University of Waterloo

CO 739:
Combinatorial Commutative Algebra

Professor: Oliver Pechenikr
Notes By: Alexander Kroitor

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Contents

0 Introduction

0.1 Course Name

This is a course on combinatorial commutative algebras, however there is a strong relationship between commutative algebras and algebraic geometry – an equally apt name for this course would be combinatorial algebraic geometry.

Example 0.1. There is a correspondence between the algebraic geometric object of the circle S^1 and the commutative algebraic object of the equation $x^2 + y^2 - 1 = 0$.

This course will be following three books broadly. In order of difficulty they are:

- Cox-Little-O'Shea
- Ene-Herzog
- Miller-Sturmfels

Note that these notes are meant to provide brief results and not maximal detail.

1 Ring Theory

We first discuss rings.

1.1 Algebraic Closure

Definition (Algebraically Closed)**.** A field K is called algebraically closed if each non-constant polynomial $f \in K[x]$ has a root.

Theorem 1.1. *Each field K has an algebraic closure $\overline{K}$, the smallest algebraically closed field containing K.*

1.2 Polynomial Rings

Let $K[x_1, \cdots, x_n]$ be a polynomial ring. We write $K[x_1, \cdots, x_n] =: K[\underline{x}]$ with

$$\begin{aligned} \underline{x}^{\underline{a}} &= x_1^{a_1} \cdots x_n^{a_n} \\ f &= \sum_{\underline{a} \in \mathbb{N}^n} c_{\underline{a}} \underline{x}^{\underline{a}} \end{aligned}$$

with only finitely many non-zero $c_{\underline{a}}$.

Definition.

$$\begin{aligned} \deg(f) &= \max \{ |\underline{a}| \mid c_{\underline{a}} \neq 0 \} \\ supp(f) &= \{ \underline{a} \mid c_{\underline{a}} \neq 0 \}. \end{aligned}$$

Note that from now on we will be dropping underlines.

Remark. $K[x]$ is a graded ring.

Note that in order to find the degree of a polynomial it doesn't suffice to just examine powers in it, one must write the polynomial in canonical form.

Example 1.1.

$$\deg\left((x+1)^2 - x^2\right) = \deg(2x+1) = 1.$$

Definition. A polynomial f is homogeneous of degree i if its only support is in degree i.

1.3 Homogeneous Polynomials

Let $S = K[x]$ and $S_i \subset S$ be the subset of degree i homogeneous polynomials (a finite dimensional vector space spanned by monomials of degree i). Then

$$S = \bigoplus_{i=0}^{\infty} S_i$$

Remark. $\dim S_i = \binom{i+n-1}{n-1}$.

Proof. Stars and bars argument. ∎

Definition. The Hilbert series of S is

$$H(S;t) = \sum_{i \in \mathbb{N}} (\dim S_i) t^i.$$

Remark. The Hilbert series of a polynomial ring is

$$H(K[x];t) = \sum \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n}$$

where the last equality is from the negative binomial theorem.

2 Ideals

An ideal is the ring analogue of a normal subgroup.

Definition (Ideal)**.** An ideal is a non-empty subset $I \subseteq K[x]$ such that

1. $f, g \in I \implies f + g \in I$,
2. $f \in I, g \in K[x] \implies gf \in I$.

Note that if in the second condition $g \in K$ instead, I is a vector subspace instead of an ideal.

2.1 Generating Ideals

Definition (Ideal Generated by a Set)**.** For $F \subset K[x]$ we define (F) (or $\langle F \rangle$) to be the smallest ideal containing F and call it the ideal generated by F.

Proposition 2.1. *For $F \subset K[x]$, let I_F be the set of all finite $K[\underline{x}]$-linear combinations of elements of F, that is to say all finite sums of the form $\sum g_i f_i$ with $g_i \in K[x], f_i \in F$. Then $I_F = (F)$.*

Proof. We check in order

1. I_F satisfies conditions 1 and 2 of the definition of an ideal and is thus an ideal.
2. I_F contains F.
3. Any ideal containing F must contain I_F.

□

A natural question arises: given $f, g_1, \cdots, g_n$, how can we tell whether or not $f \in (g_1, \cdots, g_n)$? This is a non-trivial question.

Example 2.1. $K[\underline{x}] = \mathbb{Q}[x,y,z], I = (xy - z, yz - x, xz - y), f = z^3 - z$. It is not obvious yet true that $f \in I$.

Definition. An ideal is homogeneous (or graded) if its generated by homogeneous polynomials (of possibly different degrees).

Example 2.2. $(x^2, x^3 + y^3, xy, x)$ is a homogeneous ideal.

Proposition 2.2. *Let I be an ideal of $S[\underline{x}]$. TFAE:*

1. *I is homogeneous.*
2. *$f \in I \implies$ all homogeneous components of f are in I.*
3. *$I = \bigoplus_{j=0}^{\infty} I_j$; $I_j = I \cap S_j$ (recall S_j is the set of homogeneous degree j polynomials).*

Proof.

$1 \to 2$: Define G as the set of homogeneous generators, $f = \sum_i h_i g_i \in I, h_i \in S, g_i \in G$. $\deg(g_i) = d$, then $f_j = \sum_i h_i' g_i$ st h_i' is part of h_i that's homo of degree $j - d_i$. Then $f_j \in I$.

$2 \to 3$: $f \in I, f_j \in I$, so $f_j \in I \cap S_j$, so $I = \sum_{j=0}^{\infty} I_j = \bigoplus_j I_j$

$3 \to 2$: By definition.

$2 \to 1$: Let G be set of generators for I. Take the homo components of G. Then $F = \{ g_j \mid g \in G, j \in \mathbb{N} \}$.

□

3 Quotient Rings

Also known as residue class rings.

Definition. Let $I \subset S = K[\underline{x}]$ be an ideal. The residue class of $f \mod I$ is the set

$$f + I = \{ f + i \mid i \in I \} .$$

The quotient ring S/I is the st of all residue classes.

Remark. S/I is a ring with

$$(f + I) + (g + I) = (f + g) + I$$
$$(f + I) \cdot (g + I) = (fg) + I$$

Proposition 3.1. *$f + I = g + I \iff (f - g) \in I$.*

Proof.

$(\Leftarrow)$: $f - g \in I \therefore f = g + i ; i + I = 0 + I \therefore f + I = (g + i) + I = g + I$.

$(\Rightarrow)$: $f + i \in g + I \therefore f + i = g + j \therefore f - g = j - i \in I$.

□

3.1 Ideal Operations

We have the following operations between ideals (this list is non-exhaustive.

Definition.

Sum: $I + J = \{ f + g \mid f \in I, g \in J \}$,

Intersection: $I \cap J$,

Product: $IJ = (\{ fg \mid f \in I, g \in J \})$,

Colon: $I : J = \{ f \in S \mid fj \in I \, \forall j \in J \} = \{ f \in S \mid fJ \subset I \}$

Note that the colon ideal of two ideals is also known as the ideal quotient of two ideals.

Proposition 3.2. $IJ \subseteq I \cap J$

Proof. $f \in I, g \in J \implies fg \in I, fg \in J$ by the definition of an ideal. Thus $\{ fg \} \subset I \cap J \implies (\{ fg \}) \subset I \cap J$ □

Example 3.1. Let $S = \mathbb{Q}[x]; I = (x^3 + 6x^2 + 12x + 8) = ((x+2)^3); J = (x^2 + x - 2) = ((x+2)(x-1))$. Then

$$\begin{aligned} I + J &= (x+2) && \text{(gcd)} \\ I \cap J &= \Big((x+2)^3 (x-1) \Big) && \text{(lcm)} \\ IJ &= \Big((x+2)^4 (x-1) \Big) && \text{(mult)} \\ I : J &= \Big((x+2)^2 \Big) && \text{st.} (x+2)^2 \cdot (x+2)(x-1) \sim (x+2)^3 \end{aligned}$$

Definition. The radical of an ideal I

$$\sqrt{I} = \{ f \in S \mid f^k \in I, \text{ for some } k \geq 1 \}.$$

Example 3.2. $\sqrt{I} = (x+2)$.

Example 3.3. $\sqrt{J} = \big((x+2)(x-1) \big)$.

Proposition 3.3. *$\sqrt{I}$ is an ideal.*

sketch. $(fp)^k = f^k p^k \in I$. $(f+g)^{k+l} = \sum_{k=0}^{k+i} c_i f^i g^{k+l-i} \in I$. □

Definition. I is a radical ideal if $I = \sqrt{I}$.

Proposition 3.4. *For all ideals I, we have that $\sqrt{I} = \sqrt{\sqrt{I}}$.*

sketch. One side is obvious, then $(f^k)^l \in I$. □

4 Interpreting Betti Numbers

From the last section we have the notion of the Betti numbers of finitely generated S-modules. Now let us turn to analyzing these.

4.1 Betti Diagram

We start with a motivating example.

Example 4.1. Consider

$$\begin{aligned} S &= K[x,y,z,w] \\ M = I &= (x^2 - yz, z^2w, xyz, w^3) \\ V_\infty(I) &= z\text{-axis} \cup y\text{-axis(with lots of fuzz, extra fuzz at origin).} \end{aligned}$$

This has a minimal free resolution given by

$$\begin{tikzcd} 0 \arrow[r] & S(-8) \arrow[r] & S^2(-6)\oplus S^3(-7) \arrow[dl] & & \\ & S^6(-5)\oplus S(-6) \arrow[r] & S(-2)\oplus S^3(-3) \arrow[r] & I \arrow[r] & 0 \end{tikzcd}$$

We have a notion of a Betti Diagram, a table tracking all the Betti numbers of a module.

	0	1	2	3	i
2	1				
3	3				
4		6	2		
5		1	3	1	

Notice that these are shifted by $-i$, as we “expect" Betti degrees to go down by 1 each time we go down our free resolution, so this information is uninteresting.

4.2 Projective Dimension and Regularity

Definition. The projective dimension of M is

$$pd(M) = \max\{\, i \mid \beta_{ij} \neq 0 \text{ for some } j \,\}\,.$$

“How far right the Betti diagram goes".

Definition. The regularity of M is

$$reg(M) = \max\{\, j \mid \beta_{i,j+i} \neq 0 \text{ for some } i \,\}\,.$$

“How far down the Betti diagram goes".

Proposition 4.1. *For a homogeneous ideal $I \subset S$ we have*

$$\begin{aligned} pd(S/I) &= pd(S) + 1 \\ reg(S/I) &= reg(S) - 1 \end{aligned}$$

Proof. Taking the modulo adds one term to the min free resolution of I, which implies the result. $\square$

Lemma 4.1. *For all M, $pd(M) \leq n$ (this is just the Hilbert syzygy theorem).*

Lemma 4.2. *For all M, $reg(M) \geq$ degrees of the generators.*

Definition. The depth of M is

$$depth(M) = n - pd(M) = \min\{\, i \mid \beta_{n-i,j} \neq 0 \text{ for some } j \,\}.$$

4.3 Betti Functions

Definition. The Hilbert function of M is

$$\begin{aligned} H_M : \mathbb{Z} &\to \mathbb{Z}_{\geq 0} \\ i &\mapsto \dim_K M_i. \end{aligned}$$

The Hilbert series of M is (note this is a formal Laurent series)

$$H(M,t) = \sum_i H_M(i) t^i.$$

Proposition 4.2. *For M a finitely generated graded S-module we have that*

$$H(M,t) = \frac{\sum_i (-1)^i \sum_j \beta_{ij} t^j}{(1-t)^n}.$$

The numerator of this function is called the K-polynomial $K(M,t)$ of M.

Definition. Taking the K-polynomial and substituting t for $(1-t)$ gives us the Groethendieck polynomial $\mathcal{G}(M,t)$ of M.

Definition. The minimum degree in $\mathcal{G}$ is the codimension $codim(M)$ of M. The dimension $dim(M)$ of M is $n - codim(M)$.

Example 4.2. Consider

$$M = K[x,y,z,w]/(x^2 - yz, z^2w, xyz, w^3).$$

We have the following Betti table for I:

	0	1	2	3
2	1			
3	3			
4		6	2	
5		1	3	1

and the following Betti table for R/I:

	0	1	2	3	4
0	1				
1		1			
2		3			
3			6	2	
4			1	3	1

We get

$$\begin{aligned}
H(M,t) &= \frac{1-t^2-3t^3+6t^5-t^6-3t^7+t^8}{(1-t)^4} \\
K(M,t) &= 1-t^2-3t^3+6t^5-t^6-3t^7+t^8 \\
\mathscr{G}(M,t) &= 12t^3-20t^4+7t^5+6t^6-5t^7+t^8
\end{aligned}$$

And thus

$$\begin{aligned}
dim(M) &= 1 \\
codim(M) &= 3
\end{aligned}$$

which makes sense as M is a curve.

Definition. The coefficient on the lowest degree term in $\mathscr{G}$ is the degree of M (the number of lowest degree things).

4.4 Depth

Definition. $f \in S$ is a non-zero divisor (NZD) on S/I if $(f+I)(g+I) = (0+I) \implies (g+I) = (0+I)$.

Geometrically this means that f vanishes on any (entire) component of $V_\infty(I)$, even an embedded component, hence the hypersurface $V_\infty(f)$ slices each component non-trivially (note, everything is homogeneous so non-trivial).

Definition. A homogeneous S/I-sequence is a sequence $f_1, \cdots, f_d$ such that f_1 is a NZD on S/I, f_2 is a NZD on $S/(I+(f_1))$, f_3 is a NZD on $S/(I+(f_1,f_2))$, etc. and also $I+(f_1,\cdots,f_d) \neq S$.

Definition. The depth of S/I is the maximum length of a homogeneous S/I-sequence.

Theorem 4.1 (Auslander-Buchsbaum)**.** *We have that*

$$depth(S/I) = n - pd(S/I).$$

Corollary 4.1. *We have that*

$$depth(S/I) \leq \textit{dim of smallest cpt of } V_\infty(I).$$

Proof. Each NZD must not vanish on any cpt, but also slices it nontrivially. If it has dimension d you can slice at most d times. □

Example 4.3. Consider

$$\begin{aligned}
S &= K[x,y] \\
I &= (x^2, xy) \\
V_\infty(I) &= y\text{-axis} + \text{ extra fuzz at the origin.}
\end{aligned}$$

Then we have that

$$\begin{aligned}
\dim(S/I) &= 1 \text{ (by staring at } V_\infty(I)) \\
depth(S/I) &= 0 \text{ (by corollary).}
\end{aligned}$$

In addition we have the Betti table

	0	1	2
0	1		
1		2	1

This gives us

$$\begin{aligned} pd(S/I) &= 2 \\ reg(S/I) &= 1 \\ \implies depth(S/I) &= 2 - 2 = 0. \end{aligned}$$

And while we're at it then

$$\begin{aligned} H(S/I, t) &= \frac{1 - 2t^2 + t^3}{(1-t)^2} \\ K(S/I, t) &= 1 - 2t + t^3 \\ \mathscr{G}(S/I, t) &= t + t^2 - t^3 \end{aligned}$$

which gives us that

$$\begin{aligned} codim(S/I) &= 1 \\ dim(S/I) &= 1 \\ deg(S/I) &= 1 \end{aligned}$$

4.5 Cohen-Macaulayness

Definition. A finitely generated S-module is called Cohen-Macaulay (CM) if $depth(M) = dim(M)$.

This is a strange condition, as depth is algebraic and dimension is geometric.

Proposition 4.3. *If S/I is CM then $V_\infty(I)$ is equidimensional, eg all components have the same dimension.*

Proposition 4.4. *A curve $V(I)$ is CM if and only if the origin is not a component (ie there is no extra fuzz).*

Proposition 4.5. *If $V(I)$ is 0-dimensional then S/I is CM.*

Some examples are excluded.

4.6 Initial Ideals

We note that taking the initial ideal makes these properties change in predictable ways.

Theorem 4.2. *For all i, j we have that $\beta_{ij}(S/I) \leq \beta_{ij}(S/in\ I)$. In particular we have*

- $reg(S/I) \leq reg(S/in\ I)$
- $pd(S/I) \leq pd(S/in\ I)$
- $depth(S/I) \geq depth(S/in\ I)$
- *if $S/in\ I$ is CM, so is S/I*

Definition. A Betti number β_{ij} is extremal if there are no non-0 Betti numbers to its southeast.

Theorem 4.3. *If $\operatorname{in} I$ is square-free, then all the extremal Betti numbers of S/I and $S/\operatorname{in} I$ coincide. In particular we have*

- $reg(S/I) = reg(S/in\ I)$
- $pd(S/I) = pd(S/in\ I)$
- $depth(S/I) = depth(S/in\ I)$
- *$S/in\ I$ is CM if and only if S/I is CM*

5 Simplicial Complexes

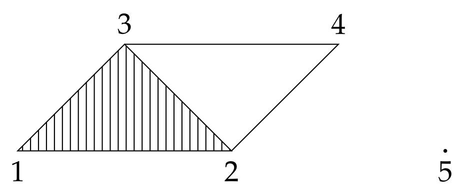
We now have a notion for Cohen-Macaulayness. We want to find a class of schemes that are CM.

5.1 Simplical Complexes

Definition. An abstract simplical complex is a collection Δ of subsets closed under taking subsets. That is to say that if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$.

- The elements of Δ are called faces.
- The dimension of $F \in \Delta$ is $dimF = |F| - 1$.
- If $F \in \Delta$ and $\nexists G \in \Delta$ with $F \subsetneq G$ then F is a facet.

Example 5.1. Consider the complex Δ with facets $\{1,2,3\}, \{2,4\}, \{3,4\}, \{5\}$.



Definition. We say that Δ is generated by its facets

$$\Delta = \langle F_1, \cdots, F_k \rangle = \langle F(\Delta) \rangle$$

where $F(\Delta)$ is the set of all the facets of Δ.

Definition. We say that $\dim \Delta = \max \{ \dim F \mid F \in F(\Delta) \}$.

Definition. We say that Δ is pure if all of its facets have the same dimension.

Remark. There are two degenerate examples:

- $\Delta = \{\}$, the void complex; no faces, $\dim = -\infty$
- $\Delta = \{\emptyset\}$, the irrelevant complex; one face, $\dim = -1$

Definition. Now let $f_i =$ the number of faces of dim i. Then we define the f-vector as

$$f(\Delta) = (f_{-1}, f_0, f_1, \cdots, f_{\dim \Delta}).$$

Example 5.2. The n-simplex is $\Delta = \langle [n] \rangle$.

$$f(\Delta) = \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots, \binom{n}{n-1}, \binom{n}{n} \right).$$

The boundary of the n-simplex is

$$\partial\Delta = \langle [n] \setminus 1, [n] \setminus 2, \cdots, [n] \setminus n \rangle$$
$$f(\partial\Delta) = \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots, \binom{n}{n-1} \right)$$

5.2 Stanley-Reisner Ideals

Definition. Now let $S = K[x_1, \cdots, x_n]$. For $F \subset [n]$ let

$$x_F = \prod_{i \in F} x_i$$

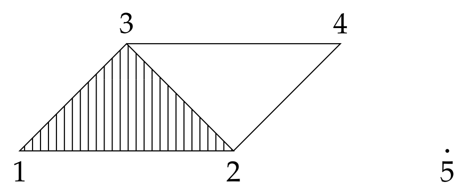
be a square-free monomial. For Δ a simplical complex we define the Stanley-Reisner ideal of Δ as

$$I_\Delta = (x_F \mid F \notin \Delta).$$

We define the Stanley-Reisner ring of Δ to be S/I_Δ.

Note that in the definition of the SR ideal we can just take minimal non-faces as generators.

Example 5.3. Consider Δ as before



Then we have

$$I_\Delta = (x_2x_5, x_4x_5, x_1x_5, x_3x_5, x_2x_3x_4, x_1x_4).$$

Theorem 5.1. *The correspondence*

$$\Delta \to S/I_\Delta$$

is a bijection between simplical complexes on $[n]$ and square-free monomial ideals in $K[x_1, \cdots, x_n]$. Moreover

$$I_\Delta = \bigcap_{F \in F(\Delta)} (x_i \mid i \notin F).$$

Example 5.4. With Δ as above

$$\begin{aligned} I_\Delta &= (x_2x_5, x_4x_5, x_1x_5, x_3x_5, x_2x_3x_4, x_1x_4) \\ &= (x_4, x_5) \cap (x_1, x_3, x_5) \cap (x_1, x_2, x_5) \cap (x_1, x_2, x_3, x_4) \end{aligned}$$

Now it's quite easy to see $V_\infty(I_\Delta)$.

Lemma 5.1. *$V_\infty(I_\Delta)$ is equidimensional iff Δ is pure.*

Definition. Δ is CM if S/I_Δ is CM.

Lemma 5.2. *Δ CM implies Δ pure.*

Proof. If Δ is not pure, then $V_\infty(I_\Delta)$ isn't equidimensional, so S/I_Δ isn't. $\square$

Definition. Let Δ be pure. A shelling of Δ is an ordering $F_1, \cdots, F_k$ of its facets such that each $\langle F_i \rangle \cap \langle F_1, \cdots, F_{i-1} \rangle$ is generated by a nonempty set of maximal proper faces of F_i (for $i > 1$).

Theorem 5.2. *If Δ is shellable then Δ is CM (over every field).*

Theorem 5.3. *CMness of simplical complexes is a topological property, eg Δ is CM iff*

$$\tilde{H}_i(\Delta; K) = 0 = H_i(\Delta, \Delta - p; K) \qquad \forall p \in \Delta; \forall i < \dim \Delta.$$

Theorem 5.4. *There is a triangulation of a tetrahedron that is not shellable (but a tetrahedron is CM).*

Definition. For $F \in \Delta$ we have the operations:

- deletion is $del(F, \Delta) = \{ G \in \Delta \mid G \cap F = \emptyset \}$
- link is $link(F, \Delta) = \{ G \in del(F, \Delta) \mid G \cup F \in \Delta \}$.

Definition. A pure simplical complex Δ is vertex-decomposable if either

1. $\Delta = \emptyset$
2. $\exists$ vertex $v \in \Delta$ such that both $del(v, \Delta)$ and $link(v, \Delta)$ are vertex-decomposable.

Theorem 5.5. *If Δ is vertex-decomposable then Δ is shellable, and hence CM.*

5.3 Class of Varieties - Generalized Determinental Varieties

We start on our quest to think about nice varieties.

Definition. We define the matrix of variables

$$Z = Z_n = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}$$

Definition. Consider a rank matrix

$$r = (r_{ij}) \qquad\qquad 1 \leq i, j \leq n$$

with each $r_{ij} \in \mathbb{N} \cup \{+\infty\}$. Then the northwest rank variety is

$$X_r = \{ M \in Mat(n \times n) \mid rank(M_{[i],[j]}) \leq r_{ij}, \forall i, j \}$$

where $M_{[i],[j]}$ is the northwest justified submatrix of M.

The ∞ is used to translate rectangular matrices into square matrices by putting $+\infty$ in the missing spots.

Remark. Lots of rank matrices define the same space of varieties.

Example 5.5.

$$X_{\left[\begin{smallmatrix} 4 & 1 \\ 3 & 7 \end{smallmatrix}\right]} = X_{\left[\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix}\right]} = X_{\left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right]} = X_{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]}$$

We move to a new form of matrices.

Definition. An alternating sign matrix (ASM) is an $n \times n$ matrix with $\{ 0, -1, 1 \}$ entries such that

1. each row and column adds up to 1

2. in each row and column, nonzero entries alternate in sign

The set of alternating sign matrices of size $n \times n$ is written $ASM(n)$.

Example 5.6. Any permutation matrix is an ASM. Another ASM is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Theorem 5.6.

$$n! \leq \left|ASM(n)\right| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Note that the left inequality comes from the fact that $Perm(n) \subset ASM(n)$.

To each ASM we associate a cornersum matrix.

Definition. Let A be an ASM. Then the cornersum matrix $r(A)$ is defined as

$$r(A)_{a,b} = \sum_{i=1}^{a} \sum_{j=1}^{b} A_{ij}.$$

Example 5.7. We have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow r(A) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow r(B) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Theorem 5.7. *For every rank matrix r there's a $d \in \mathbb{N}$ and an ASM A such that*

$$X_r \times K^d \cong X_{r(A)}.$$

Definition. A variety of the form $X_{r(A)}$ is called an ASM variety.

Remark. We can make $ASM(n)$ into a poset by adding the relation $A \geq B$ iff $r(A)_{ij} \leq r(B)_{ij} \forall i, j$.

Example 5.8. For all the elements of $ASM(3)$ we have this poset structure.

$$
\begin{tikzcd}
 & \underset{321}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dl, dash] \arrow[dr, dash] & \\
\underset{231}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dr, dash] & & \underset{312}{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dl, dash] \\
 & \underset{ASM}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dl, dash] \arrow[dr, dash] & \\
\underset{213}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dr, dash] & & \underset{132}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}} \arrow[dl, dash] \\
 & \underset{123}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}} &
\end{tikzcd}
$$

Theorem 5.8. *The poset $ASM(n)$ is a lattice, ie for any $A, B \in ASM(n)$ there's a least upper bound $A \vee B$ and a greatest lower bound $A \wedge B$. Here we have*

$$
\begin{aligned}
r(A \vee B)_{ij} &= \min \{ r(A)_{ij}, r(B)_{ij} \} , \\
r(A \wedge B)_{ij} &= \max \{ r(A)_{ij}, r(B)_{ij} \} .
\end{aligned}
$$

Definition. For $A \in ASM(n)$ we define

$$
Perm(A) = \{ \omega \in S_n \mid \omega \geq A \text{ and if } \omega \geq \nu \geq A \text{ for some } \nu \in S_n \text{ then } \omega = \nu \} .
$$

Definition. For $A \in ASM(n)$ we define the variety X_A to be the scheme of the ideal

$$
I_A = \sum_{i,j=1}^{n} I_{r(A)_{ij}+1}(Z_{[i][j]})
$$

where

$$I_k(Z_{[i][j]}) = (\text{determinants of all } k \times k \text{ submatrices of } Z_{[i][j]}).$$

Definition. A term order on Z is (anti)diagonal if the leading term of each minor is the product of the variables on the main (anti)diagonal.

Theorem 5.9. *For any antidiagonal order, the defining equations of I_ω are a Grobner basis for it.*

Corollary 5.1. *For any $\omega \in S_n$, I_ω is radical.*

Proof. By theorem, there is a Grobner degeneration from $V_\infty(I_\omega)$ to $V_\infty(inI_\omega) = V_\infty(I_\Delta)$ for some simplical complex Δ. Since I_Δ is radical, so is I_ω. □

5.4 Symmetric Stuff

The symmetric group S_n has generators

$$S_i = (i \quad i+1) \qquad \text{for } 1 \leq i \leq n-1$$

Now let Q be a string $q_1 \cdots q_m$ in the alphabet $[n-1]$. We can think of a substring of Q as a face in a simplical complex $([n])$.

Definition. We say a string $P = P_1, \cdots, P_k$ represents $\omega \in S_n$ if $\omega = S_{P_1} \cdots S_{P_k}$ and k is minimal (no smaller product for ω). We say that P contains ω if some substring of P represents ω.

Definition. The subword complex is the simplical complex

$$\Delta(Q, \omega) = \{\, Q \setminus P \mid P \text{ represents } \omega \,\}$$

(ie the facets are $F(\Delta) = \{\, Q \setminus P \mid P \text{ represents } \omega \,\}$).

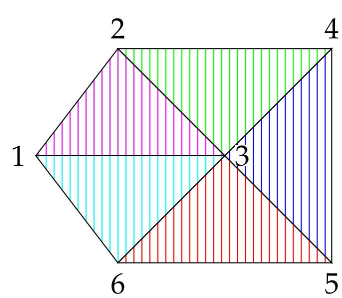
Example 5.9. We consider

$$Q = 123121 \in S_4; \omega = S_1 S_2 S_1 = 3214 = S_2 S_1 S_2$$

We have

1	2	3	4	5	6
1	2	3	1	2	1
1	2	3	1	2	1
1	2	3	1	2	1
1	2	3	1	2	1
1	2	3	1	2	1

$\longrightarrow$



Theorem 5.10. *Every subword complex $\Delta(Q, \omega)$ is vertex decomposable, thus shellable, thus CM.*

5.5 Pipe Dreams

As usual we start with a motivating example.

Example 5.10. Consider

$$\omega = 2143 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$r_w = \begin{bmatrix} \underline{0} & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & \underline{2} & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Note that only the underlined 0 and 2 are useful.

$$\begin{aligned} I_\omega &= \left(z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right) \\ inI_\omega &= (z_{11}, z_{13}z_{22}z_{31}) \qquad \text{(the last term is the antidiagonal)} \\ &= (z_{11}, z_{13}) \cap (z_{11}, z_{22}) \cap (z_{11}, z_{31}) \end{aligned}$$

This leads to 3 diagrams:

$$(z_{11}, z_{13}) \to \begin{array}{|ccc} \hline + & \cdot & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$(z_{11}, z_{22}) \to \begin{array}{|ccc} \hline + & \cdot & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$(z_{11}, z_{31}) \to \begin{array}{|ccc} \hline + & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ + & \cdot & \cdot \end{array}$$

This leads to the notion of a pipe dream.

Definition. A pipe dream is an arrangement P of $+$'s in an $n \times n$ grid. We label cells

$$\begin{array}{|ccccc} \hline 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n+1 & n+2 & \cdots & 2n-1 \end{array}$$

For each pipe dream P, we let $Q(P)$ be the string recording the labels of the $+$ positions, right to left, top to bottom.

Definition. The square word is

$$\begin{aligned} Q_{\square} &= Q(\text{pipe dream with all } +\text{'s}) \\ &= [(n)(n-1)(n-2)\cdots(2)(1)][(n+1)(n)(n-1)\cdots(3)(2)]\cdots[(2n-1)\cdots(n)]. \end{aligned}$$

A subword of $Q_{\square}$ corresponds to a pipe dream P. A pipe dream P with k $+$'s represents $\omega \in S_n$ if $\omega = S_{Q(P)_1} S_{Q(P)_2} \cdots S_{Q(P)_k}$ and ω can't be written as a product of fewer than k generators.

Example 5.11. With the pipe dreams from before we get

$$\begin{array}{|ccc} \hline + & \cdot & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \to 31 \to S_3 S_1 = 2143 = \omega$$

$$\begin{array}{|ccc} \hline + & \cdot & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & \cdot \end{array} \to 13 \to S_1 S_3 = 2143 = \omega$$

$$\begin{array}{|ccc} \hline + & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ + & \cdot & \cdot \end{array} \to 13 \to S_1 S_3 = 2143 = \omega$$

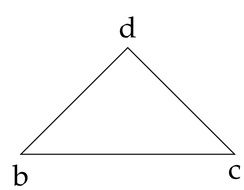
In fact these are exactly the three pipe dreams that represent $\omega = 2143$. Note that

$$\begin{array}{|ccc} \hline + & \cdot & + \\ \cdot & + & \cdot \\ + & \cdot & \cdot \end{array}$$

does not represent ω as it isn't minimal.

Now $\Delta(Q_{\square}, 2143)$ is 13 dimensional (with $16-2$ elements per facet) and has 3 facets. But all the info is contained in

$$\begin{array}{|ccc} \hline a & \cdot & b \\ \cdot & c & \cdot \\ d & \cdot & \cdot \end{array} =$$



but add 12 more vertices to each face.

Theorem 5.11. *For $\omega \in S_n$ and any antidiagonal term order*

$$\begin{aligned} inI_{\omega} &= I_{\Delta(Q_{\square},\omega)} \\ &= \bigcap_{P \in PD(\omega)} (z_{ij} \mid P \text{ has a } + \text{ in position } (i,j)) \\ &= \Big(\prod_{P \text{ has a } + \text{ in pos } (i,j)} z_{ij} \mid P \in PD(\omega) \Big). \end{aligned}$$

Corollary 5.2. *inI_ω is CM, thus I_ω is CM.*

Proof. inI_ω is CM as it's a Stanley-Reisner ideal for a vertex decomposable simplical complex. Then I_ω is CM as it Grobner degenerates to inI_ω. □

Corollary 5.3. *We have that*

- $codim\, R/I_\omega = codim\, R/I_{\Delta(Q_\square,\omega)} =$ *number of* $+$*'s in a PD representing* ω
- $deg\, R/I_\omega = deg\, R/I_{\Delta(Q_\square,\omega)} = \left|PD(\omega)\right| = \frac{1}{(l(\omega))!}\sum_{S_{a_1}\cdots S_{a_{l(\omega)}}=\omega \text{ reduced}} a_1\cdots a_{l(w)}.$

Example 5.12. $\omega = 2143$. Keeping in mind the three PDs for ω we get

$$deg(R/I_\omega) = \left|PD(2143)\right| = \frac{1}{2!}(1\cdot 3 + 3\cdot 1) = 3.$$

Example 5.13. $\omega = 3214 = S_1S_2S_1 = S_2S_1S_2$

$$deg(R/I_\omega) = \frac{1}{3!}(1\cdot 2\cdot 1 + 2\cdot 1\cdot 2) = 1$$

and indeed we only have one pipe dream (212) for ω.

Theorem 5.12. *For any simplical complex Δ the K-polynomial of S/I_Δ is*

$$\begin{aligned} K(S/I_\Delta) &= \sum_{\sigma\in\Delta} t^{|\sigma|}(1-t)^{n-|\sigma|} \\ &= \sum_{i=0}^{n} f_{i-1}t^i(1-t)^{n-i} \end{aligned}$$

where the f_{i-1} comes as i dimensional faces have $i-1$ elements.

Corollary 5.4. *We have that*

$$codim\, R/I_\omega = codim\, R/I_{\Delta(Q_\square,\omega)} = \left|PD(\omega)\right| = \frac{1}{(l(\omega))!}\sum_{S_{a_1}\cdots S_{a_{l(\omega)}}=\omega \text{ reduced}} a_1\cdots a_{l(w)}$$

The enumeration of pipe dreams is due to Macdonald, but the best proof is due to Pechenik.

5.6 Regularity

Theorem 5.13. *If S/I is CM then*

$$reg(S/I) = deg(K(S/I)) - codim(S/I).$$

Corollary 5.5. *We have that*

$$reg(S/I_\omega) = deg(K(S/I_\omega)) - l(\omega).$$

Definition. For Δ a simplical complex, the Alexander dual ideal is

$$\begin{aligned} I_\Delta^* &= (x^{[n]\setminus\sigma} \mid \sigma\in\Delta) = (x^{[n]\setminus F} \mid F\in F(\Delta)) \\ &= \bigcap_{\sigma\notin\Delta}(x_i \mid i\in\sigma). \end{aligned}$$

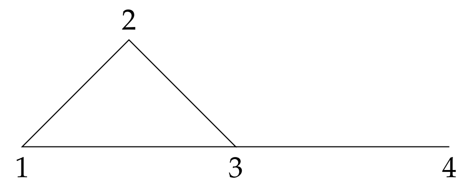
Definition. The Alexander dual complex to Δ is

$$\Delta^* = \{ [n] \setminus \sigma \mid \sigma \notin \Delta \}.$$

Lemma 5.3. $I_\Delta^* = I_{\Delta^*}$

Example 5.14. Consider the simplical complex given by

$$\Delta = \langle \{1,2\}, \{2,3\}, \{1,3\}, \{3,4\} \rangle$$

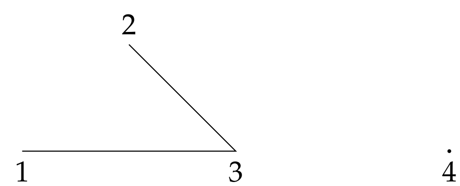


Then we have

$$\begin{aligned} I_\Delta &= (x_F \mid F \notin \Delta) = (x_1x_4, x_2x_4, x_1x_2x_3) \\ &= \bigcap_{F \in F(\Delta)} (x_i \mid i \notin F) = (x_3, x_4) \cap (x_2, x_4) \cap (x_1, x_4) \cap (x_1, x_2). \end{aligned}$$

Correspondingly we have

$$\Delta^* = \langle \{1,3\}, \{2,3\}, \{4\} \rangle$$



with

$$\begin{aligned} I_\Delta^* &= (x_{[n] \setminus F} \mid F \in \Delta) = (x_1x_4, x_2x_4, x_3x_4, x_1x_2) \\ &= \bigcap_{F \notin F(\Delta)} (x_i \mid i \in F) = (x_1, x_4) \cap (x_2, x_4) \cap (x_1, x_2, x_3). \end{aligned}$$

Definition. The 0-Hecke algebra H_n is a vector space with basis $\{ \Pi_\omega \mid \omega \in S_n \}$ (so is $n!$ dimensional) and with multiplication given by

$$\Pi_\omega \Pi_{S_i} = \begin{cases} \Pi_{\omega S_i} & \text{if } l(\omega S_i) > l(\omega) \\ \Pi_\omega & \text{else} \end{cases}$$

For a string $P = P_1 \cdots P_k$ in the alphabet $[n]$, $\delta(P) = \omega$ if $\Pi_{S_{p_1}} \cdots \Pi_{S_{p_k}} = \omega$.

Theorem 5.14. *For any Q and any ω we have that $\Delta(Q, \omega)$ is homeomorphic to either a ball or a sphere. Furthermore it is a sphere if $\delta(\omega) = \omega$, and is otherwise a ball. A face $Q \setminus P$ is in the boundary of the ball if $\delta(P) \neq \omega$.*

Corollary 5.6. *$\Delta(Q_\square, \omega)$ is always a ball (all PD's are balls; $Q_\square$ is too big to be a sphere).*

Lemma 5.4. *Let $\Delta = \Delta(Q, \omega)$. Then*

$$K(I_\Delta^*) = \sum_{P \subset Q; \delta(p)=\omega} (-1)^{|P|-l(\omega)} t^{|P|}.$$

Theorem 5.15. *For any simplical complex Δ*

$$K(S/I_\Delta) = G(I_\Delta^*)$$

Definition. Suppose that $P \subset Q$ represents ω, and $q_i \in Q \setminus P$ satisfies $\omega = \delta(P) = \delta(P \cup \{q_i\})$. Then there is a unique $p_j \in P$ such that

$$\delta(P) = \omega = \delta(P \cup \{q_i\} \setminus \{p_j\}).$$

We say that q_i is absorbable ("interior faces" ie faces where two things are glued together) if $j > i$. We define

$$abs(P) = \text{ number of absorbable letters of } Q \setminus P.$$

Theorem 5.16. *Let $\Delta = \Delta(Q, \omega)$. Then*

$$K(S/I_\Delta) = \sum_{P \subset Q; P \text{ reps } \omega} (1-t)^{|P|} t^{abs(P)}.$$

Corollary 5.7. *For any $\Delta = \Delta(Q, \omega)$*

$$reg(S/I_\Delta) = \max \{ abs(P) \mid P \subset Q; P \text{ reps } \omega \}.$$

Corollary 5.8. *We have that*

$$reg(S/I_\omega) = reg(S/I_{\Delta(Q_\square, \omega)}) = \max_{P \in PD(\omega)} \{ abs(\omega) \}.$$

Example 5.15. $reg(S/I_{2143}) = 2$.

Definition. Let $\omega \in S_n$. For each $i \in [n]$ find a largest increasing subsequence of $\omega(i)\omega(i+1)\cdots\omega(n)$ containing $\omega(i)$. Let r_i be the number of entries omitted. Then the Rajchgot index of ω is $raj(\omega) = \sum_{i=1}^n r_i$

Example 5.16. $raj(2143) = 2 + 1 + 1 + 0 = 4$.

Theorem 5.17. *For any $\omega \in S_n$ we have*

$$reg(S/I_\omega) = raj(\omega) - l(\omega).$$

Remark. Summarizing what we know for I_ω:

- S/I_ω is always CM
- $codim(S/I_\omega) = pd(S/I_\omega) = l(\omega)$
- $dim(S/I_\omega) = depth(S/I_\omega) = n^2 - l(\omega)$
- $deg(S/I_\omega) = |PD(\omega)| =$ some formula

- $reg(S/I_\omega) = raj(\omega) - l(\omega)$

Remark. Some open questions are:

- What about other Betti numbers of S/I_ω?
- What about the regularity of other complexes?
- For ASM which I_A are CM?

  Example 5.17. show example here

- What are Grobner bases for I_ω for other term orders?

For $a = a_1 \cdots a_k$ of distinct numbers, let $flat(n) = f_1 \cdots f_k \in S_k$ such that $f_i < f_j \iff a_i < a_j$ for all i, j. This is basically just labelling the elements in order using $[k]$.

Example 5.18. $a = 2, 7, -4, \pi, 0, 18 \to flat(a) = 3, 5, 1, 4, 2, 6.$

Definition. A permutation $\omega = \omega_1 \cdots \omega_n$ contains a permutation $p = p_1 \cdots p_k$ if ω has a subsequence $\omega' = \omega_{i_1} \cdots \omega_{i_k}$ such that $flat(\omega') = p$. If ω doesn't contain p then ω avoids p.

Theorem 5.18. *For any diagonal term order, the defining equations of I_ω are a Grobner basis iff ω avoids 2143.*

Definition. The CDG generators of I_ω are the 1×1 minors of the defining equations together with the minors of size $r(\omega)_{ij} + 1$ in ?????

Example 5.19. LATER

Theorem 5.19. *Fox any diagonal term order. Then the CDG generators of I_ω are a diagonal Grobner basis iff ω avoids*

$$\underbrace{13254, 21543}_{S_5}, \underbrace{214635, 215364, 315264, 215634}_{S_6}, \underbrace{4261735}_{S_7}$$

Remark. We don't know about

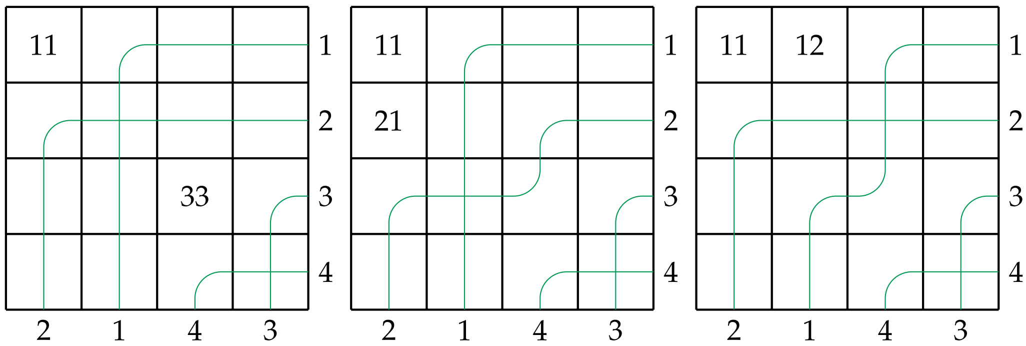
- other diagonal Grobner bases
- other term orders

Proposition 5.1. *Suppose all degree > 1 defining equations of I_ω come from a single position (i, j) in $r(\omega)$. Suppose further that $r(i, j) = \min(i, j) - 1$. Then the CDG generators of I_ω are a Grobner bases for every term order.*

5.7 Bumpless Pipe Dreams

Definition. A bumpless pipe dream is a tiling of the $n \times n$ grid with pipes, such that no tile has two pipes at once (except crossing in a plus), there are n pipes that start at the right and exit out the bottom and pairwise cross at most once.

Example 5.20. We have exactly three following BPDs representing 2413:



Notice that (this appeals to the example that i havent done yet)

$$\begin{aligned} in_{diag} &= (z_{11}, z_{33}z_{21}z_{12}) \\ &= (z_{11}, z_{33}) \cap (z_{11}, z_{21}) \cap (z_{11}, z_{12}) \end{aligned}$$

and the correspondence between the empty spaces in the BPDs and the ideals.

Theorem 5.20. *For many diagonal orders, the radical of the initial ideal*

$$\sqrt{in_{diag}(I_\omega)} = \bigcap_{P \in BPD(\omega} (z_{ij} \mid P \text{ has empty square in position } (i,j)).$$

Moreover, the degree of a component equals the number of BPDs with an empty square in those positions.

Example 5.21. For $\omega = 321654$ there are exactly two BPDs with the top left 6 boxes empty, and so one component has degree 2.

Remark. Some questions we still have:

- When is $in_{diag}(I_\omega)$ radical?
  - Probably pattern avoidance.
- When are there embedded components?
  - Possibly only when top degree fuzz but who knows.

We have reached the state of the art for this family of determinental varieties, now we will move to related topics.

5.8 Skew-Symmetric Matrices

We can consider skew-symmetric matrices (SSM) with similar NW rank conditions:

$$\{ X \in Mat_{n\times n} \mid X^t = -X \text{ and } rank(X_{[i],[j]}) \leq r(i,j) \} .$$

We require the diagonals to be all zero to deal with the $char(F) = 2$ edge case. The irreducibles are indexed by fixed point free involutions, eg $\omega \in S_n$ such that $\omega(\omega(i)) = i \neq \omega(i)$ for all i.

Theorem 5.21. *We have that*

- *The defining determinants don't generate a radical ideal. The radical is generated by a set of "Pfaffians" (square of determinants).*

- *There is a more complicated set of Pfaffians that is a GB for the radical with respect to a particular antidiagonal order.*
- *For this term order, the initial ideal is square-free and CM (as it is vertex-decomposable, which implies our original space is CM).*

For this term order, the prime decomposition of the initial ideal is governed by "fpf involution pipe dreams".

Remark. Some questions:

- What about other orders?
- What about regularity?
  - St Denis has some results in some cases.
- What about SSM with SW rank conditions?
  - This is super hard.
- What about symmetric matrices with NW rank conditions?
  - These aren't CM (351624)! These aren't "normal" either (ie super bad). - What about symmetric matrices with SW rank justification?

Theorem 5.22. *For symmetric matrices with SW rank conditions, the defining equations are a diagonal Grobner basis.*

Remark. The diagonal initial ideal is square-free.

Theorem 5.23. *CM since diagonal, in_{ideal} is Stanley-Reisner ideal of a type C subword complex.*

Remark. The facets here are given by type C pipe dreams. What about regularity?

Definition. Double determinental varieties are

$$D_{m,n,r,s,t} = \{\, (X_1, \cdots, X_r) \in Mat^r_{m\times n} \mid rank(X_1 \mid \cdots \mid X_r) \leq s, rank \begin{pmatrix} X_1 & \\ \vdots & \leq t \\ X_r & \end{pmatrix} \}\,.$$

Theorem 5.24. *The defining equations of $D_{m,n,r,s,t}$ generate a prime ideal. The defining equations are a GB under any (anti)diagonal term order. The initial ideal is a Stanley-Reisner ideal of a vertex decomposable simplical complex, so both ideals are CM.*

Definition. Let G be a simple graph on $[n]$. Let

$$I_G = (x_i x_j \mid ij \in E(G)) \subset K[x_1, \cdots, x_n]$$

be the edge ideal of G.

Proposition 5.2. *We have that*

$$\dim(S/I_G) = \alpha(G).$$

Definition. Let

$$reg'(S/I_G) = deg(K((S/I_G)) - codim(S/I_G).$$

Theorem 5.25. *For any $r, r' \geq 1$, there is a monomial ideal I with $reg(S/I) = r$ and $reg'(S/I) = r'$.*

Theorem 5.26. *In previous theorem, we can restrict to edge ideals and the result still holds. However if* $|V(G)| = n$ *then* $reg + reg' \leq n$*.*

Remark. Some questions:

- Fix $|V(G)| = n$. What pairs (reg, reg') can occur? Is it the integer points of a convex lattice polygon?
- Fix n, r, r'. What percentage of n-vertex graph ideals have $reg = r, reg' = r'$?