A LOGICAL ANALYSIS OF THE GENERALIZED BANACH CONTRACTIONS PRINCIPLE

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Abstract. Let $(\mathcal{X}, d)$ be a complete metric space, $m \in \mathbb{N} \setminus \{0\}$, and $\gamma \in \mathbb{R}$ with $0 \leq \gamma < 1$. A g-contraction is a mapping $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ such that for all $x, y \in \mathcal{X}$ there is an $i \in [1, m]$ with $d(T^i x, T^i y) <_{\mathbb{R}} \gamma^i d(x, y)$.

The generalized Banach contractions principle states that each g-contraction has a fixed point. We show that this principle is a consequence of Ramsey's theorem for pairs over, roughly, $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$.

In this paper we will show that a generalization of the Banach contraction mapping principle—the generalized Banach contractions principle—follows from Ramsey's theorem for pairs over a weak basis theory.

Let $(\mathcal{X}, d)$ be a complete metric space, let $m \in \mathbb{N} \setminus \{0\}$, and let $\gamma \in \mathbb{R}$ with $0 \leq \gamma < 1$. We call a function $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ a (m, γ)-*g-contraction* if for all $x, y \in \mathcal{X}$ there is an $i \in [1, m]$, such that $d(T^i x, T^i y) <_{\mathbb{R}} \gamma^i d(x, y)$.

The ordinary Banach contraction mapping theorem states that every (m, γ)-g-contraction with $m = 1$ has a fixed point. The generalized Banach contraction mapping principle is the statement that *every* (m, γ)-g-contraction has a fixed point.

First results on the generalized Banach contraction mapping principle have been established in Jachymski, Schroder and Stein [JSS99], where it is shown that this principle is true for g-contraction where $m = 2$. In Jachymski and Stein [JS99] it was show that the principle is true for all m if the g-contraction is uniformly continuous. Later in Merryfield, Rothschild and Stein [MRS02] it was shown, that this principle is true for all continuous g-contractions and for $m = 3$ without this continuity assumption. This proof uses Ramsey's theorem. However, it also uses full arithmetical comprehension, which is—as we will see below—much stronger than this contraction principle. Therefore, this proof is not suitable for a faithful formalization. The principle in its full generality was finally proved in Arvanitakis [Arv03] and independently in Merryfield and Stein [MS02].

1. Logical Systems

We will work in the second-order system RCA_0 and its extension to all finite types RCA_0^ω. This use of RCA_0^ω is necessary since the operator T might be non-continuous and has to be presented by a type 2 object in general. (In [JSS99] it was shown that there exists a $(3, \gamma)$-g-contraction which is not continuous.)

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The set of all finite types $\mathbf{T}$ is defined to be the smallest set that satisfies

$$0 \in \mathbf{T}, \qquad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

The type 0 denotes the type of natural numbers and the type $\tau(\rho)$ denotes the type of functions from ρ to τ. The type $0(0)$ is abbreviated by 1 the type $0(0(0))$ by 2. The type of a variable will sometimes be written as superscript of a term.

Equality $=_0$ for type 0 objects will be added as a primitive notion to the systems together with the usual equality axioms. Higher type equality $=_{\tau\rho}$ will be treated as abbreviation:

$$x^{\tau\rho} =_{\tau\rho} y^{\tau\rho} :\equiv \forall z^\rho \, xz =_\tau yz.$$

Define the recursor R_ρ of type ρ to be the functional satisfying

$$R_\rho 0yz =_\rho y, \qquad R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x.$$

Let *Gödel's system T* be the $\mathbf{T}$-sorted set of closed terms that can be build up from 0^0, the successor function S^1, λ-abstraction, and the recursors R_ρ for all finite types ρ. Denote by T_0 the subsystem of Gödel's system T where primitive recursion is restricted to recursors R_0. The system T_0 corresponds to the extension of Kleene's primitive recursive functionals to mixed types, see [Kle59], whereas full system T corresponds to Gödel's primitive recursive functionals, see [Göd58].

The system RCA_0^ω is defined to be the extension of the term system T_0 by Σ^0_1-induction, the extensionality axioms

$$(\mathsf{E}_{\rho,\tau})\colon \forall z^{\tau\rho}, x^\rho, y^\rho \, (x =_\rho y \rightarrow zx =_\tau zy)$$

for all $\tau, \rho \in \mathbf{T}$, and the schema of quantifier free choice restricted to choice of numbers over functions ($\mathsf{QF\text{-}AC}^{1,0}$), i.e.

$$\forall f^1 \, \exists x^0 \, \mathsf{A}_{qf}(f,x) \rightarrow \exists F^2 \, \forall f^1 \, \mathsf{A}_{qf}(f, F(f)).$$

This schema is the higher order equivalent to recursive comprehension (Δ^0_1-CA). (Strictly speaking the system RCA_0^ω was defined in [Koh05] to contain only quantifier free induction instead of Σ^0_1-induction. Since Σ^0_1-induction is provable in that system, we may also add it directly.)

It is clear the RCA_0 can be embedded into RCA_0^ω. The system RCA_0^ω is conservative over its second-order counterpart, where the second-order part is given by functions instead of sets. This second-order system can then be interpreted in RCA_0. See [Koh05].

A complete separable metric space $(\hat{\mathcal{X}}, \hat{d})$ is represented as completion of a countable metric space $(\mathcal{X}, d)$. A point in $\hat{\mathcal{X}}$ is given by a Cauchy sequence of elements of $\mathcal{X}$ having a fixed Cauchy-rate. Thus, a point in $\hat{\mathcal{X}}$ is represented by a type 1 object. The metric $\hat{d}$ is the continuous extension of d to $\hat{\mathcal{X}}$. Two points $x, y \in \hat{\mathcal{X}}$ are defined to be equal ($x =_{\hat{\mathcal{X}}} y$) if $\hat{d}(x,y) =_{\mathbb{R}} 0$. A function $T\colon \hat{\mathcal{X}} \longrightarrow \hat{\mathcal{X}}$ can then be represented by a type 2 object. To build the iteration T^n of T we, in general, require the recursor R_1, we will therefore work over $\mathsf{RCA}_0^\omega + (R_1)$, where (R_1) is the axiom that states that the recursor R_1 exists. See [Koh08, Chapter 4]. Note that over RCA_0^ω the axiom (R_1) implies Σ^0_2-IA and that the provably recursive functions of $\mathsf{RCA}_0^\omega + (R_1)$ are that same as for Σ^0_2-IA, see [Par72].

In case the function T is continuous in the sense of reverse mathematics, i.e. T has a continuous modulus of continuity, then T can be represented by a type 1 object (or a set), see [Sim09]. One can prove the totality of the iteration T^n in Σ^0_2-IA and, in fact, it is equivalent to Σ^0_2-IA, see [FSY93, Theorem 4.3]. Thus, if one is only interested in such T one could weaken the basis theory to $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$. (If one additionally assumes that $\hat{\mathcal{X}}$ is compact then one could also use WKL instead of $\Sigma^0_2\text{-}\mathsf{IA}$, see [FSY93, Theorem 4.5].)

We are now in the position to define formally the generalized Banach contraction mapping principle. This definition is relative to $\mathsf{RCA}^\omega_0 + (R_1)$.

Definition 1. Let GBCC_m be that statement that each presentable complete separable metric space space $(\mathcal{X}, d)$ and each function $T\colon \mathcal{X} \longrightarrow \mathcal{X}$, which is a (m,γ)-g-contraction for a γ with $0 \le \gamma < 1$ there exists a fixed point of T.

Further, let $\mathsf{GBCC} :\equiv \forall m\, \mathsf{GBCC}_m$ and let GBCC^{cont}_m, GBCC^{cont} be the restriction of those principles to continuous functions T. (GBCC is an abbreviation for "Generalized Banach contraction conjecture".)

The definition of GBCC^{cont}_m, GBCC^{cont} also makes sense in the weaker system $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$.

Definition 2 (Ramsey's theorem for pairs)**.** Let $[X]^2$ be the set of all unordered pairs of X. *Ramsey's theorem for pairs and n colors* (RT^2_n) is the statement that for each coloring of pairs of $\mathbb{N}$ using n colors $c\colon [\mathbb{N}] \longrightarrow [0,n[$ there exists an infinite, homogeneous set X, i.e., X is infinite and the restriction of c to $[X]^2$ is constant. *Ramsey's theorem for pairs and arbitrary many colors* ($\mathsf{RT}^2_{<\infty}$) is defined to be $\forall n\, \mathsf{RT}^2_n$.

It is easy to see that for each $n \ge 2$ we have $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{RT}^2_n$. However, $\mathsf{RT}^2_{<\infty}$ is stronger than RT^2_2. Therefore, we can restrict our attention to RT^2_2 and $\mathsf{RT}^2_{<\infty}$. It is known that neither RT^2_2 nor $\mathsf{RT}^2_{<\infty}$ imply arithmetical comprehension, [SS95]. For more details on the strength of these principles, see [CJS01, HS07].

We will show the following theorem.

Theorem 3.

(i) $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA} \vdash \mathsf{RT}^2_2 \to \mathsf{GBCC}^{cont}_m$ *for each m,*
(ii) $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA} \vdash \mathsf{RT}^2_{<\infty} \to \mathsf{GBCC}^{cont}$,
(iii) $\mathsf{RCA}^\omega_0 + (R_1) \vdash \mathsf{RT}^2_2 \to \mathsf{GBCC}_m$ *for each m,*
(iv) $\mathsf{RCA}^\omega_0 + (R_1) \vdash \left(\mathsf{RT}^2_{<\infty} \wedge \Sigma^0_3\text{-}\mathsf{IA}\right) \to \mathsf{GBCC}$.

Theorem 3 is established by formalizing the proof of the generalized Banach contraction mapping principle of Fremlin [Fre02] and by using some ideas of the proof of [Arv03].

We will first prove the case where T is continuous and then extend it to the general case. Before we can do this, we provide some facts on Ramsey's theorem for pairs and some combinatorial lemmata.

2. Combinatorial lemmata

A coloring $c\colon [\mathbb{N}]^2 \longrightarrow [0,n[$ is *called* stable if $c(\{x,\cdot\})$ eventually becomes constant. The restriction of Ramsey's theorem for pairs to stable colorings is called *stable Ramsey's theorem for pairs* and denoted by SRT^2_2 resp. $\mathsf{SRT}^2_{<\infty}$.

A set X is called *cohesive* for a sequence $(R_i)_{i\in\mathbb{N}}$ of subsets of $\mathbb{N}$ if

$$\forall i\, \left(X \subseteq^* R_i \vee X \subseteq^* \overline{R_i}\right),$$

where $X \subseteq^* Y :\equiv (X \setminus Y \text{ is finite})$.

The *cohesive principle* (COH) states that for every $(R_i)_{i\in\mathbb{N}}$ an infinite, cohesive set exists. The following proposition shows that COH is the counterpart to the stable Ramsey's theorem.

Proposition 4 ([CJS01, CJS09]).

- $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \land \mathsf{SRT}^2_2$
- $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_{<\infty} \leftrightarrow \mathsf{COH} \land \mathsf{SRT}^2_{<\infty}$

Proposition 5 ([CJS01, Lemmas 7.10, 7.12], [CLY10, CJS09]). *Over* RCA_0 *the principle* SRT^2_2 *is equivalent to the statement that for every* Δ^0_2*-set* A *there exists an infinite set* X *such that* $X \subseteq A$ *or* $X \subseteq \overline{A}$*.*

The principle $\mathsf{SRT}^2_{<\infty}$ *is equivalent to the statement that for every finite* Δ^0_2*-partition* $(A_i)_{i<n}$ *of* $\mathbb{N}$ *there exists an* $i < n$ *and an infinite set* X *such that* $X \subseteq A_i$*. (If* n *is uniformly bounded this principle follows from* SRT^2_2 *by induction on the metalevel.)*

Remark 6 (COH as partial non-principal ultrafilter). Let $(R_i)_{i\in\mathbb{N}}$ be a sequence of sets $R_i \subseteq \mathbb{N}$ and let S be an infinite cohesive set for this sequence.

Define $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ by

$$X \in \mathcal{F} \quad \text{iff} \quad S \subseteq^* X.$$

Then as long as one is only concerned about sets in $(R_i)_i$ the usual properties of a non-principal ultrafilter hold; i.e. let $i, j \in \mathbb{N}$ then

- $R_i \subseteq R_j \land R_i \in \mathcal{F} \Rightarrow R_j \in \mathcal{F}$,
- $R_i, R_j \in \mathcal{F} \Rightarrow R_i \cap R_j \in \mathcal{F}$,
- $R_i \in \mathcal{F} \lor \overline{R_i} \in \mathcal{F}$ (by cohesiveness of S),
- $R_i \in \mathcal{F} \Rightarrow R_i$ is infinite.

In other words, $\mathcal{F}$ defines a non-principal ultrafilter in the algebra of sets created by $(R_i)_i$. Hence, if one can fix in advance a countable number of sets, for which the properties of a non-principal ultrafilter are needed, the ultrafilter may be replaced by the filter $\mathcal{F}$.

Note that the statement $X \in \mathcal{F}$ is $\Delta^0_2(S)$ for $X \in (R_i)$.

2.1. Syndetic sets.

Definition 7 (Syndetic).

- Let $m \geq 1$. A set $I \subseteq \mathbb{N}$ is called *m-syndetic* if for all $k \in \mathbb{N}$ the set $I \cap [k, k+m[$ is not empty.
- A set $I \subseteq \mathbb{N}$ is called *piecewise m-syndetic* if there exists arbitrary large intervals $[j_1, j_2]$, such that for all $k \in [j_1, j_2 - m]$ the set $I \cap [k, k+m[$ is not empty.

Lemma 8 ($\mathsf{RCA}_0 + \mathsf{SRT}^2_{<\infty}$). *Let* $n \in \mathbb{N}$ *and* $(A_i)_{i<n}$ *be a finite sequence of disjoint* Δ^0_2*-subsets of* $\mathbb{N}$*, such that* $I := \bigcup_{i<n} A_i$ *is* m*-syndetic for an* m*, then there exists an infinite set* X *such that* $X \subseteq A_i$ *for an* i*.*

This lemma requires SRT^2_2 *if* n *and* m *are fixed and* $\mathsf{SRT}^2_{<\infty}$ *otherwise.*

Proof. Define a Δ^0_2-function $f : \mathbb{N} \longrightarrow [0, n]$, via a Δ^0_2-formula for its graph, denoting to which set a number belongs, by

$$f(x) := \begin{cases} i & \text{if } x \in A_i, \\ n & \text{otherwise.} \end{cases}$$

We now divide the natural numbers into blocks of size m, and define the Δ^0_2-function g assigning to each of those blocks the sequence of values of f on it:

$$g(x) := \langle f(x \cdot m), \dots, f(x \cdot m + m - 1) \rangle$$

Note that because I is m-syndetic $g(x) \neq \langle n, \dots, n \rangle$ for all x. The function $g(x)$ defines a Δ^0_2-partition $(B_i)_{i<n'}$ of $\mathbb{N}$ with

$$B_i := \{\, x \mid g(x) = i \,\}, \qquad n' := \langle \underbrace{n, \dots, n}_{m \text{ times}} \rangle.$$

By Proposition 5 we can find an infinite set Y on which g is constant. Since $g(Y) \neq \langle n, \dots, n \rangle$ there is an $j < m$ such that $\big(g(Y)\big)_j \neq n$. Let $X := \{x \cdot m + j \mid x \in Y\}$. By definition f is constant on X and $f(X) \neq n$. Thus, $X \subseteq A_{f(X)}$. $\square$

The original proof of Arvanitakis uses the well known fact that piecewise syndetic is partition stable property. This was proved by Brown in [Bro71] and others later, see for instance [Fur81, Theorem 1.23]. These proofs use the Bolzano-Weierstraß principle for the Cantor space and hence comprehension and are, therefore, not faithful. Luckily we only need the following weaker facts about partitions of even syndetic sets and not piecewise syndetic sets.

The following two lemmas are based on [Bro71, Lemma 1].

Lemma 9 (RCA_0)**.** *Let X be an m-syndetic set. If X is partitioned into 2 parts $A_0, A_1 = X \setminus A_0$ then either each A_i is piecewise m-syndetic or there are $i < 2$ and k such that A_i is k-syndetic.*

Proof. Suppose that there is no k such that A_0 is k-syndetic then there are intervals I of arbitrary length such that $A_0 \cap I = \emptyset$. This means that $A_1 \cap I = X \cap I$ hence A_1 is piecewise m-syndetic. Same for A_0. $\square$

Corollary 10 ($\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$)**.** *Let X be an m-syndetic set. If X is partitioned into finitely many parts $(A_i)_{i<n}$ then there is an $J \subseteq [0, n[$ and an k such that each A_i with $i \in J$ is piecewise k-syndetic and $Y := \bigcup_{i \in J} A_i$ is k-syndetic.*

If the numbers of partitions n is uniformly bounded no Σ^0_2-IA is needed.

Proof. Note that being k-syndetic is a Π^0_1-statement $(\forall x\, \exists y < x + k\ (y \in X))$. We search for a $\subseteq$-minimal set $J \subseteq [0, n[$, such that there is a k with $\bigcup_{i \in J} A_i$ is k-syndetic. To do so we build by finite Σ^0_2-comprehension a finite sequence s such that

$$(s)_j = 0 \qquad \text{iff} \qquad \begin{array}{l} \text{there is an } k, \text{ such that if } j \text{ codes the set } J \\ \text{then } \bigcup_{i \in J} A_i \text{ is } k\text{-syndetic} \end{array}$$

and then search for a minimal set. This finite comprehension requires Σ^0_2-induction if greatest index of a set $J \subseteq [0, n[$ is not fixed, i.e. if n is not uniformly bounded.

The case $J = \emptyset$ is ruled out because then $\bigcup_{i \in J} A_i = \emptyset$ and thus would not be syndetic. Let $j \in J$. Heading for a contradiction suppose that A_j is not piecewise k-syndetic. Then by Lemma 9 the set $\bigcup_{i \in J \setminus \{j\}} A_i$ must be m-syndetic for an m. Thus, J is not $\subseteq$-minimal with this property which contradicts our choice of J. $\square$

Combining Lemma 8 and Corollary 10 we obtain the following proposition.

Proposition 11 ($\mathsf{RCA}_0 + \mathsf{SRT}^2_{<\infty} + \Sigma^0_3$-$\mathsf{IA}$)**.** *Let X be an m-syndetic set partitioned into Δ^0_2-sets $(A_i)_{i<n}$, then there exists an i such that A_i is piecewise k-syndetic and*

an infinite set I such that $I \subseteq A_i$. Note that we do not require I to be piecewise syndetic.

If n is uniformly bounded only SRT^2_2 is needed. Otherwise, Σ^0_3-IA and $\mathsf{SRT}^2_{<\infty}$ is needed.

Proof. By Corollary 10 we can find a set J such that $(A_i)_{i\in J}$ is syndetic and each A_i with $i \in J$ is piecewise syndetic. Note that Σ^0_3 is needed since the partition is Δ^0_2. An application of Lemma 8 now proves the proposition. ☐

3. The proof of GBCC

3.1. **The continuous case.** Fix a provably presentable complete separable metric space $(\mathcal{X}, d)$ and a (m,γ)-g-contraction $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ which is continuous.

Lemma 12 ($\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$, [Fre02, Lemma 2])**.** *For all points $x, y \in \mathcal{X}$ the set*

$$I := \left\{\, i \in \mathbb{N} \;\middle|\; d(T^i x, T^i y) <_{\mathbb{R}} \gamma^i d(x,y) \,\right\}$$

is m-syndetic.

Proof. By the g-contraction property $I \cap [1,m] \neq \emptyset$ and for each $i \in I$ there is an $j \in [1,m]$ such that $i + j \in I$. ☐

Lemma 13 ($\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$, [MRS02, Lemma 1], [Fre02, Lemma 5])**.** *For each $x \in \mathcal{X}$ there exists an $M >_{\mathbb{R}} 0$ such that the set*

$$I := \left\{\, i \in \mathbb{N} \;\middle|\; d(T^i x, x) <_{\mathbb{R}} M \,\right\}$$

is m-syndetic.

Proof. Let $M = \frac{2}{1-\gamma} \max_{i\in[0,m]} d(T^i x, x)$. (We assume that $Tx \neq_{\mathcal{X}} x$ here, otherwise we would be done.) It is clear that $0 \in I$. For each $i \in I$ there is $j \in [1,m]$ such that $d(T^{j+i}x, T^j x) <_{\mathbb{R}} \gamma^j d(T^i x, x) <_{\mathbb{R}} \gamma M$ and hence

$$d(T^{i+j}x, x) <_{\mathbb{R}} \gamma M + d(T^j x, x) <_{\mathbb{R}} \gamma M + (1-\gamma)M = M$$

and thus $i + j \in I$. ☐

Remark 14. It is clear that the Lemmas 12 and 13 also hold for non-continuous T if the theory $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$ is replaced by $\mathsf{RCA}^\omega_0 + (R_1)$.

Lemma 15 ($\mathsf{RCA}_0 + \Sigma^0_3$-$\mathsf{IA} + \mathsf{RT}^2_{<\infty}$, [Arv03, 4.2], [Fre02, Lemma 4])**.**
Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be such that

(i) the set $\{\, i \mid (i,0) \in R \,\}$ is m-syndetic,
(ii) for every $(i,j) \in R$ the set $\{\, k \mid (i+k, j+k) \in R \,\}$ is m-syndetic.

Then there exists an infinite set I and a piecewise syndetic Δ^0_2-set $\tilde{I}$ such that $I \subseteq \tilde{I} \subseteq \mathbb{N}$ and for every $i, j \in \tilde{I}$ there is a k with $(k,i) \in R$ and $(k,j) \in R$.

If m is fixed then RT^2_2 suffices. If the existence of I is sufficient (in other words the Δ^0_2-set $\tilde{I}$ is not needed) then $\mathsf{RT}^2_{<\infty}$ suffices. Otherwise, $\mathsf{RT}^2_{<\infty}$ and Σ^0_3-IA is needed.

Proof. We claim that R meets $[l, l+2m[\times [k, k+m[$ for all $k, l \in \mathbb{N}$ with $k \leq l$. To prove this claim note that by (i) there is an $i \in [l-k, l-k+m[$ such that $(i,0) \in R$, and by (ii) there is now an $j \in [k, k+m[$ such that $(i+j, j) \in R$ and that also $(i+j, j) \in [l, l+2m[\times [k, k+m[$.

For each $i \in \mathbb{N}$ and $j < 2m$ let $L_{ij} := \{\, l \mid (l+j, i) \in R \,\}$. Using the cohesive principle (which follows from RT^2_2, see Proposition 4) we find a cohesive set S for

$(L_{ij})_{i,j}$ and a non-principal ultrafilter $\mathcal{F} := \{ X \mid S \subseteq^* X \}$ in the algebra created by (L_{ij}). The ultrafilter is Δ^0_2, see Remark 6.

By the claim it follows that

$$\bigcup_{\substack{i\in[k,k+m[\\ j<2m}} L_{ij} \supseteq [k,\infty[.$$

Hence by the ultrafilter property of $\mathcal{F}$ there is for each k some $i \in [k, k+m[$ and $j < 2m$ such that $L_{ij} \in \mathcal{F}$.

Now for $j < 2m$ define $I_j := \{ i \mid L_{ij} \in \mathcal{F} \}$. Observe that by the previous argument the set $\bigcup_{j<2m} I_j$ is m-syndetic. The sets I_j are Δ^0_2-set since $\mathcal{F}$ is.

Using Proposition 11 we can find an infinite set I and a j such that I_j is piecewise syndetic and $I \subseteq I_j$.

If $i, i' \in I_j$, then L_{ij} and $L_{i'j}$ belong to $\mathcal{F}$, so they cannot be disjoint. Thus, there is some l such that $(l+j, i)$ and $(l+j, i')$ belong to R. Hence, I and $\tilde{I} = I_j$ satisfies the lemma. If one is only interested in I then Lemma 8 instead of Proposition 11 suffices. □

We are now in the position to show Theorem 3 restricted to the continuous case, i.e.

(i) $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA} \vdash \mathsf{RT}^2_2 \to \mathsf{GBCC}^{cont}_m$ for each m,
(ii) $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA} \vdash \mathsf{RT}^2_{<\infty} \to \mathsf{GBCC}^{cont}$.

Proof of Theorem 3 for the continuous case. Fix an arbitrary $x \in \mathcal{X}$. By Lemma 13 an $M >_{\mathbb{R}} 0$ exists, such that $\left\{ i \;\middle|\; d(T^i x, x) <_{\mathbb{R}} M \right\}$ is m-syndetic. We further may assume that $M >_{\mathbb{R}} d(Tx, x)$. Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be the relation

$$R := \left\{ (i,j) \;\middle|\; d(T^i x, T^j x) <_{\mathbb{R}} M\gamma^j \right\}$$

By definition $\{ i \mid (i,0) \in R \}$ is m-syndetic. If $(i,j) \in R$ then

$$\{ k \mid (i+k, j+k) \in R \} \supseteq \left\{ k \;\middle|\; d(T^{i+k}x, T^{j+k}x) <_{\mathbb{R}} \gamma^k d(T^i x, T^j x) \right\}$$

and hence is by Lemma 12 also m-syndetic.

The set R satisfies the assumptions of Lemma 15. This lemma is not directly applicable since the set R is just a Σ^0_1-set because $<_{\mathbb{R}}$ is a Σ^0_1-statement. However, we can easily build a recursive set $R' \subseteq R$ satisfying also the assumptions of Lemma 15: By $\mathsf{QF}\text{-}\mathsf{AC}^{0,0}$ and the properties of R we can find a function $f_1(i,w)$ such that if w is a witness for $(i,0) \in R$ then $f_1(i,w) = (k,w')$ with $k < m$ and w' witnesses that $(i+k+1, 0) \in R$. Similarly there exists a function $f_2(i,j,w) = (k,w')$ for the second property. Now let w be a witness for the fact that $(1,0)$ is in R. Let

$$\begin{aligned} R'_0 &:= \{(1,0,w)\}, \\ R'_{n+1} &:= \{(i+k+1, 0, w') \mid (i,0,w) \in R'_n \text{ and } f_1(i,w) = (k',w')\} \\ &\quad \cup \{(i+k+1, j+k+1, w') \mid (i,j,w) \in R'_n \text{ and } f_2(i,j,w) = (k',w')\}, \end{aligned}$$

and let R' be the projection of $\bigcup_n R'_n$ to the first two components. The membership in R' is decidable, since the first component of the elements of the sets (R'_n) always increases and thus $(i,j) \in R'$ iff $\exists w\, (i,j,w) \in \bigcup_{n\leq i} R'_n$. The $\exists$-quantifier here is decidable since the sets (R_n) are finite. By definition R' satisfies the assumptions of Lemma 15 and is a subset of R.

Hence there is an infinite set $I \subseteq \mathbb{N}$ such that for all $i, j \in I$ there is a $k \in \mathbb{N}$ such that $(k, i), (k, j) \in R' \subseteq R$. By definition of R we have

$$d(T^i x, T^j x) \leq_{\mathbb{R}} d(T^k x, T^i x) + d(T^k x, T^j x) \leq_{\mathbb{R}} M\gamma^i + M\gamma^j \underset{i,j\to\infty}{\longrightarrow} 0.$$

Thus, the sequence $(T^i x)_{i \in I}$ is a Cauchy-sequence with Cauchy-rate $2M\gamma^i$ and admits a limit point, call it z.

Note that by continuity of T for all k we have

$$\lim_{i \in I} T^{i+k} x = T^k z.$$

Since $(1, 0) \in R'$, the set $L := \{ k \mid (1 + k, k) \in R' \} \subseteq \{ k \mid (1 + k, k) \in R \}$ is m-syndetic and so we can find for every $i \in I$ an $j_i \in [0, m[$ such that $i + j_i \in L$, i.e.

$$d(T^{i+j_i+1} x, T^{i+j_i} x) \leq_{\mathbb{R}} M\gamma^{i+j_i}.$$

By the infinite pigeonhole principle there is a j and an infinite set $J \subseteq I$ on which $j_i = j$ is constant. For every $i \in J$ then holds

$$\begin{aligned} d(T^j z, T^{j+1} z) &\leq d(T^j z, T^{i+j} x) + d(T^{i+j} x, T^{i+j+1} x) + d(T^{i+j+1} x, T^{j+1} z) \\ &\leq d(T^j z, T^{i+j} x) + M\gamma^{i+j} + d(T^{i+j+1} x, T^{j+1} z) \end{aligned}$$

The last expression tends to 0 as $i \in J$ tends to infinity. This yields that $T^j z$ is a fixed-point.

The proof formalizes in $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$ except for Lemma 15, where we need RT^2_2 is m is uniformly bounded and $\mathsf{RT}^2_{<\infty}$ otherwise. Hence, the statement follows. □

3.2. **Proof of the general case.** Now let $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ be an arbitrary mapping.

Lemma 16 ($\mathsf{RCA}^\omega_0 + (R_1)$, [Fre02, Lemma 3])**.** *Let $x \in \mathcal{X}$. If there exists an $n \geq 1$ such that $T^n x = x$ then already $Tx =_{\mathcal{X}} x$.*

Proof. Assume that n is minimal with $T^n x =_{\mathcal{X}} x$. Since $x =_{\mathcal{X}} y$ is Π^0_1 one can find such an n using $\Sigma^0_1\text{-}\mathsf{IA}$.

If $n \geq 2$ take $i < j \in [1, n[$ such that $d(T^i x, T^j x)$ is minimal. Again $\Sigma^0_1\text{-}\mathsf{IA}$ proves that such i, j exists.

By the (m, γ)-g-contraction property there is a $k \in [1, m]$ such that

$$\gamma^k d(T^i x, T^j x) >_{\mathbb{R}} d(T^{i+k} x, T^{j+k} x).$$

By the assumption $T^n(x) = x$ the right side is equal to $d(T^{(i+k) \bmod n} x, T^{(j+k) \bmod n} x)$ which is a contradiction to the minimality.

Hence $n = 1$ and $Tx =_{\mathcal{X}} x$. □

Lemma 17 (RCA_0, [Arv03, Lemma 3.2])**.** *Let N be a given multiple of m. Then for all $u, v \in \mathbb{N}$ there exists a number $p(u, v) \in \mathbb{N}$ such that whenever $R \in [1, p(u, v)] \times [0, \infty[$ is a relation satisfying*

(1) the set $\{ i \mid (i, 0) \in R \}$ meets every sets $[k + 1, k + N] \subseteq [1, p(u, v)]$,
(2) if $i + m \leq p(u, v)$ and $(i, j) \in R$, then there are $1 \leq i', j' \leq J$ such that $(i + i', j + j') \in R$,

then there exists a subinterval $[k + 1, k + N] \subseteq [1, p(u, v)]$ and $k_1, \ldots, k_u \in \mathbb{N}$ such that

(1) $k_{r+1} - k_r \geq m$ for $1 \leq r < u$,
(2) for every k_r there exists a $q \in [k + 1, k + N]$ such that $(q, k_r) \in R$.

Proof. The proof of Arvanitakis in [Arv03, Lemma 3.2] uses only quantifier free induction and can be formalized even in elementary arithmetic. □

Lemma 18 ($\mathsf{RCA}_0^\omega + (R_1)$, [Arv03, Lemma 3.1])**.** *Assume that no power of T has a fixed-point, then for every $N \in \mathbb{N}$ there exists a $p(N) \in \mathbb{N}$ such that for every point $z \in X$ there exists an $\varepsilon >_{\mathbb{R}} 0$ with the property that for every $y \in X$ one finds N successive iterates of T in the set $y, Ty, \ldots, T^{p(N)-1}y$ whose distance to z is bigger than ε.*

Proof. This lemma is an elementary application of the previous lemma. The proof of Arvanitakis ([Arv03, Lemma 3.1]) can also be formalized in this system. □

Proof of Theorem 3. Like in the continuous case we construct using Lemma 15 an infinite set I. We now use that this lemma also provides a piecewise N-syndetic Δ_2^0-set $\tilde{I}$, such that $I \subseteq \tilde{I} \subseteq \mathbb{N}$. Again $(T^i x)_{i\in\tilde{I}}$ is a Cauchy-sequence with Cauchy-rate $2M\gamma^i$ and limit point z. Note that the sequence restricted to the elements in I converges to z, too. Hence, z is definable in the system.

Assume for a contradiction that T has no fixed point. By Lemma 16 no power of T has a fixed point and hence by Lemma 18 for a given N there are $p(N), \varepsilon$, such that for every point $y \in \mathcal{X}$ in $(T^i y)_{i\in[1,p(N)]}$ there are N successive elements, which are more than ε apart from z.

By the convergence of $(T^i x)_{i\in\tilde{I}}$ there exists an i_0 such that

$$d(T^i x, z) < \varepsilon \quad \text{for } i \in \tilde{I} \text{ and } i \geq i_0.$$

The Δ_2^0-set $\tilde{I}_0 := \tilde{I} \cap [i_0, \infty[$ is evidently also piecewise N-syndetic.

Using the piecewise N-syndetic property of $\tilde{I}_0$ one can find a subset of size $p(N)$ where at least every N-th element is ε-close to z, contradicting the conclusion of Lemma 18 and thus the assumption that T has no fixed-point.

This proves the theorem.

Again, the proof formalizes in $\mathsf{RCA}_0^\omega + (R_1)$ except for Lemma 15, where we need RT_2^2 is m if uniformly bounded and $\mathsf{RT}_{<\infty}^2$ and Σ_3^0-IA otherwise. Hence, the statement follows. □

4. Final remarks

We showed that the generalized Banach contractions principle follows from Ramsey's theorem for pairs. The proof depends essentially on the fact that Ramsey's theorem splits into stable Ramsey's theorem and the cohesive principle. This shows, that this split is not only useful as a technical tool to investigate the strength of RT_2^2 but also is (implicitly) used outside of logic.

Moreover, our formalization shows that program extraction techniques we developed in [KK] are applicable. It is a first step to extract quantitative content—like rates of asymptotic regularity—from the generalized Banach contractions principle.

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