What follows is a computation of the recursion for $S_{n+1,\underline{k}}$ when the initial conditions are '1D'. That is, all $S_{0,\underline{k}}$ are zero except possibly $S_{0,\underline{0}}$ and $S_{0,(1,0)}$. We suppress the second coordinate in the indicies (that is, $S_{0,(a,b)} = S_{0,a}$).

Notice that

$$
\begin{aligned}
S_{n,2} &= \frac{S_{0,1}^2 \pi^{2n} 2^{3n}}{(2\pi)^2 n!},\\
S_{n,3} &= \frac{S_{0,1}^3 \pi^{2n} 2^{3n}}{(2\pi)^4 n!}(-2^n \frac{3}{2!} + 3^n),\\
S_{n,4} &= \frac{S_{0,1}^4 \pi^{2n} 2^{3n}}{(2\pi)^6 n!}(3^n \frac{4}{3!} + 4^n \frac{4}{8} - 5^n \frac{4}{2} + 6^n),\\
S_{n,5} &= \frac{S_{0,1}^5 \pi^{2n} 2^{3n}}{(2\pi)^8 n!}(-4^n \frac{5}{4!} - 6^n \frac{5}{12} + 7^n \frac{5}{6} + 8^n \frac{5}{4} - 9^n \frac{5}{2} + 10^n),\\
S_{n,6} &= \frac{S_{0,1}^6 \pi^{2n} 2^{3n}}{(2\pi)^{10} n!}(5^n \frac{6}{5!} + 8^n \frac{6}{48} - 9^n \frac{6}{36} - 11^n \frac{6}{6} + 12^n \frac{6}{8} + 13^n \frac{6}{8/3} - 14^n \frac{6}{2} + 15^n),\\
S_{n,7} &= \frac{S_{0,1}^7 \pi^{2n} 2^{3n}}{(2\pi)^{12} n!}(-6^n \frac{7}{6!} - 10^n \frac{7}{240} + 11^n \frac{7}{120} - 12^n \frac{7}{144} + 14^n \frac{7}{24} - 15^n \frac{7}{72} + 16^n \frac{7}{24} - 17^n \frac{7}{4} + 18^n \frac{7}{24} + 19^n \frac{7}{2} - 20^n \frac{7}{2} + 21^n)
\end{aligned}
$$

These identities have been verified for n up to 50. The general form seems to be

$$
S_{n,i} = \frac{S_{0,1}^i \pi^{2n} 2^{3n}}{(2\pi)^{2i-2} n!} \sum_{j=i-1}^{\binom{i}{2}} c_{n,i,j} j^n
$$