Assignment 1 of the course,
***Simulation of Ocean Atmosphere and Climate* (SOAC)**

This document describes the first assignment of SOAC 2020 and the background theory.

This assignment is concerned with building a model, which can simulate the time evolution of the horizontal wind velocity vector at the surface at the coast, under influence of an imposed pressure gradient force, the Coriolis force and frictional forces. For simplified boundary conditions and excluding the frictional term the model equations have an analytical solution, which can be used to test the accuracy of the numerical approximations of the time–derivatives in the governing differential equations.

Here, the model is used to simulate the sea/land breeze at the Dutch coast on 2 consecutive days in May. The model-performance is evaluated by comparing the output with hourly observations at the coastal measurement site of IJmuiden (west of Amsterdam). The quality of the simulation (or forecast) is quantified by correlating the simulated time series with the observational time series, by calculating the ratio of the standard deviations of the two time series and the root-mean-square difference between the two time series.

The performance of the model depends on the initial conditions and on the value of certain coefficients, like the frictional damping coefficient, which can be chosen freely and can be used to optimize the performance of the model.

First, the background theory is described (**section 1.17** of the lecture Notes on Atmospheric Dynamics), which should be familiar to all those who have done the course, Dynamical Meteorology.

The actual SOAC-assignment corresponds to **problems 1.14 and 1.15** at the end of this document.

A "skeleton Python script" is provided in **Box 1.9**.

A file containing the observational data, which is needed to compare the model output with reality, is provided separately.

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1.17 (D) The influence of earth's rotation, as revealed by the sea breeze

The climate-dynamicist, Victor Starr (1909-1976)[1], distinguished **two modes of convection**. The first mode of convection is **buoyancy driven** (first term on the r.h.s. of eq. 1.9), which means that it occurs when the driving force is along the vertical, as defined by the direction of gravity. The second mode of convection is brought about by **horizontal contrasts in diabatic heating**. In the second type of convection there need not necessarily be any vertical or hydrostatic instability, due to potentially colder air being found above potentially warmer air. Here, the potentially colder air tends to underrun the potentially warmer air and spread from the colder source to the warmer source. The potentially warmer air is displaced upwards, while maintaining a stable vertical stratification.

An example of the second type of "convection" is the **sea breeze circulation** [2]. On summer mornings in a coastal region, with cloudless and calm weather, the temperature of the air in the lowest kilometer of the atmosphere increases more rapidly over land than over sea. This is because the heat capacity of the upper layer of the sea, which is heated by solar radiation, is larger than the heat capacity of the layer of soil, which is heated by solar radiation. Moreover, over sea more solar heat is used to evaporate water instead of to raise the temperature. Therefore, the temperature of the land surface rises much faster than the temperature of the sea surface. The atmosphere aloft is heated principally by absorption of radiation, emitted by the earth's surface. Assuming hydrostatic equilibrium[3], this leads to a larger *vertical* pressure gradient over sea than over land. If pressure at the surface is initially the same over land and over sea, a horizontal pressure gradient perpendicular to the coast at upper levels, with *higher* pressure over land, arises. This pressure gradient drives air *from land to sea* (middle diagram in **figure 1.30**). The associated mass divergence over land and mass convergence over sea leads to a pressure decrease at the surface over land and a simultaneous pressure increase at the surface over sea. This, subsequently, initiates the sea breeze at the earth's surface.

Figure 1.31 shows observations of the sea breeze circulation over Jakarta, at the north coast of the island of Java, very close to the equator. The sea breeze is about 1 km thick, while the weaker return current is somewhat thicker. There is no appreciable land breeze during night. Farther away from the equator, the effect of the Coriolis force will become evident after a few hours. In the Netherlands this force will induce a clockwise turning of the wind vector. This interesting phenomenon can be studied with a very simple model and illustrated with observations of the wind at a coastal station.

Let us adopt an idealised model configuration in which the coast lies at x=0, i.e. parallel to the y-axis. Land lies at x>0 and sea at x<0. The sea-level pressure gradient, perpendicular to the coast, on a summer day is approximated ("**parametrised**") by a time dependent cosine with amplitude, A, as

$$\frac{\partial p}{\partial x} = A\cos(\Omega t + \varphi) \ . \tag{1.103}$$

Here, Ω is Earth's angular velocity ($=\pi/12$ hr^{-1}=7.2792×10^{-5} s^{-1}) and φ is a phase angle. If we assume that the minimum pressure gradient occurs at t=12 hrs (midday), then φ=0. The amplitude, A, is of the order of 0.1-1 hPa per 100 km (10^{-3} Pa m^{-1}) (see **figure 1.32**),

The horizontal wind at the coast is governed by eq. 7 and 8 of **box 1.6**, which are repeated in the following. For simplicity, we neglect the "tan ϕ" –terms.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \ , \tag{1.104a}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \ , \tag{1.104b}$$

where the **Coriolis parameter** is defined as

[1] V. P. Starr, 1968: **Physics of Negative Viscosity Phenomena.** MacGraw-Hill, 256 pp. (pages 26-27)

[2] This is not to say that the first, buoyancy-driven, type of convection does not play a role in the sea breeze circulation.

[3] The process of "hydrostatic adjustment", involving **sound waves** (**waves of expansion**), is treated in **chapter 3**.

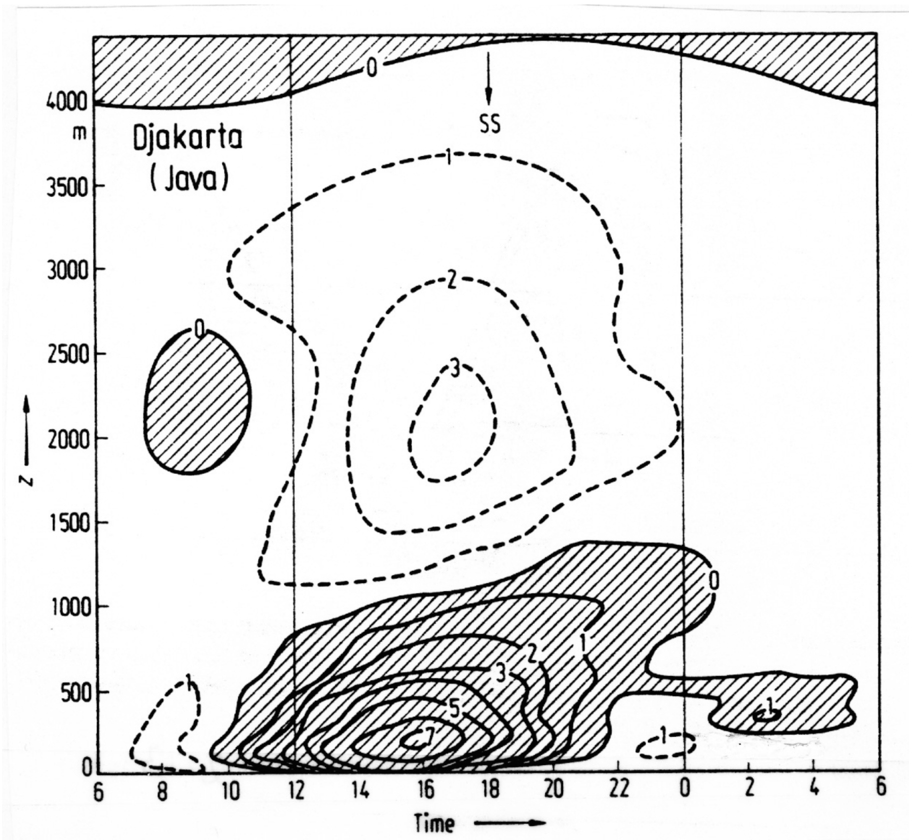

FIGURE 1.31. The "classical" analysis by van Bemmelen. Isolines of the north-south component of the wind velocity (in m/s) above Batavia (now Jakarta, Java) are obtained from hourly mean values evaluated from balloon observations from May to November 1909 to 1915. Wind from the sea (sea breeze) is hatched. There is a return current with a maximum shortly before sunset, at t=18 hours, and in height of 2 km (Van Bemmelen, W., 1922: Land-und seebrise in Batavia. **Beitr. Phys. frei. Atmos.**, **10**, 169-177).

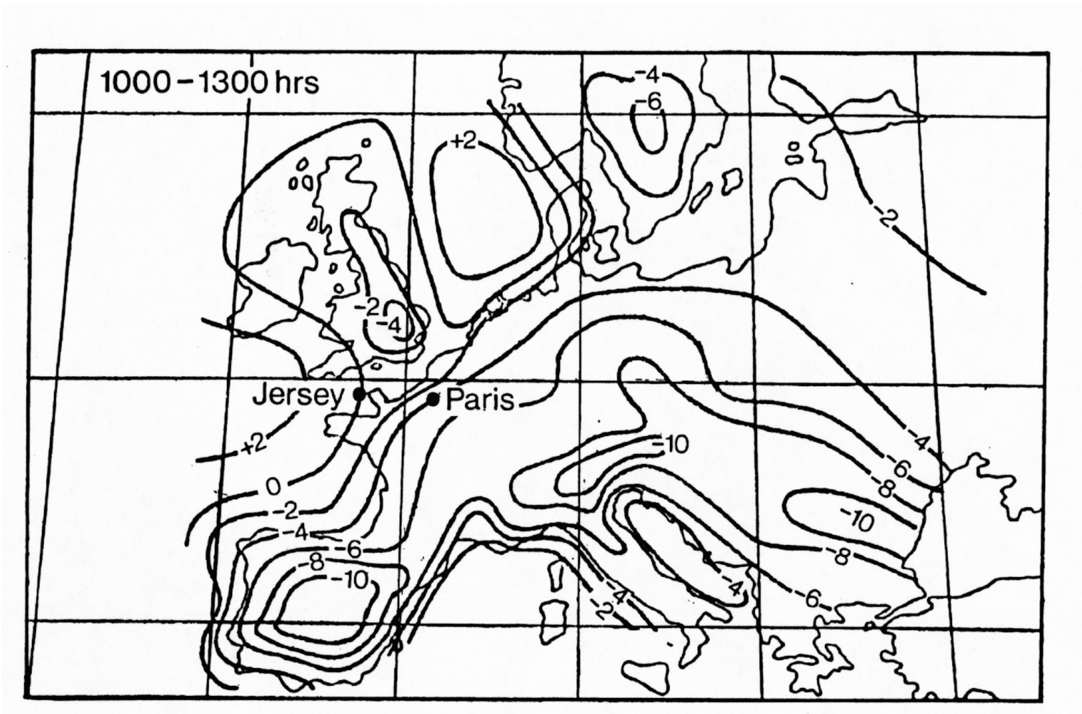

FIGURE 1.32: Surface pressure change (tenths of hPa) over Europe between 1000 and 1300 UTC in June (from Simpson, J.E., 1994: **Sea Breeze and local wind**. Cambridge University Press).

$$f \equiv 2\Omega \sin\phi$$

(also defined in eq. 9, **Box 1.6**). For the moment, we neglect frictional forces.

Let us assume that the pressure-gradient, parallel to the coast, $\partial p/\partial y$=0. We apply these equations to a point exactly at the coast. Substituting eq. 1.103 into eqs. 1.104a, and expanding the total derivative (eq. 1.5), eqs. 1.104a,b become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = fv - \frac{A}{\rho}\cos(\Omega t + \varphi) \ , \tag{1.105a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = -fu \ , \tag{1.105b}$$

Here we have, in fact, neglected all derivatives with respect to y and assumed that w=0 at the coast. The non-linear advection terms pose a serious problem to the general solution of these equations, but at the coast (x=0) it is reasonable to assume that

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial x} = 0 \ ,$$

so that eqs. 105a,b become linear:

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho}\cos(\Omega t + \varphi) \ , \tag{1.106a}$$

$$\frac{\partial v}{\partial t} = -fu \ , \tag{1.106b}$$

This set of equations can be simplified easily to

$$\boxed{\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho}\sin(\Omega t + \varphi)} \ , \tag{1.107}$$

which has the following solution:

$$\boxed{u = C_1 \sin(ft) + C_2 \cos(ft) + \frac{A\Omega}{\rho\left(f^2 - \Omega^2\right)}\sin(\Omega t + \varphi)} \ . \tag{1.108a}$$

The constants C_1 and C_2 are determined by the initial conditions (see below). The solution of eq. 1.107 consists of oscillations with two different frequencies, one, which is associated with the thermal forcing, and the other, which is associated with the natural ("eigen"-) response of the physical system. The "natural" frequency is isolated by putting A=0. We thus find that the Coriolis parameter, f, represents the natural frequency of the system. The associated oscillation is called an **inertial oscillation**. The Coriolis parameter is the **inertial frequency**. During one inertial oscillation the wind vector turns clockwise (if f>0), completing a full circle (a so-called **inertial circle**) in one **inertial period**, $2\pi/f$.

The solution (eq. 1.108a) for u is easily supplemented with the solution for v. Assuming that u=v=0 at t=0, and assuming φ=0, yields C_2=0. Substituting v=0 at t=0 in eq. 1.106a and using 1.108a yields

$$\boxed{v = C_1 \cos(ft) + \frac{fA}{\rho\left(f^2 - \Omega^2\right)}\cos(\Omega t)} \tag{1.108b}$$

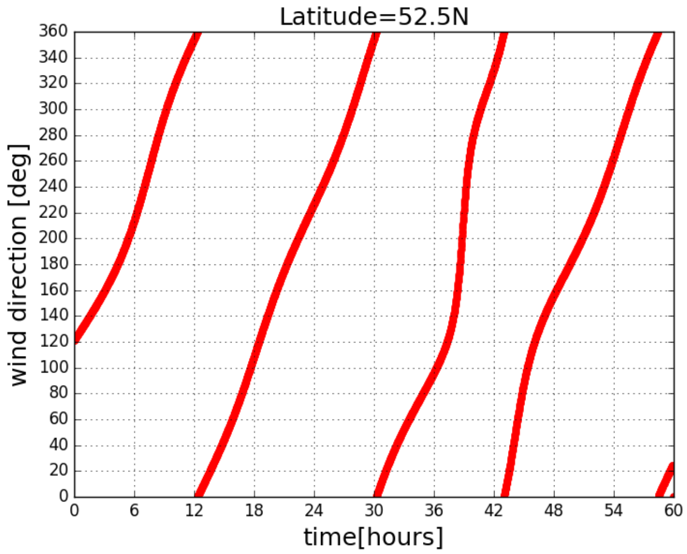

FIGURE 1.33. Wind direction, according to the analytical solution (eq. 1.108a,b), for latitude= 52.5°N, A=0.001 Pa m^{-1}, φ=0 and ρ=1.16 kg m^{-3}, assuming that u=v=0 at t=0. A wind direction equal to 270° implies a sea breeze perpendicular to the coast. The wind direction does not follow the same daily cycle each day and changes more quickly in theory (**this figure**) than in practice (**figure 1.34**). Model and observation can be brought closer together by adding friction to the model (**problem 1.15**).

with

$$C_1 = -\frac{Af}{\rho\left(f^2 - \Omega^2\right)}$$

Figure 1.33 shows the solution, in terms of wind direction, for a latitude of 52.5°N. Wind direction is defined as the direction from which the wind is blowing. A direction equal to 0° (or 360°) corresponds to a wind blowing from the north, parallel to the coast, and 270° corresponds to a wind blowing from sea to land, perpendicular to the coast (the "sea breeze"). The wind direction changes at a rate, which varies slightly with time. Between t=12 hours and t=30 hours the wind vector turns clockwise through a full circle, i.e. in 18 hours. The following two circles are completed in 12.5 hours and in 16 hours, respectively. The average of 18, 12.5 and 16 hours is 15.5 hours, which is slightly larger than the inertial period, $2\pi/f$, at 52.5°N, which is 15.1 hours.

Figure 1.34 shows hourly observations of the wind direction at a coastal station in The Netherlands, (Ijmuiden, **Figure 1.35**), during two days in early May 1976. The latitude of IJmuiden is 52.47°N. The wind at Ijmuiden is measured at a height of 10 m at the end of a pier, at a distance of about 1 km from the coast. The surface temperature of the North Sea in early May typically is about 10°C. During this period in 1976 summer came very early, with temperatures tipping 30°C at distances of about 50 km from the coast (**Figure 1.36**).

At IJmuiden the wind blows from the sea during the day and from the land during the night (**Figure 1.34**), but hardly never really perpendicular to the coast (either 300° or 120°). In accordance with the analytical solution (**figure 1.33**), the wind vector changes direction at an *approximate constant* rate. After t=41 hours the wind vector turns clockwise through a full circle within 20 hours. The turning of the wind, therefore, appears somewhat slower than according to the analytical solution. In reality, the forcing, i.e. the time-varying pressure gradient, which has a period 24 hours, determines the wind direction more strongly than is predicted by the analytical theory. By introducing the effect friction, we may be able to reconcile these differences (**Problem 1.15**).

At a latitude of 30° one inertial circle of the wind vector is completed in 24 hours. At this latitude we, therefore, expect a resonance between the thermal forcing, due to the land-sea temperature contrast, and the natural response of the atmosphere (the inertial oscillation). Indeed, according to eq. 1.108, u

beomes very large when ($f^2 - \Omega^2$) approaches zero. This is a manifestation of resonance. This occurs when sin ϕ approaches ±1//2, i.e near a latitude of ±30°. In reality resonance does not seem to occur at these latitudes, principally due to frictional effects.

Cooling at night is frequently not sufficient to reverse the sea-level pressure gradient perpendicular to the coast. Instead, the pressure gradient perpendicular to the coast tends to attain a component of constant sign, which is denoted here by B, such that eq. 1.103 becomes

$$\frac{\partial p}{\partial x} = A\cos(\Omega t + \varphi) + B\,. \tag{1.109}$$

This is observed, for example, over the Iberian Peninsula (**figure 1.32**), where a "**thermal low**" is observed nearly permanently over land in summer. In order to incorporate this effect, we use eq. 1.109 instead of eq. 1.103. For completeness, we also assume that a constant (large scale) pressure gradient, D, parallel to the coast exists. We thus rewrite eqs. 1.106a,b as

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho}\cos(\Omega t + \varphi) - \frac{B}{\rho}\ , \tag{1.110a}$$

$$\frac{\partial v}{\partial t} = -fu - \frac{D}{\rho}\ , \tag{1.110b}$$

This set of equations has a steady state solution, called "**geostrophic balance**":

$$u = u_g = -\frac{D}{f\rho};\ v = v_g = \frac{A}{f\rho}\cos(\Omega t + \varphi) + \frac{B}{f\rho} \tag{1.111}$$

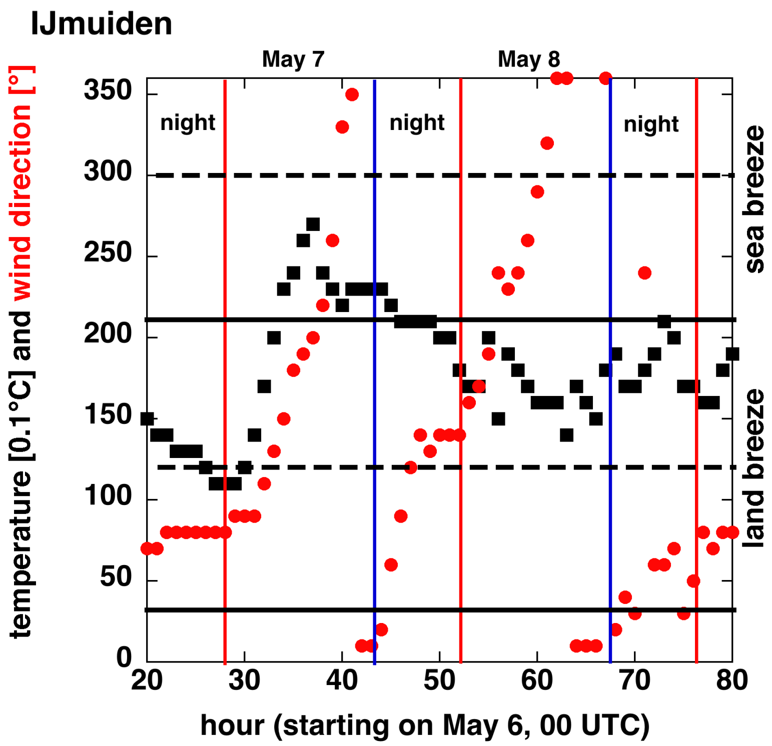

FIGURE 1.34. Hourly temperature (black squares) and wind direction (red circles) at a point on the coast of the The Netherlands (IJmuiden, see **figure 1.35**) on May 7 and 8, 1976 (hour=20 corresponds to 20 UTC on May 6). Wind direction is defined in the text. The coastline around IJmuiden is rotated clockwise by about 30°.The horizontal dashed line indicates the direction perpendicular to the coast; the horizontal solid line indicates the direction parallel to the coast. The wind has a landward component (i.e is a sea breeze) between 14 UTC and 20 UTC (shortly after sun-set) on 7 May and between 8 UTC and 20 UTC on 8 May.

FIGURE 1.35. **Left panel**: Map of Holland (on the right) and the North Sea (on the left), showing the positions of measuring stations. A wind direction of 360° (or 0°) corresponds to wind blowing from the north (indicated by the letter "N"). **Right panel**: photograph of the pier at IJmuiden, looking in south-easterly direction, with the position of the measuring site indicated by an arrow. Photograph made by Aarnout van Delden on Saturday, 1 October 2016, 15:00

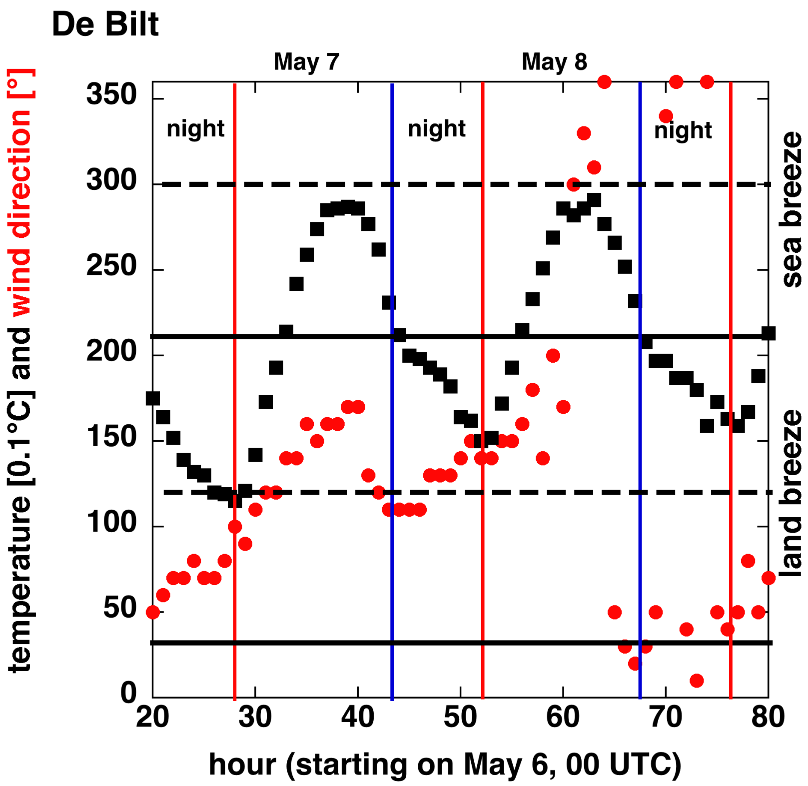

FIGURE 1.36. The hourly temperature (black squares) and wind direction (red circles) at De Bilt (left panel of **figure 1.35**), on May 7 and 8, 1976. The horizontal dashed line indicates the direction perpendicular to the coast; the horizontal solid line indicates the direction parallel to the coast. The northerly direction is indicated in **figure 1.35**. A wind direction of 270° implies that the wind is blowing eastward.

Where u_g and v_g are the components of the geostrophic wind perpendicular and parallel to the coast, respectively. By definition, the full wind is equivalent to the sum of the **geostrophic wind** (subscript, g) and the "**ageostrophic wind**" (subscript, a). We can now rewrite eq. 1.110ab as

$$\boxed{\frac{\partial u}{\partial t} = f\left(v - v_g\right) \equiv f v_a}\ , \tag{1.112a}$$

$$\boxed{\frac{\partial v}{\partial t} = -f\left(u - u_g\right) \equiv -f u_a}\ . \tag{1.112b}$$

In theory, the land-sea breeze is identified with the ageostrophic wind. The land-sea breeze, therefore, has a component parallel to the coast, $v_a \equiv (v - v_g)$, next to a component perpendicular to the coast, $u_a \equiv (u - u_g)$.

The geostrophic wind components, u_g and v_g perpendicular and parallel to the coast, respectively, are given by

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \text{ and } v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}. \tag{1.113}$$

These definitions are used in **problems 1.13-1.15** in which an attempt is made to apply the theory of this section to explain the daily cycle of the wind at the west coast of The Netherlands on May 6 to May 8, 1976.

We will return to this analysis of the sea breeze at the Dutch coast in **part G** (**section 1.31**) of these lecture notes, which is concerned with the state of "quasi-balance" ($|u_a| << |u_g|$ and $|v_a| << |v_g|$).

Box 1.9 A linear numerical model of the sea/land breeze

Equations 1.112a,b, together with the geostrophic relations (1.113) constitute a Eulerian model of the sea breeze at a fixed point at the coast. For a relatively simple case in which the pressure gradient, $\partial p/\partial x$, perpendicular to the coast, follows a daily cycle according to eq. 1.103 with φ=0 and the pressure gradient parallel to the coast, $\partial p/\partial y$=0, so that u_g=0, an **analytical solution** is available:

$$u(t) = \frac{A}{\rho(f^2 - \Omega^2)}\left(\Omega \sin(\Omega t) - f \sin(ft)\right)\ ; \tag{1a}$$

$$v(t) = \frac{Af}{\rho\left(f^2 - \Omega^2\right)}\left(\cos(\Omega t) - \cos(ft)\right)\ . \tag{1b}$$

The geostrophic wind in this special case is

$$u_g = 0\ ;\ v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} = \frac{A}{\rho f}\cos\Omega t\ . \tag{2}$$

The governing equations in the slightly more general case with $\varphi \neq 0$ are

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho}\cos\left(\Omega t + \varphi\right) \tag{3a}$$

$$\frac{\partial v}{\partial t} = -fu \tag{3b}$$

These equations can be solved numerically by first dividing the time-axis into discrete intervals, Δt, and then approximating the time-derivative by a Taylor series as follows:

$$u(t+\Delta t)=u(t)+\Delta t\frac{\partial u}{\partial t}(t)+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(t)+O\left((\Delta t)^3\right). \quad (4)$$

So that (dividing by Δt):

$$\frac{u(t+\Delta t)-u(t)}{\Delta t}=\frac{\partial u}{\partial t}(t)+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(t)+O\left((\Delta t)^2\right). \quad (5)$$

The left hand side of this equation is a numerical approximation to the first term on the right hand side if the remaining terms on the right hand side are "sufficiently small". This depends on Δt. The numerical error is said to be "first order in Δt", because the first error term is proportional to Δt. This numerical approximation to the time derivative is called "**Euler-forward scheme**".

A scheme, which, at first sight, is more accurate is the second order accurate "**leap-frog scheme**". The time-derivative is approximated by a Taylor series in both backward and forward direction as follows:

$$u(t+\Delta t)=u(t)+\Delta t\frac{\partial u}{\partial t}(t)+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(t)+\frac{(\Delta t)^3}{6}\frac{\partial^3 u}{\partial t^3}(t)+O\left((\Delta t)^4\right) \quad (6a)$$

$$u(t-\Delta t)=u(t)-\Delta t\frac{\partial u}{\partial t}(t)+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(t)-\frac{(\Delta t)^3}{6}\frac{\partial^3 u}{\partial t^3}(t)+O\left((\Delta t)^4\right) \quad (6b)$$

Subtracting these equations and dividing by $2\Delta t$ yields

$$\frac{u(t+\Delta t)-u(t-\Delta t)}{2\Delta t}=\frac{\partial u}{\partial t}(t)+\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}(t)+O\left((\Delta t)^3\right). \quad (7)$$

The left hand side of this equation is a numerical approximation to the first term on the right hand side if the remaining terms on the right hand side are "sufficiently small". This depends on $(\Delta t)^2$. The numerical error is said to be "second order in Δt". The leap-frog scheme, however, is not very reliable because odd and even (alternating) time steps are decoupled. Obviously, a numerical integration cannot be initialized with the leap-frog scheme.

A more accurate and often used numerical approximation to the time derivative is the fourth-order **Runge-Kutta scheme**, which, in general goes as follows. The numerical approximation to an ordinary differential equation of the type,

$$\frac{dX}{dt}=F(X) \quad (8a)$$

is

$$X_0=X(t);\ X_1=X_0+\frac{1}{2}F(X_0)\Delta t;\ X_2=X_0+\frac{1}{2}F(X_1)\Delta t;\ X_3=X_0+F(X_2)\Delta t;\ X_4=X_0-\frac{1}{2}F(X_3)\Delta t;$$
$$X(t+\Delta t)=\frac{1}{3}\left(X_1+2X_2+X_3-X_4\right). \quad (8b)$$

PROBLEM 1 BOX 1.9. Resonance.
When $f^2=\Omega^2$ resonance occurs, according to eq. 1a,b (**this Box**). Where on Earth may this occur? Is this a realistic scenario?

PROBLEM 2 BOX 1.9. **Sea breeze strength**.
What is the maximum value of u if f=0, ρ=1.16 kg m^{-3}? Do you expect higher or lower *absolute* wind speeds if $f \neq 0$.

PROBLEM 3 BOX 1.9. **Turning of the sea breeze**.
How will the wind vector rotate in time in the Northern Hemisphere (clockwise or anti-clockwise), given that u=0 and v=0 at t=0?

PROBLEM 4 BOX 1.9. **Sea breeze as a forced oscillator**.
If u=0 and v=0 at t=0, at what earliest time is u=0 and v<0 at the latitude of Ijmuiden (52°N)? At what next point in time is this again the case? Does the time difference correspond to the frequency of the forcing?

PROBLEM 5 BOX 1.9. **Introducing the effect of friction**.
The sea breeze model equations 3a and 3b become more realistic with the effect of surface dag added as follows.

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho}\cos(\Omega t + \varphi) - \lambda u\,, \tag{9a}$$
$$\frac{\partial v}{\partial t} = -fu - \lambda v\,. \tag{9b}$$

In honour Lord Rayleigh, the parameter, λ, is called the "**Rayleigh-damping coefficient**". It can easily be seen that the terms with λ have the effect of reducing the absolute value of the wind velocity. Write down the "stationary solution" of this system of equations and give an interpretation of this solution.

PROBLEM 6 BOX 1.9. **Python code of a linear numerical sea-breeze model.**
A linear numerical sea breeze model, which is based on eqs. 3a,b (**this Box**), with the time derivatives approximated by the Euler forward scheme (eq. 5, **this Box**), results in the following Python-script. Run the script and investigate the influence of damping by varying the value of λ (labda).

```
# BEGINNING OF THE PYTHON SCRIPT

# https://docs.python.org/2.7/
# IMPORT MODULES
import numpy as N  # http://www.numpy.org
import matplotlib.pyplot as P   # http://matplotlib.org
import math as M  # https://docs.python.org/2/library/math.html

# Length of the simulation and time-step
time_max = 48.0   # length in hours of the simulation
dt = 30.0 # time step in s

# PARAMETER VALUES
A = 0.001 # Pa m^-1
phase = 0.0 # phase of surface pressure gradient in time
lat = 52 # latitude in degress
phi = 0.0 # phase of the pressure gradient
labda = 0.0  # Rayleigh damping coefficient

# CONSTANTS
omega = 0.000072792  # angular velocity Earth [s^-1]
pi = M.pi
ro = 1.25 # density kg m^-3
fcor = 2 * omega *  M.sin(lat * pi/180)  # Coriolis parameter
C1= - (A/(fcor*ro)) * ( (M.pow(omega,2) /(M.pow(fcor,2) - M.pow(omega,2))) + 1)
C3 = A * omega /(ro*(M.pow(fcor,2) - M.pow(omega,2)))
```

```
#  NUMBER OF TIME STEPS AND TIME-AXIS FOR PLOTTING
nt_len = int(48 * 3600 / dt) # maximum number of time steps in the integration
time = N.zeros((nt_len))
time_axis = N.zeros((nt_len))

#  DEFINE time, u, v and the analytical solution, u_ana, as arrays and fill them with zero's
u = N.zeros((nt_len))      # x-component velocity, numerical solution
v = N.zeros((nt_len))      # y-component velocity, numerical solution
u_ana = N.zeros((nt_len))  # analytical solution x-component velocity

# INITIAL CONDITION (t=0) : atmosphere in rest
nt = 0
time[nt] = 0
time_axis[nt] = 0
u[nt] = 0
v[nt] = 0
u_ana[nt] = 0

# TIME LOOP EULER FORWARD SCHEME
for nt in range(len(time)-1):
 du = dt * (-(labda * u[nt]) + (fcor*v[nt]) - ((A/ro)* M.cos((omega*time[nt])+phase)))
 dv = dt * (-(labda * v[nt]) - (fcor * u[nt]))
 time[nt+1] = time[nt]+dt
 u[nt+1] = u[nt] + du
 v[nt+1] = v[nt] + dv
 u_ana[nt+1] = (C1 * M.sin(fcor * time[nt+1])) + ( C3* M.sin((omega * time[nt+1]) + phase) )

for nt in range(len(time)):
 time_axis[nt] = time[nt] / 3600. # time axis in hours

# MAKE PLOT of evolution in time of u, v and u_ana
P.plot(time_axis, u_ana, color='black')
P.plot(time_axis, u, color='red')
P.plot(time_axis, v, color='blue')
P.axis([0,time_axis[nt_len-1],-25.0,25.0])  # define axes
P.xticks(N.arange(0,time_axis[nt_len-1]+0.1,6), fontsize=12)
P.yticks(N.arange(-25.0,25.0,5), fontsize=12)
P.xlabel('time [hours]', fontsize=14) # label along x-axes
P.ylabel('velocity [m/s]', fontsize=14) # label along x-axes
P.title('SeabreezeSimulation') # Title at top of plot
P.text(1, 23, 'u (analytical solution) (no damping): black line', fontsize=10, color='black')

P.text(1, 21, 'u (numerical solution): red line (forward time difference scheme)', fontsize=10,
color='red')

P.text(1, 19, 'v (numerical solution): blue line (forward time difference scheme)', fontsize=10,
color='blue')

P.grid(True)
P.savefig("SeabreezeSimulation.png") # save plot as png-file
P.show() # show plot on screen


# END OF THE PYTHON SCRIPT
```

Box 1.10 Measuring model performance: Taylor diagram

On a Taylor diagram (Taylor 2001) the correlation coefficient, the ratio of the standard deviations of two fields and the root-mean-square difference between two fields are indicated by a single point on a two-dimensional plot, thereby providing a short summary of the degree of correspondence of the two fields. The **correlation coefficient**, r, between two fields, A and B, is defined as

$$r = \frac{\frac{1}{N}\sum_{n=1}^{N}\left(A_n - \overline{A}\right)\left(B_n - \overline{B}\right)}{\sigma_A \sigma_B}$$

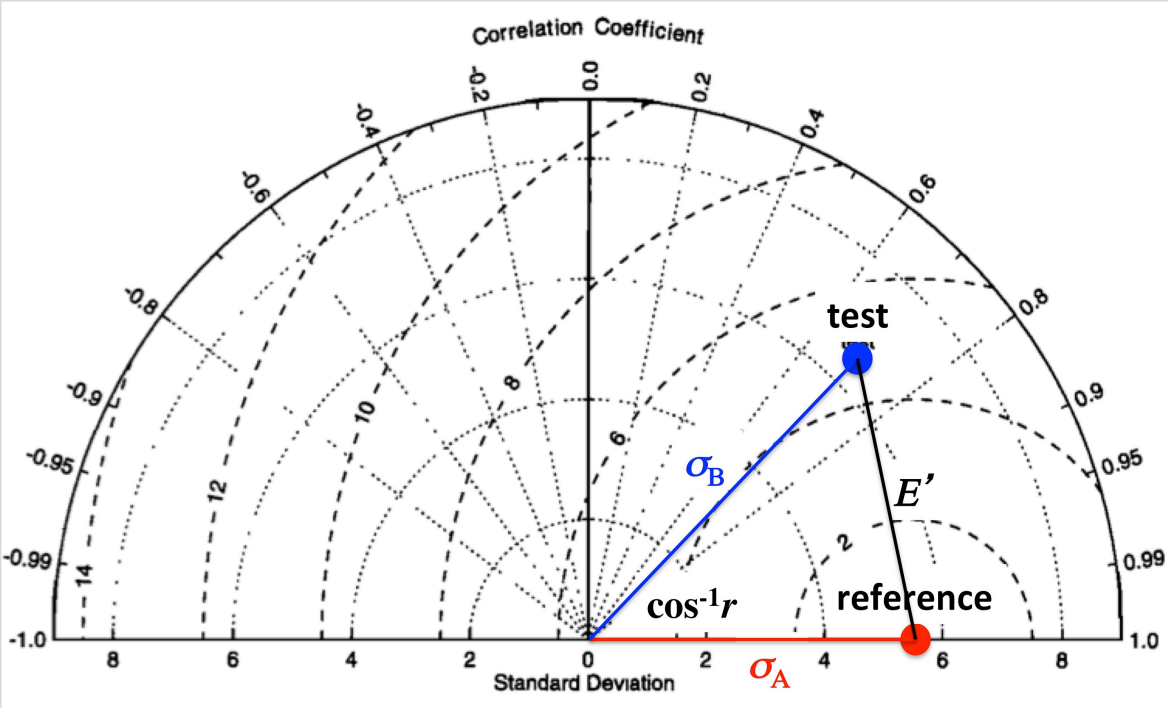

FIGURE 1 (Box 1.10). "**Taylor diagram**", for displaying pattern statistics. The radial distance from the origin is proportional to the standard deviation of a pattern (blue and red line). The centred RMS difference between the test and reference field is proportional to their distance apart (in the same units as the standard deviation). The correlation between the two fields is given by the azimuthal position of the test field.

where $\overline{A}$ and $\overline{B}$ are the mean values and σ_A and σ_B are the standard deviations of A and B, respectively. The **standard deviation** is defined as

$$\sigma_A \equiv \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(A_n - \overline{A}\right)^2}\,.$$

The maximum value of r is 1 when $\left(A_n - \overline{A}\right) = a\left(B_n - \overline{B}\right)$ for all n, where a is a positive constant. If a =1, the fields are identical. The correlation coefficient is not a measure of the relative amplitudes of variation, as determined by their variances.

To further quantify differences in two fields, the **root-mean-square (RMS) difference**, E, is introduced as,

$$E \equiv \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(A_n - B_n\right)^2}\ .$$

E can be resolved into two component: the overall **biass**,

$$\overline{E} \equiv \overline{A} - \overline{B}\,,$$

and the **centred pattern RMS**:

$$E' \equiv \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left[\left(A_n - \overline{A}\right) - \left(B_n - \overline{B}\right)\right]^2}\,.$$

The full mean square difference is

$$E^2 = \overline{E}^2 + E'^2 .$$

The statistical parameters r, E', σ_A and σ_B, which are all useful in comparisons of patterns, are related by

$$E'^2 \equiv \sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B r .$$

This relation can be interpreted in terms of the law of cosines,

$$c^2 \equiv a^2 + b^2 - 2ab\cos\phi ,$$

where a, b and c are the lengths of the sides of a triangle and ϕ is the angle opposite to side c.

We can now construct a so-called **Taylor diagram** (**figure 1** in this Box) that quantifies the statistical similarity between a reference field, A, which usually represents the observations, or reanalysis of observations, and a test field, B, which represents the model simulation. The statistical similarity is represented by a triangle in this diagram with sides of lengths E', σ_A and σ_B, where $\cos^{-1} r$ represents the angle opposite the side of length E'. The side with length σ_A coincides with the abscissa (horizontal axis) of the Taylor diagram. A Taylor diagram, which quantifies the performance of n simulations, compared to the reanalysis, should show n+1 points and n hypothetical triangles.

PROBLEM 1 BOX 1.10. Evaluating the numerical accuracy of the sea breeze model

Test the accuracy of the numerical sea breeze model with λ=0 and φ=0 and for idealised initial conditions (i.e. u=v=0 at t=0) with the Euler forward scheme (**Box 1.9**) for different time steps (e.g. for Δt=360 s, 120 s, 30 s) by correlating the numerical solution with the analytical solution. Compare the standard deviations and compute the root mean square (RMS) difference of the numerical solution with the analytical solution. Compute the centred pattern RMS. Implement the leap-frog scheme and repeat the test (start the integration with one forward step). Implement the four-step Runge Kutta Scheme (eq. 8a,b) and repreat the test. What do you conclude?

Reference

Taylor, K.E., 2001: Summarizing multiple aspects of model performance in a single diagram. **J.Geophys.Res., 106** (D7), 7183-7192.

PROBLEM 1.13. **Analysis of the sea-level pressure and wind from observations**

Observations are performed at randomly spaced points (**figure 1.35**), while theoretical analysis and/or numerical models usually require data on a regular grid of points. This requires interpolation of the observations. The most simple interpolation method is called "**piecewise linear interpolation**". Suppose that we want to interpolate observations to a point with horizontal coordinates, (x_0, y_0), e.g. IJmuiden (**figure 1.35**). The horizontal coordinates of three measuring points in the vicinity of this point are given by (x_i, y_i), with i=1, 2, 3. The value of a variable (for instance the sea level pressure) at one of these points can be *approximated* as follows.

$$p_i = p_0 + (x_i - x_0)\frac{\partial p}{\partial x} + (y_i - y_0)\frac{\partial p}{\partial y} , \qquad (1.114)$$

The desired value of the pressure at (x_0, y_0) is p_0. Applying eq. 1.115 to the three measuring points yields three equations with three unknowns, p_0, $\partial p/\partial x$ and $\partial p/\partial y$, which can be written concisely as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & (x_1 - x_0) & (y_1 - y_0) \\ 1 & (x_2 - x_0) & (y_2 - y_0) \\ 1 & (x_3 - x_0) & (y_3 - y_0) \end{pmatrix} \begin{pmatrix} p_0 \\ \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}.$$

This system of equations can be written shortly as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = M \begin{pmatrix} p_0 \\ \partial p/\partial x \\ \partial p/\partial y \end{pmatrix} \text{ which implies that } \boxed{\begin{pmatrix} p_0 \\ \partial p/\partial x \\ \partial p/\partial y \end{pmatrix} = M^{-1} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}},$$

To find, p_0, $\partial p/\partial x$ and $\partial p/\partial y$ w need to invert the matrix,

$$M = \begin{pmatrix} 1 & (x_1 - x_0) & (y_1 - y_0) \\ 1 & (x_2 - x_0) & (y_2 - y_0) \\ 1 & (x_3 - x_0) & (y_3 - y_0) \end{pmatrix}.$$

In this problem the point i=0, with coordinates, (x_0, y_0), corresponds to IJmuiden. Observations from three measuring sites are used to estimate the sea level pressure *gradient*. These measuring sites and their locations are listed in **table 1.3**. Because we need to estimate only the pressure gradient, IJmuiden is both the reference point and one of the measuring sites.

Station	Latitude [°]	Longitude [°]
IJmuiden	52.47	4.57
Schiphol	52.32	4.78
Valkenburg	52.18	4.42

Table 1.3. Coordinates of measuring stations (see **figure 1.34**). IJmuiden, Valkenburg and Schiphol form a triangle

Compute the sea level pressure-*gradient* at IJmuiden for the 60 hour period running 12 UTC on 6 May 1976 to 0 UTC on 9 May 1976 from the measurements at IJmuiden, Schiphol and Valkenburg. Note that, since the locations of stations are given in degrees latitude and longitude in **table 1.3**, it might be quicker to work in these units. Observational data can be retrieved from https://projects.knmi.nl/klimatologie/uurgegevens/. The distance from IJmuiden to Valkenburg is only about 0.3 degree in latitude. You may therefore assume that the latitude-longitude grid is Cartesian, where 1 degree in longitude corresponds to 67737 m and 1 degree in latitude corresponds to 111195 m. The coast at IJmuiden is rotated clockwise by 30° with respect to the south to north direction. Decompose the sea level ***pressure gradient perpendicular to the coast*** at IJmuiden into (1) a sinusoidal time-varying component with a period of 24 hours and (2) slowly varying or constant component (eq. 1.109 and **Box 1.9**). The second component is associated with both the slow change of the location and intensity of large-scale pressure systems, which might partially be thermally induced. The first component is associated with the diurnal thermal forcing. Estimate the value of A from the data.

PROBLEM 1.14. **Simulation of the sea breeze at the coast: parametrising the effect of surface drag (PROJECT 3, in groups of max. 3 students)**

Simulate the wind at IJmuiden on 7 and 8 May 1976 (48 hours). Use the model, which is described in **Box 1.9**. Assume that the wind is driven by a pressure gradient perpendicular to the coast, which can be expressed as

$$\frac{\partial p}{\partial x} = A\cos(\Omega t + \varphi) + B\,. \tag{1.109}$$

(**problem 1.13**). Assume that A, B and φ are constant. Find the best value A, B and φ (eq. 1.109) from the analysis (**problem 1.13**) of the observed *average daily cycle of the pressure gradient* at IJmuiden on 7 and 8 May 1976. The model is based on eqs. 1.10a,b, with Rayleigh-drag added to represent surface drag, as explained in Box 1.9:

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho}\cos(\Omega t + \varphi) - \frac{B}{\rho} - \lambda u\ , \tag{1.110a}$$

$$\frac{\partial v}{\partial t} = -fu - \lambda v\ , \tag{1.110b}$$

Initialise the model using the observed values of u and v at IJmuiden at 00 UTC on 7 May 1976. First assume that the Rayleigh damping coefficient, λ=0. Compare the outcome with the hourly observations of u and v at IJmuiden on 7 and 8 May 1976, by evaluating the correlation coefficient, standard deviation, root-mean-square (RMS) difference and centred pattern RMS (**Box 1.10**). Adjust the value of λ until you get a better result. In other words, the parameter, λ, is used to "tune" the model, to fit the observations as best as possible. Numerical models for weather prediction contain many of such tuning parameters! Repeat the exercise with the best value of λ by initialising the model with the observed values of u and v at IJmuiden at 12 UTC on 6 May 1976 and again compare the outcome with the hourly observations of u and v at IJmuiden on 7 and 8 May 1976. What do you conclude?

Try a different representation of surface drag, which depends on the square of the velocity as,

$$\frac{\partial u}{\partial t} = \ldots - C_D|u|u\ ,$$

where C_D is the drag-coefficient. Does this parametrisation work better than the linear Rayleigh damping?

PROBLEM 1.15. Initialising the model with the geostrophic wind

We now include the pressure gradient parallel to the coast by assuming that $\partial p/\partial y = D$. This pressure gradient does not have a daily cycle. It varies very slowly, so that we may assume that D=constant. The equation for v becomes

$$\frac{\partial v}{\partial t} = -fu - \frac{D}{\rho} - \lambda v$$

The constant part pf the pressure gradient is associated with a geostrophic wind as follows

$$u = u_g = -\frac{D}{f\rho};\ v = v_g = \frac{B}{f\rho}$$

Can you improve the sea breeze model performance (see **problem 1.14**) by initialising the model with the geostrophic wind?