# Asymptotics of Lattice Walks

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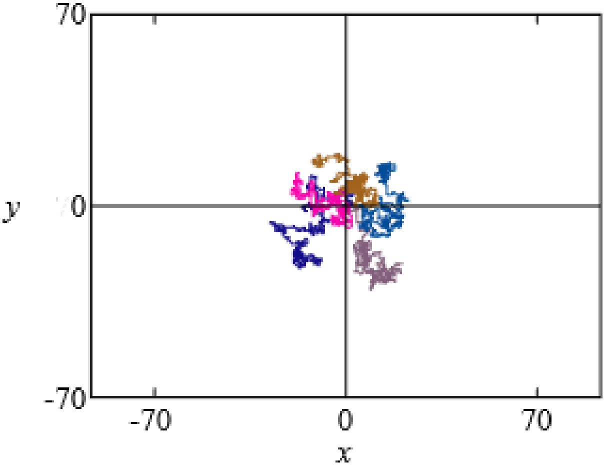

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We can use this framework to count lattice walks!

# Lattice Walks

## Definition (Lattice Walk Model)

A $d$-dimensional **lattice walk model** consists of:

- a **step set** $S \subseteq \mathbb{Z}^d$,
- a **restricting region** $R \subseteq \mathbb{Z}^d$,
- a **starting point** $p \in R$,
- a **terminal set** $T \subseteq R$.

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We always let $p = 0$ and $T = R$.

# Lattice Walks

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Reformulated in generating function language:

**Main Question**

Given a step set $S$ and a restricting region $R$, let $s_n$ be the number of lattice walks of length $n$ using $S$ and staying in $R$.

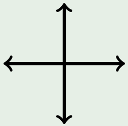
What is the coefficient of $z^n$ in $S(z) = \sum_{n \geq 0} s_n z^n$?

# Unrestricted Walks

If $R = \mathbb{Z}^d$ this is easy.

## Unrestricted NSEW steps

Let $S = \{(\pm 1, 0), (0, \pm 1)\}$ and $R = \mathbb{Z}^2$.



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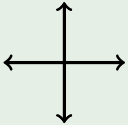

Then there are 4 possible steps to take at each time. Thus there are $4^n$ walks of length $n$ using $S$.

In general:

### Unrestricted Walks

There are $|S|^n$ unrestricted lattice walks of length $n$ using $S$.

# Halfspace Walks

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Let $S$ be any step set and $R = \mathbb{Z}^{d-1} \times \mathbb{N}$ be a halfspace. Let $s_n$ be number of walks.

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This means finding asymptotics for $s_n$ is automatic.

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This models queues (since negative people does not make sense).

This is too hard in general. How do we fix this? Restricting $S$.

# Highly Symmetric Step Sets

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$$S(\mathbf{z}) = S(z_1, \cdots, z_d) = \sum_{s \in S} \mathbf{z}^{\mathbf{s}} = \sum_{s \in S} z_1^{s_1} \cdots z_d^{s_d}.$$

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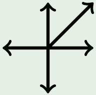
**Example**

If $S = \{(\pm 1, 0), (0, \pm 1), (1, 1)\}$



$$S(z_1, z_2) = z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2$$

# Highly Symmetric Step Sets

We also enforce some amount of *symmetry*.

**Definition (Highly Symmetric Step Sets)**

A step set $S$ is **highly symmetric** if

$$S(z_1, \cdots, z_j, \cdots, z_d) = S\left(z_1, \cdots, \frac{1}{z_j}, \cdots, z_d\right)$$

for all $1 \leq j \leq d$.

# Highly Symmetric Step Sets

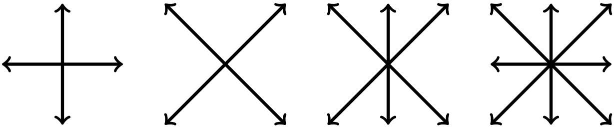
Asymptotics for highly symmetric step sets is known.

**Theorem (Melczer Mishna 2016 [MM16])**

*Suppose $S$ is a highly symmetric step set. If $s_n$ is the number of walks of length $n$ using $S$ that remains in $\mathbb{N}^d$ then*

$$s_n \sim \left[ \frac{|S|^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{|S|^n}{n^{d/2}},$$

*where $a_j$ is the number of steps in $S$ that have $j$-th coordinate 1.*



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A step set $S$ is **mostly symmetric** if

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for all $1 \leq j < d$.

This means we can write

$$S(\mathbf{z}) = \frac{1}{z_d} A(z_1, \cdots, z_{d-1}) + Q(z_1, \cdots, z_{d-1}) + z_d B(z_1, \cdots, z_{d-1})$$

where $A, Q, B$ are all highly symmetric.

# Mostly Symmetric Step Sets

Asymptotics depend on what the average direction of a step is.

**Definition (Drift)**

We say that

- $S$ has **positive drift** if $B(\mathbf{1}) - A(\mathbf{1}) > 0$,
- $S$ has **negative drift** if $B(\mathbf{1}) - A(\mathbf{1}) < 0$,
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Asymptotics for the positive and negative case are known.
Asymptotics for the zero drift case is *open*.

# Positive Drift Asymptotics

**Theorem (Melczer Wilson [MW19])**

*If $S$ is mostly symmetric with positive drift then*

$$s_n \sim \left[ \left(1 - \frac{A(1)}{B(1)}\right) \frac{S(1)^{d/2}}{(2\pi)^{d/2}} \cdot \frac{1}{(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{S(1)^n}{n^{d/2-1/2}}$$

# Negative Drift Asymptotics

**Theorem (Melczer Wilson [MW19])**

*If $S$ is mostly symmetric with negative drift and $Q(z) \neq 0$ then*

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}}.$$

*If $S$ is mostly symmetric with negative drift and $Q(z) = 0$ then*

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}} + C_{-\rho} \cdot \frac{S(1,-\rho)^n}{n^{d/2+1}}.$$

*Where $\rho$, $C_\rho$, and $C_{-\rho}$ are explicit constants.*

# Proof Overview

How is this done?
Through **analytic combinatorics in several variables** (ACSV).

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- Approximate integral using saddle-point method.

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We didn't know how to approximate this!

Now $\sim$ 4 years later we have a better idea of how to approximate.

# Zero Drift Asymptotics

**Theorem (K. Melczer [KMon])**

*Suppose $S$ is a mostly symmetric step set with zero drift. Let $s_n$ be the number of walks of length $n$ using $S$ that remain in $\mathbb{N}^d$. Then*

$$s_n \sim \left[ \frac{|S|^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{|S|^n}{n^{d/2}}$$

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This (sort of) finishes off all walks that can be done with ACSV.

# Conclusion

Thank you for listening.

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