

Complex Monge-Ampère

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Chapter 0

Introduction

This is a collection of notes based on the class MAT1502HS (Topics in Geometric Analysis: Complex Monge-Ampère equations and Kähler geometry) taught by Freid Tong at the University of Toronto in Winter 2026.

The goal of this course will be to introduce Kähler geometry from the view of complex Monge-Ampère equations. We will take a broadly historical approach:

1. Yau's resolution of the Calabi conjecture, also known as Yau's theorem. This is a fundamental result on the Ricci curvature of Kähler manifolds. Yau's main contribution here was to prove a priori estimates for the complex Monge-Ampère.
2. Generalization of Yau's theorem to singular varieties. As before, this reduces to studying the complex Monge-Ampère. Here it turns out we need sharp a priori estimates.
3. Degenerate complex Monge-Ampère equations, and constructing geodesics in the space of Kähler metrics.
4. Additional topics.

0.1 Notation

We record our notation here.

Notation (differentials). *We write interchangeably*

$$\begin{aligned}\partial_j &= \frac{\partial}{\partial z_j}, \\ \partial_{\bar{j}} &= \frac{\partial}{\partial \bar{z}_j}.\end{aligned}$$

Notation (Einstein notation). *We use Einstein notation, that is that (when unspecified) repeated lowered and raised indices are implicitly summed over:*

$$A_i B^i = \sum_i A_i B^i.$$

Notation (matrix inverses). *We use raised indices indicate the inverse of a matrix, so that*

$$A^{\mu\gamma} A_{\nu\gamma} = \delta^\mu_\nu.$$

Chapter 1

Basic Complex Geometry

Much of this should be review from Tristan's course last term.

1.1 Complex manifolds

Definition 1.1.1 (complex manifolds). A smooth manifold M is a **complex manifold** of (complex) dimension n (so real dimension $2n$) if and only if we can write $M = \bigcup_{\alpha} U_{\alpha}$ with maps

$$\phi_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}^n$$

such that its transition functions

$$\phi_{\alpha} \circ \phi_{\beta}^{-1} \Big|_{\phi_{\beta}(U_{\alpha} \cap U_{\beta})} : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

are holomorphic.

Given this data we can define holomorphic functions on M .

Definition 1.1.2 (holomorphic functions). We say a function $f : M \rightarrow \mathbb{C}$ is **holomorphic** if

$$f \circ \phi_{\alpha}^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}$$

is holomorphic for each α .

Remark 1.1.3. This definition is okay since the transition maps are holomorphic. This collection of holomorphic functions determines the complex structure of M . With this definition, the coordinate charts $\phi_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}^n$ are holomorphic coordinate charts.

Note that the local geometry of M is more or less the same as an open subset of \mathbb{C}^n . There are several important examples of complex manifolds.

Example 1.1.4 (complex projective space). $\mathbb{P}^n = \{\text{space of complex lines in } \mathbb{C}^{n+1}\}$. An element of $l \in \mathbb{P}^n$ has the form $l = [z_0 : \cdots : z_n]$, where $0 \neq (z_0, \dots, z_n) \in l$. Here we identify $[z_0 : \cdots : z_n] \sim [\lambda z_0 : \cdots : \lambda z_n]$ for any $\lambda \in \mathbb{C}^*$. The coordinate charts are $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}$ for $i = 0, \dots, n$. Here the chart maps are

$$\begin{aligned} \phi_i : U_i &\rightarrow \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i} \right). \end{aligned}$$

We can compute that the transition functions are

$$\begin{aligned} \phi_i \circ \phi_j^{-1}(w_1, \dots, w_n) &= \phi_i([w_1 : \cdots : w_{j-1} : 1 : w_j : \cdots : w_n]) \\ &= \left(\frac{w_1}{w_i}, \dots, \frac{\widehat{w_i}}{w_i}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \dots, \frac{w_n}{w_i} \right), \end{aligned}$$

which are of course all holomorphic.

To make more complex manifolds we can take distinguished subsets of complex manifolds.

Theorem 1.1.5 (holomorphic implicit function theorem). *Suppose we have a holomorphic function*

$$f : U \times V \subset \mathbb{C}_z^n \times \mathbb{C}_w^m \rightarrow \mathbb{C}^m$$

such that $f(0,0) = 0$ and $\det(\frac{\partial f}{\partial w})(0,0) \neq 0$. Then there exists a $g(z) : U' \subset U \rightarrow \mathbb{C}^m$ holomorphic such that $g(0) = 0$ and $f(z, g(z)) = 0$.

We can use this to get submanifolds of \mathbb{P}^n .

Example 1.1.6 (hypersurfaces in \mathbb{P}^n). *Let $f(z_0, \dots, z_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic homogenous (degree k) polynomial. Then $M = \{f = 0\} \subset \mathbb{P}^n$ is a complex submanifold of dimension $n - 1$ of \mathbb{P}^n at all points p where $f(p) = 0$ and $df(p) \neq 0$. Repeatedly applying the holomorphic IFT gives us projective manifolds $M^k \subset \mathbb{P}^n$ of dimension $k \leq n$.*

Surely this depends on k .

1.1.1 Local structure of complex manifolds

We define

$$\begin{aligned} T_{\mathbb{C}}M &= TM \otimes_{\mathbb{R}} \mathbb{C}, \\ \Omega_{\mathbb{C}}M &= \Omega M \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

If M is a complex manifold, then in any holomorphic coordinate chart, $T_{\mathbb{C}}M$ is locally spanned by $\{\partial_i\}_{i=1}^n$ and $\{\bar{\partial}_i\}_{i=1}^n$ with complex coefficients. One can check that the subbundles

$$\begin{aligned} T^{1,0}M &= \text{span}_{\mathbb{C}} \{\partial_i\} \subset T_{\mathbb{C}}M, \\ T^{0,1}M &= \text{span}_{\mathbb{C}} \{\bar{\partial}_i\} \subset T_{\mathbb{C}}M, \end{aligned}$$

are well-defined. This gives rise to a decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. Similarly we can define the subbundles

$$\begin{aligned} \Omega^{1,0}M &= \text{span}_{\mathbb{C}} \{dz_i\} \subset \Omega_{\mathbb{C}}M, \\ \Omega^{0,1}M &= \text{span}_{\mathbb{C}} \{d\bar{z}_i\} \subset \Omega_{\mathbb{C}}M, \end{aligned}$$

and get the decomposition $\Omega_{\mathbb{C}}M = \Omega^{1,0}M \oplus \Omega^{0,1}M$. By taking wedge products we can define

$$\Omega_{\mathbb{C}}^k M = \bigoplus_{p+q=k} \Omega^{p,q} M$$

where $\Omega^{p,q}M$ is spanned locally by $dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. The usual conjugation operation extends to $\Omega^{p,q}M \rightarrow \Omega^{q,p}M$ as

$$\overline{dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}} = d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_p} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_q},$$

and we say that $\alpha \in \Omega_{\mathbb{C}}^k M$ is real if $\alpha = \bar{\alpha}$. There is a differential structure on $\Omega^k M$ as well. We have

$$d : \Omega^k M \rightarrow \Omega^{k+1} M,$$

where if

$$\begin{aligned} \alpha &= \sum_{|I|+|J|=k} \alpha_{I\bar{J}} dz_I \wedge d\bar{z}_J, \\ I &= (i_1, \dots, i_p), \\ J &= (j_1, \dots, j_q), \\ dz_I &= dz_{i_1} \wedge \dots \wedge dz_{i_p}, \end{aligned}$$

then

$$d\alpha = \sum_{l=1}^n \sum_{|I|+|J|=k} \left(\frac{\partial \alpha_{I\bar{J}}}{\partial z_l} dz_l \wedge dz_I \wedge d\bar{z}_J + \frac{\partial \alpha_{I\bar{J}}}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_I \wedge d\bar{z}_J \right).$$

From this we can see that in fact $d : \Omega^{p,q} \rightarrow \Omega^{p+1,q} \oplus \Omega^{p,q+1}$, and so we can write $d = \partial + \bar{\partial}$ (where ∂ is d projected onto the first term in this decomposition and $\bar{\partial}$ projects d onto the second term). As an exercise one can show that $\partial^2 = \bar{\partial}^2 = 0$ and so

$$0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial.$$

Thus ∂ and $\bar{\partial}$ anticommute.

Definition 1.1.7 (ddbar operator). *We define the **ddbar operator** to be the operator $\sqrt{-1}\partial\bar{\partial}$.*

Remark 1.1.8. *The factor of $\sqrt{-1}$ makes the ddbar operator a real operator.*

1.2 Holomorphic vector bundles

As one might expect, there is a notion of holomorphic vector bundles.

Definition 1.2.1 (holomorphic vector bundles). *Let $M = \bigcup_{\alpha} U_{\alpha}$ be a complex manifold with a smooth complex vector bundle $\pi : E \rightarrow M$ of complex rank r . We say E is **holomorphic** if there exist trivializations $\{e_{\alpha,\mu}\}_{\mu=1}^r$ of E on U_{α} such that on $U_{\alpha} \cap U_{\beta}$*

$$e_{\alpha,\mu} = t_{\alpha\beta}^{\nu}{}_{\mu} e_{\beta,\nu}$$

for some holomorphic transition functions $t_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{Mat}_{\mathbb{C}}(r \times r)$.

A holomorphic structure on a vector bundle gives rise to an operator

$$\bar{\partial}_E : \Gamma(E) \rightarrow \Gamma(\Omega^{0,1} \otimes E),$$

where if $s = (s_{\alpha})^{\mu}(e_{\alpha})_{\mu}$ then

$$\bar{\partial}_E s = (\bar{\partial} s_{\alpha}^{\mu}) \otimes (e_{\alpha})_{\mu}. \tag{1.1}$$

Note 1.2.2. *We often omit the E from $\bar{\partial}_E$ and write $\bar{\partial}$.*

A priori it is not clear that $\bar{\partial}$ is well-defined. On $U_{\alpha} \cap U_{\beta}$ we have

$$\begin{aligned} \bar{\partial}s &= (\bar{\partial} s_{\beta}^{\nu}) \otimes (e_{\beta})_{\nu} \\ &= \bar{\partial}(s_{\alpha}^{\mu} t_{\alpha\beta}^{\nu}{}_{\mu}) \otimes (e_{\beta})_{\nu} \\ &= (\bar{\partial} s_{\alpha}^{\mu}) \otimes t_{\alpha\beta}^{\nu}{}_{\mu} e_{\beta,\nu} \\ &= (\bar{\partial} s_{\alpha}^{\mu}) \otimes e_{\alpha,\mu}, \end{aligned}$$

and so this operator $\bar{\partial}$ is invariant under the change of trivialization and is well-defined.

Note 1.2.3. *This does not work for ∂ , as $\partial t_{\alpha\beta}$ is not necessarily 0.*

We can define the set of holomorphic sections as

$$H^0(M, E) = \{s \in \Gamma(M, E) : \bar{\partial}s = 0\} \subset \Gamma(M, E).$$

Note that the collection of holomorphic sections determines the holomorphic structure of E .

Note 1.2.4. *Also give a holomorphic vector bundle there exist local holomorphic trivializations $\{e_{\alpha,\mu}\}$ near any point. We always compute in these holomorphic trivializations.*

1.2.1 Associated bundles to E

Given a holomorphic vector bundle $E \rightarrow M$ there are several associated holomorphic vector bundles.

1. $E^* \rightarrow M$, where we replace $e_{\alpha\mu}$ by its dual trivialization $(e_\alpha^*)^\mu$ and the transition maps $t_{\alpha\beta}$ by $(t_{\alpha\beta}^{-1})^T$.
2. $\bigwedge^s E \rightarrow M$, with trivializations $e_{i_1} \wedge \cdots \wedge e_{i_s}$ and transition functions $\bigwedge^s t_{\alpha\beta} \in \text{End}(\bigwedge^s E)$. If $s = r$ then $\bigwedge^s E$ is a holomorphic line bundle.
3. $\overline{E} \rightarrow M$ is an anti-holomorphic vector bundle with trivializations $\overline{e_{\alpha\beta}}$ and transition functions $\overline{t_{\alpha\beta}}$.

Example 1.2.5. If M is a complex manifold, $T^{1,0}M$ and $\Omega^{1,0}M$ are both holomorphic vector bundles with trivializations $\{\partial_i\}_{i=1}^n$ and $\{dz_i\}_{i=1}^n$ respectively. The transition functions are as follow: if (w_1, \dots, w_n) is another coordinate system then

$$\frac{\partial}{\partial w_i} = \frac{\partial z_j}{\partial w_i} \frac{\partial}{\partial z_j} + \frac{\partial \bar{z}_j}{\partial w_i} \frac{\partial}{\partial \bar{z}_j} = \frac{\partial z_j}{\partial w_i} \frac{\partial}{\partial z_j}$$

and

$$dw_i = \frac{\partial w_i}{\partial z_j} dz_j.$$

Note here we are using the fact that the coordinate functions z_j are holomorphic.

Example 1.2.6. The **canonical bundle** of M is $K_M = \bigwedge^n \Omega^{1,0}M = \Omega^{n,0}M$. This has trivialization $dz_1 \wedge \cdots \wedge dz_n$ and transition functions

$$dw_1 \wedge \cdots \wedge dw_n = \det \left(\frac{\partial w}{\partial z} \right) dz_1 \wedge \cdots \wedge dz_n.$$

1.2.2 Hermitian metrics

A Hermitian metric on a bundle E is a smoothly varying Hermitian inner product on the fibers of E . In a local holomorphic trivialization $\{e_\mu\}_{\mu=1}^r$ then

$$H = H_{\mu\bar{\nu}}(e^*)^\mu \otimes (\overline{e^*})^\nu \in E^* \otimes \overline{E^*},$$

where $H_{\mu\bar{\nu}}$ is a positive-definite Hermitian matrix at each point. This gives an inner product on sections

$$\langle s, t \rangle_H = H_{\mu\bar{\nu}} s^\mu \bar{t}^\nu$$

which of course gives us a norm

$$|s|_H^2 = \langle s, s \rangle_H = H_{\mu\bar{\nu}} s^\mu \bar{s}^\nu \geq 0.$$

A Hermitian metric H on E gives rise to a Hermitian structure on all associated bundles.

Example 1.2.7. On E^* the induced metric is $\tilde{H} = H^{-1}$, so that

$$\tilde{H} = H^{\mu\bar{\nu}} e_\mu \otimes \overline{e_\nu}.$$

Example 1.2.8. On a line bundle any Hermitian metric is represented by a smoothly varying positive definite 1×1 matrix, and so the Hermitian metric is just a strictly positive smooth function h . In the specific case of the line bundle $\bigwedge^r E$, there is an induced Hermitian metric

$$h = \det H_{\mu\bar{\nu}}.$$

Remark 1.2.9. Hermitian metrics always exist by a partition of unity argument.

We care about Hermitian metrics primarily because they can be used to define connections.

1.3 Connections

A connection is a way to differentiate sections of bundles to get more sections of bundles. Recall that we have a natural way to differentiate sections in anti-holomorphic directions (see (1.1)), but this naive approach to differentiating does not work for holomorphic directions (since the transition functions are not anti-holomorphic). To differentiate in holomorphic directions we must first pick a connection.

It turns out that if E is a Hermitian vector bundle (i.e., endowed with a Hermitian metric) then there exists a natural connection.

Definition 1.3.1 (Chern connection). *Let E be a Hermitian vector bundle and pick a section $s = s^\mu e_\mu$. Then we define the **Chern connection** as the connection*

$$\begin{aligned}\nabla_{\bar{j}} s^\mu &= \partial_{\bar{j}} s^\mu, \\ \nabla_j s^\mu &= H^{\mu\bar{\nu}} \partial_j (H_{\gamma\bar{\nu}} s^\mu).\end{aligned}$$

Note 1.3.2. *This looks a bit odd, but is actually natural.*

$$s^\mu \xrightarrow{\text{lower index}} H_{\gamma\bar{\nu}} s^\mu \xrightarrow{\partial_j \text{ well-defined}} \partial_j (H_{\gamma\bar{\nu}} s^\mu) \xrightarrow{\text{raise index}} H^{\mu\bar{\nu}} \partial_j (H_{\gamma\bar{\nu}} s^\mu).$$

Expanding out we see

$$\nabla_j s^\mu = \partial_j s^\mu + A_j^\mu{}_\nu s^\nu,$$

where $A_j^\mu{}_\nu = H^{\mu\bar{\gamma}} \partial_j H_{\nu\bar{\gamma}}$ are the associated connection coefficients. This definition induces the Chern connection on all tensor powers of E , E^* , and \bar{E} .

Example 1.3.3. *Consider E^* and a section $s = s_\mu (e^*)^\mu \in \Gamma(E^*)$. Then the Chern connection is*

$$\nabla_j s_\mu = \partial_j s_\mu - A_j^\nu{}_\mu s_\nu.$$

For tensor powers then ∇ gets applied to each component of the tensor product.

Example 1.3.4. *If $s = s^\mu{}_\nu (e^*)^\nu \otimes e_\mu \in \Gamma(E \otimes E^*)$ then one can show that*

$$\nabla_j s^\mu{}_\nu = \partial_j s^\mu{}_\nu + A_j^\mu{}_\gamma s^\gamma{}_\nu - A_j^\gamma{}_\nu s^\mu{}_\gamma.$$

1.3.1 Curvature

Once we have a connection we can define a notion of curvature. Unlike in Euclidean space, where all derivatives commute, this is not necessarily true for connections on an arbitrary vector bundle. The curvature associated to a specific connection measures the failure of ∇ to commute. We compute

$$\begin{aligned}[\nabla_i, \nabla_{\bar{j}}] s^\mu &= \nabla_i \nabla_{\bar{j}} s^\mu - \nabla_{\bar{j}} \nabla_i s^\mu \\ &= \nabla_i (\partial_{\bar{j}} s^\mu) - \nabla_{\bar{j}} (\partial_i s^\mu + A_i^\mu{}_\gamma s^\gamma) \\ &= \partial_i \partial_{\bar{j}} s^\mu + A_i^\mu{}_\gamma \partial_{\bar{j}} s^\gamma - \partial_{\bar{j}} \partial_i s^\mu - \partial_{\bar{j}} (A_i^\mu{}_\gamma s^\gamma) \\ &= -(\partial_{\bar{j}} A_i^\mu{}_\gamma) s^\gamma.\end{aligned}$$

Definition 1.3.5 (curvature). *In the spirit of the above computation, we define the **curvature** of a connection as*

$$F_{i\bar{j}}^\mu{}_\nu = -\partial_{\bar{j}} (A_i^\mu{}_\nu).$$

Exercise 1.3.6. *Check that $F_{ij} = F_{\bar{i}\bar{j}} = 0$, so that F is a (matrix of) $(1,1)$ -forms.*

This formula extends to tensor powers.

Exercise 1.3.7. *Show that*

$$[\nabla_i, \nabla_{\bar{j}}] s^\mu{}_\nu = F_{i\bar{j}}^\mu{}_\gamma s^\gamma{}_\nu - F_{i\bar{j}}^\gamma{}_\nu s^\mu{}_\gamma$$

and that

$$[\nabla_i, \nabla_{\bar{j}}] \bar{s}^\mu = -\overline{F_{j\bar{i}}^\mu{}_\nu} \bar{s}^\nu.$$

There is
suerly a
typo here.
The index
lowering
operation
here has
indices
that don't
match up.

1.3.2 Special Case of Line Bundles

Mostly we will be working with line bundles. In this case our formulas simplify significantly. If L is a holomorphic line bundle with a Hermitian metric h (locally we abuse notation slightly and write $h = he \otimes \bar{e}$) then

$$A_i = h^{-1}(\partial_i h) = \partial_i \log h$$

and

$$F_{i\bar{j}} = -\partial_{\bar{j}} A_i = -\partial_{\bar{j}} \partial_i \log h.$$

Then $F = \sqrt{-1} F_{i\bar{j}} dz^i \wedge d\bar{z}^j$ locally looks like

$$F = F_h = -\sqrt{-1} \partial \bar{\partial} \log h.$$

Note 1.3.8. Note that F is not $\partial \bar{\partial}$ of a global function, as we cannot take a global trivialization of L to get a global h unless L is trivial.

Exercise 1.3.9. Check that $-\sqrt{-1} \partial \bar{\partial} \log h$ is a well-defined $(1,1)$ -form, despite h not being globally defined.

In general then F_h is a closed real $(1,1)$ -form (as closedness is a local condition), but not an exact form (as h is not necessarily global). It makes sense then to think about the cohomology class of F_h .

Definition 1.3.10 (first Chern class). We define

$$c_1(L) = [F_h] \in H^2(M, \mathbb{R})$$

to be the **first Chern class** of L .

It seems that $c_1(L)$ might depend on F_h , which depends on h , but this is not the case.

Theorem 1.3.11. $c_1(L)$ is independent of h .

Proof. Any other Hermitian metric is related to h by

$$\tilde{h} = e^{-\phi} h.$$

One can show as an exercise that such a ϕ is globally defined. Then

$$\begin{aligned} F_{\tilde{h}} &= -\sqrt{-1} \partial \bar{\partial} \log \tilde{h} \\ &= -\sqrt{-1} \partial \bar{\partial} \log(e^{-\phi} h) \\ &= F_h + \sqrt{-1} \partial \bar{\partial} \phi. \end{aligned}$$

Here then ϕ is a global function, and so $[\partial \bar{\partial} \phi] = 0$ (since $\partial \bar{\partial} = d\bar{\partial}$) and so the result follows. \square