UNIT 4:

Numerical solution of Linear Equation

Trace of a matrix :
Let $A=[a_{ij}]_{nxn}$ is a square matrix of order n, then the sum of diagonal elements is called the trace of a matrix which is denoted by tr(A). $tr(A) = a_{11} + a_{22} + a_{33} + \ldots\ldots\ldots + a_{nn}$
Properties of trace of matrix:
Let A and B be any two square matrices of order n, then

1. tr(kA) = k tr(A) where k is a scalar.
2. tr(A+B) = tr(A)+tr(B)
3. tr(A-B) = tr(A)-tr(B)
4. tr(AB) = tr(BA)

Solution of a system of linear equations:
Linear equations can have three kind of possible solutions:

- No Solution
- Unique Solution
- Infinite Solution

Rank of a matrix: Rank of matrix is the number of non-zero rows in the row reduced form or the maximum number of independent rows or the maximum number of independent columns.
Let A be any m x n matrix and it has square sub-matrices of different orders. A matrix is said to be of rank r, if it satisfies the following properties:

1. It has at least one square sub-matrices of order r who has non-zero determinant.
2. All the determinants of square sub-matrices of order (r+1) or higher than r are zero.

Rank is denoted as P(A).
if A is a non-singular matrix of order n, then rank of A = n i.e. P(A) = n.
Properties of rank of a matrix:

1. If A is a null matrix then P(A) = 0 i.e. Rank of null matrix is zero.
2. If I_n is the nxn unit matrix then P(A) = n.
3. Rank of a matrix A mxn , $P(A) \leq \min(m,n)$. Thus $P(A) \leq m$ and $P(A) \leq n$.
4. $P(A_{nxn}) = n$ if $|A| \neq 0$
5. If P(A) = m and P(B)=n then $P(AB) \leq \min(m,n)$.
6. If A and B are square matrices of order n then P(AB) = P(A) + P(B) – n.

7. If $A_{m\times 1}$ is a non zero column matrix and $B_{1\times n}$ is a non zero row matrix then P(AB) = 1.
8. The rank of a skew symmetric matrix cannot be equal to one.

System of homogeneous linear equations AX = 0.

1. X = 0. is always a solution; means all the unknowns has same value as zero. (This is also called trivial solution)
2. If P(A) = number of unknowns, unique solution.
3. If P(A) < number of unknowns, infinite number of solutions.

System of non-homogeneous linear equations AX = B.

1. If P[A:B] ≠P(A), No solution.
2. If P[A:B] = P(A) = the number of unknown variables, unique solution.
3. If P[A:B] = P(A) ≠ number of unknown, infinite number of solutions.

Here P[A:B] is the rank of Gauss Elimination representation of AX = B.

There are two states of the Linear equation system:

- **Consistent State:** A System of equations having one or more solutions is called a consistent system of equations.
- **Inconsistent State:** A System of equations having no solutions is called inconsistent system of equations.

Linear dependence and Linear independence of vector:

Linear Dependence: A set of vectors X_1 ,X_2 ….X_r is said to be linearly dependent if there exist r scalars k_1 ,k_2 …..k_r such that: $k_1 X_1 + k_2X_2 +$……..$k_r X_r = 0$.

Linear Independence: A set of vectors X_1 ,X_2….X_r is said to be linearly independent if for all r scalars k_1,k_2 …..k_r such that $k_1X_1+ k_2 X_2+$……..$k_rX_r = 0$, then $k_1 = k_2 =$……. $= k_r = 0$.

How to determine linear dependency and independency ?

Let X_1, X_2 ….X_r be the given vectors. Construct a matrix with the given vectors as its rows.

1. If the rank of the matrix of the given vectors is less than the number of vectors, then the vectors are linearly dependent.
2. If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.

System of linear equations

In the numerical solution of a system of linear equations, various methods can be employed to find the solution. Here are a few commonly used methods:

1. Gaussian Elimination: Gaussian Elimination is a method that transforms the system of linear equations into an upper triangular form by performing row operations. The augmented matrix, which includes both the coefficient matrix and the right-hand side vector, is manipulated to eliminate variables and simplify the system. Once the system is in upper triangular form, back substitution is used to find the solution.
2. LU Decomposition: LU decomposition is a technique that factors the coefficient matrix of a system into the product of a lower triangular matrix (L) and an upper triangular matrix (U). The LU decomposition can be computed using various methods, such as Doolittle's method or Crout's method. Once the decomposition is obtained, forward and backward substitution can be used to find the solution.
3. Gauss-Seidel Method: The Gauss-Seidel method is an iterative method used to solve a system of linear equations. It starts with an initial guess for the solution and iteratively updates the values of the variables based on the current and updated values of neighboring variables. The method continues until convergence is achieved, typically defined by a specified tolerance.
4. Jacobi Method: Similar to the Gauss-Seidel method, the Jacobi method is an iterative technique used to solve a system of linear equations. It updates the values of variables based on the current values of all the variables from the previous iteration. The method iterates until convergence is achieved.
5. Conjugate Gradient Method: The Conjugate Gradient method is an iterative algorithm used to solve large systems of linear equations, particularly when the coefficient matrix is symmetric and positive definite. It aims to find the solution by minimizing the residual error in a conjugate direction space. The method iteratively refines the solution until a desired level of accuracy is reached.

These methods are just a few examples of the many techniques available for solving systems of linear equations numerically. The choice of method depends on factors such as the size and properties of the system, computational efficiency requirements, and desired accuracy.

Gauss elimination method

Gauss elimination method is used to solve a system of linear equations. Let's recall the definition of these systems of equations. A system of linear equations is a group of linear equations with various unknown factors. As we know, unknown factors exist in multiple equations. Solving a system involves finding the value for the unknown factors to verify all the equations that make up the system.

If there is a single solution that means one value for each unknown factor, then we can say that the given system is a consistent independent system. If multiple solutions exist, the system has infinitely many solutions; then we say that it is a consistent dependent system. If there is no solution for unknown factors, and this will happen if there are two or more equations that can't be verified simultaneously, then we say that it's an inconsistent system.

This can be summarized in a table as given below:

Name of the system of equations	Number of solutions
Consistent independent system	1
Consistent dependent system	Multiple or Infinitely many
Inconsistent system	0

Now, let's have a look at the method that can be used to find the solution(s) of the given system of equations.

What is the Gauss Elimination Method?

In mathematics, the Gaussian elimination method is known as the row reduction algorithm for solving linear equations systems. It consists of a sequence of operations performed on the corresponding matrix of coefficients. We can also use this method to estimate either of the following:

- The rank of the given matrix
- The determinant of a square matrix

- The inverse of an invertible matrix

To perform row reduction on a matrix, we have to complete a sequence of elementary row operations to transform the matrix till we get 0s (i.e., zeros) on the lower left-hand corner of the matrix as much as possible. That means the obtained matrix should be an upper triangular matrix. There are three types of elementary row operations; they are:

- Swapping two rows and this can be expressed using the notation ↔, for example, $R_2 \leftrightarrow R_3$
- Multiplying a row by a nonzero number, for example, $R_1 \rightarrow kR_2$ where k is some nonzero number
- Adding a multiple of one row to another row, for example, $R_2 \rightarrow R_2 + 3R_1$

Learn more about the elementary operations of a matrix here.

The obtained matrix will be in row echelon form. The matrix is said to be in reduced row-echelon form when all of the leading coefficients equal 1, and every column containing a leading coefficient has zeros elsewhere. This final form is unique; that means it is independent of the sequence of row operations used. We can understand this in a better way with the help of the example given below.

Gauss Elimination Method with Example

Let's have a look at the gauss elimination method example with a solution.

Question:

Solve the following system of equations:

$x + y + z = 2$

$x + 2y + 3z = 5$

$2x + 3y + 4z = 11$

Solution:

Given system of equations are:

$x + y + z = 2$

$x + 2y + 3z = 5$

$2x + 3y + 4z = 11$

Let us write these equations in matrix form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 11 \end{array}\right]$$

Subtracting R_1 from R_2 to get the new elements of R_2, i.e. $R_2 \rightarrow R_2 - R_1$.

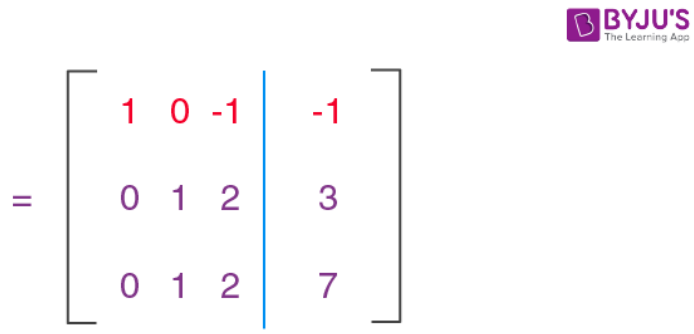
From this we get,

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 11 \end{array}\right]$$

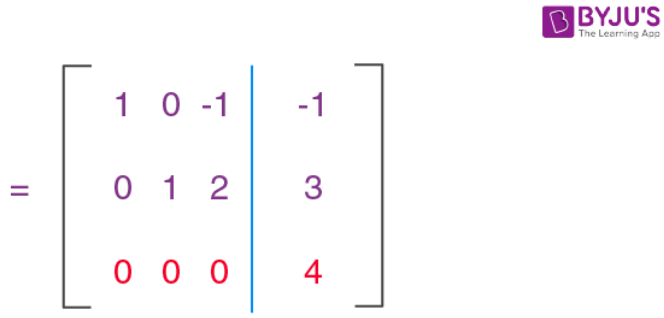
Let us make another operation as $R_3 \rightarrow R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

Subtract R_2 from R_1 to get the new elements of R_1, i.e. $R_1 \rightarrow R_1 - R_2$.



Now, subtract R_2 from R_3 to get the new elements of R_3, i.e. $R_3 \rightarrow R_3 – R_2$.



Here,

$x – z = -1$

$y + 2z = 3$

$0 = 4$

That means, there is no solution for the given system of equations.

Gauss Jordan Method

Gauss Jordan elimination consists in a sequence of elementary row operations:

1. interchanging the order of the equations, so as to make sure that the zero rows are at the bottom of the matrix;

2. multiplying (or dividing) the equations by non-zero constants, so as to make the pivots equal to ;
3. adding multiples of some equations to other equations, so as to annihilate the entries above and below the pivots.

Steps of the algorithm

The Gauss Jordan algorithm is very similar to Gaussian elimination. Therefore, we are not going to explain its steps in detail, but we are only going to comment on the differences with respect to Gaussian elimination. Please refer to the lecture on Gaussian elimination for detailed explanations.

Before laying out the algorithm, we warn the reader that the coefficient matrix of the system will be denoted by both before and after performing an elementary row operation, even if the matrix resulting from the operation is in principle a different matrix.

The steps are listed below:

1. Start from and .
2. Increment by one unit.
3. Increment by one unit.
4. Stop the algorithm if . Else proceed to the next step.
5. If for , return to step 3. Else proceed to the next step.
6. Interchange the -th equation with any equation (with) such that (if there is no need to perform an interchange).
7. Divide the -th equation by .
8. For and , subtract the -th equation multiplied by from the -th equation.
9. If , return to step 2. Else stop the algorithm.

The differences with respect to Gaussian elimination are the following:

- Step 7 is not found in the Gaussian elimination algorithm. In step 7, we divide the equation containing the current pivot by the value of the pivot

itself. This is done to ensure that all pivots are equal to in the final system.

- In step 8, we perform elementary operations to annihilate not only the elements below the pivot (as in Gaussian elimination) but also those above it.
- In step 9, we stop at the last row, unlike in Gaussian elimination, where we stop at the penultimate row. The reason is that, if there is a pivot on the last row, we need to make it equal to , and we need to annihilate the entries above it.

Examples

We present below an example of Gauss Jordan elimination.

In order to simplify the notation, we are going to use augmented matrices to represent linear systems.

Example Consider the system of three equations in four unknowns

represented by the augmented matrix $$(A|b) = \left[\begin{array}{cccc|c} -1 & 2 & 6 & 7 & 15 \\ 3 & -6 & 0 & -3 & -9 \\ 1 & 0 & 6 & -1 & 5 \end{array}\right]$$ We start from row and column . Since , we do not perform any interchange of rows. We divide the first row by and obtain

$$\left[\begin{array}{cccc|c} 1 & -2 & -6 & -7 & -15 \\ 3 & -6 & 0 & -3 & -9 \\ 1 & 0 & 6 & -1 & 5 \end{array}\right]$$ In order to annihilate the entries below the pivot , we subtract the first row multiplied by from the second and multiplied by from the third: $$\left[\begin{array}{cccc|c} 1 & -2 & -6 & -7 & -15 \\ 0 & 0 & 18 & 18 & 36 \\ 0 & 2 & 12 & 6 & 20 \end{array}\right]$$ We move to row and

column . Since , but , we interchange the second row with the third: $\left[\begin{array}{cccc|c} 1 & -2 & -6 & -7 & -15 \\ 0 & 2 & 12 & 6 & 20 \\ 0 & 0 & 18 & 18 & 36 \end{array}\right]$ We divide the second row by :

$\left[\begin{array}{cccc|c} 1 & -2 & -6 & -7 & -15 \\ 0 & 1 & 6 & 3 & 10 \\ 0 & 0 & 18 & 18 & 36 \end{array}\right]$ We add the second row multiplied by to the first:

$\left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 5 \\ 0 & 1 & 6 & 3 & 10 \\ 0 & 0 & 18 & 18 & 36 \end{array}\right]$ We move to row and column . We divide the third row by the value of its pivot : We add times the third row to the second and the third:

The matrix is now in reduced row echelon form.

Gauss–Seidel method

This is to take Jacobi's Method one step further. Where the better solution is x = (x1, x2, … , xn), if x1(k+1) is a better approximation to the value of x1 than x1(k) is, then it would better that we have found the new value x1(k+1) to use it (rather than the old value that isx1(k)) in finding x2(k+1), … , xn(k+1). So x1(k+1) is found as in Jacobi's Method, but in finding x2(k+1), instead of using the old value of x1(k) and old values of x3(k),…, xn(k), we then use the new value x1(k+1) and the old values x3(k), … , xn(k), and similarly for finding

x3(k+1), … , xn(k+1). This process to find the solution of the given linear equation is called the **Gauss-Seidel Method**
The Gauss–Seidel method is an iterative technique for solving a square system of n (n=3) linear equations with unknown x.
Given

```
Ax=B
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, to find the system of equation x which satisfy this condition.
In more detail, A, x and b in their components are :

Then the decomposition of A Matrix into its lower triangular component and its upper triangular component is given by:

The system of linear equations are rewritten as:

$$L_* \mathbf{x} = \mathbf{b} - U\mathbf{x}$$

The Gauss–Seidel method now solves the left hand side of this expression for x, using previous value for x on the right hand side. More formally, this may be written as:

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)}).$$

However, by triangular form of L*, the elements of x(k+1) can be computed sequentially using forward substitution:

This process is continuously repeated until we found the better approximated solution with least error.
Examples:

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Input :
3
4x+ y+ 2z= 4
3x+ 5y+ 1z= 7
x+ y+ 3z= 3

Output :
[0, 0, 0]
[1.0, 0.8, 0.39999999999999997]
[0.6000000000000001, 0.9599999999999997, 0.48000000000000004]
[0.52, 0.9919999999999998, 0.49600000000000005]
[0.504, 0.9983999999999998, 0.4992000000000001]
```

```
[0.5008, 0.99968, 0.49984]
[0.5001599999999999, 0.9999360000000002, 0.4999679999999999]
[0.500032, 0.9999872, 0.4999936]
[0.5000064, 0.9999974400000001, 0.49999871999999995]
[0.50000128, 0.999999488, 0.4999997439999999]
[0.500000256, 0.9999998976000001, 0.49999994880000004]
[0.5000000512, 0.9999999795199999, 0.4999999897600001]
[0.50000001024, 0.999999995904, 0.499999997952]
[0.500000002048, 0.9999999991808, 0.49999999959040003]
[0.5000000004095999, 0.9999999998361601, 0.49999999991808003]
[0.50000000008192, 0.9999999999672321, 0.49999999998361594]
[0.500000000016384, 0.9999999999934465, 0.49999999999672307]
[0.5000000000032768, 0.9999999999986894, 0.4999999999993445]
[0.5000000000006554, 0.9999999999997378, 0.49999999999986894]
[0.500000000000131, 0.9999999999999478, 0.49999999999997374]
[0.5000000000000262, 0.9999999999999897, 0.49999999999999467]
[0.5000000000000052, 0.9999999999999979, 0.49999999999999895]
[0.5000000000000011, 0.9999999999999994, 0.49999999999999983]
[0.5000000000000002, 0.9999999999999998, 0.5000000000000001]
[0.49999999999999994, 1.0, 0.5]
[0.5, 1.0, 0.5]
```

Given the three equation:

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4x + y + 2z = 4
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3x + 5y + z = 7
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```
x + y + 3z = 3
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First we assume that the solution of given equation is

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(0,0,0)
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Then first we put value of y and z in equation 1 and get value of x and update the value of x as

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(x1,0,0)
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Now, putting the updated value of x that is x1 and z=0 in equation 2 to get y1 and then updating our solution as

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(x1,y1,0)
```

Then, at last putting x1 and y1 in equation 3 to get z1 and updating our solution as

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(x1,y1,z1)
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Now repeat the same process 24 more times to get the approximate solution with minimum error.