Math 239 Fall 2023 Tutorial Questions

Various Tutors

Fall 2023

Contents

1 Week 1 – Sep. 14/15

1. (**Combinatorial Identity 1**) Give a combinatorial proof (that is, by counting the same set in two different ways), of the following identity:

$$\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{k-1}{j} = 2^n - 1.$$

2. (**Counting Two Ways**) Give a combinatorial argument (that is, by counting the same set in two different ways), of the following identity:

$$7\binom{n}{7} = n\binom{n-1}{6}.$$

3. (**Multisets and Monomials**) A *monomial* in the n variables $x_1, x_2, ..., x_n$ is an expression of the form

$$x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$$

where each $a_i \in \mathbb{N}$. The *degree* of the monomial $x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ is the sum $\sum_{i=1}^{n} a_i$.

Give a bijection between the set of monomials of degree k in the variables $x_1, x_2, ..., x_n$ and the set of k-element multisets with elements of n types.

4. (**Evens and Odds**) For $n \geq 1$, give a bijecctive proof of the following identity:

$$\sum_{\text{k is even}} \binom{n}{k} = \sum_{\text{k is odd}} \binom{n}{k}.$$

For a bijective proof, you need to describe two sets A and B whose sizes correspond to the LHS and RHS, write down a function $f : A \to B$, show that $f(x) \in B$ for all $x \in A$, write down the inverse function f^{-1}, and show that $f^{-1}(y) \in A$ for all $y \in B$. In general you need to prove that f and f^{-1} are actually inverses of each other, but you will not have to do that in this problem.

2 Week 2 – Sep. 21/22

1. (**Even and Odd**) For any integer $n \geq 0$, determine the number of compositions of n with 3 parts where exactly 2 parts are even. You need to define a relevant set, a weight function, determine a generating series, and find an explicit formula for the answer.

2. (**Power Series**) Find the power series generating functions for each of the following sequences in closed form. Each sequence is defined for all $n \geq 0$.

   - $a_n = 3^n$
   - $a_n = 3 \cdot 7^n - 5 \cdot 4^n$

3. (**Some Less-Usual Coefficient Extraction**) Let a, k, l be constants. Find the following coefficients.

   (a) $[x^n]\left[\frac{1}{2-x^2}\right]$.

(b) $[x^n]\left[\frac{1}{1-x} \cdot \frac{1}{1-3x}\right]$.

(c) $[x^n]\left[(1+ax)^k\right]$.

(d) $[x^n]\left[(1+x^l)^k\right]$.

For the last one you might need to put on your Math 135 hat!

4. (**Sum Lemma**) Let $\mathscr{S} = \{1,2,3,4,5,6\}^4$ be the set of outcomes when rolling four six-sided dice. For $(a,b,c,d) \in \mathscr{S}$, define its weight to be $\omega(a,b,c,d) = a+b+c+d$. Consider the generating series $\Phi_{\mathscr{S}}(x)$ of $\mathscr{S}$ with respect to ω.

   (a) Explain why $\Phi_{\mathscr{S}}(x) = \left(\frac{x-x^7}{1-x}\right)^4$

   (b) How many outcomes in $\mathscr{S}$ have weight 19?

   (c) Let m, d, k be positive integers. When rolling m dice, each of which has exactly d sides (numbered with $1, 2, \ldots, d$ pips, respectively), how many different ways are there to roll a total of k pips on the top faces of the dice? (Part (b) is the case $m = 4, d = 6, k = 19$.)

5. (**Bonus Material: Formal Derivatives**) This is not 239 material and will not be covered in 239. You should not use this notation or train of thought for assignments or tests.

   Consider a generating function

$$A(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

   We define the *formal derivative* of $A(x)$ to be

$$\frac{\mathrm{d}}{\mathrm{d}x} A(x) = \sum_{i=1}^{\infty} i a_i x^{i-1} = 1a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

   (a) Find $[x^n]\left[\frac{\mathrm{d}}{\mathrm{d}x} A(x)\right]$.

   (b) Find $[x^n]\left[x \frac{\mathrm{d}}{\mathrm{d}x} A(x)\right]$.

   (c) Using the *formal derivative* as a "normal derivative", ie treating the derivative like you would in a first calculus course, find the generating function for the following sequences.

      i. $a_n = n$.

      ii. $a_n = n^2$.

      iii. $a_n = n^k$ for any fixed $k \geq 1$.

   Remark. The last one might be hard/impossible, but it turns out if you care only about the *asymptotic behaviour* of a_n, that is you want a_n to be approximately n^k for large enough n, it is much easier. A result from *analytic combinatorics* says that one example of a generating function for a sequence like this is $\frac{k!}{(1-x)^{k+1}}$ (in fact by the math software Sagemath, then $[x^{1000}]\frac{4!}{(1-x)^5} \approx 1.01004513028955 \cdot (1000)^4$, which shows that at $n = 1000$ there is only about a 1% error).

3 Week 3 – Sep. 28/29

1. (**Ambiguous Expressions**) Let S be the set of all strings where each block has even length.

   (a) Come up with an unambiguous expression for S. Justify in about a sentence why this is unambiguous.

   (b) Using your unambiguous expression, find the generating function for S (with respect to length). What do you notice?

   (c) Using your unambiguous expression, find the generating function for S with the weight function $w(0) = 1$ and $w(1) = 2$.

   (d) Come up with an *ambiguous* expression for S and try to find the generating function for S (with respect to length) using it. Compare the coefficients with your coefficients from (b). What do you notice? What will you notice in general?

2. (**Unambiguous Expressions**)

   (a) Provide an unambiguous regular expression R that produces the set of all binary strings that begin and end with the same bit (this includes the empty string). What rational function does R lead to?

   (b) Let $\mathcal{S}$ be the set of all strings where any occurrence of 1 must be immediately followed by at least 5 consecutive 0's. Find an unambiguous regular expression R that produces $\mathcal{S}$, and provide the rational function that R leads to.

3. (**Compositions**) Let $\mathcal{E}$ be the set of compositions $\gamma = (c_1, c_2, \cdots, c_k)$ of any length, in which part c_i is congruent to i (modulo 2). We consider empty composition $\epsilon = ()$ in the set $\mathcal{E}$.

   Express the set in terms of disjoint unions/Cartesian products, then obtain the generating series of the set with respect to size as a rational function.

4. (**Generating Series**) Determine the generating series, with respect to length, for the number of binary strings of length n which start with a 0, end with a 1, and do not contain 0 as a block. Write out the first four non-zero terms of the generating series and the corresponding strings.

5. (**Bonus Material: A Second GF for Binomial Coefficients**) This is not 239 material and will not be covered in 239. You should not use this notation or train of thought for assignments or tests. This is intended as a challenging question for people who are interested.

Single-variate generating functions are familiar to us at this point in the course. We can similarly define bi-variate generating functions as formal power series

$$A(x, y) = \sum_{i \geq 0} \sum_{j \geq 0} a_{i,j} x^i y^j.$$

A familiar GF to us is the GF for binomial coefficients with respect to "k"

$$\sum_{k \geq 0} \binom{n}{k} x^k = (1 + x)^n.$$

Using this, and the notion of a bi-variate generating function, find a closed form for the GF for binomial coefficients with respect to "n"

$$\sum_{n \geq 0} \binom{n}{k} y^n.$$

4 Week 4 – Oct. 05/06

1. (**Warm-up Recursive Expressions**)
   (a) Write both a regular expression and a recursive decomposition for the set of binary strings, where each block of 1's has even length. Find the generating function for both of these expressions and ensure that they are the same.
   (b) Write a recursive expression for the set of strings of length n such that if there is a 1 at position j there is a 0 at position $n+1-j$ and if there is a 0 at position j then there is a 1 at position $n+1-j$.
   For example consider a string
   $$a_1 a_2 a_3 \cdots a_{n-2} a_{n-1} a_n.$$
   Then if $a_2 = 0$ we must have that $a_{n+1-2} = a_{n-1} = 1$. Find the generating function of these. Is it possible to write this using a non-recursive expression? If yes give it, if not say why.
2. (**Partial Fractions**) Use partial fraction decomposition to produce a closed-form formula for the coefficients
   $$q_n = [x^n] \frac{18 + 12x}{1 - 9x^2}$$

3. (**Avoiding Substrings**) In this question we will use recursive decomposition to count the number of binary strings with length n that do not contain 011 as a substring.

   (a) Let S be the set of binary strings avoiding 011 as a substring, and let T be the set of binary strings containing exactly one occurrence of 011 at the very end. Find two equations of sets S and T. Justify your equations.

   (b) Deduce the generating series for S with respect to the length of the strings.

   (c) (Partial Fraction Challenge Part) Express the number of binary strings with length n that do not contain 011 as a substring as a function of n in closed form.

4. (**Recurrence**) Let a_n be the sequence which satisfies

$$a_n - 2a_{n-1} - 7a_{n-2} - 4a_{n-3} = 0$$

   for $n \geq 3$ with initial conditions $a_0 = 3$, $a_1 = 7$, $a_2 = 8$. Determine an explicit formula for a_n.

5. (**Counting Binary Trees and Dyck Paths**)

   Define a full binary tree to be a binary tree where each vertex has either 0 or 2 children. Often we can realize trees and other combinatorial structures recursively. Probably the most popular example is realizing that a full binary tree is either empty, or a vertex followed by two full binary trees.

   (a) Justify that this gives us the GF relationship

$$B(x) = 1 + x \cdot B(x) \cdot B(x) = 1 + xB(x)^2.$$

   Solve this for $B(x)$ and justify which of the two solutions is the legitimate one.

   (b) Using the generalization of binomial expansion in `https://en.wikipedia.org/wiki/Binomial_series`, find a closed-form expression for the number of binary trees with n vertices (ie find $[x^n]B(x) =: C_n$).

   After a lot of algebra this is very amazingly the Catalan numbers! These numbers pop up in the wild quite a lot.

5 Week 5 – Oct. 19/20

1. (**A Routine Recurrence**) Consider the recurrence

$$a_{n+3} + 5a_{n+2} + 3a_{n+1} - 9a_n = 0$$

   where $a_0 = 1, a_1 = 3, a_2 = 5$. Find an explicit formula for a_n. What is our formula for a_n if we set $a_0 = 3, a_1 = 9, a_2 = 15$ (ie triple what they were before) instead?

Note. You can use Wolfram Alpha to factor polynomials if you want.

2. (**Another Recurrence**) Let

$$A(x) = \frac{1}{1 - 3x - 2x^3}$$

and let

$$a_n = [x^n]A(x).$$

(a) What homogeneous linear recurrence is satisfied by the coefficients a_n?

(b) What are the initial conditions of this homogeneous linear recurrence?

(c) Invent a set of combinatorial objects such that the number of objects of size n is a_n.

3. (**Degree Sequences**) Suppose that $d_1, d_2, \ldots, d_n$ are the degrees of a graph G, ordered so that $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $d_1, d_2, \ldots, d_n$ is called the **degree sequence** of G. For each of the following degree sequences, either draw a graph with the degree sequence, or explain why it can't exist.

(a) $4, 3, 2, 2, 1$

(b) $6, 5, 4, 3, 2, 1$

(c) $5, 4, 4, 3, 2, 1$

(d) $3, 3, 3, 3, 3, 3$

(e) $2, 2, \ldots, 2$ for arbitrary length n

(f) $6, 6, 4, 2, 2, 2, 1, 1$

4. (**Graph Complements**) Recall that if G is a graph $\overline{G}$, the complement of G, has $V(G) = V(\overline{G})$ and $E(\overline{G} = \{uv | uv \notin E(G), u, v \in V(\overline{G})\}$.

   (a) Let K_n be the complete graph on n vertices. Describe $\overline{K_n}$.

   (b) For any $v \in G$ what is $deg_G(v) + deg_{\overline{G}}(v)$?

   (c) Find a graph G such that $G \cong \overline{G}$.

   (d) Prove that if $|V(G)| \geq 6$ then either G or $\overline{G}$ contains a triangle.

5. (**Bonus Question: Fibonacci Recurrences**) Consider the Fibonacci recurrence given by $f_{n+2} = f_{n+1} + f_n$, where $f_0 = 0$ and $f_1 = 1$. As we know from class and partial fraction decomposition, this has the closed form

$$f_n = \frac{(1+\sqrt{5})^n}{2^n\sqrt{5}} - \frac{(1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

   Notice that the second term goes to 0 as $n \to \infty$.

   (a) Find a general formula for the Fibonacci-like recurrence given by $f_{n+2} = f_{n+1} + f_n$, where $f_0 = a$ and $f_1 = b$, where $a, b \in \mathbb{R}$. Then the set of initial conditions is $\mathbb{R}^2$. Conclude that there is a one-dimensional subspace of $\mathbb{R}^2$ such that $f_n \to 0$ when the initial conditions are in this subspace (ie there is a line of possible initial values that make f_n tend to 0). Compute the line orthogonal to this one. What happens to our sequence when our initial conditions are in this orthogonal line?

   (b) (hard) Find a closed form for the Tribonacci numbers, defined by $f_{n+3} = f_{n+2} + f_{n+1} + f_n$, where $f_0 = a_0, f_1 = a_1, f_2 = a_2$. Then the set of initial conditions is $\mathbb{R}^3$. This has a (possibly trivial) subspace of initial conditions that make $f_n \to 0$. What is the dimension of it?

6 Week 6 – Oct. 26/27

1. (**Matching Vertices**) Prove that any graph with at least two vertices has two vertices of the same degree.

   Note. It may be useful to consider the degree sequence of the graph.

2. (**Paths and Cycles**) Let G be a graph and $k \geq 2$ be the minimum degree of G (denoted $\delta(G)$.

   - Prove G has a path of length at least k.
   - Prove G has a cycle of length at least $k + 1$.

3. (**Vertices and Edges**) Let G be a graph with n vertices and m edges, where $m > \frac{(n-1)(n-2)}{2}$. Show that G does not have a vertex of degree 0

4. (**Bipartite Graph and Walks**) For each even $n \geq 2$, let G_n be the graph with vertex set

$$V(G_n) = \{\sigma \in \{0,1\}^n : \sum_{i=1}^{n} \sigma_i = n/2\}$$

and such that there is an edge between vertices $x, y \in V(G)$ if and only if y is obtained from x by replacing a substring 10 with 01 or vice versa. For example, $\{1100, 1010\}$ and $\{1010, 1001\}$ are some edges in G_4.

   (a) Prove G_n is bipartite.

   (b) Show that there exists a walk between any two vertices in G_n, and thus prove that G_n is connected.

5. (**Moore Graphs**) We often try to import the language for geometric shapes over to graphs. In this light we define the **distance** between two vertices as

$$\text{dist}(u, v) = \min_{\gamma \in path(u,v)} \text{length}(\gamma)$$

where $path(u, v)$ is the set of all paths from u to v, and the length of a path is the number of edges you cross in the path. We define the **diameter** of a graph to be

$$diam(G) = \max_{u,v \in G} \text{dist}(u, v).$$

This is a measure of how "spread out" our graph is; if our graph has lots of edges you expect the diameter to be high, and if the graph has few edges you expect a low diameter.

Let G be d-regular and have diameter k. Show that

$$|V(G)| \leq 1 + d \sum_{i=1}^{k} (d-1)^{i-1}.$$

A graph in which we achieve equality is known as a **Moore graph**. Some examples of Moore graphs are odd cycles, complete graphs, and the Petersen graph.

Math 239 Fall 2023 Tutorial Review Session
2023 Nov. 02/03

7 Midterm Review – Nov. 02/03

1. (**Bijection Question**) Let $n \geq k \geq 2$. Let

$$U = \{k\text{-element subsets of } \{1, \cdots, n\}\}$$

and

$$V = \{(a, b, A)\}$$

where

$$\begin{aligned} &a \in \{1, \cdots, n\}, \\ &b \in \{0, \cdots, n-a\}, \\ &A \text{ is a } (k-2) \text{ element subset of } \{1, \cdots, b-1\}\ . \end{aligned}$$

(I.e. V is the set of all triples like this). Find a bijection from U to V and use this to show that

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} \binom{j-1}{k-2} = \binom{n}{k}.$$

solution: Consider a subset S in U. Denote the minimal/maximal element of S as $S_{min} \in \{1, \cdots, n\}$ and $S_{max} \in \{S_{min}, \cdots, n\}$. Then

$$S \subseteq \{S_{min}, \cdots, S_{max}\}$$

with $S_{min}, S_{max} \in S$. Write

$$T = S \setminus \{S_{min}, S_{max}\} \subseteq \{S_{min} + 1, \cdots, S_{max} - 1\}.$$

T has $k-2$ elements. Now write

$$\begin{aligned} &a = S_{min} \in \{1, \cdots, n\}, \\ &b = S_{max} - S_{min} \in \{0, \cdots, n - S_{min}\} = \{0, \cdots, n-a\}, \\ &A = T - S_{min} \subseteq \{S_{min} - S_{min} + 1, \cdots, S_{max} - S_{min} - 1\} = \{1, \cdots, b-1\}. \end{aligned}$$

Where our $T - S_{min}$ is element-wise subtraction. Then (a, b, A) is a valid element of V. Note that this is clearly bijective, with the inverse being

$$(a, b, A) \mapsto \{a\} \cup (A + a) \cup \{a + b\}$$

with the $A + a$ being once again element-wise addition.

Now note that

$$|V| = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \binom{j-1}{k-2}$$

where $\binom{j-1}{k-2}$ counts number of ways to get A, $\sum_{j=0}^{n-i} \binom{j-1}{k-2}$ counts the number of ways to get b and A, and $\sum_{i=1}^{n} \sum_{j=0}^{n-i} \binom{j-1}{k-2}$ counts the ways to count elements of T. It follows that

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} \binom{j-1}{k-2} = |V| = |U| = \binom{n}{k}.$$

2. (**Coefficient Extraction**) Determine the value of the following coefficients.

   (a) $[x^{20}]3x^2(1-5x^3)^{-4}$

   (b) $[x^{20}]3x^2(1-5x^3)^{-4}(1+x^4)^{-2}$

solution: (a) Apply negative binomial theorem:

$$\begin{aligned}
[x^{20}]3x^2(1-5x^3)^{-4} &= 3[x^{18}](1-5x^3)^{-4} \\
&= 3[x^{18}]\sum_{n\geq 0}\binom{n+4-1}{4-1}(5x^3)^n \\
&= 3[x^{18}]\sum_{n\geq 0}\binom{n+3}{3}5^n x^{3n} \\
&= 3\binom{6+3}{3}5^6 \\
&= 3\binom{9}{3}\cdot 5^6
\end{aligned}$$

(b)

$$\begin{aligned}
[x^{20}]3x^2(1-5x^3)^{-4}(1+x^4)^{-2} &= 3[x^{18}](1-5x^3)^{-4}(1+x^4)^{-2} \\
&= 3[x^{18}]\left(\sum_{n\geq 0}\binom{m+4-1}{4-1}(5x^3)^m\right)\left(\sum_{n\geq}\binom{n+2-1}{2-1}(-x^4)^n\right) \\
&= 3[x^{18}]\left(\sum_{n\geq 0}\binom{m+3}{3}5^m x^{3m}\right)\left(\sum_{n\geq 0}(n+1)(-1)^n x^{4n}\right) \\
&= 3\left(\binom{6+3}{3}5^6(0+1)(-1)^0+\binom{2+3}{3}5^2(3+1)(-1)^3\right) \\
&\quad (\text{taking } 3m+4n=18, (m,n)=(6,0), \text{ or } (2,3)) \\
&= 3\left(\binom{9}{3}5^6-\binom{5}{3}5^2\cdot 4\right).
\end{aligned}$$

3. (**Weight Functions**) Are the following functions weight functions?

   (a) With $S=\mathbb{Z}$, $\omega(n)=n^2$.

   (b) With $S=\mathbb{N}$, $\omega(n)=\sqrt{n}$.

   (c) With $S=\mathbb{N}$, $w(n)=n \mod k$ for some fixed $k\in\mathbb{N}$.

solution: Recall that to be a weight function, a finite number of elements must map to each natural number.

   (a) Yes, since for each natural number at most 2 elements map to it.

   (b) No, since for example $\sqrt{2}\notin\mathbb{N}$.

   (c) No, since $0,k,2k,3k,\cdots$ all map to 0, so the pre-image of 0 is infinite.

4. (**Binary Substrings**)

(a) Give an unambiguous *regular* expression for the set X of all non-empty binary strings that start and end with 1. Use your regular expression to determine the generating series of X with respect to length.

(b) Give an unambiguous *recursive* expression for the set Y of all non-empty binary strings of at least 3 blocks, where the string starts and ends with a 0, and where the first and last blocks have the same size. You solution might make use of part (a) of this question. Use your recursive expression to determine the generating series for Y with respect to length.

solution: (a) An unambiguous regular expression for X is

$$1 \smile 1(0 \smile 1)^* 1$$

The generating series with respect to length is

$$\Phi_X(x) = \Phi_1(x) + \Phi_1(x)\Phi_{(0\smile 1)^*}(x)\Phi_1(x) = x + \frac{x^2}{1-2x} = \frac{x-x^2}{1-2x}.$$

(b) An unambiguous recursive expression for Y is

$$Y = 0(Y \smile X)0$$

The generating series with respect to length satisfies

$$\begin{aligned}\Phi_Y(x) &= \Phi_0(x)(\Phi_Y(x) + \Phi_X(x))\Phi_0(x) \\ &= x^2\left(\Phi_Y(x) + \frac{x-x^2}{1-2x}\right) \\ &= x^2\Phi_Y(x) + \frac{x^3-x^4}{1-2x}\end{aligned}$$

which implies

$$\Phi_Y(x) = \frac{x^3-x^4}{(1-2x)(1-x^2)}.$$

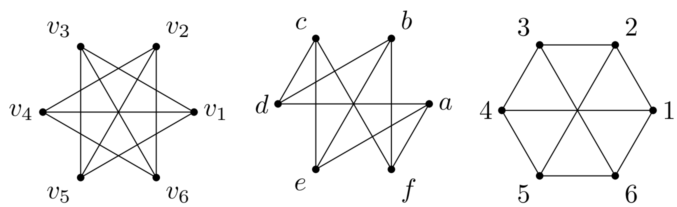
5. (**Isomorphism of Graphs**) Which of the following graphs are isomorphic? Provide justification.

   (a) G_1 is the complement of a 6-cycle.

   (b) G_2 is the graph with $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{ad, ae, af, bd, be, bf, cd, ce, cf\}$.

   (c) G_3 is the graph with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{ij | i + j$ is odd $\}$.

solution: Note: I recommend drawing out all three graphs and making the observation that G_2 is $K_{3,3}$. Here are G_1, G_2, G_3 respectively.



G_1 is not isomorphic to G_2 since G_1 contains a triangle and G_2 is bipartite. Recall that bipartite graphs contain no odd cycles. G_2 and G_3 are isomorphic. Take for example the function sending $a \to 1$, $b \to 3$, $c \to 5$, $d \to 2$, $e \to 4$, and $f \to 6$. Finally by transitivity of isomorphism G_1 and $G)_3$ are not isomorphic.

6. (**Partial Fractions**) Use partial fractions to determine and explicit formula for $[x^n]F(x)$ where $F(x) = \frac{1-6x+x^2}{(1-4x)(1-x)^2}$.

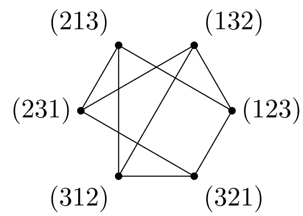
solution: Observe the denominator is already factored so we have $\frac{1-6x+x^2}{(1-4x)(1-x)^2} = \frac{A}{(1-4x)} + \frac{Bx+C}{(1-x)^2}$. Solving for the constants we get $A = \frac{-7}{9}$, $B = \frac{-4}{9}$ and $C = \frac{16}{9}$. This gives $[x^n]F(x) = \frac{-7}{9}4^n + \frac{-4}{9}n(1)^n + \frac{16}{9}(1)^n = \frac{-7}{9}4^n + \frac{-4}{9}n + \frac{16}{9}$

7. (**Graph Construction**) For $n \geq 2$, let G_n be the graph whose vertices are permutations of $\{1, \ldots, n\}$, where we have an edge between two permutations if we can obtain one from the other by swapping two elements.

   (a) Draw G_3 and label its vertices.

   (b) Show that this graph is regular, find the degree.

solution: (a) Using the notation $\sigma = (\sigma_1, \sigma_2, \sigma_3)$:



(b) Degree should be $\binom{n}{2}$ because a vertex v will be adjacent to permutations that are obtained from v by two elements swapping places, so we pick 2 elements to swap out of n.

8. (**Compositions**) How many integer compositions of n consist of either 5 or 6 parts, where each part is even?

solution: For a partition with only even parts, we want $A = \{2, 4, 6, \dots\}$ with weight function $w(a) = a$, which gives us generating series $\Phi_A(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2}$.

The set we are counting is $A^5 \cup A^6$, which is a disjoint union, so we can apply the sum lemma to get $\Phi_S(x) = \Phi_{A^5}(x) + \Phi_{A^6}(x)$.

By the product lemma, we have

$$\Phi_{A^5}(x) = (\Phi_A(x))^5 = (\tfrac{x^2}{1-x^2})^5 = \tfrac{x^{10}}{(1-x^2)^5}$$
$$\Phi_{A^6}(x) = (\Phi_A(x))^6 = (\tfrac{x^2}{1-x^2})^6 = \tfrac{x^{12}}{(1-x^2)^6}$$

So

$$\Phi_S(x) = \tfrac{x^{10}}{(1-x^2)^5} + \tfrac{x^{12}}{(1-x^2)^6} = \tfrac{x^{10}-x^{12}+x^{12}}{(1-x^2)^6} = \tfrac{x^{10}}{(1-x^2)^6}$$

The number of integer compositions of n consisting of 5 or 6 even parts is

$$[x^n]\Phi_S(x) = [x^n]\tfrac{x^{10}}{(1-x^2)^6} = [x^{n-10}](1-x^2)^{-6} = \begin{cases} 0 & n \text{ is odd} \\ \binom{\frac{n-10}{2}+5}{5} & n \text{ is even} \end{cases}$$

8 Week 8 – Nov. 09/10

1. (**Exotic Cubes**) Let $C_{n,m}$ be the graph where the vertices are the elements of $\{0, 1, \cdots, m-1\}^n$, and where there is an edge between v_1 and v_2 iff v_1 and v_2 differ by exactly 1 in exactly one coordinate. For example if $n = 2$ and $m = 3$ the vertex $(1, 2)$ to the vertices $(0, 2), (2, 2), (1, 1)$.

   (a) Draw $C_{2,3}$.

   (b) Is $C_{n,m}$ bipartite?

   (c) What is the maximum degree of a vertex in $C_{n,m}$? What about the minimum?

   (d) (hard, attempt other questions first) What is the number of edges in $C_{n,m}$?

2. (**Find the Mistake**) The following is a false statement and a false proof. Identify the error in the provided proof and consider how to avoid making this mistake in general. Can you think of a way to make the theorem true? If so, prove it.

   Non-theorem: If G is a graph with $|V(G)| \geq 4$ and minimum degree at least 2, then G contains a cycle of length at least 4.

   Non-proof: We proceed by induction on $|V(G)|$. We begin with the base case: Let G be a graph with four vertices and minimum degree at least 2. Suppose for a contradiction that G does not contain a cycle of length 4. Since G has minimum degree at least two, it follows that G contains a cycle: and since G does not contain a cycle of length 4, it therefore contains a cycle of length 3. Let v be the vertex not in this cycle of length three. Since G has minimum degree at least 2 and $|V(G)| = 4$, it follows that v is adjacent to at least two vertices in this cycle. Then G does contain a cycle of length 4, a contradiction. Thus the base case holds. To prove the inductive step, let H be a graph on $n-1$ vertices for which the theorem holds. Construct a new graph G on n vertices by adding a new vertex to H with at least two incident edges. Since H contains a cycle of length 4 by induction, G contains a cycle of length 4 and the result holds.

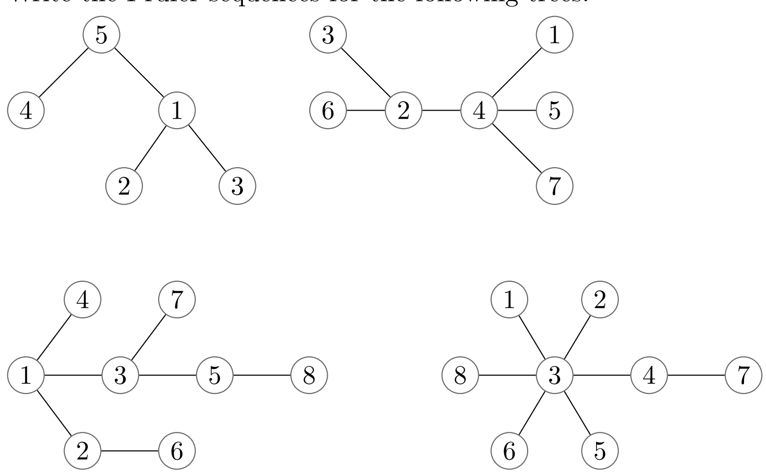
3. (**Prüfer Codes**) Let $n \geq 3$, and suppose we have a tree on vertex set $\{1, \ldots, n\}$. We can construct a sequence of elements from $\{1, \ldots, n\}$ algorithmically as follows:

   1) Check to make sure the tree has more than a single edge. If not, terminate.

   2) Find the leaf with the smallest label. Add the label of its neighbor to the sequence. Delete the leaf from the graph and repeat.

   Notice that this will produce a sequence of length $n-2$, as we're terminating the algorithm when we have a single edge (two vertices) left. The sequence we get from this algorithm is called a Prüfer sequence.

   (a) Write the Prüfer sequences for the following trees:



   (b) Can any sequence of $n-2$ numbers from $\{1, \ldots, n\}$ be the Prüfer sequence of a tree on n vertices? Construct an algorithm for building a labeled tree out of a given sequence, or explain why it's not possible.

(c) Use Prüfer sequences to count the number of labelled trees on n vertices. (Hint: The algorithm you wrote in part (b) gives you a mapping that is the inverse of the mapping from labeled trees to their Prüfer sequences. You do not need to prove that it's a bijection.)

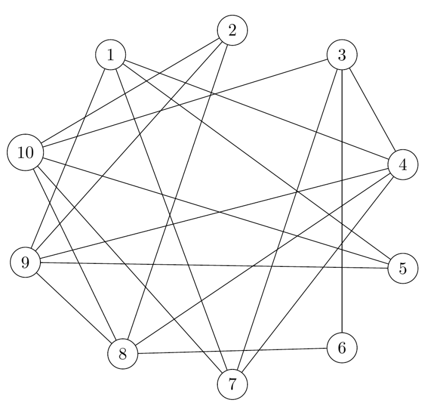
(d) (hard, attempt other questions first) Prove that the algorithm gives us a bijective mapping.

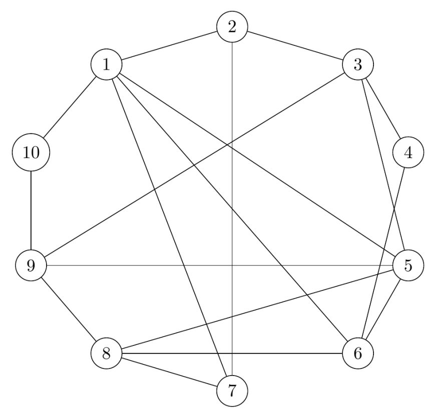
4. (**Spanning Tree**) Let e be an edge in a connected graph G. Prove that e is a bridge in G if and only if e is in every spanning tree of G.

5. (**Bonus: Complete k-partite Graphs**) We say a graph $G = (V, E)$ is **k-partite** if we can partition $V = V_1 \sqcup \cdots \sqcup V_k$ ($\sqcup$ indicates *disjoint* union, and the V_j are called *parts*) such that there are no edges between distinct elements of V_j (where $j = 1, \cdots, k$). Find a tight bound on the number of edges in a k-partite graph with kn vertices, and say when this tight bound occurs.

9 Week 9 – Nov. 16/17

1. (**Planar or Not**)For each of the following graphs, find a planar embedding or an edge subdivision of K_5 or $K_{3,3}$. In the latter case, mark the edges in the subdivision of K_5 or $K_{3,3}$.





2. (**Euler's Formula**) Suppose G is a connected planar embedding where every vertex has degree 2 or 5 and every face has degree 4.

(a) Determine a formula for the number of edges in terms of the number of vertices of degree 5.

(b) Suppose there are exactly 4 vertices of degree 5. Determine the number of edges, faces, and vertices of degree 2. Draw a planar embedding that satisfies these parameters.

3. (**Complete Planar Tripartite Graphs**) For positive integers $a \geq b \geq c$ define $K_{a,b,c}$ to be the **complete tripartite graph** with a tripartition A, B, C (with $|A| = a$ etc.) such that $uv \in E$ iff u and v are in different parts. What are the possible values of (a, b, c) such that $K_{a,b,c}$ is planar? Ignore the trivial cases where any of a, b, c are 0.

4. (**Partitioning Into Planar**) Let $n \geq 6$. Prove that the it is not possible to partition the edges of K_n into $\lfloor \frac{n}{6} \rfloor$ planar subgraphs.

5. (**Bonus Question: The Pentagon Problem**) Let G be a 5-cycle (a pentagon) with vertices enumerated $1, 2, 3, 4, 5$ in a circle. Let $c = (c_1, c_2, c_3, c_4, c_5)$ denote certain numbers we put on vertex $1, 2, 3, 4, 5$ respectively (so vertex 3 has the value c_3 on it). We let $c_j \in \mathbb{Z}$, but we requires that $c_1 + c_2 + c_3 + c_4 + c_5 \geq 1$. Now play the following game:

   > If we have all $c_j > 0$ then we have won. Otherwise pick a vertex j with $c_j < 0$. Then "activate" the vertex to get a new configuration c' with
   >
   > $$c'_i = \begin{cases} c_i & \text{if } i \text{ is not adjacent to } j, \\ c_i + c_j & \text{if } i \text{ is adjacent to } j, \\ c_i - 2c_i & \text{if } i = j. \end{cases}$$
   >
   > Then continue the game from the start with the configuration c'.

   Show that we always win, so we always reach a state where $c_j > 0$ for all initial c's.

   This problem is sourced from *The Mathematics of Chip-Firing* by Klivans (problem 2.8.17). I have no idea how to do it.

10 Week 10 – Nov. 23/24

1. (**Planarity and Graph Complements**)

   (a) For each $k \in \{5, 6\}$ give two non-isomorphic graphs G_1 and G_2 with k vertices each such that $G_1, G_2, \overline{G_1}, \overline{G_2}$ are planar. Prove they are planar(give a planar embedding) and give a one sentence proof that G_1 and G_2 are non-isomorphic.

   (b) Describe a graph G where $\overline{G}$ and G are both non-planar.

2. (**Cartesian Product of Graphs**) The *Cartesian Product* of two graphs G and H, denoted $G \square H$ is defined as

   $$V(G \square H) = V(G) \times V(H)$$

   and if two vertices (a_1, b_1) and (a_2, b_2) are adjacent in $G \square H$ if and only if either

- $a_1 = a_2$ and b_1 is adjacent to b_2 in H, or
- $b_1 = b_2$ and a_1 is adjacent to a_2 in G,

i.e.

$$E(G\Box H) = \{(a_1,b_1)(a_2,b_2) : a_1 = a_2 \text{ and } b_1b_2 \in E(H), \text{ or, } a_1a_2 \in E(G) \text{ and } b_1 = b_2\}$$

(a) Suppose that ϕ_1 is a k-coloring of G (assigning colors $\{1, \cdots, k\}$) and ϕ_2 is a k-coloring of H (assigning colors $\{1, \cdots, k\}$) for some positive integer k. Define

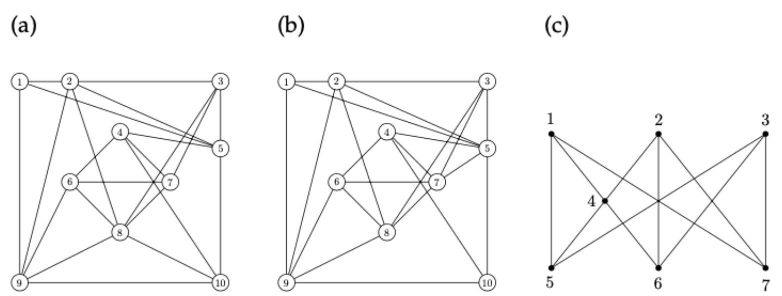
$$\phi((a,b)) := \phi_1(a) + \phi_2(b) \mod k.$$

Prove that ϕ is a k-coloring of $G\Box H$.

(b) Deduce that if G is k_1-colorable and H is k_2-colorable, then $G\Box H$ is $\max\{k_1, k_2\}$-colorable.

3. (**Graph Coloring**) Let $k \geq 1$. Let G be a graph where every nonempty subgraph has a vertex of degree at most k. Prove that G is $(k+1)$-colourable. Find an example of G where G is not k-colourable.

4. (**A Second Planarity Question**) Determine whether or not each of the following graphs is planar. If so, give a planar embed- ding. If not, exhibit an edge subdivision of K_5 or $K_{3,3}$ in the graph.



5. (**Bonus Question: Spectral Graph Theory**) The Laplacian operator on $\mathbb{R}^n$, which acts on functions $\mathbb{R}^n \to \mathbb{R}$ by sending f to $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$. The Laplacian operator notionally measures how much f at p differs from points near p. It turns out this plays a large role in a theory of mathematics called spectral theory, which is concerned with finding functions f such that $\Delta f = \lambda f$ (so-called "eigenfunctions").

   There is a graph theory equivalent of the Laplacian. Let G be a graph with n vertices, labelled $1, \cdots, n$. Then Define the **degree matrix** D_G of G as the $n \times n$ matrix

$$D_G(i,j) = \begin{cases} \deg(v_i) & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Define as well the **adjacency matrix** A_G of G as the (symmetric) $n \times n$ matrix

$$A_G(i,j) = \begin{cases} -1 & \text{if } v_i v_j \in E, \\ 0 & \text{else.} \end{cases}$$

Now finally define the **graph Laplacian** of G as $L = L_G = D_G - A_G$. Since this is an $n \times n$ real symmetric matrix, there are n real eigenvalues (counted with multiplicity). This is exactly the provenance of **spectral graph theory**. We call the multiset of eigenvalues the **spectrum** of a graph, and often write them in increasing order $\lambda_1 \leq \cdots \leq \lambda_n$ (so that λ_1 is the lowest eigenvalue of G).

(a) Argue that for any G then $L(1, \cdots, 1)^T = 0$, and so 0 is always an eigenvalue of G.

(b) Find the spectrum of K_n, $K_{n,n}$, P_n, C_n, the star graph S_n, and the n-cube Q_n.

(c) Prove that if G is connected then $\lambda_1 > 0$. Further show that if $\lambda_i = 0$ and $\lambda_{i+1} \neq 0$ then G has exactly $i+1$ connected components.

Math 239 Fall 2023 Tutorial Review Session
2023 Nov. 30/Dec. 1

11 Final Review – Nov. 30/ Dec. 01

1. (**Compositions**) Find the generating function for number of compositions with at least 3 parts where each part is an even number greater than 3.

solution: Each part has generating function

$$P(x) = x^4 + x^6 + x^8 + \cdots = \frac{x^4}{1-x^2}.$$

Then the GF for the whole thing is

$$C(x) = P(x)^3 + P(x)^4 + P(x)^5 + \cdots = \frac{P(x)^3}{1-P(x)} = \frac{x^{12}}{1-3x^2+2x^4+x^6-x^8}.$$

2. (**Sequences**) Give the generating functions of the following sequences. Give a recursion that these sequences satisfy.

(a) $a_n = 3 + 3(2^n)$.

(b) $b_n = \lfloor \frac{n}{2} \rfloor$ (so $0, 0, 1, 1, 2, 2, \cdots$).

solution: (a) $A(x) = 3\sum x^n + 3\sum (2x)^n = \frac{3}{1-x} + \frac{3}{1-2x} = \frac{6-9x}{2x^2-3x+1}$. Then the recursion with characteristic polynomial $2x^2 - 3x + 1$ is

$$a_n = 3a_{n-1} - 2a_{n-2}, \qquad a_0 = 6, a_1 = 9.$$

(b) First recall/note that $\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$.

$$\begin{aligned}
B(x) &= 0x^0 + 0x + 1x^2 + 1x^3 + 2x^4 + 2x^5 + \cdots \\
&= \left(0x^0 + 1x^2 + 2x^4 + \cdots\right) + x\left(0x^0 + 1x^2 + 2x^4 + \cdots\right) \\
&= \sum_{n\geq 0}\left(nx^{2n}\right) + x\sum_{n\geq 0}\left(nx^{2n}\right) \\
&= \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^2)^2} \\
&= \frac{x^2}{x^3 - x^2 - x + 1}
\end{aligned}$$

This gives a recursion of

$$a_n = a_{n-1} + a_{n-2} - a_{n-3}, \qquad a_0 = 0, a_1 = 0, a_2 = 1.$$

3. (**Trees**) Let T be a tree with $|V(T)| \geq 2$ and no vertices of degree two. Prove that more than half of the vertices of T are leaves.

solution: Let $n = |V(T)|$. Since T is a tree $|E(T)| = n - 1$. By the handshaking lemma,

$$\sum_{v\in V(T)} deg(v) = 2|E(T)| = 2(n-1).$$

Let L be the set of leaves in T and $|L| = l$. Since T has no vertices of degree two

$$l + \sum_{v\notin L} deg(v) = 2n - 2 \geq l + 3(n-l) = 3n - 2l.$$

Hence $2l \geq n + 2$ and thus $l \geq \frac{n}{2} + 1$ and the result follows.

4. (**Cycles**) Let G be a connected graph on $n \geq 1$ vertices with n edges.

   (a) Prove that the average degree of the vertices of G is 2.

   (b) Prove that G has a unique cycle.

solution: (a) By the handshaking lemma, $\sum_{v\in V(G)} deg(v) = 2|E(G)| = 2n = 2|V(G)|$. Thus $\frac{1}{|V(G)|}\sum_{v\in V(G)} deg(v) = 2$.

(b) Since $n > n-1$, G is not a tree. Thus G contains a cycle. For the sake of contradiction suppose G contains two distinct cycles C_1 and C_2. Since the cycles are distinct C_1 contains some edge e_1 such that e_1 is not in C_2 and C_2 contains some edge e_2 such that e_2 is not in C_1. Since e_1 and e_2 are in cycles, e_1 and e_2 are not bridges in G. Since e_1 is not a bridge $G - e_1$ is still connected. Since e_1 is not in C_2, C_2 is still a cycle in $G - e_1$, so e_2 is still not a bridge.

Thus $G' = G - e_1 - e_2$ is connected. However $|E(G')| = n - 2$ which contradicts G' being connected since the minimum number of edges a connected graph on n vertices can have is $n - 1$. Thus G contains exactly one cycle.

5. (**Planar Embedding**) Let G be a connected planar graph. Prove that if G is not bipartite, then any planar embedding of G has at least 2 faces with odd degree.

solution: Since G is not bipartite, G contains an odd cycle C. Let $F_1, \ldots, F_k$ be the faces inside C in the planar embedding. Consider the sum of the degrees of the faces.

- Each edge in C gets counted once
- Let D be the set of edges on any boundary of any F_i, but not on C.

Each edge in D is counted twice. So $\sum_i \deg(F_i) = |E(C)| + 2|D|$. Since $|E(C)|$ is odd and $2|D|$ is even, $\sum_i \deg(F_i)$ is odd, so $deg(F_i)$ must be odd for some i, and therefore we have at least one face inside C with odd degree. We can use a similar argument for the existence of a face outside C with odd degree, so there must be at least two faces with odd degree.

6. (**Matching**) Let G be a graph on n vertices where n is even. Suppose that $deg_G(v) \geq n/2$ for all $v \in V(G)$. Prove that G has a perfect matching.

Hint: Prove that if M is a matching that is not perfect, then there exists an augmenting path with respect to M of length 1 or 3.

solution: Suppose not (for a contradiction). Let M be a maximum matching of G. Let $e_1 = u_1v_1, e_2 = u_2v_2, \cdots, e_k = v_ku_k$ be the edges of M. Let $X = \{u_i, v_i : i \in \{1, \cdots, k\}\}$. Note that $|X| = 2k$.

First suppose there exists an edge $e = uv \in E(G)$ where $u, v \notin X$. But then e is an M-augmenting path of length one, contradicting Lemma 8.1.1 as M is maximum.

So we assume that for every edge $e = uv \in E(G)$, we have $\{u, v\} \cap X \neq$ (that is, X is a cover of G). Since M is not perfect, we have that $k < n/2$ and hence $|X| < n$. Since $|X|$ is even and n is even, it follows that $|X| \leq n - 2$. Thus there exists distinct $u, v \in V(G) \setminus X$.

Since X is a cover of G, we find that $N_G(u) \subseteq X$ and similarly $N_G(v) \subseteq X$.

First suppose there exists $i \in \{1, \cdots, k\}$ such that $|N_G(u) \cap \{u_i, v_i\}| + |N_G(v) \cap \{u_i, v_i\}| \geq 3$.

Without loss of generality, we assume that $u_i, v_i \in N_G(u)$; similarly without loss of generality, we assume that $v_i \in N_G(v)$. But then $P = uu_iv_iv$ is an M-augmenting path of G, contradicting Lemma 8.1.1 as M is maximum.

So we assume that for every $i \in \{1, \cdots, k\}$ we have that

$$|N_G(u) \cap \{u_i, v_i\}| + |N_G(v) \cap \{u_i, v_i\}| \le 2.$$

But then summing over i and using that $N_G(u), N_G(v) \subseteq X$, we find that

$$|N_G(u)| + |N_G(v)| \le \sum_{i=1}^{k} |N_G(u) \cap \{u_i, v_i\}| + |N_G(v) \cap \{u_i, v_i\}| \le 2k < n.$$

Yet by assumption every vertex of G has degree at least $n/2$ and hence

$$|N_G(u)| + |N_G(v)| = deg_G(u) + deg_G(v) \ge n/2 + n/2 = n,$$

contradicting the previous inequality.

7. (**Cover**) Let G be a bipartite graph with bipartition (A, B), and let C_1 and C_2 be covers of G. Define

$$C_3 := ((C_1 \cap C_2) \cap A) \cup ((C_1 \cup C_2) \cap B)$$

   (a) Show that C_3 is a cover.

   (b) Show that if C_1 and C_2 are minimum covers of G, then C_3 is a minimum cover of G.

solution: (a) By definition of cover, it suffices to show that for each $ab \in E(G)$, we have $a \in C_3$ or $b \in C_3$. Without loss of generality, we assume that $a \in A$ and $b \in B$.

Since C_1 is a cover, we have by definition of cover that $a \in C_1$ or $b \in C_1$. Similarly since C_2 is a cover, we have $a \in C_2$ or $b \in C_2$.

First suppose that $b \in C_1$. Then $b \in C_1 \cup C_2$. Since $b \in B$, we have $b \in (C_1 \cup C_2) \cap B \subseteq C_3$ as desired.

So we assume $b \notin C_1$ and hence $a \in C_1$. Similarly if $b \in C_2$, then $b \in C_3$ as desired. So we assume $b \notin C_2$ and hence $a \in C_2$.

But then $a \in C_1 \cap C_2$. Since $a \in A$, we have $a \in (C_1 \cap C_2) \cap A \subseteq C_3$ and hence $a \in C_3$ as desired.

(b) Since C_1 and C_2 are minimum covers of G, we have that $|C_1| = |C_2|$. By (a), C_3 is a cover. Let

$$C_4 := ((C_1 \cap C_2) \cap B) \cup ((C_1 \cup C_2) \cap A).$$

By applying (a) symmetrically to the bipartition (B, A) we find that C_4 is also a cover of G. Since C_1 and C_2 are minimum covers of G, we have that $|C_1| \le |C_3|$ and $|C_2| \le C_4$. Hence

$$\begin{aligned} |C_1| + |C_2| \le |C_3| + |C_4| &= |(C_1 \cap C_2) \cap A| + |(C_1 \cup C_2) \cap B| \\ &\quad + |(C_1 \cup C_2) \cap A| + |(C_1 \cap C_2) \cap B| \\ &= |C_1 \cap A| + |C_2 \cap A| + |C_1 \cap B| + |C_2 \cap B| \\ &= |C_1| + |C_2|. \end{aligned}$$

Thus equality holds through out and we find that $|C_3| + |C_4| = |C_1| + |C_2|$. Since $|C_1| \leq |C_3|$ and $|C_2| \leq |C_4|$, it follows that $|C_3| = |C_4| = |C_1| = |C_2|$. That is, C_3 and C_4 are also minimum covers of G as desired.