l_2,n-x}^{m_2}. \end{aligned}$$

*Proof.* **To be completed.** □

## 5. Flat tori

Let now $\Sigma$ be a flat torus. Thus, without any loss of generality, we can assume that there are vectors

$$\underline{e}_1 = \begin{pmatrix} L_1 \\ 0 \end{pmatrix}, \quad \& \quad \underline{e}_2 = \begin{pmatrix} L_2\cos(\theta) \\ L_2\sin(\theta) \end{pmatrix}.$$

such that, if $\Lambda := \mathbb{Z}\underline{e}_1 \oplus \mathbb{Z}\underline{e}_2$, then

$$\Sigma = \mathbb{R}^2/\Lambda.$$

Note that $A_\Sigma = L_1L_2\sin(\theta)$.

Let

$$\forall \underline{x}\in\Sigma : \forall \underline{k}\in\Lambda : \quad \Psi_{\underline{k}}(\underline{x}) := \frac{1}{\sqrt{A_\Sigma}}e^{2\pi i\underline{k}\cdot\underline{x}}.$$

Then $\left(\Psi_{\underline{k}}\right)_{\underline{k}\in\Lambda}$ is an orthonormal basis for the *complex* Hilbert space $L^2_{\mathbb{C}}(\Sigma,g)$. Furthermore, note that $\Psi_{\underline{k}} = \overline{\Psi_{-\underline{k}}}$. Finally, note that

$$\Delta\Psi_{\underline{k}} = 4\pi^2\left|\underline{k}\right|^2\Psi_{\underline{k}},$$

thus $\left(\Psi_{\underline{k}}\right)_{\underline{k}\in\Lambda}$ is an eigenbasis for the Laplacian, albeit a complex one. The corresponding spectrum is $\left(4\pi^2\left|\underline{k}\right|^2\right)_{\underline{k}\in\Lambda}$.

Function on $\Sigma$ can be viewed as $\Lambda$-periodic functions on $\mathbb{R}^2$, thus if $\rho$ is an ($L^2_{\mathbb{C}}$) function on $\Sigma$, then we use Fourier decomposition to get:

$$\forall \underline{k} \in \Lambda : \quad R_{\underline{k}} := \frac{1}{\sqrt{A_\Sigma}} \int_\Sigma e^{-2\pi i \underline{k}\cdot\underline{x}} \rho(\underline{x})\, \mathrm{d}A(\underline{x}), \quad \Leftrightarrow \quad \rho = \sum_{\underline{k}\in\Lambda} R_{\underline{k}} \Psi_{\underline{k}}.$$

If $\rho$ is real, then $R_{\underline{k}} = \overline{R_{-\underline{k}}}$. In this section we slightly deviate from our previous method and use the above complex basis and coefficients.

Nonetheless, the ideas and proofs of the previous sections still apply, and if $\rho \in C^1\Big((0,T), L^2_{\mathbb{C}}(\Sigma, g)\Big)$, then we can define the coefficients functions $R_{\underline{k}} \in C^1_{\mathbb{C}}(\Sigma)$, so that

$$\rho(t,\underline{x}) = \sum_{n\in\mathbb{N}} \sum_{\underline{k}\in\Lambda} R_{\underline{k}}(t) \Psi_{\underline{k}}(\underline{x}).$$

If, furthermore, $\rho$ is a solution to the Keller–Segel equation (2.1a), then we get (after a straightforward computation) that

$$\dot{R}_{\underline{k}} = \left(\tfrac{M}{A_\Sigma} - 4\pi^2 \left|\underline{k}\right|^2\right) R_{\underline{k}} + \sum_{\underline{l}\in\Lambda} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}}.$$

Finally, if $\rho$ is analytic in $t$, and we define $R_{n,\underline{k}} \in \mathbb{C}$ as

$$R_{\underline{k}}(t) = \sum_{n\in\mathbb{N}} R_{n,\underline{k}} t^n,$$

then we get the corresponding iteration for these coefficients to be

$$R_{n+1,\underline{k}} = \frac{1}{n+1} \left( \left(\tfrac{M}{A_\Sigma} - 4\pi^2 \left|\underline{k}\right|^2\right) R_{n,\underline{k}} + \sum_{\underline{l}\in\Lambda} \sum_{m}^{n} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{m,\underline{l}} R_{n-m,\underline{k}-\underline{l}} \right). \tag{5.1}$$

As in Section 4, let us make the following definition:

$$\forall n \in \mathbb{N} : \quad Z_n := \left\{ \underline{k} \in \Lambda \;\middle|\; R_{n,\underline{k}} \neq 0 \right\}.$$

**Proposition 5.1.** *Assume that $Z_0$ is finite. Then for all $\underline{k} \in \Lambda$ and $n \in \mathbb{N}$, $R_{n,\underline{k}}$ exists. Furthermore, $Z_n$ is also finite, and equation* (5.1) *becomes*

$$R_{n+1,\underline{k}} = \frac{1}{n+1} \left( \left(\tfrac{M}{A_\Sigma} - 4\pi^2 \left|\underline{k}\right|^2\right) R_{n,\underline{k}} + \sum_{\substack{\underline{l}\in Z_m \\ \underline{k}-\underline{l}\in Z_{n-m}}} \sum_{m}^{n} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{m,\underline{l}} R_{n-m,\underline{k}-\underline{l}} \right).$$

*Proof.* **To be completed.** $\square$

## 6. Solutions on $\mathbb{R}^2/\mathbb{Z}^2$

*If we figure this out.*

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