UNIT 4: Progression

Progression

A progression (which is also known as a sequence) is nothing but a pattern of numbers. For example, 3, 6, 9, 12, ... is a progression because there is a pattern observed where every number here is obtained by adding 3 to its previous number. But this pattern doesn't need to be the same in every progression.

The pattern of a progression depends on its type. Let us see the types of progressions along with examples and their formulas.

What is the Definition of a Progression?

Progression is a list of numbers (or items) that exhibit a particular pattern. A progression is also known as a sequence. In a progression, every term is obtained by applying a specific rule on its previous term. In other words, every term of a progression is defined by a general term (or) n^{th} term, which is denoted by a_n.

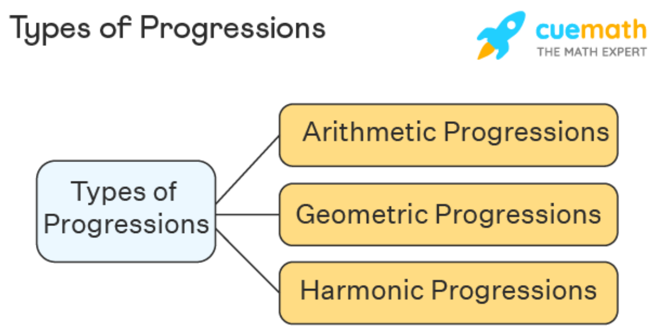
For example, the n^{th} term of a progression 3, 5, 7, 9, ... is $a_n = 2n + 1$. Substituting n = 1, 2, 3, ... here, we get the 1^{st}, 2^{nd}, 3^{rd}, ... terms. For example:

- When n = 1, first term = 2(1) + 1 = 3.
- When n = 2, second term = 2(2) + 1 = 5.
- When n = 3, third term = 2(3) + 1 = 7
  and so on.

Types of Progressions

There are mainly 3 types of progressions in math. They are:

- Arithmetic Progression (AP)
- Geometric Progression (GP)
- Harmonic Progression (HP)



Each type of progression along with a simple definition and example is tabulated below.

Progression	Definition	Example
Arithmetic Progression (AP)	The differences between any two consecutive numbers are all same.	1, 4, 7, 10, ...
Geometric Progression (GP)	The ratios of any two consecutive numbers are all same.	4, 16, 64, 256, ...
Harmonic Progression (HP)	The reciprocals of terms form an AP.	1/2, 1/4, 1/6, ...

We will learn about each progression in detail in the upcoming sections.

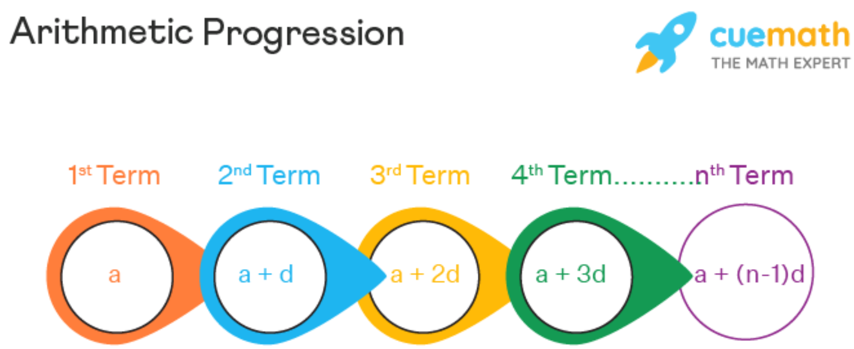
Arithmetic Progression

An arithmetic progression (AP) is a sequence of numbers in which each successive term is the sum of its preceding term and a fixed number. This fixed number is called the common difference. For example, 1, 4, 7, 10, ... is an AP as every number is obtained by adding a fixed number 3 to its previous term.

- **2^{nd} term** = 4 = 1 + 3 = **1^{st} term + 3**
- **3^{rd} term** = 7 = 4 + 3 = **2^{nd} term + 3**
- **4^{th} term** = 10 = 7 + 3 = **3^{rd} term + 3**

  and so on.

In general, an arithmetic progression looks like this:



Here,

- 'a' is the first term and
- 'd' is the common difference (fixed number)

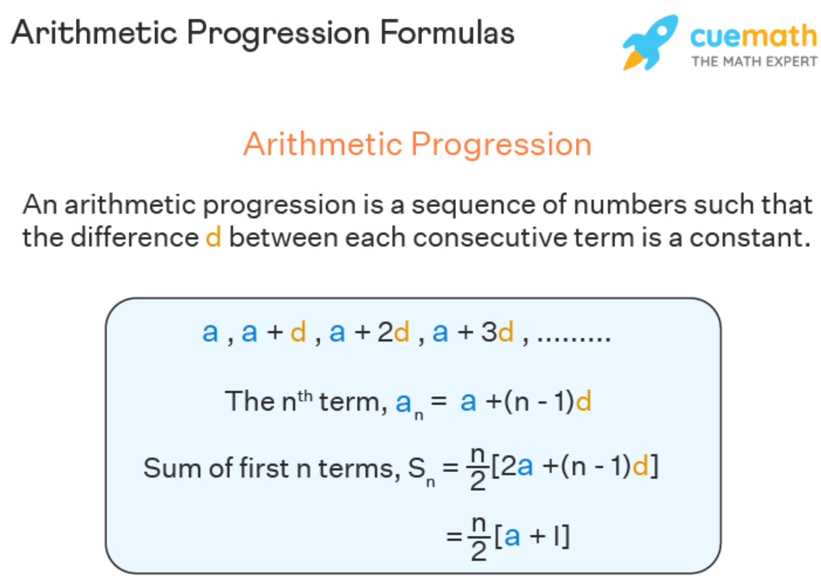
Arithmetic Progression Example

For example, Minnie put $30 in her piggy bank when she was 7 years old. She increased the amount she put in her piggy bank on each successive birthday by $3. So, the amount in her piggy bank follows the pattern of $30, $33, $36, and so on. The succeeding terms are obtained by adding a fixed number, that is, $3. This fixed number is called the common difference (It can be positive, negative, or zero). Hence the progression 30, 33, 36, ... is an AP.

Arithmetic Progression Formulas

Let the first term of the progression be a, the common difference be d, and the n^{th} term be a_n. Then, the arithmetic progression formulas are given by:

- $a_n = a + (n - 1) d$
- $d = a_n - a_{n-1}$
- Sum of the first n terms, $S_n = n/2(2a+(n-1)d)$ (or) $S_n = n/2(a + l)$, where l = the last term = T_n.

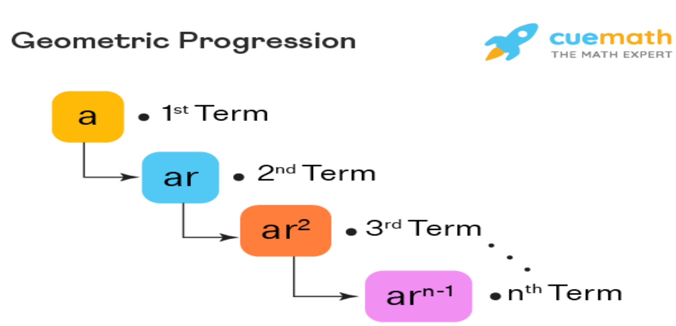


Geometric Progression

A geometric progression (GP) is a sequence of numbers in which each successive term is the product of its preceding term and a fixed number. This fixed number is called the common ratio. For example, 4, 16, 64, 256, ... is a GP as every number is obtained by multiplying a fixed number 4 to its previous term.

- **2^{nd} term** = 16 = 4(4) **= 4(1^{st} term)**
- **3^{rd} term** = 64 = 4(16) **= 4(2^{nd} term)**
- **4^{th} term** = 256 = 4(64) **= 4(3^{rd} term)**
  and so on.

In general, a geometric progression looks like this:



Here,

- 'a' is the first term and
- 'r' is the common ratio (fixed number)

Geometric Progression Example

Consider an example of a geometric progression: 1, 4, 16, 64, ... Observe that 4/1 = 16/4 = 64/16 = ... = 4. All the ratios are same. Hence it is a GP.

Geometric Progression Formulas

Let the first term of the progression be a, the common ratio be r, and the n^{th} term be a_n. Then, the geometric progression formulas are given by:

- $a_n = ar^{n-1}$
- Sum of the first 'n' terms, $S_n = a(r^n - 1) / (r - 1)$ when $r \neq 1$ and $S_n = na$ when $r = 1$.
- Sum of infinite geometric series, $S_\infty = a / (1 - r)$ when $|r| < 1$ and S_∞ diverges when $|r| \geq 1$.

Geometric Progression Formulas

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For a geometric progression a , ar , ar^2 ,........

- n^{th} term ,

$$a_n = ar^{n-1}$$

- Sum of n terms,

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} & \text{, when } r \neq 1 \\ na & \text{, when } r = 1 \end{cases}$$

- Sum of infinite terms,

$$S_\infty = \begin{cases} \dfrac{a}{1 - r} & \text{, when } |r| < 1 \\ \text{diverges} & \text{, when } |r| \geq 1 \end{cases}$$

Harmonic Progression

A harmonic progression is a sequence obtained by taking the reciprocal of the terms of an arithmetic progression. The sequence of natural numbers is an arithmetic progression. So, taking reciprocals of each term we get 1,1/2,1/3,1/4,... This is harmonic progression.

Harmonic Progression Example

When a ball is dropped, the initial height reached by the ball is 1/2 units and after the first impact, the height attained by the ball is 1/4 units. After the second impact, the height attained by the ball is 1/6 units. After the third impact, the height attained by the ball is 1/8 units, and so on. Now the sequence of heights formed by the ball is: 1/2, 1/4, 1/6, 1/8, ...

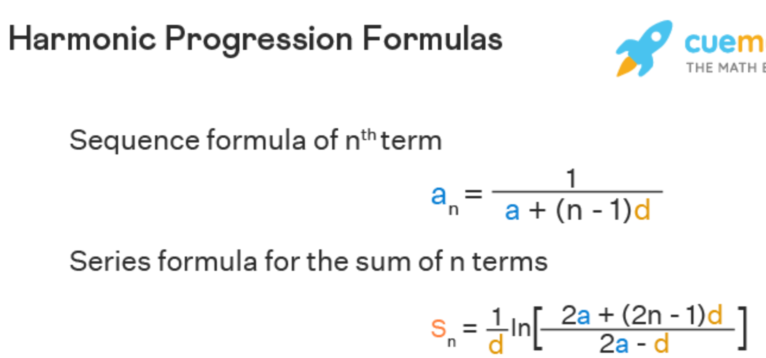
This sequence is a harmonic progression because the reciprocal of all the terms of this progression form an arithmetic progression.

- Reciprocal of the sequence: 2, 4, 6, 8, ... → AP
- Hence, the sequence: 1/2, 1/4, 1/6, 1/8, ... → HP

Harmonic Progression Formulas

For a harmonic progression 1/a, 1/(a+d), 1/(a+2d), ...

- n^{th} term, $a_n = 1 / (a + (n - 1) d)$
- Sum of the first n terms, $S_n = 1/d \ln [(2a + (2n - 1) d] / (2a - d)]$



Important Notes on Progression:

- In an arithmetic progression, each successive term is obtained by adding the common difference to its preceding term.
- In a geometric progression, each successive term is obtained by multiplying the common ratio to its preceding term.
- The reciprocal of terms in harmonic progression form an arithmetic progression.

Arithmetic Mean

Arithmetic mean represents a number that is achieved by dividing the sum of the values of a set by the number of values in the set. If a_1, a_2, a_3,….,a_n, is a number of group of values or the Arithmetic Progression, then;

$AM=(a_1+a_2+a_3+\ldots,+a_n)/n$

Geometric Mean

The Geometric Mean for a given number of values containing n observations is the nth root of the product of the values.

$GM = n\sqrt{}(a_1a_2a_3\ldots.a_n)$

Or

$GM = (a_1a_2a_3\ldots.a_n)^{1/n}$

Harmonic Mean

HM is defined as the reciprocal of the arithmetic mean of the given data values. It is represented as:

$HM = n/[(1/a_1) + (1/a_2) + (1/a_3) + \ldots.+ (1/a_n)]$

What is the Relation between AM, GM and HM?

The relationship between AM, GM and HM is given by:

AM x HM = GM²

Now let us understand how this relation is derived;

First, consider a, AM, b is an Arithmetic Progression.

Now the common difference of Arithmetic Progression will be;

AM – a = b – AM

$a + b = 2\ AM$ …………..(1)

Secondly, let a, GM, b is a Geometric Progression. Then, the common ratio of this GP is;

$GM/a = b/GM$

$ab = GM^2$…………..(2)

Third, is the case of harmonic progression, a, HM, b, where the reciprocals of each term will form an arithmetic progression, such as:

1/a, 1/HM, 1/b is an AP.

Now the common difference of the above AP is;

$1/HM – 1/a = 1/b – 1/HM$

$2/HM = 1/b + 1/a$

$2/HM = (a + b)/ab$ …………(3)

Substituting eq. 1 and eq.2 in eq. 3 we get;

$2/HM = 2AM/GM^2$

$GM^2 = AM \times HM$

Hence, this is the relation between Arithmetic, Geometric and Harmonic mean.