Machine Learnig Homework 5

Problem 1

There are four names in the hat: Martin, Alexi, Igor, and Valery. The name that is randomly drawn wins the lottery. We define three different events:

X_1 Martin or Alexi wins.

X_2 Alexi or Igor wins.

X_3 Igor or Martin wins.

We have $P(X_1) = P(X_2) = P(X_3) = 0.5$ and $P(X_1, X_2) = P(X_1, X_3) = P(X_2, X_3) = 0.25$, so the events are pairwise independent. But $\mathrm{P}(X_1, X_2, X_3) = 0$, not 0.125, as it would if the events were all mutual independent.

Problem 2

We use the following identities:

$$Var[X] = E[X^2] + E[X]^2 \tag{1}$$

$$E[X + Y] = E[X] + E[Y] \tag{2}$$

Thus we have

$$\begin{aligned}
Var[X + Y] &\overset{(1)}{=} E\left[(X + Y)^2\right] - E[X + Y]^2 \\
&\overset{(2)}{=} E\left[X^2 + 2XY + Y^2\right] - (E[X] + E[Y])^2 \\
&\overset{(2)}{=} E\left[X^2\right] + 2E[XY] + E\left[Y^2\right] - E[X]^2 - 2E[X]E[Y] - E[Y]^2 \\
&= Var[X] + Var[Y] + 2\,(E[XY] - E[X]E[Y]) = Var[X] + Var[Y] + 2Cov(X, Y).
\end{aligned}$$

Problem 3

$$Var[X+Y] = E[(X+Y)^2]-(E[X]+E[Y])^2 = E[X^2]-E[X]^2+E[Y]^2-E[Y]^2+2E[XY]-2E[X]E[Y] \tag{3}$$

And hence,

$$Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]. \tag{4}$$

Problem 4

$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathrm{N}(\mu_Z, \Sigma_Z)$, with $\mu_Z = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ and $\Sigma_Z = \begin{pmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(X,Y) & \sigma_Y^2 \end{pmatrix}$. Since $\rho(X,Y) = 0$ also $Cov(X,Y) = 0$ and therefore $\Sigma_Z = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix}$ is diagonal. We can compute the conditional probabilities $p(X|Y) \sim \mathrm{N}(\mu_{X|Y}, \Sigma_{X|Y})$ and $p(Y|X) \sim \mathrm{N}(\mu_{Y|X}, \Sigma_{Y|X})$ using the formula (4.69) from Murphy p. 111. We get:

$$\begin{aligned} \mu_{X|Y} &= \mu_X + Cov(X,Y)\frac{1}{\sigma_y^2}(Y - \mu_Y) = \mu_X \\ \Sigma_{X|Y} &= \sigma_X^2 - Cov(X,Y)\frac{1}{\sigma_Y^2}Cov(Y,X) = \sigma_X^2 \end{aligned} \tag{5}$$

thus $P(X|Y) = P(X)$, proves that X and Y are independent.

$$\tag{6}$$