Complete Black-Scholes Proofs: From First Principles

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1 Part I: Mathematical Foundations and Definitions

This section establishes all the mathematical machinery needed for the Black-Scholes proofs. We assume knowledge of measure theory and basic stochastic calculus but will define all financial and probabilistic concepts explicitly.

1.1 Financial Market Framework

Definition 1.1 (Financial Market)

A financial market consists of:

1. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω represents all possible market scenarios
2. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ representing information available at time t
3. A finite collection of traded assets with price processes $\{S_t^i\}_{i=0,1,\ldots,n}$

Definition 1.2 (Brownian Motion)

A stochastic process $\{W_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard Brownian motion if:

1. $W_0 = 0$ almost surely
2. W has independent increments: for $0 \leq s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$
3. $W_t - W_s \sim N(0, t-s)$ for all $0 \leq s < t$
4. W has continuous sample paths almost surely

Definition 1.3 (Natural Filtration of Brownian Motion)

The natural filtration of Brownian motion is

$$\mathcal{F}_t^W = \sigma(W_s : 0 \leq s \leq t)$$

the σ-algebra generated by the Brownian motion up to time t.

Definition 1.4 (Adapted Process)

A stochastic process $\{X_t\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition 1.5 (Geometric Brownian Motion)

A process $\{S_t\}_{t\geq 0}$ follows geometric Brownian motion with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ if it satisfies the stochastic differential equation:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

with initial condition $S_0 > 0$.

Remark 1.1

The explicit solution to geometric Brownian motion is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

This can be verified using Itô's lemma on $f(t, x) = \ln x$.

1.2 Portfolio and Trading Strategy Concepts

Definition 1.6 (Trading Strategy)

A trading strategy is a pair of adapted processes (ϕ_t^0, ϕ_t^1) where:

- ϕ_t^0 represents the number of units of the bond held at time t
- ϕ_t^1 represents the number of shares of stock held at time t

Both processes must be adapted to the filtration $\{\mathcal{F}_t\}$.

Definition 1.7 (Portfolio Value)

The value of a portfolio with trading strategy (ϕ_t^0, ϕ_t^1) at time t is:

$$V_t = \phi_t^0 B_t + \phi_t^1 S_t$$

where B_t is the bond price and S_t is the stock price.

Definition 1.8 (Self-Financing Strategy)

A trading strategy (ϕ_t^0, ϕ_t^1) is self-financing if:

$$dV_t = \phi_t^0 dB_t + \phi_t^1 dS_t$$

This means no money is added or withdrawn from the portfolio; changes in value come only from price movements of held assets.

Definition 1.9 (Arbitrage Opportunity)

An arbitrage opportunity is a self-financing trading strategy with:

1. Initial value $V_0 = 0$
2. $\mathbb{P}(V_T \geq 0) = 1$ for some time $T > 0$
3. $\mathbb{P}(V_T > 0) > 0$

Definition 1.10 (Arbitrage-Free Market)

A market is arbitrage-free if no arbitrage opportunities exist.

1.3 Martingale Theory for Finance

Definition 1.11 (Martingale)

An adapted process $\{M_t\}_{t\geq 0}$ is a martingale with respect to filtration $\{\mathcal{F}_t\}$ and probability measure $\mathbb{P}$ if:

1. $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$
2. $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for all $0 \leq s \leq t$

Definition 1.12 (Equivalent Probability Measures)

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ are equivalent (written $\mathbb{P} \sim \mathbb{Q}$) if they have the same null sets:

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0 \text{ for all } A \in \mathcal{F}$$

Definition 1.13 (Radon-Nikodym Derivative)

If $\mathbb{Q} \ll \mathbb{P}$ (Q is absolutely continuous with respect to P), then there exists a non-negative $\mathcal{F}$-measurable random variable Z such that:

$$\mathbb{Q}(A) = \int_A Z \, d\mathbb{P} \text{ for all } A \in \mathcal{F}$$

We write $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and call Z the Radon-Nikodym derivative.

Theorem 1.1 (Girsanov's Theorem - Statement)

Let θ be an adapted process with $\int_0^T \theta_s^2 \, ds < \infty$ almost surely. Define:

$$Z_t = \exp\left(-\int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right)$$

If $\mathbb{E}[Z_T] = 1$, then:

1. The process Z_t is a martingale
2. The measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ is a probability measure equivalent to $\mathbb{P}$
3. Under $\mathbb{Q}$, the process $\tilde{W}_t = W_t + \int_0^t \theta_s \, ds$ is a Brownian motion

Definition 1.14 (Risk-Neutral Measure)

In a financial market with bond $B_t = e^{rt}$ and stock following

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is called risk-neutral if the discounted stock price $e^{-rt} S_t$ is a $\mathbb{Q}$-martingale.

1.4 Options and Derivatives

Definition 1.15 (European Option)

A European option is a financial contract that gives the holder the right (but not obligation) to:

- **Call option**: Buy an asset at strike price K at maturity time T
- **Put option**: Sell an asset at strike price K at maturity time T