THE COHESIVE PRINCIPLE AND THE BOLZANO-WEIERSTRASS PRINCIPLE

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Abstract. Let $\mathrm{BW}_{\mathrm{weak}}$ be the principle stating that every bounded sequence of real numbers contains a Cauchy subsequence (a sequence converging but not necessarily fast). We show that $\mathrm{BW}_{\mathrm{weak}}$ is equivalent to the (strong) cohesive principle (StCOH) and – using this – obtain a classification of the computational and logical strength of $\mathrm{BW}_{\mathrm{weak}}$. Especially we show that $\mathrm{BW}_{\mathrm{weak}}$ does not solve the halting problem and does not lead to more than primitive recursive growth. Therefore it is strictly weaker than the usual Bolzano-Weierstraß principle BW. We also discuss possible uses of $\mathrm{BW}_{\mathrm{weak}}$.

In this paper we show that the variant of the Bolzano-Weierstraß principle stating only the existence of a Cauchy subsequence is strictly weaker than the full Bolzano-Weierstraß principle. More precisely we show that this weak variant is equivalent to the strong cohesive principle.

We proceed by first presenting the cohesive principle and the Bolzano-Weierstraß principle. Following this we will show that these principles are equivalent and discuss the consequences.

1. Cohesive Principle

Definition 1. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$.

- A set S is *cohesive* for $(R_n)_{n\in\mathbb{N}}$ if $\forall n \left(S \subseteq^* R_n \vee S \subseteq^* \overline{R_n}\right)$,[1] i.e.
$$\forall n\, \exists s\, (\forall j \geq s\, (j \in S \to j \in R_n) \vee \forall j \geq s\, (j \in S \to j \notin R_n)).$$
- A set S is *strongly cohesive* for $(R_n)_{n\in\mathbb{N}}$ if
$$\forall n\, \exists s\, \forall i < n\, (\forall j \geq s\, (j \in S \to j \in R_i) \vee \forall j \geq s\, (j \in S \to j \notin R_i)).$$
- A set is called (*p-cohesive*) *r-cohesive* if it is cohesive for all (primitive) recursive sets.

Definition 2. The *cohesive principle* (COH) is the statement that for every sequence of sets an infinite cohesive set exists. Similarly, the *strong cohesive principle* (StCOH) is the statement that for every sequence of sets an infinite strongly cohesive set exists.

Hirschfeldt and Shore showed that StCOH is equivalent to COH $\wedge$ Π^0_1-CP, see [HS07, 4.4]. Hence there is no recursion theoretic difference between these principles.

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[1] $A \subseteq^* B$ stands for $A \setminus B$ is finite.

The recursion theoretic strength of the cohesive principle is well understood, its reverse mathematical strength is a topic of active research mainly in the context of the classification of Ramsey's theorem for pairs, see [HS07] for a survey.

The most important results for COH are

Theorem 3 ([JS93, JS97], see also [CJS01, theorem 12.4]). *For any degree d the following are equivalent:*

- *There is an r-cohesive (p-cohesive) set with jump of degree d,*
- $d \gg 0'$.[2]

Especially there exists a low$_2$ r-cohesive set.[3]

Theorem 4. COH *is* Π^1_1*-conservative over* RCA_0, $\mathrm{RCA}_0 + \Pi^0_1$-CP, $\mathrm{RCA}_0 + \Sigma^0_2$-IA.

This result for RCA_0 and $\mathrm{RCA}_0 + \Sigma^0_2$-IA is due to Cholak, Jockusch, Slaman, see [CJS01], the result for $\mathrm{RCA}_0 + \Pi^0_1$-CA is due to Chong, Slaman, Yang, see [CSY].

Corollary 5. $\mathrm{RCA}_0 + \mathrm{StCOH}$ *is* Π^0_2*-conservative over* PRA.

Proof. Theorem 4 together with the fact that Π^0_1-CP is Π^0_2-conservative over PRA. □

2. Bolzano-Weierstrass principle

Let BW be the statement that every sequence $(y_i)_{i\in\mathbb{N}}$ of rational numbers in the interval $[0,1]$ admits a subsequence converging with speed 2^{-n}. This principle covers the full strength of Bolzano-Weierstraß, i.e. one can take a bounded sequence of real numbers.

Let $\mathrm{BW}_{\mathrm{weak}}$ be the statement that every sequence $(y_i)_{i\in\mathbb{N}}$ of rational numbers in the interval $[0,1]$ admits a Cauchy subsequence (a sequence converging but not necessarily fast), more precisely

$$(\mathrm{BW}_{\mathrm{weak}}):$$
$$\forall (y_i)_{i\in\mathbb{N}} \subseteq \mathbb{Q}\cap[0,1]\,\exists f \text{ strictly monotone}\,\forall n\,\exists s\,\forall v,w \geq s\ |y_{f(v)} - y_{f(w)}| <_{\mathbb{Q}} 2^{-n}.$$

The statement $\mathrm{BW}_{\mathrm{weak}}$ also implies that every bounded sequence of real numbers contains a Cauchy subsequence. Just continuously map the bounded sequence into $[0,1]$ and take a diagonal sequence of rational approximations of the elements of the original sequence.

Moreover BW and $\mathrm{BW}_{\mathrm{weak}}$ also imply the corresponding Bolzano-Weierstraß principle for the Cantor space $2^{\mathbb{N}}$:

Lemma 6. *Over* RCA_0

- BW *implies the Bolzano-Weierstraß principle for the Cantor space* $2^{\mathbb{N}}$ *and*
- $\mathrm{BW}_{\mathrm{weak}}$ *implies the weak Bolzano-Weierstraß principle for the Cantor space* $2^{\mathbb{N}}$*, i.e. for every sequence in* $2^{\mathbb{N}}$ *there exists a slowly converging Cauchy subsequence.*

These implications are instance-wise, this means that for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$ *there exists (provably in* RCA_0*) a sequence* $(y_n)_{n\in\mathbb{N}} \subseteq [0,1]$*, such that from a (slowly) converging subsequence of* $(y_n)_n$ *one can compute a (slowly) converging subsequence of* $(x_n)_n$*.*

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[2] $a \gg b$ denotes that the Turing degree a contains an infinite computable branch for every b-computable 0/1-tree. Note that by the low basis theorem ([JS72]) for every b there exists a degree $a \gg b$ which is *low* over b, i.e. $a' \equiv b'$.

[3] A degree d is *low*$_2$ if $d'' = 0''$.

Proof. Define the mapping $h\colon 2^{\mathbb{N}} \to [0,1]$ as

$$h(x) = \sum_{i=0}^{\infty} \frac{2x(i)}{3^{i+1}}.$$

The image of h is the Cantor middle-third set.

One easily establishes

$$dist_{2^{\mathbb{N}}}(x,y) < 2^{-n} \quad \text{iff} \quad dist_{\mathbb{R}}(h(x),h(y)) < 3^{-(n+1)}.$$

Therefore (slow) Cauchy sequences of $2^{\mathbb{N}}$ primitive recursively correspond to (slow) Cauchy sequences of the Cantor middle-third set. The lemma follows. □

The full Bolzano-Weierstraß principle (BW) results from $\mathrm{BW}_{\mathrm{weak}}$, if we additionally require an effective Cauchy-rate, e.g. $s = 2^{-n}$ in the above definition of $\mathrm{BW}_{\mathrm{weak}}$. One also obtains full BW if one uses an instance of Π^0_1-comprehension (or Turing jump) to thin out the Cauchy sequence making it fast converging.

The weak version of the Bolzano-Weierstraß principle is for instance considered in computational analysis, see [LRZ08, section 3].

$\mathrm{BW}_{\mathrm{weak}}$ is also interesting in the context of proof-mining or hard analysis, i.e. the extraction of quantitative information for analytic statements, for an introduction to hard analysis see [Tao08, §1.3], for proof-mining see [Koh08]:
For instance if one uses $\mathrm{BW}_{\mathrm{weak}}$ to prove that a sequence converges, by theorem 9 below one can expect a primitive recursive rate of metastability, in the sense of Tao. Such proofs occur in fixed-point theory, for example Ishikawa's fixed-point theorem uses such an argument, see [Koh05, Ish76].

Note that in this case only a single instance of the Bolzano-Weierstraß principle is used and the accumulation point is not used in a Σ^0_1-induction, therefore one obtains the same results using Kohlenbach's elimination of Skolem functions for monotone formulas, see for instance [Koh00, theorem 1.2]. Nested uses of BW imply arithmetic comprehension and thus lead to non-primitive recursive growth. In contrast to that, even nested uses of $\mathrm{BW}_{\mathrm{weak}}$ in a context with full Σ^0_1-induction do not result in more than primitive recursive growth.

3. Results

Theorem 7.

$$\mathrm{RCA}_0 \vdash \mathrm{BW} \leftrightarrow \Sigma^0_1\text{-WKL},$$

where Σ^0_1-WKL *is weak König's lemma for trees given by* Σ^0_1*-predicates.*

These principles are moreover equivalent instance-wise, i.e. for each sequence in $[0,1]$ *there is (provably in* RCA_0*) a* Σ^0_1*-tree such that each infinite branch computes an accumulation point. Similar, for each* Σ^0_1*-tree there is an sequence in* $[0,1]$ *such that each accumulation point computes an infinite branch.*

Proof. For the right to left direction see [SK] and [Koh98, section 5.4].

For the other direction note that Σ^0_1-WKL is equivalent to Σ^0_2-separation, i.e. the statement that for two Σ^0_2-sets A_0, A_1 with $A_0 \cap A_1 = \emptyset$ there exists a set S, such that $A_0 \subseteq S \subseteq \overline{A_1}$.

Let B_i for $i < 2$ be a quantifier free formula such that

$$n \in \overline{A_i} \equiv \forall x\, \exists y\, B_i(x,y;n).$$

We assume that y is unique; one can always achieve this by requiring y to be minimal. Note that $\forall x\, \exists y\, B_0(x,y;n) \ \vee\ \forall x\, \exists y\, B_1(x,y;n)$.

Then define

$$f_i(n,k) := \max \left\{ s < k \mid \forall x < \mathrm{lth}\, s \ \left(B_i(x,(s)_x;n)\right) \right\}.$$

If for fixed n, i the statement $\forall x\, \exists y\, B_i(x, y; n)$ holds and f_y is the choice function for y, i.e. the function satisfying $\forall x\, B_i(x, f_y(x); n)$, then

$$f_i(n, \bar{f}_y(m) + 1) = \bar{f}_y(m).$$

If $\forall x\, \exists y\, B_i(x, y; n)$ does not hold then $\lambda k. f_i(n, k)$ is bounded. Hence

$$\text{the range of } g_i(n) := \lambda k.\, \mathrm{lth}(f_i(n, k)) \text{ is } \mathbb{N} \quad \text{iff} \quad \forall x\, \exists y\, B_i(x, y; n).$$

Therefore it is sufficient to find a set S obeying

$$\forall n\ (rng(g_0(n)) \neq \mathbb{N} \to n \in S \wedge rng(g_1(n)) \neq \mathbb{N} \to n \notin S)\,. \tag{1}$$

Define a sequence $(h_k)_{k\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$ by

$$h_k(n) := \begin{cases} 0 & \text{if } g_0(n, k) \geq g_1(n, k), \\ 1 & \text{otherwise.} \end{cases}$$

If for a fixed n there is exactly one $i < 2$, such that the range of $g_i(n)$ is $\mathbb{N}$ then $\lim_{k\to\infty} h_k(n) = i$. In this case (1) is satisfied for this n if

$$n \in S \quad \text{iff} \quad \lim_{k\to\infty} h_k(n) = 1.$$

If for each $i < 2$ the range $g_i(n)$ is $\mathbb{N}$ then (1) is trivially satisfied for this n.

Since for any accumulation point h of $(h_k)_k$ it is true that

$$h(n) = \lim_{k\to\infty} h_k(n) \quad \text{if the limit exists,}$$

h describes a characteristic function of a set S obeying (1).

This proves the theorem. □

Theorem 8.

$$\mathrm{RCA}_0 \vdash \mathrm{StCOH} \leftrightarrow \mathrm{BW}_{\mathrm{weak}}$$

Moreover these principles imply each other instance-wise.

Proof. To prove $\mathrm{BW}_{\mathrm{weak}}$ for a fixed sequence $(y_i)_{i\in\mathbb{N}}$ define

$$R_i := \left\{ j \in \mathbb{N} \;\middle|\; y_j \in \bigcup_{k \text{ even}} \left[\frac{k}{2^i}, \frac{k+1}{2^i}\right] \right\}$$

and

$$R^x := \bigcap_{i<\mathrm{lth}(x)} \begin{cases} R_i & \text{if } (x)_i = 0, \\ \overline{R_i} & \text{otherwise.} \end{cases}$$

Let f be a strictly increasing enumeration of a strongly cohesive set for $(R_i)_i$. Then by definition it follows, that

$$\forall i\, \exists x, s\ (\mathrm{lth}(x) = i \wedge \forall w > s\ \ f(w) \in R^x)\,.$$

This statement is equivalent to

$$\forall i\, \exists k, s\, \forall w > s\ \left(y_{f(w)} \in \left[\frac{k}{2^i}, \frac{k+1}{2^i}\right]\right),$$

which implies $\mathrm{BW}_{\mathrm{weak}}$.

For the other direction, let $(R_i)_{i\in\mathbb{N}}$ be a sequence of sets. Let $(y_i)_{i\in\mathbb{N}} \subseteq 2^\omega$ be the sequence defined by

$$y_i(n) := \begin{cases} 1 & \text{if } i \in R_n \\ 0 & \text{if } i \notin R_n \end{cases}.$$

Applying $\mathrm{BW}_{\mathrm{weak}}$ and lemma 6 to $(y_i)_i$ yields a slowly converging subsequence $(y_{f(i)})_i \in \mathbb{N}$, i.e.

$$\forall n\, \exists s\, \forall j, j' \geq s\, dist(y_{f(j)}, y_{f(j')}) < 2^{-n}.$$

By spelling out the definition of *dist* and y_i we obtain

$$\forall n\, \exists s\, \forall j, j' \geq s\, \forall i < n\ (f(j) \in R_i \leftrightarrow f(j') \in R_i)\,,$$

which implies that the set strictly monotone enumerated by f is strongly cohesive. □

Hence all results for StCOH carry over to $\mathrm{BW}_{\mathrm{weak}}$:

Theorem 9. $\mathrm{BW}_{\mathrm{weak}}$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + \Pi^0_1\text{-CP}$*,* $\mathrm{RCA}_0 + \Sigma^0_2\text{-IA}$*. Especially* $\mathrm{RCA}_0 + \mathrm{BW}_{\mathrm{weak}}$ *is* Π^0_2*-conservative over* PRA.

Proof. Theorem 8 and theorem 4. □

Theorem 10.

1. *Every recursive sequence of real numbers contains a* low_2 *Cauchy subsequence (a sequence converging but not necessarily fast).*
2. *There exists a recursive sequence of real numbers containing no computable Cauchy subsequence.*
3. *There exists a recursive sequence of real numbers containing no converging subsequence computable in* $0'$*.*

Proof. Theorem 8 and theorem 3. For (3) note that in the jump of a slowly converging Cauchy sequence computes a fast converging subsequence. □

Theorem 7 gives rise to another prove of this theorem and theorem 3: Let d be a degree containing solutions to all recursive instances of BW. Since BW is equivalent to Σ^0_1-WKL any degree $d \gg 0'$ suffices. Thus we may assume that d is *low* over $0'$, i.e. $d' \equiv 0''$. Now let e be a degree containing solutions to all recursive instances of $\mathrm{BW}_{\mathrm{weak}}$. Since the choice of a fast convergent subsequence of a slow convergent subsequence is equivalent to the halting problem, e may be chosen such that $e' \equiv d$. Thus $e'' \equiv 0''$ or in other words e is low_2.

Theorem 10.1 improves a result obtained by Le Roux and Ziegler in [LRZ08, section 3], which only considers integral Turing degrees.

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