Math 239 Fall 2023 Tutorial Question Solutions

Various Tutors

Fall 2023

Contents

1 Week 1 – Sep. 14/15

This week was using an old sheet of questions, and so no answers were written.

2 Week 2 – Sep. 21/22

1. (**Even and Odd**) Let $E = \{2, 4, 6, \cdots\}$, $O = \{1, 3, 4, \cdots\}$. The set of all compositions of 3 parts with exactly 2 even parts is

$$S = (E \times E \times O) \cup (E \times O \times E) \cup (O \times E \times E).$$

Define the weight function $w(a, b, c) = a + b + c$ for each $(a, b, c) \in S$. Consider the weight function $\alpha(a) = a$ on E and O. The generating series for E and O with respect to α is given by

$$\Phi_E^\alpha(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2}\ \Phi_O^\alpha(x) = x^1 + x^3 + x^5 \cdots = \frac{x}{1-x^2}.$$

Using the product lemma,

$$\Phi^w_{E\times E\times O}(x) = (\Phi_E^\alpha(x))^2 \Phi_O^\alpha(x) = \frac{x^5}{(1-x^2)^3}.$$

Similarly,

$$\Phi^w_{E\times O\times E}(x) = \Phi^w_{O\times E\times E}(x) = \frac{x^5}{(1-x^2)^3}.$$

By the sum lemma,

$$\Phi^w_S(x) = \Phi^w_{E\times E\times O}(x) + \Phi^w_{E\times O\times E}(x) + \Phi^w_{O\times E\times E}(x) = \frac{3x^5}{(1-x^2)^3}$$

Applying the negative binomial theorem,

$$\Phi^w_S(x) = 3x^5 \sum_{l\geq 0} \binom{l+2}{2} x^{2l} = \sum_{l\geq 0} 3\binom{l+2}{2} x^{5+2l}. \tag{1}$$

The number of compositions of $n \geq 0$ with the stated properties is given by $[x^n]\Phi^w_S(x)$.

Using equation (1):
$[x^n]\Phi^w_S(x) = 0$ for $0 \leq n \leq 4$ or for n even,
and for $n \geq 5$, n odd, let $n = 5+2l$ (or $l = (n-5)/2$), with $[x^n]\Phi^w_S(x) = 3\binom{l+2}{2} = 3\binom{(n-1)/2}{2}$.

2. (**Power Series**)

   - $\sum_{n\geq 0} n^2x^n = x\sum_{n\geq 0} \frac{d}{dx} nx^n = x\frac{d}{dx}\sum_{n\geq 0} xn^n = \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$
   - $\sum_{n\geq 0} 3^n x^n = \sum_{n\geq 0}(3x)^n = \frac{1}{1-3x}$
   - $\sum_{n\geq 0}(3\cdot 7^n - 5\cdot 4^n)x^n = \sum_{n\geq 0} 3\cdot 7^n x^n - 5\cdot 4^n x^n = \sum_{n\geq 0} 3\cdot 7^n x^n - \sum_{n\geq 0} 5\cdot 4^n x^n = 3\cdot\sum_{n\geq 0} 7^n x^n - 5\cdot\sum_{n\geq 0} 4^n x^n = 3\cdot\sum_{n\geq 0}(7x)^n - 5\cdot\sum_{n\geq 0}(4x)^n = \frac{3}{1-7x} - \frac{5}{1-4x}$

3. (**Some Less-Usual Coefficient Extraction**) The first two are straightforward and the last two are less so (but still straightforward).

   (a)

$$[x^n]\left[\frac{1}{2-x^2}\right] = \frac{1}{2}[x^n]\left[\frac{1}{1-(\frac{x^2}{2})}\right] = \frac{1}{2}[x^n]\sum_{i\geq 0}\left(\frac{x^2}{2}\right)^i = \begin{cases}\frac{1}{2}\cdot\frac{1}{2^k} & n = 2k,\\ 0 & n \neq 2k.\end{cases}$$

(b)

$$[x^n]\left[\frac{1}{1-x}\cdot\frac{1}{1-3x}\right] = [x^n]\sum_{i\geq 0} x^i \cdot \sum_{j\geq 0} 3^j x^j.$$

Now think about how many ways we can get x^n from the first sum and the second sum. Either we get all our x^n from the first, or x^{n-1} from the first and $3x$ from the second etc. It follows that

$$[x^n]\sum_{i\geq 0} x^i \cdot \sum_{j\geq 0} 3^j x^j = 1 + 3 + \cdots + 3^n.$$

One may note also that this can be seen as a partial "geometric sum" to give

$$1 + 3 + \cdots + 3^n = \frac{3^{n+1}-1}{3-1}.$$

(c)

$$[x^n]\left[(1+ax)^k\right] = [x^n]\sum_{i=0}^{k}\binom{k}{i}a^i x^i = \binom{k}{n}a^n.$$

Note here that $\binom{k}{n} = 0$ if $n \notin \{0, \cdots, k\}$.

(d)

$$\begin{aligned}
[x^n]\left[(1+x^l)^k\right] &= [x^n]\sum_{i=0}^{k}\binom{k}{i}x^{il} \\
&= [x^n]\left[\binom{k}{0}x^0 + \binom{k}{1}x^l + \binom{k}{2}x^{2l} + \cdots + \binom{k}{k}x^{kl}\right] \\
&= \begin{cases} \binom{k}{n/l} & n = 0 \mod l, \\ 0 & n \neq 0 \mod l. \end{cases}
\end{aligned}$$

4. (**Sum Lemma**)

(a) Consider for one dice situation. $\mathcal{A} = \{1,2,3,4,5,6\}$, with weight function $\omega(a) = a$, then the generating series of $\Phi_{\mathcal{A}}$ with respect to ω is

$$\Phi_{\mathcal{A}}(x) = \sum_{k=1}^{6} x^k = x\sum_{k=0}^{5} x^k = x\frac{1-x^6}{1-x} = \frac{x-x^7}{1-x}$$

Since $\mathscr{S} = \mathcal{A}^4$,

$$\Phi_{\mathscr{S}}(x) = \Phi_{\mathcal{A}}^4 = (\frac{x-x^7}{1-x})^4$$

(b)

$$\begin{aligned}
[x^{19}]\Phi_{\mathcal{A}}(x) &= [x^{19}](\frac{x-x^7}{1-x})^4 \\
&= [x^{19}](x-x^7)^4\frac{1}{(1-x)^4} \\
&= \sum_{k=0}^{19}[x^k]\left(x^4(1-x^6)^4\right)[x^{19-k}]\frac{1}{(1-x)^4} \\
&= \sum_{k=4}^{19}[x^{k-4}](1-x^6)^4[x^{19-k}]\sum_{n=0}^{\infty}\binom{n+4-1}{4-1}x^n \\
&= \sum_{k=4}^{19}[x^{k-4}](1-x^6)^4\binom{19-k+3}{3} \\
&= \sum_{k=4}^{19}[x^{k-4}]\sum_{i=0}^{4}\binom{4}{i}x^{6i}\binom{22-k}{3} \\
&= \sum_{k=4}^{19}\sum_{6|(k-4)}\binom{4}{(k-4)/6}\binom{22-k}{3} \\
&= \binom{4}{0}\binom{22-4}{3} + \binom{4}{1}\binom{22-10}{3} + \binom{4}{2}\binom{22-16}{3}
\end{aligned}$$

(c) The number of outcomes is characterized by coefficients of x^k of the generating series,

$$\begin{aligned}
[x^k]\left(\frac{x-x^{d+1}}{1-x}\right)^m &= \sum_{i=0}^{k}[x^{i-m}](1-x^d)^m[x^{n-i}]\frac{1}{(1-x)^m} \\
&= \sum_{i=0}^{k}\sum_{d|(i-m)}\binom{m}{(i-m)/d}\binom{n-i+m-1}{m-1}
\end{aligned}$$

5. (**Bonus Material: Formal Derivatives**) Answers are

   (a) $[x^n]\left[\frac{\mathrm{d}}{\mathrm{d}x}A(x)\right] = (n+1)a_{n+1}$.

   (b) $[x^n]\left[x\frac{\mathrm{d}}{\mathrm{d}x}A(x)\right] = na_n$.

   (c) i. Here we just let $a_n = 1$ to get that $[x^n]\left[x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x}\right] = n$. Taking a derivative in the normal way and then multiplying by x gives that the generating function for n is

   $$\frac{x}{(1-x)^2}.$$

   ii. If we let $a_n = n$ we get that applying $x\frac{\mathrm{d}}{\mathrm{d}x}$ to $\frac{1}{1-x}$ twice gives us the generating function for n^2. Thus it is

   $$\frac{x(x+1)}{(1-x)^3}.$$

   iii. The first few powers of n are

   $$\begin{aligned} n^0 &\rightarrow \frac{1}{1-x}, \\ n^1 &\rightarrow \frac{x}{(1-x)^2}, \\ n^2 &\rightarrow \frac{x(x+1)}{(1-x)^3}, \\ n^3 &\rightarrow \frac{x(x^2+4x+1)}{(1-x)^4}, \\ n^4 &\rightarrow \frac{x(x^3+11x^2+11x+1)}{(1-x)^5}. \end{aligned}$$

   I know of no general formula or pattern, and factoring these does not give a whole lot.

3 Week 3 – Sep. 28/29

1. (**Ambiguous Expressions**) This is fairly easy warm-up question.

   (a) A string with only even blocks has a natural decomposition given by

   $$S = \{00, 11\}^* = \{00\}^*\big(\{11\}\{11\}^*\{00\}\{00\}^*\big)^*\{11\}^*.$$

   (b) We use the second one. This gives us

   $$S(x) = \frac{1}{1-x^2} \cdot \left(\frac{1}{1 - x^2 \cdot \frac{1}{1-x^2} \cdot x^2 \cdot \frac{1}{1-x^2}}\right) \cdot \frac{1}{1-x^2} = \frac{1}{1-2x^2},$$

   where the last step follows by Wolfram Alpha. This is not surprising, since we are basically just taking normal binary strings with double the weight.

(c) We use the second one. This gives us

$$S(x) = \frac{1}{1-x^2} \cdot \left(\frac{1}{1 - x^4 \cdot \frac{1}{1-x^4} \cdot x^2 \cdot \frac{1}{1-x^2}} \right) \cdot \frac{1}{1-x^4} = \frac{1}{1-x^2-x^4},$$

where the last step follows by Wolfram Alpha. This is again not surprising.

(d) This should work with any ambiguous representation. When you have a GF from an ambiguous rep, you are overcounting things, so you end up with something like each ambiguous coefficient being larger than each unambiguous coefficient.

2. (**Unambiguous Expressions**)

(a) We can construct the regular expression as follows

$$\varepsilon \smile 1 \smile 0 \smile 1(0 \smile 1)^*1 \smile 0(0 \smile 1)^*0$$

and in fact, if K is an unambiguous regular expression that produces all possible binary strings, R can be chosen to be

$$\varepsilon \smile 1 \smile 0 \smile 1\mathrm{K}1 \smile 0\mathrm{K}0,$$

leading to the rational function

$$1 + x + x + x\frac{1}{1-2x}x + x\frac{1}{1-2x}x = 1 + 2x + \frac{2x^2}{1-2x}.$$

(b) We can construct the regular expression as follows

$$0^*(1000000^*)^*$$

leading to the rational function

$$\frac{1}{1-x}\frac{1}{1-\frac{x^6}{1-x}} = \frac{1}{1-x-x^6}.$$

3. (**Compositions**) For different parity of the length of compositions, we have

$$\mathcal{E}_{2k+1} = \overbrace{O \times E \times O \times \cdots \times O}^{2k+1}$$
$$\mathcal{E}_{2k} = \underbrace{O \times E \times O \times \cdots \times E}_{2k}$$

where $E = \{2, 4, 6, \dots\}, O = \{1, 3, 5, \dots\}$. Then

$$\Phi_E(x) = x^2 + x^4 + x^6 + \dots + = \frac{x^2}{1-x^2}$$

$$\Phi_O(x) = x + x^3 + x^5 + \dots + = \frac{x}{1-x^2}$$

So we have

$$\Phi_{\mathcal{E}_{2k+1}} = \Phi_O(x)^{k+1}\Phi_E(x)^k = \frac{x^{3k+1}}{(1-x^2)^{2k+1}}$$

$$\Phi_{\mathcal{E}_{2k}} = \Phi_O(x)^k\Phi_E(x)^k = \frac{x^{3k}}{(1-x^2)^{2k}}$$

$$\begin{aligned}
\mathcal{E} &= \bigcup_{k=0}^{\infty} \mathcal{E}_{2k} \cup \bigcup_{k=0}^{\infty} \mathcal{E}_{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(1-x^2)^{2k+1}} + \frac{x^{3k}}{(1-x^2)^{2k}} \\
&= \sum_{k=0}^{\infty} \frac{x^{3k}}{(1-x^2)^{2k}}(\frac{x}{1-x^2} + 1) \\
&= \frac{x+1-x^2}{1-x^2} \sum_{k=0}^{\infty} \left(\frac{x^3}{(1-x^2)^2}\right)^k \\
&= \frac{1+x-x^2}{1-x^2} \cdot \frac{1}{1-\frac{x^3}{(1-x^2)^2}} \\
&= \frac{(1+x-x^2)(1-x^2)}{(1-x^2)^2 - x^3} \\
&= 1 + \frac{x}{1-2x^2-x^3+x^4}
\end{aligned}$$

4. (**Generating Series**) We start out by trying to satisfy the requirements: The string starts with 0, can't have blocks of 0 so we add a second 0, and then we can add however many more zeroes we want to that. After that, the sequence has to end with 1, so maybe we allow any number of 1's and then a terminating 1, which will give us 000^*1^*1.

   However, this doesn't allow for, say, 001001, which satisfies the requirements. So we modify our attempt a bit by allowing the pattern to repeat arbitrarily many times, like so: $000^*1^*1(000^*1^*1)^*$.

   So our unambiguous pattern for our set S is $000^*1^*1(000^*1^*1)^*$. Now we look for the generating function by first finding it for 000^*1^*1:

   $$\Phi_{000^*1^*1}(x) = \Phi_{00}(x) \cdot \Phi_{0^*}(x) \cdot \Phi_{1^*}(x) \cdot \Phi_1(x) = \\ \Phi_{00}(x) \cdot \tfrac{1}{1-\Phi_0(x)} \cdot \tfrac{1}{1-\Phi_1(x)} \cdot \Phi_1(x) = x^2 \cdot \tfrac{1}{1-x} \cdot \tfrac{1}{1-x} \cdot x = \tfrac{x^3}{(1-x)^2}$$

   Now we plug this in to get:

   $$\Phi_S(x) = \Phi_{000^*1^*1}(x) \cdot \Phi_{(000^*1^*1)^*}(x) = \tfrac{x^3}{(1-x)^2} \cdot \tfrac{1}{1-\frac{x^3}{(1-x)^2}} = \\ \tfrac{x^3}{(1-x)^2} \cdot \tfrac{(1-x)^2}{(1-x)^2-x^3} = \tfrac{x^3}{(1-x)^2-x^3} = \tfrac{x^3}{(1-x)^2-2x+x^2-x^3} = \tfrac{x^3}{(1-x)^2-(2x-x^2+x^3)}$$

   We'll expand to find the first 4 terms. Note that we don't need to worry about terms with power 7 or higher, so we just ignore the rest of the sum.

   $$\tfrac{x^3}{(1-x)^2-(2x-x^2+x^3)} = x^3 \textstyle\sum_{k\geq 0}(2x-x^2+x^3)^k = x^3 + x^3(2x-x^2+x^3) + \\ x^3(2x-x^2+x^3)^2 + x^3(2x-x^2+x^3)^3 + x^3 \textstyle\sum_{k\geq 4}(2x-x^2+x^3)^k = \\ x^3 + 2x^4 - x^5 + x^6 + x^3(4x^2 - 2x^3 + 2x^4 - 2x^3 + x^4 - x^5 + 2x^4 - x^5 + \\ x^6) + x^3(8x^3 + \cdots) + \cdots = x^3 + 2x^4 + 3x^5 + 5x^6 + \cdots$$

   Check with some examples to verify, e.g. there's only one string of length 3: 001.

5. (**Bonus Material: A Second GF for Binomial Coefficients**) This is from Wilf. Write

   $$\begin{aligned} \sum_n \binom{n}{k} y^n &= [x^k]\sum_n \sum_k \binom{n}{k} x^k y^n = [x^k]\sum_n (1+x)^n y^n \\ &= [x^k]\frac{1}{1-y(1+x)} = \frac{1}{1-y}[x^k]\frac{1}{1-(\frac{y}{1-y})x} \\ &= \frac{1}{1-y}\frac{y^k}{(1-y)^k} = \frac{y^k}{(1-y)^{k+1}}. \end{aligned}$$

   Pretty nifty!

4 Week 4 – Oct. 05/06

1. (**Warm-up Recursive Expressions**)

   (a) Consider the representation for all binary strings given by

   $$\{0\}^*\left(\{1\}\{1\}^*\{0\}\{0\}^*\right)^*\{1\}^*.$$

   Changing this by writing 11 instead of 1 gives

   $$\{0\}^*\left(\{11\}\{11\}^*\{0\}\{0\}^*\right)^*\{11\}^*.$$

   Note a shorter expression is $\{0, 11\}^*$. A recursive expression can be given by

   $$T = \epsilon \cup \{0, 11\}T.$$

   Finding the generating function of the first expression gives $\frac{1}{1-x-x^2}$. Finding the generating function of the second expression gives $T(x) = 1 + (x + x^2)T(x)$, which indeed simplifies to $T(x) = \frac{1}{1-x-x^2}$.

   (b) This is simply $T = \epsilon \cup 0T1 \cup 1T0$. Solving this gives $T(x) = \frac{1}{1-2x^2}$. It is not possible to write a non-recursive expression for this, since regular expressions are memory-less, and we need a memory to write this down left-to-right.

2. (**Partial Fractions**) The denominator factors as

   $$1 - 9x^2 = (1 - 3x)(1 + 3x).$$

   Hence we have
   $$\frac{18 + 12x}{1 - 9x^2} = \frac{A}{1 - 3x} + \frac{B}{1 + 3x}$$
   for some constants A and B. We have the relation

   $$A(1 + 3x) + B(1 - 3x) = 18 + 12x,$$

   yielding the system of equations

   $$\begin{aligned} A + B &= 18 \\ 3A - 3B &= 12 \end{aligned}$$

   which can be solved to find that $A = 11$ and $B = 7$.

   Thus

   $$\begin{aligned} q_n &= [x^n]\frac{18 + 12x}{1 - 9x^2} \\ &= [x^n]\frac{11}{1 - 3x} + [x^n]\frac{7}{1 + 3x} \\ &= 11 \cdot 3^n + 7 \cdot (-3)^n \end{aligned}$$

3. (**Avoiding Substrings**) (a) We claim

$$S\{0,1\} \cup \{\epsilon\} = S \cup T$$
$$S\{011\} = T$$

To see that $S\{0,1\} \cup \{\epsilon\} \subseteq S \cup T$, note that $\epsilon \in S \cup T$. If $\alpha \in S$, then $\alpha 0 \in S$, and $\alpha 1$ can contain at most one occurrence of 011, and if it occurs it must be at the end of the string. Thus, $\alpha 1 \in S \cup T$. This verifies $S\{0,1\} \cup \{\epsilon\} \subseteq S \cup T$. To see the other direction, for any $\alpha \in S \cup T$, either $\alpha = \epsilon$, or $\alpha = \beta\gamma$ where $\gamma \in \{0,1\}$. Moreover, by the definition of S and T, β cannot contain 011 as a substring. So $\beta \in S$. This verifies the first equation above.

For the second equation, it is obvious that $T \subseteq S\{011\}$ by the definition of S and T. For the other direction, for any $\alpha \in S$, α is 011-free. Moreover, by looking at the last three bits of $\alpha 0$, $\alpha 01$, it is clear that $\alpha 0$ and $\alpha 01$ are both 011-free. Thus, $\alpha 011$ has exactly one occurrence of 011 at its very end. Hence, $S\{011\} \subseteq T$. This verifies the second equation.

(b) First we show that the equations in (a) are unambiguous. For the first equation, $S\{0,1\}$ is obviously unambiguous. Since $\epsilon \notin S\{0,1\}$, the union on the LHS of the equation is disjoint, it follows that the set expression on the LHS is unambiguous. For the RHS, since $S \cap T = \emptyset$, the set expression is also unambiguous. The second equation is unambiguous because $S\{011\}$ is unambiguous. Let $\Phi_S(x)$ and $\Phi_T(x)$ denote the generating series for S and T respectively with respect to the lengths of the strings. It follows from the equations above, and the sum and product lemmas, that

$$2x\Phi_S(x) + 1 = \Phi_S(x) + \Phi_T(x)$$

$$x^3\Phi_S(x) = \Phi_T(x).$$

Therefore,

$$\Phi_S(x) = \frac{1}{1 - 2x + x^3} = \frac{1}{(1-x)(1-x-x^2)}.$$

(c) Using partial fraction, we have

$$\Phi_S(x) = -\frac{1}{1-x} + \frac{1 - 2\sqrt{5}/5}{1 + 2x/(\sqrt{5}+1)} + \frac{1 + 2\sqrt{5}/5}{1 - 2x/(\sqrt{5}-1)}$$

$$= \sum_{n \geq 0} (-1 + (1 - \frac{2\sqrt{5}}{5})(-\frac{2}{\sqrt{5}+1})^n$$

$$+(1 + \frac{2\sqrt{5}}{5})(\frac{2}{\sqrt{5}-1})^n)x^n$$

Thus, the number of binary strings of length n that do not contain 011 as a substring is

$$-1+\left(1-\frac{2\sqrt{5}}{5}\right)\left(-\frac{2}{\sqrt{5}+1}\right)^n+\left(1+\frac{2\sqrt{5}}{5}\right)\left(\frac{2}{\sqrt{5}-1}\right)^n$$

for every nonnegative integer n.

4. (**Recurrence**) The characteristic polynomial is

$$1-2x-7x^2=(1+x)^2(1-4x).$$

So we have $\lambda_1=-1$, $\lambda_2=4$, $d_1=2$, and $d_2=1$. This gives us that our general solution is of the form

$$a_n=P_1(n)(-1)^n+P_2(n)\cdot 4^n$$

where $\deg(P_1)<d_1=2$ and $\deg(P_2)<d_2=1$. So

$$a_n=(A+B)(-1)^n+C\cdot 4^n$$

for constants A,B,C. Substitute using initial conditions:

$$\begin{aligned} a_0=3&=A+C\\ a_1=7&=(A+B)(-1)+C\cdot 4=-A-B+4C\\ a_2=8&=(A+2B)+C\cdot 4^2=A+2B+16C \end{aligned}$$

Solving for the constants, we get

$$A=2,\quad B=-5,\quad C=1$$

Thus

$$a_n=(2-5n)(-1)^n+4^n$$

5 Week 5 – Oct. 19/20

1. (**A Routine Recurrence**) We use theorems 3.2.1 and 3.2.2 from the IC (Introduction to Combinatorics notes) on Learn. If one is not familiar with the notation or the theorems, one should read over them.

   The characteristic polynomial of this recurrence is

$$x^3+5x^2+3x-9=(x-1)(x+3)^2.$$

By theorem 3.2.2 then this implies that we have the closed form

$$a_n = A \cdot (1)^n + (B + Cn) \cdot (-3)^n$$

for some constants A, B, C. Plugging in initial values of a_n gives us

$$\begin{aligned} a_0 &= A + B \\ a_1 &= A - 3B - 3C \\ a_2 &= A + 9B + 18C. \end{aligned}$$

Solving this system of linear equations gives us

$$\begin{aligned} A &= 2, \\ B &= -1, \\ C &= \frac{2}{3}. \end{aligned}$$

This implies that

$$a_n = 2 + (-1 + \tfrac{2}{3}n)(-3)^n.$$

Multiplying the initial conditions by 3 just scales A, B, C by 3, and so we get that

$$\begin{aligned} a_n &= 6 + (-3 + 2n)(-3)^n \\ &= 3(2 + (-1 + \tfrac{2}{3}n)(-3)^n). \end{aligned}$$

2. (**Another Recurrence**)

   (a)
$$a_n = 3a_{n-1} + 2a_{n-3}.$$

   (b)
$$a_0 = 1, a_1 = 3, a_2 = 9.$$

   (c) The answers varies. One possibility is coloured compositions of n where all parts are 1 or 3; each part can be coloured independently red, green, or yellow; and each 3 part can be coloured independently blue or purple.

3. (**Degree Sequences**)

(a) $4, 3, 2, 2, 1$ Graph exists, draw one vertex adjacent to the other four, then pick another vertex to be adjacent to two others.

(b) $6, 5, 4, 3, 2, 1$ Not possible because there aren't enough vertices for the first vertex to be adjacent to.

(c) $5, 4, 4, 3, 2, 1$ Not possible because odd number of odd vertices (contradicts handshaking lemma)

(d) $3, 3, 3, 3, 3, 3$ This is just $K_{3,3}$

(e) $2, 2, \ldots, 2$ for arbitrary length n is just cycle of length n

(f) $6, 6, 4, 2, 2, 2, 1, 1$ The only way to have two vertices of degree 1 in this graph is to have the two vertices of degree 6 adjacent to each other, as otherwise forces all of the other vertices to have degree at least 2. However, even if the two vertices of degree 6 are adjacent to each other, we have a graph where two vertices have degree 1 and every other vertex has degree 2, so there's no way to add edges to increase the degree of just one vertex to 4 unless we add a loop. Havel-Hakimi algorithm verifies that this isn't the degree sequence of a graph.

4. (**Graph Complements Solution**) Recall that if G is a graph $\overline{G}$, the complement of G, has $V(G) = V(\overline{G}$ and $E(\overline{G} = \{uv | uv \notin E(G), u, v \in V(\overline{G})\}$.

   (a) Let K_n be the complete graph on n vertices. Describe $\overline{K_n}$.
   Solution: $\overline{K_n}$ is the graph of n isolated vertices.

   (b) For any $v \in G$ what is $deg_G(v) + deg_{\overline{G}}(v)$?
   Solution: $|V(G)| - 1$

   (c) Find a graph G such that $G \cong \overline{G}$.
   Solution: The two simplest example are P_4 and C_5.

   (d) Prove that if $|V(G)| \geq 6$ then either G or $\overline{G}$ contains a triangle.
   Solution: Let $v \in V(G)$. Since $\deg_G(v) + \deg_{\overline{G}}(v) = n - 1 \geq 5$, either $\deg_G(v) \geq 3$ or $\deg_{\overline{G}}(v) \geq 3$. Suppose $\deg_G(v) \geq 3$. Let $N_G(v)$ be the set of neighbours of v in G. Thus, $|N_G(v)| \geq 3$. If there are two vertices in $N_G(v)$ that are adjacent, then these two vertices together with v form a triangle in G. Suppose every pair of vertices in $N_G(v)$ are not adjacent, then all vertices in $N_G(v)$ are pairwise adjacent in $\overline{G}$, implying that $\overline{G}$ has a triangle, as $|N_G(v)| \geq 3$. The proof for the case $\deg_{\overline{G}}(v) \geq 3$ is similar, with G and $\overline{G}$ interchanged. □

6 Week 6 – Oct. 26/27

1. (**Matching Vertices**) Write down the degree sequence. This is a sequence $d_n, \cdots, d_1$ where $0 \leq d_j \leq n - 1$.

   Suppose that G is connected, so that each vertex has degree at least one. Then $1 \leq d_j \leq n - 1$. This means that since there are only $n - 1$ possible values for n vertices, there must be some repeated value (by pigeonhole). Thus at least one value in the degree sequence is repeated, and so two vertices have the same degree.

   Now suppose that G has some vertex of degree 0. Then $0 \leq d_j \leq n - 2$. Similarly to before then by pigeonhole there must be two vertices with the same degree.

2. (**Paths and Cycles**) Let G be a graph and $k \geq 2$ be the minimum degree of G (denoted $\delta(G)$).

   - Prove G has a path of length at least k.
     Solution: Let P be the longest path in G. Let v_0 be the first vertex in the path. Observe that every neighbor of v_0 is in P or we could build a longer path. Thus P has at least $k+1$ vertices and hence k edges.

   - Prove G has a cycle of length at least $k+1$.
     Solution: Suppose $P = v_0 v_1 ... v_n$ is the longest path in G. Observe that all neighbors of v_0 are in the path or we could make the path longer. Let v_j be a neighbor of v_0 such that j is as large as possible. That is to say that v_j is the kth neighbor we encounter of v_0 along the path. Then $P v_j v_0$, which means taking the path P from the v_0 to the vertex v_j and then returning to v_0 is a cycle of length $k+1$.

3. (**Vertices and Edges**) Assume, for contradiction, that G can have a vertex of degree 0 given the above conditions. This means that within G, there are at least two components: one with a single vertex and no edges, and one with $n-1$ vertices and $m > \frac{(n-1)(n-2)}{2}$ edges. We can attempt to construct the component with $n-1$ vertices by starting to add edges: For the first vertex, we are able to add $n-2$ edges, for the second vertex, we are able to add $n-3$ edges, etc. The total number of edges we can add to a graph of $n-1$ vertices is equal to $\sum_{i=1}^{n-2} i$ which is equal to $\frac{(n-1)(n-2)}{2}$. This presents a contradiction, as we assumed that $m > \frac{(n-1)(n-2)}{2}$. Therefore, there cannot be a vertex in G with degree 0.

4. (**Bipartite Graph and Walks**)

   (a) For each vertex $z = z_1 z_2 \cdots z_n \in \{0,1\}^n$, consider the sum of the bits with even indices:
   $$E(z) = \sum_{j=1}^{n/2} z_{2j}.$$
   The vertex set of G_n can be partitioned as $A \cup B$, where
   $$A = \{z \in G_n : E(z) \text{ is even}\} \quad B = \{z \in G_n : E(z) \text{ is odd}\}.$$
   Now suppose that $x, y \in G_n$ are connected by an edge. Then, using the definition of the graph G_n we may write $x = asb$ and $y = a\bar{s}b$ with $s \in \{01, 10\}$. Using the definition of E and the form of x and y we see that
   $$E(x) - E(y) = \begin{cases} 1, & \text{if } a \text{ has even length,} \\ -1, & \text{if } a \text{ has odd length.} \end{cases}$$
   Therefore if $E(x)$ is even then $E(y)$ is odd and vice versa, and all edges of the graph G_n have one endpoint in A and one endpoint in B. Thus G_n is bipartite.

   (b) To show that G_n is connected, note that two vertices $x \neq y \in V(G_n)$ are connected by an edge if and only if x is obtained from y by swapping two adjacent bits. Suppose we are given a binary string $\sigma \in V(G_n)$. Since σ has $n/2$ ones, we can construct a walk from σ to the string $1^{n/2}0^{n/2}$ by walking along edges that implement nearest-neighbour swaps to move the leftmost 1 to the first bit, then using swaps of the bits $2, \ldots, n$ to move the next 1 to the second bit, and so on. Thus, given any two strings $\sigma, \sigma' \in V(G_n)$ we can construct a walk between them by concatenating the walk from σ to $1^{n/2}0^{n/2}$ with the walk from $1^{n/2}0^{n/2}$ to σ'. This shows that G_n is connected.

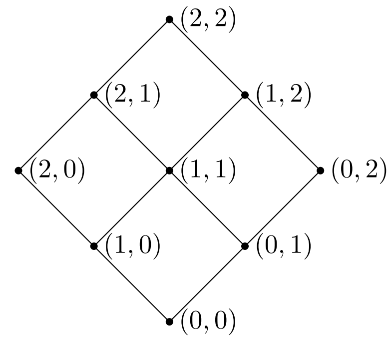
7 Midterm Review – Nov. 02/03

This week was a review session and so answers were not written in this file.

8 Week 8 – Nov. 09/10

1. (**Exotic Cubes**)

   (a) There are $3^2 = 9$ vertices in this graph (m^n in general).



(b) G_p is always bipartite. This is given by the usual bipartition $V_1 \sqcup V_2$ where V_1 has all vertices with even coordinate sum and V_2 has all vertices with odd coordinate sum. Since two vertices are adjacent only their coordinate sums differ by one, this is a good bipartition.

(c) The degree of a vertex is maximized if $v \pm e_j$ exists for all e_j, ie if we can change each coordinate by 1 while keeping our coordinates in $\{0, \cdots, m-1\}$. It follows that the maximum degree is $2n$ (attained by, for example, the point $(1, \cdots, 1)$). Minimum degree is attained when we have as few directions to move in general. This happens when we cannot remove or add 1 from as many coordinates as possible. Since we can still either add or remove from one coordinate then the minimum degree is n (for example the point $(0, \cdots, 0)$).

(d) I would recommend not writing this part of the question. From my scratch-work we should have

$$|E| = \frac{1}{2}\sum_{j=0}^{n}(2n-j)(m-2)^{n-j}2^j\binom{n}{j}.$$

I am not sure if this simplifies in any nice way. For more details on my thinking consult the bonus questions document.

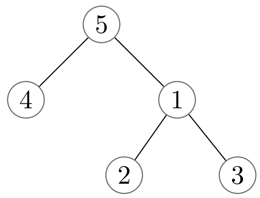
2. (**Find the Mistake**) To see the non-theorem is false observe that the graph consisting of two disjoint three cycles satisfies the theorem conditions but does not contain a cycle of length four. The non-proof falsely assume that all graphs satisfying the statement can be constructed by taking a graph that does and adding a new vertex. This is not true. To see this consider two disjoint four cycles as a graph. This graph satisfies that statement and cannot be constructed in this manner.

   Theorem: If G is a graph with $|V(G)| \geq 3$ and minimum degree at least 2, then G contains a cycle.

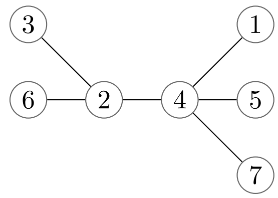
   Proof: Suppose not. Then there exists a graph G such that $|V(G)| \geq 3$, the minimum degree of G is at least 2 and G does not contain a cycle. Since G does not contain a cycle G is a tree. Since G is a tree G has a leaf, which is a vertex of degree one. A contradiction. Hence the theorem holds.

3. (**Prüfer Codes**)

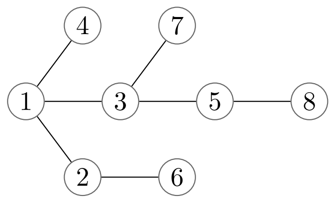
(a) The codes are



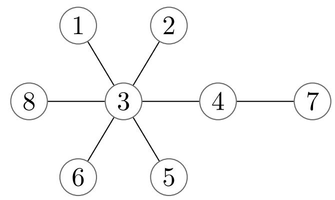
$(1, 1, 5)$



$(4, 2, 4, 2, 4)$



$(1, 2, 1, 3, 3, 5)$



$(3, 3, 3, 3, 4, 3)$

(b) Suppose we have sequence $(\sigma_1, \ldots, \sigma_{n-2})$. Start with an empty graph on n vertices.

The inverse algorithm is roughly:

i. Let x be the largest number in $\{1, \ldots, n\}$ that doesn't appear in $(\sigma_k, \ldots, \sigma_{n-2})$

ii. While the sequence is non-empty, repeat the following:

A. Let v be the lowest number of $\{1, \ldots, n\}$ that doesn't appear in $(\sigma_k, \ldots, \sigma_{n-2})$ and that has no edges in our graph.

B. Draw an edge between v and σ_k.

C. Delete σ_k from the beginning of the sequence.

iii. Draw an edge between x and σ_{n-2}.

(c) If we count the number of possible sequences of length $n - 2$ of $\{1, \ldots, n\}$, we get n^{n-2}. We take as given that we have a bijection, so that's the number of labeled trees.

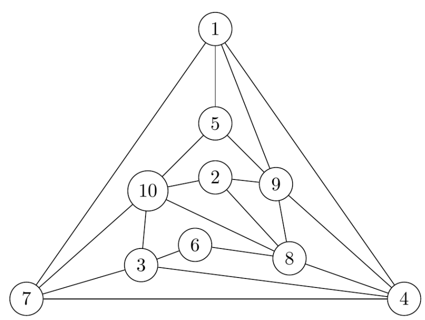
(d) **Bonus:** This will be a bit trickier to prove, but we have surjectivity from part (b). For injectivity it's an induction argument but for the induction hypothesis, you need to weaken the constraint that elements are strictly from $\{1, \ldots, n\}$ and instead just assume you have n elements of $\mathbb{N}$. There's also an argument that needs to be made regarding the relationship between degree and how many times the vertex appears in the sequence.

4. (**Spanning Tree**) First, assume that $e = \{v, w\}$ is a bridge in a connected graph G. Arguing for a contradiction, suppose that T is a spanning tree of G that does that not contain e. Since T is a spanning subgraph of G, both v and w are vertices of T. Since T is a tree, there is a unique path P in T from v to w (Lemma 5.1.3 in IC notes). But now $C = T \cup \{e\}$ is a cycle in G that contains the edge e. Non of the edge in a cycle can be a bridge (Lemma 4.10.3 in IC notes), so e is not a bridge. This contradiction shows that if e is a bridge then e is contained in every spanning tree of G.

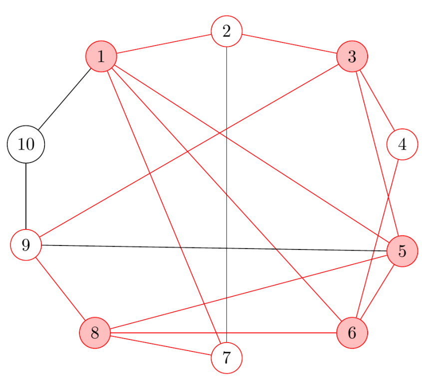
   Conversely, assume that e is not a bridge. Then $H = G \setminus e$ is still connected. Every connected graph has a spanning tree (Theorem 5.2.1 in IC notes), then the graph H has a spanning tree T. Since H is a spanning subgraph of G, this spanning tree T of H is also a spanning tree of G. Since T does not contain e, this completes the proof.

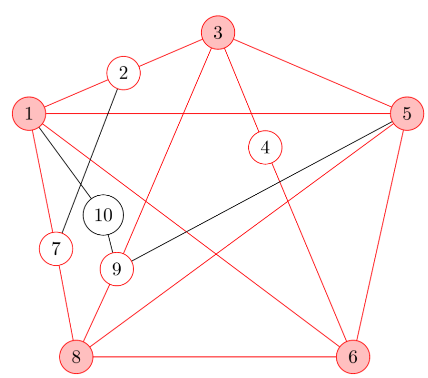
9 Week 9 – Nov. 16/17

1. (**Planar or Not**) The graph on the left is planar:



   The graph on the right contains a subdivision of K_5





2. (**Euler's Formula**)

(a) Let p_2, p_5, q and s be the number of vertices of degree 2, number of vertices of degree 5, number of edges and faces in G. By Euler's formula, $p_2 + p_5 - q + s = 2$. By the two versions of handshake lemmas, $2p_2 + 5p_5 = 2q = 4s$. From the first equation, we obtain that $q = p_2 + p_5 + s - 2$. Substituting this to the other two equations, we obtain that

$$2p_2 + 5p_5 = 2(p_2 + p_5 + s - 2) = 4s.$$

From the first equation above, we obtain that $3p_5 = 2s - 4$ and thus $2s = 3p_5 + 4$. Substituting this into the equation $2p_2 + 5p_5 = 4s$ we have $2p_2 + 5p_5 = 6p_5 + 8$. Hence $p_2 = (p_5 + 8)/2$. Now,

$$q = (2p_2 + 5p_5)/2 = \frac{2}{5}p_5 + \frac{1}{2}(p_5 + 8) = 3p_5 + 4.$$

(b) Given $p_5 = 4$, we immediately have $p_2 = (4+8)/2 = 6; q = 3\times 4 + 4 = 16; a = 3 \times 4/2 + 2 = 8$.
Drawing

3. (**Complete Planar Tripartite Graphs**) Consider (a, b, c) and remember that $a \geq b \geq c$. Now consider $K_{a,b,c}$. Then deleting all the edges from part B and part C gives you $K_{a,b+c}$ as a subgraph. Then we can think about the various cases for a.

$a \geq 3$: Since $K_{a,b+c}$ is a subgraph then we must have that $b + c \leq 2$ (otherwise we have some $K_{3,3}$). Then the only possibility is $b = c = 1$, which gives $K_{a,1,1}$ for all $a \geq 1$. Drawing this graph shows it is indeed planar.

$a \leq 2$: the only possibilities are

$$(2, 2, 2), (2, 2, 1), (2, 1, 1), (1, 1, 1).$$

Then $K_{2,2,2}$ is planar from the following drawing. Then

$$K_{2,2,1}, K_{2,1,1}, K_{1,1,1}$$

are all planar as subgraphs of $K_{2,2,2}$.

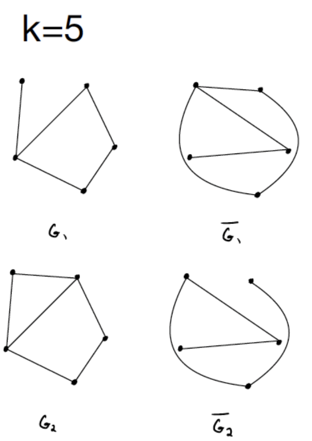
This gives exactly $K_{a,1,1}, K_{2,2,2}, K_{2,2,1}, K_{2,1,1}, K_{1,1,1}$.

4. (**Partitioning Into Planar**) Assume for contradiction that we can partition K_n into $\lfloor \frac{n}{6} \rfloor$ planar subgraphs. Each subgraph has at most n vertices, and we know from class that any planar graph on n vertices can have at most $3n - 6$ edges. So in our partitioned graph, we'll have at most $\lfloor \frac{n}{6} \rfloor (3n - 6) \leq \frac{n}{6}(3n - 6) = \frac{1}{2}(n^2 - 2n) = \frac{n(n-2)}{2}$. But our complete graph K_n has $\frac{n(n-1)}{2}$ edges, which is more than $\frac{n(n-2)}{2}$. Therefore, we can't partition K_n into $\lfloor \frac{n}{6} \rfloor$ planar subgraphs.

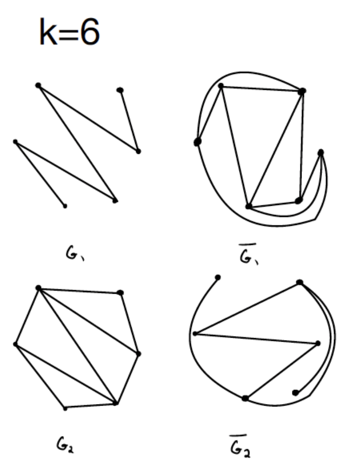
10 Week 10 – Nov. 23/24

1. (**Planarity and Graph Complements**)

   (a) There are numerous solutions. Here are some examples.



The graphs are not isomorphic because one has a degree one vertex and the other does not.



The graphs are not isomorphic because one is bipartite and the other is not.

   (b) Let G be $K_{3,3}$ plus five isolated vertices.

2. (**Cartesian Product of Graphs**)

   (a) Let $(a_1, b_1)(a_2, b_2) \in E(G\Box H)$. That is, by definition, either $a_1 = a_2$ and $b_1 b_2 \in E(H)$, or $a_1 a_2 \in E(G)$ and $b_1 = b_2$. Suppose the former. Then $\phi_1(a_1) = \phi_1(a_2)$ since $a_1 = a_2$. On the other hand $\phi_2(b_1) \neq \phi_2(b_2)$ since $b_1 b_2 \in E(H)$ and ϕ_2 is a k-coloring of H. Thus, since ϕ_2 assigns colors $\{1, \cdots, k\}$, we find that

$$\phi_2(b_1) \neq \phi_2(b_2) \mod k.$$

It follows that

$$\phi((a_1, b_1)) = \phi_1(a_1) + \phi_2(b_1) \mod k \neq \phi_1(a_2) + \phi_2(b_2) \mod k = \phi((a_2, b_2)).$$

[Note one could just say what is written below follows by symmetry, but we include it for completeness] So we assume the latter case. Then $\phi_2(b_1) = \phi_2(b_2)$ since $b_1 = b_2$. On the other hand, $\phi_1(a_1) \neq \phi_1(a_2)$ since $a_1a_2 \in E(G)$ and ϕ_1 is a k-coloring of G. Thus, since ϕ_1 assigns colors $\{1, \cdots, k\}$, we find that

$$\phi_1(a_1) \neq \phi_1(a_2) \mod k$$

It follows that

$$\phi((a_1, b_1)) = \phi_1(a_1) + \phi_2(b_1) \mod k \neq \phi_1(a_2) + \phi_2(b_2) \mod k = \phi((a_2, b_2)).$$

Hence ϕ is a k-coloring of $G \square H$ as desired.

(b) Let $k = \max\{k_1, k_2\}$. Since G is k_1-colorable, G has a k_1-coloring ϕ_1. Since $k \geq k_1$, we have that ϕ_1 is also a k-coloring of G. Similarly, since H is k_2-colorable, H has a k_2-coloring ϕ_2. Since $k \geq k_2$, we have that ϕ_2 is also a k-coloring of H.

Let ϕ be as defined in part (a). By part (a), ϕ is a k-coloring of $G \square H$. Hence $G \square H$ is k-colorable as desired.

3. (**Graph Coloring**) We will prove this by induction on the number of vertices, n.

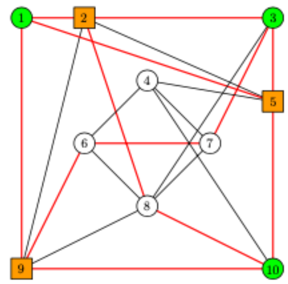
Base Case: $n = 1$. G is a single vertex, which is $k+1$-colorable. **Induction Hypothesis:** Assume every graph on $n-1$ vertices where every non-empty subgraph contains a vertex of degree at most k is $(k+1)$-colorable.

Inductive Step: Let G be such a graph with n vertices. Let v be a vertex of degree at most k (it must exist, as G is a subgraph of itself). Let $G' = G - v$. Since G' is a subgraph of G, it also contains a vertex of degree at most k. By the induction hypothesis, G' is $(k+1)$-colorable. We can extend this to a coloring of G by coloring v with a color that isn't used by any of its neighbors, which much exist since v has at most k neighbors and we have $k+1$ colors to choose from.

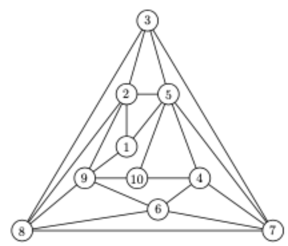
Graphs that satisfy the description that are not k-colorable include K_n and odd cycles.

4. (**A Second Planarity Question**)

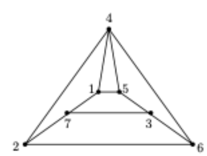
(a) This is non-planar with a $K_{3,3}$ edge subdivision.



(b) This is planar with the following planar embedding.



(c) This is planar with the following planar embedding. This is an important example! An edge subdivision CANNOT use one of the vertices in two edges.



Math 239 Fall 2023 Tutorial Review Session
This week was a review session and so answers were not written in this file.