UNIT 1:

Sets and Relations:

Sets, in mathematics, are an organized collection of objects and can be represented in set-builder form or roster form. Usually, sets are represented in curly braces {}, for example, A = {1,2,3,4} is a set. Also, check the set symbols here.

In sets theory, you will learn about sets and it's properties. It was developed to describe the collection of objects. You have already learned about the classification of sets here.
The set theory defines the different types of sets, symbols and operations performed.

Definition of Sets

Sets are represented as a collection of well-defined objects or elements and it does not change from person to person. A set is represented by a capital letter. The number of elements in the finite set is known as the cardinal number of a set.

What are the Elements of a Set

Let us take an example:

A = {1, 2, 3, 4, 5 }

Since a set is usually represented by the capital letter. Thus, A is the set and 1, 2, 3, 4, 5 are the elements of the set or members of the set. The elements that are written in the set can be in any order but cannot be repeated. All the set elements are represented in small letter in case of alphabets. Also, we can write it as 1 ∈ A, 2 ∈ A etc. The cardinal number of the set is 5. Some commonly used sets are as follows:

- N: Set of all natural numbers
- Z: Set of all integers
- Q: Set of all rational numbers

- R: Set of all real numbers
- Z^+: Set of all positive integers

Order of Sets

The order of a set defines the number of elements a set is having. It describes the size of a set. The order of set is also known as the **cardinality**.

The size of set whether it is is a finite set or an infinite set, said to be set of finite order or infinite order, respectively.

Representation of Sets

The sets are represented in curly braces, {}. For example, {2,3,4} or {a,b,c} or {Bat, Ball, Wickets}. The elements in the sets are depicted in either the Statement form, Roster Form or Set Builder Form.

Statement Form

In statement form, the well-defined descriptions of a member of a set are written and enclosed in the curly brackets.

For example, the set of even numbers less than 15.

In statement form, it can be written as {even numbers less than 15}.

Roster Form

In Roster form, all the elements of a set are listed.

For example, the set of natural numbers less than 5.

Natural Number = 1, 2, 3, 4, 5, 6, 7, 8,……….

Natural Number less than 5 = 1, 2, 3, 4

Therefore, the set is N = { 1, 2, 3, 4 }

Set Builder Form

The general form is, A = { x : property }

Example: Write the following sets in set builder form: A={2, 4, 6, 8}

Solution:

2 = 2 x 1

4 = 2 x 2

6 = 2 x 3

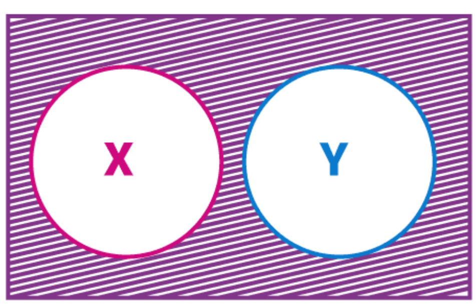
8 = 2 x 4

So, the set builder form is A = {x: x=2n, n ∈ N and 1 ≤ n ≤ 4}

Also, Venn Diagrams are the simple and best way for visualized representation of sets.

What is a Venn Diagram?

A diagram used to represent all possible relations of different sets. A Venn diagram can be represented by any closed figure, whether it be a Circle or a Polygon (square, hexagon, etc.). But usually, we use circles to represent each set.



In the above figure, we can see a Venn diagram, represented by a rectangular shape about the universal set, which has two independent sets, X and Y. Therefore, X and Y are disjoint sets. The two sets, X and Y, are represented in a circular shape. This diagram shows that set X and set Y have no relation between each other, but they are a part of a universal set.

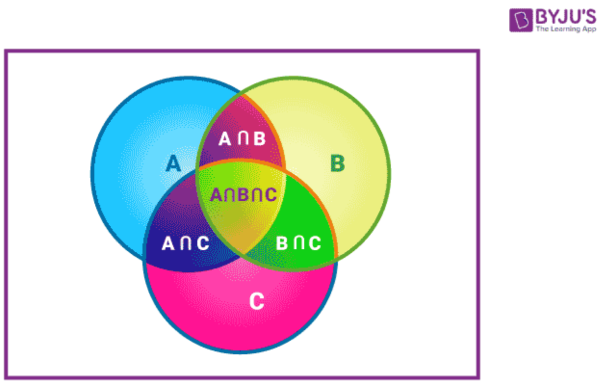
For example, set X = {Set of even numbers} and set Y = {Set of odd numbers} and Universal set, U = {set of natural numbers}

We can use the below formula to solve the problems based on two sets.

$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

Venn Diagram of Three Sets

Check the Venn diagram of three sets given below.



The formula used to solve the problems on Venn diagrams with three sets is given below:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Venn Diagram Symbols

The symbols used while representing the operations of sets are:

- Union of sets symbol: ∪
- Intersection of sets symbol: ∩
- Complement of set: A’ or A^c

How to draw a Venn diagram?

To draw a Venn diagram, first, the universal set should be known. Now, every set is the subset of the universal set (U). This means that every other set will be inside the rectangle which represents the universal set.

So, any set A (shaded region) will be represented as follows:

Figure 1:

Where U is a universal set.

We can say from fig. 1 that

A ∪ U = U

All the elements of set A are inside the circle. Also, they are part of the big rectangle which makes them the elements of set U.

Types of Sets

We have several types of sets in Maths. They are empty set, finite and infinite sets, proper set, equal sets, etc. Let us go through the classification of sets here.

Empty Set

A set which does not contain any element is called an empty set or void set or null set. It is denoted by { } or Ø.

A set of apples in the basket of grapes is an example of an empty set because in a grapes basket there are no apples present.

Singleton Set

A set which contains a single element is called a singleton set.

Example: There is only one apple in a basket of grapes.

Finite set

A set which consists of a definite number of elements is called a finite set.

Example: A set of natural numbers up to 10.

A = {1,2,3,4,5,6,7,8,9,10}

Infinite set

A set which is not finite is called an infinite set.

Example: A set of all natural numbers.

A = {1,2,3,4,5,6,7,8,9……}

Equivalent set

If the number of elements is the same for two different sets, then they are called equivalent sets. The order of sets does not matter here. It is represented as:

n(A) = n(B)

where A and B are two different sets with the same number of elements.

Example: If A = {1,2,3,4} and B = {Red, Blue, Green, Black}

In set A, there are four elements and in set B also there are four elements. Therefore, set A and set B are equivalent.

Equal sets

The two sets A and B are said to be equal if they have exactly the same elements, the order of elements do not matter.

Example: A = {1,2,3,4} and B = {4,3,2,1}

A = B

Disjoint Sets

The two sets A and B are said to be disjoint if the set does not contain any common element.

Example: Set A = {1,2,3,4} and set B = {5,6,7,8} are disjoint sets, because there is no common element between them.

Subsets

A set ‘A’ is said to be a subset of B if every element of A is also an element of B, denoted as A ⊆ B. Even the null set is considered to be the subset of another set. In general, a subset is a part of another set.

Example: A = {1,2,3}

Then {1,2} ⊆ A.

Similarly, other subsets of set A are: {1},{2},{3},{1,2},{2,3},{1,3},{1,2,3},{}.

Note: The set is also a subset of itself.

If A is not a subset of B, then it is denoted as A⊄B.

Proper Subset

If A ⊆ B and A ≠ B, then A is called the proper subset of B and it can be written as A⊂B.

Example: If A = {2,5,7} is a subset of B = {2,5,7} then it is not a proper subset of B = {2,5,7}

But, A = {2,5} is a subset of B = {2,5,7} and is a proper subset also.

Superset

Set A is said to be the superset of B if all the elements of set B are the elements of set A. It is represented as A ⊃ B.

For example, if set A = {1, 2, 3, 4} and set B = {1, 3, 4}, then set A is the superset of B.

Universal Set

A set which contains all the sets relevant to a certain condition is called the universal set. It is the set of all possible values.

Example: If A = {1,2,3} and B {2,3,4,5}, then universal set here will be:

U = {1,2,3,4,5}

Operations on Sets

In set theory, the operations of the sets are carried when two or more sets combine to form a single set under some of the given conditions. The basic operations on sets are:

- Union of sets
- Intersection of sets
- A complement of a set
- Cartesian product of sets.
- Set difference

Basically, we work more on union and intersection of sets operations, using Venn diagrams.

Union of Sets

If set A and set B are two sets, then A union B is the set that contains all the elements of set A and set B. It is denoted as A ∪ B.

Example: Set A = {1,2,3} and B = {4,5,6}, then A union B is:

A ∪ B = {1,2,3,4,5,6}

Intersection of Sets

If set A and set B are two sets, then A intersection B is the set that contains only the common elements between set A and set B. It is denoted as A ∩ B.

Example: Set A = {1,2,3} and B = {4,5,6}, then A intersection B is:

A ∩ B = { } or Ø

Since A and B do not have any elements in common, so their intersection will give null set.

Complement of Sets

The complement of any set, say P, is the set of all elements in the universal set that are not in set P. It is denoted by P'**.**

Properties of Complement sets

1. $P \cup P' = U$
2. $P \cap P' = \Phi$
3. Law of double complement : $(P')' = P$
4. Laws of empty/null set(Φ) and universal set(U), $\Phi' = U$ and $U' = \Phi$.

Cartesian Product of sets

If set A and set B are two sets then the cartesian product of set A and set B is a set of all ordered pairs (a,b), such that a is an element of A and b is an element of B. It is denoted by $A \times B$.

We can represent it in set-builder form, such as:

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Example: set A = {1,2,3} and set B = {Bat, Ball}, then;

A × B = {(1,Bat),(1,Ball),(2,Bat),(2,Ball),(3,Bat),(3,Ball)}

Difference of Sets

If set A and set B are two sets, then set A difference set B is a set which has elements of A but no elements of B. It is denoted as A – B.

Example: A = {1,2,3} and B = {2,3,4}

A – B = {1}

Sets Formulas

Some of the most important set formulas are:

For any three sets A, B and C
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
If $A \cap B = \emptyset$, then $n(A \cup B) = n(A) + n(B)$

n(A – B) + n(A ∩ B) = n(A)
n(B – A) + n(A ∩ B) = n(B)
n(A – B) + n (A ∩ B) + n(B – A) = n (A ∪ B)
n (A ∪ B ∪ C) = n(A) + n(B) + n(C) – n (A ∩ B) – n (B ∩ C) – n (C ∩ A) + n (A ∩ B ∩ C)

What is Duality?

Duality is known to be a very general as well as a broad concept, without a strict definition that captures all those uses. There usually is a precise definition when duality is applied to specific concepts, for just that context. The common idea is that there are two things that basically are just two sides of the same coin.

Principle of Duality in Discrete Mathematics

The principle of duality is a type of pervasive property of algebraic structure in which two concepts are interchangeable only if all results held in one formulation also hold in another. This concept is known as dual formulation. We will interchange unions(∪) into intersections(∩) or intersections() into the union() and also interchange universal set into the null set(∅) or null set into universal(U) to get the dual statement. If we interchange the symbol and get this statement itself, it will be known as the self-dual statement.

For example:

The dual of (X ∩ Y) ∪ Z is (X ∪ Y) ∩ Z

Duality can also be described as a property that belongs to the branch of algebra. This theory can be called lattice theory. This theory has the ability to involve order and structure, which are common to different mathematical systems. If the mathematical system has the order in a specified way, this structure will be known as lattice.

The principle of duality concept should not be avoided or underestimated. It has the ability to provide several sets of theorems, concepts, and identities. To explain the duality principle of sets, we will assume S be any identity that involves sets, and operation complement, union, intersection. Suppose we obtain the S* from S with the help of substituting $\cup \rightarrow \cap \Phi$. In this case, the statement S* will also be true, and S* can also be known as dual statement S.

Examples of Duality:

Examples 1:

$A \cup (B \cap A) = A$

When we perform duality, then the union will be replaced by intersection, or the intersection will be replaced by the union.

$A \cap (B \cup A) = A$

Example 2:

$A \cup ((B^C \cup A) \cap B)^C = U$

When we perform duality, then the union will be replaced by intersection, or intersection will be replaced by the union. The universal will also be replaced by null, or null will be replaced by universal.

$A \cap ((B^C \cap A) \cup B)^C = \Phi$

Example 3:

$(A \cup B^C)^C \cap B = A^C \cap B$

When we perform duality, then the union will be replaced by intersection, or intersection will be replaced by the union.

$(A \cap B^C)^C \cup B = A^C \cup B$

Partitioning of a Set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say P_1, P_1, ... P_n that satisfies the following three conditions −

- P_i does not contain the empty set.
  $[P_i \neq \{ \emptyset \}$ for all $0 < i \leq n]$
- The union of the subsets must equal the entire original set.
  $[P_1 \cup P_2 \cup ... \cup P_n = S]$
- The intersection of any two distinct sets is empty.
  $[P_a \cap P_b = \{ \emptyset \}$, for $a \neq b$ where $n \geq a, b \geq 0]$

Example

Let S = { a, b, c, d, e, f, g, h }

One probable partitioning is { a }, { b, c, d }, { e, f, g, h }

Another probable partitioning is { a, b }, { c, d }, { e, f, g, h }

Relations in Set

Relation refers to a relationship between the elements of 2 sets A and B. It is represented by R. We say that R is a relation from A to A, then $R \subseteq A \times A$. A relation from set A to set B is a subset of $A \times B$. i.e aRb $\leftrightarrow (a,b) \subseteq R \leftrightarrow R(a, b)$.

9 Important Properties Of Relations In Set Theory

1. Identity Relation: Every element is related to itself in an identity relation. It is denoted as $I = \{(a, a), a \in A\}$.

2. Empty relation: There will be no relation between the elements of the set in an empty relation. It is the subset $\emptyset$.

3. Reflexive relation: Every element gets mapped to itself in a reflexive relation. A relation R in a set A is reflexive if $(a, a) \in R$ for all $a \in R$.

4. Irreflexive relation: If any element is not related to itself, then it is an irreflexive relation.

5. Inverse relation: When a set has elements which are inverse pairs of another set, then the relation is an inverse relation. For example, if A = {(p,q), (r,s)}, then R^{-1} = {(q,p), (s,r)}. Inverse relation is denoted by R^{-1} = {(b, a): (a, b) ∈ R}.

6. Symmetric relation: A relation R is a symmetric relation if (b, a) ∈ R is true when (a,b) ∈ R. For example R = {(3, 4), (4, 3)} for a set A = {3, 4}. Symmetric relation is denoted by
aRb ⇒ bRa, ∀ a, b ∈ A.

7. Transitive relation: A relation is transitive, if (a, b) ∈ R, (b, c) ∈ R, then (a, c) ∈ R. It is denoted by aRb and bRc ⇒ aRc ∀ a, b, c ∈ A

8. Equivalence relation: A relation is called equivalence relation if it is reflexive, symmetric, and transitive at the same time.

9. Universal relation: A relation is said to be universal relation, If each element of A is related to every element of A, i.e. R = A × A.

Binary Relation

Let P and Q be two non- empty sets. A binary relation R is defined to be a subset of P x Q from a set P to Q. If (a, b) ∈ R and R ⊆ P x Q then a is related to b by R i.e., aRb. If sets P and Q are equal, then we say R ⊆ P x P is a relation on P e.g.

1. (i) Let A = {a, b, c}
2. B = {r, s, t}
3. Then R = {(a, r), (b, r), (b, t), (c, s)}
4. is a relation from A to B.
5.
6. (ii) Let A = {1, 2, 3} and B = A
7. R = {(1, 1), (2, 2), (3, 3)}
8. is a relation (equal) on A.

Example1: If a set has n elements, how many relations are there from A to A.

Solution: If a set A has n elements, A x A has n^2 elements. So, there are 2^{n^2} relations from A to A.

Example2: If A has m elements and B has n elements. How many relations are there from A to B and vice versa?

Solution: There are m x n elements; hence there are $2^{m \times n}$ relations from A to A.

Equivalence Relation

A relation R on a set A is said to be an **equivalence relation** if and only if the relation R is reflexive, symmetric and transitive. The equivalence relation is a relationship on the set which is generally represented by the symbol "∼".

Reflexive: A relation is said to be reflexive, if (a, a) ∈ R, for every a ∈ A.

Symmetric: A relation is said to be symmetric, if (a, b) ∈ R, then (b, a) ∈ R.

Transitive: A relation is said to be transitive if (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R.

In terms of equivalence relation notation, it is defined as follows:

A binary relation ∼ on a set A is said to be an equivalence relation, if and only if it is reflexive, symmetric and transitive.

(i.e) For all x, y, z in set A,

- x ∼ x (Reflexivity)
- x ∼ y if and only if y ∼ x (Symmetry)
- If x∼y and y∼z, then x∼z (Transitivity)

Equivalence relations can be explained in terms of the following examples:

- The sign of 'is equal to (=)' on a set of numbers; for example, 1/3 = 3/9.
- For a given set of triangles, the relation of 'is similar to (~)' and 'is congruent to (≅)' shows equivalence.
- For a given set of integers, the relation of 'congruence modulo n (≡)' shows equivalence.
- The image and domain are the same under a function, which shows the relation of equivalence.
- For a set of all angles, 'has the same cosine'.
- For a set of all real numbers,' has the same absolute value'.

Equivalence Relation Proof

Here is an equivalence relation example to prove the properties.

Let us assume that R be a relation on the set of ordered pairs of positive integers such that ((a, b), (c, d))∈ R if and only if ad=bc. Is R an equivalence relation?

In order to prove that R is an equivalence relation, we must show that R is reflexive, symmetric and transitive.

The Proof for the given condition is given below:

Reflexive Property

According to the reflexive property, if (a, a) ∈ R, for every a∈A

For all pairs of positive integers,

((a, b),(a, b))∈ R.

Clearly, we can say

ab = ab for all positive integers.

Hence, the reflexive property is proved.

Symmetric Property

From the symmetric property,

if (a, b) ∈ R, then we can say (b, a) ∈ R

For the given condition,

if ((a, b),(c, d)) ∈ R, then ((c, d),(a, b)) ∈ R.

If ((a, b),(c, d))∈ R, then ad = bc and cb = da

since multiplication is commutative.

Therefore ((c, d),(a, b)) ∈ R

Hence symmetric property is proved.

Transitive Property

From the transitive property,

if (a, b) ∈ R and (b, c) ∈ R, then (a, c) also belongs to R

For the given set of ordered pairs of positive integers,

((a, b), (c, d))∈ R and ((c, d), (e, f))∈ R,

then ((a, b),(e, f) ∈ R.

Now, assume that ((a, b), (c, d))∈ R and ((c, d), (e, f)) ∈ R.

Then we get, ad = cb and cf = de.

The above relation implies that a/b = c/d and that c/d = e/f,

so a/b = e/f we get af = be.

Therefore ((a, b),(e, f))∈ R.

Hence transitive property is proved.

Composition of Relations

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. Then R and S give rise to a relation from A to C indicated by R◦S and defined by:

1. a (R◦S)c **if for** some b ∈ B we have aRb and bSc.
2. is,
3. R ◦ S = {(a, c)| there exists b ∈ B **for** which (a, b) ∈ R and (b, c) ∈ S}

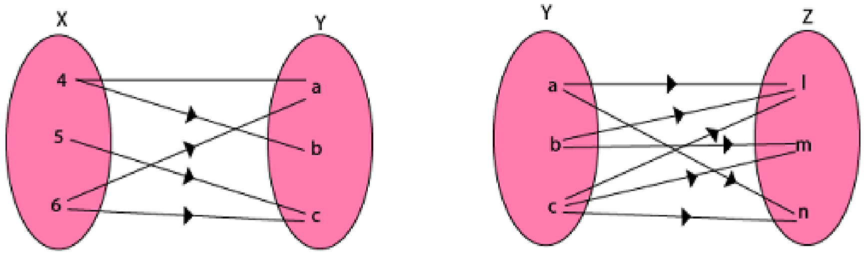
The relation R◦S is known the composition of R and S; it is sometimes denoted simply by RS.

Let R is a relation on a set A, that is, R is a relation from a set A to itself. Then R◦R, the composition of R with itself, is always represented. Also, R◦R is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R$ = R◦R◦R, and so on. Thus R^n is defined for all positive n.

Example1: Let X = {4, 5, 6}, Y = {a, b, c} and Z = {l, m, n}. Consider the relation R_1 from X to Y and R_2 from Y to Z.

$R_1 = \{(4, a), (4, b), (5, c), (6, a), (6, c)\}$

$R_2 = \{(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)\}$



Find the composition of relation **(i)** R_1 o R_2 **(ii)** R_1o R_1^{-1}

Solution:

(i) The composition relation R_1 o R_2 as shown in fig:

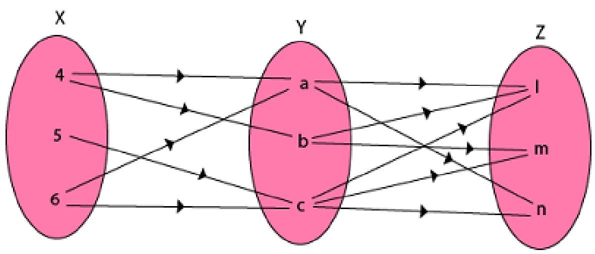


Fig : R_1 o R_2

R_1 o R_2 = {(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)}

(ii) The composition relation R_1o R_1^{-1} as shown in fig:

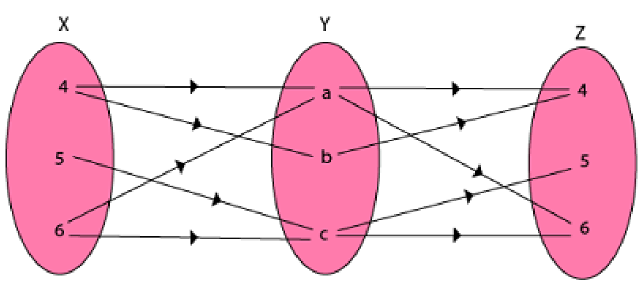


Fig : R_1 o R_1^{-1}

R_1o R_1^{-1} = {(4, 4), (5, 5), (5, 6), (6, 4), (6, 5), (4, 6), (6, 6)}