Analytic Combinatorics via Fourier-like Integrals

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Outline

Generating Functions

Take a sequence $\{a_n\}_{n\in\mathbb{N}}$ and consider $A(z) = \sum_{i\in\mathbb{N}} a_i z^i$.

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Goal: find asymptotics of a_n.

Starting point is

Theorem (Cauchy's Residue Theorem)

Let $A(z) = \sum_{n \in \mathbb{N}} a_n z^n$. Then

$$a_n = \frac{1}{2\pi i} \int_C A(z) \frac{\mathrm{d}z}{z^{n+1}} - \sum_j \mathrm{Res}_{z=a_j} \left[\frac{A(z)}{z^{n+1}} \right]$$

where C is some closed curve containing 0 and non-0 singularities of $A(z)$ labelled a_j (and Res is some complex analytic tool that is usually quite computable).

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where C *is some closed curve containing* 0 *and non-*0 *singularities of* $A(z)$ *labelled* a_j *(and* Res *is some complex analytic tool that is usually quite computable).*

Idea is to expand C and pick up more and more singularities, while the integral term vanishes.

Let a_n = number of alternating permutations. Then

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$$= 2 \left(\frac{2}{\pi} \right)^{n+1} + O(2^{-n}) \qquad (n \text{ odd}).$$

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If we make C bigger we get more residues and smaller O term.

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How do we deal with this integral? **Saddle points**.

Definition (Saddle Points)

A saddle point of $f(z)$ is a point z where $df(z) = 0$, but z is neither a local maximum or minimum.

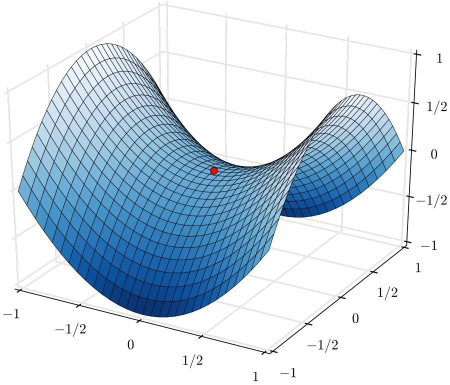

Figure: Saddle Point [kos23]

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3. Approximate integral

Stirling's Approximation

We derive Stirling's Approximation.

$$\frac{1}{n!} = [z^n]e^z = \frac{1}{2\pi i}\int_C e^z \frac{dz}{z^{n+1}} = \frac{1}{2\pi i}\int_C e^{z-(n+1)\log(z)}\,dz.$$

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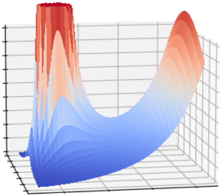

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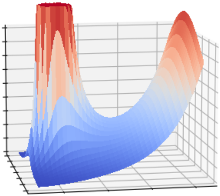
Saddle point at (around) $z = n$. In polar coords,

$$\left(\frac{e}{n}\right)^n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ne^{i\theta}-1-i\theta}\, d\theta \sim \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{2\pi n}}$$



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- These integrals are deeply connected with saddle-point techniques.
- Well studied in a single variable, what about in several variables?

Generating Functions in Several Variables

Take a sequence $\{a_{n_1,n_2,\cdots,n_d}\}_{n_i\in\mathbb{N}}$ and consider

$$A(\mathbf{z}) = \sum_{n_1,\cdots,n_d\in\mathbb{N}} a_{n_1,\cdots,n_d} z_1^{n_1}\cdots z_d^{n_d}$$
$$= \sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$$

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What are asymptotics of $a_{\mathbf{n}}$? Not as clear now.

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$$a_{\mathbf{r}n} = a_{r_1 n, \cdots, r_d n} = [\mathbf{z}^{\mathbf{r}n}]A(\mathbf{z}) = [z_1^{r_1 n} \cdots z_d^{r_d n}]A(\mathbf{z}),$$

where $\mathbf{r} \in \mathbb{N}^d$.

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From now on we normalize by dividing $\mathbf{r}$ by $\|\mathbf{r}\|_1$.

For the remainder of this talk we assume that our generating function is of the form

$$A(\mathbf{z}) = \frac{1}{\prod_{j=1}^{m} \ell_j(\mathbf{z})^{p_j}},$$

- $\ell_j(\mathbf{z})$ is a real-linear function $1 - \mathbf{b}^{(j)} \cdot \mathbf{z}$,
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The set $\mathbb{V}$ is the singular set of A.

Similar to before

$$a_{\mathbf{r}n} = \left(\frac{1}{2\pi i}\right)^d \int_T A(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{r}n+\mathbf{1}}},$$

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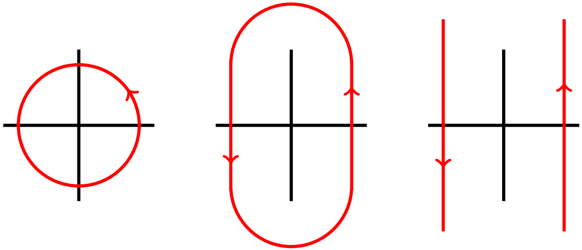

In several dimensions we get something similar, deforming T to

$$\sum_{\alpha \in \{-1,1\}^d} \operatorname{sign}(\alpha) C_{\alpha\epsilon}$$

where $C_{\alpha\epsilon} = (\alpha_1\epsilon, \cdots, \alpha_d\epsilon) + i\mathbb{R}^d$.

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In two complex dimensions T deforms to

$$C_{(\epsilon,\epsilon)} - C_{(-\epsilon,\epsilon)} - C_{(\epsilon,-\epsilon)} + C_{(-\epsilon,-\epsilon)}.$$

Definition (Strata)

A **stratum** is an intersection of hyperplanes with all smaller intersections removed.

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This is achieved via **Morse theory**.

Pick a direction $\mathbf{r} = (r_1, \cdots, r_d)$. Define the height function

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$$\left|\frac{1}{\mathbf{z}^{\mathbf{r}n}}\right| = \left|\exp\left[\log \mathbf{z}^{-\mathbf{r}n}\right]\right| = \exp\left[-n\sum_{i=1}^{d} r_i \log |z_i|\right]$$

Definition (Critical Points)

Let S be a stratum. Then $\mathbf{p} \in S$ is a **critical point** of S (relative to $\mathbf{r}$) if

$$-\nabla h_{\mathbf{r}}\big|_S(\mathbf{p}) = 0.$$

These are our saddle points!

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\begin{aligned}
& -\nabla h_{\mathbf{r}}(\mathbf{p}) \parallel -\nabla \ell_j(\mathbf{z}) \\
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- For a critical point $\boldsymbol{\sigma}$ on the intersection of multiple hyperplanes this is equivalent to

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-\nabla h_{\mathbf{r}}(\mathbf{p}) \in \left\{ \sum_j \lambda_j \mathbf{b}^{(j)} \mid \lambda_j \in \mathbb{R} \right\}
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We want to reduce height! Height going up is counterproductive. Define the **positive normal cone** at $\boldsymbol{\sigma}$ as

$$N(\boldsymbol{\sigma}) = \left\{ \sum_j \lambda_j \mathbf{b}^{(j)} \mid \lambda_j \geq 0 \right\},$$

where j runs over planes $\boldsymbol{\sigma}$ is on.

Setup is finally done!

Definition (Genericity of Direction)

We say a direction is **generic** if for all critical points, $\lambda_j > 0$. We say a direction is **non-generic** if some $\lambda_j = 0$.

Taking Residues

If we are at a critical point, we can use the same logic used to write

$$T = C_\epsilon - C_{-\epsilon}$$

to write

$$\int_{C_\epsilon} = \int_T + \int_{C_{-\epsilon}} .$$

$\int_T$ is computable and if $\lambda > 0$ for this hyperplane, our remaining integral is smaller.

Consider $A(z) = \frac{1}{(1-\frac{2x+y}{3})(1-\frac{x+2y}{3})}$ with $\mathbf{r} = (1,1)$. Draw critical points Then

$$\begin{aligned}
[\mathbf{z}^{\mathbf{r}n}]A(\mathbf{z}) &= \int_T A(\mathbf{z}) \frac{\mathrm{d}x\,\mathrm{d}y}{x^{n+1}y^{n+1}} \\
&= \int_{\sigma+(-\epsilon,-\epsilon)+i\mathbb{R}^2} A(\mathbf{z}) \frac{\mathrm{d}x\,\mathrm{d}y}{x^{n+1}y^{n+1}} \\
&= \int_{T_\sigma} - \int_{\sigma+(\epsilon,\epsilon)+i\mathbb{R}^2} \\
&\quad + \int_{\sigma+(\epsilon,-\epsilon)+i\mathbb{R}^2} + \int_{\sigma+(-\epsilon,\epsilon)+i\mathbb{R}^2}
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- Number of $\lambda_j = 0$ is the number of dimensions we cannot kill by taking residues,
- Case where exactly one of $\lambda_j = 0$ was already done by Baryshnikov, Melczer, Pemantle 2023 [BMP23].

Consider $A(x,y,z) = \frac{1}{(1-2x-y-z)(1-x-2y-z)(1-x-y-2z)}$ in the direction $\mathbf{r} = (1,1,2)$.

$\boldsymbol{\sigma} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is a critical point with

$$-\nabla h_{\mathbf{r}}(\boldsymbol{\sigma}) = (4,4,8) = 0\mathbf{b}^{(1)} + 0\mathbf{b}^{(2)} + 4\mathbf{b}^{(3)}$$

After some work

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[\mathbf{z}^{\mathbf{r}n}]A(\mathbf{z}) &= \frac{1}{(2\pi i)^3}\int_{\boldsymbol{\sigma}+(-\epsilon,-\epsilon,-\epsilon)+i\mathbb{R}^3} A(x,y,z)\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^{n+1}y^{n+1}z^{2n+1}} \\
&\sim \frac{1}{(2\pi i)^2}\int \frac{1}{(y+3z-1)(z-y)}\,\frac{\mathrm{d}y\,\mathrm{d}z}{(1-y-2z)^n y^n z^{2n}} \\
&= \frac{64^n}{(2\pi i)^2}\int \frac{1}{AB}\,\frac{\mathrm{d}A\,\mathrm{d}B}{(1-3A+B)^n(1+A-3B)^n(1+A+B)^{2n}} \\
&\sim \frac{64^n}{(2\pi i)^2}\int_D \frac{1}{AB}\exp\left[-n(6A^2-4AB+6B^2)\right]\mathrm{d}A\,\mathrm{d}B
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where $D = \mathbb{R}^2 + i(\epsilon,\epsilon)$.

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How to evaluate?

We have a need to evaluate integrals like

$$\int_{\mathbb{R}^2+i(\epsilon,\epsilon)} \frac{1}{x^{k_1} y^{k_2}} \exp\left[-n\phi(x,y)\right] \mathrm{d}x\, \mathrm{d}y.$$

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These are negative Gaussian moments!

In the numerator case we use the Morse lemma.

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Theorem

Let $\phi(\mathbf{0}) = 0$. If $\phi(\mathbf{x})$ has vanishing gradient and non-singular Hessian at $\mathbf{0}$, there is a change of variables $\mathbf{x} = \psi(\mathbf{y})$ such that

$$\phi(\psi(\mathbf{y})) = \sum_i y_i^2.$$

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Theorem

Let $\phi(\mathbf{0}) = 0$. If $\phi(\mathbf{x})$ has vanishing gradient and non-singular Hessian at $\mathbf{0}$, there is a change of variables $\mathbf{x} = \psi(\mathbf{y})$ such that

$$\phi(\psi(\mathbf{y})) = \sum_i y_i^2.$$

$$\int A(\mathbf{x}) \exp\left[-n\phi(\mathbf{x})\right] \mathrm{d}\mathbf{x} = \int A(\psi(\mathbf{y})) \det \mathrm{d}\psi(\mathbf{y}) \exp\left[-n \sum y_i^2\right] \mathrm{d}\mathbf{y}$$

Then $A(\phi(\mathbf{y})) \det \mathrm{d}\psi(\mathbf{y})$ is a power series; use Fubini to separate

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$$\int \frac{1}{x+y} \exp\left[-n(x^2+y^2)\right] \mathrm{d}x\,\mathrm{d}y,$$

did not get anywhere.

In the 1-variable case we differentiate.

$$I(n) = \int \frac{1}{y^k} \exp\left[-ny^2\right] \mathrm{d}y$$
$$\implies \frac{\mathrm{d}}{\mathrm{d}n} I(n) = \int \frac{-1}{y^{k-2}} \exp\left[-ny^2\right] \mathrm{d}y$$

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Eventually get some positive power, which we can evaluate.

This also does not work for us.

$$
\begin{aligned}
I(n) &= \int \frac{1}{xy} \exp\left[-n(x^2 + xy + y^2)\right] \mathrm{d}x\,\mathrm{d}y \\
\implies \frac{\mathrm{d}}{\mathrm{d}n} I(n) &= -\int \frac{x^2}{xy} \exp\left[-n(x^2 + xy + y^2)\right] \mathrm{d}x\,\mathrm{d}y \\
&\quad - \int \frac{xy}{xy} \exp\left[-n(x^2 + xy + y^2)\right] \mathrm{d}x\,\mathrm{d}y \\
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Differentiating $\int \frac{x}{y}$ gets us nowhere!

Morse lemma is inductive; first does x substitution, then y etc.

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Idea: differentiate until we get positive power on top, then apply Morse lemma to only the variable on top.

Let

$$
\begin{aligned}
I(n) &= \iint \frac{1}{x^2 y} \exp\left[-n(x^2 + xy + y^2)\right] \mathrm{d}x \, \mathrm{d}y \\
I'(n) &= -\iint \frac{1}{y} - \iint \frac{1}{x} - \iint \frac{y}{x^2} \\
&= I_1(n) + I_2(n) + I_3(n)
\end{aligned}
$$

$$
\begin{aligned}
I_3(n) &= -\iint \frac{y}{x^2} \exp\left[-n(x^2+xy+y^2)\right] \mathrm{d}x\,\mathrm{d}y \\
&= -\frac{1}{2}\iint \frac{1}{x} \exp\left[-n(b^2+\tfrac{3}{4}x^2)\right] \mathrm{d}x\,\mathrm{d}b \\
&= -\frac{1}{2}\int \exp\left[-n(b^2)\right] \mathrm{d}b \int \frac{1}{x} \exp\left[-n(\tfrac{3}{4}x^2)\right] \mathrm{d}x \\
&= \frac{i\pi^{3/2}}{2} n^{-1/2}
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$$\Longrightarrow \; I(n) = 3i\pi^{3/2} n^{1/2}$$

Applying same procedure gives

$$
\begin{aligned}
[\mathbf{z}^{\mathbf{r}n}]A(\mathbf{z}) &\sim \frac{64^n}{(2\pi i)^2}\int_{\mathbb{R}^2+i\epsilon}\frac{1}{AB}\exp\left[-n(6A^2-4AB+6B^2)\right]\mathrm{d}A\,\mathrm{d}B \\
&\sim \frac{64^n}{(2\pi i)^2}\frac{\sqrt{2}-6\sqrt{3}}{2}\pi\log n \\
&= \frac{6\sqrt{3}-\sqrt{2}}{8\pi}64^n\log n
\end{aligned}
$$

Conclusion

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$\rightarrow$ Implement

Bibliography

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