A QUANTUM ALGORITHM TO FIND THE MAXIMUM OF A PAIR OF (SIGNED) INTEGERS

The idea for the code came from a video by Anant Vigyan, see [1].

Comments are welcome (email to akosnagymath@gmail.com).

1. The $U_<$ gate

First, let's construct a quantum gate, $U_<$, on 3 quantum qubits with the following property: given two classical bits, $a, b \in \{0,1\}$, the effect of $U_<$ on $|ab0\rangle (=: |a\rangle|b\rangle|0\rangle)$ is

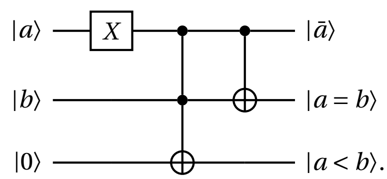
$$U_<(|ab0\rangle) = |\bar{a}(a=b)(a<b)\rangle,$$

where $\bar{a} := \text{NOT } a$.

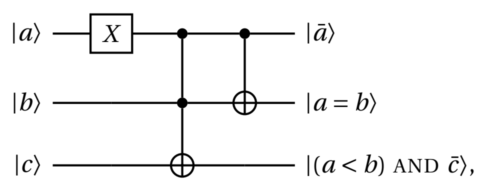
For the rest of this document, $\text{CNOT}_{i,j}$ will denote the CNOT gate with control qubit i and controlled qubit j. Similarly $\text{TOFF}_{i,j,k}$ will denote the Toffoli/TOFF gate with control qubits i and j, and controlled qubit k. Abstractly, the gate can be given as

$$U_< = \left(\text{CNOT}_{1,2} \otimes \mathbb{1}_2\right)\left(\text{TOFF}_{1,2,3}\right)\left(X \otimes \mathbb{1}_2 \otimes \mathbb{1}_2\right),$$

where $\mathbb{1}_2$ is the 2-by-2 identity matrix. Schematically $U_<$ is



More generally, the effect of $U_<$ for any $a, b, c \in \{0,1\}$ bits is



or, more abstractly, $U_<(|abc\rangle) = |\bar{a}(b+\bar{a})(c+\bar{a}b)\rangle$, where addition and multiplication is understood modulo 2, and thus, in the lexicographically ordered computational basis[1], the matrix of $U_<$ is

$$\left(U_<\right) = \begin{pmatrix} 0_4 & \mathbb{1}_4 \\ X \otimes \mathbb{1}_2 & 0_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

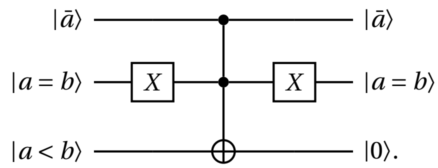
The claim about the effect of $U_<$ is now a matter of simple computation.

[1] that is, in $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

Remark 1.1. *Note that $U_<$ returns two important pieces of information: 1. which bit is larger (or equal to) than the other one, and 2. whether the bits are equal. The algorithm makes use of this by first implementing $U_<$ (with $c=0$) to each digit (including the sign digit) of $\mathtt{number}_1$ and $\mathtt{number}_2$.*

2. The U_0 gate

The purpose of this gate is to reset the third qubit to $|0\rangle$ in the output of $U_<$. This can easily be implemented with



3. The classical algorithm

Next, I describe the idea behind the algorithm.

We represent a nonnegative integer, $\mathtt{number} \in \mathbb{N}$ via

$$\mathtt{number} =_n 1c_1 \dots c_n,$$

where $n \geqslant \max\left(1, \lceil \log_2(\mathtt{number}) \rceil\right)$, $c_i \in \{0,1\}$, and $(c_1 \dots c_n)_2 = \mathtt{number}$. For negative integers, $\mathtt{number} \in -\mathbb{N}_+$, use

$$\mathtt{number} = 0c_1 \dots c_n,$$

where $n \geqslant \max\left(1, \lceil \log_2(|\mathtt{number}|) \rceil\right)$, $c_i \in \{0,1\}$, and $(c_1 \dots c_n)_2 = 2^n - |\mathtt{number}|$. This is **not** the usual binary representation of signed integers, but it is useful for the problem at hand. In particular the Python code:

```
bits = []
sign = number >> 31
while number != sign:
   bits.append([number \& 1])
   number >>= 1

bits = bits[::-1]
```

produces the array $\mathtt{bits} = [c_0, c_1, \dots, c_n]$, with $n = \lceil \log_2(|\mathtt{number}|) \rceil$. This has $O\left(\log_2(|\mathtt{number}|)\right)$ space and time complexity. If `number` is given in other binary representations, the conversion can also be done with the same space and time complexities.

Let now $n := \max\left(1, \log_2(\max(|\mathtt{number}_1|, |\mathtt{number}_2|))\right)$ and assume that the inputs, $\mathtt{number}_1$ and $\mathtt{number}_2$, are given the forms

$$\mathtt{number}_1 = a_0 a_1 \dots a_n,$$

$$\mathtt{number}_2 = b_0 b_1 \dots b_n,$$

defined above.

Since nonnegative numbers are larger than negatives, the statement $((\texttt{number}_1 < \texttt{number}_2)\ \text{AND}\ (b_0 < a_0))$ is `False`. Thus we have the following identity, by the virtue of our binary presentation

$$\begin{aligned}(\texttt{number}_1 < \texttt{number}_2) &= (a_0 < b_0)\ \text{OR}\ (a_1 \dots a_n < b_1 \dots b_n) \\ &= (a_0 < b_0) \\ &\qquad \text{OR} \\ &((a_1 < b_1)\ \text{XOR}\ (a_1 = b_1)\ \text{AND}\ (a_2 \dots a_n < b_2 \dots b_n)\ \text{XOR}\ \dots)\end{aligned}$$

In the next, final section, I construct a quantum circuit and show that a particular qubit is always found in the eigenstate $|\texttt{number}_1 < \texttt{number}_2\rangle$.

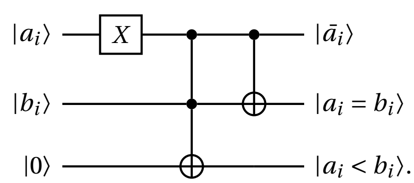
4. The quantum circuit

Again assume that the inputs, $\texttt{number}_1, \texttt{number}_2 \in \mathbb{Z}$, are given the forms

$$\texttt{number}_1 = a_0 a_1 \dots a_n,$$
$$\texttt{number}_2 = b_0 b_1 \dots b_n,$$

where $n := \max\big(1, \lceil \log_2(\max(|\texttt{number}_1|, |\texttt{number}_2|)) \rceil\big)$. The algorithm can be described as follows:

Step 0: Prepare $3 * n$ quantum registers and a classical register. For each $i \in \{0, 1, \dots, n\}$, initialize the $(3 * i + 1)$st qubit as $|a_i\rangle$, the $(3 * i + 2)$nd qubit as $|b_i\rangle$, and the $(3 * i + 3)$rd register as $|0\rangle$. Attach a $U_<$ gate to each such triplet:



Since $n \geqslant 1$, This generates at least 6 registers.

Step 1: Attach a $\text{TOFF}_{2,6,3}$ gate, a U_0 gate to the registers 4, 5, and 6, and finally a $\text{TOFF}_{2,5,6}$ gate.

Note that right after this gate, the third register is in the state $|(a_0 < b_0)\ \text{OR}\ (a_1 < b_1)\rangle$.

Step $\geqslant 2$: For $i \in \{2, \dots, n\}$ (in the normal order), attach a $\text{TOFF}_{3*i, 3*i+3, 3}$ gate, then a U_0 gate to the registers $(3 * i + 1)$, $(3 * i + 2)$, and $(3 * i + 3)$.

Note that right after this gate, the third register is in the state $|(a_0 < b_0)\ \text{OR}\ (a_1 \dots a_i < b_1 \dots b_i)\rangle$.

Last step: Measure the third qubit, which is not in the state $|(a_0 < b_0)\ \text{OR}\ (a_1 \dots a_n < b_1 \dots b_n)\rangle = |\texttt{number}_1 < \texttt{number}_2\rangle$.

Remark 4.1. *The attachment of the last U_0 gate does not change the result of the measurement.*

References

[1] Anant Vigyan, *Quantum algorithm 2 quantum comparison (new, to be published)*, available at https://www.youtube.com/watch?v=AmqvNmzeRkA. ↑1