UNIT 2: Calculus

Differentiation is a fundamental concept in calculus that measures how a function changes as its input variable changes. It involves finding the derivative of a function, which gives the rate at which the function value is changing at any given point.

Elaborating on differentiation involves understanding its key components and applications. Here are some aspects to consider:

1. Derivative: The derivative of a function represents its rate of change. It measures how the function output varies concerning the input variable. Geometrically, it corresponds to the slope of the tangent line to the function's graph at a particular point. The derivative is denoted by various notations, such as f'(x), dy/dx, or df(x)/dx.
2. Differentiability: A function is said to be differentiable at a point if its derivative exists at that point. Differentiability implies that the function is smooth and has no abrupt changes or sharp corners at that particular point.
3. Differentiation Rules: Differentiation follows several rules that help compute derivatives efficiently. These rules include the power rule, product rule, quotient rule, chain rule, and the rules for trigonometric, exponential, and logarithmic functions. These rules provide a systematic approach to finding derivatives of various functions.
4. Applications of Differentiation: a. Rates of Change: Differentiation allows us to determine rates of change in various real-world situations. For example, it can be used to calculate the velocity of an object, the growth rate of a population, or the rate at which a chemical reaction occurs.

   b. Optimization: Differentiation helps find optimal solutions. By analyzing the derivative of a function, we can identify critical points, such as maximum or minimum values, which are crucial in optimization problems. This has applications in fields like economics, engineering, and physics.

   c. Curve Sketching: Derivatives provide information about the shape and behavior of a function. By analyzing the derivative, we can determine the intervals where the function is increasing or decreasing, identify points of inflection, and analyze concavity. These insights aid in sketching accurate graphs of functions.

d. Physics and Engineering: Differentiation is extensively used in physics and engineering to analyze motion, acceleration, electric and magnetic fields, fluid dynamics, and more. It helps describe and understand physical phenomena and provides tools for modeling and prediction.
e. Economics and Finance: Differentiation plays a vital role in economic analysis, particularly in calculating marginal costs, revenues, and profits. It helps determine supply and demand curves, elasticity, and optimization of economic variables. In finance, differentiation is used to assess risk, analyze option pricing models, and calculate portfolio sensitivities.
f. Science and Medicine: Many scientific and medical disciplines utilize differentiation for data analysis, modeling, and prediction. It aids in understanding complex biological systems, analyzing medical images, predicting drug interactions, and analyzing experimental data.

Overall, differentiation is a powerful mathematical tool that enables us to study rates of change, optimize functions, sketch graphs, and solve various real-world problems across a wide range of fields.

Derivative of a Function of One Variable

The derivative of a function of one variable represents the rate at which the function changes with respect to that variable. It measures the instantaneous rate of change of the function at each point.

Mathematically, let's consider a function f(x) where x is the independent variable. The derivative of f(x), denoted as f'(x) or dy/dx, represents the rate of change of f(x) with respect to x.

There are several methods to find the derivative of a function depending on the complexity of the function and available differentiation rules. Here are some common cases:

1. Power Rule: If f(x) = x^n, where n is a constant, the derivative is given by: f'(x) = nx^(n-1)
2. Constant Rule: If f(x) = c, where c is a constant, the derivative is zero since a constant function has no variable dependence: f'(x) = 0
3. Sum/Difference Rule: If f(x) = g(x) ± h(x), where g(x) and h(x) are differentiable functions, the derivative is calculated separately for each function: f'(x) = g'(x) ± h'(x)
4. Product Rule: If f(x) = g(x) * h(x), where g(x) and h(x) are differentiable functions, the derivative is given by: f'(x) = g'(x) * h(x) + g(x) * h'(x)

5. Quotient Rule: If f(x) = g(x) / h(x), where g(x) and h(x) are differentiable functions and h(x) ≠ 0, the derivative is given by: f'(x) = (g'(x) * h(x) - g(x) * h'(x)) / [h(x)]^2
6. Chain Rule: If f(x) = g(h(x)), where g(u) is differentiable and h(x) is differentiable, the derivative is given by: f'(x) = g'(h(x)) * h'(x)

These are just a few of the commonly used rules for differentiation. For more complex functions, other rules and techniques such as the logarithmic, exponential, and trigonometric differentiation rules, as well as implicit differentiation or related rates, may be necessary.

It's important to note that the derivative provides information about the rate of change and slope of the function but not its absolute values or specific behavior at certain points. Additional analysis may be required to determine maximum or minimum points, inflection points, or other features of the function.

Differentiation by method of Substitution

Differentiation by the method of substitution, also known as the chain rule, is a technique used to find the derivative of a composite function. It is particularly useful when dealing with functions that are composed of inner and outer functions. The chain rule allows us to differentiate such composite functions by breaking them down into simpler parts.

Here's how the method of substitution works:

1. Identify the composite function: Start with a given function that can be expressed as the composition of two or more functions. For example, let's consider the function f(x) = g(u(x)), where u(x) is an intermediate function and g(u) is the outer function.
2. Differentiate the outer function: Take the derivative of the outer function g(u) with respect to its variable u, treating u as the independent variable. This derivative is denoted as g'(u).
3. Differentiate the inner function: Take the derivative of the inner function u(x) with respect to x, treating x as the independent variable. This derivative is denoted as u'(x).
4. Apply the chain rule: Multiply the derivative of the outer function (g'(u)) by the derivative of the inner function (u'(x)). This accounts for the chain rule's principle that the derivative of a composite function is the product of the derivatives of its individual components.
5. Simplify the result: If necessary, simplify the expression obtained in step 4 by substituting the original function's inner function (u(x)) back in place of u.

6. Finalize the derivative: The resulting expression gives the derivative of the composite function f(x) = g(u(x)). You can further manipulate or evaluate it depending on the requirements of the problem.

It's important to note that the method of substitution or the chain rule can be extended to cases where there are multiple nested functions or even more complex compositions. In such cases, you apply the chain rule iteratively, starting from the outermost function and working your way inwards.

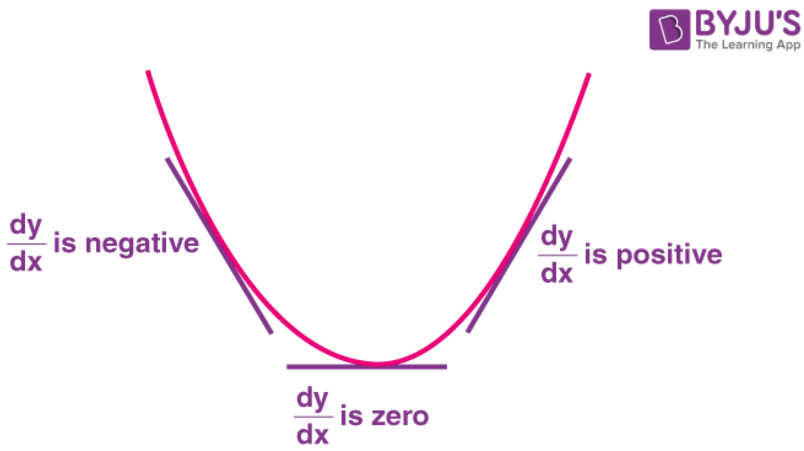
By using the method of substitution and applying the chain rule correctly, you can differentiate composite functions and calculate their derivatives efficiently. This technique is valuable in various areas of mathematics, physics, engineering, and other disciplines where composite functions arise.

Maxima and Minima

Maxima and minima in calculus are found by using the concept of derivatives. As we know, the concept of the derivatives gives us information regarding the gradient/slope of the function, we locate the points where the gradient is zero, and these points are called turning points/stationary points. These are points associated with the largest or smallest values (locally) of the function.

Maxima and Minima Points

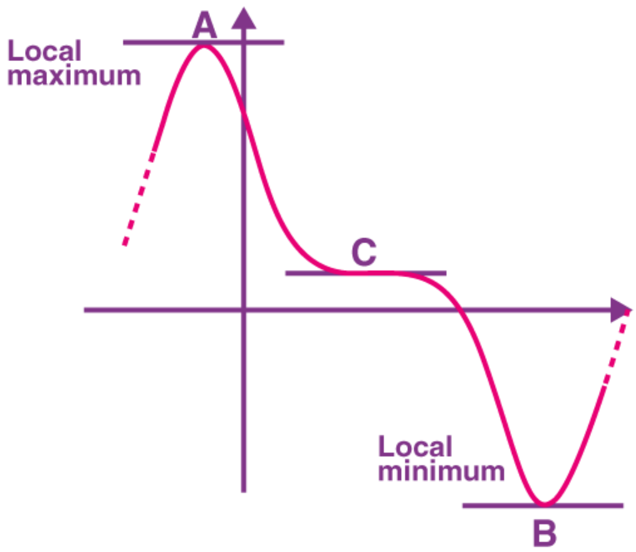
The figure for the curve with stationary points is shown below. Thus, it can be seen from the figure that before the slope becomes zero, it was negative; after it gets zero, it becomes positive. It can be said dy/dx is -ve before the stationary point dy/dx is +ve after the stationary point. Hence, it can be said that d^2y/dx^2 is positive at the stationary point shown below. Therefore, it can be said wherever the double derivative is positive, it is the point of minima. Vice versa, wherever the double derivative is negative is the point of maxima on the curve. This is also known as the second derivative test.



Let f be a function defined on an open interval I.
Let f be continuous at a critical point c in I.
If f'(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflection.

Stationary Points vs Turning Points

Stationary points are the points where the slope of the graph becomes zero. In other words, the tangent of the function becomes horizontal, and dy/dx = 0. All the stationary points, A, B and C, are given in the figure shown below. And the points in which the function changes its path, if it was going upward; it will go downward and vice versa, i.e., points A and B are turning points since the curve changes its path. But point C is not a turning point, although the graph is flat for a short period of time but continues to go down from left to right.



Derivative Tests

The derivative test helps to find the maxima and minima of any function. Usually, the first-order derivative and second-order derivative tests are used. Let us have a look in detail.

First Order Derivative Test

Let f be the function defined in an open interval I. And f be continuous at critical point c in I such that f'(c) = 0.

1. If f'(x) changes sign from positive to negative as x increases through point c, then c is the point of local maxima, and the f(c) is the maximum value.

2. If f'(x) changes sign from negative to positive as x increases through point c, then c is the point of local minima, and the f(c) is the minimum value.

3. If f'(x) doesn't change sign as x increases through c, then c is neither a point of local nor a point of local maxima. It will be called the point of inflection.

Second Derivative Test

Let f be the function defined on an interval I, and it is two times differentiable at c.

i. x = c will be point of local maxima if $f'(c) = 0$ and $f''(c)<0$. Then, f(c) will be having local maximum value.

ii. x = c will be point of local minima if $f'(c) = 0$ and $f''(c) > 0$. Then, f(c) will be having local minimum value.

iii. When both $f'(c) = 0$ and $f''(c) = 0$, the test fails, and the first derivative test will give you the value of local maxima and minima.

Properties of maxima and minima

1. If f(x) is a continuous function in its domain, then at least one maximum or one minimum should lie between equal values of f(x).

2. Maxima and minima occur alternately, i.e., between two minima, there is one maxima and vice versa.

3. If f(x) tends to infinity as x tends to a or b and $f'(x) = 0$ only for one value x, i.e., c between a and b, then f(c) is the minimum and the least value. If f(x) tends to $-\infty$ as x tends to a or b, then f(c) is the maximum and the highest value.

Indefinite Integrals

In Calculus, the two important processes are differentiation and integration. We know that differentiation is finding the derivative of a function, whereas integration is the inverse process of differentiation. Here, we are going to discuss the important component of integration called "integrals" here.

Suppose a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I. In that case, a simple question arises: Can we determine the function for obtained f' at each point? The functions that could have provided function as a derivative are called antiderivatives (or primitive). The formula that gives all these antiderivatives is called the **indefinite integral** of the function. And such a process of finding antiderivatives is called integration.

The integrals are generally classified into two types, namely:

- Definite Integral
- Indefinite Integral

Here, let us discuss one of the integral types called "Indefinite Integral" with definition and properties in detail.

Indefinite Integrals Definition

An integral which is not having any upper and lower limit is known as an indefinite integral.

Mathematically, if F(x) is any anti-derivative of f(x) then the most general antiderivative of f(x) is called an indefinite integral and denoted,

$\int f(x)\,dx = F(x) + C$

We mention below the following symbols/terms/phrases with their meanings in the table for better understanding.

Symbols/Terms/Phrases	Meaning
$\int f(x)\,dx$	Integral of f with respect to x
f(x) in $\int f(x)\,dx$	Integrand
x in $\int f(x)\,dx$	Variable of integration
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

Anti derivatives or integrals of the functions are not unique. There exist infinitely many antiderivatives of each of certain functions, which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason, C is customarily referred to as an arbitrary constant. C is the parameter by which one gets different antiderivatives (or integrals) of the given function.

Get Indefinite Integral calculator here.

Indefinite Properties

Let us now look into some properties of indefinite integrals.

Property 1: The process of differentiation and integration are inverses of each other in the sense of the following results:

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

And

$$\int f'(x)\,dx = f(x) + \mathrm{C}$$

where C is any arbitrary constant.

Let us now prove this statement.

Proof: Consider a function f such that its anti-derivative is given by F, i.e.

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

Then,

$$\int f(x)\,dx = \mathrm{F}(x) + \mathrm{C}$$

On differentiating both the sides with respect to x we have,

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left(\mathrm{F}(x) + \mathrm{C}\right)$$

As we know, the derivative of any constant function is zero. Thus,

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left(F(x)+C\right)$$

$$= \frac{d}{dx}F(x)$$

$$= f(x)$$

The derivative of a function f in x is given as f'(x), so we get;

$$f'(x) = \frac{d}{dx}\; f(x)$$

Therefore,

$$\int f'(x)\,dx = f(x) + C$$

Hence, proved.

Property 2: Two indefinite integrals with the same derivative lead to the same family of curves, and so they are equivalent.

Proof: Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$$

or

$$\frac{d}{dx}\left[\int f(x)\,dx - \int g(x)\,dx\right] = 0$$

Now,

$$\int f(x)\,dx - \int g(x)\,dx = C$$

or

$$\int f(x)\,dx = \int g\,(x)\,dx + C$$

where C is any real number.

From this equation, we can say that the family of the curves of [∫ f(x)dx + C_3, C_3 ∈ R] and [∫ g(x)dx + C_2, C_2 ∈ R] are the same.

Therefore, we cay say that, ∫ f(x)dx = ∫ g(x)dx

Property 3: The integral of the sum of two functions is equal to the sum of integrals of the given functions, i.e.,

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Proof:

From the property 1 of integrals we have,

$$\frac{d}{dx}\left[\int [\,f(x) + g(x)]\,dx\right] = f(x) + g\,(x) \quad \dots (1)$$

Also, we can write;

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx = f(x) + g(x) \quad \dots (2)$$

From (1) and (2),

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Hence proved.

Property 4: For any real value of p,

$$\int pf(x)\,dx = p\int f(x)\,dx$$

Proof: From property 1 we can say that

$$\frac{d}{dx}\int pf(x)\,dx = pf(x)$$

Also,

$$\frac{d}{dx}\left[p\int f(x)\,dx\right] = p\frac{d}{dx}\int f(x)\,dx = pf(x)$$

From property 2 we can say that

$$\int pf(x)\,dx = p\int f(x)\,dx$$

Property 5:

For a finite number of functions f_1, f_2…. f_n and the real numbers p_1, p_2…p_n,

$\int[p_1f_1(x) + p_2f_2(x)….+p_nf_n(x)]dx = p_1\int f_1(x)dx + p_2\int f_2(x)dx + ….. + p_n\int f_n(x)dx$

Indefinite Integral Formulas

The list of indefinite integral formulas are

- $\int 1\, dx = x + C$
- $\int a\, dx = ax + C$
- $\int x^n\, dx = ((x^{n+1})/(n+1)) + C$; $n \neq 1$
- $\int \sin x\, dx = – \cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \text{cosec}^2 x\, dx = -\cot x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$
- $\int (1/x)\, dx = \ln |x| + C$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = (a^x/\ln a) + C$; $a > 0$, $a \neq 1$

Indefinite Integrals Examples

Go through the following indefinite integral examples and solutions given below:

Example 1:

Evaluate the given indefinite integral problem: $\int 6x^5 - 18x^2 + 7\, dx$

Solution:

Given,

$\int 6x^5 - 18x^2 + 7\, dx$

Integrate the given function, it becomes:

$\int 6x^5 - 18x^2 + 7\, dx = 6(x^6/6) - 18(x^3/3) + 7x + C$

Note: Don't forget to put the integration constant "C"

After simplification, we get the solution

Thus, $\int 6x^5 - 18x^2 + 7\, dx = x^6 - 6x^3 + 7x + C$

Example 2:

Evaluate f(x), given that $f'(x) = 6x^8 - 20x^4 + x^2 + 9$

Solution:

Given,

$f'(x) = 6x^8 - 20x^4 + x^2 + 9$

We know that, the inverse process of differentiation is an integration.

Thus, $f(x) = \int f'(x)\, dx = \int [6x^8 - 20x^4 + x^2 + 9]\, dx$

$f(x) = (2/3)x^9 - 4x^5 + (1/3)x^3 + 9x + C$

Integration by Substitution

Let's learn what is Integration before understanding the concept of Integration by Substitution. The integration of a function f(x) is given by F(x) and it is represented by:

$$\int f(x)dx = F(x) + C$$

Here R.H.S. of the equation means integral of f(x) with respect to x.

- F(x) is called anti-derivative or primitive.
- f(x) is called the integrand.
- dx is called the integrating agent.
- C is called constant of integration or arbitrary constant.
- x is the variable of integration.

The anti-derivatives of basic functions are known to us. The integrals of these functions can be obtained readily. But this integration technique is limited to basic functions and in order to determine the integrals of various functions, different methods of integration are used. Among these methods of integration let us discuss integration by substitution.

Integration by Substitution Method

In this method of integration by substitution, any given integral is transformed into a simple form of integral by substituting the independent variable by others.

Take for example an equation having an independent variable in x, i.e. $\int \sin (x^3).3x^2.dx$————————(i),

In the equation given above the independent variable can be transformed into another variable say t.

Substituting $x^3 = t$ ———————-(ii)

Differentiation of above equation will give-

$3x^2.dx = dt$ ———————-(iii)

Substituting the value of (ii) and (iii) in (i), we have

$\int \sin (x^3).3x^2.dx = \int \sin t \, . \, dt$

Thus the integration of the above equation will give

$\int \sin t \,.\, dt = -\cos t + c$

Again putting back the value of t from equation (ii), we get

$\int \sin (x^3).3x^2.dx = -\cos x^3 + c$

The General Form of integration by substitution is:

$$\int \mathbf{f(g(x)).g'(x).dx = f(t).dt, \text{ where } t = g(x)}$$

Usually the method of integration by substitution is extremely useful when we make a substitution for a function whose derivative is also present in the integrand. Doing so, the function simplifies and then the basic formulas of integration can be used to integrate the function.

When to Use Integration by Substitution Method?

In calculus, the integration by substitution method is also known as the “Reverse Chain Rule” or “U-Substitution Method”. We can use this method to find an integral value when it is set up in the special form. It means that the given integral is of the form:

$\int f(g(x)).g'(x).dx = f(u).du$

Here, first, integrate the function with respect to the substituted value (f(u)), and finish the process by substituting the original function g(x).

Integration by Substitution Example

To understand this concept better, let us look into the examples.

Example 1:

Find the integration of

$$\int \frac{e^{tan^{-1}x}}{1+x^2}$$

Solution:

Given :

$$\int \frac{e^{tan^{-1}x}}{1+x^2}$$

Let t = $\tan^{-1}x$ …… (1)

dt = $(1/ 1+x^2)$. dx

I = ∫ e^t . dt

= e^t + C …….(2)

Substituting the value of (1) in (2), we have I = $e^{\tan^{-1}x}$ + C. This is the required integration for the given function.

Example 2:

Integrate 2x cos (x^2 – 5) with respect to x .

Solution:

I = ∫2xcos(x^2 – 5).dx

Let x^2 – 5 = t …..(1)

2x.dx = dt

Substituting these values, we have

I = ∫cos(t).dt

= sin t + c …..(2)

Substituting the value of 1 in 2, we have

= sin (x^2 – 5) + C

This is the required integration for the given function.

Integration by Parts

Integration by part is the technique used to find the integration of the product of two or more functions where the integration can not be performed using normal techniques. Suppose we have two functions f(x) and g(x) and we have to find the integration of their product i.e., ∫ f(x).g(x) dx where it is not possible to further solve the product of this product f(x).g(x).

This integration is achieved using the formula,

$$\int f(x).g(x)\,dx = f(x) \int g(x)\,d(x) - \int [f'(x) \{\int g(x)\,dx\}\,dx]\,dx + c$$

where f'(x) is the first differentiation of f(x).

This formula is read as,

Integration of the First Function multiplied by the Second Function is equal to (First Function) multiplied by (Integration of Second Function) – Integration of (Differentiation of First Function multiplied by Integration of Second Function).

From the above formula, we can easily observe that choosing the first function and the second function is very important for the success of this formula, and how we choose the first function and the second function is discussed further in this article.

Integration By Parts Formula

Integration by parts formula is the formula that helps us to achieve the integration of the product of two or more functions. Suppose we have to integrate the product of two functions as

$\int u.v\,dx$

where u and v are the functions of x, then this can be achieved using,

$$\int u.v\,dx = u \int v\,d(x) - \int [u' \{\int v\,dx\}\,dx]\,dx + c$$

The order to choose the First function and the Second function is very important and the concept used in most of the cases to find the first function and the second function is ILATE concept.

Using the above formula and the ILATE concept we can easily find the integration of the product of two functions. The integration by part formula is shown in the image below,

ILATE Rule

The ILATE rule tells us about how to choose the first function and the second function while solving the integration of the product of two functions. Suppose we have two functions of x u and v and we have to find the integration of their product then we choose the first function and the by ILATE rule.

The ILATE full form is discussed in the image below,

The ILATE rules give us the hierarchy of taking the first function, i.e. if in the given product of the function, one function is a Logarithmic function and another function is a Trigonometric function. Now we take the Logarithmic function as the first function as it comes above in the hierarchy of the ILATE rule similarly, we choose the first and second functions accordingly.

NOTE: It is not always appropriate to use the ILATE rule sometimes other rules are also used to find the first function and the second function.

Integration By Parts Formula Derivation

Integration By Parts Formula is derived using the product rule of differentiation. Suppose we have two functions **u** and **v** and x then the derivative of their product is achieved using the formula,

d/dx (uv) = u (dv/dx) + v (du/dx)

Now to derive the integration by parts formula using the product rule of differentiation.

Rearranging the terms

u (dv/dx) = d/dx (uv) – v (du/dx)

Integrating both sides with respect to x,

∫ u (dv/dx) (dx) = ∫ d/dx (uv) dx – ∫ v (du/dx) dx

simplifying,

∫ u dv = uv – ∫ v du

Thus, the integration by parts formula is derived.

How to Find Integration by Part

Integration by part is used to find the integration of the product of two functions. We can achieve this using the steps discussed below,

Suppose we have to simplify ∫uv dx

Step 1: *Choose the first and the second function according to the ILATE rule. Suppose we take u as the first function and v as the second function.*
Step 2: *Differentiate u(x) with respect to x that is,* ***Evaluate du/dx.***
Step 3: *Integrate v(x) with respect to x that is,* ***Evaluate ∫v dx.***
Use the results obtained in Step 1 and Step 2 in the formula,

$\int uv\,dx = u\int v\,dx - \int((du/dx)\int v\,dx)\,dx$
Step 4: *Simplify the above formula to get the required integration.*

Applications of Integration by Parts

Integration by Parts has various applications in integral calculus it is used to find the integration of the function where normal integration techniques fail. We can easily find the integration of inverse and logarithmic functions using the integration by parts concept.

We will find the Integration of the Logarithmic function and Arctan function using integration by part rule,

Integration of Logarithmic Function (log x)

Integration of Inverse Logarithmic Function (log x) is achieved using Integration by part formula. The integration is discussed below,

$\int \log x.dx = \int \log x.1.dx$

Taking log x as the first function and 1 as the second function.

Using $\int u.v\,dx = u \int v\,d(x) - \int [u' \{\int v\,dx\}\,dx]\,dx$

$\Rightarrow \int \log x.1.dx = \log x. \int 1.dx - \int ((\log x)'.\int 1.dx).dx$

$\Rightarrow \int \log x.1.dx = \log x.x - \int (1/x .x).dx$

$\Rightarrow \int \log x.1.dx = x\log x - \int 1.dx$

$\Rightarrow \int \log x.dx = x \log x - x + C$

Which is the required integration of logarithmic function.

Integration of Inverse Trigonometric Function ($\tan^{-1}$ x)

Integration of Inverse Trigonometric Function ($\tan^{-1}$ x) is achieved using Integration by part formula. The integration is discussed below,
$\int \tan^{-1}x.dx = \int \tan^{-1}x.1.dx$
Taking $\tan^{-1} x$ *as the first function and 1 as the second function.*
Using $\int u.v\,dx = u \int v\,d(x) - \int [u' \{\int v\,dx\}\,dx]\,dx$

$\Rightarrow \int \tan^{-1}x.1.dx = \tan^{-1}x.\int 1.dx - \int((\tan^{-1}x)'.\int 1.dx).dx$
$\Rightarrow \int \tan^{-1}x.1.dx = \tan^{-1}x.\ x - \int(1/(1 + x^2).x).dx$
$\Rightarrow \int \tan^{-1}x.1.dx = x.\ \tan^{-1}x - \int 2x/(2(1 + x^2)).dx$
$\Rightarrow \int \tan^{-1}x.dx = x.\ \tan^{-1}x - ½.\log(1 + x2) + C$
Which is the required integration of Inverse Trigonometric Function.

Formulas Related to Integration by Parts

We can derive the integration of various functions using the integration by parts concept. Some of the important formulas derived using this technique are

- $\int e^x(f(x) + f'(x)).dx = e^x f(x) + C$
- $\int\sqrt{(x^2 + a^2)}.dx = ½ . x.\sqrt{(x^2 + a^2)}+ a^2/2. \log|x + \sqrt{(x^2 + a^2)}| + C$
- $\int\sqrt{(x^2 – a^2)}.dx =½ . x.\sqrt{(x^2 – a^2)} – a^2/2. \log|x +\sqrt{(x^2 – a^2)}| \; C$
- $\int\sqrt{(a^2 – x^2)}.dx = ½ . x.\sqrt{(a^2 – x^2)} + a^2/2. \sin^{-1} x/a + C$

Solved Examples on Integration By Parts

Example 1: Find $\int e^x x\, dx$.

Solution:

Let $I = \int e^x x\, dx$

Choosing u and v using ILATE rule

$u = x$

$v = e^x$

Differentiating u

$u'(x) = d(u)/dx$

$\Rightarrow u'(x) = d(x)/dx$

$\Rightarrow u'(x) = 1$

$\int v\, dx = \int e^x\, dx = e^x$

Using the Integration by part formula,

$\Rightarrow I = \int e^x x\, dx$

$\Rightarrow I = x \int e^x dx − \int 1 (\int e^x dx)\, dx$

$\Rightarrow I = xe^x − e^x + C$

$\Rightarrow I = e^x(x − 1) + C$

Integration by the Partial Fractions Method

We can observe rational functions as integrands in the process of integration. In this case, we must reduce the integration of any rational function to the integration of a proper rational function. The rational functions we shall consider here for integration purposes will be those whose denominators can be factored into linear and quadratic factors. The method of writing the integrand, an improper rational function as a sum of simpler rational functions, is called partial fraction decomposition. Finding the integral in such cases is called integration by partial fraction method. This method of integration is simple and can be done using easy steps and formulas.

Integration by Partial Fractions Formula

The list of formulas used to decompose the given improper rational functions is given below. Using these expressions, we can quickly write the integrand as a sum of proper rational functions.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
	• where $x^2 + bx + c$ cannot be factorised further	

Also, get: Partial Fractions Decomposition Calculator

How to do Integration by Partial Fractions?

Go through the steps given below to understand the integration process by partial fractions.

Step 1: Check whether the given integrand is a proper or improper rational function.

Step 2: If the given function is an improper rational function, identify the type of denominator.

Step 3: Decompose the integrand using a suitable expression by comparing it with the five different forms given above.

Step 4: Now, divide the integration into parts and integrate the individual functions.