PROGRAM EXTRACTION FOR 2-RANDOM REALS

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Abstract. Let 2-RAN be the statement that for each real X a real 2-random relative to X exists. We apply program extraction techniques we developed in [KK12, Kre12] to this principle.

Let WKL_0^ω be the finite type extension of WKL_0. We obtain that one can extract primitive recursive realizers from proofs in $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + 2\text{-}\mathsf{RAN}$, i.e., if $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + 2\text{-}\mathsf{RAN} \vdash \forall f\, \exists x\, A_{qf}(f,x)$ then one can extract from the proof a primitive recursive term $t(f)$ such that $A_{qf}(f,t(f))$.

As a consequence, we obtain that $\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CP} + 2\text{-}\mathsf{RAN}$ is Π^0_3-conservative over RCA_0.

Let n-RAN be the statement

$$\forall X\, \exists Y\ (Y \text{ is } n\text{-random relative to } X)\,.$$

It is known that 1-RAN is equivalent to weak weak König's lemma (WWKL). That is the restriction of weak König's Lemma to infinite binary trees T, which additionally satisfy

$$\lim_{i\to\infty} \frac{|\{s \in T \mid \mathrm{lth}(s) = i\}|}{2^i} > 0, \tag{1}$$

see [YS90].

Avigad, Dean, and Rute showed that, relative to $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$, the principle 2-$\mathsf{RAN}$ is equivalent to WWKL for trees computable in the first Turing jump (of the parameters), see [ADR12]. This principle is denoted by 2-WWKL. Recently, Conidis and Slaman showed that 2-RAN is Π^1_1-conservative over $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$, see [CS].

In this paper we will prove a program extraction result along this lines which additionally deals with WKL. In detail, we will show the following theorem:

Theorem 1. *The system* $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{CAC} + 2\text{-}\mathsf{RAN}$ *is conservative over* RCA_0^ω *for sentences of the for* $\forall f\, \exists x\, A_{qf}(f,x)$. *Moreover, from a proof one can extract a primitive recursive realizer* $t[f]$ *for* y.

The ω superscript at WKL_0 and RCA_0 indicates that we use the finite type variant of these systems. This means they are not sorted into two types for $\mathbb{N}$ and subsets of $\mathbb{N}$, but into countable many types for $\mathbb{N}$, $\mathbb{N}^\mathbb{N}$, $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ etc. These systems are conservative over their second order counterpart. See [Koh05]. Below we will also need the finite type variants of the systems WKL_0^* and RCA_0^*. These systems are defined to be RCA_0 resp. WKL_0 where Σ^0_1-induction is replaced by the exponential

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function and quantifier-free induction, see [Sim09, X.4]. The finite type variants will be denote by $\mathsf{WKL}_0^{\omega*}$ and $\mathsf{RCA}_0^{\omega*}$.

Theorem 1 also deals with the chain antichain principle (CAC). This principle states that each partial ordering contains an infinite chain or an infinite antichain. In [CSY12] is was show that CAC is Π^1_1-conservative over $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$. We established a program extraction for CAC in [Kre12]. Theorem 1 extends this result.

Since Π^0_3-statements are equivalent to statements of the form $\forall f\, \exists x\, A_{qf}(f,x)$, we immediately obtain from Theorem 1 the following corollary.

Corollary 2. $\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{CAC} + 2\text{-}\mathsf{RAN}$ *is* Π^0_3*-conservative over* RCA_0*.*

The proof of Theorem 1 is based on the techniques we developed in [KK12, Kre12]. There we introduced the notion *proofwise low*. Roughly speaking, this notion covers the computational content of *low*$_2$-ness but also keeps track of the induction used in the proof. A Π^1_2-principle P of the form

$$\forall X\, \exists Y\, P'(X,Y) \tag{2}$$

is called proofwise low over a system, say $\mathsf{WKL}_0^{\omega*}$, if for each term ϕ a term ξ exists such that

$$\mathsf{WKL}_0^{\omega*} \vdash \forall X\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi X) \to \exists Y\, \left(P'(X,Y) \wedge \Pi^0_1\text{-}\mathsf{CA}(\phi XY)\right)\right).$$

Here $\Pi^0_1\text{-}\mathsf{CA}(t) :\equiv \exists f\, \forall n\, (f(n) = 0 \leftrightarrow \forall x\, t(n,x) = 0)$.

We showed that for principles P of the form (2) where P' is Π^0_1 and that are proofwise low relative to $\mathsf{WKL}_0^{\omega*}$, a program extraction result of the form of Theorem 1 holds, see [Kre12, Corollary 3.4]. We will prove Theorem 1 by showing that 2-WWKL, and hence 2-RAN, is (equivalent to) such a principle and these results are applicable.

Proof of Theorem 1

Let $\mathcal{K}$-WWKL be weak weak König's Lemma where the tree is given by a formula of the class $\mathcal{K}$. Using this notation 2-WWKL is the same as Δ^0_2-WWKL. The following lemma shows that we can restrict our attention to Σ^0_1-WWKL.

Lemma 3.

(i) $\mathsf{RCA}_0^* \vdash \Pi^0_1\text{-}\mathsf{WWKL} \leftrightarrow \mathsf{WWKL}$
(ii) $\mathsf{RCA}_0^* \vdash \Pi^0_2\text{-}\mathsf{WWKL} \leftrightarrow \Sigma^0_1\text{-}\mathsf{WWKL}$

Proof. Let $T = \{s \in 2^{\mathbb{N}} \mid \forall k\, f(s,k) = 0\}$ be a Π^0_1-tree such that (1) holds. Then the tree $T' := \{s \in 2^{\mathbb{N}} \mid \forall s' \sqsubseteq s\, \forall k \leq \mathrm{lth}(s)\, f(s',k) = 0\}$ is recursive, has the same infinite branches as T, and satisfies (1) since $T' \supseteq T$. Thus WWKL suffices to find an infinite branch of T.

Now let $T = \{s \in 2^{\mathbb{N}} \mid \forall k\, \exists n\, f(s,k,n) = 0\}$ be a Π^0_2-tree such that again (1) holds. Then $T' := \{s \in 2^{\mathbb{N}} \mid \forall s' \sqsubseteq s\, \forall k \leq \mathrm{lth}(s)\, \exists n\, f(s',k,n) = 0\}$ is a Σ^0_1-tree and again has the same infinite branches as T and satisfies (1). Therefore, Σ^0_1-WWKL yields an infinite branch. □

Proposition 4. *For each term ϕ and each m there exists a closed term ξ such that* $\mathsf{RCA}_0^{\omega*} + \Pi^0_1\text{-}\mathsf{CA}(\xi)$ *proves that there exists a tree T with*

$$\lim_{i\to\infty} \frac{|\{s \in T \mid \mathrm{lth}(s) = i\}|}{2^i} \geq 1 - 2^{-m} \tag{3}$$

and for each infinite branch b of T the statement Π^0_1-$\mathsf{CA}(\phi b)$ is provable.

The proof of this proposition make use of the concept of an associate. An associate is a representation of a continuous functional on $\mathbb{N}^{\mathbb{N}}$. For a continuous functional $F(g)$ a function α_F satisfying the following statement is called an associate for F.

$$\forall f\, \exists n\, \alpha_F(\overline{g}n) \neq 0, \quad \forall f, n\ (\alpha_F(\overline{g}n) \neq 0 \to \alpha_F(\overline{g}n) - 1 = F(g))\,,$$

where $\overline{g}$ denotes the course-of-value function for g. Note that the functional F is determined by the values of α_F. The closed terms of the finite type systems we consider here are provably continuous and have associates, see [Tro73, Koh05].

Before we come to the proof we define the shorthand

$$\mu_i(X) := \frac{\left|X \cap 2^i\right|}{2^i}.$$

With this, condition (1) can be rephrased as $\lim_{i\to\infty} \mu_i(T) \geq 1 - 2^{-m}$.

Proof of Proposition 4. Let $\alpha_\phi(s, n, k)$ be an associate of $\phi(b, n, k)$. Then we have

$$\forall k\, \phi(b, n, k) = 0 \leftrightarrow \forall k, k'\, \alpha_\phi(\overline{b}(k'), n, k) \leq 1.$$

For each n the full binary tree $2^{<\mathbb{N}}$ decomposes into the sets

$$X_n := \{s \in 2^{<\mathbb{N}} \mid \forall k\, \alpha_\phi(s, n, k) \leq 1\} \quad \text{and} \quad Y_n := \{s \in 2^{<\mathbb{N}} \mid \exists k\, \alpha_\phi(s, n, k) > 1\}.$$

The sets X_n is by the properties of an associated closed under prefix. Therefore, it forms a tree. The sets Y_n can be approximated with the sets $Y_{n,l} := \{s \in 2^{\leq l} \mid \exists k < l\, \alpha_\phi(s, n, k) > 1\}$ in the sense that $Y_n = \bigcup_{l\in\mathbb{N}} Y_{n,l}$ and $Y_{n,l} \subseteq Y_{n,l'}$ for $l < l'$.

Since $\alpha_\phi(s, n, k) > 1$ implies $\alpha_\phi(s * \langle x\rangle, n, k) > 1$ for any $x < 2$ we have that

$$\mu_i(Y_n) \leq \mu_j(Y_n) \quad \text{and} \quad \mu_i(Y_{n,l}) \leq \mu_j(Y_{n,l}) \qquad \text{if } i < j. \tag{4}$$

With this we obtain

$$\lim_{i\to\infty} \mu_i(Y_n) = \lim_{i\to\infty} \lim_{l\to\infty} \mu_i(Y_{n,l}) \overset{(4)}{\leq} \lim_{l\to\infty} \mu_l(Y_{n,l}) \leq \lim_{i\to\infty} \mu_i(Y_n).$$

We conclude that all the expressions are equal and thus

$$\forall n, k\, \exists l\, \forall i > l\ |\mu_i(Y_n) - \mu_l(Y_{n,l})| < 2^{-k}. \tag{5}$$

A choice function $g(n, k)$ that outputs for each n, k such an l, exists by a suitable instance of Π^0_1-AC which follows from Π^0_1-$\mathsf{CA}(\xi)$ for a suitable choice of ξ, see [Koh98], [Koh08, Chapter 13.4].

Let $Y^{\sqsubseteq}_{n,l}$ be the set of all branches going thought $Y_{n,l}$, i.e.

$$Y^{\sqsubseteq}_{n,l} := \{s \in 2^{\mathbb{N}} \mid \exists s' \in Y_{n,l}\ (s' \sqsubseteq s \vee s \sqsubseteq s')\}.$$

By definition we have

$$\mu_l(Y_{n,l}) = \mu_i(Y^{\sqsubseteq}_{n,l}) \qquad \text{for all } i \geq l.$$

Since the set $Y_{n,l}$ is finite and decidable, it is clear that $Y^{\sqsubseteq}_{n,l}$ is also decidable. The set $Y^{\sqsubseteq}_{n,l}$ is obviously closed under prefix and therefore is a tree.

Consider $X_n \cup Y^{\sqsubseteq}_{n,g(n,k)}$. This set is a union of trees and, hence, a tree. Moreover

$$\begin{aligned}\lim_{i\to\infty} \mu_i(X_n \cup Y^{\sqsubseteq}_{n,g(n,k)}) &= \lim_{i\to\infty} \mu_i(X_n) + \lim_{i\to\infty} \mu_i(Y^{\sqsubseteq}_{n,g(n,k)}) \\ &= \lim_{i\to\infty} \mu_i(X_n) + \mu_{g(n,k)}(Y^{\sqsubseteq}_{n,g(n,k)}) \\ &\overset{(5)}{\geq} \lim_{i\to\infty} \mu_i(X_n) + \lim_{i\to\infty} \mu_i(Y_n) - 2^k = 1 - 2^k\end{aligned}$$

By definition of the sets X_n and Y_n we have that for each infinite branch b of the tree $X_n \cup Y^{\sqsubseteq}_{n,g(n,k)}$ we have that

$$\forall k\, \phi(b,n,k) = 0 \tag{6}$$

if and only if b is an infinite branch through X_n which is only the case if b does not go through $Y^{\sqsubseteq}_{n,g(n,k)}$. Since this is decidable, we can decide (6). The tree $X_n \cup Y^{\sqsubseteq}_{n,g(n,k)}$ is Π^0_1 since X_n is Π^0_1.

Now consider the tree $T = \bigcap_{n\in\mathbb{N}}(X_n \cup Y^{\sqsubseteq}_{n,g(n,m+n+1)})$. Since T is an intersection of trees, it is again a tree. One checks that

$$\lim_{i\to\infty} \mu_i(T) \geq 1 - \sum_{n=0}^{\infty} 2^{m+n+1} \geq 1 - 2^m.$$

Let b be any infinite branch of T. Since T is contained in $X_n \cup Y^{\sqsubseteq}_{n,g(n,m+n+1)}$ for each n the property (6) is decidable and thus Π^0_1-$\mathsf{CA}(\phi b)$ provable.

The tree T is Π^0_1. Using the construction described in the proof of Lemma 3 one obtains a recursive tree which has the desired properties. □

This proof is inspired by [Kau91], [DH10, Theorem 8.14.1].

In order to show that Σ^0_1-WWKL can be written as a principle of the form (2) with $P' \in \Pi^0_1$ we first observe that the sequence under the limit in (1) is decreasing, if T is a tree. Thus this limit is > 0 if and only if there exists an m such that each element of the sequence is $\geq 2^{-m}$. With this Σ^0_1-WWKL can be written in the following form

$$\forall f, m \left(\mathsf{T}_{\Sigma^0_1}(f) \land \forall n\, \frac{|\{s \in 2^n \mid \exists k\, f(s,k) = 0\}|}{2^n} \geq_{\mathbb{Q}} 2^{-m} \to \exists b\, \forall n\, \exists k\, f(\overline{b}(n), k) = 0 \right)$$

where b is a function, $\overline{b}$ is the course-of-value function of b and $\mathsf{T}_{\Sigma^0_1}(f)$ denotes the statement that f describes a binary Σ^0_1-tree, i.e.

$$\forall s\, \left(\exists k\, f(s,k) = 0 \to \forall s' \sqsubseteq s\, \exists k\, f(s,k) = 0 \land s \in 2^{<\mathbb{N}}\right).$$

Let $f'(s,k) := \min_{k'\leq k} f(s,k')$. By taking a choice function for the first k and a maximum we obtain the following, equivalent statement

$$\forall f, g, m \left(\mathsf{T}_{\Sigma^0_1}(f) \land \forall n\, \frac{|\{s \in 2^n \mid f'(s,g(n)) = 0\}|}{2^n} \geq_{\mathbb{Q}} 2^{-m} \to \exists b\, \forall n\, \exists k\, f(\overline{b}(n), k) = 0 \right) \tag{7}$$

We define the following constructions: Let

$$\hat{f}(s,k) := \begin{cases} 0 & \text{if } s \in 2^{<\mathbb{N}} \text{ and } \forall s' \sqsubseteq s\, f'(s',k) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$f_{g,m}(s,k) := \begin{cases} f(s,k) & \text{if } \forall n \le \mathrm{lth}(s) \left(\frac{1}{2^n} \left|\{s \in 2^n \mid f'(s,g(n)) = 0\}\right| \ge_{\mathbb{Q}} 2^{-m}\right), \\ 0 & \text{otherwise.} \end{cases}$$

These constructions can be defined in $\mathsf{RCA}_0^{\omega*}$ and it is easy to see that $\forall f\, \mathsf{T}_{\Sigma^0_1}(\hat{f})$ and $\forall f\, \mathsf{T}_{\Sigma^0_1}(f) \to f =_1 \hat{f}$. Also by construction (provably in $\mathsf{RCA}_0^{\omega*}$)

$$\forall f, g\, \forall m, n \left(\frac{1}{2^n} \left| \left\{ s \in 2^n \mid \widehat{(\hat{f})_{g,m}}(s, g(n)) = 0 \right\} \right| \ge_{\mathbb{Q}} 2^{-m} \right)$$

and $(\hat{f})_{g,m} = \hat{f}$ if f, g, m satisfy $\forall n\, \frac{1}{2^n} \left| \left\{ s \in 2^n \mid \hat{f}(s, g(n)) = 0 \right\} \right| \ge_{\mathbb{Q}} 2^{-m}$.

Thus (7) is equivalent to

$$\forall f, g, m\, \exists b\, \forall n\, \exists k\, \widehat{\hat{f}_{g,m}}(\bar{b}(n), k) = 0.$$

By an application of $\mathsf{QF\text{-}AC}^{0,0}$ this is equivalent to

$$\forall f, g, m\, \exists b, h\, \forall n\, \widehat{\hat{f}_{g,m}}(\bar{b}(n), h(n)) = 0,$$

which is the desired form. We will call this principle $\Sigma^0_1\text{-}\widehat{\mathsf{WWKL}}(\langle f,g,m\rangle, \langle b,h\rangle)$.

Theorem 5. *The principle $\Sigma^0_1\text{-}\widehat{\mathsf{WWKL}}$ is proofwise low over $\mathsf{WKL}_0^{\omega*}$, i.e. for all terms ϕ there exists an ξ such that*

$$\begin{aligned} \mathsf{WKL}_0^{\omega*} \vdash \forall f, g, m \Big(& \Pi^0_1\text{-}\mathsf{CA}(\xi(f,g,m)) \\ & \to \exists b, h \left(\Sigma^0_1\text{-}\widehat{\mathsf{WWKL}}(\langle f,g,m\rangle, \langle b,h\rangle) \wedge \Pi^0_1\text{-}\mathsf{CA}(\phi(f,g,m,b,h)) \right) \Big). \end{aligned}$$

Proof. Fix f, g, m and assume that that f describes a Σ^0_1-tree

$$T = \{ s \in 2^{\mathbb{N}} \mid \exists k\, f(s,k) = 0 \}$$

and satisfies premise of (7). Otherwise we could replace f by $\widehat{\hat{f}_{g,m}}$. We may also assume that for each s there is at most one k such that $f(s,k) = 0$.

Let $\alpha_{\phi(f,g,m)}$ be that associate of ϕ with respect to the parameters b, h. Then

$$\begin{aligned} \forall k\, \phi(f,g,m,b,h,n,k) = 0 & \leftrightarrow \forall k\, \forall k', k''\, \alpha_{\phi(f,g,m)}(\bar{b}(k'), \bar{h}(k''), n, k) = 0 \\ & \leftrightarrow \forall k\, \forall k'\, \forall s'' \left(\forall i < \mathrm{lth}(s'')\, f(\bar{b}(i), (s'')_i) = 0 \to \alpha_{\phi(f,g,m)}(\bar{b}(k'), s'', n, k) = 0 \right) \end{aligned}$$

Thus, we many disregard the parameter h and just prove $\Pi^0_1\text{-}\mathsf{CA}(\phi'(f,g,m,b))$ for a given ϕ'.

By Proposition 4 there exists a term $\xi_1(f,g,m)$ a tree T' such that $\Pi^0_1\text{-}\mathsf{CA}(\xi_1(f,g,m))$ proves that T' exists, for each infinite branch b of T' the statement $\Pi^0_1\text{-}\mathsf{CA}(\phi'(f,g,m,b))$ is provable, and $\lim_{i\to\infty} \mu_i(T') \ge 1 - 2^{-(m+1)}$.

Let $\xi_2(f,g,m,n,k) := f(n,k)$. Then $\Pi^0_1\text{-}\mathsf{CA}(\xi_2(f,g,m))$ decides $\exists k\, f(s,k) = 0$ and thus relative to this statement T is recursive. By the properties of T we have that $\lim_{i\to\infty} \mu_i(T) \ge 2^{-m}$.

Consider the tree $T \cap T'$. For this tree $\lim_{i\to\infty} \mu_i(T \cap T') \ge 2^{-m+1}$. Therefore, it is infinite. By WKL it has an infinite branch b, and by definition $\Pi^0_1\text{-}\mathsf{CA}(\phi'(f,g,m,b))$ is provable.

Noting that Π^0_1-$\mathsf{CA}(\xi_1(f,g,m))$ and Π^0_1-$\mathsf{CA}(\xi_2(f,g,m))$ can be coded into one instance $\xi(f,g,m)$ of Π^0_1-CA, see [Koh98, Remark 3.8.2], proves the theorem. $\square$

Proof of Theorem 1. The theorem without CAC follows from Corollary 3.4 of [Kre12], Theorem 5, and the fact that Σ^0_1-$\widehat{\mathsf{WWKL}}$ and 2-RAN are equivalent over $\mathsf{WKL}^\omega_0 + \Pi^0_1$-$\mathsf{CP}$.

The full statement of Theorem 1 follows from the fact that CAC is proof-wise low over a suitable system, see also [Kre12], and one can code two proofwise low principle into one. $\square$

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