Week 1

Vector

- For this course remember vector means rows or columns of numbers

Matrix

Types of Matrices

Row Matrix

$$\begin{pmatrix} a & b & c \end{pmatrix}$$

Column Matrix
Vector Matrix

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Zero Matrix
Null Matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonal Matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Scalar Matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

Unit Matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Upper Triangular Matrix

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

Lower Triangular Matrix

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

Properties of matrix addition and multiplication

1. $(A + B) + C = A + (B + C)$ (Associativity of addition)
2. $(AB)C = A(BC)$ (Associativity of multiplication)
3. $A + B = B + A$ (Commutativity of addition)
4. $AB \neq BA$ (In General)

Week 2 P1

$L2.1$ Determinants (Part3)

Important properties and Identities

Property 1 : Determinant of a product is product of the determinants

1. det(AB) = det(A)*det(B)
2. det(AB) = det(BA) here A and B are square matrices of size n
3. det($A^T A$) $= det(A)^2$ here ($det(A^t)$ = $det(A)$)

Property 2: Switching two rows or columns changes the sign of the determinant

$$det(A) \ = \ - \ det(A^{\sim})$$

Property 3: Adding multiple of a row to another row leaves the determinant unchanged (Same for columns)

Property 4: Scalar multiplication of a row by a constant t multiplied the determinant by t. (same for columns)

L2.2 Cramer's Rule

Crammer's Rule 3x3

$$\begin{aligned} 2x + 3y - 5z &= 1 \\ x + y - z &= 2 \\ 2y + z &= 8 \end{aligned} \qquad x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad z = \frac{D_z}{D}$$

$$D = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -7 \qquad D_y = \begin{vmatrix} 2 & 1 & -5 \\ 1 & 2 & -1 \\ 0 & 8 & 1 \end{vmatrix} = -21$$

$$D_x = \begin{vmatrix} 1 & 3 & -5 \\ 2 & 1 & -1 \\ 8 & 2 & 1 \end{vmatrix} = -7 \qquad D_z = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 8 \end{vmatrix} = 14$$

Week 2 P2

L2.3 Solution to a system of linear equations with an invertible coefficient matrix

- If $Ab = BA = I_{N\times N}$ and is denoted by A^{-1} , B is called inverse of A
- Inverse of a matrix exist iff $det(A) \neq 0$

Adjugate of a square matrix

- $adj(A) = C^T \, where \, C_{ij} = (-1)^{i+j}M_{ij}$ cofactor
- $A^{-1} = \frac{1}{det(A)}adj(A)$ A is nxn square matrix and det(A) is not 0

The solution of a system of linear equations with an invertible coefficient matrix

$x = A^{-1}b$ here A is invertible i.e det(A) is not 0

Solutions of a homogeneous system of linear equations

$Ax = 0$

- Unique solution which is 0 if $det(A) \neq 0$
- Infinite solution if $det(A) = 0$

Properties of Adjugate of a square matrix

- $adj(AB) = adj(B) \times adj(A)$
- $adj(A + B) = adj(A) + adj(B)$
- $adj(A^T) = adj(A)^T$
- $adj(A^{-1}) = adj(A)^{-1}$

The echelon form

A matrix is a **row echelon form** if

- The first non-zero element in each row, called the leading entry is 1

Week 2 P3

- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any are below rows having a non-zero element
- For a non-zero row, the leading entry in the row is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced row echelon form

Row Reduction

1. Interchange two rows e.g $R1 \leftrightarrow R2$
2. Scalar multiplication of a row by a constant e.g $R1/3$
3. Adding multiple of a row to another row e.g $R1 - 3R2$

Recall from determinants

$A \sim\sim\sim B$

1. $Ri \leftrightarrow Rj \Rightarrow det(A) = -det(B)$
2. $Ri * c \Rightarrow det(A) = c * det(B)$
3. $Ri + cRj \Rightarrow det(A) = det(B)$

L2.6 Gaussian Elimination method

1. Augmented matrix is denoted by $[A \mid B]$ $we\ denote\ the\ like\ this$

System of Equations

Associated Augmented Matrix

$$\begin{Bmatrix} x_1 + x_2 + x_3 = 3 \\ 2x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{Bmatrix} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

Week 3 P1

L3.1 Introduction to vector spaces

Properties of addition and scalar multiplication

Let $v, w\ and\ v`$ be vectors in $\Re$ and $a, b \in \Re$

i. $v + w = w + v$
ii. $(v + w) + v` = v + (w + v`)$
iii. The $0^{\rightarrow}$ vector satisfies that $v + 0 = 0 + v = v$
iv. The vector $-v$ satisfies that $v + (-v) = 0$
v. $1v = v$
vi. $(ab)v = a(bv)$
vii. $a(v + w) = av + aw$
viii. $(a + b)v = av + bv$

A **vector space** is a set with two operations (called addition and scalar multiplication
With the above properties (i) to (viii).

Definition of a vector space

A vector space V over $\Re$ is a set along with two functions

$+ : V \times V \rightarrow V$ and $\bullet : \Re \times V \rightarrow V$

(i.e for each pair of elements $v_1 and\ v_2\ in\ V$, there is a unique element $v_1 + v_2\ in\ V$ and for each $c \in \Re\ and\ v \in V$ there is a unique element $c.v$ in V)

It is standard to suppress the . and only write cv instead of c.v
The function + and • are required to satisfy the above mentioned rules.(i-viii)

L3.2 Some properties of vector spaces

Cancellation law of Vector addition

⇒If $v_1, v_2, v_3 \in V\ such\ that\ v_1 + v_3 = v_2 + v_3\ then\ v_1 = v_2$
⇒ The vector 0 described in (iii) is unique
⇒ The vector v` described in (iv) is unique and it is standard to refer to it as -v
In any Vector space V the following statements are true

- $0v = 0$ for each $v \in V$ and $c0 = 0$ for each $c \in \Re$

Week 3 P2

L3.3 Linear dependence

Linear Combination

Let V be a vector space and $v_1, v_2, v_3, \ldots v_n \in V$. The ***linear combination*** of $v_1, v_2, v_3, \ldots v_n$ with coefficients $a_1, a_2, a_3, \ldots a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$

A vector $v \in V$ is a ***linear combination*** of $v_1, v_2, v_3, \ldots v_n$ if there exist some $a_1, a_2, a_3, \ldots a_n \in \mathbb{R}$

So that v = $\sum_{i=1}^{n} a_i v_i$

Definition of Linear Dependence

A set of vectors $v_1, v_2, v_3, \ldots v_n$ from a vector space V is said to be ***linearly dependent***, if there exist scalars $a_1, a_2, a_3, \ldots a_n$ *not all zero* such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n = 0$$

Equivalently , the 0 vector is a linear combination of $v_1, v_2, v_3, \ldots v_n$ with non-zero coefficients.

- If a set is linearly dependent, then so is every superset of it.

L3.4 Linear Independence Part-1

Definition of linear independence

A set of vectors $v_1, v_2, v_3, \ldots v_n$ from a vector space V is said to be ***linearly independent*** if $v_1, v_2, v_3, \ldots v_n$ are not ***linearly dependent .***

Equivalently : A set of vectors $v_1, v_2, v_3, \ldots v_n$ from a vector space V is said to be ***linearly independent*** if the only linear combination of $v_1, v_2, v_3, \ldots v_n$ which equals 0 is the linear combination with all coefficients 0.

Week 3 P3

The **0** vector

Let $v_1, v_2, v_3, \ldots v_n$ be a set of vectors containing the **0** vector. Suppose $v_i = 0$. Then we can choose $a_i = 1$ and $a_j = 0$ for $j \neq i$

Then the linear combination of $a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n$ is 0 but not all coefficients are 0.

Hence, a set of vectors $v_1, v_2, v_3, \ldots v_n$ containing the **0** vectors is always a linearly dependent set.

- **Two non-zero vectors are *linearly independent* precisely when they are *not multiples of each other.***
- **If three vectors are *linearly independent* then *none of these vectors is a linear combination of the other two.***
- **To check $v_1, v_2, v_3, \ldots v_n \in \mathbb{R}^m$ are linearly independent we have to check that homogeneous system of linear equations $Vx = 0$ has only the trivial solution, where the j^{th} column of V is v_j.**
- **Any set of n vectors in $\mathbb{R}^2$ with $n \geq 3$ are linearly dependent.**
- **Any set of r vectors in $\mathbb{R}^n$ with $r \geq n$ are linearly dependent.**

- **If $det(A) \neq 0$ then vectors are linearly independent.**

Subspace

Def .

A subset W of a vector space V over $\mathbb{R}$ is called subspace of V if W is also a Vector space over $\mathbb{R}$ with the same operations , defined over V.

Suppose W is a given subset of V

Test for becoming a subspace

i. $0 \in W$
ii. $x \in W, y \in W \Rightarrow x + y \in W$ (closed under addition)
iii. $c \in \mathbb{R}, x \in W \Rightarrow cx \in W$ (closed under scalar multiplication)

Then we say W is subspace of V

Week 4 P1

L4.1 What is a basis for a vector space

Span of a set of Vectors

The span of a set S (of Vectors) is defined as the set of all finite linear combinations of elements (vectors) of S, and denoted by Span(S)

i.e $$Span(S) = \left\{ \sum_{i=1}^{n} a_i v_i \in V \mid a_1, a_2, ... a_n \in \mathbb{R} \right\}$$

Example:

Let $S = \{(1, 0)\} \in \mathbb{R}^2$ Then

$Span(S) = \{a(1, 0) \mid a \in \mathbb{R} \} = \{(a, 0) \mid a \in \mathbb{R} \}$

Thus $Span(S)$ is the X-axis in $\mathbb{R}^2$

Spanning set for a vector space

Let V be a vector space. A set $S \subseteq V$ is a spanning set for V if $Span(S) = V$

Example:

If $S = \{(1, 0), (0, 1)\}$ then $Span(S) = \mathbb{R}^2$

If $S = \{(1, 0), (0, 1), (1, 2)\}$ then $Span(S) = \mathbb{R}^2$

If $S = \{(1, 1), (0, 1)\}$ then $Span(S) = \mathbb{R}^2$

If $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ then $Span(S) = \mathbb{R}^3$

What is the basis ?

A basis B of a vector space V is a linearly independent subset of V that spans V.
Example:

Let $e_i \in \mathbb{R}^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0 e.g.

$e_1 = (1, 0, 0,...., 0)$

The set $\epsilon = \{e_1, e_2, e_3,... e_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$

Week 4 P2

L4.2: Finding bases for vector spaces

Equivalent conditions for B to be a basis

The following conditions are equivalent to a subset $B \subseteq V$ being a basis:
i. B is linearly independent and Span(B) = V
ii. B is a maximal linearly independent set
iii. B is a minimal spanning set.

How do we find a basis ?

i) Start with the ϕ and keep appending vectors which are not in the span of the set thus far obtained , until we obtain a spanning set.

ii) Take a spanning set and keep deleting vectors which are linear combinations of the other vectors, until the remaining vectors satisfy that they are not a linear combination of the other remaining ones.

L4.3: What is the rank / dimension for a vector space.

The dimension (or rank) of a vector space is the **size (or cardinality) of a basis of the vector space.**

For this course: If B is a basis of V, then the rank is the number of elements in B.

For every vector space there exists a basis, and all bases of a vector space have the same number of elements (or cardinality); hence, the dimension (or rank) of a vector space (say V) is uniquely defined and denoted by $dim(V)$ (or $rank(V)$) respectively.

Dimension of $\mathbb{R}^n$

Recall the i-th standard basis vector is $\mathbb{R}^n$.
$e_i = (0, 0, 0..., 0, 1, 0... 0)$
i.e. the i-th coordinate is 1 and 0 elsewhere
Recall that the set $\{e_1, e_2, e_3,..., e_n\}$ is a basis of $\mathbb{R}^n$ called the standard basis.
Hence the dimension of $\mathbb{R}^n$ is n.

Week 4 P3

L4.3:

Rank of a matrix

Let A be an $m \times n$ matrix.

⇒ The **column space** of a A is the subspace of $\mathbb{R}^m$ spanned by the column vector of A.
⇒ The **row space** of a A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
⇒ The dimension of the column space of A is defined as the **column rank** of A.
⇒ The dimension of row space of A is defined as the **row rank** of A.

Fact: **Column rank = row rank** and this number is called the rank of A.

L4.4: Rank and dimension using Gaussian elimination

Finding dimension and basis with a given spanning set.

e.g. Let us consider the vector space W spanned by the set S = {(1,0,1),(-2,-3,1),(3,3,0)}.

We will use the following steps to find the dimension and a basis for W and carry out the steps for our example.

Reduce to a matrix in row echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \sim\sim\sim\sim> \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

⇒ The number of non-zero rows is the dimension of the vector
⇒ The vectors corresponding to the non-zero rows form the basis of the vector space W.
In the example , the final matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence dimension of the vector space spanned by {(1,0,1),(-2,-3,1),(3,3,0)} is 2
And a basis is given by (1,0,1),(0,1,-1)

Week 4 P4

L4.4:

An alternative to the row-based method.

Column method

Example
Let us consider the vector space W spanned by the set S = {(1,0,1),(-2,-3,1),(3,3,0)}.
We will use the fact (see notes or slides) to find a basis for W which is a subset of S.
Form the matrix with the vectors in S as the columns.

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

Row reduce this matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form and columns with pivot entries (leading 1s) are the first and second columns.
Therefore (1,0,1),(-2,-3,1) which are the first and second vectors in S respectively, form a basis for W.