Inertia Wheel Inverted Pendulum

Ashwin Krishna
Harvard University
Email: ashwin_krishna@college.harvard.edu

Nao Ouyang
Harvard University
Email: nouyang@g.harvard.edu

***Abstract*—We explore the classic nonlinear controls problem, inverting a pendulum, using analyses learned in this class. Specifically, we look at using a flywheel to stabilize the pendulum. We build a physical system from scratch. In simulation, we derive the equations of motion and apply LQR and region of attraction analyses for our system. We additionally perform controllability analysis to the case where the full state may not be measurable, as happened in our physical system. In hardware, we successfully implement downward stabilization and swingup controls using both a PD controller an a bang-bang controller. We discuss the design decisions, design iterations, and present work toward implementing a controller for the inverted state.**

I. Introduction

Pendulum is inherently unstable around upright fixed point.

Example write-up:

This demo file is intended to serve as a "starter file" for IEEE conference papers produced under LaTeX using IEEEtran.cls version 1.8b and later. I wish you the best of success.

mds

August 26, 2015

II. LQR Analysis

A. Equations of Motion

To derive the equations of motion (EOM), we use the Lagrangian method. Let L equal to the kinetic energy plus the potential energy of the system.

$$L = KE - PE \tag{1}$$

By Lagrange's method,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = \sum_{i=0}^{n} F_i \tag{2}$$

for $i = 1, 2, 3...n$ forces.

Thus, we need to write out the KE, the PE, the derivative of L with respective to each state q, the derivative of L with respect to the (time) derivative of each state q, and then the time derivative of that last term.

Let us first consider the unaltered case, from the problem set, where here we will derive the equations of motion by hand but otherwise simply explain the derivation in detail. Later, we will consider a system that more closely matches our real-life system. We will not be able to compare the model with reality, since we were unable to implement the full state measurement so LQR cannot apply. Instead, we show another example as applied to a modified system where the reaction wheel pendulum is put on an (unpowered) cart.

1) Write the KE of the system. We can decompose this into the translational and rotational components.

Let us consider (abstractly) the translational KE of a point mass m rotating around the origin on a massless string of length l. θ is defined as angle from the downward vertical point, increasing counterclockwise (diagram not provided). The position of the point mass is $x = lcos\theta$ and $y = lsin\theta$. KE is $\frac{1}{2}m \cdot q^2$, where q is the position.

$$KE_x = 0.5m(l \cdot \frac{d}{dt}\sin\theta)^2 = \frac{1}{2}m(l\dot{\theta}\cos\theta)^2 \tag{3}$$

$$KE_y = 0.5m(l \cdot \frac{d}{dt}\cos\theta)^2 = \frac{1}{2}m(-l\dot{\theta}\sin\theta)^2 \tag{4}$$

$$KE = KE_x + KE_y = \frac{1}{2}ml^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \tag{5}$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 \tag{6}$$

$$\tag{7}$$

where on the last step we used the trig identity $\cos^2 + sin^2 = 1$.

Now applying this to the stick and flywheel components of our system, we calculate 1) the stick around the origin 2) the flywheel around the origin. Note that the KE of the stick acts at l_1, the center-of-mass of the stick, not l_2.

$$KE_{\text{translational}} = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2(l_2\dot{\theta}_1)^2 \tag{8}$$

$$\tag{9}$$

Additionally we have the inertial component of KE since we have angular velocities here and our stick has mass and our previous point mass is instead a rotating flywheel. The general formula is $KE = \frac{1}{2}I_2\dot{\theta}^2$. Noting that angular velocities "add", and applying this to each component of our system; we calculate 1) inertial KE of the stick 2) inertial KE of the flywheel.

$$KE_{\text{inertial}} = \frac{1}{2}m_1(I_1\dot{\theta}_1)^2 + \frac{1}{2}m_2(I_2(\dot{\theta}_1 + \dot{\theta}_2))^2 \tag{10}$$

$$\tag{11}$$

The total KE of the system is the sum of the above.

2) The potential energy (PE) of the system is more straightforward. Gravitationally speaking, (and with a bit of geometry

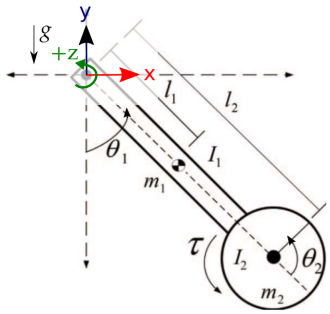

Fig. 1. Free body diagram

TABLE I
SYSTEM CONSTANTS

Property	Measurement
$m_{stick} = m_1$	115 g
$m_{flywheel}$	546 g
m_{motor}	450 g
$l_{stick} = l_2$	21 cm
$r_{flywheel}$	8.5 cm

- note that our theta is defined from vertical and increasing counterclockwise)

$$PE = m_1 g(1 - l_1 \cos\theta_1) + m_2 g(1 - l_2 \cos\theta_1) \tag{12}$$

3) Now we have the lagrangian $L = KE - PE$ and must take the partial of the lagrangian with respect to each state variable, in our case θ_1 and θ_2.

Using sympy, we calculate

$$\frac{\partial L}{\partial q_i} = BLAH \tag{13}$$

4) We also calculate the

$$\frac{\partial L}{\partial \dot{q}_i} = BLAH \tag{14}$$

5) Finally, we calculate the time derivative of the last term

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = BLAH \tag{15}$$

6) Subtracting, we set the equation equal to our input torque.
TODO
This gives us our final manipulator equations.

B. Find the Fixed Points

To find the fixed points of the system, we calculate $s.t. f() = 0$

C. Linearization Around Fixed Point

We can further use sympy to linearize our fixed points.

D. Constants

E. K values

F. Region of Attraction via Lyapunov

G. Energy-based Controller for Swingup

III. LQR FOR "SYSTEM ON WHEELS"

For a detour (in order to demonstrate understanding of the problem set material) we imagine sticking the whole thing on wheels and redo the same analysis, although for sanity we run the calculations through sympy instead of by hand.

A. Controllability

We begin by analyzing the controllability of the system.

1) Equations of Motion:

2) Linearization:

IV. CONCLUSION

The conclusion goes here.

ACKNOWLEDGMENT

The authors would like to thank...

REFERENCES

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.