UNIT 5: Network Analysis and Game Theory

Network Analysis

Networks are diagrams easily visualized in transportation system like roads, railway lines, pipelines, blood vessels, etc.

A project will consist of a number of jobs and particular jobs can be started only after finishing some other jobs. There may be jobs which may not depend on some other jobs. Network scheduling is a technique which helps to determine the various sequences of jobs concerning a project and the project completion time. There are two basic planning and control techniques that use a network to complete a pre-determined schedule. They are Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM). The critical path method (CPM) was developed in 1957 by JE Kelly of Ramington R and M.R. Walker of Dupon to help schedule maintenance of chemical plants. CPM technique is generally applied to well known projects where the time schedule to perform the activities can exactly be determined.

Network is a technique used for planning and scheduling of large projects in the fields of construction, maintenance, fabrication, purchasing, computer system instantiation, research and development planning etc. There is multitude of operations research situations that can be modeled and solved as network. Some recent surveys reports that as much as 70% of the real-world mathematical programming problems can be represented by network related models. Network analysis is known by many names _PERT (Programme Evaluation and Review Technique), CPM (Critical Path Method), PEP (Programme Evaluation Procedure),

LCES (Least Cost Estimating and Scheduling), SCANS (Scheduling and Control by

Automated Network System), etc

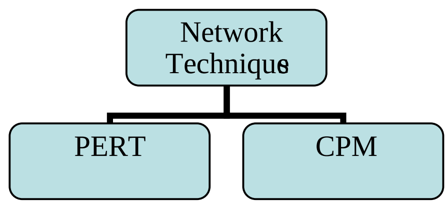
This chapter will present three of algorithms.

1. PERT & CPM
2. Shortest- route algorithms
3. Maximum-flow algorithms

The Basic Terminology

Network

It is a graphical representation of logical and sequentially connected activities and events of a project. Network is also called **arrow diagram**. PERT (Programme Evolution Review Technique) and (Critical Path Method) are the two most widely applied techniques.



Project

A project is defined as a combination of interrelated activities which must be executed in a certain order in for its completion.

Project Management Process

Network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects

Activity

Any individual operation, which utilizes resources and has an end and a beginning, is called activity.

- A task or a certain amount of work required in the project
- Requires time to complete
- Represented by an arrow

These are usually classified into four categories:

- Predecessor activity
- Successor activity

- Concurrent activity
- Dummy activity

Dummy Activity

It Indicates only precedence relationships and does not require any time of effort

PERT(Program Evaluation and Review Technique) is a method to analyze the involved tasks in completing a given project, especially the time needed to complete each task, and identifying the minimum time needed to complete the total project.

PERT is based on the assumption that an activity's duration follows a probability distribution instead of being a single value

Three time estimates are required to compute the parameters of an activity's duration distribution:

1. **Pessimistic time** (t_p) - the time the activity would take if things did not go well
2. **Most likely time** (t_m) - the consensus best estimate of the activity's duration
3. **Optimistic time** (t_o) - the time the activity would take if things did go well.

Mean (expected time) = $\frac{(t_p - 4t_m + t_o)}{6}$

$\sigma^2 \quad \left(\frac{t_p - t_o}{6}\right)^2$

Probability computation Determine probability that project is completed within specified

$$= \frac{X - \mu}{\sigma}$$

Where μ = project mean t ime

σ = project standard mean time

x = (proposed) specified time

Variance ( )=

time Z

Float:

Float of an activity represents the excess of available time over its duration.

Total Float (F_t)

The amount of time by which the completion of an activity could be delay beyond the earliest expected completion time without affecting the overall project duration.

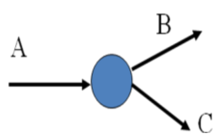
i.e. Tf= (Latest start-Earliest start) for activity(i-j), or, $(Tf)_{ij}=(LS)_{jj}-(ES)_{ij}$

Free Float (F_f)

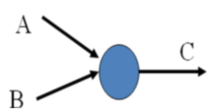
The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent (succeeding) activities.

Situations in network diagram

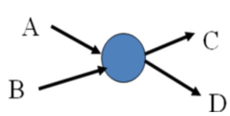
1. A must finish before either B or C can start



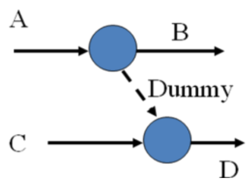
2. Both A and B must finish before C can start



3. Both A and C must finish before either of B or D can start



4. A must finish before B can start both A and C must finish before D can start



Benefits of CPM/PERT

1) Useful at many stages of project management
2) Mathematically simple
3) Give critical path and slack time
4) Provide project documentation
5) Useful in monitoring costs

Distinguish Between PERT and CPM?

PERT (Programme Evaluation Review Technique)	**CPM** (Critical Path Method)
1. PERT is event oriented.	1. CPM is activity oriented.
2. PERT is probabilistic.	2. CPM is deterministic.
3. PERT is primarily concerned with time only.	3. CPM places dual emphasis on project time as well cost.
4. PERT is generally used for projects where time required to complete the activities is not known a priori. Thus PERT is used for large, R&D type of projects.	4. CPM is used for projects which are repetitive in nature and comparatively small in size.
5. Three time estimates are possible for activities linking up two events.	5.One time estimate is possible for activities (No allowance is made for uncertainty)

Example: 02. The following table gives the activities of a construction project and other relevant information.

Activities (i-j)	Normal duration (days)	Crash duration (days)	Crashing cost (Rs. per day)
1-2	9	6	20
1-3	8	5	25
1-4	15 5	10	30
2-4	10	3	10
3-5	2	6	15
4-5		1	40

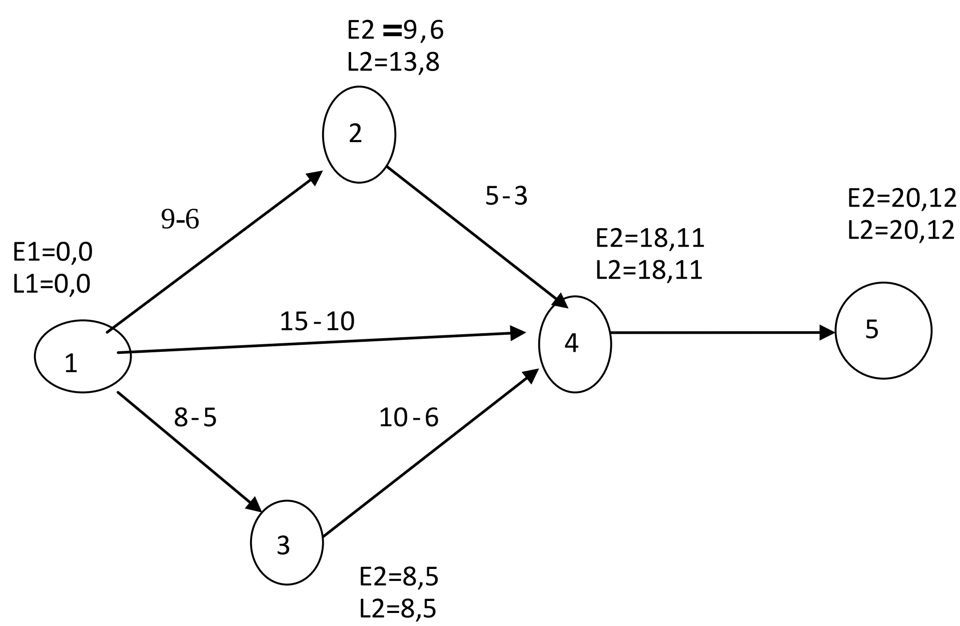
A. What is the normal project length and minimum project length?

B. Determine the minimum crashing costs of schedule ranging from length down to and the minimum length schedule.

C. What is the optimal length schedule duration of each job for your solution?

Given that over head cost total Rs. 60 per day.

Solution:



A. The Critical path is 1→3→4→5 with normal duration 20 days and minimum project length is 12 days.

Normal Project length(days)	Crashing Cost (day/Rs.)	Overhead cost @ Rs. 60/day	Total Cost(Rs.)
20	------	20 [redacted] 60=1200	1200
19	1×15=:	19 [redacted] 60=1140	1155
18	15+1 15=30	18 [redacted] 60=1080	1110
17	30+1 13=45	17 [redacted] 60=1020	1065
16	45+1 40=85	16 [redacted] 60=960	1045
15	85+1 40+1 30=145	15 [redacted] 60=900	1030
14	145+1 30+1 10+1 25=195	14 [redacted] 60=840	1035

B. Total cost increasing for 14 days duration, the minimum total cost Rs. 1030 occurs 15 days duration.
C. Optimum duration of each job is as follows:
D.

Job:	(1,2)	(1,3)	(1,4)	(2,4)	(3,4)	(4,5)
Optimum duration days:	9	8	14	4	6	1

Example-3: The time estimates (in hours) for the activities of a PERT network are given below:

Activity	t_0	t_m	t_p
1-2	1	1	7
1-3	1	4	7
1-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

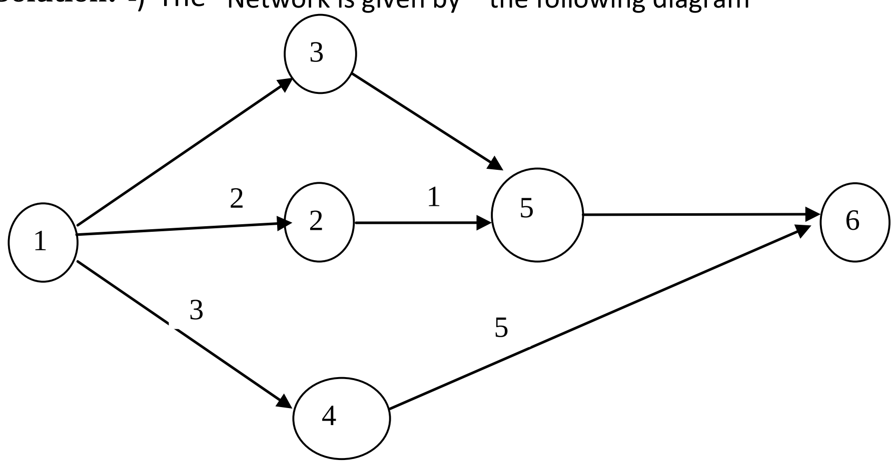
Where t_0 is the optimistic time t_p is the pessimistic time and t_m is most likely time

(a) Draw the project network
(b) Identify all paths through it and write critical path.
(c) Determine the expected project length
(d) Calculate standard deviation and variance of the project length
(e) What is the percentage of confidence that the project will complete
    I. at least 4 weeks earlier then expected time
    II. not more than 4 weeks than the expected time

What should be the scheduled complication times for the probability of complication are 90% confidence and 100% confidence?

Given data $P(Z \le 1.33) = 0.9082, P(Z \le 1.28) = 0.9, P(Z \le 5) = 0.99999.$ **WBUT-08**

Solution: i) The Network is given by the following diagram



The expected time and variance of each activity is shown below

Activity	t_0	t_m	t_p	$t_e = \frac{t_0 - t_m - t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$
1-2	1	1	7	2	1
1-3	1	4	7	4	1
1-4	2	2	8	3	1
2-5	1	1	1	1	0
3-5	2	5	14	6	4
4-6	2	5	8	5	1
5-6	3	6	15	7	4

b) Determination of project paths

Length of the path 1→ 2→5→6 = 2+1+7=10

Length of the path 1→ 3→5→6 = 4+6+7=17

Length of the path 1→ 4→6 = 5+3= 8

Since 1→ 3→5→6 has largest duration. Therefore the critical path is1→ 3→5→6.

c) The expected project length duration is = 17 weeks

d) Standard deviation of project length (σ^2)=sum of the standard deviations of the activities on the critical paths=1+4+4 =9

e) I) the probability of completing the project with in 4 weeks earlier than expected is given by

$P(Z \leq D)$, where $D = \frac{T_S - T_E}{\sigma}$

$D = \frac{13-17}{3} = -1.33$. Given that $P(Z \leq 1.33) = 0.9082$. Therefore

$$P(Z \leq -1.33) = 0.5 - \varphi(1.33)$$
$$= 0.5 - 0.4082 = 0.0918$$
$$= 9.18\%$$

ii) The probability of completing the project not more than 4 weeks than the expected time is given by

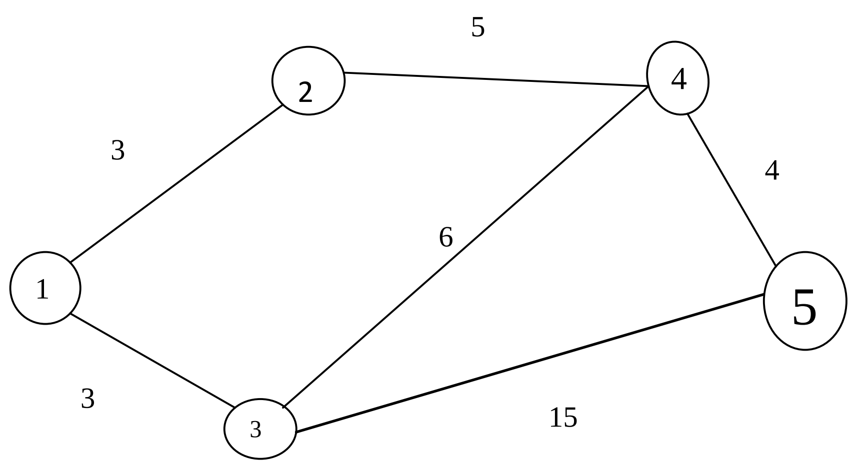
$P(Z \leq D)$, where $D = \frac{T_S - T_E}{\sigma}$

$= \frac{21-17}{3} = 1.33$. Given that $P(Z \leq D = 1.33) = 0.9082$. Therefore

$$P(Z \leq 1.33) = 0.5 - \varphi(1.33)$$
$$= 0.5 + 0.4082$$
$$= 0.9082 = 90.82\%$$

f) Value of Z for p=90% i.e. $\frac{T_S - 17}{3} = 1.28$, Therefore T_S=1.28×3+17=20.84 weeks

Example-06: Find the shortest path between every two nodes by Floyd's algorithms?



Solution: Using Floyd's algorithms, we have

Iteration-0, k=0

D_0

	1	2	3	4	5
1	—	3	10	α	α
2	3	—			α
3	10	α	—	6	15
4	α	5	6	—	4
5	α	α	α	4	—

S_0

	1	2	3	4	5
1	—	2	3	4	5
2	1	—	3	4	5
3	1	2	—	4	5
4	1	2	3	—	5
5	1	2	3	4	—

Iteration-1,k=1

D_1

	1	2	3	4	5
1	—	3	10	α	α
2	3	—	13	5	α
3	10	15	—	6	15
4	α	5	6	—	4
5	α	α	α	4	—

S_1

	1	2	3	4	5
1	—	2	3	4	5
2	1	—	1	4	5
3	1	1	—	4	5
4	1	2	3	—	5
5	1	2	3	4	—

Iteration-2,k=2 D_2

	1	2	3	4	5
1	—	3	10	α	α
2	3	—	13	5	α
3	10	15	—	6	15
4	α	5	6	—	4
5	α	α	α	4	—

S_2

	1	2	3	4	5
1	—	2	3	2	4
2	1	—	1	4	5
3	1	1	—	4	5
4	2	2	3	—	5
5	1	2	3	4	—

Iteration-3,k=3 D_3

	1	2	3	4	5
1	—	3	10	α	α
2	3	—	13	5	α
3	10	15	—	6	15
4	α	5	6	—	4
5	α	α	α	4	—

S_3

	1	2	3	4	5
1	—	2	3	2	3
2	1	—	1	4	3
3	1	1	—	4	5
4	2	2	3	—	5
5	1	2	3	4	—

Iteration-4,k=4			D_4		
	1	2	3	4	5
1	—	3	10	α	α
2	3	—	13	5	α
3	10	15	—	6	15
4	α	5	6	—	4
5	α	α	α	4	—

S_4					
	1	2	3	4	5
1	—	2	3	2	4
2	1	—	1	4	4
3	1	1	—	4	4
4	2	2	3	—	5
5	4	4	4	4	—

Shortest distance is given by d_{15}=12 units and path is

=1→5

=1 →4→5

=1 →2→4→ 5, Therefore required path S_{15} is 1→2 →4 →5.

Therefore Maximum flow in the network is F=f1+f2+f3+f4+f5=20+10+10+10+10=60 units.

Fulkerson's Rule for Numbering the Events

Steps to be followed as per the rule are discussed below:

(1) The starting event, the event having no predecessor activity is numbered J'. Other events are numbered in increasing order from event to rightwards. If there are more than one initial event, found in diagram, anywhere they are to be numbered from top to bottom in increasing order. No two events can have the same number in any case.

(2) Over sight all the activities emerging out from event J' in the diagram, one or more initial events having no predecessor activities are found. Number these events according to rule (1)

(3) Follow the rule (2) for newly numbered events and so on till the event having no activity emerging out from it is found. That event is numbered as highest one in diagram.

Example

Number the events of the network showing Fig. 23.6 with the help of Fulkerson rule:

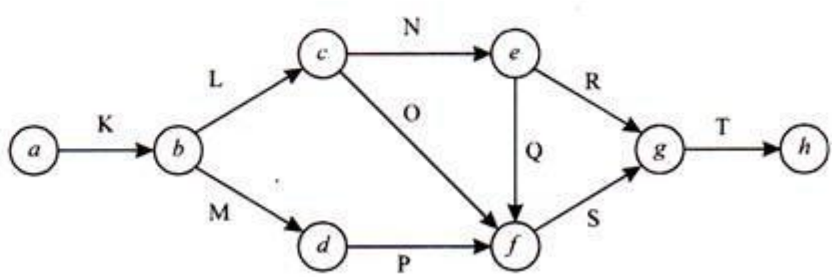


Fig. 23.6

Solution:

1. Event a is the starting or initial event; hence number it as 1.

2. Due to the activity K emerging out of a and ending at event h the end of activity will be the new initial event and number it as 2.

3. There are two arrows L and M emerging out of event 2. Now by neglecting ends these activities c and d, two more new initial events 3 and 4 are obtained

4. Following the same procedure and neglecting ends e, f, g, h of activities N, O, F, Q, R, S and T new events 5, 6, 7 and 8 are entered in circles and the numbered network diagram is shown in the Fig. 23.7.

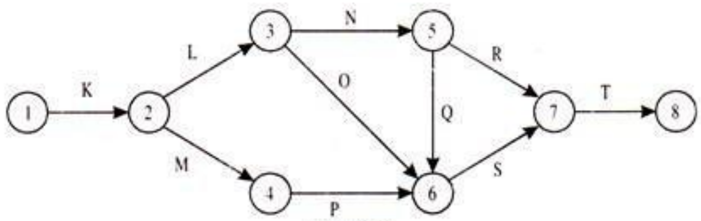


Fig. 23.7

PERT and CPM Method

1. Project Evaluation and Review Technique (PERT) **:**
PERT is appropriate technique which is used for the projects where the time required or needed to complete different activities are not known. PERT is majorly applied for scheduling, organization and integration of different tasks within a project. It provides the blueprint of project and is efficient technique for project evaluation .

2. Critical Path Method (CPM) **:**
CPM is a technique which is used for the projects where the time needed for completion of project is already known. It is majorly used for determining the approximate time within which a project can be completed. Critical path is the largest path in project management which always provide minimum time taken for completion of project

Difference between PERT and CPM :

S.No.	PERT	CPM
1.	PERT is that technique of project management which is used to manage uncertain (i.e., time is not known) activities of any project.	CPM is that technique of project management which is used to manage only certain (i.e., time is known) activities of any project.
2.	It is event oriented technique which means that network is constructed on the basis of event.	It is activity oriented technique which means that network is constructed on the basis of activities.
3.	It is a probability model.	It is a deterministic model.

S.No.	PERT	CPM
4.	It majorly focuses on time as meeting time target or estimation of percent completion is more important.	It majorly focuses on Time-cost trade off as minimizing cost is more important.
5.	It is appropriate for high precision time estimation.	It is appropriate for reasonable time estimation.
6.	It has Non-repetitive nature of job.	It has repetitive nature of job.
7.	There is no chance of crashing as there is no certainty of time.	There may be crashing because of certain time boundation.
8.	It doesn't use any dummy activities.	It uses dummy activities for representing sequence of activities.
9.	It is suitable for projects which required research and development.	It is suitable for construction projects.

Game Theory

Game theory is a Mathematical subject that is commonly used in practical life. It is applied to various other non-mathematical fields too. Game theory explains how a strategic game is played. It determines the way or order in which the players should make moves. It considers the information for the players at each decision point.

In-game theory, the interdependence of actions of players is the essence of the game. The game has two kinds of strategic interdependence – one is sequential, and the other is simultaneous. In sequential interdependence, players act in a sequence, aware of other players actions. While, in simultaneous interdependence, players act at the same time, ignoring other players' actions. The game theory is all about such strategies. Let us go ahead and learn more about game theory.

Game Theory Definition

The game theory is said to be the science of strategies which comes under the probability distribution. It determines logical as well as mathematical actions that should be taken by the players in order to obtain the best possible outcomes for themselves in the games. The games studied in game theory may range from chess to tennis and from child-rearing to takeovers. But there is one thing common that such an array of games is interdependent, i.e. outcome for each player depends upon the strategies of all.

In other words, game theory deals with mathematical models of cooperation and conflicts between rational decision-makers. Game theory can be defined as the study of decision-making in which the players must make strategies affecting the interests of other players.

Zero-Sum Game Theory

There is a special kind of game studied in game theory, called zero-sum games. They are constant-sum games. In such games, the available resources can neither be increased nor decreased. Also, the total benefit in zero-sum games for all combination of strategies, always adds to zero. We can say that in zero-sum games, one wins and exactly one opponent loses. The sum of benefits of all the players for any outcome is equal to zero is called a zero-sum game. Thus, the interest of the two players is opposed.

Several games, game theory are non-zero-sum games, since net result of outcome is less than or greater than zero. So, when one player's gain does not correspond to other's loss, it is called a non-zero sum game.

Game Theory Applications

The game theory is widely applied to study human as well as animal behaviours. It is utilized in economics to understand the economic behaviours, such as behaviours of consumers, markets and firms. Game theory has been commonly used in social sciences as well. It is applied in the study of sociological, political and psychological behaviours. The use of analysis based on game theory is seen in biology too. In addition to behavioural prediction, game theory utilized in the development of theories of normative or ethical behaviour.

Here are some common applications of game theory in operations research:

1. **Competitive Strategy:** Game theory helps in understanding and formulating competitive strategies in industries where firms interact strategically, such as pricing decisions, market entry, and product positioning. By modeling the interactions between competitors, researchers and businesses can anticipate and optimize their strategic moves.
2. **Supply Chain Management:** Game theory can be applied to analyze and optimize decision-making in supply chain networks involving multiple suppliers, manufacturers, and retailers. The interactions between these entities can be modeled as games to find stable and efficient solutions for the entire supply chain.
3. **Bidding and Auctions:** In procurement processes, auctions, or contract bidding, game theory can be used to design optimal bidding strategies for the participants to maximize their chances of winning while minimizing costs.
4. **Resource Allocation:** When resources are limited and need to be allocated among competing entities, game theory helps in understanding the best allocation strategies to ensure fairness and efficiency.
5. **Negotiation and Coalition Formation:** Game theory can aid in understanding negotiation strategies and coalition formation in multi-party scenarios. It helps in predicting the outcomes of negotiations and identifying potential alliances between different parties.
6. **Project Management:** In projects involving multiple stakeholders with conflicting interests, game theory can be used to model the decision-making process and to devise strategies that lead to successful project completion.
7. **Game Theory in Queuing Systems:** Game theory can be applied to study customer behavior in queuing systems, such as service centers or call centers, to improve service efficiency and customer satisfaction.
8. **Game Theory in Environmental Management:** When dealing with shared resources like water, forests, or fisheries, game theory can provide insights into sustainable management by analyzing the strategic interactions between different parties involved.

By using game theory in operations research, businesses and researchers can gain valuable insights into complex decision-making scenarios, enabling them to make more informed and effective choices to achieve their objectives.

Game Theory Example

The best example of game theory is a classical hypothesis called “Prisoners Dilemma”. According to this situation, two people are supposed to be arrested for stealing a car. They have to serve 2-year imprisonment for this. But, the police also suspects that these two people have also committed a bank robbery. The police placed each prisoner in a separate cell. Both of them are told that they are suspects of being bank robbers. They are inquired separately and are not able to communicate with each other.

The prisoners are given two situations:

- If they both confess to being bank robbers, then each will serve 3-year imprisonment for both car theft and robbery.
- If only one of them confesses to being a bank robber and the other does not, then the person who confesses will serve 1-year and others will serve 10-year imprisonment.

According to game theory, the prisoners will either confess or deny the bank robbery. So, there are four possible outcomes :

	2-Confess	**2-Deny**
1-Confess	Both punished 3 years	Prisoner 1 punished 1 year Prisoner 2 punished 10 years
1-Deny	Prisoner 1 punished 10 year Prisoner 2 punished 1 year	Both punished 2 years

Here, the best option is to deny. In this case, both will have to serve 2 years sentence. But it cannot be guaranteed that others would not confess, therefore most likely both of them would confess and serve the 3-year sentence.

Fundamental theorems of game

In game theory, there are several fundamental theorems that provide important insights and results about the existence and properties of equilibria in different types of games. These theorems are crucial in understanding the strategic interactions between players and the stability of outcomes. Here are three fundamental theorems in game theory:

1. **Nash's Existence Theorem:** Named after mathematician John Nash, this theorem states that every finite game with a finite number of players has at least one mixed strategy Nash equilibrium. A mixed strategy Nash equilibrium is a set of strategies, one for each player, such that no player can benefit by unilaterally changing their strategy while the other players keep their strategies unchanged.

2. **Nash's Bargaining Theorem:** Also known as the Nash Bargaining Solution, this theorem addresses the problem of how to fairly divide the gains from cooperation between two players engaged in a cooperative game. Nash showed that there exists a unique solution that maximizes the product of the players' individual gains, subject to their individual rationality. This solution is called the Nash bargaining solution and provides an equitable allocation of the gains from cooperation.
3. **Minimax Theorem:** The Minimax theorem, developed by John von Neumann, is used in two-player zero-sum games, where the gain of one player is exactly balanced by the loss of the other player. It states that in such games, there exists a value, known as the minimax value, which represents the maximum guaranteed payoff for the player making the first move, assuming that both players play optimally. The minimax value is also equal to the minimum guaranteed payoff for the player making the second move.

These fundamental theorems are essential tools in the analysis of various game types and have significant implications for decision-making in operations research. They provide valuable insights into equilibrium concepts, cooperative solutions, and optimal strategies in games, which can be applied to real-world scenarios, such as competitive situations in business, negotiation processes, and resource allocation problems in operations research.