THE KELLER–SEGEL EQUATION COMPACT SURFACES

1. The Keller–Segel equation on the torus

Consider a distribution of germs $\rho(t,x_1,x_2)=\rho(t,\underline{x})$ and food $c(t,x_1,x_2)=c(t,\underline{x})$. We impose, as a model of nature,

$$\partial_t\rho = \partial_a^2\rho - \partial_a(\rho\partial_a c), \tag{1.1a}$$

$$\partial_a^2 c = -\rho. \tag{1.1b}$$

When a appears as an index, summation over $a\in\{1,2\}$ is implied. We take the Fourier transform of ρ: For all $\underline{k}\in\mathbb{Z}^2$, let $f_{\underline{k}}(\underline{x})=e^{2\pi i\underline{k}\cdot\underline{x}}$. Note that $f_{\underline{k}}$ is an eigenfunction of the Laplacian; i.e. $\partial_a^2 f_{\underline{k}}=-4\pi^2|k|^2 f_{\underline{k}}$. Also note that $\partial_a f_{\underline{k}}=2\pi i k_a f_{\underline{k}}$.

Let us write

$$\rho(t,\underline{x})=\sum_{\underline{k}\in\mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}). \tag{1.2}$$

Now any solution, c, to equation (1.1b) has the form

$$c(t,\underline{x}) = c_0 + \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{1}{4\pi^2|\underline{l}|^2} R_{\underline{l}}(t) f_{\underline{l}}(\underline{x}),$$

where $c_0\in\mathbb{C}$ can be chosen arbitrarily.

Using equations (1.1a) and (1.2) we get that

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^2}\dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2|\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} &= -\partial_a\left(\sum_{\underline{m}\in\mathbb{Z}^2} R_{\underline{m}} f_{\underline{m}} \sum_{\substack{\underline{l}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{R_{\underline{l}}}{4\pi^2|\underline{l}|^2}\partial_a f_{\underline{l}}\right)\\ &= -\sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2\\ \underline{l}\neq\underline{0}}} \frac{2\pi i l_a}{4\pi^2|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} \partial_a\Big(f_{\underline{l}} f_{\underline{m}}\Big).\end{aligned}$$

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Using that $f_{\underline{m}} f_{\underline{l}} = f_{\underline{m}+\underline{l}}$ and substituting $\underline{k} = \underline{l} + \underline{m}$ on the right-hand side, we get

$$\sum_{\underline{k}\in\mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k}\in\mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} = -i^2 \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{l_a(l_a+m_a)}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}}$$
$$= \sum_{\substack{\underline{l},\underline{m}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot(\underline{l}+\underline{m})}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{m}} f_{\underline{l}+\underline{m}}$$
$$= \sum_{\substack{\underline{l},\underline{k}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}} f_{\underline{k}}.$$

After pairing with $f_{\underline{k}}$ for any $\underline{k} \in \mathbb{Z}^2 - \{\underline{0}\}$ and separating out $R_{\underline{k}}$ terms, we get

$$\dot{R}_{\underline{k}} = \left(R_0 - 4\pi^2|\underline{k}|^2\right) R_{\underline{k}} + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0},\underline{k}}} \frac{\underline{l}\cdot\underline{k}}{|\underline{l}|^2} R_{\underline{l}} R_{\underline{k}-\underline{l}}.$$

It remains to be shown that there are such $R_{\underline{k}}$ which satisfy the equations.

2. The general case

Let Σ now some compact surface, with (positive definite) Laplace operator Δ (we can discuss what that means at some point), and assume that $f_0, f_1, f_2, \ldots, f_n, \ldots$ are an orthonormal basis of eigenvectors for $L^2(\Sigma)$, that is there are numbers $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ so that for all $n, o \in N$ we have

$$\Delta f_n = \lambda_n f_n, \quad \& \quad \langle f_n | f_o \rangle_{L^2(\Sigma)} = \delta_{n,o}.$$

For all $n, o, p \in \mathbb{N}$, let

$$\varphi_{n,o,p} := \int_\Sigma f_n f_o f_p \,\mathrm{d}A.$$

With that in mind, at some point prove the following: If $\rho \in C^1\big([0,T]; L^2(\Sigma)\big)$ solve the Keller–Segel equations on Σ and $R_n(t) := \langle f_n | \rho(t,\cdot)\rangle_{L^2(\Sigma)}$, then R_0 is constant and

$$\forall n \in \mathbb{N} - \{0\}: \quad \dot{R}_n = (R_0 - \lambda_n) R_n + \sum_{o,p\in\mathbb{N}-\{0\}} \frac{\lambda_m - \lambda_o + \lambda_p}{\lambda_p} \varphi_{n,o,p} R_o R_p.$$

Remark 2.1. *Note how the sign of the first term changes depending on whether $R_0 = \int_\Sigma \rho \,\mathrm{d}A$ is small or greater than λ_n!*

3. Banach Business

Consider the map $\mathcal{K} : \{R_{\underline{k}}\}_{\underline{k}\in\mathbb{Z}^2} \to C$ (C is something, perhaps $\{R_{\underline{k}}\}_{\underline{k}\in\mathbb{Z}^2}$) defined by

$$\mathcal{K}(R_{\underline{k}}) = R_{\underline{k}}(0) + \int_0^t (R_0 - 4\pi^2|\underline{k}|^2)R_{\underline{k}}(\tau) + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2} R_{\underline{k}-\underline{l}}(\tau)R_{\underline{l}}(\tau)\,\mathrm{d}\tau$$

Collections of fourier coefficients with $\mathcal{K}(R) = R$ satisfy equation (1.1a). We seek, therefore, conditions (on $(R_{\underline{k}})_{\underline{k}\in\mathbb{Z}^2}$ and t) which yield fixed points of $\mathcal{K}$. It would be sufficient to bound

$$\|\mathcal{K}(R) - \mathcal{K}(S)\| \leq \theta\|R - S\|$$

where $\|\cdot\|$ is perhaps

$$\sup_{t\in[0,t]} \sum_{\underline{k}\in\mathbb{Z}^2} |R_k(t)|^2.$$

and $0 \leq \theta < 1$ is some constant. We might begin to consider $\|\mathcal{K}(R) - \mathcal{K}(S)\| =$

$$\left\| \int_0^t (R_0 - 4\pi^2|\underline{k}|^2)\Big(R_{\underline{k}}(\tau) - S_{\underline{k}}(\tau)\Big) + \sum_{\substack{\underline{l}\in\mathbb{Z}^2 \\ \underline{l}\neq\underline{0}}} \frac{\underline{k}\cdot\underline{l}}{|\underline{l}|^2}\Big(R_{\underline{k}-\underline{l}}(\tau)R_{\underline{l}}(\tau) - S_{\underline{k}-\underline{l}}(\tau)S_{\underline{l}}(\tau)\Big)\,\mathrm{d}\tau \right\|.$$

It may be fruitful to bound the summands of $\|\cdot\|$ corresponding to values of $\underline{k}$, sharply enough that the infinite sum converges.