Linear-Time Algorithms for Scattering Number and Hamilton-Connectivity of Interval Graphs

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Vulnerability of Graphs

- Study of a graph when some of its elements are removed
- Studies resistance of network to disruption of operations
- Well known measures of vulnerability
  - Usual : Connectivity, Domination Number, Domatic Number, Independence Number
  - Recents: Toughness, Binding Number, Scattering Number, Integrity, Tenacity, Rupture Degree

Toughness and Scattering Number

Toughness (Chvátal '73) : $\tau(G) = min\left\{ \frac{|S|}{c(G-S)} : S \subset V \ and \ c(G-S) \geq 2 \right\}$

- c(G) - number of connected components
- G is hamiltonian $\Rightarrow \tau(G) \geq 1$

Scattering Number (Jung '78) : $sc(G) = max\left\{ c(G-S) - |S| : S \subset V \ and \ c(G-S) \geq 2 \right\}$

- *Scattering Set* : S on which sc(G) is attained
- G is hamiltonian $\Rightarrow sc(G) \leq 0$

Connectivity Properties and Decision Problems

Graph G = (V, E)

- *Hamiltonian* - If G has a Hamilton cycle i.e. cycle containing all vertices of G
- *Traceable* - If G contains a Hamilton path i.e. a path containing all vertices of G
- *{1, 2}-Hamilton* - A Hamilton path starting at a vertex u or that is between two given vertices u and v, respectively.
- *Hamilton Connected* - For each two distinct vertices s and t, there is a Hamilton path with end vertices s and t.
- *k-Hamilton-Connected* - If G - S is **Hamilton-connected** for every set $S \subset V$ with |S| at most k.
- *Path Cover* - Set of mutually vertex-disjoint paths $P_1, P_2, \ldots, P_k$ with $\bigcup P_i = V(G)$

Scattering Number

- G is a complete graph -> sc(G) = $-\infty$
- G is Hamiltonian -> sc(G) <= 0
- G is Traceable -> sc(G) <= 1 (Shih et.al. $sc(G) \leq \pi(G)$)
- $sc(G) = 0 \Leftrightarrow \tau(G) = 1$

Scattering Number - Complexity

- $sc(G) = 0 \Leftrightarrow \tau(G) = 1$
- Bauer et.al - Deciding $\tau(G) = 1$ is coNP-complete.
- Thus, deciding $sc(G) = 0$ is coNP-complete.
- Zhang *et. al.* - Computing the scattering set is NP-complete. Is there a S subset of V such that (c(G-S) > |S| + s)?

Motivation

- Chen etal. - Proper INT graph is Hamiltonian ***iff*** it is 2-connected and Hamilton-connected ***iff*** it is 3-connected

- Proper INT graph has scattering number at most 2 - k ***iff*** it is k-connected. [Chen and Chang]

- Testing **Hamilton Cycles** and **Hamilton paths** - O(n+m) algorithm of Keil and can be done in O(nlog n) if an interval representation is given.

- If a sorted interval representation is given then it can be solved in O(n) [Chang, Peng and Liaw]. This holds for **Path Cover** too.

Motivation

When no Hamilton path exists

- Longest Path and Path Cover are natural problems to consider.
- $O(n^4)$ for Longest Path - [Ioannidou, Mertzios and Nikolopoulos]
- $O(n+m)$ for Path Cover - [Arikati and Pandu Rangan, Damaschke]
- 1-Path Cover and hence 1-Hamilton path - $O(n^3)$ [Asdre and Nikolopoulos]. Improved to $O(n+m)$ by Li and Wu.
- For all k >=1 , any cocomparibility graph has a path cover of size at most k ***iff*** sc(G) at most k. Hence, holds for INT graph too. [Deogun, Kratsch and Steiner]
- Cocomparability graph is hamiltonian ***iff*** sc(G) is at most 0 (equivalently , toughness at least 1).
- $O(n+m)$ algorithm finding a scattering set of INT if sc(G) >= 0.

Motivation

When Hamilton path does exist

- Hamilton Connectivity is a natural problem to consider.
- Issak used a closed variant of touchness called k-path toughness to characterize INT graph that contain the k^{th} power of a Hamiltonian path.
- Deogun et.al. - Characterizing k-Hamilton-connectivity in terms of <u>scattering number</u> of an Interval Graph may be more appropriate than doing this in terms of its <u>toughness</u>.
- Current paper confirms this by showing - an INT graph is k-Hamilton-connected ***iff*** its scattering number is at most -(k+1) for all k>=0.

This Paper

Theorem 1. *Let G be an interval graph. Then* $\mathrm{sc}(G) \leq k$ *if and only if*

(i) G has a path cover of size at most k when $k \geq 1$
(ii) G has a Hamilton cycle when $k = 0$
(iii) G is $-(k+1)$-Hamilton-connected when $k \leq -1$.

- Hamilton Connectivity in O(n+m)
- Scattering Number for INT - O(n+m)
- Proper INT graph is k-Hamilton-connected ***iff*** it is (k+3)-connected

Input: A clique-path $C_1, \ldots, C_s$ in an interval graph G.
Output: An optimal spanning stave $\mathcal{P}$ between u_1 and u_n, if it exists.

begin
  let $p = \deg(u_1)$;
  let $R_i = u_1$ for all $i = 1, \ldots, p$;
  let $\mathcal{P} = \{R_1, \ldots, R_p\}$;
  let $\mathcal{Q} = \emptyset$;
  for $t := 1$ *to* $s - 1$ **do**
    choose a $P \in \mathcal{P}$ whose terminal has the smallest end point among all terminals;
    if $C_t \setminus (C_{t+1} \cup \bigcup(\mathcal{P} \cup \mathcal{Q})) \neq \emptyset$ **then** extend P by attaching vertices of $C_t \setminus (C_{t+1} \cup \bigcup(\mathcal{P} \cup \mathcal{Q}))$ in an arbitrary order
    for *every path* $R \in \mathcal{P}$ **do**
      if *the terminal of* R *is not in* C_{t+1} **then**
        try to extend R by a new vertex u from $(C_t \cap C_{t+1}) \setminus \bigcup(\mathcal{P} \cup \mathcal{Q})$ with the smallest end point;
        if *such* u *does not exist* **then**
          remove R from $\mathcal{P}$;
          insert R into $\mathcal{Q}$;
          decrement p;
          if $p = 0$ **then** report that G has no spanning 1-stave between u_1 and u_n and **quit**
        end
      end
    end
  end
  choose any $P \in \mathcal{P}$;
  extend P by attaching vertices of $C_s \setminus \bigcup(\mathcal{P} \cup \mathcal{Q})$ in an arbitrary order;
  let $P = \mathrm{merge}(P, \mathcal{Q})$;
  for *every path* $R \in \mathcal{P} \setminus P$ **do** extend R by u_n;
  report the optimal spanning p-stave $\mathcal{P}$.
end

Algorithm 1. Finding an optimal spanning stave

Algorithm

1. Gradually builds up a set P of internally disjoint monotone paths

2. Each path starts at u_1 and pass through vertices of $C_t \backslash C_{t+1}$ and then moves to C_t intersect C_{t+1} for t = 1,2,..,s-1

3. If some path cannot be extended, continue without it.

4. At the end merge these early finished path with any path that stayed till the end.

5. The number of the paths of the resulting spanning stave gives the scattering number : sc(G) = 2 - k

Theorem: spanning p-stave

Theorem 2. *An interval non-complete graph G contains a spanning p-stave between u_1 and u_n if and only if* $\mathrm{sc}(G) \leq 2 - p$.

Lemma 2. *Suppose that Algorithm 1 terminates at line 16 or finishes an iteration of the loop at lines 6–20. Let the current value of the variable t be also denoted by t. If there is at least one depleted vertex during the interval $(t, t+1)$, then there exists an integer $t' < t$ with the following properties:*

(i) $C_{t'+1,t} \setminus (C_{t'} \cup C_{t+1}) \neq \emptyset$,

(ii) *a unique vertex $u_i \in C_{t'} \cap C_{t+1}$ is active during $(t', t'+1)$ and is depleted during $(t, t+1)$,*

(iii) *all vertices that are active during $(t, t+1)$ are also active during $(t', t'+1)$, with the only possible exception of the last descendant of u_i (which we denote by v) that can be free during $(t', t'+1)$,*

(iv) *all vertices that are depleted during $(t, t+1)$ and distinct from u_i are also depleted during $(t', t'+1)$,*

(v) *all vertices that are active during $(t', t'+1)$ are also active during $(t, t+1)$, with the only exception of u_i, and*

(vi) *all vertices that are free during $(t', t'+1)$ are also free during $(t, t+1)$, with the only possible exception of v if it is active during $(t, t+1)$.*

Algorithm

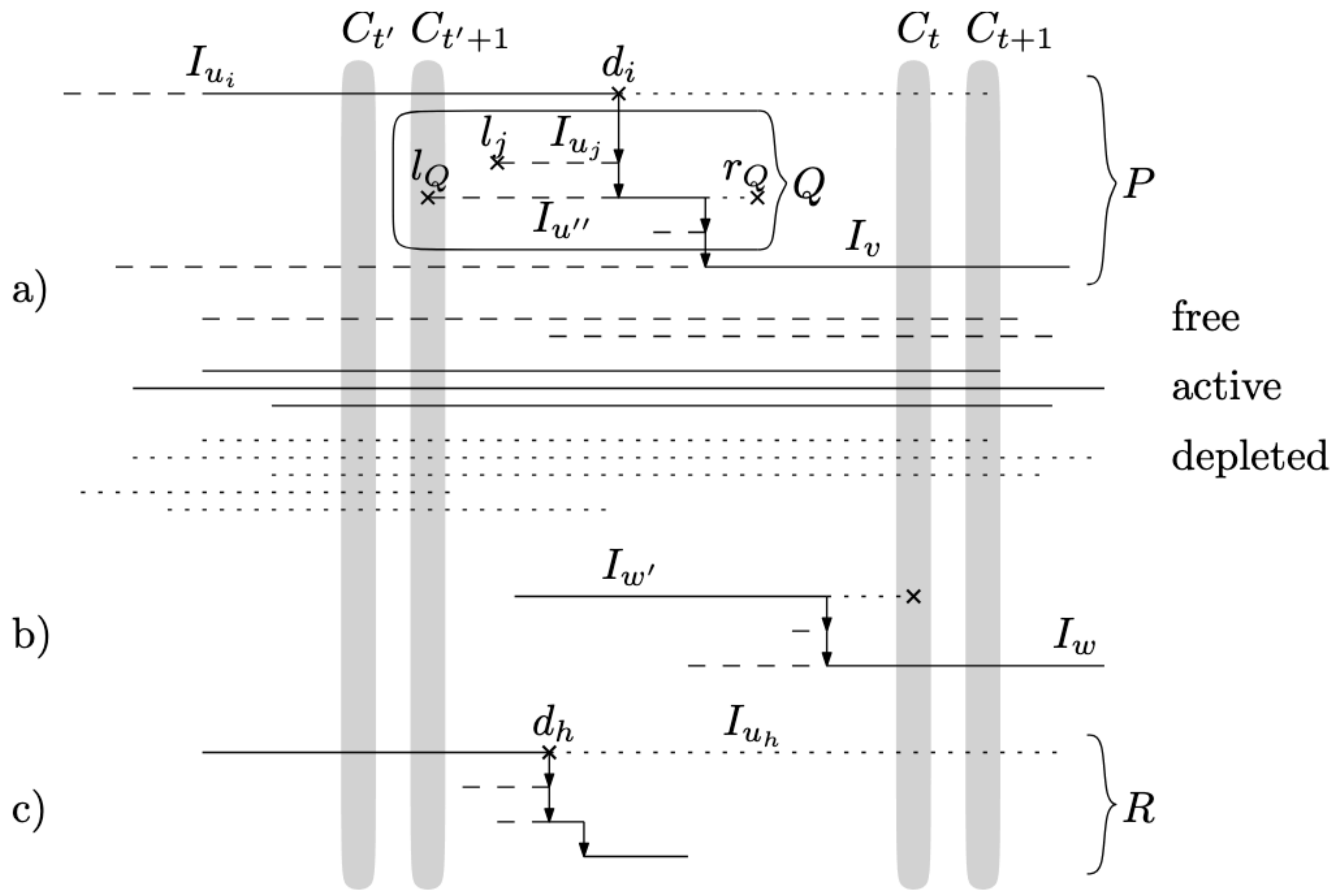

Fig. 1. A path system as described in Lemma 2. The vertical arrows indicate successors in the paths and the time of activation and deactivation.

Open Directions

- 2-Hamilton Path
- Can Toughness be solved in linear time?
- Hamilton Connectivity on circular-arc graphs?
- Scattering number known to be in $O(n^4)$ for circular-arc graphs. Can it be done in linear time?

Where does this algorithm fail?

- Method fails for any graph class that contains all complete birpartite graphs $K_{n,n}$
- For example, cocomparability graph , permutation graphs and convex graphs
- Not clear which two vertices to be chosen as “leftmost” and “rightmost”.

Recent Results

- Linear time algorithm for scattering number and scattering set for strictly Chordal graphs [Markenzon and Waga 2022]

- Polynomial time algorithm for weighted scattering number of INT graphs [Zhang and Broersma 2019]

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