UNIT 3:
Matrix and Determinants

Matrices Definition

Matrices are the ordered rectangular array of numbers, which are used to express linear equations. A matrix has rows and columns. we can also perform the mathematical operations on matrices such as addition, subtraction, multiplication of matrix. Suppose the number of rows is m and columns is n, then the matrix is represented as m × n matrix.

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_m & \cdots & a_{mn} \end{bmatrix}$$

Properties of Matrices

The following properties of matrices help in easily performing numerous operations on matrices.

Addition Property of Matrics

- Commutative Law. For the given two matrixes, matrix A and matrix B of the same order, say m x n, then A + B = B + A.
- Associative law: For any three matrices, A , B, C of the same order m x n, we have (A + B) + C = A + (B + C)
- Existence of additive identity Let A be a matrix of order m × n, and O be a zero matrix or a null matrix of the same order m × n , then A + O = O + A = A. In other words, O is the additive identity for matrix addition.
- Existence of additive inverse Let A be a matrix of order m × n. and let -A be another matrix of order m × n such that A + (– A) = (– A) + A= O. So the matrix – A is the additive inverse of A or the negative of matrix A.

Scalar Multiplication Property of Matrices

- The product of a constant with the sum of matrices is equal to the sum of the individual product of the constant and the matrix. $k(A + B) = kA + kB$
- The product of the sum of the constants with a matrix is equal to the sum of the product of each of the constants with the matrix. $(k + l)A = kA + lA$

Multiplication Property of Matrices

- Associative Property: For any three matrices A, B, C following the matrix multiplication conditions, we have $(AB)C = A(BC)$. Here both sides of the matrix multiplication are defined.
- Distributive Property: For any three matrices A, B, C following the matrix multiplication conditions, we have $A(B + C) = AB + AC$.
- The existence of multiplicative identity. For a square matrix A, having the order $m \times n$, and an identity matrix I of the same order we have $AI = IA = A$. Here the product of the identity matrix with the given matrix results in the same matrix.

Transpose Property of Matrices

- The transpose of a matrix on further taking a transpose for the second time results in the original matrix. $(A')' = A$
- The transpose of the product of a constant and a matrix is equal to the product of the constant and the transpose of the matrix. $(kA)' = kA'$
- The transpose of the sum of two matrices is equal to the sum of the transpose of the individual matrices. $(A + B)' = A' + B'$
- The transpose of the product of two matrices is equal to the product of the transpose of the second matrix and the transpose of the first matrix. $(AB)' = B'A'$

Other Properties of Matrices

- For a square matrix with real number entries, $A + A'$ is a symmetric matrix, and $A - A'$ is a skew-symmetric matrix.
- A square matrix can be expressed as a sum of symmetric and skew-symmetric matrix. $A = 1/2(A + A') + 1/2(A - A')$.
- The inverse of a matrix if it exists is unique. $AB = BA = I$.
- If matrix A is the inverse of matrix B, then matrix B is the inverse of matrix A.
- If A and B are invertible matrices of the same order $m \times n$, then $(AB)^{-1} = B^{-1}A^{-1}$.

Matrix Operations

There are various operations that can be performed on the matrix, which are described as follows:

1. Matrix addition
2. Matrix subtraction

3. Matrix equality
4. Matrix multiplication

Now we will explain them one by one.

Matrix Addition

Suppose there are two matrices, A and B, where A = [a_{ij}] and B = [b_{ij}]. Here i and j are used to indicate any value. This matrix is in the form m∗n. With the help of symbol A+B, we can indicate the sum of A and B. If A = a_{ij} and B = b_{ij}, then the addition of A and B can be described as follows:

$A + B = a_{ij} + b_{ij}$

Suppose there are two matrices, A and B, which contain the following element:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The addition of A and B will be C, which contains the following matrix:

$$C = \begin{bmatrix} a11 + b11 & a12 + b12 \\ a21 + b21 & a22 + b22 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Properties

There are some properties that are contained by matrix addition, which are described as follows:

Suppose matrix A, B, and C are conformable. Then

It will contain the commutative law, which contains the following relation:

A+B = B+A

It will contain the associative law, which contains the following relation:

A+(B+C) = (A+B)+C

It will contain the distributive law, which contains the following relation:

λ(A+B) = λA + λB, where λ is used to indicate a scalar.

Matrix Subtraction

In the subtraction process, we will subtract elements of same position in the given matrices. If there are two matrices, A and B, which have the same dimension or have the same order, only then we can subtract them. If two given matrices have two different orders, then we cannot perform the subtraction operation on them. With the help of symbol A-B, we can indicate the subtraction of A and B. If A = a_{ij} and B = b_{ij}, then the subtraction of A and B can be described as follows:

A - B = a_{ij} - b_{ij}

Here i and j can have any value.

Example 1:

$$\begin{bmatrix} 8 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -7 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 8-2 & 3-(-7) \\ 4-6 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -2 & 1 \end{bmatrix}$$

Matrix equality

Suppose there are two matrices, A and B. These two matrices will be equal if there are the same number of rows and columns in the matrices, and every element at each position in A will have the same element at the corresponding position in B. In this type of matrix, matrices will have the same corresponding entries.

For example:

$$\begin{bmatrix} 5 & 0 \\ -\frac{4}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 0.75 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 0 & -3 \end{bmatrix} \neq \begin{bmatrix} -2 & 6 \\ 3 & -2 \end{bmatrix}$$

Matrix Multiplication:

If there are two matrices, A and B, where A has an m∗k matrix and B has a k∗n matrix. We can perform the multiplication operation of two given matrices only if the number of columns in 1^{st} matrix and the number of rows in 2^{nd} matrix are similar to each other. With the help of symbol A∗B, we can indicate the multiplication of A and B. In the multiplication process, we will multiply rows by columns. If there is a matrix A, which is an a∗b matrix, and another matrix B which is a b∗c matrix, then the multiplication of A and B can be described as follows:

C = AB

Here matrix C is used to indicate the resultant matrix.

Suppose matrix C contains elements x and y, then it will be defined as

$$C_{xy} = A_{x1}B_{y1} + A_{x2}B_{y2} + ... + A_{xb}B_{by} = \sum^{b}_{k=1} A_{xk}B_{ky}$$

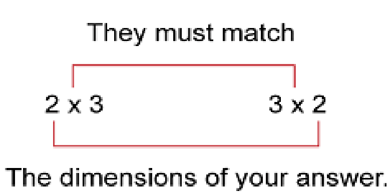
For x = 1....a and y = 1....c

In this process, the addition of multiplication of the ith row of matrix A to the corresponding element of jth column of matrix B will be equal to the product of A∗B matrix with its (i, j)th entry.

For example: In this example, we have two matrices, which are described as follows:

$$\begin{bmatrix} -3 & 2 & 5 \\ 7 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} -8 & 2 \\ 1 & 5 \\ 0 & -3 \end{bmatrix}$$

The dimension of first matrix is 2∗3, and the dimension of the second matrix is 3∗2. So we can see that number of rows in the 1^{st} matrix is equal to the number of column in the 2^{nd} matrix. Similarly, the number of columns in the 1^{st} matrix is equal to the number of rows in the 2^{nd} matrix. So we can multiply the given matrices.



Example of matrix multiplication

There are various examples of matrix multiplication, and some of them are described as follows:

Example 1: In this example, we have two matrices, A and B where

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Now we have to show whether AB is equal to BA or not.

Solution: By multiplying AB, we will get:

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

By multiplying BA, we will get:

$$BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

So we can see that AB ≠ BA.

This is because matrix multiplication is not commutative.

What Are Determinants and How Do They Work?

In matrices, determinants are a form of scaling factor. They can be thought of as matrices expanding out and contracting in functions. Determinants accept a square matrix as input and produce a single number as output. A square matrix is a matrix with the same rows and columns on both sides.

Definition of Determinants

A determinant can be defined as a scalar value that is a real or complex integer for every square matrix of order nn, C = [c], where c is the (i,j)th member of matrix C. The determinant can be written as det(C) or |C|; instead of using square brackets, the determinant is expressed by taking the grid of integers and putting it inside the absolute-value bars.

Characteristics of Determinants

- The determinant's value remains unchanged if the rows and columns are switched around.
- If any two rows or columns of a determinant are swapped, the sign of the determinant changes.
- The determinant is 0 if any two rows or columns of the determinant are the same.
- If a variable k is multiplied by any column or row of the determinant, its value is multiplied by k.

- If some or all of the constituents of a row or column can be written as the sum of two or more terms, the determinant can also be stated as a combination of two or more determinants.

Rules of Determinants

The following rules must be followed to perform row and column operations on determinants:

- The determinant's value remains unchanged if the columns and rows are switched.
- The sign of the determinant changes when two rows or two columns are switched.
- If any row or column of a matrix is equal, the determinant has 0 value.
- When each element of a row or column is multiplied by the constant, the determinant's value is multiplied by the constant.
- If the row or column items are represented as a sum of elements, the determinant can be computed as the sum of determinants.
- When the elements of one row or column are added or subtracted with the matching multiples of elements from another row or column, the determinant's value remains unchanged.

Determinant of the Matrix

The determinant's value has a lot of ramifications for the matrix. The presence of a determinant of 0 indicates that the matrix is unique and so invertible. Cramer's rule can be used to solve a system of linear equations by making a matrix out of the coefficients and taking the determinant; this method can only be utilised when the determinant is not equal to 0. From a geometric standpoint, the determinant reflects the signed area of the parallelogram generated by the column vectors in Cartesian coordinates.

The determinant can be computed using a variety of methods. The determinants of some matrices, like diagonal or triangular matrices, can be obtained by multiplying the components on the major diagonal. The Leibniz formula is used to get the determinant of a 2-by-2 matrix by subtracting the reversed diagonal from the main diagonal. Since the determinant of the product of matrices is equal to the product of determinants of those matrices, decomposing a matrix into simpler matrices, calculating the individual determinants, then multiplying the results may be advantageous. The QR, LU, and Cholesky decomposition are some useful decomposition methods. The determinant must be calculated using the Laplace formula, Gaussian elimination, or other procedures for more complicated matrices.

Evaluating the Determinant of the Matrix

To begin with, the matrix must be square. After that, it's just arithmetic.

For a 2×2 Matrix

For a 2×2 matrix (2 columns and 2 rows):

The determinant is:

|A| = ad − bc

"A's determinant equals a times d minus b times c,"

The determinant for a 22 matrix is ad – bc.

To evaluate determinants, multiply a by the determinant of the 22 matrix that is not in a row or column for a 33 matrix, and b and c by the determinant of the 22 matrix that is not in a's row or column for b and c, but note that b has a negative sign.

For larger matrices, repeat the process: multiply a by the determinant of the matrix, which is not in a row or column. Continue in this manner across the entire row while remembering the pattern, and you can also use an online determinant calculator.

Minors and cofactors

Minors and cofactors are two of the most important concepts in matrices, as they are crucial in finding the adjoint and the inverse of a matrix. To find the determinants of a large square matrix (like 4×4), it is important to find the minors of the matrix and then the cofactors of the matrix. Below is a detailed explanation of "What minors and cofactors are", along with steps to find them.

What are Minors?

Minor is defined as the value calculated from the determinant of a **square matrix**. It is calculated by crossing the rows and columns corresponding to the given element. Minor of an element such as a_{yz} of a determinant can be finding out by deleting its yth row and zth column in which the element a_{yz} lies. Minor of an element a_{yz} can be represented as M_{yz}.

Minor of an element of a determinant of the order $n(n \geq 3)$ is the determinant of the order n-2.

Example 1.

Find the minor of an element 5 in the determinant: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution. Element 5 lies in the 2^{nd} row and 2^{nd} column. So, its Minor will be M_{22}.

M_{22} = $\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ = 9-21 = -12(obtained by deleting R_2 and C_2 in the ?)

Example 2.

Find the minors of the elements in the determinant: $\begin{bmatrix} 1 & -2 & 4 \\ -1 & 3 & -4 \\ 2 & -3 & 5 \end{bmatrix}$

Solution. Minor of the element M_{11} = 15 – 12 = 3

Minor of the element M_{12} = -5 + 8 = 3

Minor of the element M_{13} = 3 – 6 = -3

Minor of the element M_{21} = -10 + 12 = 2

Minor of the element M_{22} = 5 – 8 = -3

Minor of the element M_{23} = -3 + 4 = 1

Minor of the element M_{13} = 3 – 6 = -3

Minor of the element M_{23}= -3 + 4 = 1

Minor of the element M_{33}= 3 – 2 = 1

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What are Cofactors?

Cofactor is known as the signed minor. Cofactor of an element, a_{yz}, denoted by A_{yz}, is defined by A= (-1)$^{y+z}$M, where M stand for Minor of a_{yz}.

Some important rules:

- If the sum of y+z is even, then $A_{yz} = M_{yz}$.
- If the sum of y+z is odd, then $A_{yz} = -M_{yz}$.
- It shows that the difference between the related minors and cofactors are only in the terms of sign.

The value of determinant of order three can be in the terms of minors and cofactors can be defined as:

D= $a_{11}M_{11}$ - $a_{12}M_{12}$ + $a_{13}M_{13}$ or

$D = a_{11}C_{11} - a_{12}C_{12} + a_{13} C_{13}$

A determinant having order 3 will result in 9 minors and each minor will be a determinant of order 2. Similarly, a determinant of order 4 will have 16 minors, and the determinant will be of order 3.

If cofactor is multiplied to different rows/columns, their sum will be zero.

Example.1: Find minors and cofactors of the determinant**:** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Solution. Minor of the element ayz is M_{yz}.

Here, a_{11}= 1. So, minor of M_{11} = 3

a_{12} = 2, M_{12} = 4

M_{21} = 2, M_{22} = 1

Now, cofactor of a_{yz} is A_{yz}.

$A_{11}=(-1)^{1+1} = (-1)^2(3)= 3$

$A_{12} = (-1)^{1+2} = (-1)^3(4) = -4$

$A_{21} =(-1)^{2+1} = (-1)^3 (2) = -2$

$A_{22} = (-1)^{2+2} = (-1)^4 (1) = 1$

Things to Remember

- Minors and cofactors are one of the most important concepts of the determinants.
- Minors and Cofactors are important as they help to find out the determinant of large square matrix.
- The knowledge of Minors and Cofactors helps to determine the adjoint as well the inverse while calculating the determinant of a square matrix.
- Cofactor Expansion is the term defined for such computation of the determinant.
- Cofactor is known as the signed minor.
- Cofactor of an element, a_{yz}, denoted by A_{yz}, is defined by $A= (-1)^{y+z}M$, where M stand for Minor of a_{yz}.
- Minor of an element of a determinant of the order $n(n \geq 3)$ is the determinant of the order n-2.
- If the sum of y+z is even, then $A_{yz} = M_{yz}$.
- If the sum of y+z is odd, then $A_{yz} = -M_{yz}$.

- It shows that the difference between the related minors and cofactors are only in the terms of sign.

What Are the Properties of Determinants?

The features of determinants aid in quickly calculating the value of a determinant with the fewest steps and calculations possible. The following are the 7 most important properties of determinants.

1. Property of Interchange:

When the rows or columns of a determinant are swapped, the determinant's value remains unchanged.

A = $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$ A' = $\begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$ Det(A) = Det(A')

As a result of this property, if the rows and columns of the matrix are swapped, the matrix is transposed, and the determinant value and the determinant of the transposition are both equal.

2. Sign Property: If any two rows or columns are swapped, the sign of the determinant's value changes.

A = $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

R1 ⇔ R3

B = $\begin{bmatrix} 1 & 5 & -7 \\ 6 & 0 & 4 \\ 2 & -3 & 5 \end{bmatrix}$

If the row or column is swapped once, the determinant's value changes the sign. To obtain matrix B, the first row of matrix A has been swapped with the third row, and we have Det(A) = -Det(A) (B). If the determinant's value is D and the rows or columns are swapped n times, the new determinant's value is (-1)nD.

3. Zero Property: If the elements in any two rows or columns are the same, the determinant's value is zero.

A = $\begin{bmatrix} 1 & 5 & 7 \\ 1 & 5 & 7 \\ 2 & -3 & 5 \end{bmatrix}$

The items in the first and second rows are identical in this case. As a result, the determinant's value is zero.

Det(A) = 0

4. Multiplication Property: If each of the elements of a given row or column is k times the earlier value of the determinant, the deciding value becomes k times the earlier value of the determinant. If each element of a given row or column is multiplied by a constant k, the determining value becomes k times the earlier value of the determinant.

A = $\begin{bmatrix} 1 & 5 & 7 \\ 6 & 0 & 4 \\ 2 & -3 & 5 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 5 & 7 \\ 24 & 0 & 16 \\ 2 & -3 & 5 \end{bmatrix}$

Det(B) = k× Det(A)

The second row of A is multiplied with a constant k. Here, k = 4.

The second row's elements are multiplied by a constant k, and the determinant value is multiplied by the same constant. This characteristic aids in the extraction of a common factor from each determinant row or column. In addition, the value of the determinant is 0 if the corresponding elements of any two rows or columns are equal.

5. Sum Property: The determinant can be expressed as a sum of two or more determinants if a few items of a row or column are expressed as a sum of words.

6. Property of Invariance: When each element of a determinant's row and column is multiplied by the equimultiples of the elements of another determinant's row or column, the determinant's value remains unchanged. This can be stated as a formula as follows:

$R_i \rightarrow R_i + \alpha R_j + \beta R_k$ or $C_i \rightarrow C_i + \alpha C_j + \beta C_k$

7. Triangular Property: The value of the determinant is equal to the product of the components of the diagonal of the matrix if the elements above and below the main diagonal are both equal to zero.

$\begin{bmatrix} a1 & a2 & a3 \\ 0 & b1 & b3 \\ 0 & 0 & c3 \end{bmatrix} = \begin{bmatrix} a1 & 0 & 0 \\ a2 & b1 & 0 \\ a3 & b2 & c3 \end{bmatrix}$ = a1.b2.c3