Diffusion Model

CMSC 25025 Final Project

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Question 1

As given in the paper, $q_{t|t-1}(x_t|x_{t-1}) = \mathcal{N}\left(\sqrt{\alpha_t}x_{t-1}, (1-\alpha_t)I_d\right)$. But through the reparameterization trick, we can write the above as

$$q_{t|t-1}(x_t|x_{t-1}) = \sqrt{\alpha_t}x_{t-1} + \varepsilon_1\sqrt{(1-\alpha_t)}$$
$$q_{t-1|t-2}(x_{t-1}|x_{t-2}) = \sqrt{\alpha_{t-1}}x_{t-2} + \varepsilon_2\sqrt{(1-\alpha_{t-1})}$$
$$\Rightarrow q_{t|t-2}(x_t|x_{t-2}) = \sqrt{\alpha_t}\sqrt{\alpha_{t-1}}x_{t-2} + \sqrt{\alpha_t}\sqrt{1-\alpha_{t-1}}\varepsilon_2 + \sqrt{1-\alpha_t}\varepsilon_1$$

where $\varepsilon_i \in \mathcal{N}(0, I)$. Furthermore, we have that

$$q_{t-2|t-3}(x_{t-2}|x_{t-3}) = \sqrt{\alpha_{t-2}}x_{t-3} + \varepsilon_3\sqrt{1-\alpha_{t-2}}$$
$$\Rightarrow q_{t|t-3}(x_t|x_{t-3}) = \sqrt{\alpha_t\alpha_{t-1}\alpha_{t-2}}x_{t-3} + \sqrt{\alpha_t\alpha_{t-1}(1-\alpha_{t-2})}\varepsilon_3 + \sqrt{\alpha_t(1-\alpha_{t-1})}\varepsilon_2 + \sqrt{1-\alpha_t}\varepsilon_1$$

In this form a clear expression forms for $q_{t|0}$ from the xpatterns above (with the substitution that $\overline{a_t} = \prod_{s=1}^t \alpha_s$):

$$q_{t|0}(x_t|x_0) = \sqrt{\overline{\alpha_t}}x_0 + \sum_{i=1}^t \beta_i\varepsilon_i \text{ where } \beta_i = \sqrt{(1-\alpha_{t-i+1})\prod_{j=0}^{i-2}\alpha_{t-j}}$$

The sum is simply the sum of several standard normal distributions with variance β_i^2 which is equivalent to a single normal distribution with mean 0 and the variance equal to the sum of all the individual variances.

$$\sum_{i=1}^3 \beta_i^2 = \alpha_t\alpha_{t-1}(1-\alpha_{t-2}) + \alpha_t(1-\alpha_{t-1}) + (1-\alpha_t) = 1 - \alpha_t\alpha_{t-1}\alpha_{t-2}$$
$$\Rightarrow \sum_{i=1}^t \beta_i^2 = 1 - \prod_{s=1}^t \alpha_s = 1 - \overline{\alpha_t}$$

Hence, we can then write $q_{t|0}(x_t|x_0) = \sqrt{\overline{\alpha_t}}x_0 + \sqrt{1-\overline{\alpha_t}}\varepsilon$ where $\varepsilon \sim \mathcal{N}(0, (1-\overline{\alpha_t})I_d)$ which implies that $q_{t|0} \sim \mathcal{N}\left(\sqrt{\overline{\alpha_t}}x_0, (1-\overline{\alpha_t})I_d\right)$ by the reparameterization trick.

Question 2

We are given that Eq 1 is

$$\int_{x_1,\dots,x_T} \log\left[\frac{\prod_{s=1}^T p_{t-1|t}(x_{t-1}|x_t;\theta)p_T(x_T)}{\prod_{t=1}^T q_{t|t-1}(x_t|x_{t-1})}\right]\prod_{t=1}^T q_{t|t-1}(x_t|x_{t-1})dx_1\dots dx_T$$

Letting $z = (x_1, \dots, x_T)$ for a given x_0, we know that $p(z) = \prod_{t=1}^T q_{t|t-1}(x_t\ |x_{t-1})$. Hence the above expression can be viewed as an expected value function:

$$= \mathbb{E}_{z\sim(x_1,\dots x_T)}\left[\log\left[\frac{\prod_{s=1}^T p_{t-1|t}(x_{t-1}|x_t;\theta)p_{T(x_T)}}{}\right]\right]$$