UNIT 4:

Recursion and recurrence

Recursive functions in discrete mathematics

A recursive function is a function that its value at any point can be calculated from the values of the function at some previous points. For example, suppose a function f(k) = f(k-2) + f(k-3) which is defined over non negative integer. If we have the value of the function at k = 0 and k = 2, we can also find its value at any other non-negative integer. In other words, we can say that a recursive function refers to a function that uses its own previous points to determine subsequent terms and thus forms a terms sequence. In this article, we will learn about recursive functions along with certain examples.

What is Recursion?

Recursion refers to a process in which a recursive process repeats itself. Recursive is a kind of function of one and more variables, usually specified by a certain process that produces values of that function by continuously implementing a particular relation to known values of the function.

Here, we will understand the recursion with the help of an example.

Suppose you are going to take a stair to reach the first floor from the ground floor. So, to do this, you have to take one by one steps. There is only a way to go to the second step that is to the steeped first step. Suppose you want to go to the third step; you need to take the second step first. Here, you can clearly see the repetition process. Here, you can see that with each next step, you are adding the previous step like a repeated sequence with the same difference between each step. This is the actual concept behind the recursive function.

Step 2: Step 1 + lowest step.

Step 3: Step 2 + Step 1 + lowest step.

Step 4: Step 3 + step 2 + step 1+ lowest step, and so on.

A set of natural numbers is the basic example of the recursive functions that start from one goes till infinity, 1,2,3,4,5,6,7,8, 9,…….infinitive. Therefore, the set of natural numbers shows a recursive function because you can see a common difference between each term as 1; it shows each time the next term repeated itself by the previous term.

What is a recursively defined function?

The recursively defined functions comprise of two parts. The first part deals with the smallest argument definition, and on the other hand, the second part deals with the nth term definition. The smallest argument is denoted by f (0) or f (1), whereas the nth argument is denoted by f (n).

Follow the given an example.

Suppose a sequence be 4,6,8,10

The explicit formula for the above sequence is f (n)= 2n + 2

The explicit formula for the above sequence is given by

f (0) = 2

f(n) = f (n-1) + 2

Now, we can get the sequence terms applying the recursive formula as follows f(2) f (1) + 2

f(2) = 6

f (0) = 2

f(1) = f(0) + 2

f (1) = 2 + 2 = 4

f(2) = f (1) + 2

f(2) = 4 + 2 = 6

f(3) = f (2) + 2

f(3) = 6 + 2 = 8

With the help of the above recursive function formula, we can determine the next term.

What makes the function recursive?

Making any function recursive needs its own term to calculate the next term in the sequence. For example, if you want to calculate the nth term of the given sequence, you first need to know the previous term and the term before the previous term. Hence, you need to know the previous term to find whether the sequence is recursive or not recursive. So we can conclude that if the function needs the previous term to determine the next term in the sequence, the function is considered a recursive function.

The formula of the Recursive function

If a_1, a_2, a_3, a_4, a_5, a_6, ……..a_n,……is many sets or a sequence, then a recursive formula will need to compute all the terms which are existed previously to calculate the value of an

$\mathbf{a_n = a_{n-1} + a_1}$

The above formula can also be defined as Arithmetic Sequence Recursive Formula. You can see clearly in the sequence mentioned above that it is an arithmetic sequence, which comprises the first term followed by the other terms and a common difference between each term. The common difference refers to a number that you add or subtract to them.

A recursive function can also be defined as the geometric sequence, where the number sets or a sequence have a common factor or common ratio between them. The formula for the geometric sequence is given as

$\mathbf{a_n = a_{n-1} * r}$

Usually, the recursive function is defined in two parts. The first one is the statement of the first term along with the formula, and another is the statement of the first term along with the rule related to the successive terms.

How to write a Recursive formula for arithmetic sequence

To write the Recursive formula for arithmetic sequence formula, follow the given steps

Step 1:

In the first step, you need to ensure whether the given sequence is arithmetic or not (for this, you need to add or subtract two successive terms). If you get the same output, then the sequence is taken as an arithmetic sequence.

Step 2:

Now, you need to find the common difference for the given sequence.

Step 3:

Formulate the recursive formula using the first term and then create the formula by using the previous term and common difference; thus you will get the given result

$\mathbf{a_n = a_{n-1} + d}$

now, we understand the given formula with the help of an example

suppose 3,5,7,9,11 is a given sequence

In the above example, you can easily find it is the arithmetic sequence because each term in the sequence is increases by 2. So, the common difference between two terms is 2. We know the formula of recursive sequence

$\mathbf{a_n = a_{n-1} + d}$

Given,

d = 2

$a_1 = 3$

so,

$a_2 = a_{(2-1)} + 2 = a_1 + 2 = 3+2 = 5$

$a_3 = a_{(3-1)} + 2 = a_2 + 2 = 5+2 = 7$

$a_4 = a_{(4-1)} + 2 = a_3 + 2 = 7+2 = 9$

$a_5 = a_{(5-1)} + 2 = a + 2 = 9+2 = 11$, and the process continues.

How to write a Recursive formula for Geometric sequence?

To write the Recursive formula for Geometric sequence formula, follow the given steps:

Step 1

In the first step, you need to ensure whether the given sequence is geometric or not (for this, you need to multiply or divide each term by a number). If you get the same output from one term to the next term, the sequence is taken as a geometric sequence.

Step 2

Now, you need to find the common ratio for the given sequence.

Step 3

Formulate the recursive formula using the first term and then create the formula by using the previous term and common ratio; thus you will get the given result

$\mathbf{a_n = r * a_{n-1}}$\

Now, we understand the given formula with the help of an example

suppose 2,8,32, 128,.is a given sequence

In the above example, you can easily find it is the geometric sequence because the successive term in the sequence is obtained by multiplying 4 to the previous term. So, the common ratio between two terms is 4. We know the formula of recursive sequence

$\mathbf{a_n = r * a_{n-1}}$\

$a_n = 4$

$a_{n-1} = ?$

Given,

$r = 4$

$a_1 = 2$

so,

$a_2 = a_{(2-1)} * 4 = a_1 + * 4 = 2 * 4 = 8$

$a_3 = a_{(3-1)} * 4 = a_2 * 4 = 8 * 4 = 32$

$a_4 = a_{(4-1)} * 4 = a_3 * 4 = 32 * 4 = 128$, and the process continues.

Example of recursive function

Example 1:

Determine the recursive formula for the sequence 4,8,16,32,64, 128,….?

Solution:

Given sequence 4,8,16,32,64,128,…..

The given sequence is geometric because if we multiply the preceding term, we get the successive terms.

To determine the recursive formula for the given sequence, we need to write it in the tabular form

Term Numbers	Sequence Term	Function Notation	Subscript Notation
1	4	f(1)	a_1
2	8	f(2)	a_2
3	16	f(3)	a_3
4	32	f(4)	a_4
5	64	f(5)	a_5
6	128	f(6)	a_6
n	.	f(n)	a_n

Hence, the recursive formula in function notion is given by

f(1) = 4, f(n) . f(n- 1)

In subscript notation, the recursive formula is given by

$a_1 = 4$, $a_n = 2. a_{n-1}$

Closed-form expression

A **closed-form expression** is a mathematical process that can be completed in a finite number of operations. Closed-form expressions are of interest when trying to develop general solutions to problems.

A **closed-form solution** is a general solution to a problem in the form of a closed-form expression. A closed-form solution is nearly always desirable because it means that a solution can be found efficiently. Unfortunately, closed-form solutions are not always possible. Many mathematicians concern themselves with finding closed-form solutions to open problems, and in lieu of that, proving whether or not a closed-form solution is possible.

Of course, an operation can be defined to "cheat" the requirements of a closed-form expression. For example, define operation ¤¤ to solve any problem in existence. *Voila*! Now every single problem in existence has a closed-form solution! Of course, this would be absurd.

It is for this reason that closed-form expressions and solutions are typically restricted to having the following operations:

- Addition, subtraction, multiplication, and division
- Exponents and logarithms
- Trigonometric functions and inverse trigonometric functions

Limits and other operations that involve an infinite number of terms or operations are not permitted. Finite Sums and Products (using the $\Sigma\Sigma$ and $\Pi\Pi$ symbols, respectively) are also generally avoided.

Recurrence Relation

A **recurrence relation** is an equation which represents a sequence based on some rule. It helps in finding the subsequent term (next term) dependent upon the preceding term (previous term). If we know the previous term in a given series, then we can easily determine the next term. Since a standard pattern is developed now, we can find the set of new terms. This is also applicable for arithmetic and geometric sequence.

Recurrence Relation Definition

When we speak about a standard pattern, all the terms in the relation or equation have the same characteristics. It means if there is a value of ‘n’, it can be used to determine the other values by just entering the value of ‘n’.

The value of n should be organised and accurate, which is known as the Simplest form. In case of the simplest form of any such relation, the next term is dependent only upon the previous term. The sequence or series generated by recurrence relation is called a **Recurrence Sequence**.

Also, read:

- Sequence And Series
- Sequences And Series Class 11
- Factorial

Recurrence Relation Formula

Let us assume x_n is the nth term of the series. Then the recurrence relation is shown in the form of;

$$x_n + 1 = f(x_n) \; ; \; n>0$$

Where $f(x_n)$ is the function.

We can also define a recurrence relation as an expression that represents each element of a series as a function of the preceding ones.

$$x_n = f(n, x_{n-1}) \; ; \; n>0$$

To write the recurrence relation of first-order, say order k, the above formula can be represented as;

$$x_n = f(n, x_{n-1}, x_{n-2}, \ldots\ldots, x_{n-k}) \; ; \; n-k>0$$

Examples of Recurrence Relation

In Mathematics, we can see many examples of recurrence based on series and sequence pattern. Let us see some of the examples here.

Factorial Representation

We can define the factorial by using the concept of recurrence relation, such as;

$n!=n(n-1)!$; $n>0$

When n = 0,

0! = 1 is the initial condition.

To find the further values we have to expand the factorial notation, where the succeeding term is dependent on the preceding one.

Fibonacci Numbers

In Fibonacci numbers or series, the succeeding terms are dependent on the last two preceding terms. Therefore, this series is the best example of recurrence. As we know from the definition of the Fibonacci sequence,

$F_n = F_{n-1} + F_{n-2}$

Now, if we take the initial values;

$F_0 = 0$ and $F_1 = 1$

So, $F_2 = F_1 + F_0 = 0 + 1 = 1$

In the same way, we can find the next succeeding terms, such as;

$F_3 = F_2 + F_1$

$F_4 = F_3 + F_2$

And so on.

Thus, the Fibonacci series is given by;

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …∞

In the same way, there are other examples of recurrence such as a logical map, binomial coefficients where the same concept is applicable. Also, arithmetic and geometric series could be called a recurrence sequence.

Solving Recurrence Relations

To solve given recurrence relations we need to find the initial term first. Suppose we have been given a sequence;

$a_n = 2a_{n-1} - 3a_{n-2}$

- Now the first step will be to check if initial conditions $a_0 = 1$, $a_1 = 2$, gives a closed pattern for this sequence.
- Then try with other initial conditions and find the closed formula for it.
- The result so obtained after trying different initial condition produces a series.
- Check the difference between each term, it will also form a sequence.
- We need to add all the terms of the new sequence, to understand which sequence is formed
- After understanding the pattern we can now identify the initial condition of the recurrence relation.

Recurrence Relation Problem

Now let us solve a problem based on the solution provided above.

Question: Solve the recurrence relation $a_n = a_{n-1} - n$ with the initial term $a_0 = 4$.

Solution: Let us write the sequence based on the equation given starting with the initial number.

The sequence will be 4,5,7,10,14,19,…..

Now see the difference between each term.

$a_1 - a_0 = 1$

$a_2 - a_1 = 2$

$a_3 - a_2 = 3$

………

$a_n – a_{n-1} = n$

and so on.

Now adding all these equations both at the right-hand side, we get;

1 +2+3+4+…..n = 1/2 [n(n+1)]

Whereas on the left-hand side we get;

$(a_1 – a_0) + (a_2 – a_1) + (a_3 – a_2) + ……. + (a_{n-1} – a_{n-2}) + (a_n – a_{n-1})$

So you can see, all the terms get cancelled but $– a_0$ and a_n

Therefore, $a_n – a_0 = 1/2 [n(n+1)]$

or

$a_n = 1/2 [n(n+1)] + a_0$

Hence, the solution to the recurrence relation with initial condition $a_0 = 4$, is;

$a_n = 1/2 [n(n+1)] + 4$

Types of recurrence relations

- **First order Recurrence relation :-** A recurrence relation of the form : **$a_n = ca_{n-1} + f(n)$** for n>=1
  where c is a constant and f(n) is a known function is called linear recurrence relation of first order with constant coefficient. If f(n) = 0, the relation is homogeneous otherwise non-homogeneous.
  Example :- $x_n = 2x_{n-1} – 1$, $a_n = na_{n-1} + 1$, etc.
  Question :- Solve the recurrence relation $T(2^k) = 3T(2^{k-1}) + 1$, $T(1) = 1$.
  Let $T(2^k) = a_k$. Therefore, $a_k = 3a_{k-1} + 1$
  Multiplying by x^k and then taking sum,
  $\Sigma a_k x^k = 3\Sigma a_{k-1}x^k + \Sigma x^k$ ——> (1)
  $\Sigma a_{k-1}x^k = [a_0x + a_1x^2 + ……]$
  $= x[a_0 + a_1x + ……] = x[G(x)]$
  (1) becomes

$G(x) – 3xG(x) – x/(1-x) = 0$
$G(x)(1-3x) – x/(1-x) = 0$
$G(x) = x/[(1-x)(1-3x)] = A/(1-x) + B/(1-3x)$
–> $A = -1/2$ and $B = 3/2$
$G(x) = (3/2)\Sigma(3x)^k – (1/2)\Sigma(x)^k$
Coefficient of x^k is, $a_k = (3/2)3^k – (1/2)1^k$
So, $a_k = [3^{k+1} – 1]/2$.

- **Second order linear homogeneous Recurrence relation :-** A recurrence relation of the form
  $\mathbf{c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0}$ ——> (1)
  for n>=2 where c_n, c_{n-1} and c_{n-2} are real constants with c_n != 0 is called a second order linear homogeneous recurrence relation with constant coefficients.
  Solution to this is in form $a_n = ck^n$ where c, k!=0
  Putting this in (1)
  $c_n ck^n + c_{n-1} ck^{n-1} + c_{n-2} ck^{n-2} = 0$
  $c_n k^2 + c_{n-1} k + c_{n-2} = 0$ ——> (2)
  Thus, $a_n = ck^n$ is solution of (1) if k satisfies quadratic equation (2). This equation is called characteristic equation for relation (1).
  Now three cases arises,
  Case 1 : If the two roots k_1, k_2 of equation are real and distinct then, we take
  $\mathbf{a_n = A(k_1)^n + B(k_2)^n}$ as general solution of (1) where A and B are arbitrary real constants.
  Case 2 : If the two roots k_1, k_2 of equation are real and equal, with k as common value then, we take
  $\mathbf{a_n = (A + Bn)k^n}$ as general solution of (1) where A and B are arbitrary real constants.
  Case 3 : If the two roots k_1 and k_2 of equation are complex then, k_1 and k_2 are complex conjugate of each other i.e $k_1 = p + iq$ and $k_2 = p – iq$ and we take
  $\mathbf{a_n = r^n(A\cos n\theta + B\sin n\theta)}$ as general solution of (1) where A and B are arbitrary complex constants, $r = |k_1| = |k_2| = \sqrt{p^2 + q^2}$ and $\theta = \tan^{-1}(q/p)$.

Question :- Solve the recurrence relation $a_n + a_{n-1} – 6a_{n-2} = 0$ for n>=2 given that $a_0 = -1$ and $a_1 = 8$.
Here coefficients of a_n, a_{n-1} and a_{n-2} are $c_n = 1$, $c_{n-1} = 1$ and $c_{n-2} = -6$ respectively. Hence, characteristic equation is
$k^2 + k – 6$ or $(k + 3)(k – 2) = 0$ ——> (1)
The roots of (1) are $k_1 = -3$ and $k_2 = 2$ which are real and distinct. Therefore, general solution is
$a_n = A(-3)^n + B(2)^n$
where A and B are arbitrary constants. From above we get, $a_0 = A + B$ and $a_1 = -3A + 2B$
$A + B = -1$
$-3A + 2B = 8$

Solving these we get A = -2 and B = 1
Therefore, $a_n = -2(-3)^n + (2)^n$

Generating Functions

Generating function is a method to solve the recurrence relations.

Let us consider, the sequence $a_0, a_1, a_2....a_r$ of real numbers. For some interval of real numbers containing zero values at t is given, the function G(t) is defined by the series

$G(t)= a_0, a_1t+a_2 t^2+\cdots+a_r t_r+$...........equation (i)

This function G(t) is called the generating function of the sequence a_r.

Now, for the constant sequence 1, 1, 1, 1.....the generating function is

$$G(t) = \frac{1}{(1-t)}$$

It can be expressed as

$G(t) =(1-t)^{-1}=1+t+t^2 +t^3+t^4+\cdots$[By binomial expansion]

Comparing, this with equation (i), we get

$a_0=1,a_1=1,a_2=1$ and so on.

For, the constant sequence 1,2,3,4,5,..the generating function is

$G(t) = \frac{1}{(1-t)^2}$ because it can be expressed as
$G(t) =(1-t)^{-2}=1+2t+3t^2 +4t^3+\cdots+(r+1) t^r$

Comparing, this with equation (i), we get
$a_0=1,a_1=2,a_2=3,a_3=4$ and so on.

The generating function of Z^r,($Z\neq0$ and Z is a constant)is given by
$G(t)= 1+Zt+Z^2 t^2+Z^3 t^3+\cdots+Z^r t^r$

$G(t)=\frac{1}{(1 - Zt)}$ [Assume $|Zt|<1$]

So, $G(t)=\frac{1}{(1 - Zt)}$ generates $Z^r, Z\neq0$

Also,If $a^{(1)}_r$ has the generating function $G_1(t)$ and $a^{(2)}_r$ has the generating function $G_2(t)$, then $\lambda_1 a^{(1)}_r+\lambda_2 a^{(2)}_r$ has the generating function $\lambda_1 G_1(t)+ \lambda_2 G_2(t)$. Here λ_1 and λ_2 are constants.