THE KELLER–SEGEL EQUATION ON THE TORUS

Consider a distribution of germs $\rho(t, x_1, x_2) = \rho(t, \underline{x})$ and food $c(t, x_1, x_2) = c(t, \underline{x})$. We impose, as a model of nature,

$$\begin{cases} \partial_t \rho = \partial_a^2 \rho - \partial_a(\rho \partial_a c), \\ \partial_a^2 c = -\rho. \end{cases}$$

When a appears as an index, summation over $a \in \{x_1, x_2\}$ is implied. We take the fourier transform of ρ: with $f_{k_1,k_2}(x_1, x_2) = f_{\underline{k}}(\underline{x}) = e^{2\pi i \underline{k} \cdot \underline{x}}$ eigenfunctions,

$$\rho(t, \underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}).$$

Now we may explicitly write c to satisfy the shorter equation:

$$c(t, \underline{x}) = \sum_{\substack{\underline{k} \in \mathbb{Z}^2 \\ \underline{k} \neq \underline{0}}} \frac{1}{4\pi^2 |\underline{k}|^2} R_{\underline{k}}(t) f_{\underline{k}}(\underline{x}).$$

Let's write the remaining equation more explicitly:

$$\sum_{\underline{k} \in \mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} = -\sum_{\underline{k} \in \mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} - \partial_a \left(\sum_{\underline{k} \in \mathbb{Z}^2} \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{R_{\underline{k}} f_{\underline{k}}}{4\pi^2 |\underline{l}|^2} 2\pi i k_a R_{\underline{l}} f_{\underline{l}} \right).$$

Next we perform the outer derivative and rearrange:

$$\sum_{\underline{k} \in \mathbb{Z}^2} \dot{R}_{\underline{k}} f_{\underline{k}} + \sum_{\underline{k} \in \mathbb{Z}^2} 4\pi^2 |\underline{k}|^2 R_{\underline{k}} f_{\underline{k}} = -i^2 \sum_{\underline{k} \in \mathbb{Z}^2} \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{R_{\underline{k}} R_{\underline{l}} f_{\underline{k}+\underline{l}}}{|\underline{l}|^2} k_a(k_a + l_a).$$

We substitute $k \mapsto k - l$ in the right sum so that

$$\sum_{\underline{k} \in \mathbb{Z}^2} \left(\dot{R}_{\underline{k}} + 4\pi^2 |\underline{k}|^2 R_{\underline{k}} \right) f_{\underline{k}} = \sum_{\underline{k} \in \mathbb{Z}^2} \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{R_{\underline{k}-\underline{l}} R_{\underline{l}}}{|\underline{l}|^2} \underline{k} \cdot (\underline{k} - \underline{l}) f_{\underline{k}}.$$

We use the fact, as we have implicitly already used to assert the form of ρ, that $L^2(\mathbb{T})$ is a vector space with basis $\{f_k\}$ to assert that the coefficients are equal

$$\forall \underline{k} \quad \dot{R}_{\underline{k}} + 4\pi^2 |\underline{k}|^2 R_{\underline{k}} = \sum_{\substack{\underline{l} \in \mathbb{Z}^2 \\ \underline{l} \neq \underline{0}}} \frac{R_{\underline{k}-\underline{l}} R_{\underline{l}}}{|\underline{l}|^2} \underline{k} \cdot (\underline{k} - \underline{l}).$$

It remains to be shown that there are such $R_{\underline{k}}$ which satisfy the equations.