An Elementary Construction of Constant-Degree Expanders

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Definition

A d-regular graph $G(V,E)$ is δ-edge expander if $\forall S \subseteq V$ of atmost size $\frac{|V|}{2}$,

$$e(S,\overline{S}) \geq \delta d|S|$$

Definition

We will say that a graph is $[n, d, \delta]$-expander if it is an n-vertex d-regular δ-expander.

Motivation

Why do we need constant degree expanders?

Motivation

If we have constant -sized object then we can freely use it without having to find a nice description for it. Also we can always find constant size in constant time by brute force approach.

Main Result

There exists a fixed $\delta > 0$ such that for any integer $q = 2^t$ and for any $\frac{q^4}{40} \leq r \leq \frac{q^4}{2}$, there is a polynomial time construtible $[q^{4r+12}, 12, \delta] - \textit{expander}$.

Construction

Theorem[Replacement Product]

If E_1 is an $[n, D, \delta_1]$-expander and E_2 is an $[D, d, \delta_2]$-expander then $E_1 \circ E_2$ is $[nD, 2d, \frac{1}{80}\delta_1^2\delta_2]$-expander

Construction

1. Take graph E_1 as $[q^2, 3, \delta]$-expander (By enumerating all possible graphs)
2. Take graph E_2 as $[q^6, q^2, \frac{1}{4}]$-expander
3. Obtain $E' = E_2 \circ E_1$ which is $[q^8, 6, \delta']$-expander
4. Take E_3 as $[q^{4r+4}, q^8, \frac{1}{4}]$-expander
5. Finally obtain $E_4 = E_3 \circ E'$ which will be $[q^{4r+12}, 12, \delta']$-expander

Replacement Product

Definition

Let G be a D-regular D-edge colorable graph on n-vertices and H be a d-regular graph on D-vertices. Replace every vertex v_i of G with a cluster of D vertices , which we denote $C_i = \{v_1^i, v_2^i, \dots v_D^i\}$. For $1 \leq i \leq n$, we will put an edge between (v_p^i, v_q^i) , iff $(p, q) \in E(H)$. For every $(p, q) \in E(G)$, which is colored t , we will put d parallel edges between (v_t^p, v_t^q)

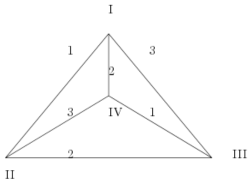
Replacement Product

Claim : The resultant graph $G \circ H$ is $2d$-regular on nD-vertices

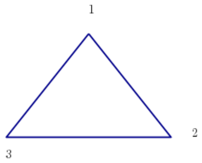
Replacement Product

Example

$G[4, 3, \delta_1]$

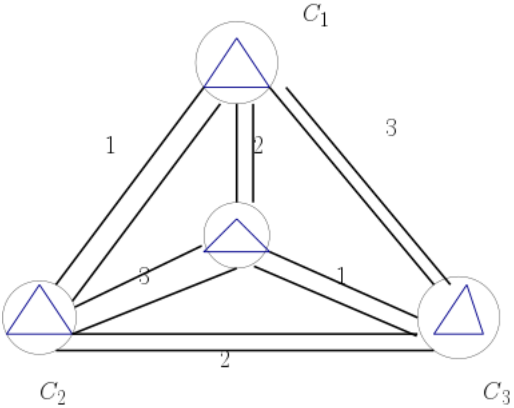


$H[3, 2, \delta_2]$



Replacement Product

$G \circ H$



Replacement Product

Theorem

If E_1 is an $[n, D, \delta_1]$-expander and E_2 is an $[D, d, \delta_2]$-expander then $E_1 \circ E_2$ is $[nD, 2d, \frac{1}{80}\delta_1^2\delta_2]$-expander

Replacement Product

Proof Sketch : Take a set of vertices $X \subseteq V$ of atmost size $\frac{nD}{2}$ and show that $e(X, \overline{X})$ has either many edges within the same cluster C_i or between them.

Replacement Product

Proof : Now we can see all the vertices as n-clusters of vertices $C_1, C_2 \dots C_n$ each of size D.

$$\text{Goal: } e(X, \overline{X}) \geq \tfrac{1}{80}\delta_1^2\delta_2.2d.|X|$$

Replacement Product

Definition Let $X \subseteq V$ of size atmost $\frac{nD}{2}$

$$X_i = C_i \cap X$$

$$I' = \{i \mid |X_i| \leq (1 - \tfrac{1}{4}\delta_1)D\}$$

$$I'' = \{i \mid |X_i| > (1 - \tfrac{1}{4}\delta_1)D\}$$

$$X' = \textstyle\bigcup_{i \in I'} X_i$$

$$X'' = \textstyle\bigcup_{i \in I''} X_i$$

Replacement Product

- Consider an X_i , $i \in I'$
- Since E_2 is an δ_2 expander , we have

$$e(X, C_i \setminus X_i) \geq \frac{1}{4}\delta_1\delta_2 d|X_i|$$

- $e(X', \overline{X}) \geq \frac{1}{4}\delta_1\delta_2 d|X'|$

Observation

If $|X'| \geq \frac{1}{10}\delta_1|X|$ then $e(X, \overline{X}) \geq \frac{1}{80}\delta_1^2\delta_2.2d.|X|$ and we are done.

Suppose

$$|X'| < \frac{1}{10}\delta_1 X$$

then we will show that there are enough number of edges between the clusters.

Proof Contd..

- We have $|X''| \geq (1 - \frac{1}{10}\delta_1)|X|$
- $|X''| \leq |X| \leq \frac{1}{2}nD$
- $|I''| \leq \frac{2}{3}n$
- There is a set of edges $|M| \geq \frac{1}{2}\delta_1 D|I''|$ connecting vertices from I'' to I'

Proof Contd....

- The corresponding $d|M| \geq \frac{1}{2}\delta_1 dD|I''|$ edges connect vertices from $\bigcup_{i \in I''} C_i$ with vertices of $\bigcup_{i \in I'} C_i$
- At most $\frac{1}{4}\delta_1 dD|I''|$ of $d|M|$ many edges connect $C_i \setminus X_i$ with vertices of $\bigcup_{i \in I'} C_i$. Why??
- $Size(|C_i \setminus X_i|) \geq \frac{1}{4}\delta_1 D$ $(times\ d.|I''|)$-many edges
- Hence at least $\frac{1}{4}\delta_1 dD|I''|$ of $d|M|$ edges connect $\bigcup_{i \in I''} X_i$ to $\bigcup_{i \in I'} C_i$

Proof Contd...

- No of $d|M|$ edges that connects vertices from $\bigcup_{i\in I''} C_i$ with vertices of X' is clearly at most $d|X'|$
- Since $|X'| \leq \frac{1}{10}\delta_1|X| \leq \frac{1}{6}\delta_1 D|I''|$, there are $\frac{1}{6}\delta_1 dD|I''|$ edges from $\bigcup_{i\in I''} C_i$ to X'
- $\frac{1}{4}\delta_1 dD|I''| - \frac{1}{6}\delta_1 dD|I''| = \frac{1}{12}\delta_1 dD|I''|$ edges connect vertices of $\bigcup_{i\in I''} X_i$ with vertices of $\bigcup_{i\in I'} C_i \setminus X_i$.

Proof Contd..

- AS $|I''| \geq \frac{|X''|}{D}$ and $|X''| \geq \frac{1}{2}|X|$, we have atleast $\frac{1}{48}\delta_1 2d|X|$ edges as required.

Conclusion

We have constructed a constant degree expanders by applying replacement product twice.

THANK YOU ALL

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