UNIT 2: Functions

Functions are an important part of discrete mathematics. This article is all about functions, their types, and other details of functions. A function assigns exactly one element of a set to each element of the other set. Functions are the rules that assign one input to one output. The function can be represented as f: A ⇢ B. A is called the *domain of the function* and B is called the *codomain function*.

Functions:

- A function assigns exactly one element of one set to each element of other sets.
- A function is a rule that assigns each input exactly one output.
- A function f from A to B is an assignment of exactly one element of B to each element of A (where A and B are non-empty sets).
- A function f from set A to set B is represented as **f: A ⇢ B** where A is called the domain of f and B is called as codomain of f.
- If b is a unique element of B to element a of A assigned by function F then, it is written as f(a) = b.
- Function f maps A to B means f is a function from A to B i.e. f: A ⇢ B

Domain of a function:

- If f is a function from set A to set B then, A is called the domain of function f.
- The set of all inputs for a function is called its domain.

Codomain of a function:

- If f is a function from set A to set B then, B is called the codomain of function f.
- The set of all allowable outputs for a function is called its codomain.

Pre-image and Image of a function:

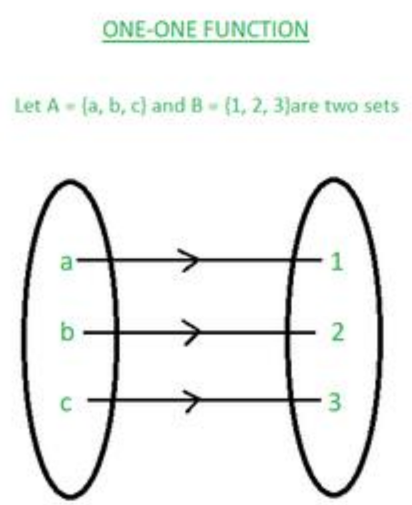
A function f: A ⇢ B such that for each a ∈ A, there exists a unique b ∈ B such that (a, b) ∈ R then, a is called the pre-image of f and b is called the image of f.

Types of function:

One-One function (or Injective Function):

A function in which one element of the domain is connected to one element of the codomain.

A function f: A ⇢ B is said to be a one-one (injective) function if different elements of A have different images in B.
f: A ⇢ B is one-one
⇒ $a \neq b \Rightarrow f(a) \neq f(b)$ for all a, b ∈ A
⇒ $f(a) = f(b) \Rightarrow a = b$ for all a, b ∈ A

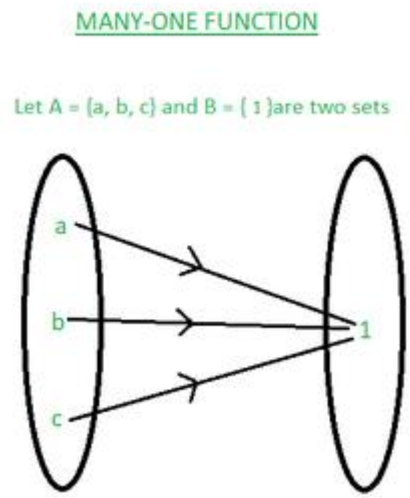


ONE-ONE FUNCTION

Many-One function:

A function f: A ⇢ B is said to be a many-one function if two or more elements of set A have the same image in B.
A function f: A ⇢ B is a many-one function if it is not a one-one function.

f: A ⇢ B is many-one
⇒ a ≠ b but f(a) = f(b)   for all a, b ∈ A



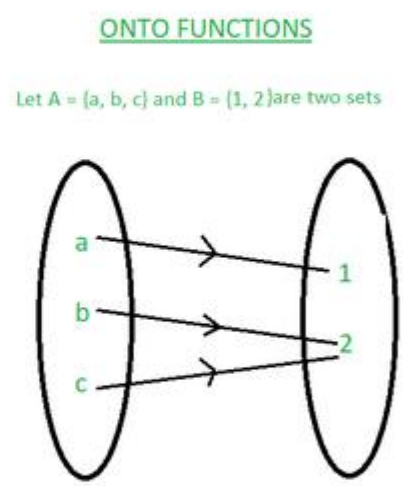
MANY-ONE FUNCTION

Onto function(or Surjective Function):

A function f: A -> B is said to be onto (surjective) function if every element of B is an image of some element of A i.e. f(A) = B or range of f is the codomain of f.

A function in which every element of the codomain has one pre-image.

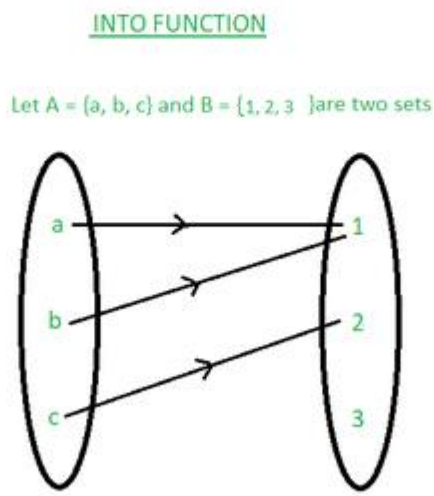
f: A ⇢ B is onto if for each b∈ B, there exists a∈ A such that f(a) = b.



ONTO FUNCTION

Into Function:

A function f: A ⇢ B is said to be an into a function if there exists an element in B with no pre-image in A.
A function f: A⇢ B is into function when it is not onto.



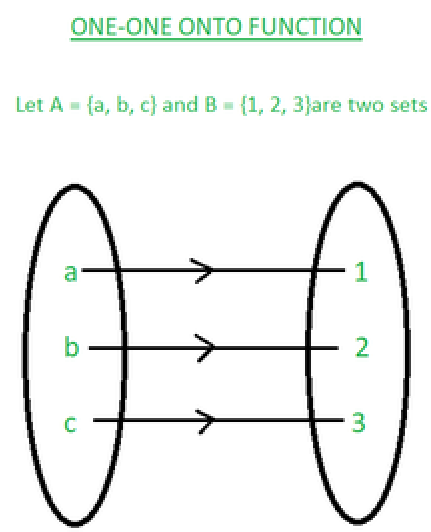
INTO FUNCTION

One-One Correspondent function(or Bijective Function or One-One Onto Function):

A function which is both one-one and onto (both injective and surjective) is called one-one correspondent(bijective) function.

f : A ⇢ B is one-one correspondent (bijective) if:

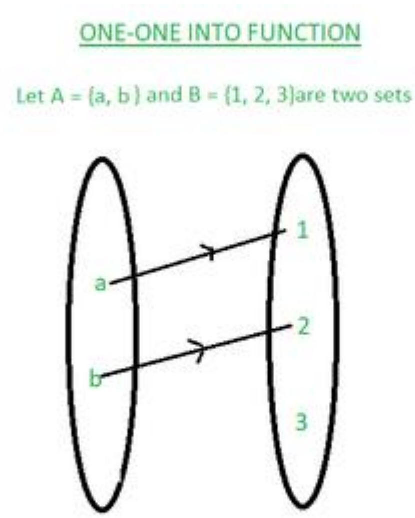
- **one-one i.e. f(a) = f(b) ⇒ a = b for all a, b ∈ A**
- **onto i.e. for each b ∈ B, there exists a ∈ A such that f(a) = b.**



ONE-ONE CORRESPONDENT FUNCTION

One-One Into function:

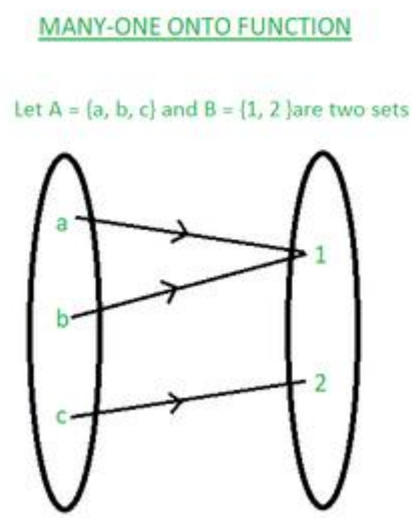
A function that is both one-one and into is called one-one into function.



ONE-ONE INTO FUNCTION

Many-one onto function:

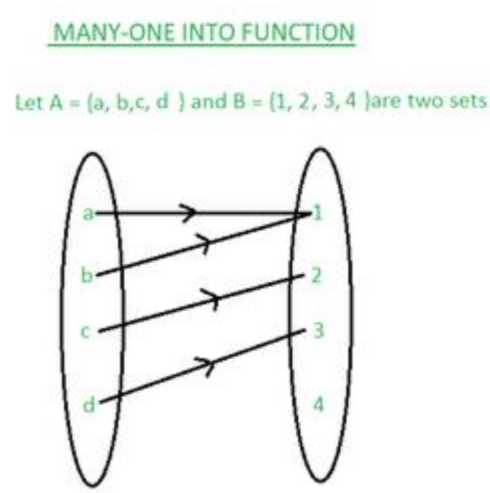
A function that is both many-one and onto is called many-one onto function.



MANY-ONE ONTO FUNCTION

Many-one into a function:

A function that is both many-one and into is called many-one into function.



MANY-ONE INTO FUNCTION

Inverse of a function:

Let f: A ⇢ B be a bijection then, a function g: B ⇢ A which associates each element b ∈ B to a different element a ∈ A such that f(a) = b is called the inverse of f.
f(a) = b ⟷ g(b) = a

Composition of functions :-

Let f: A ⇢ B and g: B ⇢ C be two functions then, a function gof: A ⇢ C is defined by
(gof)(x) = g(f(x)), for all x ∈ A
is called the composition of f and g.

Note:

Let X and Y be two sets with m and n elements and a function is defined as f : X->Y then,

- Total number of functions = n^m
- Total number of one-one function = ${}^nP_m$
- Total number of onto functions = $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - \ldots\ldots\ldots\ldots + (-1)^{n-1}\,{}^nC_{n-1}1^m$ if $m \geq n$.

For the composition of functions f and g be two functions :

- fog ≠ gof
- If f and g both are one-one function then fog is also one-one.
- If f and g both are onto function then fog is also onto.
- If f and fog both are one-one function then g is also one-one.
- If f and fog both are onto function then it is not necessary that g is also onto.
- $(fog)^{-1} = g^{-1}o\ f^{-1}$
- $f^{-1}o\ f = f^{-1}(f(a)) = f^{-1}(b) = a$
- $fof^{-1} = f(f^{-1}(b)) = f(a) = b$

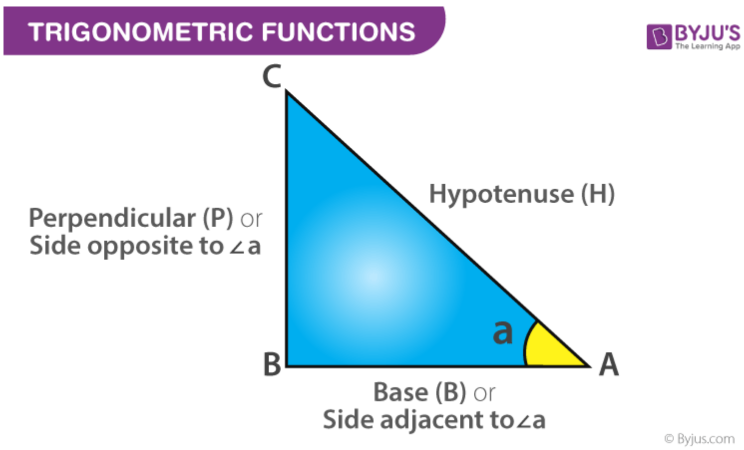
Trigonometric functions

Trigonometric functions are also known as **Circular Functions** can be simply defined as the functions of an angle of a triangle. It means that the relationship between the angles and sides of a triangle are given by these trig functions. The basic trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant.

There are a number of trigonometric formulas and identities that denotes the relation between the functions and help to find the angles of the triangle.

Six Trigonometric Functions

The angles of **sine, cosine, and tangent** are the primary classification of functions of trigonometry. And the three functions which are cotangent, secant and cosecant can be derived from the primary functions. Basically, the other three functions are often used as compared to the primary trigonometric functions. Consider the following diagram as a reference for an explanation of these three primary functions. This diagram can be referred to as the sin-cos-tan triangle. We usually define trigonometry with the help of the right-angled triangle.



Sine Function

Sine function of an angle is the ratio between the opposite side length to that of the hypotenuse. From the above diagram, the value of sin will be:

- **Sin a =Opposite/Hypotenuse = CB/CA**

Cos Function

Cos of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. From the above diagram, the cos function will be derived as follows.

- **Cos a = Adjacent/Hypotenuse = AB/CA**

Tan Function

The tangent function is the ratio of the length of the opposite side to that of the adjacent side. It should be noted that the tan can also be represented in terms of sine and cos as their ratio. From the diagram taken above, the tan function will be the following.

- **Tan a = Opposite/Adjacent = CB/BA**

Also, in terms of sine and cos, tan can be represented as:

Tan a = sin a/cos a

Secant, Cosecant and Cotangent Functions

Secant, cosecant (csc) and cotangent are the three additional functions which are derived from the primary functions of sine, cos, and tan. The reciprocal of sine, cos, and tan are cosecant (csc), secant (sec), and cotangent (cot) respectively. The formula of each of these functions are given as:

- **Sec a = 1/(cos a) = Hypotenuse/Adjacent = CA/AB**
- **Cosec a = 1/(sin a) = Hypotenuse/Opposite = CA/CB**
- **cot a = 1/(tan a) = Adjacent/Opposite = BA/CB**

Note: Inverse trigonometric functions are used to obtain an angle from any of the angle's trigonometric ratios. Basically, inverses of the sine, cosine, tangent, cotangent, secant, and cosecant functions are represented as arcsine, arccosine, arctangent, arc cotangent, arc secant, and arc cosecant.

Formulas

Let us discuss the formulas given in the table below for functions of trigonometric ratios(sine, cosine, tangent, cotangent, secant and cosecant) for a right-angled triangle.

Formulas for Angle θ	**Reciprocal Identitie**s
sin θ = Opposite Side/Hypotenuse	sin θ = 1/cosec θ

cos θ = Adjacent Side/Hypotenuse	cos θ = 1/sec θ
tan θ = Opposite Side/Adjacent	tan θ = 1/cot θ
cot θ = Adjacent Side/Opposite	cot θ = 1/tan θ
sec θ = Hypotenuse/Adjacent Side	sec θ = 1/cos θ
cosec θ = Hypotenuse/Opposite	cosec θ = 1/sin θ

Binomial Theorem

Binomial Theorem is used to solve binomial expressions in a simple way. This theorem was first used somewhere around 400 BC by Euclids a famous Greek mathematician.

It gives an expression to calculate the expansion of algebraic expression $(a+b)^n$. The terms in the expansion of the following expression are exponent terms and the constant term associated with each term is called the coefficient of terms.
The Binomial theorem for the expansion of $(a+b)^n$ is stated as,

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + ... + {}^nC_r a^{n-r} b^r + ... + {}^nC_n a^0 b^n$$

where n>0 and the ${}^nC_k$ is the binomial coefficient.
Example: Find the expansion of $(x + 5)^6$ using the binomial theorem.
Solution:
We know that,

$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + ... + {}^nC_r a^{n-r} b^r + ... + {}^nC_n a^0 b^n$
Thus, $(x + 5)^6 = {}^6C_0 x^6 5^0 + {}^6C_1 x^{6-1} 5^1 + {}^6C_2 x^{6-2} 5^2 + {}^6C_3 x^{6-3} 5^3 + {}^6C_4 x^{6-4} 5^4 + {}^6C_5 x^{6-5} 5^5 + {}^6C_6 x^{6-6} 5^6$
$= {}^6C_0 x^6 + {}^6C_1 x^5 5 + {}^6C_2 x^4 5^2 + {}^6C_3 x^3 5^3 + {}^6C_4 x^2 5^4 + {}^6C_5 x^1 5^5 + {}^6C_6 x^0 5^6$
$= x^6 + 30x^5 + 375x^4 + 2500x^3 + 9375x^2 + 18750x + 15625$

Binomial Expansion

Binomial Theorem is used to expand the algebraic identity $(x + y)^n$. Hence it is also called the binomial expansion. The binomial expansion of $(x + y)^n$ with the help of the binomial theorem is,

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

Using this expansion we get each term in the expansion of $(x+y)^{n!}$
Example: Find the value of $(x+y)^2$ and $(x+y)^3$ using Binomial expansion.
Solution:
$(x+y)^2 = {}^2C_0 x^2 y^0 + {}^2C_1 x^{2-1} y^1 + {}^2C_2 x^{2-2} y^2$

$\Rightarrow (x+y)^2 = x^2 + 2xy + y^2$

And $(x+y)^3 = {}^3C_0\, x^3y^0 + {}^3C_1\, x^{3-1}y^1 + {}^3C_2\, x^{3-2}\, y^2 + {}^3C_3\, x^{3-3}\, y^3$

$\Rightarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$\Rightarrow (x+y)^3 = x^3 + 3xy(x+y) + y^3$

Binomial Theorem Formula

The binomial theorem formula gives the expansion of the algebraic identities in the form of a series. This formula is used to find the expansion up to n terms of the $(a+b)^n$. The binomial expansion of $(a+b)^n$ can easily be represented with the summation formula.

The Binomial Theorem Formula for the expansion of $(a+b)^n$ is,

$$(a+b)^n = \sum^n{}_r\, {}^nC_r\, a^{n-r}\, b^r$$

Where,

- ***n** is a positive integer,*
- ***a, b** are real numbers, and **0 < r ≤ n***

We can easily find the expansion of the various identities such as $(x+y)^7$, $(x+9)^{11}$, and others using the Binomial Theorem Formula. We can also find the expansion of $(ax + by)^n$ using the Binomial Theorem Formula,

The expansion formula for $(ax + by)^n$ is,

$$(ax + by)^n = \sum^n{}_r\, {}^nC_r\, (ax)^{n-r}\, (by)^r$$

Where $0 < r \leq n$.

Also using the combination formula we know that,

$${}^nC_r = n! / [r!\,(n - r)!]$$

Mathematical Induction

Mathematical Induction is one of the fundamental methods of writing proofs and it is used to prove a given statement about any well-organized set. Generally, it is used for proving results or establishing statements that are formulated in terms of **n**, where n is a natural number. Suppose P(n) is a statement for n natural number then it can be proved using the Principle of Mathematical Induction, Firstly we will prove for P(1) then let P(k) is true then prove for P(k+1). If P(k+1) holds true then we say that P(n) is true by the principle of mathematical induction.

We can compare mathematical induction to falling dominoes. When a domino falls, it knocks down the next domino in succession. The first domino knocks down the second one, the second one knocks down the third, and so on. In the end, all of the dominoes will be bowled over. But there are some conditions to be fulfilled:

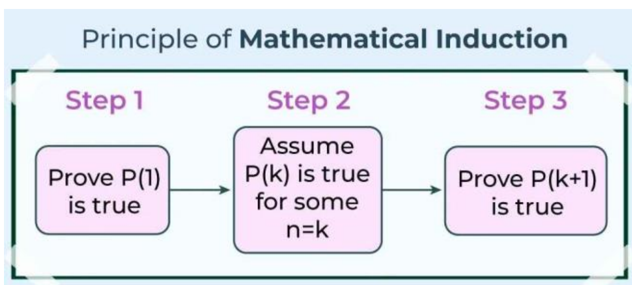
- The starting domino must fall to set the knocking process in action. This is the base step.

- The distance between dominoes must be equal for any two adjacent dominoes. Otherwise, a certain domino may fall down without bowling over the next. Then the sequence of reactions will stop. Maintaining the equal inter-domino distance ensures that P(k) ⇒ P(k + 1) for each integer k ≥ a. This is the inductive step.

Principle of Mathematical Induction Statement

Any statement P(n) which is for "n" natural number can be proved using the Principle of Mathematical Induction by following the below steps,

Step 1: *Verify if the statement is true for trivial cases (**n = 1)** i.e. check if P(1) is true.*
Step 2: *Assume that the statement is true for n = k for some k ≥ 1 i.e. P(k) is true.*
Step 3: *If the truth of P(k) implies the truth of P(k + 1), then the statement P(n) is true for all **n ≥ 1**.*
The image added below contains all the steps of Mathematical Induction



The first statement is the fact and if it is not possible for all P(n) to hold true at n = 1 then these statements are true for some other values of n say n = 2, n = 3, and others.

If the statement is true for P(k) then if P(k+1) is proven to be true then we say that P(n) is true for all n belonging to Natural Numbers (N)

Mathematical Induction Steps

Various steps used in Mathematical Induction are named accordingly. The names of the various steps used in the principle of mathematical induction are,

- **Base Step:** Prove P(k) is true for k =1
- **Assumption Step:** Let P(k) is true for all k in N and k > 1
- **Induction Step:** Prove P(k+1) is true using basic mathematical properties.

If the above three steps are proved then we can say that "By the principle of mathematical induction, P(n) is true for all n belonging to N".

Mathematical Induction Example

Mathematical induction is used to prove various statements we can learn this with the help of the following example.

For any positive integer number n, prove that $n^3 + 2n$ is always divisible by 3

Solution:

Let P(n): $n^3 + 2n$ is divisible by 3 be the given statement.

Step 1: Basic Step

Firstly we prove that P(1) is true. Let n = 1 in $n^3 + 2n$

$= 1^3 + 2(1)$

$= 3$

As 3 is divisible by 3. Hence, P(1) is true.

Step 2: Assumption Step

Let us assume that P(k) is true

Then, $k^3 + 2k$ is divisible by 3

Thus, we can write it as $k^3 + 2k = 3n$, (where n is any positive integer)….(i)

Step 3: Induction Steps

Now we have to prove that algebraic expression $(k + 1)^3 + 2(k + 1)$ is divisible by 3

$= (k + 1)^3 + 2(k + 1)$

$= k^3 + 3k^2 + 5k + 3$

$= (k^3 + 2 k) + (3k^2 + 3k + 3)$

from eq(i)

$= 3n + 3(k^2 + k + 1)$

$= 3(n + k^2 + k + 1)$

As it is a multiple of 3 we can say that it is divisible by 3.

Thus, P(k+1) is true i.e. $(k + 1)^3 + 2(k + 1)$ is be divisible by 3. Now by the Principle of Mathematical Induction, we can say that, P(n): $n^3 + 2n$ is divisible by 3 is true.