On Mathematics of Chemotactic Collapse

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Chemotaxis

Chemotaxis is a directed movement of organisms in response to the concentration gradient of an external chemical signal and is common in biology.

The chemical signals either come from external sources or are secreted by the organisms themselves. The latter situation leads to aggregation of organisms and to the formation of patterns.

Chemotaxis underlies many social activities of micro-organisms, e.g. social motility, fruiting body development, quorum sensing and biofilm formation.

Chemotaxis: Examples

- Aggregation of bacteria (say, E. coli) under starvation conditions;
- Formation of multicellular structures of $\sim 10^5$ cells by single cell bacterivores, when challenged by adverse conditions;
- Formation of blood vessels.

Reduced Keller-Segel equations

Natural assumptions:
(a) the organism population is large and the individuals are small relative to the domain where they move
(b) the chemical diffuses much faster than the organisms do

$\Longrightarrow$ the simplest model of the gradient detection mechanism:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \Delta \rho - \nabla \cdot (\rho \nabla c)\,, \\ 0 &= \Delta c + \rho. \end{aligned} \tag{1}$$

Here $\rho(x,t)$ and $c(x,t)$ are the organism density and chemical concentration.

Eq. (1) also appear in

- physics (e.g. in description of stellar collapse)
- social sciences (e.g. in description of crime patterns or riot dynamics)

Properties Keller-Segel equations

- Positivity preserv.: $\rho_0(x) \geq 0 \Longrightarrow \rho(x,t) \geq 0$ (max. pr.).
- Mass (number) conservation:

$$\int_\Omega \rho(x,t)\,dx = \int_\Omega \rho(x,0)\,dx.$$

- Gradient property: $\partial_t \rho = -\operatorname{grad} \mathcal{F}(\rho)$, with the energy

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^2} \left[-\frac{1}{2}\rho(-\Delta)^{-1}\rho + \rho \ln \rho \right] dx. \tag{2}$$

  and the metric $\langle v, w \rangle_J := \left\langle v, J^{-1} w \right\rangle_{L^2}$, $J := -\nabla \cdot \rho \nabla$.
- Scaling invariance: if a pair $\rho(x,t)$ and $c(x,t)$ is a solution to KS, then for any $\lambda > 0$ so is the pair

$$\lambda^{-2}\rho\left(\lambda^{-1}x, \lambda^{-2}t\right) \text{ and } c\left(\lambda^{-1}x, \lambda^{-2}t\right). \tag{3}$$

Key previous results

The most interesting is the critical dimension $d = 2$, which we consider from now on.

For $d = 2$, KS has a radially symmetric static solution (Wol, Vel):

$$R(x) := \frac{8}{(1 + |x|^2)^2}, \tag{4}$$

The total mass of R is $M = \int R(x) d^2x = 8\pi$. This mass turns out to be the threshold separating a regular behavior and a breakdown of the solution:

- (Blanchet, Dolbeault, Perthame) If the initial total mass satisfies $M := \int_{\mathbb{R}^2} \rho_0 \, dx \leq 8\pi$, then the solution to KS exists globally and, for $M := \int_{\mathbb{R}^2} \rho_0 \, dx < 8\pi$, converges to 0, as $t \to \infty$;
- (Biler) If the initial total mass satisfies $M := \int_{\mathbb{R}^2} \rho_0 \, dx > 8\pi$, then the solution to KS breaks down in finite time.

$M < 8\pi$: Existence argument

For the long-time behaviour, compute the change in the entropy

$$\partial_t \int \rho \ln \rho = \underbrace{-4 \int |\nabla \sqrt{\rho}|^2}_{\text{entropy dissipation}} + \underbrace{\int \rho^2}_{\text{entropy production}} .$$

Depending on whether the entropy dissipation or the entropy production wins we expect either dissipation or the collapse.

The Nirenberg - Gagliardo inequality (the dimension $n = 2$),

$$\int \rho^2 \leq c_{\mathrm{gn}} \int |\nabla \sqrt{\rho}|^2 \int \rho$$

shows the dissipation wins if $Mc_{\mathrm{gn}} < 4$ (here $M = \int \rho$).

Using the logarithmic Hardy-Littlewood-Sobolev inequality for the free energy gives $M < 8\pi$.

Virial relation

Differentiating the second moment, $W(t) := \int_{\mathbb{R}^2} x^2 \rho(x,t)\,dx$, of the density ρ, one arrives at the virial relation

$$\partial_t W = 4M(1 - \frac{1}{8\pi}M),$$

which shows how the mass threshold $M_* = 8\pi$ enters the dynamics:

If $M > 8\pi$, then the right hand side is constant and negative, and hence, W becomes negative in finite time $\Rightarrow$ contradiction, since $\rho(t) \geq 0$. Thus, if $M > 8\pi$, then the solution ρ breaks down in a finite time.

Our goal now is to investigate how this breakdown takes place.

$M > 8\pi$: Rescaling

Recall that KS has the manifold of static solutions

$$\mathcal{M}_{\text{stat}} := \{\frac{1}{\lambda^2} R(x/\lambda) \mid \lambda > 0\}. \tag{5}$$

Assuming this manifold is stable, one can slide along it either in the direction $\lambda \to \infty$ (dissipation) or in the direction $\lambda \to 0$ (collapse).

To analyze the long time dynamics, we pass to the reference frame evolving (say, collapsing) with the solution, by introducing the *adaptive blowup variables*,

$$y = \frac{x}{\lambda(t)} \text{ and } \tau = \int_0^t \frac{1}{\lambda^2(s)}\, ds, \tag{6}$$

and $\lambda : [0, T) \to [0, \infty)$, $T > 0$ (*compression* or *dilatation* parameter), such that $\lambda(t) \to 0$ and $\tau \to \infty$, as $t \uparrow T$.

Rescaled Equation

The advantage of passing to blowup variables is that the function

$$u(y,\tau) = \lambda^2(t)\rho(x,t) \quad \text{with} \quad y = \frac{x}{\lambda(t)} \ \& \ \tau = \int_0^t \frac{1}{\lambda^2(s)}\, ds,$$

is expected to be *bounded* and the blowup time is mapped to ∞ and is eliminated from consideration: $\tau \to \infty$ as $t \to T$.

Writing KS in blowup variables and letting $v(y,\tau) = c(x,t)$, we find the equation for the rescaled bio mass density function

$$\partial_\tau u = \Delta_y u - \nabla \cdot (u \nabla v) - a \nabla_y \cdot (yu), \tag{7}$$

where $a := -\dot{\lambda}\lambda$. The blowup problem for KS is mapped into the problem of *asymptotic dynamics of solitons* for the equation (7).

The rescaled KS is a gradient flow for the modified free energy functional

$$\mathcal{F}_a(u) = \int_{\mathbb{R}^2} \left[-\frac{1}{2} u(-\Delta)^{-1} u + u \ln u - \frac{a}{2} |y|^2 u \right] dy \tag{8}$$

Stability analysis for $R(y)$

The rescaled KS involves two unknowns, u and a, and has the static solution $(R(y), a = 0)$. (To find λ solve $-\dot{\lambda}\lambda = a$.)

To investigate stability of R, we linearize the rescaled KS around (R, a) $\Rightarrow$ the linear operator (cf. Wolansky)

$$L_a = -\Delta - \nabla \cdot \left((\nabla \Delta^{-1} R) + R \nabla \Delta^{-1} \right) + a \nabla \cdot y. \qquad (9)$$

L_a is the hessian of the modified free energy functional $\mathcal{F}_a(u)$ at (R, a) and therefore it is self-adjoint in the inner product $\langle v, w \rangle_{J_R} := \langle v, J_R^{-1} w \rangle_{L^2}$, where $J_R := -\nabla \cdot R \nabla > 0$.

The operator L_a commutes with the rotations $\Longrightarrow$

$$L_a = \oplus_{m \geq 0} L_{am}.$$

Dejak-Lushnikov-Ovchinnikov-Sigal: Besides the zero and negative eigenvalues due to breaking translation and scaling symmetries, $\exists$ non-positive EVs $L_a \Rightarrow \mathcal{M}_{\mathrm{stat}} := \{ \frac{1}{\lambda^2} R(x/\lambda) \mid \lambda > 0 \}$ is unstable!

Dressing up the leading term

As $\mathcal{M}_{\rm stat} := \{\frac{1}{\lambda^2}R(x/\lambda) \mid \lambda > 0\}$ is unstable (with one unstable direction), we construct a one-parameter deformation of it.

Recalling $R(y) := \frac{8}{(1+|y|^2)^2}$, we let

$$R_{bc}(y) := \frac{8b}{(c+|y|^2)^2}, \tag{10}$$

with $b > 1$, b and c close to 1 and with an extra relation between the parameters a, b and c $\Longrightarrow$
a two-param. family of approx. solutions to the rescaled KS $\Longrightarrow$
the deformation (or *almost center-unstable*) manifold

$$\mathcal{M}_{\text{stat deform}} := \{(1/\lambda^2)R_{bc}(x/\lambda) \mid \lambda > 0,\ b, c\}. \tag{11}$$

This manifold absorbs all unstable/neutral degrees of freedom.
The previous result gives the *linear stability* of $\mathcal{M}_{\text{stat deform}}$ in the *radially symmetric* case.

Splitting the solution

We expect that the solution to the rescaled KS approaches the manifold $\mathcal{M}_{\text{stat deform}}$, as $\tau \to \infty$

$\Longrightarrow$ decompose the solution $u(y,\tau)$ to the rescaled KS as

$$u(y,\tau) = \underbrace{R_{b(\tau)c(\tau)}(y)}_{\text{leading term, finite dim}} + \underbrace{\phi(y,\tau)}_{\text{fluctuation, infinite dim}}, \qquad (12)$$

and require that the fluctuation $\phi(y,\tau)$ is orthogonal to the tangent space of $\mathcal{M}_{\text{stat deform}}$ at $R_{b(\tau)c(\tau)}(y)$,

$$\langle \partial_{b,c} R_{b(\tau)c(\tau)}(\cdot), \phi(\cdot,\tau)\rangle = 0.$$

The leading term, $R_{b(\tau)c(\tau)}(y)$, and the fluctuation, $\phi(y,\tau)$, evolve on a *different spatial scales*, as R_{bc} can rewritten as

$$R_{bc}(y) = R_{\frac{b}{c^2},1}\left(\frac{y}{\sqrt{c}}\right).$$

Collapse dynamics

Restrict to the *radially symmetric* case. Substituting the splitting

$$u = R_{bc} + \phi$$

into the rescaled KS and projecting the resulting equation onto $T\mathcal{M}_{\text{stat deform}}$, we arrive at

$$\begin{cases} c_\tau = 2a - \frac{4(b-1)}{\ln(\frac{1}{a})} + R_1(\phi, a, b, c), \\ \frac{b_\tau}{a} = -\frac{2(b-1)}{\ln(\frac{1}{a})} + R_2(\phi, a, b, c), \end{cases} \tag{13}$$

$|R_i(\phi, a, b, c)| \lesssim \frac{a}{\ln^2(\frac{1}{a})} \frac{1}{\ln(\frac{1}{a})}[(b-1)\|\phi\|_{L^2} + \|(1+|y|^2)^{-1}\phi\|^2_{L^2}]$,
and an eqn for the fluctuation ϕ. To eliminate a large term on the r.h.s. we choose

$$b - 1 = \frac{1}{2} a \ln \frac{1}{a}.$$

then ignoring R_i, these equations give the differential eq. for a

$$a_\tau = -\frac{2a^2}{\ln(\frac{1}{a})} + O(\frac{a^2}{\ln^2(\frac{1}{a})}). \tag{14}$$

Solving the equation

$$a_\tau = -\frac{2a^2}{\ln(\frac{1}{a})} + O(\frac{a^2}{\ln^2(\frac{1}{a})}).$$

in the leading order and recalling that

$$\lambda(t)\dot{\lambda}(t) = -a(\tau)$$

where

$$\tau = \int_0^t \frac{1}{\lambda^2(s)}\, ds,$$

we obtain the scaling law

$$\lambda(t) = (T-t)^{\frac{1}{2}} e^{-|\frac{1}{2}\ln(T-t)|^{\frac{1}{2}}}(c_1 + o(1)).$$

Numerical simulations

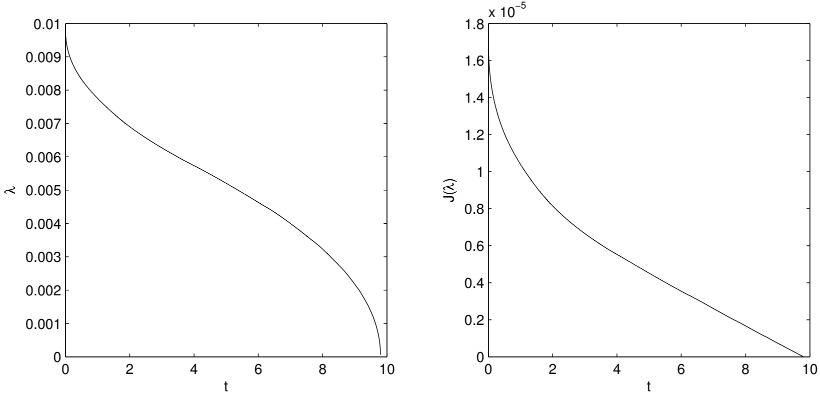

Figure: The right pane plots the quantity $J(\lambda) := e^{\sqrt{4\ln\frac{\lambda_0}{\lambda}}} / \sqrt{\ln\frac{\lambda_0}{\lambda}}(\frac{\lambda}{\lambda_0})^2$ against time, which according to the $\lambda-$equation should be linear as the blowup time is approached.

Related Results and Conclusions

Related results: Velázquez, Herrero - Velázquez, Lushnikov, Dyachenko - Lushnikov - Vladimirova

Rigorous results: Rafael and Schweyer

Conclusions: We described the universal profile of the collapse, which depends in a 'self-similar' way on a single time-dependent parameter, λ, whose dynamics is is given by the equations

$$a_\tau = -\frac{2a^2}{\ln(\frac{1}{a})} + O(\frac{a^2}{\ln^2(\frac{1}{a})}), \tag{15}$$

$$\lambda(t)\dot{\lambda}(t) = -a(\tau(t)),$$

where the new and old times are related as $\tau = \tau(t) \equiv \int_0^t \frac{1}{\lambda^2(s)}\, ds$.

Perspectives

- Trigger: aggregation $\Longrightarrow$ traveling wave/front
- Many agents (e.g. $c_\pm$ are positive/negative charge densities) and angiogenesis (formation of blud vessels)
- Kinetic and stochastic theory

$$\partial_t f + v \cdot \nabla_x f = \lambda T f, \tag{16}$$

  where $T = T(c)$ is a turning operator
- Microscopic models and derivation of the full Keller - Segel equations
- Social sciences: crime patterns and riot dynamics (density of physical events is coupled with an fictitious 'interaction' field, like a 'rumour mill' (crime patterns), or social tensions (riot dynamics), or social influences (illegal or uncivil activity))

Thank-you for your attention.

Comparison with Yang-Mills and Wave Maps Equations

Compare the dynamics for the scaling parameter $\lambda(t)$ for (MCF) and the critical Yang-Mills equation

$$\lambda\ddot{\lambda} = \frac{3}{4}\dot{\lambda}^4,$$

which gives

$$\lambda \approx \sqrt{\frac{2}{3}}\frac{t_* - t}{\sqrt{-\ln(t_* - t)}}.$$

and the critical wave map equation

$$\dot{\lambda}^2 = \lambda\ddot{\lambda}\ln\frac{a}{\lambda\ddot{\lambda}}\ , \ a = 0.122.$$