Machine Learnig Homework 11

Problem 1

Consider a neural network with an input layer, a hidden layer and an output layer. Assume a linear activation function in the output layer and let the output layer take form of a single neuron. The output of the network can be expressed as

$$y(x) = V^T a(x) + c = V^T f(z) = V^T f(W^T x + b) \tag{1}$$

where V weight matrix from the hidden layer to the output, a the activation of the hidden layer, f the hidden layer's nonlinearity, which acts on its input element wise, W the weight matrix from the input to the hidden layer, b and c are biases and x the input vector.

Sigmoid is a scaled and translated version of tanh, since

$$2\sigma(2x-1) = \frac{2}{1+e^{-2x}} - 1 = \frac{2e^{2x}}{e^{2x}+1} - 1 = \frac{e^{2x}-1}{e^{2x}+1} = tanh(x) \tag{2}$$

therefore, if we take $W = 2\widetilde{W}$ and $b = 2\widetilde{b}$ it is obvious that

$$2\sigma(W^T x + b) - 1 = tanh(\widetilde{W}^T x + \widetilde{b}) \tag{3}$$

Weights V and the bias c of the final layer can easily account for scaling and translation of the resulting hidden's layer activation.

Problem 2

$$\frac{d}{dx}\sigma(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1+e^{-x}}{(1+e^{-x})^2} - \frac{1}{(1+e^{-x})^2} = \sigma(x) - \sigma^2(x) = \sigma(x)(1-\sigma(x)) \tag{4}$$

$$
\begin{aligned}
\frac{d}{dx}tanh(x) &= \frac{2e^{2x}\left((e^{2x}+1) - (e^{2x}-1)\right)}{(e^{2x}+1)^2} \\
&= \frac{\left((e^{2x}+1) + (e^{2x}-1)\right)\left((e^{2x}+1) - (e^{2x}-1)\right)}{(e^{2x}+1)^2} \\
&= \frac{(e^{2x}+1)^2 - (e^{2x}-1)^2}{(e^{2x}+1)^2} \\
&= 1 - \frac{(e^{2x}-1)^2}{(e^{2x}+1)^2} = 1 - tanh^2(x)
\end{aligned}
\tag{5}
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