

TERM EXTRACTION AND RAMSEY'S THEOREM FOR PAIRS

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ABSTRACT. In this paper we study with proof-theoretic methods the function(al)s provable recursive relative to Ramsey's theorem for pairs and the cohesive principle (COH).

Our main result on COH is that the type 2 functionals provable recursive from $\text{RCA}_0 + \text{COH} + \Pi_1^0\text{-CP}$ are primitive recursive. This also provides a uniform method to extract bounds from proofs that use these principles. As a consequence we obtain a new proof of the fact that $\text{WKL}_0 + \Pi_1^0\text{-CP} + \text{COH}$ is Π_2^0 -conservative over PRA.

Recent work of the first author showed that $\Pi_1^0\text{-CP} + \text{COH}$ is equivalent to a weak variant of the Bolzano-Weierstraß principle. This makes it possible to use our results to analyze not only combinatorial but also analytical proofs.

For Ramsey's theorem for pairs and two colors (RT_2^2) we obtain the upper bounded that the type 2 functionals provable recursive relative to $\text{RCA}_0 + \Sigma_2^0\text{-IA} + \text{RT}_2^2$ are in T_1 . This is the fragment of Gödel's system T containing only type 1 recursion — roughly speaking it consists of functions of Ackermann type. With this we also obtain a uniform method for the extraction of T_1 -bounds from proofs that use RT_2^2 . Moreover, this yields a new proof of the fact that $\text{WKL}_0 + \Sigma_2^0\text{-IA} + \text{RT}_2^2$ is Π_3^0 -conservative over $\text{RCA}_0 + \Sigma_2^0\text{-IA}$.

The results are obtained in two steps: in the first step a term including Skolem functions for the above principles is extracted from a given proof. This is done using Gödel's functional interpretation. After this the term is normalized, such that only specific instances of the Skolem functions are used. In the second step this term is interpreted using Π_1^0 -comprehension. The comprehension is then eliminated in favor of induction using either elimination of monotone Skolem functions (for COH) or Howard's ordinal analysis of bar recursion (for RT_2^2).

1. INTRODUCTION

The aim of this paper is to develop a technique of program extraction for proofs that use Ramsey's theorem for pairs, the cohesive principle and other principle weaker than Ramsey's theorem for pairs. As a consequence it also gives a proof theoretic account of conservation results for those principles. This paper extends our previous treatment of Ramsey's theorem for pairs in [34], where only single instances of Ramsey's theorem are discussed, to the full second order closure of those principles.

Ramsey's theorem for pairs (RT_n^2) is the statement that every coloring of pairs of natural numbers ($[\mathbb{N}]^2$) with n colors has an infinite homogeneous set. A simple colorblindness argument shows that

$$\text{RT}_2^2 \leftrightarrow \text{RT}_n^2 \quad \text{for every fixed } n.$$

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Ramsey's theorem for pairs and arbitrary large colorings ($\text{RT}_{<\infty}^2$) is defined as $\forall n \text{RT}_n^2$. This principle is proof-theoretically stronger than RT_2^2 , whereas from the viewpoint of computation there is no difference in strength.

A coloring c of pairs is called *stable* if $c(\{x, \cdot\})$ eventually becomes constant for every x . The restriction of RT_n^2 to stable colorings is denoted by SRT_n^2 . Here a similar colorblindness argument can be applied.

A set G is called *cohesive* for a sequence $(R_i)_{i \in \mathbb{N}}$ of subsets of \mathbb{N} if

$$\forall i \left(G \subseteq^* R_i \wedge G \subseteq^* \overline{R_i} \right),$$

where $X \subseteq^* Y \equiv (X \setminus Y \text{ is finite})$. The *cohesive principle* (COH) states that for every $(R_i)_{i \in \mathbb{N}}$ a cohesive set exists. It is in some way the counterpart to SRT_n^2 since

$$\text{RCA}_0 \vdash \text{RT}_n^2 \leftrightarrow \text{SRT}_n^2 \wedge \text{COH}$$

for $2 \leq n \leq \infty$, see [7, 8].

The computational strength of Ramsey's theorem has been investigated since the early 70's. Specker showed 1971 that there exists a computable coloring of $[\mathbb{N}]^2$ that has no computable homogeneous set, see [46]. Jockusch improved this 1972 by showing that in general there is not even a Σ_2^0 infinite homogeneous set. He also provided an upper bound on the strength of Ramsey's theorem for pairs and showed that each computable coloring of pairs admits an infinite homogeneous set H with $H' \leq_T 0''$, see [21]. Seetapun and Slaman showed in [42] that RT_2^2 does not solve the halting problem. Cholak, Jockusch and Slaman improved both results by showing that an infinite homogeneous low_2 set exists for every computable coloring of pairs, i.e. a set H satisfying $H'' \leq_T 0''$, see [7].

From Specker's results it is clear that $\text{RCA}_0 \not\vdash \text{RT}_2^2$. Seetapun's and Slaman's results immediately yield an upper bound on the proof-theoretic strength, it implies that RT_2^2 does not prove Π_1^0 -comprehension or ACA_0 . Hirst showed 1987 that RT_2^2 implies the infinite pigeonhole principle ($\text{RT}_{<\infty}^1$) which is equivalent to the Π_1^0 -bounded collection principle ($\Pi_1^0\text{-CP}$)¹, see [17]. Cholak, Jockusch and Slaman showed along their recursion theoretic proof that RT_2^2 is Π_1^1 -conservative over $\text{RCA}_0 + \Sigma_2^0\text{-IA}$.

This leaves the question whether RT_2^2 implies $\Sigma_2^0\text{-IA}$. Despite of many efforts in the last years this question could not be settled yet.

Ramsey's theorem for triples and bigger tuples is equivalent to ACA_0 and hence fully classified in the sense of reverse mathematics, see [44].

The cohesive principle has been originally considered in recursion theory, see for instance [45]. Its computational strength has been fully determined in [19]. Cholak, Jockusch and Slaman observed in [7] that Ramsey's theorem for pairs splits nicely into a stable part and the cohesive principle. They also showed that it is Π_1^1 -conservative over RCA_0 and $\text{RCA}_0 + \Sigma_2^0\text{-IA}$. In the course of the classification of Ramsey's theorem the cohesive principle's logical strength received attention in the last years, see for instance [10] and [9]. In [9] it was shown that the cohesive principle is Π_1^1 -conservative over $\text{RCA}_0 + \Pi_1^0\text{-CP}$. We recently showed that over $\text{RCA}_0 + \Pi_1^0\text{-CP}$ the cohesive principle is equivalent to a weak form of the Bolzano-Weierstraß principle, see [36]. Thus the cohesive principle also shows up in analytic proofs.

For an extensive survey on the current status of Ramsey's theorem for pairs and weaker principles, see [15] and [43].

The purpose of this paper is to give an account to the above mentioned conservation results from the perspective of proof mining and program extraction. We provide new proofs for these conservation results which additionally yield realizing

¹In first order context this principle is usually denoted by $B\Pi_1^0$ which is equivalent to $B\Sigma_2^0$.

terms. Since the types of these terms rise with the complexity of the formula this is naturally bounded to Π_3^0 -sentences.

Proofwise low. Define Π_1^0 -comprehension as

$$(\Pi_1^0\text{-CA}): \forall X \exists Y \forall u (u \in Y \leftrightarrow \forall v \langle u, v \rangle \in X).$$

This covers the full strength of Π_1^0 -comprehension since $\forall v \langle u, v \rangle \in X$ is a universal Π_1^0 -statement (relative to the parameter u). Full arithmetical comprehension (ACA_0) follows by iteration. For a primitive recursive term t we will write $\Pi_1^0\text{-CA}(t)$ if X is instantiated with the set $\{n \mid t(n) = 0\}$.² For a closed term t the principle $\Pi_1^0\text{-CA}(t)$ is also called an *instance of Π_1^0 -comprehension*.

The union of $\Pi_1^0\text{-CA}(t)$ for all terms t containing only number variables free is the same as light-face Π_1^0 -comprehension. In particular, this does not prove ACA_0 .

Let \mathcal{P} be a second order principle stating the existence of a set G relative to a set parameter S — that is a principle of the form

$$(\mathcal{P}): \forall S \exists G P(S, G).$$

Definition 1 (proofwise low). Call a principle of the form \mathcal{P} *proofwise low* over a system \mathcal{T} if for every provably continuous³ term φ a provably continuous term ξ exists such that

$$(1) \quad \mathcal{T} \vdash \forall S (\Pi_1^0\text{-CA}(\xi S) \rightarrow \exists G (P(S, G) \wedge \Pi_1^0\text{-CA}(\varphi SG))).$$

If we additionally can prove this for a sequence of solutions, i.e.

$$(2) \quad \mathcal{T} \vdash \forall (S_i)_{i \in \mathbb{N}} (\Pi_1^0\text{-CA}(\xi(S_i)_i) \rightarrow \exists (G_i)_{i \in \mathbb{N}} (\forall i P(S_i, G_i) \wedge \Pi_1^0\text{-CA}(\varphi(S_i)_i(G_i)_i)))$$

then we call \mathcal{P} *proofwise low in sequence* over the system \mathcal{T} .

The notion of proofwise low is comparable to low_2 in the recursion theoretic setting: take for instance $\mathcal{T} = \text{WKL}_0$, then a proofwise low statement in \mathcal{T} satisfies

$$\text{RCA}_0 \vdash \forall S (\text{WKL} \wedge \Pi_1^0\text{-CA}(\xi S) \rightarrow \exists G (P(S, G) \wedge \Pi_1^0\text{-CA}(\varphi SG))).$$

The analogous recursion theoretic statement would be that relative to an oracle of Turing degree $d \gg 0'$ — this resembles the premise — a set G satisfying the statement $P(S, G)$ and its Turing jump G' can be computed. From this follows that $G'' \equiv_T 0''$ or in other word that G is low_2 .

The main results of this paper are divided into two parts:

- (i) We show roughly that
 - RT_2^2 is proofwise low over WKL_0 and that
 - COH is proofwise low in sequence over WKL_0^* .⁴
- (ii) We show for principles \mathcal{P} where $P(S, G)$ is Π_3^0 that
 - if \mathcal{P} is proofwise low over WKL_0 , the system $\text{WKL}_0 + \Sigma_2^0\text{-IA} + \mathcal{P}$ is Π_3^0 -conservative over Σ_2^0 -induction. (This covers RT_2^2 .)
 - if \mathcal{P} is proofwise low in sequence over WKL_0^* the system $\text{WKL}_0 + \Pi_1^0\text{-CP} + \mathcal{P}$ is Π_3^0 -conservative over RCA_0 . (This covers COH .)

²Strictly speaking RCA_0 does not contain terms. Here and in the following we silently assume that we work in the conservative extension of RCA_0 by all primitive recursive functions.

³Continuous means her continuous in the sense of Baire space, i.e. φ is continuous if

$$\forall f \exists n \forall g (\forall x < n f(x) = g(x) \rightarrow \varphi(f) = \varphi(g)).$$

Such functionals can be coded into primitive recursive functions. For details see definitions 8 and 9 below.

⁴The system RCA_0^* is defined to be RCA_0 where Σ_1^0 -induction is replaced by quantifier-free induction plus the exponential function. The system WKL_0^* is RCA_0^* plus weak König's lemma. See [44, X.4.1].

This simplifies the results slightly. The actual results require a suitable finite type extension of WKL_0 and WKL_0^* , see below.

The first part of this paper is based on the proofs by “first jump control” for SRT_2^2 and COH of Cholak, Jockusch and Slaman, see [7], showing that these principles have low_2 solutions. To our knowledge these proofs have not been used before to obtain conservativity results for RT_2^2 . Cholak, Jockusch and Slaman developed in this paper a different, more complicated proof needing Π_2^0 -comprehension that can be used in a forcing construction to show conservativity of RT_2^2 over Σ_2^0 -induction.

For the second part we use Gödel’s functional interpretation (always combined with a negative translation) to extract a term t from a proof of an arbitrary statement of the following form

$$\mathcal{P} \rightarrow \forall x \exists y A(x, y),$$

where A is quantifier-free and \mathcal{P} is a proofwise low principle. For an oracle solution \mathcal{P} of the functional interpretation of \mathcal{P} this term will then satisfy

$$\forall x A(x, t(\mathcal{P}, x)).$$

We normalize t so that every application of \mathcal{P} in the proof is of a specific form and one can read off from the term and the proof how much of \mathcal{P} is used. The functional \mathcal{P} is then eliminated from t by interpreting every specific application of \mathcal{P} . This is done either by (2) or the functional interpretation of (1) in a way that retains the instance of comprehension. If this retained instance of comprehension is used for the next interpretation of \mathcal{P} then an inductive treatment of every application of \mathcal{P} yields that

- (i) in the first case one instance of the functional interpretation of Π_1^0 -CA suffices to prove totality of t and hence $\forall x \exists y A(x, y)$,
- (ii) in the second case one instance of Π_1^0 -CA proves the totality of t and hence $\forall x \exists y A(x, y)$.

The instance of comprehension is then eliminated in favor of induction:

In the case (i) the solution to this functional interpreted instance of comprehension is provided by an instance of Spector’s bar recursion (in fact by an application of the rule of bar recursion). This usage of bar recursion is then eliminated using Howard’s ordinal analysis of bar recursion in favor of Σ_2^0 -induction (section 10).

In the case (ii) the instance of comprehension is eliminated through elimination of Skolem functions for monotone formulas, see [27], yielding that $\forall x \exists y A(x, y)$ is provable in primitive recursive arithmetic. For this it is crucial that \mathcal{P} is proofwise low over a system that does *not* contain Σ_1^0 -induction, for instance WKL_0^* .

These techniques of elimination of instances of comprehension can be viewed as a proof-theoretic refinement of the arithmetical conservativity of ACA_0 over PA , see [4], [11], [49] and [44, IX.1.6].

Comparison to conservation results by syntactic forcing. Syntactic forcing is a method to prove conservativity result. It is commonly used in reverse mathematics.

To show that a second order principle \mathcal{P} is conservative over \mathcal{T} it proceeds by first taking an arbitrary countable model of \mathcal{T} . This model is then extended through a forcing argument to include sets solving all instances of \mathcal{P} without altering the first order part. The conservativity then follows by Gödel’s completeness theorem. For details and further information see [2].

The elimination of monotone Skolem functions and Howard’s elimination of bar recursion are constructive: a careful analysis of the proofs would yield a uniform method to obtain a term of \mathcal{T} for each function provable total using \mathcal{P} . Whereas the forcing argument essentially uses a reductio ad absurdum argument (if \mathcal{P} would not be conservative then by the completeness theorem there would be a model that could not be extended).

Forcing yields in many cases full Π_1^1 -conservativity whereas the functional interpretation usually stops at Π_3^0 -conservativity. This is a consequence of the way the functional interpretation works: it transforms every statement in a functional, where for every additional quantifier alternation the type-level rises, making it more complex to analyze. For instance, Π_3^0 -statements correspond to type 2 functionals (i.e. functionals essentially of the form $\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$).

This makes it easier to handle principles implying the Π_1^0 -bounded collection principle (Π_1^0 -CP). Due to the well-known fact that Π_1^0 -CP is Π_3^0 -conservative over Σ_1^0 -IA the base theory for the functional interpretation does not change. This circumvents the problems forcing experiences when proving conservativity over Π_1^0 -CP, see [15, §6].

The original proof that RT_2^2 or COH is Π_1^1 -conservative over Σ_2^0 -induction uses syntactic forcing, also the proof that COH is Π_1^1 -conservative over Σ_1^0 -induction uses it, see [7]. The original proof of the fact that COH is Π_1^1 -conservative over Π_1^0 -CP is done using a complicated double forcing, see [9]. Our proof of the fact that $\text{COH} + \Pi_1^0$ -CP is Π_3^0 -conservative over Σ_1^0 -IA is similar to the proof of [7] since we show conservativity over RCA_0 (without Π_1^0 -CP) and therefore do not face the problems forcing experiences with Π_1^0 -CP and that Chong, Slaman and Yang in [9] deal with. Additionally, our proof is open for proof mining that means it provides a method for program extraction.

Possible extensions. The question arises whether RT_2^2 also is proofwise low in sequence over WKL_0^* and hence does not imply Σ_2^0 -induction.

The first obstacle to this is that the proof of the lowness-property crucially depends on full Σ_1^0 -induction which renders RCA_0^* or WKL_0^* insufficient. The other obstacle is following property of RT_2^2 which makes this even without the use of Σ_1^0 -induction provably impossible:

Proposition 2.

$$\text{iRCA}_0^* \vdash \text{RT}_2^2 \rightarrow \Pi_2^0\text{-LEM},$$

where iRCA_0^* is the intuitionistic system corresponding to RCA_0^* and $\Pi_2^0\text{-LEM}$ is the Π_2^0 -law of excluded middle.

More precisely, for every Π_2^0 -statement $\forall x \exists y A_{qf}(x, y)$ there is a coloring such that one can decide constructively from a homogeneous set whether the Π_2^0 -statement is true or not.

Proof. We show for an arbitrary quantifier-free formula A_{qf} that

$$\forall x \exists y A_{qf}(x, y) \vee \exists x \forall y \neg A_{qf}(x, y).$$

First note that over iRCA_0^*

$$\forall x \exists y A_{qf}(x, y) \leftrightarrow \forall x \exists y \forall x' \leq x \exists y' \leq y A_{qf}(x', y').$$

Hence we may assume that A_{qf} is monotone in the sense that

$$A_{qf}(x, y) \rightarrow \forall u \leq x \forall v \geq y A_{qf}(u, v).$$

Now color each pair $\{x, y\}$ with $x < y$ red if $A_{qf}(x, y)$ holds and blue otherwise. It is easy to see that there exists an infinite red homogeneous set iff $\forall x \exists y A_{qf}(x, y)$ is true. \square

Remark 3. The coloring in the proof of proposition 2 is actually stable and transitive in the color red, i.e.

$$(x < y < z \wedge \{x, y\} \text{ red} \wedge \{y, z\} \text{ red}) \rightarrow \{x, z\} \text{ red}.$$

Hence one may replace RT_2^2 by the principle SCAC, which is equivalent to RT_2^2 restricted to stable coloring that are transitive in all but one color. For a definition see [15].

An immediate consequence of this proposition is that a sequence of homogeneous sets computes an instance of Π_2^0 -comprehension. But this is impossible to derive having only instances of Π_1^0 -comprehension available.

In the proof of the proofwise low property of RT_2^2 — proposition 36 — this manifests itself in the essential use of the law of exclude middle.

This is also the reason why we have to use the functional interpretation of the proofwise low property of RT_2^2 to obtain the conservativity result. Under the functional interpretation the law of excluded middle vanishes and one can pass without any additional reasoning to sequences of instances and solutions.

Besides COH also the atomic model theorem can be shown to be proofwise low in sequence, for a definition see [16]. The question arises whether there are besides these principle others weaker than RT_2^2 which are proofwise low in sequence. The preceding argumentation and proposition 2 renders this impossible for any principle implying SCAC. This settles this question for many principle considered in reverse mathematics surrounding RT_2^2 , see [15, 43].

An exception to this is SADS, which is equivalent to RT_2^2 restricted to stable and transitive colorings. For a definition see also [15]. Another exception is the infinite Erdős-Moser principle (EM) considered by Bovykin and Weiermann, see [6]. This principle is equivalent to the statement that every 2-coloring of pairs admits an infinite set on which the coloring is transitive. The logical strength of this principle is in general unclear even though Bovykin and Weiermann provided some evidence that EM might be weak. The authors are not aware of any fact that would make it impossible that SADS or EM is proofwise low in sequence over for instance WKL_0^* .

2. LOGICAL SYSTEMS

We will work in a setting based on fragments of Heyting and Peano arithmetic in all finite types introduced in [50], for details see also [32].

2.1. Finite types. The set of all finite types \mathbf{T} is inductively defined as

$$0 \in \mathbf{T}, \quad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T},$$

where 0 denotes the type of natural numbers and $\tau(\rho)$ the type of functions from ρ to τ . The set of pure types $\mathbf{P} \subset \mathbf{T}$ is defined as

$$0 \in \mathbf{P}, \quad \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

They will often be denoted by natural numbers:

$$0(n) := n + 1,$$

e.g. $0(0) = 1$. The level or degree $\text{deg}(\rho)$ of a type ρ is inductively defined as

$$\text{deg}(0) := 0, \quad \text{deg}(\tau(\rho)) := \max(\text{deg}(\tau), \text{deg}(\rho) + 1).$$

We will often denote the type of a term or variable by a superscribed index.

Equality $=_0$ for type 0 objects will be added as primitive notion to the systems. Higher type equality $=_{\tau\rho}$ will be treated as abbreviation:

$$x^{\tau\rho} =_{\tau\rho} y^{\tau\rho} \equiv \forall z^\rho \, xz =_\tau yz.$$

2.2. Gödel's system T . Define the λ -combinators $\Pi_{\rho,\sigma}, \Sigma_{\rho,\sigma,\tau}$ to be the functionals satisfying

$$\Pi_{\rho,\sigma} x^\rho y^\sigma =_\rho x, \quad \Sigma_{\rho,\sigma,\tau} x^{\tau\sigma\rho} y^{\sigma\rho} z^\rho =_\tau xz(yz).$$

Similar define the recursor R_ρ of type ρ to be the functional satisfying

$$R_\rho 0 yz =_\rho y, \quad R_\rho (Sx^0) yz =_\rho z(R_\rho xyz)x.$$

Let *Gödel's system T* be the \mathbf{T} -sorted set of closed terms that can be build up from 0^0 , the successor function S^1 , the λ -combinators and, the recursors R_ρ for

all finite types ρ . Using the λ -combinators one easily sees that T is closed under λ -abstraction, see [50].

T_n denotes the subsystem of Gödel's system T , where primitive recursion is restricted to recursors R_ρ with $\deg(\rho) \leq n$. The system T_0 corresponds to the extension of Kleene's primitive recursive functionals to mixed types, see [24], whereas full system T corresponds to Gödel's primitive recursive functionals, see [13].

2.3. Heyting and Peano arithmetic. Define the *neutral Heyting/Peano arithmetic* ($\mathbf{N}\text{-HA}^\omega$, $\mathbf{N}\text{-PA}^\omega$) to be the extension of the term system T to a \mathbf{T} -sorted intuitionistic resp. classical logical system plus the schema of full induction and the equality axioms for type 0, i.e.

- $x =_0 x$, $x =_0 y \rightarrow y =_0 x$, $x =_0 y \wedge y =_0 z \rightarrow x =_0 z$,
- $x_1 =_0 y_1 \wedge \dots \wedge x_n =_0 y_n \rightarrow t(x_1, \dots, x_n) =_0 t(y_1, \dots, y_n)$ for any n -ary term t of suitable type,

and substitution schemata for λ -combinators and the recursors, i.e.

$$(\text{SUB}): \begin{cases} t[\Pi xy] =_0 t[x] \\ t[\Sigma xyz] =_0 t[xz(yz)] \\ t[R0yz] =_0 t[y] \\ t[R(Sx)yz] =_0 t[z(Rxyx)] \end{cases} \quad \text{for all } t \text{ of type 0.}$$

For a formal definition see [51, I.1.6.15] (there $\mathbf{N}\text{-HA}^\omega$ is called HA^ω).

These theories are neutral with respect to an intensional or an extensional interpretation of higher type objects. However, for type 0 object the usual equality axioms hold. Higher type equality is of no effect except for the λ -combinators and the recursors. Later we will add functionals yielding cohesive and homogeneous set which are not extensional (in the presence of extensionality they would prove full arithmetical comprehension, see [31]) and therefore can only be analyzed in a neutral context.

Let *weakly extensional Heyting/Peano arithmetic* ($\mathbf{WE}\text{-HA}^\omega$, $\mathbf{WE}\text{-PA}^\omega$) be $\mathbf{N}\text{-HA}^\omega$ resp. $\mathbf{N}\text{-PA}^\omega$ plus the quantifier-free rule of extensionality, i.e.

$$(\text{QF-ER}): \frac{A_{qf} \rightarrow s =_\rho t}{A_{qf} \rightarrow r[s/x^\rho] =_\tau r[t/y^\rho]},$$

where A_{qf} is quantifier-free and s^ρ, t^ρ, r^τ are terms of $\mathbf{WE}\text{-HA}^\omega$. Note that the addition of SUB here is redundant, since QF-ER together with the axioms for Π, Σ, R proves it. The systems with *full extensionality*, i.e. $\mathbf{N}\text{-HA}^\omega$, $\mathbf{N}\text{-PA}^\omega$ plus the extensionality axioms

$$(\text{E}_{\rho, \tau}): \forall z^{\tau\rho}, x^\rho, y^\rho (x =_\rho y \rightarrow zx =_\tau zy)$$

for all $\tau, \rho \in \mathbf{T}$, will be denoted by $\mathbf{E}\text{-HA}^\omega$ and $\mathbf{E}\text{-PA}^\omega$. For a detailed definition of these systems, see [32, section 3].

The weakly extensional and neutral theories allow functional interpretation in themselves, which is not possible in the presence of full extensionality. Later we will eliminate the usage of extensionality (see proposition 7 below), hence neither the interpretation of constants yielding cohesive/homogeneous sets nor the functional interpretation will lead to problems. For a discussion of these systems and the connection to functional interpretation we refer to [50].

It is also important to note that in presence of only QF-ER the deduction theorem in general fails, see [32, theorem 9.11]. To overcome this we will restrict the use of principles in premises of QF-ER. This will be denoted by the \oplus -sign, e.g. $\mathbf{WE}\text{-PA}^\omega \oplus \text{WKL}$ denote the system $\mathbf{WE}\text{-PA}^\omega + \text{WKL}$, where WKL may not be used in the premise of QF-ER. The weak extensional systems satisfy the deduction theorem with respect to \oplus .

We now introduce fragments of neutral and (weakly) extensional Heyting/Peano arithmetic corresponding to T_n :

Define $\text{N-HA}_n^\omega \upharpoonright$ to be the logical system extending T_n plus Σ_{n+1}^0 -IA and plus the case-distinction functionals $(\text{Cond}_\rho)_{\rho \in \mathbf{T}}$ and its substitution axioms

$$(\text{SUB}_{\text{Cond}}): \begin{cases} t[\text{Cond}_\rho(0^0, x^\rho, y^\rho)] =_0 t[x] \\ t[\text{Cond}_\rho(Su, x^\rho, y^\rho)] =_0 t[y] \end{cases} \quad \text{for all } t \text{ of type } 0.$$

These case distinction functionals are needed for the functional interpretation and cannot be defined in these fragments of N-HA^ω , see [40, 3]. In the full system T they can be simulated by the recursors. Instead of $\text{N-HA}_0^\omega \upharpoonright$ we also write $\widehat{\text{N-HA}}^\omega \upharpoonright$. The classical systems $\text{N-PA}_n^\omega \upharpoonright$, $\widehat{\text{N-PA}}^\omega \upharpoonright$ are defined similarly. In the same way also the (weakly) extensional systems $(\text{W})\text{E-HA}_n^\omega \upharpoonright$, $(\text{W})\widehat{\text{E-HA}}^\omega \upharpoonright$, $(\text{W})\text{E-PA}_n^\omega \upharpoonright$, $(\text{W})\widehat{\text{E-PA}}^\omega \upharpoonright$ are defined.⁵ However for the classical systems defined here one does not need to add Cond to the system since it is provably definable with the λ -combinators and R_0 , see [40]. Note that Σ_{n+1}^0 -induction is provable with the recursor R_n and quantifier-free induction and full QF-AC in all types (definition below) over the classical systems defined here. Hence the addition of it to the classical systems is actually superfluous. This follows from [40] and Kreisel's characterization theorem, see [32, proposition 10.13].

2.4. Grzegorzczuk arithmetic. We moreover need weaker fragments of Heyting and Peano arithmetic containing only quantifier-free induction.

Let *weakly extensional Grzegorzczuk arithmetic of level n in all finite types* $\text{G}_n\text{A}_{(i)}^\omega$ be the (intuitionistic) system containing $=_0$ -axioms, QF-ER, λ -abstraction, the n -th branch of the Ackermann-function, bounded search and bounded primitive recursion. For a detailed definition see [26].⁶ The neutral variant will be denoted by $\text{N-G}_n\text{A}^\omega$, the extensional one by $\text{E-G}_n\text{A}^\omega$.

Let $\text{G}_\infty\text{A}^\omega$ be the union of all these systems. This system contains all primitive recursive functions but not all primitive recursive functionals (in the sense of Kleene). For instance R_0 is not contained in $\text{G}_\infty\text{A}^\omega$. Thus it also contains no Σ_1^0 -induction. The set of all closed terms of $\text{G}_n\text{A}^\omega$ is called $\text{G}_n\text{R}^\omega$. See [26] and [32, Chapter 3] for all of this.

2.5. Quantifier-free axiom of choice. Let QF-AC be the schema

$$\forall x \exists y A_{qf}(x, y) \rightarrow \exists f \forall x A_{qf}(x, f(x)),$$

where A_{qf} is a quantifier-free formula. If the types of x, y are restricted to α, β we write $\text{QF-AC}^{\alpha, \beta}$.

The scheme $\text{QF-AC}^{0,0}$ corresponds to recursive comprehension (Δ_1^0 -CA) in a second order context. Thus $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{1,0}$ and RCA_0 share the same proof theoretic strength. RCA_0 can easily be embedded into $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{1,0}$ and $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{1,0}$ is conservative over RCA_0 modulo this embedding, see [31]. For this reason $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{1,0}$ is called RCA_0^ω .

In the same way RCA_0^* can be embedded into $\text{G}_3\text{A}^\omega + \text{QF-AC}^{1,0}$ and both systems are Π_2^0 -conservative over Kalmar elementary arithmetic.

⁵For a formal definition let $(\text{W})\widehat{\text{E-HA}}^\omega \upharpoonright$ be defined as in [32, section 3.4] and define $(\text{W})\text{E-HA}_n^\omega \upharpoonright$ to be $(\text{W})\widehat{\text{E-HA}}^\omega \upharpoonright$ plus Σ_{n+1}^0 -IA and the defining axioms and constants for the recursors R_ρ with $\text{deg}(\rho) \leq n$. The neutral variants are defined in the same way but without the rule of extensionality.

⁶In [32] the system $\text{G}_n\text{A}^\omega$ is defined to include all $\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N}^{\mathbb{N}^\mathbb{N}}$ -true \forall -sentences. In a pure proof-mining context these sentences do not matter because they have no impact on the provable recursive functions in the system. We only add quantifier-free induction (QF-IA), to be able to later establish conservativity over PRA.

In ordinary mathematics higher types usually do not occur and second order arithmetic is sufficient to formalize most of it. We require here a system containing all finite types to be able to carry out a functional interpretation and thus cannot use second order systems.

2.6. The quantifier-free subsystems. In order to exploit the full subtlety of the functional interpretation we will also need the *quantifier-free subsystems of* $\text{N-G}_n\text{A}_1^\omega$ and $\text{N-HA}_n^\omega \upharpoonright$. The quantifier-free subsystems are denoted by $\text{qf-N-G}_n\text{A}^\omega$ resp. $\text{qf-N-PA}_n^\omega \upharpoonright$. (The quantifier free subsystems satisfy the law of excluded middle and are therefore classical.)

They are obtained from the full systems as follows:

- The quantifier-rules and -axioms are dropped from logic.
- For all axioms of the form $A(x_1^{\rho_1}, \dots, x_n^{\rho_n})$, where A is quantifier-free, the following axioms are added to the system:

$$A(t_1^{\rho_1}, \dots, t_n^{\rho_n}),$$

where t_i are arbitrary terms.

- The induction schema is replaced by the (quantifier-free) induction rule:

$$\frac{A(0^0), \quad A(x^0) \rightarrow A(Sx^0)}{A(t^0)},$$

where A is quantifier-free, x does not occur free in the assumption and t is an arbitrary term.

These quantifier-free systems contain only prime formulas of the form

$$t_0 =_0 t_1,$$

where t_0, t_1 are terms in $\text{N-G}_n\text{A}_1^\omega$ resp. $\text{N-HA}_n^\omega \upharpoonright$. Formulas are logical combinations of these predicates. Obviously, $\text{qf-N-G}_n\text{A}^\omega$ and $\text{qf-N-PA}_n^\omega \upharpoonright$ are subsystems of $\text{N-G}_n\text{A}_1^\omega$ resp. $\text{N-HA}_n^\omega \upharpoonright$. (For a detailed discussion of these systems we also refer the reader to [50, 1.6.5]. For technical reason we use here the variant of the systems described remark 1.5.8.)

Observe, that in these system we can only instantiate type 0 variables (via the induction rule) and not higher type variables, hence we immediately obtain the following lemma:

Lemma 4. *Let A be a sentence and*

$$\mathcal{T} \vdash A,$$

where $\mathcal{T} = \text{qf-N-G}_n\text{A}^\omega, \text{qf-N-PA}_n^\omega \upharpoonright$.

Then there exists a derivation of A in \mathcal{T} that contains only the variables of A plus some fresh variables of type 0.

Proof. In a derivation of A in \mathcal{T} replace every variable not of type 0 and not occurring in A by constant 0^ρ of suitable type. Since higher type variables cannot be instantiated the derivation remains valid. \square

2.7. Functional interpretation. *Functional interpretation* will denote in this paper a negative translation followed by Gödel's Dialectica translation.

Gödel's Dialectica translation is a proof interpretation that translates proofs from (a fragment of) WE-HA^ω or N-HA^ω into its quantifier-free subsystem, see [13].

Let \mathcal{T} be such a system. The Dialectica translation associates to each formula A of \mathcal{T} a $\exists\forall$ -formula

$$A^D := \exists x \forall y A_D(x, y),$$

where A_D is quantifier-free. In particular, for a Σ_2^0 sentence A the formula A_D is the quantifier-free matrix of A .

From a proof of A one then can extract a term t , such that

$$\text{qf-}\mathcal{T} \vdash A_D(t, x).$$

A *negative translation* is a proof translation that translates classical proofs into intuitionistic proofs. It also proceeds by associating each formula A a formula A^N such that

$$\mathcal{S} \vdash A \leftrightarrow A^N \quad \text{and} \quad \mathcal{S} \vdash A \implies \mathcal{S}_i \vdash A^N.$$

Here \mathcal{S} is any of $(\text{W})\text{E-PA}^\omega$, $(\widehat{\text{W}})\text{E-PA}^\omega$, $\text{G}_n\text{A}^\omega$ or its neutral variants and \mathcal{S}_i is its intuitionistic counterpart. (To be specific, Kuroda's negative translation A^N is obtained from A by inserting $\neg\neg$ after each \forall and in front of the whole formula.)

Thus we denote by functional interpretation a proof translation from (a fragment of) WE-PA^ω or N-PA^ω into its quantifier-free part. We abbreviate the functional interpretation by ND. The ND-translation of a formula A will be denoted by A^{ND} and the quantifier-free matrix of it by A_{ND} .

The functional interpretation in particular has the property to extract a term for each provable recursive function, i.e. from a proof of a $\forall\exists$ -statement (in WE-PA^ω or any other fragment for which the functional interpretation holds)

$$\text{WE-PA}^\omega \vdash \forall u \exists v A_{qf}(u, v)$$

it extracts a term t such that

$$\text{qf-WE-HA}^\omega \vdash \underbrace{A_{qf}(u, tu)}_{\equiv A_{ND}(t, u)}.$$

For an introduction to the functional interpretation see [32, 3, 50].

2.8. Additional notation and definitions. We denote sets by capital letters. If not otherwise noted they are represented by characteristic functions. Sometimes capital letters also denote higher type functionals. It will be clear from the context what is meant.

It is important to note that in systems not containing Σ_1^0 -induction it is in general not provable that every infinite set — that is a set X satisfying $\forall k \exists n > k \ n \in X$ — can be strictly increasingly enumerated, i.e. there exists a strictly monotone function f such that $\text{rng}(f) = X$. The system $\widehat{\text{WE-HA}}^\omega + \text{QF-AC}^{0,0}$ proves that the first statement implies the second. The converse — every strictly increasingly enumerable set is infinite — is already provable without Σ_1^0 -induction, for instance $\text{G}_3\text{A}^\omega$ suffices.

Sequence codes are denoted by $\langle x_0, \dots, x_n \rangle$. The corresponding projection functions and length function are denoted by $(\cdot)_i$ and $\text{lth}(\cdot)$. We encode sequences using a bijective and monotone (in each component) sequence-coding based on the Cantor pairing, see [32, definition 3.30]. This coding is definable in every system containing $\text{qf-N-G}_3\text{A}^\omega$.

We model in our systems n -colorings of $[\mathbb{N}]^2$ as functions $c: \mathbb{N} \times \mathbb{N} \rightarrow n$ with $c(x, y) = c(y, x)$.

Further we define the following notions:

- \bar{f} denotes the course-of-value function of f^1 .
- $x \sqsubseteq X$ iff x is an initial segment of a strictly monotone enumeration of X .
- $x \sqsubseteq^{fin} X$ iff x is an code for a finite subset of X .

Definition 5 (Bounded type 1 recursor, \tilde{R}_1). The bounded type 1 recursor \tilde{R}_1 is defined as

$$\begin{aligned} \tilde{R}_1 0 y z h u &=_0 \min(y(u), h(0, u)) \\ \tilde{R}_1 (x+1) y z h u &=_0 \min(z(\tilde{R}_1 x y z h) x u, h(x, u)). \end{aligned}$$

We will denote by (\tilde{R}_1) the defining axioms. Note that they are purely universal and that \tilde{R}_1 can be trivially majorized.

Definition 6 (Uniform weak König's lemma, UWKL, [30]). Uniform weak König's lemma is the statement

$$\exists \Phi \leq_{1(1)} 1 \forall f (T^\infty(f) \rightarrow \forall x^0 f(\overline{\Phi f}x) = 0),$$

where T^∞ expresses that f describes an infinite 0/1-tree.

We can modify (in $G_\infty A^\omega$) every function f such that it describes an infinite 0/1-tree and is not altered if it already described such a tree. We will write \check{f} for this modification, see [25, 32].

With this we can restate UWKL equivalently as

$$\exists \Phi \leq_{1(1)} 1 \forall f^1 \forall x^0 \check{f}(\overline{\Phi \check{f}}x) = 0.$$

Note that the condition $\leq_{1(1)}$ is superfluous because the modified tree contains only 0/1-sequences.

By Skolemization we add a weak König's Lemma functional constant \mathcal{B} described by the (purely universal) axiom

$$(3) \quad \forall f \forall x^0 \check{f}(\overline{\mathcal{B} \check{f}}x) = 0.$$

This axiom will be denoted by \mathcal{B} . Note that \mathcal{B} can be trivially majorized.

In a system containing full extensionality UWKL implies Π_1^0 -CA, see [30], hence it is too strong for our purpose. But in a weakly extensional system it often can be handled like WKL, for instance it vanishes under a monotone functional interpretation like WKL and can be added to the elimination of monotone Skolem functions, see [30].

Proposition 7 (Elimination of extensionality, [37]). *Let A be a formula containing only free variables and quantification of degree ≤ 1 .*

If

$$\text{E-PA}^\omega + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} \vdash A$$

then

$$\text{N-PA}^\omega + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} \vdash A.$$

The same holds also for the fragments $\widehat{\text{N-PA}^\omega}$ and $\text{N-G}_n A^\omega$.

Proof. Proposition 10.45 and lemma 10.41 of [32]. These lemma and proposition actually do not make use of weak extensionality and therefore show conservativity even over a neutral theory. \square

Recall that a type 2 functional φ is continuous if

$$(4) \quad \forall g^1 \exists n^0 \forall h^1 (\bar{g}n = \bar{h}n \rightarrow \varphi(g) = \varphi(h)).$$

Definition 8 (Associate, [23, 33]). For every continuous type 2 functional φ we will denote by α_φ an associate of φ , i.e. a type 1 function with the properties

$$(5) \quad \begin{aligned} & \forall f \exists n \alpha_\varphi(\bar{f}n) \neq 0, \\ & \forall f, n (\alpha_\varphi(\bar{f}n) \neq 0 \rightarrow \varphi(f) = \alpha_\varphi(\bar{f}n) \div 1). \end{aligned}$$

The value of φ is uniquely determined through α_φ . For every continuous functional there exists an associate, though it is not uniquely determined. For details see also [38].

Definition 9. A functional given by a closed term φ^ρ of \mathcal{T} is called *provably continuous* if for some term α_φ (containing at most the free variables of φ) of type 1 (if $\rho > 0$) resp. 0 (if $\rho = 0$), the following holds:

$$\mathcal{T} \vdash \varphi \approx_\rho \alpha_\varphi.$$

Here, for general x^ρ and $\alpha^{0/1}$, the relation $x \approx_\rho \alpha$ is defined by induction on ρ :

$$x \approx_0 \alpha := x =_0 \alpha,$$

$$x \approx_{\tau\rho} \alpha := \alpha \in \text{ECF}_{\tau\rho} \wedge \forall y^\rho \forall \beta \in \text{ECF}_\rho (y \approx_\rho \beta \rightarrow xy \approx_\tau \alpha \upharpoonright \beta),$$

where ECF is the model of extensional hereditarily continuous functionals formalized in \mathcal{T} and \upharpoonright denotes the application in ECF. (See [24, 33, 50], for a definition see also [32, definitions 3.58, 3.59].)

Especially, in the case of $\rho = 2$ a functional φ is provably continuous in \mathcal{T} if it has an associate α_φ in \mathcal{T} and (5) is provable.

Proposition 10. *For every term $t^2 \in G_n R^\omega, T_0, T_1$ there exists provably in $G_n A^\omega$ resp. $\widehat{\text{WE-PA}}^\omega \upharpoonright$, $\text{WE-PA}_1^\omega \upharpoonright$ a (primitive recursive) associate α_t . In other words t is provably continuous.*

Proof. We first consider the case of $\widehat{\text{WE-PA}}^\omega \upharpoonright = \text{WE-PA}_0^\omega \upharpoonright$ and $G_n A^\omega$. Here the only functional constants having no trivial associate are the λ -combinators and R_0 (in the case of $\widehat{\text{WE-PA}}^\omega \upharpoonright$) and the course-of-value functional (in the case of $G_n A^\omega$). The associates of R_0 and the course-of-value functional can easily be computed and (5) be proven in the respective systems. By normalization one can find a term $\hat{t} =_2 t$ that does not include λ -abstraction of degree ≥ 1 . The proposition for $\widehat{\text{WE-PA}}^\omega \upharpoonright$ and $G_n A^\omega$ follows from this.

In the case of $\text{WE-PA}_1^\omega \upharpoonright$ we prove by induction over the structure of t that t is provably continuous. For this it is sufficient to prove that every functional constant is provably continuous and to observe that this property is retained under composition. The associates for the λ -combinators are easily definable and provable in these systems, see [50].

Here we only show that the existence of an associate for R_1 is provable in $\text{WE-PA}_1^\omega \upharpoonright$, since we are only interested in this case. For the other recursors the proof is similar. Let

$$\alpha_{R_1}(0, y', z', u) := \begin{cases} (y')_u + 1 & \text{if } u < \text{lth } y', \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_{R_1}(x + 1, y', z', u) := \begin{cases} (z') \langle x, (\lambda k. \alpha_{R_1}(x, y', z', k) \dot{-} 1)k \rangle & \text{if } \exists k < \text{lth } y', \text{ such that} \\ & \alpha_{R_1}(x, y', z', k) > 0 \\ & \text{and this is } > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using Π_2^0 -induction one shows that

$$\forall x (\forall u \exists n \alpha_{R_1}(x, \bar{y}n, \overline{\alpha_{\lambda r. \bar{z} r} n}, u) = R_1(x, y, z, u) + 1)$$

and hence that α_{R_1} is an associate of R_1 . \square

2.9. Properties of instances of comprehension.

Remark 11. A sequence of Π_1^0 -comprehension instances $(\Pi_1^0\text{-CA}(f_i))_i$ may be reduced to the single instance of $\Pi_1^0\text{-CA}(f')$ with $f'xy := f_{(x)_1}(x)_2y$, see [27, remark 3.8].

Lemma 12 ([27, 28]). *For suitable terms ξ_i of $G_3 A^\omega$ we have*

$$(i) \ G_3 A^\omega + \text{QF-AC}^{0,0} \vdash \forall f (\Pi_1^0\text{-CA}(\xi_1 f) \rightarrow \Pi_1^0\text{-AC}(f)),$$

- (ii) $G_3A^\omega + \text{QF-AC}^{0,0} \vdash \forall f (\Pi_1^0\text{-CA}(\xi_2 f) \rightarrow \Delta_2^0\text{-CA}(f)),$
- (iii) $G_3A^\omega + \text{QF-AC}^{0,0} \vdash \forall f (\Pi_1^0\text{-CA}(\xi_3 f) \rightarrow \Delta_2^0\text{-IA}(f)),$
- (iv) $G_3A^\omega + \text{QF-AC}^{0,0} \vdash \forall f (\Pi_1^0\text{-CA}(\xi_4 f) \rightarrow \Pi_1^0\text{-CP}(f)),$
- (v) $G_3A^\omega + \text{QF-AC}^{0,0} + \text{WKL} \vdash \forall f (\Pi_1^0\text{-CA}(\xi_5 f) \rightarrow \Pi_2^0\text{-WKL}(f)).$

Here the principle $\mathcal{K}\text{-AC}$ denotes the scheme of axiom of choice, where the base formula is of type \mathcal{K} . Similarly $\mathcal{K}\text{-WKL}$ denotes weak König's lemma where the tree is given by a predicate of type \mathcal{K} . The principles $\mathcal{K}\text{-IA}$ and $\mathcal{K}\text{-CA}$ are defined likewise.

If $\mathcal{K} = \Pi_n^0, \Sigma_n^0$ then an instance of those principles is given by a function f coding the quantifier-free part of the Π_n^0 resp. Σ_n^0 formula. For instance

$$\Pi_1^0\text{-AC}(f) \equiv \forall x \exists y \forall z f(x, y, z) = 0 \rightarrow \exists Y \forall x \forall z f(x, Y(x), z) = 0.$$

Similar a Δ_2^0 -formula is given by an f coding a function for a Π_n^0 and a function for a Σ_n^0 formula.

Proof of lemma 12. For (i), (ii) see [28, lemma 4.2]. The statements (iii), (iv) are immediate consequences of these. Note that we require here G_3A^ω and not only G_2A^ω as in the reference, since we do not add the true universal sentences to the system, see footnote 6.

For (v) let ξ_5 be such that the instance of $\Pi_1^0\text{-CA}$ yields the comprehension function for the innermost quantifier of the tree predicate reducing $\Pi_2^0\text{-WKL}$ to $\Pi_1^0\text{-WKL}$. This is equivalent to WKL and thus included in the system, see for instance [44]. \square

For the ordinal analysis of terms we will need the following abbreviation:

$$\omega_0^\mu = \mu \quad \text{and} \quad \omega_{k+1}^\mu = \omega^{\omega_k^\mu},$$

where $k \in \mathbb{N}$ and μ is an ordinal.

Lemma 13. *Let $n \in \mathbb{N}$ and let $t[g]$ be a type 1 term with the only free variable g such that $\lambda g.t[g] \in T_n$. Then for every term φ in T_{n-1} or in $G_\infty R^\omega$ if $n = 0$ there exists a term ξ in the same system such that $\text{WE-PA}_{n-1}^\omega \upharpoonright + \text{QF-AC}$ or $G_\infty A^\omega + \text{QF-AC}$ in the case of $n = 0$ proves*

$$\forall g (\Pi_1^0\text{-CA}(\xi g) \rightarrow \exists f^1 (f \text{ satisfies the defining axioms of } t[g] \wedge \Pi_1^0\text{-CA}(\varphi f g))).$$

Proof. First fix a suitable encoding for ordinals smaller than ε_0 in this system, see for instance [14].

Every term $t^1 \in T_n$ can be defined through (unnested) ordinal recursion of order $< \omega_{n+1}^\omega$; the totality of such a recursion can be proven using a suitable instance of $\Sigma_{n+1}^0\text{-IA}$, see [39] and theorem 44 below. Such an instance is included in the system because a suitable instance of $\Pi_1^0\text{-CA}$ reduces it to $\Sigma_n^0\text{-IA}$. This proves the claim that there is a total function f satisfying the definition of $t[g]$.

For the second part note that the defining axioms of unnested ordinal primitive recursion of order type α are given by

$$(6) \quad \begin{aligned} f(0) &:= f_0, \\ f(n) &:= h(n, f(l(n))), \end{aligned}$$

where l satisfies

$$(7) \quad l(n) \prec n \quad \text{for } n > 0$$

and \prec defines a well-ordering on \mathbb{N} of order type α .

We say a finite sequence s satisfies the defining axioms (6) up to n if

$$(s)_0 = f_0, \quad (s)_i = h(i, (s)_{l(i)}) \quad \text{for all } i \in \bigcup_{n' \leq n} \bigcup_k \{l^k(n')\} \setminus \{0\}$$

For notational ease we assume here that $l(0) = 0$. Note that because of (7) the set $\bigcup_k \{l^k(n')\}$ defines an \prec -descending chain and is therefore provably finite.

For the second part we have to prove a comprehension of the form

$$(8) \quad \exists H \forall k (k \in H \leftrightarrow \forall x \varphi(f, g, k, x) = 0).$$

We use the imposed instance of comprehension to prove the following comprehension

$$\begin{aligned} \exists H \forall k (k \in H \leftrightarrow \forall x \forall s, n (s \text{ satisfies the defining axioms of } t[g] \text{ up to } n \\ \rightarrow \alpha_{\lambda f, \varphi(f, g, k, x)}(s) \leq 1)). \end{aligned}$$

Note that this comprehension is equivalent to (8) if f is total. \square

The proof of the comprehension above is similar to the construction of a 1-generic set: If the statement

$$\forall x \varphi(f, g, k, x) = 0$$

for a fixed k fails, then there is an x such that $\varphi(f, g, k, x) \neq 0$. Since φ is continuous this depends only on an initial segment of f . We express this by using associates, i.e. this statement is equivalent to

$$\exists n \alpha_{\lambda f, \varphi(f, g, k, x)}(\bar{f}n) > 1.$$

Hence it suffices to consider only finite initial segments.

We will use this technique in most proofs of instances of comprehension in this paper. This is the reason why we require φ to be provably continuous in the definition of proofwise low.

3. COHESIVE PRINCIPLE (COH)

Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of subsets of \mathbb{N} . A set G is *cohesive* for $(R_n)_{n \in \mathbb{N}}$ if $\forall n (G \subseteq^* R_n \vee G \subseteq^* \overline{R_n})$, i.e.

$$\forall n \exists s (\forall j \geq s (j \in G \rightarrow j \in R_n) \vee \forall j \geq s (j \in G \rightarrow j \notin R_n)).$$

A set G is *strongly cohesive* for $(R_n)_{n \in \mathbb{N}}$ if

$$\forall n \exists s \forall i < n (\forall j \geq s (j \in G \rightarrow j \in R_i) \vee \forall j \geq s (j \in G \rightarrow j \notin R_i)).$$

The *cohesive principle* (COH) is the statement that for every sequence of sets an infinite cohesive set exists. Similarly the *strong cohesive principle* (StCOH) is the statement that for every sequence of sets an infinite strongly cohesive set exists.

We denote by $(\text{St})\text{COH}(r, G)$ the statement that G is a set that satisfies the (strong) cohesive principle for the sets given by the characteristic functions $(\lambda x.r(i, x))_i$ where $r: \mathbb{N} \times \mathbb{N} \rightarrow 2$.

Proposition 14 ([15, 4.4]).

- (i) $G_3A^\omega \vdash \text{StCOH} \rightarrow \text{COH}$
- (ii) $G_3A^\omega \vdash \text{StCOH} \rightarrow \Pi_1^0\text{-CP}$
- (iii) $G_3A^\omega \vdash \text{StCOH} \leftrightarrow \text{COH} \wedge \Pi_1^0\text{-CP}$

Proof. The first statement is clear and the third statement is an immediate consequence of the first and second.

For the second we prove the infinite pigeonhole principle $\text{RT}_{<\infty}^1$ from StCOH. The infinite pigeonhole principle is equivalent to $\Pi_1^0\text{-CP}$, over Σ_1^0 -induction. This was shown in [17]. The proof can even be carried out in G_3A^ω , see [35]:

Let $f: \mathbb{N} \rightarrow n$ be a coloring. Define $R_i := \{x \mid f(x) = i\}$. Let G be an infinite, strongly cohesive set for R_i . By definition there is an s with

$$\forall i < n (\forall j \geq s (j \in G \rightarrow j \in R_i) \vee \forall j \geq s (j \in G \rightarrow j \notin R_i)).$$

By the totality of f there is exactly one i such that the first disjunction holds, i.e. the color i occurs infinitely often on G and thus on \mathbb{N} . \square

Lemma 15. G_3A^ω proves that a countable number of instances of (St)COH is uniformly equivalent to a single instance of (St)COH.

Proof. Let $(R_{n,i})_{n,i \in \mathbb{N}}$ be a double indexed sequence of sets. A solution to

$$(\text{St})\text{COH}((R_{(k)_1, (k)_2})_k)$$

is also a solution to

$$\forall i (\text{St})\text{COH}((R_{n,i})_n).$$

□

Proposition 16.

$$G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL} \vdash \forall r: \mathbb{N} \times \mathbb{N} \rightarrow 2 \ (\Pi_1^0\text{-CA}(\xi r) \rightarrow \exists G \text{StCOH}(r, G)),$$

where ξ is a suitable term.

Proof. Define

$$R_n := \lambda x. r(n, x), \quad R^x := \bigcap_{i < \text{lt}(x)} \begin{cases} R_i & \text{if } x_i = 0, \\ \overline{R_i} & \text{otherwise.} \end{cases}$$

Here and in the following let x be the code of the sequence $\langle x_0, \dots, x_{\text{lt}(x)-1} \rangle$.

For every n the set (of sets) $\{R^x \mid x \in 2^n\}$ is a partition of \mathbb{N} , i.e.

$$(9) \quad \forall n \forall z \exists! x \in 2^n \ z \in R^x.$$

This statement can be proved with an instance of quantifier-free induction (the tuple $\langle x_0, \dots, x_{n-1} \rangle$ is bounded by $\bar{1}n$ and z is a parameter).

We construct an infinite Π_2^0 -0/1-tree T deciding at level n whether for the solution set G either $G \subseteq^* \overline{R_n}$ or $G \subseteq^* R_n$ holds: Let

$$T(\langle x_0, \dots, x_n \rangle) \text{ iff } R^{\langle x_0, \dots, x_n \rangle} \text{ is infinite.}$$

The statement “ R^x is infinite” is Π_2^0 . The predicate T clearly defines a tree. The tree is infinite because otherwise

$$\exists n \forall x \in 2^n \exists y \forall z > y \ z \notin R^x$$

and this together with an instance of Π_1^0 -CP yields a contradiction to (9). (x can be bounded by $\bar{1}n$.)

With an application of an instance of Σ_1^0 -induction we prove

$$\forall x (R^x \text{ infinite} \rightarrow \forall n \exists \langle l_0, \dots, l_{n-1} \rangle (\forall i < n-1 \ l_i < l_{i+1} \wedge \forall i < n \ l_i \in R^x))$$

and then conclude

$$(10) \quad \forall n \forall x (\text{lt}(x) = n \wedge R^x \text{ infinite} \rightarrow \exists \langle l_0, \dots, l_{n-1} \rangle \forall i < n-1 \ l_i < l_{i+1} \wedge \forall i < n \ l_i \in R^x).$$

An instance of Π_2^0 -WKL yields an infinite branch b of T , i.e. $\forall n \ (R^{b(n)} \text{ infinite})$.

Using (10) we obtain

$$(11) \quad \forall n \exists \langle l_0, \dots, l_{n-1} \rangle (\forall i < n-1 \ l_i < l_{i+1} \wedge \forall i < n \ l_i \in R^{b(n)} \subseteq R^{b(i)}).$$

An application of QF-AC yields an enumeration $n \mapsto \langle l_0, \dots, l_{n-1} \rangle$ of finite tuples. Searching for the least code of a tuple and the properties of (11) assure that every tuple is extended by the following. Hence we may diagonalize to obtain an the set $G := \{l_0, l_1, \dots\}$. This set is strongly cohesive and solves the proposition.

Note that the instances of Σ_1^0 -IA, Π_1^0 -CP and Π_2^0 -WKL can be reduced to an instance of Π_1^0 -CA using lemma 12 and remark 11 yielding a suitable term ξ . □

We now strengthen this proposition to

Proposition 17. *For every term φ one can construct a term ξ such that*

$$\begin{aligned} & \mathsf{G}_\infty \mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC} \oplus \mathsf{WKL} \vdash \\ & \forall r: \mathbb{N} \times \mathbb{N} \rightarrow 2 \left(\Pi_1^0\text{-CA}(\xi r) \rightarrow \exists G \left(\mathsf{StCOH}(r, G) \wedge \Pi_1^0\text{-CA}(\varphi r G) \right) \right). \end{aligned}$$

Proof. We construct an infinite Π_2^0 -0/1-tree, in which we decide at level

- $2n$ whether $G \subseteq^* \overline{R_n}$ or $G \subseteq^* R_n$ and at level,
- $2n + 1$ the n -th value of the instance of Π_1^0 -comprehension, i.e. whether $\forall k (\varphi r G)nk = 0$ is true.

We assign to every element of the tree a finite (potential) initial segment L^x of G . At level $2n$ we add — as in the previous proposition — the next element of R^x ; at level $2n + 1$ we only add the smallest counterexample (extending our old initial segment of G with elements from R^x) to the statement $\forall k (\varphi r G)nk = 0$ if it is false and nothing otherwise. Define:

$$\begin{aligned} T(\langle x_0, \dots, x_{2n} \rangle) & \text{ iff } R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \text{ is infinite,} \\ T(\langle x_0, \dots, x_{2n}, 0 \rangle) & \text{ iff } \forall l \subseteq^{fin} R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \forall k \alpha_\varphi(L^{\langle x_0, \dots, x_{2n} \rangle} * l, n, k) \leq 1, \\ T(\langle x_0, \dots, x_{2n}, 1 \rangle) & \text{ iff } \exists l \subseteq^{fin} R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \exists k \alpha_\varphi(L^{\langle x_0, \dots, x_{2n} \rangle} * l, n, k) > 1, \\ L^{\langle \rangle} & := \langle \rangle, \\ L^{\langle x_0, \dots, x_{2n} \rangle} & := L^{\langle x_0, \dots, x_{2n-1} \rangle} * \left\langle \min \left\{ x \in R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \mid x > \max L^{\langle x_0, \dots, x_{2n-1} \rangle} \right\} \right\rangle, \\ L^{\langle x_0, \dots, x_{2n}, 0 \rangle} & := L^{\langle x_0, \dots, x_{2n} \rangle}, \\ L^{\langle x_0, \dots, x_{2n}, 1 \rangle} & := L^{\langle x_0, \dots, x_{2n} \rangle} * l, \\ k^{\langle x_0, \dots, x_{2n}, 1 \rangle} & := k, \\ k^x & := 0 \quad \text{for all } x \text{ not of this form,} \end{aligned}$$

where $\langle l, k \rangle$ minimal with

$$l \sqsubset R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \wedge \alpha_\varphi(L^{\langle x_0, \dots, x_{2n} \rangle} * l, n, k) > 1.$$

For notational simplification we omitted the requirements to make T closed under prefix, but we can simply add the conditions of the previous levels to the definition of T making it a tree.

L^x and k^x is clearly defined if $T(x)$ is true (use an instance of Σ_1^0 -induction to show this — weaken the Π_2^0 -statement “ R^x is infinite” in the definition of T to $\exists z \in R^x$).

Using the same argument as in the previous proposition we see that the tree is infinite. But we cannot apply Σ_1^0 -WKL(ξr), because this instance contains L , which is in general not computable in r (in the sense of $\mathsf{G}_\infty \mathsf{A}^\omega$).

The graph of $x \mapsto (L^x, k^x)$ is definable and Δ_1^0 . For notational ease we define the graph of its course-of-value function:

$$\begin{aligned} & (\langle x_0, \dots, x_n \rangle, \langle L_0, \dots, L_n \rangle, \langle k_0, \dots, k_n \rangle) \in \mathcal{G}_{\bar{L}, \bar{k}} \quad \text{iff} \\ n = 0: & L_n = \langle \rangle, k_n = 0, \\ n \text{ even:} & L_n = L_{n-1} * \langle y \rangle, k_n = 0 \\ & \text{where } y \text{ minimal with } y \in R^{\langle x_0, \dots, x_{2m-1} \rangle} \wedge y > \max(L_{n-1}), \\ n \text{ odd and } x_n = 0: & L_n = L_{n-1}, k_n = 0, \\ n \text{ odd and } x_n = 1: & L_n = L_{n-1} * l \text{ and } \langle l, k_n \rangle \text{ minimal with } l \sqsubset R^{\langle x_0, x_2, \dots, x_{2n} \rangle} \wedge \\ & \alpha_\varphi(L_n * l, (n-1)/2, k_n) > 2. \end{aligned}$$

(Note that equations like $L_n = L_{n-1} * l$ we omitted for notational ease the bounded quantifier $\exists l < L_n$ for l .) So we can replace every reference to L^x in the definition of T by

$$\exists k, y (x, (y, k)) \in \mathcal{G}_{L, k} \quad \text{or} \quad \forall k, y (x, (y, k)) \in \mathcal{G}_{L, k}.$$

The resulting tree is still Π_2^0 so we may apply an instance of Π_2^0 -WKL and obtain an infinite branch b .

Setting $G := \bigcup_n L^{\bar{b}(n)}$ now enumerates an infinite strongly cohesive set and from b we can decide $\forall k (\varphi r G)nk = 0$ for every n . \square

Corollary 18 (to the proof). *For every system \mathcal{T} containing $G_\infty A^\omega$ and every provably continuous term φ there exists a term ξ , such that*

$$\mathcal{T} + \text{QF-AC} \oplus \text{WKL} \vdash$$

$$\forall r: \mathbb{N} \times \mathbb{N} \rightarrow 2 \left(\Pi_1^0\text{-CA}(\xi r) \rightarrow \exists G \left(\text{COH}(r, G) \wedge \Pi_1^0\text{-CA}(\varphi r G) \right) \right).$$

Corollary 19. *(St)COH is proofwise low in sequence over $G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL}$.*

Proof. Lemma 15 and proposition 17. \square

Comparison to the Bolzano-Weierstraß principle. Recall that BW is the statement that every bounded sequence $(y_i)_{i \in \mathbb{N}}$ of real numbers contains a subsequence $(y_{f(i)})_{i \in \mathbb{N}}$ converging with the rate 2^{-n} , i.e. $\forall n \forall i, j \geq n |y_{f(i)} - y_{f(j)}| < 2^{-n}$, see [44]. It turns out that StCOH is equivalent to a natural variant of this principle, namely the statement that each bounded sequence $(y_i)_{i \in \mathbb{N}}$ of reals contains a Cauchy subsequence $(y_{f(i)})_{i \in \mathbb{N}}$. This means a sequence which converges but possibly without a computable rate of convergence, i.e. $\forall n \exists k \forall i, j \geq k |y_{f(i)} - y_{f(j)}| < 2^{-n}$, see [36]. Hence the term extraction results we obtain below for StCOH also apply to this variant of the Bolzano-Weierstraß principle.

4. TERM-NORMALIZATION

Denote by $T_{(k)}[F_0, \dots, F_{n-1}]$ the extension of the system T_k resp. T with the constants F_0, \dots, F_{n-1} . Further we treat here R_ρ as an unspecified constant (without R_ρ axioms) in the case of $\text{qf-}G_n A^\omega$.

In the following we will call the reduction of

$$\text{Cond}_{\rho(\tau)}(x, y, z)u^\tau \quad \text{to} \quad \text{Cond}_\rho(x, yu, zu)$$

a *Cond-reduction*. These Cond-reductions are provably valid in $\text{qf-N-}G_n A^\omega$.

Theorem 20 (term-normalization for degree 2). *Let F_i be constants of degree ≤ 2 .*

For every term $t^1 \in T_0[(\text{Cond}_\rho)_{\rho \in \mathbf{T}}, F_0, \dots, F_{n-1}]$ there is provably in $\text{qf-N-}G_3 A^\omega$ a term $\tilde{t} \in T_0[\text{Cond}_0, F_0, \dots, F_{n-1}]$ for which

$$\forall x \, tx =_0 \tilde{t}x$$

and where every occurrence of an F_i is of the form

$$F_i(\tilde{t}_0[y^0], \dots, \tilde{t}_{k-1}[y^0]).$$

Here k is the arity of F_i , and $\tilde{t}_j[y^0]$ are fixed terms whose only free variable is y^0 .

Proof. Without loss of generality we take the system $T_0[F]$ where F is of type 2. For notational simplification we assume that the recursor R_0 can be obtained from F . This can always be achieved with coding.

Let t^1 be a term in $T_0[F]$. The term tx , where x is a fresh variable, is $=_0$ -equal to a term $t'[x]$ where t' results from tx by carrying out all possible Π -, Σ -, and Cond-reductions. The outermost symbol of t' cannot be Π , Σ , or Cond_ρ with $\rho \neq 0$, since otherwise in t' either not all Π -, Σ -reductions had been carried out or t' would not be of type 0.

Hence one of the following holds:

- 1) $t'[x] = 0^0$
- 2) $t'[x] = S(t_a^0[x])$
- 3) $t'[x] = F(t_b^1[x])$

$$4) \ t'[x] = \text{Cond}_0(t_c^0[x], t_d^1[x], t_e^1[x])$$

In the first case we are done, $\lambda x.t'[x]$ satisfies the theorem. In the second case we proceed the same way with the term t_a . In the third case we proceed with the term $t_b y^0$ where y^0 is a new variable making t_b to type 0 and in the fourth case we proceed with the terms $t_c, t_d y^0, t_e y^0$. Note that we can code the variables x and y in on type 0 variable. Also note that since we applied all Cond-reductions only Cond_0 occurs.

By the strong normalization theorem this process stops, yielding the desired term, see e.g. [12]. \square

Theorem 21 (term-normalization for degree 3). *Now let G_i be constants of degree ≤ 3 . For every term $t^1 \in T_0[(\text{Cond}_\rho)_{\rho \in \mathbf{T}}, G_0, \dots, G_{n-1}]$ there is provably in $\text{qf-N-G}_3\text{A}^\omega$ a term $\tilde{t} \in T_0[\text{Cond}_0, G_0, \dots, G_{n-1}]$ for which*

$$\forall x \ tx =_0 \tilde{t}x$$

and where every occurrence of an G_i is of the form

$$G_i(\tilde{t}_0[f^1], \dots, \tilde{t}_{k-1}[f^1]).$$

Here k is the arity of G_i , and $\tilde{t}_j[f^1]$ are fixed terms whose only free variable is f^1 .

Proof. Analogous to proof of theorem 20. See also [29, proof of proposition 4.2]. \square

Note that the equality between t, \tilde{t} is only pointwise. Therefore one needs (weak) extensionality to conclude that $s[t] =_0 s[\tilde{t}]$ for an arbitrary term s .

Application to proofs in quantifier-free systems. For a term t call the term where every maximal type 0 subterm is replaced by a fresh type 0 variable *skeleton*. Obviously, t can be regained from its skeleton by substitution of type 0 terms.

Lemma 22. *Let \mathcal{T} be $\text{qf-N-G}_n\text{A}^\omega$ with $n \geq 3$ or $\widehat{\text{qf-N-PA}^\omega}$ augmented with arbitrary constants H_0, H_1, \dots , let $t_0, t_1 \in T_0[\text{Cond}_0, H_0, H_1, \dots]$ and in t_0, t_1 all possible Π -, Σ -reductions have been carried out.*

Then the following are equivalent:

- (i) *The terms t_0, t_1 are provable equal in every term context ($\mathcal{T} \vdash s[t_0] =_0 s[t_1]$ for every term s).*
- (ii) *$\mathcal{T} \vdash P(t_0) =_0 P(t_1)$, where P is a variable of suitable type.*
- (iii) *The terms t_0, t_1 have the same skeleton (modulo renaming of type 0 variables) and t_0, t_1 are obtained from the skeleton by substitution of $=_0$ -equal terms.*

Proof. (i) \Rightarrow (ii) is clear. (ii) \Rightarrow (i) follows from the fact that one can replace P by any term in the derivation and so in particular by $\lambda x.s[x]$. By definition of the axioms of a quantifier-free system the axioms of this new derivation are also in \mathcal{T} . (iii) \Rightarrow (i) follows from the $=_0$ -axioms.

For (ii) \Rightarrow (iii) observe that the only way to prove the equality in (ii) are the SUB rule, the SUB_{Cond} rule for Cond_0 , or the $=_0$ -axioms. The Π -, and Σ -reductions commute with applications of $=_0$ -axioms and in t_0, t_1 all possible Π - and Σ -reductions have been carried out we may assume that only the $=_0$ -axioms, SUB_{Cond}-axioms for Cond_0 , and the SUB-axioms for R_0 are used. These axioms only change type 0 values and, therefore, the skeletons have to be the same. The lemma follows. \square

Proposition 23. *Let \mathcal{T} be $\text{qf-N-G}_n\text{A}^\omega$ where $n \geq 3$ or $\widehat{\text{qf-N-PA}^\omega}$ augmented by a type 2 constant F . Further let A be a formula containing only type 0 variables free and satisfying $\mathcal{T} \vdash A$.*

Then there exists a formula \tilde{A} such that the weakly extensional intuitionistic system \mathcal{T}_{WE} corresponding to \mathcal{T} (i.e. $\text{G}_n\text{A}_1^\omega$ or $\widehat{\text{WE-HA}^\omega}$) proves $A \leftrightarrow \tilde{A}$ and that

there is a derivation $\tilde{\mathcal{D}}$ of $\mathcal{T} \vdash \tilde{A}$ where every occurring term is normalized according to theorem 20, i.e. each occurrence of F is of the form $F(t_i[x])$.

Moreover, these applications of F can be chosen independently from each other in the sense that

$$\mathcal{T} \not\vdash P[F(t')] =_0 P[F(t'')] \quad \text{for a fresh variable } P$$

for all type 0 substitution instances t', t'' of t_i resp. t_j with $i \neq j$. (In other words, the theory \mathcal{T} does not see that the $F(t'), F(t'')$ are applications of F and not just an arbitrary term of suitable type and with the same free variables. Hence they may be replaced independently.)

Using coding we may also allow finitely many constants F_i of degree 2.

Proof. Let \mathcal{D} be a derivation of $\mathcal{T} \vdash A$. By lemma 4 we may assume that only the variables of A and some free type 0 variables occur in \mathcal{D} . Hence every term showing up in \mathcal{D} satisfies the premise of theorem 20.

We obtain a new derivation $\tilde{\mathcal{D}}$ by replacing every term in \mathcal{D} with its normal form as defined in the proof of theorem 20 (in particular all possible Π -, and Σ -reductions have been carried out and only Cond_0 occurs in \tilde{t}). The derivation $\tilde{\mathcal{D}}$ is still valid because the used logical axioms and rules, SUB-axioms for the recursor and Cond , $=_0$ -axioms, and quantifier-free induction rule are translated into other instances of themselves. The used SUB-axioms for Π and Σ become trivial since in all terms all possible Π - and Σ -reductions have been carried out.

Let \tilde{A} be the result of $\tilde{\mathcal{D}}$. Each term occurring in \tilde{A} is just the normal form of the term at the same position in A and therefore weakly extensional equal to it. Hence

$$\mathcal{T}_{\text{WE}} \vdash A \leftrightarrow \tilde{A}.$$

Obviously, the derivation $\tilde{\mathcal{D}}$ contains only finitely many applications t_i of F . Each of the t_i contains only type 0 variables free. However, these applications of F are not independent from each other because there might be equalities between them provable in \mathcal{T} .

Passing to the skeletons of t_i we obtain applications of F which are by lemma 22 pairwise independent or literally equal and which still contain only type 0 parameters. \square

Remark 24. If in the above theorem one adds a type 3 constant G instead of F to the system and uses theorem 21 instead of 20 one obtains a similar result with the exception that the applications t_i now also depend on function variables f_i . (These variables result from the normalization defined in theorem 21. They can be coded together into one variable f such that the derivation $\tilde{\mathcal{D}}$ may be contains only the variables occurring in \tilde{A} plus some fresh type 0 variables.)

5. ELIMINATION OF MONOTONE SKOLEM FUNCTIONS

Let Δ be a set of sentences of the form $\forall a \exists b < ra \forall c^0 B_{qf}(a, b, c)$, where r is a closed term and B_{qf} is quantifier-free and contains any further free variables than those shown. Let $\tilde{\Delta}$ be the corresponding set of Skolem normal form of the sentence of Δ , i.e. the corresponding formulas of the form $\exists B < r \forall a, c^0 B_{qf}(a, Ba, c)$.

Theorem 25 ([27, 3.8]). *Let γ be an arbitrary type and let A_{qf} be a quantifier-free statement where only the shown variables are free and let s be a term in $G_\infty \mathbf{R}^\omega$. If*

$$G_\infty \mathbf{A}^\omega + \text{QF-AC} \oplus \Delta \vdash \forall u^1 \forall v \leq_\gamma su \ (\Pi_1^0\text{-CA}(\xi uv) \rightarrow \exists w^0 A_{qf}(u, v, w))$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\text{WE-HA}^\omega} \upharpoonright \oplus \tilde{\Delta} \vdash \forall u^1 \forall v \leq_\gamma su \exists w \leq_0 tu A_{qf}(u, v, w).$$

Epecially, in case that $A_{qf} \in \mathcal{L}(\text{PRA})$, u of type 0, v absent and $\Delta = \emptyset$ we have

$$\text{PRA} \vdash \forall u^0 A_{qf}(u, tu).$$

Corollary 26. *Let γ, ξ, s, A_{qf} be as in theorem 25. However ξ may contain \mathcal{B} but s and A_{qf} must not. Then the following holds: If*

$$G_\infty A^\omega + \text{QF-AC} \oplus (\mathcal{B}) \vdash \forall u \forall v \leq_\gamma su \left(\Pi_1^0\text{-CA}(\xi uv) \rightarrow \exists w^0 A_{qf}(u, v, w) \right)$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\text{WE-HA}^\omega} \upharpoonright \vdash \forall u^1 \forall v \leq_\gamma su \exists w \leq_0 tu A_{qf}(u, v, w).$$

Proof. First note that due to [27, remark 2.10] we may add the (majorizable) constant \mathcal{B} to $G_\infty A^\omega$ in theorem 25.

Apply this theorem to $\Delta := \{\forall f \forall x \check{f}(\mathcal{B}\check{f}x) = 0\}$, cf. definition 6 and (3) on p. 11. The premise of the corollary implies that

$$G_\infty A^\omega + \text{QF-AC} \oplus \Delta \vdash \forall u \forall v \leq_\gamma su \left(\Pi_1^0\text{-CA}(\xi uv) \rightarrow \exists w^0 A_{qf}(u, v, w) \right).$$

Theorem 25 and noticing that $\Delta \equiv \tilde{\Delta}$ yields

$$\widehat{\text{WE-HA}^\omega} \upharpoonright \oplus \Delta \vdash \forall u^1 \forall v \leq_\gamma su \exists w \leq_0 tu A_{qf}(u, v, w)$$

and so

$$\widehat{\text{WE-HA}^\omega} \upharpoonright \vdash \Delta \rightarrow \forall u^1 \forall v \leq_\gamma su \exists w \leq_0 tu A_{qf}(u, v, w).$$

Since the constant \mathcal{B} only occurs in Δ , we may replace it with a new variable and so replace Δ with UWKL. The corollary now follows from [32, corollary 10.34]. \square

6. RESULTS FOR THE COHESIVE PRINCIPLE

Our goal is now to interpret consequences (of the form $\forall x^1 \exists y^0 A_{qf}(x, y)$) of a principle \mathcal{P} that is proofwise low in sequence. For this we will strengthen \mathcal{P} to the statement that there exists a uniform solution functional \mathcal{P} for \mathcal{P} . The functional \mathcal{P} must be of degree ≤ 2 , such that after extracting terms using the functional interpretation one can normalizing them with the tools of Section 4. With this we will see that \mathcal{P} is only used finitely many times and can be replaced using the lowness property in favor of an instance of $\Pi_1^0\text{-CA}$.

The properties of the solution functional \mathcal{P} must be axiomatizable universally, since they will become an implicative assumption. After prenexation they will become purely existential and the functional interpretation will extract terms witnessing them. Existential quantifier in the axiomatization of \mathcal{P} would become universal after prenexation and therefore would need to be presented afterward.

If \mathcal{P} is of the form

$$(12) \quad \forall S \exists G \underbrace{\forall x P_{qf}(S, G, x)}_{\equiv: P(S, G)},$$

where P_{qf} is quantifier-free. Then one can take for \mathcal{P} the Skolem functional for G , i.e. a functional \mathcal{P} satisfying

$$\forall S \forall x P_{qf}(S, \mathcal{P}(S), x).$$

With the help of the following lemma we can obtain a functional for \mathcal{P} where P is a Π_3^0 formula. This is sufficient for StCOH .

Lemma 27. *Let \mathcal{P} be a principle proofwise low in sequence over $G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL}$, that has the form*

$$(13) \quad (\mathcal{P}): \forall S \exists G \underbrace{\forall x \exists y \forall z P_{qf}(S, G, x, y, z)}_{\equiv: P(S, G)},$$

where P_{qf} is quantifier-free.

Then the principle

$$(14) \quad (\mathcal{P}') : \forall S \exists G, Y \forall Z^1 \forall x \ P_{qf}(S, G, x, Y(x, Z), Z(Y(x, Z)))$$

is proofwise low in sequence, in the sense that for every closed term φ a closed term ξ exists, such that $\Pi_1^0\text{-CA}(\xi(S_i)_i(Z_i)_i)$ proves

$$\begin{aligned} \exists (G_i)_i, (Y_i)_i (\forall i, Z', x \ P_{qf}(S_i, G_i, x, (Y_i)_i(x, Z'), Z'(Y_i(x, Z')))) \wedge \\ \Pi_1^0\text{-CA}(\varphi(S_i)(Z_i)(G_i)(\lambda x. Y_i(x, Z_i)_i)). \end{aligned}$$

Proof. The lowness of \mathcal{P} provides that for every term φ' an instance of Π_1^0 -comprehension $\Pi_1^0\text{-CA}(\xi SZ)$ proves

$$\exists G (\forall x^0 \exists y^0 \forall z^0 \ P_{qf}(S, G, x, y, z) \wedge \Pi_1^0\text{-CA}(\varphi' SZG)).$$

Hence it also proves

$$\exists G (\forall x, Z \exists y \ P_{qf}(S, G, x, y, Z(y)) \wedge \Pi_1^0\text{-CA}(\varphi' SZG)).$$

By searching for the least y we may assume that there exists a unique y for each x, Z . Let $Y(x, Z)$ be the choice function for y obtained using QF-AC. To show that \mathcal{P}' is proofwise low it suffices to show for every closed φ that there is a closed φ' (and thus a closed ξ) such that $\Pi_1^0\text{-CA}(\varphi SZG(\lambda x. Y(x, Z)))$ is provable from $\Pi_1^0\text{-CA}(\varphi SZ)$.

Since Y is computable in S, G a suitable φ can easily be constructed with the same generic construction used in the proof of lemma 13.

One also easily verifies that the whole argumentation is stable under sequences and hence that \mathcal{P}' is proofwise low in sequence. \square

It is easy to see that \mathcal{P}' is equivalent to \mathcal{P} over QF-AC^{0,0}. For such principle we could then use a solution functional $\mathcal{P} = (\mathcal{P}_G, \mathcal{P}_Y)$ that codes together the Skolem functions for G, Y in (14), i.e.

$$(15) \quad \forall S \forall Z \forall x \underbrace{P_{qf}(S, \mathcal{P}_G(S), x, \mathcal{P}_Y(G, x, Z), Z(\mathcal{P}_G(G, x, Z)))}_{\equiv: P_S(\mathcal{P}, (Z, x))}.$$

Proposition 28. Let $A_{qf} \in \mathcal{L}(G_\infty A^\omega)$ be a quantifier-free formula that contains only the shown variables free and let \mathcal{P} be a principle proofwise low in sequence over $G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL}$ of the form (13). If

$$\widehat{\text{E-PA}^\omega} \upharpoonright + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} + \Pi_1^0\text{-CP} + \mathcal{P} + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y),$$

then one can find a term ξ such that

$$G_\infty A^\omega + \text{QF-AC} \oplus (\mathcal{B}) \vdash \forall x^1 (\Pi_1^0\text{-CA}(\xi x) \rightarrow \exists y^0 A_{qf}(x, y)).$$

Proof. We first prove the proposition without $\Pi_1^0\text{-CP}$.

Note that due to

- the deduction theorem (which holds for $\widehat{\text{E-PA}^\omega} \upharpoonright$),
- the elimination of extensionality (proposition 7),
- the strengthening of WKL to UWKL and
- the strengthening of \mathcal{P} to the Skolem normal-form of \mathcal{P}' , i.e. the statement there exists an \mathcal{P} satisfying (15)

we obtain

$$\text{N-}G_\infty A^\omega + \text{QF-AC} \vdash (\exists \mathcal{P} \forall u^1 P_S(\mathcal{P}, u)) \wedge (R_0) \wedge (\mathcal{B}) \rightarrow \forall x^1 \exists y^0 A_{qf}(x, y),$$

where u codes the pair (Z, x) from (15) and (R_0) are the defining axioms for the recursor R_0 . Note that also the formulas (R_0) , (\mathcal{B}) can be written in the form $\exists R_0 \forall u^1 (R_0)_{qf}(R_0, u)$ resp. $\exists \mathcal{B} \forall u^1 (\mathcal{B})_{qf}(\mathcal{B}, u)$ for quantifier free $(R_0)_{qf}$, $(\mathcal{B})_{qf}$.

Applying the functional interpretation to this yields terms $t_y, t_P, t_{R_0}, t_{\mathcal{B}} \in G_\infty R^\omega$ such that

$$(16) \quad \text{qf-N-}G_\infty A^\omega \vdash (P_S(\mathcal{P}, t_P(x, \mathcal{P}, R_0, \mathcal{B})) \wedge (R_0)_{qf}(R_0, t_{R_0}(x, \mathcal{P}, R_0, \mathcal{B})) \wedge (\mathcal{B})_{qf}(\mathcal{B}, t_{\mathcal{B}}(x, \mathcal{P}, R_0, \mathcal{B})) \rightarrow A_{qf}(x, t_y(x, \mathcal{P}, R_0, \mathcal{B}))),$$

see [50, 26].

The terms $t_P(x, \mathcal{P}, R_0, \mathcal{B})$, $t_{R_0}(x, \mathcal{P}, R_0, \mathcal{B})$, $t_{\mathcal{B}}(x, \mathcal{P}, R_0, \mathcal{B})$, $t_y(x, \mathcal{P}, R_0, \mathcal{B})$ have degree ≤ 1 . By proposition 23 we obtain a new derivation in $\text{qf-N-}G_\infty A^\omega$ of a sentence which is equivalent to (16) over $\text{qf-}G_\infty A^\omega$ and where each application of \mathcal{P} is of the form $\mathcal{P}(t_i[z^0])$ or a substitution instance of $\mathcal{P}(t_i[z^0])$ and $\mathcal{P}(t_j[z^0])$ and $\mathcal{P}(t_j[z^0])$ are independent (in the sense of proposition 23). Same for R_0, \mathcal{B} .

Our goal is now to replace these occurrences of \mathcal{P} , R_0 , and \mathcal{B} in the normalized derivation of (16) by a low solution to those principles, such that the premise of (16) becomes provable.

We proceed by inductively over the nesting-depth of \mathcal{P} , R_0 , and \mathcal{B} replacing the applications (and their substitution instances) with low solutions retaining the instance of comprehension. This operation leaves the derivation valid since the different applications are independent. Concretely we replace $\mathcal{P}, R_0, \mathcal{B}$ by the following:

- $R_0(t_i[z^0])$ simply defines a primitive recursive function, which is provably total using an instance of Σ_1^0 -induction. This instance can be obtained from QF-IA and an instance of Π_1^0 -comprehension. Then lemma 13 yields a new instance of comprehension (which allows $R_0(t_i[z^0])$ as parameter).
- $\mathcal{P}(t_i[z^0])$ can be handled using the assumption that \mathcal{P} is proofwise low in sequence (lemma 27)
- $\mathcal{B}(t_i[z^0])$ can trivially be handled because it is present in the verifying system.

For the construction of these replacements we work in the system $G_\infty A^\omega$, i.e. with weak extensionality and quantifiers. After this the premise of (16) becomes provable. Quantifying over all x and coding x, z together into a new variable x , yields the proposition without Π_1^0 -CP.

To prove the full proposition note that we can add StCOH to the system since it is proofwise low in sequence, see corollary 19, and that StCOH implies Π_1^0 -CP, see proposition 14. This completes the proof. \square

Theorem 29 (Conservation for proofwise low in sequence). *Let \mathcal{P} be a principle of the form (13) that is proofwise low in sequence over $G_\infty A^\omega + \text{QF-AC} \oplus \text{WKL}$. In particular, this includes all principles of this form proofwise low in sequence over WKL_0^* .*

If

$$\widehat{\text{E-PA}^\omega} \upharpoonright + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} + \Pi_1^0\text{-CP} + \mathcal{P} + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

then one can extract a primitive recursive term t such that

$$\widehat{\text{WE-HA}^\omega} \upharpoonright \vdash \forall x^1 A_{qf}(x, tx).$$

Especially if $A_{qf} \in \mathcal{L}(\text{PRA})$ and x is of type 0 we have

$$\text{PRA} \vdash \forall x A_{qf}(x, tx).$$

Proof. We assume that $A_{qf} \in \mathcal{L}(G_\infty A^\omega)$. Otherwise it would contain R_0 . If this is the case we normalize every term occurring in A_{qf} and replace every occurrence of R_0uvw by a fresh variable that will be \exists -quantified. There are no other occurrence of R_0 in A_{qf} since it contains (beside Π, Σ) no constant of degree > 2 . These fresh

variables will hold the value of R_0uvw . This values exists provably with Σ_1^0 -IA and can be expressed in a quantifier-free way.

Apply now elimination of Skolem function for monotone formulas (corollary 26) to the result of proposition 28. \square

Corollary 30. *Especially from a proof of*

$$\widehat{\text{E-PA}}^\omega \upharpoonright + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} + \Pi_1^0\text{-CP} + \text{COH} + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

one can extract a primitive recursive term t such that

$$\widehat{\text{WE-HA}}^\omega \upharpoonright \vdash \forall x^1 A_{qf}(x, tx).$$

Proof. Theorem 29 and corollary 19. \square

Corollary 31. *The system*

$$\text{WKL}_0^\omega + \Pi_1^0\text{-CP} + \text{COH}$$

is Π_2^0 -conservative over PRA. Additionally, for every Π_2^0 -sentence one can extract uniformly a primitive recursive (provably) realizing term.

Further $\text{WKL}_0^\omega + \Pi_1^0\text{-CP} + \text{COH}$ is Π_3^0 -conservative over RCA_0^ω .

Proof. The first statement is clear from the preceding corollary and the definition of WKL_0 . The second statement follows also from this corollary by noting that over $\text{QF-AC}^{0,0}$ a Π_3^0 -formula

$$\forall x^0 \exists y^0 \forall z^0 A_{qf}(x, y, z)$$

is equivalent to

$$\forall x^0, Z^1 \exists y^0 A_{qf}(x, y, Zy).$$

\square

This in some sense is the best possible result since $\text{RCA}_0 + \Pi_1^0\text{-CP}$ is not Σ_3^0 -conservative over a theory containing only Σ_1^0 -induction, see [1].

7. STABLE RAMSEY'S THEOREM FOR PAIRS (SRT_2^2)

An n -coloring $c: [\mathbb{N}]^2 \rightarrow n$ is called *stable* if

$$\forall x \exists k \forall y > k \ c(x, k) = c(x, y).$$

The point k is called a *stability point* for x .

We call an n -coloring *strongly stable* if

$$\forall x \exists k \forall y > k \forall x' \leq x \ c(x', k) = c(x', y).$$

Over $\Pi_1^0\text{-CP}$ strongly stable and stable coincide. Even an instance of the collection principle of the form $\Pi_1^0\text{-CP}(\xi c)$ where ξ is a suitable term and c the coloring suffices to prove this equivalence.

Let SRT_n^2 be the statement expressing that every *stable* n -coloring of pairs has an infinite homogeneous set and let $\text{SRT}_{<\infty}^2 := \forall n \text{SRT}_n^2$. For a stable n -coloring c the statement $\text{SRT}_n^2(c, H)$ denotes that H is a homogeneous set for c .

The principle SRT_2^2 is over Σ_1^0 -induction equivalent to the statement that for every Δ_1^0 -set X there exists an infinite set Y with $Y \subseteq X$ or $Y \subseteq \bar{X}$, see [7, 8].

Before we go on with the main result we need some auxiliary lemmata:

Lemma 32 ([7, lemma 4.2]). *For every fixed n , let $(\xi_{k,i})_{k < n, i \in \mathbb{N}}$ be a sequence of Π_1^0 -sentences of the form $\xi_{k,i} \equiv \forall x A(k, i, x)$ for a quantifier-free A such that $\forall i \exists k < n \ \xi_{k,i}$. Then WKL proves that there exists a choice function $g: \mathbb{N} \rightarrow n$ satisfying $\forall i \ \xi_{g(i),i}$.*

If WKL is replaced by $\Sigma_1^0\text{-WKL}$ the same holds for Π_2^0 -sentences.

Proof. Define

$$f(\langle x_0, \dots, x_n \rangle) = 0 \quad \text{iff} \quad \bigwedge_{i \leq n} \xi_{x_i, i}.$$

The function f clearly defines a Π_1^0 -0- n -tree and is by assumption infinite.

Via the equivalence of 0- n -trees and 0/1-trees and of Π_1^0 -WKL and WKL (see [44]), weak König's lemma yields a infinite branch g solving the lemma. \square

Lemma 33 (and definition, Π_1^0 -class, [22]). *A Π_1^0 -class \mathcal{A} of 2^ω is a set of functions of the form*

$$\mathcal{A} = \{f \in 2^\omega \mid \forall n \, A(\bar{f}n)\},$$

where A is a quantifier-free formula.

WKL proves that a Π_1^0 -class \mathcal{A} is not empty if

$$(17) \quad \forall n^0 \exists s \in 2^n \forall s' \sqsubseteq s \, A(s').$$

(The definition of Π_1^0 -class induces an infinite tree in which every $f \in \mathcal{A}$ codes an infinite path through it.) The statement (17) is equivalent to a Π_1^0 -statement.

Note that one may also allow A to be a Π_1^0 -formula as the \forall -quantifier can be coded into the quantification over n (see for instance [44]).

Remark 34 (Treatment of Π_1^0 -0/1-trees). Let $T(w) := (\forall k \, T_{qf}(w, k) = 0)$ be a Π_1^0 -predicate. Using the UWKL functional \mathcal{B} one can define the functional

$$\mathcal{B}_{\Pi_1^0}(T_{qf}) := \mathcal{B} \left(\min_{w' \sqsubseteq w, k \leq \text{lh } w} T_{qf}(w', k) \right)$$

that yields an infinite branch of T , if T defines an infinite 0/1-tree.

Furthermore, an instance of Π_1^0 -CA decides whether the tree T is infinite, since

$$\forall n \exists w \in 2^n \forall k \, T_{qf}(w, k)$$

is equivalent a Π_1^0 -statement (over $G_\infty A^\omega + \text{QF-AC}$).

Hence one can treat Π_1^0 -0/1-trees mostly like quantifier free trees.

Proposition 35.

$$G_\infty A^\omega + \text{QF-AC} \vdash \forall c: \mathbb{N} \times \mathbb{N} \rightarrow 2 \, (\Pi_1^0\text{-CA}(\xi c) \rightarrow \exists H \, \text{SRT}_2^2(c, H)),$$

where ξ is a suitable term.

Proof. Assume that the coloring c is stable. Define for $i < 2$

$$A_i := \{x \mid \forall k \exists y \geq k \, c(x, y) = i\}.$$

By stability

$$A_i = \{x \mid \exists k \forall y \geq k \, c(x, y) = i\}.$$

Hence each A_i is a Δ_2^0 -set.

At least for one i the set A_i is infinite (by RT_2^1). Fix such an i . With an instance of Π_1^0 -CP we obtain strong stability, i.e.

$$\forall x \exists k \forall y > k \forall x' \leq x \, c(x', k) = c(x', y).$$

This instance of Π_1^0 -CP follows from a suitable instance of Π_1^0 -CA, see lemma 12.(iv).

Together with the infinity of A_i we get

$$\forall x \exists k \in A_i \forall x' \leq x \, (x' \in A_i \rightarrow c(x', k) = i).$$

Define the set H inductively by

$$x \in H \quad \text{iff} \quad x \in A_i \text{ and } c(x', x) = i \text{ for all } x' < x \text{ with } x' \in H.$$

This definition only uses bounded course-of-value recursion in the characteristic function of A_i which can be obtained from a suitable instance of Π_1^0 -CA, see

lemma 12.(ii). (The characteristic function χ_H of H is clearly bounded and hence also its course-of-value function $\overline{\chi_H}$, which is actually defined in the recursion.)

The set H is clearly infinite and homogeneous. (The two instances of Π_1^0 -CA can be coded into one term ξ , see remark 11.) \square

Proposition 36. *Let φcH be a term that is provably continuous in H , where $\alpha_{\varphi c}(\cdot, n, k)$ is an associate for $\lambda H. \varphi(c, H, n, k)$. Then there exists a term ξ , such that*

$$\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ \forall c: \mathbb{N} \times \mathbb{N} \rightarrow 2 \left(\Pi_1^0\text{-CA}(\xi c) \rightarrow \exists H \text{SRT}_2^2(c, H) \wedge \Pi_1^0\text{-CA}(\varphi cH) \right).$$

If φcH is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous.

Sketch of proof. We assume that each A_i is unbounded, otherwise we are done.

We will build a set G such that $G \cap A_0$ and $G \cap A_1$ are infinite, homogeneous and at least for one $i < 2$ the comprehension $\Pi_1^0\text{-CA}(\varphi c(G \cap A_i))$ is decided. The set $H := G \cap A_i$ then solves this proposition.

We will construct the set G in steps such that at each step n we will assure that

$$|G \cap A_i| \geq n \quad \text{for every } i < 2$$

and for some $i < 2$ the comprehension for $G \cap A_i$ at the position $(n)_i$ will be decided, i.e. whether the statement

$$(18) \quad \forall k (\varphi c(G \cap A_i)(n)_i)k = 0$$

holds. More precisely, we will construct functions $I, J: \mathbb{N} \rightarrow 2$, such that

$$\exists I, J \forall n \left(\forall k (\varphi c(G \cap A_{I(n)})(n)_{I(n)})k = 0 \leftrightarrow J(n) = 0 \right).$$

With these functions we can then obtain a comprehension function for one of the sets $G \cap A_i$, because either

$$(19) \quad \forall m \exists n \left(m = (n)_{I(n)} \wedge I(n) = 0 \right)$$

and then $J(N(m))$, where $N(m)$ is some choice function for n obtained by QF-AC, decides the comprehension for $G \cap A_0$ or

$$(20) \quad \exists m \forall n \left(m \neq (n)_{I(n)} \vee I(n) = 1 \right).$$

By choosing $n = \langle m, m' \rangle$ we obtain $\forall m' I(\langle m, m' \rangle) = 1$ and therefore the function $\lambda m'. J(\langle m, m' \rangle)$ decides the comprehension for $G \cap A_1$.

The set G and the functions I, J will be constructed by recursion. We will first give a sketch of the argument and later show that R_0 and the imposed comprehension suffice for the construction.

By induction we construct (d_n, L_n) , such that the sequence (d_n) is an ascending sequence of finite sets and (L_n) is a descending sequence of infinite sets of possible candidates to extend d_n (i.e. $d_{n+1} \setminus d_n \subset L_n$ and $\min(L_n)$ is greater than the stability point of d_n). Each set L_n is *low*, in the sense that it can be described by a term containing \mathcal{B} and \tilde{R}_1 . The set G will be given by $\bigcup_n d_n$.

We start with (\emptyset, \mathbb{N}) . Assume (d_n, L_n) is already defined. We distinguish two cases:

Case i) A partition Z_0 and Z_1 of L_n exists such that

$$(21) \quad \forall z \subseteq^{fin} Z_i \left(z \text{ is } i\text{-homogeneous} \rightarrow \forall k \alpha_{\varphi c}(d_n^i \cup z)(n)_i k \leq 1 \right),$$

where $d_n^i = d_n \cap A_i$, holds for all $i < 2$. (If we extend the initial segment d_n with elements from Z_i the comprehension remains true.)

At least one of Z_0 and Z_1 is infinite because L_n is infinite. We take this set as L_{n+1} , forcing (18) to be true for this i on all further extensions and let $d_{n+1} := d_n$.

Case ii) No partition satisfying (21) exists.

We know then that especially $L_n \cap A_0$ and $L_n \cap A_1$ is no such partition. So we can find for one i a finite i -homogeneous set $d' \subseteq^{fin} A_i$ such that

$$\exists k \alpha_{\varphi_c}(d_n^i \cup d')(n)_i k > 1.$$

Setting $d_{n+1} := d_n \cup d'$ and $L_{n+1} := \{x \in L_n \mid x > \max d'\}$ forces the comprehension function to be $\neq 0$ at $(n)_i$.

Note that (21) defines a Π_1^0 -class of 2^ω . (We view here a partition of \mathbb{N} into two sets Z_0, Z_1 as a function $f \in 2^\omega$ with $f(n) = i$ iff $n \in Z_i$.) Thus we may assume that the Z_i are low and we can decide which case holds by asking if a certain 0/1-tree is infinite (this is a Π_1^0 -statement).

The size requirements are met by extending d_{n+1} with suitable elements of L_n .

The set $G := \bigcup_n d_n$ then satisfies the proposition. \square

Proof. Define

$$(22) \quad L^{\langle \rangle}(w) := 0, \\ L^{\langle x_0, \dots, x_{n-1}, (d, k, y) \rangle}(w) := \begin{cases} 1 & \text{if } w \leq y, \\ sg \left| \mathcal{B}_{\Pi_1^0}(\theta(L^{\langle x_0, \dots, x_{n-1} \rangle}, d)) - (k-1) \right| & \text{if } k \geq 1 \text{ and } w > y, \\ L^{\langle x_0, \dots, x_{n-1} \rangle} w & \text{if } k = 0 \text{ and } w > y. \end{cases}$$

(d is just an auxiliary parameter used to build the tree, it will be set to d_{n-1} defined below; k denotes the case, $k = 0$ for case ii), $k \geq 1$ for case i) and Z_{k-1} infinite in the sketch; y is a lower bound for L .)

Here $\theta(B, d^0, d^1)wk$ will be the characteristic function of the predicate

$$(23) \quad \forall i < 2 \quad \forall y \subseteq^{fin} B \cap \{x < \text{lth}(w) \mid (w)_x = i\} \\ (y \text{ is } i\text{-homogeneous} \rightarrow \alpha_{\varphi_c}(d^i \cup y)(n)_i k \leq 1),$$

where the variables w, y are numerals coding finite sets. The statement

$$T_{B, d^0, d^1}(w) := \forall k \theta(B, d^0, d^1)wk = 0$$

defines the Π_1^0 -0/1-tree build in (21) in the sketch.

We will write $T_{B, d}$ and $\theta(B, d)wk$ for T_{B, d^0, d^1} resp. $\theta(B, d \cap A_0, d \cap A_1)wk$. This will not lead to problems because $d \cap A_i$ is just a number computable from d relative to the imposed instance of comprehension. Note that L^x can be defined in \mathcal{B} and θ using the bounded iterator \tilde{R}_1 . Thus the function L^x can be described by a term in this system.

We assume that for all x and i the set $L^x \cap A_i$ is infinite if L^x is infinite. Otherwise the set $L^x \cap [k, \infty]$ for a suitable k would be an infinite subset of A_{1-i} and therefore solve the proposition.

Using this and an instance of Δ_2^0 -comprehension (over L) we generate functions g_i such that

$$(24) \quad g_i(x) := \min(L^x \cap A_i).$$

With an application of Π_1^0 -AC and taking a maximum we obtain a function $h(\langle x_1, \dots, x_n \rangle)$ giving a common stability point of x_1, \dots, x_n .

We now define (d_n, l_n) by recursion. (L^{l_n} should match L_n from sketch above.) We use primitive recursion in the sense of Kleene, i.e. the recursion can be defined with the recursors R_0 .

Let $d_0 := \langle \rangle$ and $l_0 := \langle \rangle$. For the recursion step we distinguish the cases:

Case i) The tree $T_{L^{l_n}, d_n}(w)$ is infinite, i.e.

$$\forall m \exists w \in 2^m \forall k \theta(L^{l_n}, d_n)wk = 0.$$

By RT_2^1 there is at least one $j < 2$ such that $\{x \in \mathbb{N} \mid \mathcal{B}_{\Pi_1^0}(\theta(L^{l_n}, d_n))x = j\}$ is infinite. An index j can be chosen constructively relative to Σ_1^0 -WKL, see lemma 32. Set

$$d'_{n+1} := d_n \quad \text{and} \quad k'_{n+1} := j + 1.$$

Case ii) The tree $T_{L^{l_n}, d_n}(w)$ is finite, i.e.

$$\exists m \forall w \in 2^m \exists k \theta(L^{l_n}, d_n)wk \neq 0.$$

Then especially the set A_0 does not code a path through the tree, i.e. for this m

$$\exists k \theta(L^{l_n}, d_n)(\overline{\chi_{A_0}}m)k \neq 0,$$

where χ_{A_0} is the characteristic function of A_0 . So there is an i and a finite i -homogeneous set $y \subseteq^{fin} A_i \cap \{0, \dots, m-1\} \cap L^{l_n}$ such that

$$\exists k \alpha_{\varphi c}(d^i \cup y)(n)_i k > 1.$$

Set

$$d'_{n+1} := d \cup y \quad \text{and} \quad k'_{n+1} := 0.$$

Note that this case distinction is constructive relative to the given instance of comprehension (the second quantifier of the formula is bounded).

Now we extend d'_{n+1} with suitable elements, such that the size requirements are met:

$$\begin{aligned} d_{n+1} &:= d_n \cup \bigcup_{i < 2} \{g_i(l_n * \langle d_n, l', h(d'_{n+1}) + 1 \rangle)\} \\ l_{n+1} &:= l_n * \langle d_n, k'_{n+1}, h(d_{n+1}) + 1 \rangle \end{aligned}$$

Applying RT_2^1 yields an i such that all comprehension instances are decided. From the d_n and the given comprehension one can easily obtain an enumeration of the set $G \cap A_i =: H$.

This solves the proposition. The term ξc is continuous in c because the only discontinuous functional in this system is \mathcal{B} but it is only used to define L^x and to prove WKL. Hence ξ can be chosen such that c does not occur as a parameter to \mathcal{B} . More precisely ξc is of the form $\xi'[t_1 c, \lambda x. L^x]$ with $\xi', t \in T_0$ and therefore continuous. \square

Proposition 37. *Let $\varphi c H$ be a term that is provably continuous in H and let $\alpha_{\varphi c}$ be as in proposition 36. Then there exists a term ξ such that*

$$\begin{aligned} \widehat{\text{WE-PA}}^\omega \upharpoonright + \Sigma_2^0\text{-IA} + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ \forall c: \mathbb{N} \times \mathbb{N} \rightarrow n \left(\Pi_1^0\text{-CA}(\xi c) \rightarrow \exists H \text{SRT}_{<\infty}^2(c, H) \wedge \Pi_1^0\text{-CA}(\varphi c H) \right). \end{aligned}$$

If φ is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous in c .

Proof. Analogous to Proposition 36.

The applications of RT_2^1 become applications of $\text{RT}_{<\infty}^1$, which is equivalent to $\Pi_1^0\text{-CP}$ and thus provable using $\Sigma_2^0\text{-IA}$. The 0/1-trees will become 0- n -trees; but these trees can be constructively transformed into 0/1-trees, see [44].

The only difficult part is adopt the assumption that

$$(25) \quad \forall x \forall i < n \left(L^x \text{ infinite} \rightarrow L^x \cap A_i \text{ infinite} \right),$$

which leads to the definition of g_i in (24) because we cannot simply deduce the existence of a solution from the failure of (25).

First note that (25) due to the minimal element parameter (y in (22)) is equivalent to

$$(26) \quad \forall x \forall i < n \ (L^x \text{ infinite} \rightarrow L^x \cap A_i \text{ not empty}).$$

If (25) resp. (26) does not hold, our goal is to find a set L^x on which — provided we neglect colors that do not occur — the assumption holds. This can be done by finding a maximal set $K \subseteq n$, such that there is an x with $L^x \cap \bigcup_{k \in K} A_k$ is empty. Then for all $x' \supseteq x$ and $i \notin K$ the sets $A_i \cup L^x$ are not empty. Thus if we relativize our argumentation to L^x and the colors $n \setminus K$ the condition (25) holds.

To find such a K and x define

$$\eta(\langle s_0, \dots, s_{n-1} \rangle) := \exists x \left(L^x \text{ infinite} \wedge \bigwedge_i (s_i = 0 \rightarrow L^x \cap A_i = \emptyset) \right).$$

η is clearly Σ_3^0 . Finding a minimal tuple $\langle s_0, \dots, s_{n-1} \rangle$ satisfying η yields a suitable solution. A minimal tuple can be obtained using an instance of Σ_3^0 -induction, which is provable from Σ_2^0 -IA and an instance of Π_1^0 -comprehension. \square

Corollary 38. *Let φcH be a term that is provably continuous in H . Then there exists a term ξ such that*

$$(27) \quad \widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ \forall c: \mathbb{N} \times \mathbb{N} \rightarrow n \ (\Pi_1^0\text{-CA}(\xi c) \rightarrow \exists H \text{RT}_2^2(c, H) \wedge \Pi_1^0\text{-CA}(\varphi cH)).$$

The term ξ can be chosen such that c does not occur as a subterm of a parameter of \mathcal{B} .

If Σ_2^0 -IA is added to the system, RT_2^2 may be replaced by $\text{RT}_{<\infty}^2$.

Hence RT_2^2 is proofwise low over $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1)$ and $\text{RT}_{<\infty}^2$ is proofwise low over $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) + \Sigma_2^0\text{-IA}$.

Proof. Let $R_i = \{x \in \mathbb{N} \mid c(i, x) = 0\}$ and let g be a strictly increasing enumeration of a cohesive set for R_i . The coloring $c'(x, y) := c(gx, gy)$ is stable and for each homogeneous set H' of c' the set gH' is homogeneous for c . See [7].

Hence the corollary follows from corollary 18 and proposition 36 resp. proposition 37. \square

Due to the use of Π_2^0 -LEM in the proof of proposition 36 it is not possible to show that RT_2^2 is proofwise low in sequence, see proposition 2. The application of Π_2^0 -LEM is actually hidden in the use of $\text{RT}_2^1 + \Pi_1^0\text{-CA}(t)$ which over $\widehat{\text{WE-HA}}^\omega \upharpoonright$ implies it for suitable terms t .⁷

8. ND-INTERPRETATION

Later we want to use the previous theorem to interpret the usages of RT_2^2 in proofs of $\forall\exists$ -statements. The problem is that with corollary 38 we can only interpret applications of \mathcal{R} to fixed closed terms. We already mentioned that it is not possible to interpret the application of \mathcal{R} to even a term containing only free variable of type 0, because this would imply Π_2^0 -comprehension, see the discussion below proposition 2.

The principle Π_2^0 -LEM is needed in the proof of proposition 36 to show either (19) or (20) holds. But if we go to the functional interpretation (i.e. ND-interpretation) the need for Π_2^0 -LEM vanishes and we can interpret the solutions to the functional

⁷ RT_2^1 is equivalent to $\forall f (\forall n \exists k > n f(k) = 0 \vee \forall n \exists k > n f(k) \neq 0)$. One easily sees that this implies (over an intuitionistic theory) that for a monotone function f the following holds $\forall n f(n) = 0 \vee \exists n f(n) \neq 0$. Since $\forall n f(n) = 0$ is even for monotone f universal for Π_1^0 one obtains $\text{RT}_2^1 \rightarrow \Pi_1^0\text{-LEM}$.

interpretation if it is applied to terms that *may contain free variable of type 1*. By remark 24 this suffices.

For notational simplification we sometimes will not apply the last application of QF-AC to the ND-interpretation. This corresponds to the so-called Shoenfield translation, see [48]. For RT_2^2 we use the formalization

$$\text{RT}_2^2 := \forall c: [\mathbb{N}]^2 \rightarrow 2 \exists H \forall u < v \ c(Hu, Hv) = c(H0, H1).$$

The ND-interpretation then yields

$$(28) \quad \text{RT}_2^{2ND} := \forall c: [\mathbb{N}]^2 \rightarrow 2 \forall U < V \exists H \underbrace{c(H(UH), H(VH)) = c(H0, H1)}_{\equiv: \text{RT}_{2ND}^2(H; c, U, V)}.$$

Here the set H is given as an enumeration, i.e. H is strictly monotone and Hn is the n -th element of H , and $U < V$ is defined pointwise.⁸ Sometimes the parameters c, U, V in $\text{RT}_{2ND}^2(H; c, U, V)$ will be coded into a single parameter.

For the ND-interpretation of Π_1^0 -comprehension we use an ε -calculus like formulation:

$$(29) \quad \Pi_1^0\text{-}\widehat{\text{CA}}(\varphi) := \exists f \forall x, y \underbrace{(\varphi(x, f(x)) = 0 \vee \varphi(x, y) \neq 0)}_{\equiv: (\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi))_{Q_F(f, x, y)}}.$$

This leads to following ND-interpretation (modulo a last application of QF-AC)

$$(\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi))^{ND} \equiv \forall X, Y \exists f \ (\varphi(Xf, f(Xf)) = 0) \vee \varphi(Xf, Yf) \neq 0.$$

Because RT_2^2 and $\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi)$ are only $\forall\exists\forall$ -statements, the ND-interpretation coincides with the no-counterexample interpretation. So one might view a solution to RT_2^{2ND} , i.e. a term $t(c, U, V)$ that yields for every c, U, V a set H that may not be homogeneous in total but for which $c(H0, H1) = c(H(UH), H(VH))$ holds, as a procedure that disproves every possible counterexample to RT_2^2 . Same for $\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi)$.

Proposition 39 ([47], [32, 41]). *The solution to $(\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi))^{ND}$ can be defined with a single use of Φ_0 , this is Spector's bar recursor for type 0:*

$$t_f := \Phi_0 Xu0(\lambda k^0.0), \quad unv := \begin{cases} 1 & \text{if } \varphi(n, Y(v1)), \\ Y(v1) & \text{otherwise.} \end{cases}$$

The bar recursor Φ_0 is defined as in [32]. It is primitive recursively and instance-wise definable in the bar recursor $B_{0,1}$, see definition 46 below.

The statement from corollary 38 spelled out is

$$\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \left(\exists f_\xi \forall x_\xi, y_\xi \left(\Pi_1^0\text{-}\widehat{\text{CA}}(\xi c) \right)_{Q_F(f_\xi, x_\xi, y_\xi)} \rightarrow \exists H \left(\forall u < v \ c(Hu, Hv) = c(H0, H1) \wedge \exists f_\varphi \forall x_\varphi, y_\varphi \left(\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi cH) \right)_{Q_F(f_\varphi, x_\varphi, y_\varphi)} \right) \right).$$

An ND-interpretation leads then to

⁸Officially, quantification over functions like $c: [\mathbb{N}]^2 \rightarrow 2$ or strictly monotone increasing functions like H are not included in our system as primitive notions, but we can enforce the same behavior by quantifying over $c: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $H: \mathbb{N} \rightarrow \mathbb{N}$ and replacing every occurrence of c, H with

$$\tilde{c}(x, y) := \min \left(1, \begin{cases} c(x, y) & \text{if } x < y \\ c(y, x) & \text{otherwise} \end{cases} \right), \quad \tilde{H}(x) := x + \sum_{k \leq x} H(k).$$

Theorem 40 (ND-interpretation of corollary 38). *For every provably continuous (in c, H) term $\varphi \in T_0[\mathcal{B}, \tilde{R}_1]$ a term $\xi \in T_0[\mathcal{B}, \tilde{R}_1]$ (that is continuous in c) exists such that*

$$(30) \quad \widehat{\text{WE-HA}}^\omega \upharpoonright \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \forall f_\xi \forall U < V \forall X_\varphi, Y_\varphi \exists x_\xi, y_\xi \exists H \exists f_\varphi \\ \left((\Pi_1^0\text{-}\widehat{\text{CA}}(\xi c))_{Q_F}(f_\xi, x_\xi, y_\xi) \rightarrow (c(H(UHf_\varphi), H(VHf_\varphi)) = c(H0, H1) \right. \\ \left. \wedge \Pi_1^0\text{-}\widehat{\text{CA}}(\varphi cH))_{Q_F}(f_\varphi, X_\varphi Hf_\varphi, Y_\varphi Hf_\varphi) \right).$$

Moreover, there exist terms $t_{x_\xi}, t_{y_\xi}, t_H, t_{f_\varphi} \in T_0[\mathcal{B}, \tilde{R}_1]$ (with the given parameters) satisfying this formula.

Proof. The system $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}$ has an ND-interpretation into $\widehat{\text{WE-HA}}^\omega \upharpoonright$. This also extends to additions of new constants and universal axioms. See e.g. [3, 32]. \square

The term t_H and t_{f_φ} can be seen as procedures transforming the no-counterexample interpretation of the premise to the no-counterexample interpretation of the conclusion; the terms t_{x_ξ} and t_{y_ξ} yield which instance of the premise is needed to prove the conclusion.

Note that the counter-functions of RT_2^2 and $\Pi_1^0\text{-}\widehat{\text{CA}}$ have access to both t_H and t_{f_φ} . The proof of proposition 50 below will use this.

To show that the no-counterexample interpretation of the conclusion (and hence the conclusion) holds we have to provide an f_ξ that satisfies $(\Pi_1^0\text{-}\widehat{\text{CA}}(\xi c))_{Q_F}(f_\xi, t_{x_\xi}, t_{y_\xi})$. This can be done using $B_{0,1}$, see proposition 39.

Note that here the application of $(\Pi_1^0\text{-}\widehat{\text{CA}}(\varphi))^{ND}$ in the premise is not fully interpreted. We obtain this form by applying logical simplifications after the negative translation. This leads to fixed terms in the second and third parameter of the premise and will reduce the need for the bar recursor $B_{0,1}$ to the rule of $B_{0,1}$.

9. MAJORIZING THE BAR RECURSOR

Definition 41 (bar induction of type 0). Let bar induction of type 0 be

$$(\text{BI}_0): \begin{cases} \forall x^1 \exists n_0^0 \forall n \geq n_0 Q(\overline{x, n}; n) \wedge \\ \forall x^1, n^0 (\forall d Q(\overline{x, n} * d; n+1) \rightarrow Q(\overline{x, n}; n)) \\ \rightarrow \forall x^1, n^0 Q(\overline{x, n}; n), \end{cases}$$

where

$$(\overline{x, n})k := \begin{cases} x(k), & \text{if } k < n, \\ 0, & \text{otherwise,} \end{cases} \quad (\overline{x, n} * d)k := \begin{cases} x(k), & \text{if } k < n, \\ d, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

If Q is restricted to formulas in \mathcal{K} , we write $\mathcal{K}\text{-BI}_0$.

Lemma 42.

$$\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{0,0} \vdash \Pi_1^0\text{-BI}_0$$

Proof. Let $Q(\overline{x, n}; n) \equiv \forall k Q_{gf}(\overline{x, n}; n; k)$. Suppose that $\Pi_1^0\text{-BI}_0$ does not hold, i.e. the premises of $\Pi_1^0\text{-BI}_0$ are true and

$$\exists x_0^1, n_0^0 \neg \forall k_0^0 Q_{gf}(\overline{x_0, n_0}; n_0; k_0),$$

which is equivalent to

$$(31) \quad \exists x_0^1, n_0^0, k_0^0 \neg Q_{gf}(\overline{x_0, n_0}; n_0; k_0).$$

The second premise yields

$$\forall x^1, n^0, k^0 \exists d, k' (\neg Q_{qf}(\bar{x}, \bar{n}; n; k) \rightarrow \neg Q_{qf}(\bar{x}, \bar{n} * d; n + 1; k')).$$

Since the whole statement only depends on an initial segment of x^1 , it can be coded in a type 0 object x'^0 . For instance let $x' := \bar{x}n$ then $\overline{\lambda i.(x')_i}, n = \bar{x}, \bar{n}$.

Using QF-AC^{0,0} we then obtain functions $D(x, n, k)$, $K(x, n, k)$ with

$$(32) \quad \forall x^0, n, k \left(\neg Q_{qf}(\overline{\lambda i.(x)_i}, n; n; k) \rightarrow \neg Q_{qf}(\overline{\lambda i.(x)_i}, n * D(x, n, k); n + 1; K(x, n, k)) \right).$$

Then define using simultaneous course-of-value recursion (n_0, x_0, k_0 are from (31)) the functions D_0, K_0 :

$$\left. \begin{array}{l} D_0(n) := x_0(n) \\ K_0(n) := k_0 \end{array} \right\} \text{ for } n \leq n_0, \quad \left. \begin{array}{l} D_0(n) := D(\overline{D_0}, n, n, K_0(n-1)) \\ K_0(n) := K(\overline{D_0}, n, n, K_0(n-1)) \end{array} \right\} \text{ for } n > n_0.$$

The definition of D_0 and (31),(32) yield

$$\forall n \geq n_0 \neg Q(\overline{D_0}, n; n)$$

and hence a contradiction to the first premise of $\Pi_1^0\text{-BI}_0$. \square

Proposition 43. $\widehat{\text{WE-PA}}^\omega \upharpoonright + \text{QF-AC}^{0,0}$ proves that there exists a majorant $B_{0,1}^*$ of $B_{0,1}$.

Proof. Define $B_{0,1}^*$ like in [32, proof of theorem 11.17]. By the cited proof it suffices to show $\Pi_1^0\text{-BI}_0$. (Note that in that proof Q is a Π_1^0 formula in the case where $\rho = 0$.) Hence the proposition is an immediate consequence of lemma 42. See also [5]. \square

10. ORDINAL ANALYSIS OF TERMS

Ordinal Peano/Heyting arithmetic. In this section we will investigate the strength of induction along ordinals the systems $\widehat{\text{WE-HA}}^\omega \upharpoonright, \widehat{\text{WE-PA}}^\omega \upharpoonright$.

We will code ordinals using the ordinal coding of [14, II.3.a]. (This coding uses the Cantor normal form for ordinals to define primitive recursive codes for ordinals.) For convenience we repeat the definition of ω_k^μ :

$$\omega_0^\mu = \mu \quad \text{and} \quad \omega_{k+1}^\mu = \omega_k^{\omega_k^\mu}$$

Here $k \in \mathbb{N}$ and μ is an arbitrary ordinal number. Let $\text{IA}(\sigma)$ be the scheme of induction along ordinals $< \sigma$.

Theorem 44 ([39]). *The functions and functionals of level 2 that are ordinal recursive (unnested) in an ordering $< \omega_{k+1}^\omega$ are exactly the functions and functionals in T_k .*

Theorem 45 ([14, II.3.18]).

$$\widehat{\text{WE-HA}}^\omega \upharpoonright + \Sigma_{m+k-1}^0\text{-IA} \vdash \Sigma_m^0\text{-IA}(\omega_k^\omega)$$

for every $m, k \in \mathbb{N}$.

Proof. See [14, II.3.18]. This proof is formulated to prove the least element property, which is classically equivalent to induction. But mutatis mutandis this proof also works in Heyting arithmetic. \square

Application to bar recursion. Our goal is now to use the equivalences between ordinal induction and Σ_k^0 -induction and an ordinal analysis of bar recursion to establish conservation results of bar recursion over induction along ω .

Definition 46 (Howard's bar recursor). Define the bar recursor $B_{\rho,\tau}$ as

$$B_{\rho,\tau}AFGt :=_{\tau} \begin{cases} Gt, & \text{if } A[t] < \text{lth } t, \\ Ft(\lambda u^\rho.B_{\rho,\tau}AFG(t * u)), & \text{otherwise,} \end{cases}$$

where $[t] := \lambda x.(t)_x$.

Definition 47 (restricted bar recursor).

$$\Phi'_\tau AFGt :=_{\tau} \begin{cases} Gt, & \text{if } A[t] < \text{lth } t, \\ Ft(\Phi'_\tau AFG(t * 0))(\Phi'_\tau AFG(t * 1)), & \text{otherwise.} \end{cases}$$

The bar recursor Φ'_0 can be used to solve the functional interpretation of WKL, see [18]. (Φ'_τ is the restricted bar recursor schema 1 from there.)

We call a term *semi-closed* if it contains only variables of type ≤ 1 free.

Theorem 48 ([18, 2.2, 2.3]). *Let $\Phi'_0 AFGc$ resp. $B_{0,1} AFGc$ be a semi-closed term and let A, F, G have the computational sizes a, f, g then*

- (i) $\Phi'_0 AFGc$ has computational size $\sigma := (f + g + h)\omega + \omega(h + 1)$,
where $h := \omega a + \omega$ and,
- (ii) $B_{0,1} AFGc$ has computational size $\sigma := \omega^{g+f2^h}$, where $h := \omega a + \omega$.

This equivalence can be proven in $\Sigma_1^0\text{-IA}(\sigma)$.

Proof. See the proofs in [18, 2.2, 2.3]. Note that these proofs actually define a counting function for the computation-tree through transfinite recursion. This recursion is essentially a transfinite primitive recursion over σ . Hence this proof may be carried out in $\Sigma_1^0\text{-IA}(\sigma)$. \square

Remark 49. If we apply the rule of bar recursion to semi-closed, primitive recursive terms (in the sense of Kleene, i.e. terms of computation size ω^n for $n \in \omega$) we obtain a term with computation size $< \omega^{m\omega}$ for an $m \in \omega$ and therefore a term that is provably definable already in $\widehat{\text{WE-HA}}^\omega \upharpoonright_{\omega_2^l}$ for an $l \in \omega$ or in $\widehat{\text{WE-HA}}^\omega \upharpoonright + \Sigma_2^0\text{-IA}$. We can carry out the proof of the equivalence, theorem 48, in the same system, see theorem 44. Hence in each of these systems we can also proof the equivalence of both terms.

If we apply the rule of restricted bar recursion to primitive recursive terms, which contain only free variable of type 0, we even end up with a primitive recursive term.

11. APPLICATION TO RAMSEY'S THEOREM

Proposition 50. *Let $t^1[g]$ be a term such that $\lambda g.t^1[g] \in T_0[\mathcal{R}]$, where \mathcal{R} is a functional solving RT_2^{2ND} , and every occurrence of \mathcal{R} is of the form*

$$\mathcal{R}(t_c[g], t_u[g], t_v[g]).$$

Then there exist terms $t_x, t_y, \xi \in T_0[\tilde{R}_1, \mathcal{B}]$, such that one can inductively replace every occurrence of \mathcal{R} in t with a new term

$$r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g])$$

(here r is a term and $\tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]$ are the results of replacing \mathcal{R} in $t_c[g], t_u[g], t_v[g]$), such that

$$\begin{aligned} \widehat{\text{WE-HA}}^\omega \upharpoonright + \text{QF-AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall g^1, f \left(\Pi_1^0\text{-}\widehat{\text{CA}}(\xi g) \right)_{QF} (f, t_x g, t_y g) \\ \rightarrow \text{RT}_{2ND}^2(r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]); \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]). \end{aligned}$$

The formula RT_{2ND}^2 denotes the quantifier-free part of RT_2^{2ND} , see (28) on p. 29.

Proof. We use theorem 40 to inductively interpret the term t . For convenience we repeat (30), the existential quantified variables are replaced by their realizing terms constructed in that theorem:

$$(33) \quad \widehat{\text{WE-HA}}^\omega \upharpoonright \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \forall f_\xi \forall U < V \forall X_\varphi, Y_\varphi \\ \left((\Pi_1^0\text{-CA}(\xi c))_{QF}(f_\xi, t_{x_\xi}, t_{y_\xi}) \rightarrow c(t_H(U t_H t_{f_\varphi}), t_H(V t_H t_{f_\varphi})) = c(t_H 0, t_H 1) \right. \\ \left. \wedge (\Pi_1^0\text{-CA}(\varphi c t_H))_{QF}(t_{f_\varphi}, X_\varphi(t_H t_{f_\varphi}), Y_\varphi(t_H t_{f_\varphi})) \right)$$

It is clear that in case of $t_c, t_u, t_v \in T_0$, i.e. there are no nested applications of \mathcal{R} , every application of \mathcal{R} in the term t can be interpreted using (33). (Just set $c = t_c$, $U = \lambda f_\varphi. t_u$, $V = \lambda f_\varphi. t_v$ and the others variable to 0.) Using contraction of Π_1^0 -comprehension, see remark 11, a term containing multiple such occurrence of \mathcal{R} can be interpreted.

If the term t_c contains a single occurrence of \mathcal{R} then we first interpret this inner \mathcal{R} but now we will take advantage of φ and set $\varphi, X_\varphi, Y_\varphi$ so that the resulting instance of ND-comprehension suffices to interpret the outer occurrence of \mathcal{R} in t .

Iterating this process allows us to interpret all terms $t \in T_0[\mathcal{R}]$ where every occurrence of \mathcal{R} is of the form $\mathcal{R}(t_c[g], t_u[g], t_v[g])$ with $t_u, t_v \in T_0$.

Now inductively assume that t_u, t_v are terms for which this proposition holds, i.e. there exists terms \tilde{t}_u, \tilde{t}_v equal to t_u, t_v modulo a given instance of ND-comprehension with the parameter H . The problem is now that the instances of comprehension cannot be generated parallel to t_c because they include the parameter H . But we take advantage of the argument t_{f_φ} of U and V . Coding the instances of ND-comprehension together (ND-interpretation of remark 11) we can find $\varphi', X'_\varphi, Y'_\varphi$ such that

$$(\Pi_1^0\text{-CA}(\varphi' c H))_{QF}(f_\varphi, X'_\varphi(H f_\varphi), Y'_\varphi(H f_\varphi))$$

proves the original ND-instance of $\Pi_1^0\text{-CA}$ for φ and those needed for t_u, t_v .

This proves the proposition. \square

Corollary 51 (Extension to R_1, Φ'_0). *The statement of proposition 50 also holds for terms $t^1[g]$ with $\lambda g. t[g] \in T_0[\mathcal{R}, R_1, \Phi'_0] = T_1[\mathcal{R}, \Phi'_0]$, where every occurrence of \mathcal{R} is of the form required in proposition 50 and every occurrence of R_1 or Φ'_0 is of the form*

$$R_1(t_1[g], t_2[g], t_3[g]) \quad \text{resp.} \quad \Phi'_0(t_1[g], t_2[g], t_3[g]).$$

Proof. The proof proceeds like in proposition 50:

To interpret R_1 while retaining the instance of ND-comprehension, we will essentially use a functional interpretation of the proof of lemma 13 (for $n = 1$). First note that $s := R_1(t_1[g], t_2[g], t_3[g])$ defines a type 1 function in $T_1[g]$. Arguing as in lemma 13, it is clear that over $\widehat{\text{WE-PA}}^\omega \upharpoonright$ a suitable instance of $\Pi_1^0\text{-CA}$ with the parameter g proves that s is total ($\forall x \exists y \langle x, y \rangle \in \mathcal{G}_s[g]$, where \mathcal{G}_s is the graph of s). An ND-interpretation of this statement yields that even an instance of the ND-interpretation of $\Pi_1^0\text{-CA}$ is sufficient to prove that s is total. Another instance of ND-comprehension proves the ND-interpretation of the $\Pi_1^0\text{-CA}$ -instance in (8) on p. 14. This instance is modulo the totality of s equivalent to an instance of ND-comprehension with the parameter s . The two instances of ND-comprehension used can be coded together, see remark 11.

The functional Φ'_0 can be replaced by a function in $T_1[g]$, see theorem 48 and remark 49, and hence can also be interpreted. \square

Proposition 52. *Let A_{qf} be a quantifier-free formula that contains only the shown variables free. If*

$$(34) \quad \widehat{\text{N-PA}}^\omega \vdash + \text{QF-AC} + \Sigma_2^0\text{-IA} + \text{RT}_2^2 + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

then one can find a terms $t_y, t_u, t_v, \xi \in T_0[\mathcal{B}, \tilde{R}_1]$ such that

$$\begin{aligned} \widehat{\text{WE-HA}}^\omega \vdash \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \\ \vdash \forall x^1 \forall f \left((\Pi_1^0\text{-CA}(\xi x))_{QF}(f, t_u f x, t_v f x) \rightarrow A_{qf}(x, t_y f x) \right). \end{aligned}$$

Proof. A functional interpretation of the statement (34) yields closed terms resp. term tuples $t_y, t_{R_1}, t_{\mathcal{R}}, t_{\Phi'_0} \in T_0$, such that

$$\begin{aligned} \text{qf-}\widehat{\text{N-PA}}^\omega \vdash \vdash ((R_1)_{ND}(R_1, t_{R_1} R_1 \mathcal{R} \Phi'_0 x) \wedge \text{RT}_{2ND}^2(\mathcal{R}, t_{\mathcal{R}} R_1 \mathcal{R} \Phi'_0 x) \\ \wedge \text{WKL}_{ND}(\Phi'_0, t_{\Phi'} R_1 \mathcal{R} \Phi'_0 x)) \rightarrow A_{qf}(x, t_y R_1 \mathcal{R} \Phi'_0 x). \end{aligned}$$

Here we use that $(\Sigma_2^0\text{-IA})^{ND}$ can be solved by R_1 , see [40].

Apply now proposition 23 and remark 24 to this derivation to normalize it such that only finitely many independent applications of $\mathcal{R}, R_1, \Phi'_0$ occur, where each of them is of the form

$$\mathcal{R}^*(t_1[g], t_2[g], t_3[g]) \quad \text{resp.} \quad R_1(t_1[g], t_2[g], t_3[g]), \quad \Phi'_0(t_1[g], t_2[g], t_3[g])$$

and t_1, t_2, t_3 are semi-closed.

The terms occurring in this normalized derivation can be interpreted using corollary 51. (Applications to literally equal terms are replaced by the same interpretation.)

The instances of ND-comprehension needed for corollary 51 can be coded together in one instance using remark 11. \square

The application of $\Pi_1^0\text{-CA}$ can be interpreted by the rule of bar-recursion — this means we substitute f with a solution t_f to $(\Pi_1^0\text{-CA})^{ND}$:

$$\begin{aligned} \widehat{\text{WE-HA}}^\omega \vdash \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus (B_{0,1}) \\ \vdash \forall x^1 \left((\Pi_1^0\text{-CA}(\xi x))_{QF}(t_f[x], t_u t_f[x]x, t_v t_f[x]x) \rightarrow A_{qf}(x, t_y t_f[x]x) \right) \end{aligned}$$

The term $t_f \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$ is defined as in proposition 39. Note that t_f depends on ξ, t_u, t_v and that it is of type 2 containing only *one application* of $B_{0,1}$ to semi-closed terms defining a type 2 object.

Since t_f solves the instance of comprehension we obtain:

$$\widehat{\text{WE-HA}}^\omega \vdash \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus (B_{0,1}) \vdash \forall x^1 A_{qf}(x, t_y t_f[x]x).$$

The term $t := \lambda x. t_y t_f[x]x \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$, contains only majorizable constants; the majorants to \mathcal{B}, \tilde{R}_1 are trivial and $B_{0,1}$ is essentially majorized by itself, see proposition 43, hence we can find a majorant $t^* \in T_0[B_{0,1}]$ to t containing also only one application of $B_{0,1}$ to semi-closed terms. Now we can apply bounded search to obtain a new realizer t' for y not containing \mathcal{B} or \tilde{R}_1 :

$$t'x := \begin{cases} \text{minimal } y \leq t^*x \text{ with } A_{qf}(x, y), & \text{if such a } y \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Since t' now does not contain \mathcal{B} anymore we may weaken (\mathcal{B}) to UWKL and then eliminate it from the system using a monotone functional interpretation, see [25, 32]. Hence we obtain a term $t' \in T_0[B_{0,1}]$ containing after normalization only

one occurrence of $B_{0,1}$ defining a type 2 object, such that with the rule $R-(B_{0,1})$ of $B_{0,1}$

$$\widehat{\text{WE-HA}}^\omega \upharpoonright \oplus (\tilde{R}_1) \oplus R-(B_{0,1}) \vdash \forall x^1 A_{qf}(x, t'x).$$

Using ordinal analysis of the $B_{0,1}$ -rule (cf. theorem 48 and remark 49) yields a new term t'' definable with ordinal primitive recursion up to ω_2^ω such that

$$\widehat{\text{WE-HA}}^\omega \upharpoonright_{\omega_2^\omega} \oplus (\tilde{R}_1) \vdash \forall x^1 A_{qf}(x, t''x).$$

Combining this with theorem 44 and noting that \tilde{R}_1 is included in $\text{WE-HA}_1^\omega \upharpoonright$ we obtain the following theorem:

Theorem 53 (Conservation for RT_2^2). *If*

$$\widehat{\text{N-PA}}^\omega \upharpoonright + \text{QF-AC} + \Sigma_2^0\text{-IA} + \text{RT}_2^2 + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

then one can extract a term $t \in T_1$ such that

$$\text{WE-HA}_1^\omega \upharpoonright \vdash \forall x^1 A_{qf}(x, tx).$$

11.0.1. *Extension to $\text{RT}_{<\infty}^2$* . Proposition 50 holds analogously for $\text{RT}_{<\infty}^2$ if one adds R_1 and $\Sigma_2^0\text{-IA}$ to the verifying system; corollary 51 holds if one replaces R_1 by R_2 .

But in contrast to the previous the technique used in remark 24 to extract terms that meet the requirements of these propositions can only be applied to terms in $T_1[\mathcal{R}_\infty]$ and not to terms $T_2[\mathcal{R}_\infty]$, because $\deg(R_2) = 4$ and therefore we could not apply the term normalization. The mathematical reason is that R_2 is strong enough to iterate $B_{0,1}$ and \mathcal{R}_∞ .

This will hinder us to achieve full conservativity for full $\Sigma_3^0\text{-IA}$ over a system in all finite types but a restricted variant of $\Sigma_3^0\text{-induction}$ can be handled. Define the rule of $\Sigma_3^0\text{-induction}$ $\Sigma_3^0\text{-IR}$ as

$$(\Sigma_3^0\text{-IR}): \frac{\forall n (\exists x \forall y \exists z A_{qf}(n, x, y, z, \underline{a}) \rightarrow \exists u \forall v \exists w A_{qf}(n+1, u, v, w, \underline{a}))}{\forall n \exists x \forall y \exists z A_{qf}(n, x, y, z, \underline{a})},$$

where A_{qf} is quantifier-free and contains only the variables shown, u, v, w, x, y, z, n are type 0 variables and \underline{a} denotes an arbitrary tuple of parameters. Let $\Sigma_3^0\text{-IR}_2$ be the restriction of $\Sigma_3^0\text{-IR}$ to parameters \underline{a} of degree ≤ 2 then

Theorem 54 (Conservation for $\text{RT}_{<\infty}^2$). *If*

$$(35) \quad \text{N-PA}_1^\omega \upharpoonright + \text{QF-AC} + \Sigma_3^0\text{-IR}_2 + \text{RT}_{<\infty}^2 + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

then one can extract a term $t \in T_2$ such that

$$\text{WE-HA}_2^\omega \upharpoonright \vdash \forall x^1 A_{qf}(x, tx).$$

Proof. The ND-interpretation of the conclusion of $\Sigma_3^0\text{-IR}_2$ is given by

$$\forall n^0 \forall Y^2 \exists x^0, Z^1 A_{qf}(n, x, YxZ, Z(YxZ), \underline{a}^2).$$

One immediately see that $\Sigma_3^0\text{-IR}_2$ introduces only degree 3 terms (t_Z, t_x ranging over $n^0, Y^2, \underline{a}^2$). Hence we can ND-interpret (35) in

$$\text{qf-N-PA}_1^\omega \upharpoonright + (G_1) + \dots + (G_n)$$

where (G_i) are defining axioms and constants of degree ≤ 3 introduced by the rule $\Sigma_3^0\text{-IR}_2$. The terms occurring in the derivation can be viewed as terms in $T_1[\mathcal{R}_\infty, \Phi'_0, G_1, \dots, G_n]$. The requirements of theorem 21 in remark 24 are met and we obtain a normalized derivation.

By [40], $(\Sigma_3^0\text{-IA})^{ND}$ can be solved by R_2 . Since $\Sigma_3^0\text{-IA}$ implies $\Sigma_3^0\text{-IR}_2$ the constants G_i may be chosen to be in $T_2[\mathcal{R}_\infty, \Phi'_0]$. These terms can be handled like in proposition 52.

This completes the proof. \square

Corollary 55. *If*

$$\widehat{\text{E-PA}}^\omega \upharpoonright + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} + \Sigma_2^0\text{-IA} + \text{RT}_2^2 + \text{WKL} \vdash \forall x^1 \exists y^0 A_{qf}(x, y)$$

one can extract a term $t \in T_1$ such that

$$\text{WE-HA}_1^\omega \upharpoonright \vdash \forall x^1 A_{qf}(x, tx).$$

If $\text{RT}_{<\infty}^2 + \Sigma_3^0\text{-IR}_2$ is added to the above system then one can extract a term $t \in T_2$ realizing y provably in $\text{WE-HA}_2^\omega \upharpoonright$ instead of $\text{WE-HA}_1^\omega \upharpoonright$.

Proof. Apply elimination of extensionality (proposition 7) and use theorem 53.

For the second statement use theorem 54. To be able to use the elimination of extensionality the induction rule $\Sigma_3^0\text{-IR}_2$ has to be altered to include the premise that the parameters are extensional. Since this is a formula of the form $\forall u^1 \exists v^0 B_{qf}(u, v)$, the functional interpretation does not introduce terms of degree > 3 and the rule which still follows from $\Sigma_3^0\text{-IA}$ can be interpreted like in the proof of theorem 54. \square

Corollary 56.

- $\text{WKL}_0^\omega + \Sigma_2^0\text{-IA} + \text{RT}_2^2$ is conservative over $\text{RCA}_0^\omega + \Sigma_2^0\text{-IA}$ for sentences of the form $\forall x^1 \exists y^0 A_{qf}(x, y)$. Moreover one can extract a term $t \in T_1$ realizing y .
- $\text{WKL}_0^\omega + \Sigma_2^0\text{-IA} + \Sigma_3^0\text{-IR}_2 + \text{RT}_{<\infty}^2$ is conservative over $\text{RCA}_0^\omega + \Sigma_3^0\text{-IA}$ for sentences of the form $\forall x^1 \exists y^0 A_{qf}(x, y)$. Moreover one can extract a term $t \in T_2$ realizing y .

Since every sentence of the form $\forall x^1 \exists y^0 \forall z^0 B_{qf}(x, y, z)$ is over $\text{QF-AC}^{0,0}$ equivalent to a sentence of the form $\forall x^1 \exists y^0 A_{qf}(x, y)$ also Π_3^0 -conservativity is obtained.

REFERENCES

1. Jeremy Avigad, *Notes on Π_1^1 -conservativity, ω -submodels, and collection schema*, Tech. report, Carnegie Mellon Department of Philosophy, 2002.
2. Jeremy Avigad, *Forcing in proof theory*, Bull. Symbolic Logic **10** (2004), no. 3, 305–333. MR 2083287
3. Jeremy Avigad and Solomon Feferman, *Gödel’s functional (“Dialectica”) interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam, 1998, pp. 337–405. MR 1640329
4. Jon Barwise and John Schlipf, *On recursively saturated models of arithmetic*, Model theory and algebra (A memorial tribute to Abraham Robinson), Springer, Berlin, 1975, pp. 42–55. Lecture Notes in Math., Vol. 498. MR 0409172
5. Marc Bezem, *Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals*, J. Symbolic Logic **50** (1985), no. 3, 652–660. MR 805674
6. Andrey Bovykin and Andreas Weiermann, *The strength of infinitary Ramseyan principles can be accessed by their densities*, accepted for publication in Ann. Pure Appl. Logic, <http://logic.pdmi.ras.ru/~andrey/research.html>, 2005.
7. Peter A. Cholak, Carl G. Jockusch, Jr., and Theodore A. Slaman, *On the strength of Ramsey’s theorem for pairs*, J. Symbolic Logic **66** (2001), no. 1, 1–55. MR 1825173
8. ———, *Corrigendum to: “On the strength of Ramsey’s theorem for pairs”*, J. Symbolic Logic **74** (2009), no. 4, 1438–1439. MR 2583829
9. Chit Tat Chong, Theodore Slaman, and Yue Yang, *Π_1^0 -conservation of combinatorial principles weaker than Ramsey’s theorem for pairs*, preprint, available at http://www.math.nus.edu.sg/~chongct/COH_10.pdf.
10. Chit Tat Chong, *Nonstandard methods in Ramsey’s theorem for pairs*, Computational prospects of infinity. Part II. Presented talks, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 15, World Sci. Publ., Hackensack, NJ, 2008, pp. 47–57. MR 2449459

11. Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 235–242. MR 0429508
12. Robin O. Gandy, *Proofs of strong normalization*, To H. B. Curry: essays on combinatory logic, lambda calculus and formalism, Academic Press, London, 1980, pp. 457–477. MR 592815
13. Kurt Gödel, *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*, *Dialectica* **12** (1958), 280–287. MR 0102482
14. Petr Hájek and Pavel Pudlák, *Metamathematics of first-order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998, Second printing. MR 1748522
15. Denis R. Hirschfeldt and Richard A. Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, *J. Symbolic Logic* **72** (2007), no. 1, 171–206. MR 2298478
16. Denis R. Hirschfeldt, Richard A. Shore, and Theodore A. Slaman, *The atomic model theorem and type omitting*, *Trans. Amer. Math. Soc.* **361** (2009), no. 11, 5805–5837. MR 2529915
17. Jeffry L. Hirst, *Combinatorics in subsystems of second order arithmetic*, Ph.D. thesis, Pennsylvania State University, 1987.
18. William A. Howard, *Ordinal analysis of simple cases of bar recursion*, *J. Symbolic Logic* **46** (1981), no. 1, 17–30. MR 604874
19. Carl Jockusch and Frank Stephan, *A cohesive set which is not high*, *Math. Logic Quart.* **39** (1993), no. 4, 515–530. MR 1270396
20. ———, *Correction to: "A cohesive set which is not high"*, *Math. Logic Quart.* **43** (1997), no. 4, 569. MR 1477624
21. Carl G. Jockusch, Jr., *Ramsey's theorem and recursion theory*, *J. Symbolic Logic* **37** (1972), 268–280. MR 0376319
22. Carl G. Jockusch, Jr. and Robert I. Soare, Π_1^0 *classes and degrees of theories*, *Trans. Amer. Math. Soc.* **173** (1972), 33–56. MR 0316227
23. Stephen C. Kleene, *Countable functionals*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 81–100. MR 0112837
24. ———, *Recursive functionals and quantifiers of finite types. I*, *Trans. Amer. Math. Soc.* **91** (1959), 1–52. MR 0102480
25. Ulrich Kohlenbach, *Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization*, *J. Symbolic Logic* **57** (1992), no. 4, 1239–1273. MR 1195271
26. ———, *Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*, *Arch. Math. Logic* **36** (1996), no. 1, 31–71. MR 1462200
27. ———, *Elimination of Skolem functions for monotone formulas in analysis*, *Arch. Math. Logic* **37** (1998), 363–390. MR 1634279
28. ———, *On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness*, *Annals of Pure and Applied Logic* **95** (1998), 257–285. MR 1650651
29. ———, *On the no-counterexample interpretation*, *J. Symbolic Logic* **64** (1999), no. 4, 1491–1511. MR 1780065
30. ———, *On uniform weak König's lemma*, *Ann. Pure Appl. Logic* **114** (2002), no. 1-3, 103–116, Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999). MR 1879410
31. ———, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295. MR 2185441
32. ———, *Applied proof theory: Proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer Verlag, 2008. MR 2445721
33. Georg Kreisel, *Interpretation of analysis by means of constructive functionals of finite types*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 101–128. MR 0106838
34. Alexander Kreuzer and Ulrich Kohlenbach, *Ramsey's theorem for pairs and provably recursive functions*, *Notre Dame J. Formal Logic* **50** (2009), no. 4, 427–444 (2010). MR 2598872
35. Alexander P. Kreuzer, *Der Satz von Ramsey für Paare und beweisbar rekursive Funktionen*, Diploma Thesis, Technische Universität Darmstadt, 2009.
36. ———, *The cohesive principle and the Bolzano-Weierstraß principle*, *Math. Logic Quart.* **57** (2011), no. 3, 292–298.
37. Horst Luckhardt, *Extensional Gödel functional interpretation. A consistency proof of classical analysis*, Lecture Notes in Mathematics, Vol. 306, Springer-Verlag, Berlin, 1973. MR 0337512
38. Dag Normann, *Recursion on the countable functionals*, Lecture Notes in Mathematics, vol. 811, Springer, Berlin, 1980. MR 583016

39. Charles Parsons, *Proof-theoretic analysis of restricted induction schemata*, Meetings of the Association for Symbolic Logic (Gaisi Takeuti, ed.), Association for Symbolic Logic, 1971, abstract, p. 361.
40. ———, *On n -quantifier induction*, J. Symbolic Logic **37** (1972), 466–482. MR 0325365
41. Pavol Safarik and Ulrich Kohlenbach, *On the computational content of the Bolzano-Weierstraß principle*, Math. Log. Q. **56** (2010), no. 5, 508–532. MR 2742886
42. David Seetapun and Theodore A. Slaman, *On the strength of Ramsey's theorem*, Notre Dame J. Formal Logic **36** (1995), no. 4, 570–582, Special Issue: Models of arithmetic. MR 1368468
43. Richard A. Shore, *Reverse mathematics: the playground of logic*, Bull. Symbolic Logic **16** (2010), no. 3, 378–402. MR 2731250
44. Stephen G. Simpson, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR 1723993
45. Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987, A study of computable functions and computably generated sets. MR 882921
46. Ernst Specker, *Ramsey's theorem does not hold in recursive set theory*, Logic Colloquium '69 (Proc. Summer School and Colloq., Manchester, 1969), North-Holland, Amsterdam, 1971, pp. 439–442. MR 0278941
47. Clifford Spector, *Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics*, Proc. Sympos. Pure Math., Vol. V, American Mathematical Society, Providence, R.I., 1962, pp. 1–27. MR 0154801
48. Thomas Streicher and Ulrich Kohlenbach, *Shoenfield is Gödel after Krivine*, Math. Log. Q. **53** (2007), no. 2, 176–179. MR 2308497
49. Gaisi Takeuti, *Two applications of logic to mathematics*, Iwanami Shoten, Publishers, Tokyo, 1978, Kanô Memorial Lectures, Vol. 3, Publications of the Mathematical Society of Japan, No. 13. MR 0505474
50. Anne S. Troelstra (ed.), *Metamathematical investigation of intuitionistic arithmetic and analysis*, Lecture Notes in Mathematics, Vol. 344, Springer-Verlag, Berlin, 1973. MR 0325352
51. ———, *Note on the fan theorem*, J. Symbolic Logic **39** (1974), 584–596. MR 0384494