Asymptotics of Lattice Walks Via ACSV

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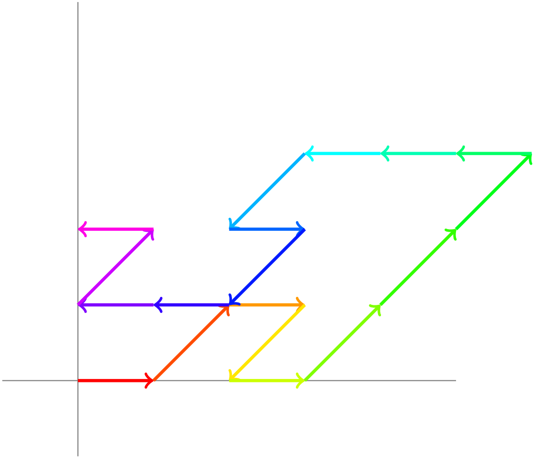

Figure: An Orthant Walk using the *Gessel* step Set

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- **exponential** behaviour of $a_n \leftrightarrow$ location of singularities
- **subexponential** behaviour of $a_n \leftrightarrow$ kind of singularities

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Here $C = \frac{1}{2 \log 2}$ is easily computed. If $H(z) = \frac{1}{(2-e^z)^2}$ then

$$\frac{h_n}{n!} \sim Cn \cdot \left(\frac{1}{\log 2} \right)^n.$$

There are two main techniques in AC:

No. 1: Residue Theorem

Let $G(z) = \sum g_n z^n$ be meromorphic in $D_r(0)$ and analytic at 0. Let $\rho_1, \cdots, \rho_k$ be a list of all the singularities of G in $D_r(0)$. Then

$$g_n = -\sum_{j=1}^{k} Res_{z=\rho_j}\left(\frac{G(z)}{z^{n+1}}\right) + \frac{1}{2\pi i}\int_{|z|=r} G(z)\frac{dz}{z^{n+1}}.$$

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If $G(z)$ has no singularities we are out of luck!

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  by replacing $G(z)z^{-n-1}$ by its leading terms,
- (compute) compute what's left.

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We also need a **direction**; we will only talk about the **main diagonal** of a generating function: given $F = \sum_{j_1,\cdots,j_d} f_{j_1,\cdots,j_d} \mathbf{z}^{\mathbf{j}}$ we consider $\Delta F = \sum_n f_{n,\cdots,n} t^n$.

We have a specialized version of the Cauchy Integral Theorem.

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Then

$$f_n = \frac{1}{(2\pi i)^d} \int_{|z_j| = \alpha_j} G(z_1, \cdots, z_d) \frac{dz_1 \cdots dz_d}{z_1^{n+1} \cdots z_d^{n+1}}$$

for α_j sufficiently small for all j.

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Definition (Minimal Critical Points)

Suppose that $F = G/H$. Then $\mathbf{z}$ is a **smooth critical point** of F if $H_{z_1}(\mathbf{z}) \neq 0$ and it is a solution to the system of equations

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Our analysis depends on finding minimal critical points.

Central Binomial Coefficients

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$$\begin{aligned}
\binom{2n}{n} &= \frac{1}{(2\pi i)^2} \int_{\substack{|x|=1/2 \\ |y|=1/2-\epsilon}} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}} \\
&\sim \frac{1}{2\pi i} \int_{\substack{|x|=1/2 \\ \arg(x)\in(-\pi/4,\pi/4)}} \frac{dx}{x^{n+1}(1-x)^{n+1}} \\
&\sim \frac{4^n}{2\pi} \int_{-\pi/4}^{\pi/4} \frac{1}{1-e^{it}/2} e^{-n(\log(2-e^{it})+it)} dt \sim \frac{4^n}{\sqrt{\pi n}}
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A d-dimensional **weighted short-step lattice walk model** consists of:

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- a **restricting region** $R \subseteq \mathbb{Z}^d$,
- a **starting point** $p \in R$,
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We always let $p = 0$ and $T = R$.

We want a convenient way to store S.

Definition (characteristic polynomial)

The **characteristic polynomial** of a step set S is

$$S(\mathbf{z}) = \sum_{s \in S} \omega_s \mathbf{z}^s = \sum_{s \in S} \omega_s z_1^{s_1} \cdots z_d^{s_d}$$

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We write $\overline{z}$ to mean z^{-1}.

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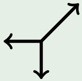
Kreweras Step Set

Let $S = \{(-1,0),(0,-1),(1,1)\}$. This is the *Kreweras* step set.



Then $S(x,y) = \overline{x} + \overline{y} + xy$.

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Note that a d-dimensional orthant walk encodes a *multiqueue system* with d queues: each point of $\mathbb{N}^d$ gives the number of people in each queue.

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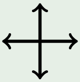

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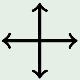

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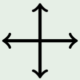

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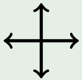

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If S is **mostly symmetric** it does.

Definition (Symmetric Step Sets)

- A step set S is **mostly symmetric** if

$$S(z_1, \cdots, z_d) = S(z_1, \cdots, \bar{z}_j, \cdots, z_d) \quad 1 \leq j \leq d-1,$$

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$$S(\mathbf{z}) = \overline{z}_d A(\hat{\mathbf{z}}) + Q(\hat{\mathbf{z}}) + z_d B(\hat{\mathbf{z}})$$

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Notation

We also write $\overline{S}(\mathbf{z}) = z_d A(\hat{\mathbf{z}}) + Q(\hat{\mathbf{z}}) + \overline{z}_d B(\hat{\mathbf{z}})$.

Theorem (Melczer Wilson [MW19])

Let S be a mostly symmetric short-step step set. If f_n is the number of orthant walks of length n using S then

$$F(t) = \sum_{n \geq 0} f_n t^n = \Delta \left(\frac{(1+z_1)\cdots(1+z_{d-1})\left(B(\hat{\mathbf{z}}) - z_d^2 A(\hat{\mathbf{z}})\right)}{(1-z_d)B(\hat{\mathbf{z}})(1 - tz_1 \cdots z_d \overline{S}(\mathbf{z}))} \right).$$

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There are 4 different cases to consider.

Theorem (Melczer Wilson [MW19])

Let S be a mostly symmetric short-step step set. If f_n is the number of orthant walks of length n using S then

$$F(t) = \sum_{n \geq 0} f_n t^n = \Delta \left(\frac{(1+z_1)\cdots(1+z_{d-1})\left(B(\hat{\mathbf{z}}) - z_d^2 A(\hat{\mathbf{z}})\right)}{(1-z_d)B(\hat{\mathbf{z}})(1 - tz_1 \cdots z_d \overline{S}(\mathbf{z}))} \right).$$

Proof.

Use the kernel method. □

There are 4 different cases to consider.
These will be demonstrated by example.

Highly Symmetric Step Set Asymptotics

In the simplest case where S is highly symmetric, this reduces to

$$F(t) = \sum_{n \geq 0} f_n t^n = \Delta \left(\frac{(1+z_1)\cdots(1+z_d)}{1 - tz_1 \cdots z_d \overline{S}(\mathbf{z})} \right).$$

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NSEW Steps

Consider

$$F(x,y,t) = \frac{G(x,y,t)}{H(x,y,t)} = \frac{(1+x)(1+y)}{1 - txyS(x,y)}$$

where $S(x,y) = x + \overline{x} + y + \overline{y}$.

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where $S(x,y) = x + \overline{x} + y + \overline{y}$. This has two minimal CPs: $\sigma_- = (-1,-1,-1/4)$ and $\sigma_+ = (1,1,1/4)$. Since G vanishes at $(-1,-1)$ then σ_+ determines dominant asymptotics.

NSWE Steps cont.

CIT implies that

$$f_n \sim \frac{1}{(2\pi i)^3} \int_{\substack{|x|=1 \\ |y|=1}} \left(\int_{|t|=1/4-\epsilon} \frac{(1+x)(1+y)}{1-txyS(x,y)} \frac{dt}{t^{n+1}} \right) \frac{dxdy}{x^{n+1}y^{n+1}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_N \frac{(1+x)(1+y)}{xy} S(x,y)^n dxdy$$

$$\sim \frac{4^n}{(2\pi)^2} \int_K 4e^{-n(x^2/4+y^2/4)} dxdy$$

$$\sim \frac{4}{\pi} \cdot \frac{4^n}{n}.$$

The techniques in this example generalize to arbitrary highly symmetric step sets.

Theorem (Melczer Mishna 2016 [MM16])

Suppose S is a highly symmetric step set. If s_n is the number of walks of length n using S that remains in $\mathbb{N}^d$ then

$$s_n \sim \left[\frac{S(\mathbf{1})^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}}\right] \cdot \frac{S(\mathbf{1})^n}{n^{d/2}},$$

where a_j is the number of steps in S that have j-th coordinate 1.

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Cardinal Directions

If $S = \{\pm e_1, \cdots, \pm e_d\}$ then

$$s_n \sim \left[\frac{(2d)^{d/2}}{\pi^{d/2}}\right] \cdot \frac{(2d)^n}{n^{d/2}}$$

Mostly Symmetric Step Set Asymptotics - Setup

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the factor of B in the denominator behaves uncontrollably. Instead we write $F(t)$ as the diagonal of

$$\frac{(1+z_1)\cdots(1+z_{d-1})\left(1-tz_1\cdots z_d(Q(\hat{\mathbf{z}})+2z_dA(\hat{\mathbf{z}}))\right)}{(1-z_d)(1-tz_1\cdots z_d\overline{S}(\mathbf{z}))(1-tz_1\cdots z_d(Q(\hat{\mathbf{z}})+z_dA(\hat{\mathbf{z}})))}.$$

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This looks awful, but it's easier to deal with the denominator.

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In each of the different cases the singular variety is differently positioned, and so the asymptotics change.

Mostly Symmetric Step Set Asymptotics - Negative Drift

The second simplest case is negative drift mostly symmetric.

Negative Drift

Let $S = \{(-1,-1),(1,-1),(0,1)\}$. Then the GF is the diagonal of

$$\frac{(1+x)(1-2t(x^2y^2+1))}{(1-y)(1-t(x^2y^2+y^2+x))(1-t(x^2y^2+1))}.$$

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There are now four minimal critical points

$$(1,1/\sqrt{2},1/2),(1,-1/\sqrt{2},1/2),(-1,\pm i/\sqrt{2},-1/2).$$

The numerator vanishes at the last two CPs, implying that the first two give dominant asymptotics. Only one factor of the denominator vanishes (the "smooth" case).

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The numerator vanishes at the last two CPs, implying that the first two give dominant asymptotics. Only one factor of the denominator vanishes (the "smooth" case). We have

$$s_n \sim \frac{16+12\sqrt{2}}{\pi}\cdot\frac{(2\sqrt{2})^n}{n^2}+\frac{-16+12\sqrt{2}}{\pi}\cdot\frac{(-2\sqrt{2})^n}{n^2}$$

In general we have the following.

Theorem (Melczer Wilson [MW19])

If S is mostly symmetric with negative drift and $Q(z) \neq 0$ then

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}}.$$

If S is mostly symmetric with negative drift and $Q(z) = 0$ then

$$s_n \sim C_\rho \cdot \frac{S(1,\rho)^n}{n^{d/2+1}} + C_{-\rho} \cdot \frac{S(1,-\rho)^n}{n^{d/2+1}}.$$

Where ρ, C_ρ, and $C_{-\rho}$ are explicit constants.

Mostly Symmetric Step Set Asymptotics - Positive Drift

In the positive drift case we encounter CPs that lie on the intersection of two factors of the denominator.

Positive Drift

Let $S = \{(-1,1),(1,1),(0,-1)\}$. Then the GF is the diagonal of

$$F(x,y,t) = \frac{(1+x)(1-2txy^2)}{(1-y)(1-t(xy^2+x^2+1))(1-txy^2)}.$$

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The only minimal CP is $(1,1,1/3)$. Two terms in the denominator vanish at this point! This means we are no longer in the "smooth" case. Since the main diagonal is "generic" we are able to take two residues.

Positive Drift cont.

$$s_n \sim \frac{1}{(2\pi i)^3} \int_{|x|=1} \left(\int_{\substack{|y|=1-\epsilon \\ |t|=1/3-\epsilon}} F(x,y,t) \frac{dydt}{y^{n+1}t^{n+1}} \right) \frac{dx}{x^{n+1}}$$

$$\sim \frac{1}{2\pi i} \int_N \frac{(x^2-x+1)(1+x)}{x(x^2+1)} (x+1+1/x)^n dx$$

$$\sim \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}.$$

Theorem (Melczer Wilson [MW19])

If S is mostly symmetric with positive drift then

$$s_n \sim \left[\left(1 - \frac{A(1)}{B(1)}\right) \frac{S(1)^{d/2}}{(2\pi)^{d/2}} \cdot \frac{1}{(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{S(1)^n}{n^{d/2-1/2}}$$

Mostly Symmetric Step Set Asymptotics - Zero Drift

In the zero drift case $A(\mathbf{1}) = B(\mathbf{1})$; we expect the numerator to vanish when $\mathbf{z} = \mathbf{1}$.

Zero Drift

Let $S = \{(-1,-1),(1,-1),2\cdot(0,1)\}$. Then the GF is the diagonal of

$$F(x,y,t) = \frac{(1+x)(1-2ty^2(x^2+1))}{(1-y)(1-txy(2/y+xy+y/x))(1-ty^2(x^2+1))}.$$

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There are four minimal critical points

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There are four minimal critical points

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It turns out the contribution from the last three CPs is negligible, and only the first matters. Two terms in the denominator vanish, but this time the main diagonal is “non-generic”, and so we are only able to take one residue.

Zero Drift cont.

$$s_n \sim \frac{1}{(2\pi i)^3} \int_{|x|=1} \left(\int_{\substack{|y|=1-\epsilon \\ |t|=1/4-\epsilon}} F(x,y,t) \frac{dydt}{y^{n+1} t^{n+1}} \right) \frac{dx}{x^{n+1}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{N(1)} \int_{N(1-\epsilon)} \frac{1+x}{2x^2 y} \cdot \frac{2x - y^2(1+x^2)}{1-y} \overline{S}(x,y)^n dxdy.$$

Zero Drift cont.

$$
\begin{aligned}
s_n &\sim \frac{1}{(2\pi i)^3} \int_{|x|=1} \left(\int_{\substack{|y|=1-\epsilon \\ |t|=1/4-\epsilon}} F(x,y,t) \frac{dydt}{y^{n+1}t^{n+1}} \right) \frac{dx}{x^{n+1}} \\
&\sim \frac{1}{(2\pi i)^2} \int_{N(1)} \int_{N(1-\epsilon)} \frac{1+x}{2x^2 y} \cdot \frac{2x - y^2(1+x^2)}{1-y} \overline{S}(x,y)^n dxdy.
\end{aligned}
$$

We cannot take any more residues, and must use saddle-point techniques. Note that at $x = y = 1$ both the numerator and denominator of $P(x,y) = \frac{2x-y^2(1+x^2)}{1-y}$ vanish.

Zero Drift cont.

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$$\sim \frac{1}{(2\pi i)^2}\int_{N(1)}\int_{N(1-\epsilon)}\frac{1+x}{2x^2y}\cdot\frac{2x-y^2(1+x^2)}{1-y}\overline{S}(x,y)^n dxdy.$$

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We cannot take any more residues, and must use saddle-point techniques. Note that at $x = y = 1$ both the numerator and denominator of $P(x,y) = \frac{2x-y^2(1+x^2)}{1-y}$ vanish. One can carefully show that $P(x,y)$ has leading term equal to 4, with other terms contributing a negligible amount to asymptotics. Thus

$$s_n \sim \frac{4}{(2\pi i)^2} \int_N \frac{1+x}{2x^2y} \overline{S}(x,y)^n dxdy \sim \frac{4^n}{n} \cdot \frac{2\sqrt{2}}{\pi}.$$

Dealing with this tricky term gives us the following result.

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Theorem (K. Melczer [KM24])

Suppose S is a mostly symmetric step set with zero drift. Let s_n be the number of walks of length n using S that remain in $\mathbb{N}^d$. Then

$$s_n \sim \left[\frac{|S|^{d/2}}{\pi^{d/2}(a_1 \cdots a_d)^{1/2}} \right] \cdot \frac{|S|^n}{n^{d/2}}$$

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Higher Order Terms

We can find the dominant term for zero drift walks. What about higher order terms?

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Need to get more terms in the expansion of $P(x, y)$.

One can show

$$\frac{1}{(2\pi i)^2}\int_{N(1)}\int_{N(1-\epsilon)}\frac{1+x}{2x^2y}\cdot\frac{2x-y^2(1+x^2)}{1-y}\overline{S}(x,y)^n dxdy$$

$$\sim\frac{1}{(2\pi)^2}\int_{K+i(0,\epsilon)}\left(4+\frac{is^2}{t}\right)e^{-n(s^2/4+t^2/2)}dsdt$$

$$\sim\frac{4^n}{n}\cdot\frac{2\sqrt{2}}{\pi}+\frac{4^n}{n^{3/2}}\cdot\frac{1}{\sqrt{\pi}}.$$

Conclusion

To find the higher order asymptotic terms of f_n we need to be able to evaluate integrals of the form

$$\int_{K+i\epsilon} \frac{A(\mathbf{z})}{\mathbf{z}^{\mathbf{k}}} \exp\left[-n\phi(\mathbf{z})\right] d\mathbf{z}.$$

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Fin.

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