Vacuum polarization and photon propagation in an electromagnetic plane wave

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The QED vacuum polarization in external monochromatic plane-wave electromagnetic fields is calculated with spatial and temporal variations of the external fields being taken into account. We develop a perturbation theory to calculate the induced electromagnetic current that appears in the Maxwell equations, based on Schwinger's proper time method, and combine it with the so-called gradient expansion to handle the variation of external fields perturbatively. The crossed field, i.e., the long wavelength limit of the electromagnetic wave, is first considered. The eigenmodes and the refractive indices, as the eigenvalues associated with the eigenmodes, are computed numerically for the probe photon propagating in some particular directions. In so doing, no limitation is imposed on the field strength and the photon energy, unlike previous studies. It is shown that the real part of the refractive index becomes less than unity for strong fields, a phenomenon that has been known to occur for high-energy probe photons. We then evaluate numerically the lowest-order corrections to the crossed field resulting from the field variations in space and time. It is demonstrated that the corrections occur mainly in the imaginary part of the refractive index.
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1. Introduction

In the quantum vacuum, virtual particles and anti-particles are produced and annihilated repeatedly in very short times as intuitively represented by bubble Feynman diagrams. When an external field is applied, even these virtual particles are affected, leading to modifications of the property of quantum vacuum. One of the interesting consequences is a deviation of the refractive index from unity accompanied by a birefringence, i.e., distinct refractive indices for different polarization modes of the photon.[1] It is a purely quantum effect that becomes remarkable when the strength of the external field approaches or even exceeds the critical value $f_c = m^2/e$, with m and e being the electron mass and the elementary charge, respectively, whereas the deviation of the refractive index from unity is proportional to the square of the field strength for much weaker fields. Photon splitting, which is another phenomenon in external fields, has also been considered [2].

Such strong electromagnetic fields are not unrealistic these days. In fact, the astronomical objects called magnetars, a subclass of neutron stars, are believed to have dipole magnetic fields of $\sim 10^{14-15}$ G [3].[2] Although the origin of such strong magnetic fields is still unknown, they are

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[1] Interestingly, this does not occur for the non-linear electrodynamics theory by Born and Infeld [1].

[2] The online catalog of magnetars is found at http://www.physics.mcgill.ca/~pulsar/magnetar/main.html, date last accessed March 5, 2018.

supposed to have implications for various activities of magnetars such as giant flares and X-ray emissions [4]. In fact, their strong magnetic fields are thought to affect the polarization properties of surface emissions from neutron stars by the quantum effect [5,6]. This phenomenon may indeed have been detected in a recent optical polarimetric observation [7]. The quantum correction may also play an important role through the so-called resonant mode conversions [8–10]. On the other hand, the progress in the high-field laser is very fast. Although the highest intensity realized so far by the HERCULES laser at CUOS [11] is still sub-critical (2×10^{22} W cm^{-2}) for the moment, we may justifiably expect that the laser intensity will reach the critical value in the not too distant future. Some theoretical studies on vacuum polarization are intended for experimental setups of the high-field laser [12–16].

The study of vacuum polarization in strong-field QED has a long history. It was pioneered by Toll in 1952 [17]. He studied in his dissertation the polarization of vacuum in stationary and homogeneous magnetic fields in detail, and many authors followed with different methods, both analytic and numerical [18–28], and obtained the refractive indices. The vacuum polarization for mixtures of constant electric and magnetic fields was also investigated [29–34]. Note that such fields can be brought to either a purely magnetic or a purely electric field by an appropriate Lorentz transformation, with so-called crossed fields being an exception.

In Ref. [17], the polarization in the crossed field was also discussed. The crossed field may be regarded as a long-wavelength limit of electromagnetic waves, having mutually orthogonal electric and magnetic fields of the same amplitude. Toll first calculated the imaginary part of the refractive index from the amplitude of pair creations and then evaluated the real part of the refractive index via the Kramers–Kronig relation. Although there was no limitation to the probe-photon energy, the external field strength was restricted to small values (weak-field limit) because he ignored the modification of the dispersion relation of the probe photon. Baier and Breitenlohner [35] obtained the refractive index for the crossed field in two different ways: they first employed the polarization tensor that had been inferred in Ref. [18] from the one-loop calculation for the external magnetic fields, and utilized in the second method the expansion of the Euler–Heisenberg Lagrangian to the lowest order of field strength. Note that both approaches are valid only for weak fields or low-energy probe photons.

The expression of the polarization tensor to the full order of field strength for the external crossed field was obtained from the one-loop calculation with the electron propagator derived either with Schwinger's proper time method [36] or with Volkov's solution [37]. In Ref. [36], the general expressions for the dispersion relations and the refractive indices of the two eigenmodes were obtained. Note, however, that the refractive indices were evaluated only in the weak-field and strong-field limits.[3] On the other hand, another expression of the polarization tensor was obtained and its asymptotic limit was derived in Ref. [37], although the refractive index was not considered.

The evaluation of the refractive index based on the polarization tensor of Ref. [37] was attempted by Heinzl and Schröder [38] in two different ways: the first one is based on the hypothesized expression of the polarization tensor in the so-called large-order expansion with respect to the probe-photon energy; in the evaluation of the real part of the refractive index, the external crossed field was taken into account only to the lowest order of the field strength in each term of the expansion and the

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[3] Although these limits are referred to as "weak-field limit" and "strong-field limit" in the literature, they may be better called "weak-field or low-energy limit" and "strong-field and high-energy limit," respectively. See Fig. 2 for the actual parameter region.

imaginary part was estimated from the hypothesized integral representation. In the second approach, the polarization tensor was expanded with respect to the product of the external field strength and the probe-photon energy, and the refractive index was evaluated; the imaginary part was calculated consistently to the leading order and the anomalous dispersion for high-energy probe photons, which had been demonstrated by Toll [17], was confirmed. Note that in these evaluations of the refractive index in the crossed field, the modification of the dispersion relation for the probe photon was again ignored as in Ref. [17], and hence the results cannot be applied to super-critical fields.

It should now be clear that the vacuum polarization and the refractive index have not been fully evaluated for supra-critical field strengths even in the crossed field. One of our goals is therefore to do just that.

It is understandable, on the other hand, that the evaluation of the refractive index in the external electromagnetic plane wave is more involved because of its non-uniformity. In fact, the refractive index has not been obtained except for some limiting cases. The polarization tensor and the refractive index in the external plane wave were first discussed by Becker and Mitter [39]. They derived the polarization tensor in momentum space from the one-loop calculation with the electron propagator obtained by Mitter [40], which is actually Volkov's propagator represented in momentum space. Although the formulation is complete, the integrations were performed only for circularly polarized plane waves as the background. The refractive indices were then evaluated at very high energies ($\gg m$) of the probe photon.

Baier et al. [41] calculated the scattering amplitudes of a probe photon, again by the circularly polarized external plane wave to the one-loop order, employing the electron propagator expressed with the proper time integral. The general expression of the dispersion relation was obtained but evaluated only in the weak-field and low-energy limit. The refractive indices for the eigenmodes of probe photons were also calculated in this limit alone. Affleck [42] treated this problem by expanding the Euler–Heisenberg Lagrangian to the lowest order of the field strength, assuming that the external field varies slowly in time and space. The refractive index was evaluated only in the weak-field limit again. Recently, yet another representation of the polarization tensor in the external plane wave was obtained from calculation of the one-loop diagram with Volkov's electron propagator [43]. Only the expression of the polarization tensor was obtained, however, and no attempt was made to evaluate it in this study.

In Ref. [13], Dinu et al. employed the light front field theory, one of the most mathematically sophisticated formulations, to derive the amplitude of photon–photon scatterings, from which the refractive index integrated over the photon path was obtained. They calculated it for a wide range of probe-photon energy and field strength. Although they gave the expression for the local refractive index, it was not evaluated. The eigenmodes of probe photons were not calculated, either.

In this paper, we also derive the expression of the polarization tensor and the refractive index for the external electromagnetic plane wave, developing a perturbation theory for the induced electromagnetic current based on the proper time method. It is similar to Adler's formulation [20] but is more general, based on the interaction picture, or Furry's picture, and not restricted to a particular field configuration. Combining it with the so-called gradient expansion, we calculate the lowest-order correction from temporal and spatial field variations to the induced electromagnetic current, and hence also to the vacuum polarization tensor, for the crossed fields. This is nothing but the WKB approximation and, as such, may be applicable not only to the electromagnetic wave but also to any slowly varying background electromagnetic fields. We then evaluate numerically the refractive indices for eigenmodes of the Maxwell equations with the modification of the dispersion relation

being fully taken into account. Note that unlike Ref. [13], our results are not integrated over the photon path but local, being obtained at each point in the plane wave.

This paper is organized as follows: We first review Schwinger's proper time method briefly, and then outline the perturbation theory based on the Furry picture to obtain the induced electromagnetic current to the linear order of the field strength of the probe photon in Sect. 2. This is not new stuff. We then apply it to the plane wave background in Sect. 3; in so doing, we also appeal to the so-called gradient expansion of the background electromagnetic wave around the crossed field. Technical details are given in appendices. Numerical evaluations are performed both for the crossed fields and for the first-order corrections in Sect. 4; we summarize the results and conclude the paper in Sect. 5.

2.   Perturbation theory in the proper time method

In this section, we briefly summarize Schwinger's proper time method and outline its perturbation theory, which will be applied to monochromatic plane waves in the next section.

2.1.   Schwinger's proper time method

The effective action of electromagnetic fields is represented as

$$\Gamma = \Gamma_{\mathrm{cl}} + \Gamma_{\mathrm{q}}, \tag{1}$$

where Γ_{cl} is the classical action and Γ_{q} is the quantum correction, which satisfies the following relation:

$$\frac{\delta \Gamma_{\mathrm{q}}}{\delta A_\mu} \equiv \langle j^\mu(x) \rangle = ie \, \mathrm{tr}[\gamma^\mu G(x,x)]. \tag{2}$$

Then, the vacuum Maxwell equation is modified as

$$-\Box A_\mu + \partial^\nu \partial_\mu A_\nu - \langle j_\mu \rangle = 0. \tag{3}$$

Although there is no electromagnetic current generated by real charged particles in the vacuum, $\langle j_\mu \rangle$ defined in this way is referred to as the induced electromagnetic current [33]. This term can be written with the electron propagator $G(x,y)$ [44] with tr in Eq. (2) being the trace, or the diagonal sum on spinor indices; the γ^μ are the gamma matrices. In this paper, the Greek indices run over 0 through 3 and the Minkowski metric is assumed to be $\eta = \mathrm{diag}(+,-,-,-)$.

The electron propagator G in the external electromagnetic field is different from the ordinary one in the vacuum, and the modification by the external field, the strength of which is close to or even exceeds the critical value f_{c}, cannot be treated perturbatively. The proper time method is a powerful tool to handle such situations. The electron propagator satisfies the Dirac equation in the external electromagnetic field A^μ:

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu(x) - m)G(x,y) = \delta^4(x-y). \tag{4}$$

It is supposed in the proper time method that there exists an operator $\hat{G}$, the x-representation of which gives the propagator as $\langle x|\hat{G}|y \rangle = G(x,y)$. Then Eq. (4) can be cast into the following equation for the operators:

$$(\gamma^\mu \hat{\Pi}_\mu - m)\hat{G} = \hat{\mathbf{1}}, \tag{5}$$

where $\hat{\mathbf{1}}$ is the unit operator and $\hat{\Pi}_\mu = i\partial_\mu - eA_\mu$. Here we used $\delta^4(x - y) = \langle x|y\rangle$. From this equation, the operator $\hat{G}$ is formally solved as

$$\hat{G} = \frac{\hat{\mathbf{1}}}{\gamma^\mu \hat{\Pi}_\mu - m}, \tag{6}$$

which can be cast into the following integral form:

$$\hat{G} = i(-\gamma^\mu \hat{\Pi}_\mu - m) \int_0^\infty ds \exp\left[-i(m^2 - (\gamma^\mu \hat{\Pi}_\mu)^2 - i\varepsilon)s\right]. \tag{7}$$

In the above expression, the parameter s is called the proper time and $-i\varepsilon$ is introduced to make the integration convergent as usual; it will be dropped hereafter for brevity. The electron propagator, being an x-representation of this operator, is obtained as

$$G(x, y) = i \int_0^\infty ds\, e^{-im^2 s} \left[\langle x| - \gamma^\mu \hat{\Pi}_\mu e^{-i(-(\gamma^\nu \hat{\Pi}_\nu)^2)s}|y\rangle - \langle x|m e^{-i(-(\gamma^\mu \hat{\Pi}_\mu)^2)s}|y\rangle\right]. \tag{8}$$

Here we had better comment on the boundary condition for the electron propagator, or the causal Green function, in the electromagnetic wave. This issue may be addressed most conveniently for finite wave trains in the so-called light front formulation (e.g. Refs. [45,46]), in which double null coordinates are employed. This is because the asymptotic states in the remote past and future (in the null-coordinate sense) are unambiguously defined [46], which is crucially important, particularly when one calculates S-matrix elements [13,46]; it is also important that the translational symmetry is manifest in one of the null coordinates. Then the causal Green function is obtained in the usual way, i.e., by the appropriate linear combination of the homogeneous Green functions with positive and negative energies according to the time ordering in the null coordinate [45,46]. On the other hand, it is a well-known fact that the Dirac equation can be solved in a closed form for an arbitrary plane wave [40,47]. It is then possible to construct the same causal Green function with these Volkov solutions [37,40]. According to Ritus [37], all that is needed is a well-known $-i\epsilon$ prescription, i.e., the introduction of an infinitesimal negative imaginary mass. It was then pointed out by Mitter [40] that this is equivalent to the same prescription in the proper time method of Schwinger, that is, the formulation we adopt in this paper [see Eq. (7)]. In this sense, the propagator we employ in this paper is the causal Green function thus obtained in the limit of the infinite wave train. As will become clear later [see Eq. (38) in Sect. 3], since we employ the gradient expansion in the local approximation, the distinction between the finite or infinite wave train will not be important in our formulation.

Returning to Eq. (8) and interpreting the operator $e^{-i(-(\gamma^\mu \hat{\Pi}_\mu)^2)s}$ as the evolution operator in proper time, one can reduce the original field-theoretic problem to one in quantum mechanics for the Hamiltonian $H = -(\gamma^\mu \hat{\Pi}_\mu)^2$. Then the transformation amplitude is given as

$$\langle x|e^{-i(-(\gamma^\mu \hat{\Pi}_\mu)^2)s}|y\rangle = \langle x|e^{-iHs}|y\rangle$$

$$= \langle x(s)|y(0)\rangle. \tag{9}$$

Here, the state $|x(s)\rangle$ is defined as the eigenstate for the operator $\hat{x}$ in the Heisenberg picture:

$$|x(s)\rangle \equiv e^{iHs}|x\rangle. \tag{10}$$

The Hamiltonian H is expressed as

$$H = -\hat{\Pi}^2 + \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}, \tag{11}$$

where we used the Clifford algebra for the gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and the commutation relation $[\Pi^\mu, \Pi^\nu] = -ieF^{\mu\nu}$ to obtain $\hat{\Pi}^2 = \hat{\Pi}_\mu \hat{\Pi}^\mu$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$; A_μ and $F_{\mu\nu}$ are the vector potential and the field tensor for the external electromagnetic field, respectively. The proper time evolutions of the operators $\hat{x}$ and $\hat{\Pi}$ are given by the Heisenberg equations:

$$\frac{d\hat{x}^\mu(s)}{ds} = 2\hat{\Pi}^\mu(s), \tag{12}$$

$$\frac{d\hat{\Pi}^\mu(s)}{ds} = 2eF^\mu{}_\nu \hat{\Pi}^\nu(s) + e^{iHs} ie \frac{\partial F^\mu{}_\nu}{\partial x_\nu} e^{-iHs} + e^{iHs} \frac{1}{2} e\sigma^{\nu\lambda} \frac{\partial F_{\nu\lambda}}{\partial x_\mu} e^{-iHs}. \tag{13}$$

Then, the induced electromagnetic current $\langle j^\mu \rangle$ in Eq. (2) is represented as follows [20]:

$$\langle j^\mu(x) \rangle = \frac{e}{2} \int_0^\infty ds\, e^{-im^2 s} \text{tr} \left[\langle x(s)|\hat{\Pi}^\mu(s) + \hat{\Pi}^\mu(0)|x(0)\rangle - i\sigma^{\mu\nu} \langle x(s)|\hat{\Pi}_\nu(s) - \hat{\Pi}_\nu(0)|x(0)\rangle \right]. \tag{14}$$

Note that this is equivalent to the one-loop approximation with the external field being fully taken into account.

In order to obtain the refractive index of the vacuum in the presence of an external electromagnetic field, we have to consider a probe photon in addition to the background electromagnetic field and apply Eq. (3) to the amplitude of the probe photon. In so doing, the induced electromagnetic current $\langle j_\mu \rangle$ needs to be evaluated to the linear order of the amplitude of the probe photon, and perturbation theory is required at this point [33]. The Heisenberg equations given above can be solved analytically for some limited cases, such as time-independent homogeneous electric or magnetic fields and single electromagnetic plane waves [44]. We will employ the latter as an unperturbed solution in the perturbative calculations in Sect. 3. It is stressed that calculating the effective action for a given plane wave background and taking its derivative with respective to the field strength is not sufficient for the evaluation of the refractive index, since the probe photon in general has a different wavelength and propagates in a different direction from those of the background electromagnetic wave. We hence need to take these differences fully into account in the perturbative calculations of the induced electromagnetic current. This was essentially done by Ref. [39] in a different framework, i.e., performing one-loop calculations in momentum space. In this paper we assume that the background wave has a long wavelength and calculate the refractive index locally in the sense of the WKB approximation. In so doing, we appeal to the gradient expansion of the background plane wave, as explained in Sect. 3.

2.2. Outline of perturbation theory

We now consider the perturbation theory in the proper time method. The purpose is to evaluate the induced electromagnetic current Eq. (14) up to the linear order of the amplitude of the probe photon, which is supposed to propagate in an external electromagnetic field. It is then plugged into Eq. (3) to derive the refractive indices. The Heisenberg equations of Eqs. (12) and (13) can be analytically solved for a single monochromatic electromagnetic plane wave [44]. The calculation of the first-order corrections to this solution is the main achievement in this paper. As explained in the next section, we further employ the gradient expansion of the background electromagnetic wave, which in turn enables us to obtain the refractive indices locally in the WKB sense. In this section, we give the outline of the generic part of this perturbation theory, which is not limited to the plane wave background. We will then proceed to its application to the monochromatic plane wave background in the next section.

In the perturbation theory, the external electromagnetic fields are divided into two pieces: the background A^μ and the perturbation b^μ. The corresponding field strengths are denoted by $F_{\mu\nu}$ and $g_{\mu\nu}$, respectively. We take the latter into account only to the first order. Then, the Hamiltonian given in Eq. (11) can be written as

$$H = -\left(i\partial_\mu - eA_\mu(\hat{x}) - eb_\mu(\hat{x})\right)^2 + \frac{1}{2}e\sigma^{\mu\nu}\left(F_{\mu\nu}(\hat{x}) + g_{\mu\nu}(\hat{x})\right)$$

$$= H^{(0)} + \delta H. \tag{15}$$

In this expression, $H^{(0)}$ is the unperturbed Hamiltonian, for which we assume that the proper time evolution is known, preferably analytically as in the time-independent homogeneous electric or magnetic fields and the single plane wave. δH is the perturbation to the Hamiltonian. It is evaluated to the first order of b_μ and expressed with $\delta\Pi_\mu = -eb_\mu$ as

$$\delta H = -\hat{\Pi}^{(0)}_\mu \delta\hat{\Pi}^\mu - \delta\hat{\Pi}^\mu \hat{\Pi}^{(0)}_\mu + \frac{1}{2}e\sigma^{\mu\nu}g_{\mu\nu}. \tag{16}$$

In the proper time method, the amplitudes of operators such as $\langle x(s)|\hat{\Pi}_\mu(s)|x(0)\rangle$ are evaluated very frequently, and in the perturbation theory they need to be calculated with perturbations to both the operators and the states being properly taken into account. In so doing, we employ the interaction picture, which is also referred to as the Furry picture [46] in the current case, rather than making full use of the properties of particular field configurations as in Ref. [20]. The relation between the operator in the Heisenberg picture $\hat{A}_H(u)$ and that in the interaction picture $\hat{A}_I(u)$ is then given by the transformation $\hat{A}_H(u) = U^{-1}(u)\hat{A}_I(u)U(u)$, where the operator $U(u)$ is written as $U(u) = e^{iH^{(0)}u}e^{-iHu}$. It also satisfies the following equation: $i\frac{\partial}{\partial u}U(u) = \delta H_I(u)U(u)$. Here the perturbation Hamiltonian in the interaction picture δH_I is given as $\delta H_I(u) \equiv e^{iH^{(0)}u}\delta H e^{-iH^{(0)}u}$. The equation of $U(u)$ can be solved iteratively as

$$U(u) = 1 + (-i)\int_0^u du_1 \delta H_I(u_1)$$

$$+ (-i)^2 \int_0^u du_1 \int_0^{u_1} du_2 \delta H_I(u_1)\delta H_I(u_2) + \cdots$$

$$+ (-i)^n \int_0^u du_1 \cdots \int_0^{u_{n-1}} du_n \delta H_I(u_1)\cdots\delta H_I(u_n)$$

$$+ \cdots. \tag{17}$$

Note that the right-hand side of this equation includes only unperturbed quantities, since the operators obey the free Heisenberg equations in the interaction picture.

The transformation amplitude $\langle x(s)|x(0)\rangle$ is also expressed with the unperturbed operators and states as

$$\langle x(s)|x(0)\rangle \simeq \langle x^{(0)}(s)|\left[1 - i\int_0^s du\,\delta H_I(u)\right]|x(0)\rangle. \tag{18}$$

In this expression, the index (0) attached to the state indicates its proper time evolution by $H^{(0)}$:

$$|x^{(0)}(s)\rangle = e^{iH^{(0)}s}|x(0)\rangle. \tag{19}$$

Since we assume that the interaction and Heisenberg pictures are coincident with each other at $u = 0$, we have

$$|x^{(0)}(0)\rangle = |x(0)\rangle. \tag{20}$$

The operators $\hat{\Pi}(u)$ and the states $|x(u)\rangle$ at an arbitrary proper time u are expressed with the unperturbed counterparts $\hat{\Pi}^{(0)}$, $\hat{x}^{(0)}$, and $|x^{(0)}\rangle$ via the operator $U(u)$ given in Eq. (17) in a similar way.

The amplitudes that appear in Eq. (14) for $\langle j^\mu(x)\rangle$ can be represented as

$$\langle x(s)|\hat{\Pi}^\mu(s)|x(0)\rangle = \langle x^{(0)}(s)|\hat{\Pi}_I^\mu(s)U(s)|x(0)\rangle, \tag{21}$$

$$\langle x(s)|\hat{\Pi}^\mu(0)|x(0)\rangle = \langle x^{(0)}(s)|U(s)\hat{\Pi}_I^\mu(0)|x(0)\rangle, \tag{22}$$

with the operators and states in the interaction picture. The calculations of these amplitudes are accomplished by the permutations of operators $\hat{x}(s)$ and $\hat{x}(0)$ with the employment of their commutation relations so that $\hat{x}(s)$ should always sit to the left of $\hat{x}(0)$.

3. Application to the single plane wave

In this section, we apply the perturbation theory outlined above to the calculation of the induced electromagnetic current in the monochromatic plane wave; see Fig. 1. It is stressed that the distinction between the electromagnetic wave train having a finite or infinite length is not important in our calculations, since they employ only local information of the electromagnetic wave in the background thanks to the gradient expansion. We first summarize the well-known results for the unperturbed background [44]. The plane wave is represented as

$$F_{\mu\nu} = f_{\mu\nu}F(\Omega\xi), \tag{23}$$

with $f_{\mu\nu}$ being a constant tensor that sets the typical amplitude of the wave and $F(\Omega\xi)$ being an arbitrary function of $\Omega\xi = \Omega n_\mu x^\mu$; Ω is a frequency of the wave and n^μ is a null vector that specifies
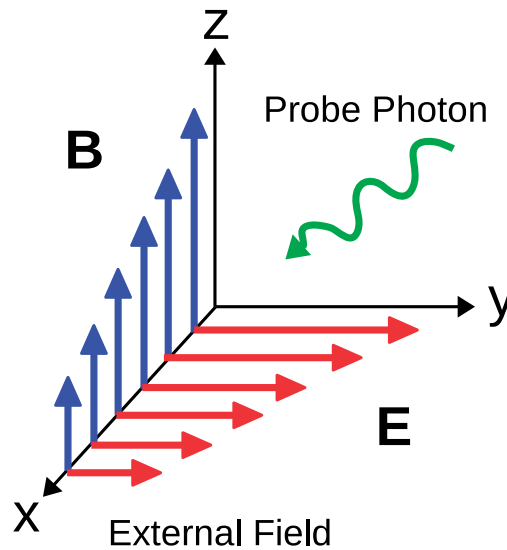
Fig. 1. Schematic picture of the system considered in this paper. The external field consists of non-uniform electric and magnetic fields denoted by E and B, respectively, and the probe photon.

the direction of wave propagation. The Heisenberg equations are written in this case as

$$\frac{d\hat{x}^\mu(s)}{ds} = 2\hat{\Pi}^\mu(s), \tag{24}$$

$$\frac{d\hat{\Pi}^\mu(s)}{ds} = 2eF^\mu{}_\nu(s)\hat{\Pi}^\nu(s) + \frac{e}{2}n^\mu f_{\nu\lambda}\sigma^{\nu\lambda}\frac{dF(\Omega\xi(s))}{d\xi(s)}. \tag{25}$$

Note that the phase $\xi(s) = n_\mu \hat{x}^\mu(s)$ in these equations is an operator and a function of the proper time s. The term that contains $\partial^\nu F^\mu{}_\nu$ vanishes in the equation of $\hat{\Pi}$ because it is written as $\partial_\nu F_\mu{}^\nu = f_\mu{}^\nu n_\nu [dF(\Omega\xi(s))/d\xi(s)]$ and the relation $f_\mu{}^\nu n_\nu = 0$ holds for the plane wave.

To solve these equations, one introduces

$$C^\mu = f^\mu{}_\nu \hat{\Pi}^\nu(s) - ef^2 n^\mu A(\Omega\xi(s)), \tag{26}$$

which one can show is a constant of motion. In this expression, $f^2 = f_{\mu\nu}f^\nu{}_\lambda/n_\mu n_\lambda$ is the amplitude squared of the plane wave and $A(\Omega\xi(s))$ is defined as a quantity that satisfies the following relation: $F(\Omega\xi(s)) = dA(\Omega\xi(s))/d\xi(s)$. Then, the operators $\hat{\Pi}^\mu(s)$ and $\hat{\Pi}^\mu(0)$ are obtained as follows:

$$\begin{aligned}
\hat{\Pi}^\mu(s) = {} & \frac{\hat{x}^\mu(s) - \hat{x}^\mu(0)}{2s} \\
& + \frac{s}{\xi(s) - \xi(0)}\left[2C^\mu eA(\Omega\xi(s)) + n^\mu e^2 f^2 A^2(\Omega\xi(s)) + \frac{1}{2}e\sigma^{\nu\lambda}f_{\nu\lambda}n^\mu F(\Omega\xi(s))\right] \\
& - \frac{s}{(\xi(s) - \xi(0))^2}\int_{\xi(0)}^{\xi(s)} d\xi(u)\left[2C^\mu eA(\Omega\xi(u)) + n^\mu e^2 f^2 A^2(\Omega\xi(u))\right. \\
& \left. \qquad\qquad\qquad\qquad\qquad + \frac{1}{2}e\sigma^{\nu\lambda}f_{\nu\lambda}n^\mu F(\Omega\xi(u))\right],
\end{aligned} \tag{27}$$

$$\begin{aligned}
\hat{\Pi}^\mu(0) = {} & \frac{\hat{x}^\mu(s) - \hat{x}^\mu(0)}{2s} \\
& + \frac{s}{\xi(s) - \xi(0)}\left[2C^\mu eA(\Omega\xi(0)) + n^\mu e^2 f^2 A^2(\Omega\xi(0)) + \frac{1}{2}e\sigma^{\nu\lambda}f_{\nu\lambda}n^\mu F(\Omega\xi(0))\right] \\
& - \frac{s}{(\xi(s) - \xi(0))^2}\int_{\xi(0)}^{\xi(s)} d\xi(u)\left[2C^\mu eA(\Omega\xi(u)) + n^\mu e^2 f^2 A^2(\Omega\xi(u))\right. \\
& \left. \qquad\qquad\qquad\qquad\qquad + \frac{1}{2}e\sigma_{\nu\lambda}f^{\nu\lambda}n^\mu F(\Omega\xi(u))\right].
\end{aligned} \tag{28}$$

C^μ is also expressed as

$$C^\mu = \frac{f^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0))}{2s} - \frac{1}{\xi(s) - \xi(0)}\int_{\xi(0)}^{\xi(s)} d\xi(u) n^\mu f^2 eA(\Omega\xi(u)). \tag{29}$$

The amplitude $\langle x'(s)|x''(0)\rangle$ is given, on the other hand, as

$$\langle x'(s)|x''(0)\rangle$$
$$= \frac{1}{i(4\pi)^2} \exp\left[-\int_{x''}^{x'} dx_\mu eA^\mu(x)\right] \frac{1}{s^2} \exp\left\{-\frac{i}{4s}(x'-x'')^2 + \frac{if^2s}{(\xi'-\xi'')^2}\left[\int_{\xi''}^{\xi'} eA(\Omega\xi)d\xi\right]^2\right.$$
$$\left. - \frac{is}{\xi'-\xi''}\int_{\xi''}^{\xi'} d\xi\left[e^2f^2A^2(\Omega\xi) + \frac{1}{2}e\sigma_{\rho\lambda}f^{\rho\lambda}F(\Omega\xi)\right]\right\}. \tag{30}$$

To derive the induced electromagnetic current $\langle j^\mu\rangle$, we use the amplitude $\langle x(s)|x(0)\rangle$, which is immediately obtained from the above equation as

$$\langle x(s)|x(0)\rangle = \frac{1}{i(4\pi)^2 s^2} \exp\left[-\frac{i}{2}e\sigma^{\alpha\beta}f_{\alpha\beta}F(\Omega\xi)s\right]$$
$$= \frac{1}{i(4\pi)^2 s^2}\left(1 - \frac{ies}{2}F(\Omega\xi)\sigma^{\alpha\beta}f_{\alpha\beta}\right). \tag{31}$$

We then find from Eqs. (27)–(30) that $\langle j^\mu\rangle$ in Eq. (14) is vanishing, as pointed out first by Schwinger in his seminal paper [44]. This situation changes, however, if another plane wave is added.

In this paper, we consider the propagation of a probe photon through the external monochromatic plane wave, with the former being treated as a perturbation to the latter as usual. We have in mind its application to high-field lasers. Since the wavelengths of these lasers are close to optical wavelengths, we assume in the following that the wavelength of the unperturbed monochromatic plane wave is much longer than the electron's Compton wavelength, or $\Omega_0/m \ll 1$ for the wave frequency Ω_0. It may be then sufficient to consider temporal and spatial variations of the unperturbed plane wave to the first order of Ω_0. This is equivalent to the so-called gradient expansion of the unperturbed field to the first order, which can be expressed generically as $F_{\mu\nu} \simeq f_{\mu\nu}(1 + \Omega\xi)$. In fact, the plane wave field given as $F_{\mu\nu} = f_{0\mu\nu}\sin(\Omega_0 n_\alpha x^\alpha)$ is Taylor-expanded at a spacetime point x_0^μ as $f_{0\mu\nu}\sin(\Omega_0 n_\alpha x_0^\alpha)(1 + \cos(\Omega_0 n_\beta x_0^\beta)\Omega_0 n_\gamma(x^\gamma - x_0^\gamma))$ to the first order. This can be recast into $F_{\mu\nu} \simeq f_{\mu\nu}(1 + \Omega n_\alpha x^\alpha)$ after shifting coordinates by x_0^α and employing the local amplitude and the gradient of the background field at x^α as $f_{\mu\nu} = f_{0\mu\nu}\sin(\Omega_0 n_\alpha x_0^\alpha)$ and $\Omega = \cos(\Omega_0 n_\beta x_0^\beta)\Omega_0$, respectively. Note that the above assumption on Ω_0 implies $\Omega/m \ll 1$.

Gusynin and Shovkovy [48] developed a covariant formulation to derive the gradient expansion of the QED effective Lagrangian, employing the world line formalism under the Fock–Schwinger gauge. Although their method is systematic and elegant indeed, the results obtained in their paper cannot be applied to our current problem of interest, since the actual calculations were done only for the following field configurations: $F_{\mu\nu} = \Phi(x_\alpha)f_{\mu\nu}$, where $\Phi(x_\alpha)$ is an arbitrary slowly varying function of x_α while $f_{\mu\nu}$ is a constant tensor; the former gives a field variation in space and time, and the latter specifies a field configuration. Although it appears quite generic, it does not include the configurations of our concern, i.e., those consisting of two electromagnetic waves propagating in different directions, unfortunately. Note that if the background and probe plane waves are both traveling in the same direction and having the identical polarization, then one may regard the sum of their amplitudes as Φ and apply the gradient expansion of Gusynin and Shovkovy [48] to them; in this case, however, Schwinger [44] already showed that there is no quantum correction to the effective Lagrangian.

In our method, the probe photon, which is also treated as a classical electromagnetic wave, is assumed to be monochromatic locally. Strictly speaking, it has neither a constant amplitude nor

a constant frequency because the external field changes temporally and spatially. As long as the wavelength of the external field is much longer than that of the probe photon, which we assume in the following, the above assumption that the probe field can be regarded as monochromatic locally may be justified. We need to elaborate on this issue a bit further, though. As Becker and Mitter developed in Ref. [39], the polarization tensor $\Pi^{\mu\nu}(x_1, x_2)$ depends not only on the difference of the two coordinates $x_1 - x_2$ but also on each of them separately and, as a result, its Fourier transform has two momenta corresponding to these coordinates. If the electromagnetic wave in the background is monochromatic, then the Floquet theorem dictates that the difference between them should be equal to some multiple of the wave vector of the electromagnetic wave in the background [49]. It follows then that eigenmodes of the probe photon are not diagonal in momentum in general. In fact, they should satisfy the following Maxwell equation:

$$\Box_{x_1} b^\mu(x_1, x_2) - \partial_{x_1}^\nu \partial_{x_1}^\mu b_\nu(x_1, x_2) = \int dx' \Pi^{\mu\nu}(x_1, x') b_\nu(x', x_2). \tag{32}$$

Becker and Mitter Fourier-transformed this equation and attempted to solve it in momentum space. Although they showed analytically that the momenta of the probe photon were indeed mixed in the expected way, they ignored the mixing in actual evaluations of the refractive index, since the effect is of higher order in the coupling constant.

We take another approach in this paper. Assuming, as mentioned above, that the electromagnetic wave in the background varies slowly in time and space, and hence the probe photon can "see" the local field strength and its gradient alone, we expand the above equation in the small gradient. In so doing, we employ the Wigner representations of variables:

$$\Pi^{\mu\nu}(x_1 - x_2 : X) = \int \frac{d^4 p}{(2\pi)^4} \tilde{\Pi}^{\mu\nu}(p, X) e^{ip(x_1 - x_2)}, \tag{33}$$

$$b^\mu(x_1 - x_2 : X) = \int \frac{d^4 p}{(2\pi)^4} \tilde{b}^\mu(p, X) e^{ip(x_1 - x_2)}, \tag{34}$$

where $X = (x_1 + x_2)/2$ is the center-of-mass coordinates and $\Pi^{\mu\nu}$ and b^μ are regarded in these equations as functions of the relative coordinates $x_1 - x_2$ and X instead of x_1 and x_2. Inserting these expressions into the right-hand side of Eq. (32) and Fourier-transforming it with respective to the relative coordinates $x_1 - x_2$, we obtain

$$\int d^4(x_1 - x_2) \int d^4 x' \Pi^{\mu\nu}(x_1, x') b_\nu(x', x_2) e^{-ip(x_1 - x_2)}$$

$$= \int d^4(x_1 - x_2) \int d^4 x' \Pi^{\mu\nu}(x_1 - x' : X$$

$$+ (x' - x_2)/2) b_\nu(x' - x_2 : X + (x' - X_1)/2) e^{-ip(x_1 - x_2)}$$

$$= \tilde{\Pi}^{\mu\nu}(p, X) \exp\left(-\frac{1}{2} \partial_p^\Pi \partial_X^b\right) \exp\left(\frac{1}{2} \partial_p^b \partial_X^\Pi\right) \tilde{b}_\nu(p, X)$$

$$\sim \tilde{\Pi}^{\mu\nu}(p, X) \tilde{b}_\nu(p, X), \tag{35}$$

in which ∂_p^Π is a partial derivative with respective to p acting on $\tilde{\Pi}$; other ∂'s should be interpreted in similar ways; the juxtapositions of two ∂'s stand for four-dimensional contractions; the last expression is the approximation to the lowest order with respective to the gradient in X, which is justified by

our assumption. In deriving the third line of the above equations, we employ the following relations:

$$\Pi^{\mu\nu}(x_1 - x' : X + (x' - x_2)/2) = \exp\left(\frac{x' - x_2}{2}\partial_X^\Pi\right)\Pi^{\mu\nu}(x_1 - x' : X), \qquad (36)$$

$$b^\mu(x' - x_2 : X + (x' - x_1)/2) = \exp\left(\frac{x' - x_1}{2}\partial_X^b\right)b^\mu(x' - x_2 : X). \qquad (37)$$

Fourier-transforming the left-hand side of Eq. (32) also with respective to the relative coordinates $x_1 - x_2$, we finally obtain the "local" Maxwell equation as follows:

$$-p^2\tilde{b}^\mu(p, X) + p^\nu p^\mu \tilde{b}_\nu(p, X) = \tilde{\Pi}^{\mu\nu}(p, X)\tilde{b}_\nu(p, X). \qquad (38)$$

Note that we also ignore the derivative with respect to X in the kinetic part of the Maxwell equation, which is again valid under the current assumption. We then consider the dispersion relation for the probe photon in point-wise fashion, plugging the polarization tensor obtained locally this way. This is nothing but the WKB approximation for the propagation of the probe photon. Note that the momentum of the probe photon is hence not the one in the asymptotic states [13] but the local one defined at each point in the background electromagnetic wave. It should also be stressed that the derived refractive index is a local quantity. Although such a quantity may not be easy to detect in experiments, this is regardless the main accomplishment in this paper.

We now proceed to the actual calculations. The induced electromagnetic current is written as

$$\langle j^\mu \rangle = \frac{e}{2}\int_0^\infty ds\, e^{-im^2 s}$$
$$\times \text{tr}\left[\langle x(s)|\hat{\Pi}^\mu(s)U(s) + U(s)\hat{\Pi}^\mu(0)|x(0)\rangle - i\sigma^{\mu\nu}\langle x(s)|\hat{\Pi}_\nu(s)U(s) - U(s)\hat{\Pi}_\nu(0)|x(0)\rangle\right]. \qquad (39)$$

In this expression we drop for brevity the superscript $^{(0)}$, which means the unperturbed states, and the subscript $_\text{I}$, which stands for the operators in the interaction picture. We use these notations in the following.

The proper time evolution operator $U(s)$ is given as

$$U(s) = 1 - i\int_0^s du\,\delta H(u)$$
$$= 1 - i\int_0^s du\left\{e\hat{\Pi}^\alpha(u)b_\alpha \exp\left[-ik_\delta\hat{x}^\delta(u)\right]\right.$$
$$\left. + eb_\alpha \exp\left[-ik_\delta\hat{x}^\delta(u)\right]\hat{\Pi}^\alpha(u) + \frac{1}{2}e\sigma^{\alpha\beta}(u)g_{\alpha\beta}\exp\left[-ik_\delta\hat{x}^\delta(u)\right]\right\} \qquad (40)$$

for the present case. Note that $\sigma^{\alpha\beta}(u)$ is a proper-time-dependent operator in the interaction picture, which is defined as

$$\sigma^{\alpha\beta}(u) = e^{iHu}\sigma^{\alpha\beta}e^{-iHu}. \qquad (41)$$

The explicit expression of $\sigma^{\alpha\beta}(u)$ can be easily obtained to the first order of perturbation for the current Hamiltonian evaluated at $u = 0$[4] as $H = -\hat{\Pi}^2(0) + \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(0)$:

$$\sigma^{\alpha\beta}(u) \simeq \left[1 + \frac{i}{2}eu(\sigma f)(1 + \Omega\xi(0))\right]\sigma^{\alpha\beta}\left[1 - \frac{i}{2}eu(\sigma f)(1 + \Omega\xi(0))\right], \qquad (42)$$

where we employ the abbreviation $(\sigma f) \equiv \sigma^{\mu\nu}f_{\mu\nu}$. Note that although $\xi(0)$ does not have a spinor structure and commutes with $(\sigma g) \equiv \sigma^{\alpha\beta}g_{\alpha\beta}$, $\xi(s)$ may have a non-trivial spinor structure induced by the proper time evolution.

The amplitudes of $\langle x(s)|\hat{\Pi}^{\mu}(s)U(s)|x(0)\rangle$ and $\langle x(s)|U(s)\hat{\Pi}^{\mu}(0)|x(0)\rangle$ can now be expressed with the unperturbed operators and states. The calculations are involved, though, and given in Appendix A. The final expressions are given as

$$\langle x(s)|\hat{\Pi}^{\mu}(s)U(s)|x(0)\rangle$$

$$= \langle x(s)|\hat{\Pi}^{\mu}(s)|x(0)\rangle$$

$$- i\int_0^s du\langle x(s)|2eb_{\alpha}\hat{\Pi}^{\mu}(s)\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$$

$$- i\int_0^s du\langle x(s)| - eb_{\alpha}k^{\alpha}\hat{\Pi}^{\mu}(s)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$$

$$+ \int_0^s du\langle x(s)|\hat{\Pi}^{\mu}(s)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$$

$$\times \left(-\frac{ie}{2}\right)\left[(\sigma g) + \frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\} + \frac{e^2u^2}{4}(\sigma f)(\sigma g)(\sigma f)\right]$$

$$+ \int_0^s du\langle x(s)|\hat{\Pi}^{\mu}(s)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$$

$$\times \left(-\frac{ie}{2}\right)\left[\frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}(\Omega\xi) + \frac{e^2u^2}{2}(\sigma f)(\sigma g)(\sigma f)(\Omega\xi)\right], \qquad (43)$$

$$\langle x(s)|U(s)\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$= \langle x(s)|\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$- i\int_0^s du\langle x(s)|2eb_{\alpha}\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$- i\int_0^s du\langle x(s)| - eb_{\alpha}k^{\alpha}\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$+ \int_0^s du\langle x(s)|\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$\times \left(-\frac{ie}{2}\right)\left[(\sigma g) + \frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\} + \frac{e^2u^2}{4}(\sigma f)(\sigma g)(\sigma f)\right]$$

$$+ \int_0^s du\langle x(s)|\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$\times \left(-\frac{ie}{2}\right)\left[\frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}(\Omega\xi) + \frac{e^2u^2}{2}(\sigma f)(\sigma g)(\sigma f)(\Omega\xi)\right]$$

---

[4] The Hamiltonian is proper time independent and can be evaluated at any time.

$$+ \int_0^s du \langle x(s)| \exp\left[-ik_\delta \hat{x}^\delta(u)\right] |x(0)\rangle$$

$$\times \left(-\frac{ie}{2}\right)(-in^\mu)\left[\frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}\Omega + \frac{e^2 u^2}{2}(\sigma f)(\sigma g)(\sigma f)\Omega\right]. \quad (44)$$

Then the induced electromagnetic current can be calculated by inserting these amplitudes in Eq. (39). The operators $\hat{\Pi}^\mu(s)$, $\hat{\Pi}^\mu(0)$, $\hat{\Pi}^\mu(u)$, and $\exp[-ik_\alpha \hat{x}^\alpha(u)]$ that appear in these expressions can be written in terms of the operators $\hat{x}^\mu(s)$ and $\hat{x}^\mu(0)$ as given in Eqs. (A.16), (A.17), (A.19) and (A.26). Since the operators $\hat{x}^\mu(s)$ and $\hat{x}^\mu(0)$ do not commute with each other, we need to permute them with the help of the commutation relations for these operators so that all $\hat{x}^\mu(s)$ should sit to the left of all $\hat{x}^\mu(0)$. The details are given in Appendix B. Note that the commutators such as $[\hat{x}^\mu(s), \hat{x}^\nu(0)]$ are operators and hence we need to calculate commutation relations like $[\hat{x}^\mu(s), [\hat{x}^\nu(s), \hat{x}^\lambda(0)]]$. After these permutations, various amplitudes can be easily obtained from the following relations:

$$\langle x(s)|\hat{x}^\mu(s) = x^\mu \langle x(s)|, \quad (45)$$

$$\langle x(s)|\xi(s) = \xi \langle x(s)|, \quad (46)$$

$$\hat{x}^\mu(0)|x(0)\rangle = x^\mu |x(0)\rangle, \quad (47)$$

$$\xi(0)|x(0)\rangle = \xi |x(0)\rangle. \quad (48)$$

In deriving Eqs. (42)–(44), we consider only the neighborhood of the coordinate origin, the linear size of which is much shorter than the wavelength of the background plane wave but larger than the wavelength of the probe photon. As mentioned earlier, however, the origin is arbitrary and one can shift the coordinates so that the point of interest should coincide with the origin. Hence the results are actually applicable to any point. More discussion on this point will be found in Appendix C. Note also that Furry's theorem dictates that the number of external fields that appear in the expression of the induced electromagnetic current should be even; the details can be found in Appendix D.

After all these considerations and calculations, the induced electromagnetic current is given to the lowest order of the perturbation and Ω. The details are presented in Appendix E. Since the induced electromagnetic current $\langle j_\mu \rangle$ is vanishing in the absence of the probe photon, it is generated by its presence and should be proportional to it:

$$\langle j_\mu(x)\rangle = \Pi_\mu{}^\nu(k,x)b_\nu(k,x)\exp(-ik_\alpha x^\alpha). \quad (49)$$

Note that we employ the local approximation here again.

In this expression, the probe photon is given as $b_\nu \exp(-ik_\alpha x^\alpha)$ and the proportionality coefficient $\Pi_\mu{}^\nu$ is nothing but the polarization tensor at each point. Following Ritus [37], we decompose the polarization tensor so obtained as

$$\Pi_\mu{}^\nu = \int_0^\infty ds \int_0^s du \left[\Pi_1 (fk)_\mu (fk)^\nu + \Pi_2 (\tilde{f}k)_\mu (\tilde{f}k)^\nu + \Pi_3 G_\mu G^\nu\right], \quad (50)$$

with three mutually orthogonal vectors

$$(fk)_\mu = f_\mu{}^\nu k_\nu, \quad (\tilde{f}k)_\mu = \tilde{f}_\mu{}^\nu k_\nu, \quad G_\mu = \frac{k_\alpha k^\alpha}{k_\beta f^\beta{}_\gamma f^\gamma{}_\delta k^\delta} f_\mu{}^\nu f_\nu{}^\lambda k_\lambda, \quad (51)$$

where $\tilde{f}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}/2$ is the dual tensor of $f^{\mu\nu}$. Here, $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita antisymmetric symbol, which satisfies $\varepsilon^{0123} = 1$. The abbreviations $(kk) = k_\mu k^\mu$ and $(kffk) = k_\mu f^\mu{}_\nu f^\nu{}_\lambda k^\lambda$ are

also used [37]. Then the coefficients are given as follows:

$$
\Pi_1 = \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3 (kffk)} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk)\right]\left(-1 + \exp\left[\frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]\right)(-18i - 9s(kk))
$$

$$
+ \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right] e^2
$$

$$
\left(-\frac{18}{s}\right) u(s^3 - 3s^2 u + 4su^2 - 2u^3)
$$

$$
+ \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]
$$

$$
\times \left[\left(\frac{2}{s^2}e^2(\Omega kn)(s-u)u(6s^4 + 22s^3 u - 79s^2 u^2 + 78su^3 - 36u^4)\right)\right.
$$

$$
+ (-2i)\, e^2 (\Omega kn)(kk)(s-u)^2 u(3s^2 - su - 3u^2)
$$

$$
\left.+ \left(-\frac{4i}{s^2}\right) e^4 (kffk)(\Omega kn)(s-u)^3 u^2 (3s^4 - 7s^3 u + 5s^2 u^2 + 4su^3 - 6u^4)\right], \tag{52}
$$

$$
\Pi_2 = \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3 (kffk)} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk)\right]\left(-1 + \exp\left[\frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]\right)(-18i - 9s(kk))
$$

$$
+ \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right] e^2 (-18)su(s-u)
$$

$$
+ \frac{e^2 e^{-im^2 s}}{72\pi^2 s^3} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]
$$

$$
\times \left[\left(\frac{2}{s}e^2(\Omega kn)(s-u)u(6s^3 + 4s^2 u - 7su^2 + 6u^3)\right)\right.
$$

$$
+ (-2i)\, e^2 (\Omega kn)(kk)(s-u)^2 u(3s^2 - su - 3u^2)
$$

$$
\left.+ (-4i)\, e^4 (kffk)(\Omega kn)(s-u)^3 u^2 (3s^2 - su - 3u^2)\right], \tag{53}
$$

$$
\Pi_3 = \frac{e^2 e^{-im^2 s}}{72\pi^2 s^4 (kk)} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk)\right]\left(-1 + \exp\left[\frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]\right)
$$

$$
\times (-6is + (-3s^2 + 16su - 16u^2)(kk))
$$

$$
+ \frac{e^4 (kffk) e^{-im^2 s}}{324\pi^2 s^4 (kk)^2} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]
$$

$$
\times \left[3i(s^3 - 6s^2 u + 6su^2) - 2u^2(s^2 - 3su + 2u^2)^2 e^2 (kffk) - 12(s - 2u)^2 (s - u)u(kk)\right]
$$

$$
- \frac{ie^2 e^{-im^2 s}}{27(kk)^2 \pi^2 s^4} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]
$$

$$
\times (\Omega kn)e^2 (kffk)(s-u)u(s^2 - 5su + 5u^2)
$$

$$
- \frac{e^2 e^{-im^2 s}}{972(kk)^2 \pi^2 s^5} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s}\right]
$$

Table 1. Eigenmodes of the probe photons with different four-momenta.

Probe momentum k_μ	Eigenmode	Mode name
$(k_0, k_1, 0, 0)$	$(0, 0, 1, 0)_\mu$	x2 mode
	$(0, 0, 0, 1)_\mu$	x3 mode
$(k_0, 0, k_2, 0)$	$(k_2, -k_2, k_0, 0)_\mu$	y1 mode
	$(0, 0, 0, 1)_\mu$	y3 mode
$(k_0, 0, 0, k_3)$	$(k_3, -k_3, 0, k_0)_\mu$	z1 mode
	$(0, 0, 1, 0)_\mu$	z2 mode
$(k_0, \frac{k_i}{\sqrt{3}}, \frac{k_i}{\sqrt{3}}, \frac{k_i}{\sqrt{3}})$	$(A, B, 1, 0)_\mu$	s2 mode*
	$(A, B, 0, 1)_\mu$	s3 mode

* A, B are constants written with k_0 and k_i.

$$\times (\Omega kn)e^2(kffk)(s-u)^2 u \left[2e^2(kffk)(3s^5 - 19s^4 u + 6s^3 u^2 + 87s^2 u^3 - 138su^4 + 60u^5) \right.$$

$$\left. -9(6s^3 + 9s^2 u - 46su^2 + 40u^3)(kk) \right]$$

$$-\frac{ie^2 e^{-im^2 s}}{2916(kk)^2 \pi^2 s^5} \exp\left[i\left(u - \frac{u^2}{s}\right)(kk) + \frac{i(s-u)^2 u^2 e^2 (kffk)}{3s} \right]$$

$$\times (\Omega kn)e^2(kffk)(s-u)^2 u(3s^2 - su - 3u^2)\left[4e^4(kffk)^2 u^2 (s^2 - 3su + 2u^2)^2 \right.$$

$$\left. + 24e^2(kffk)(kk)(s-2u)^2(s-u)u + 9(3s^2 - 16su + 16u^2)(kk)^2 \right], \tag{54}$$

where $(\Omega kn) = \Omega k_\mu n^\mu$ is the inner product of the momentum vectors of the external plane wave and the probe photon. The refractive indices for physical modes are related to Π_1 and Π_2.

The proper time integration in Eq. (50) or its pre-decomposition form, Eq. (E.1), has to be done numerically. The original form is not convenient for this purpose and we rotate the integral path by $-\pi/3$ in the complex plane so that the integral converges exponentially as s goes to infinity. Note that the rotation angle is arbitrary as long as it is in the range $(0, -\pi/3]$. The refractive index is then obtained by solving the Maxwell equation reduced to the following form:

$$A_\mu{}^\nu(k)b_\nu = 0, \tag{55}$$

with $A_\mu{}^\nu = -(kk)\delta_\mu{}^\nu + k_\mu k^\nu + \Pi_\mu{}^\nu$. The probe photon is hence described as a non-trivial solution of this homogeneous equation and its dispersion relation is obtained from the relation $\det A = 0$. Note that not all of them are physical. Unphysical modes are easily eliminated, however, by calculating the electric and magnetic field strengths, which are gauge invariant. It is then found that only two of them associated with Π_1 and Π_2 are physical as expected. Note that the four momenta of the probe photon thus obtained are no longer null in accordance with the refractive indices different from unity. The polarization vectors are also obtained simultaneously.

In the next section, we show the results of some numerical evaluations. As representative cases, we consider four propagating directions of the probe photon as summarized in Table 1. Since the background plane wave is assumed to have a definite propagation direction (x-direction) and linear polarization (y-direction), these four directions are not equivalent. For each propagation direction, there are two physical eigenmodes, as mentioned above, which are in general different from each other, having distinctive dispersion relations, i.e., the background is birefringent.

4.   Results

In this section, we numerically evaluate the refractive index N, which is defined as $N = |\boldsymbol{k}|/k_0$. First, the crossed fields are considered and then the first-order correction δN in the gradient expansion is calculated for the plane wave field. The eigenmodes of the probe photon depend on the propagation direction, as already mentioned. The refractive index is complex in general with the real part representing the phase velocity of the probe photon divided by the light speed and the imaginary part indicating the decay, possibly via electron–positron pair creation. Since the deviation of the refractive index from unity is usually much smaller than unity, only the deviations are shown in the following: $\mathrm{Re}[N - 1]$ and $\mathrm{Im}[N]$.

Note that for all the cases considered in this paper, the refractive indices, both real and imaginary parts, of the y1 and z2 modes are identical and so are those of the y3 and z1 modes. Although the exact reason for this phenomenon is not known to us for the moment, the following should be mentioned: the polarization tensor $\Pi^{\mu\nu}$ is expressed as the sum of three contributions proportional to $(fk)^\mu (fk)^\nu$, $(\tilde{f}k)^\mu (\tilde{f}k)^\nu$, and $G^\mu G^\nu$ given as Eq. (50); each pair of the modes that have the identical refractive index are actually eigenmodes of either $(fk)^\mu (fk)^\nu$ or $(\tilde{f}k)^\mu (\tilde{f}k)^\nu$. We will show these degenerate modes with the same color in figures hereafter.

4.1.   Crossed fields

As mentioned in Sect. 1, the vacuum polarization in the crossed fields has already been obtained by many authors. The refractive index has also been evaluated both analytically and numerically [17,35–38]. The regions in the plane of the field strength f and the probe photon energy k_0 that have been investigated in these papers are summarized in Fig. 2. It is apparent from the figure that there is still an unexplored region, which is unshaded. And that is the target of this paper. The parameter
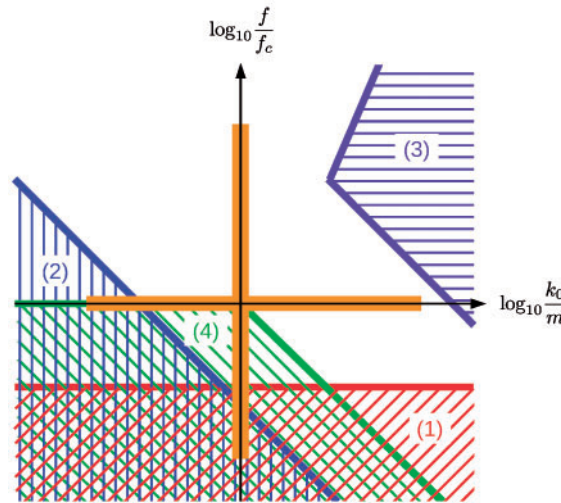
Fig. 2. Regions in the plane of the strength of external crossed fields and the probe photon energy that have already been explored. Each region is labeled as follows: region (1) is the weak-field limit $f/f_c \ll 1$ [17]; region (2) is for the weak-field or low-energy limit $e^2 k_\mu f^\mu{}_\nu f^{\nu\lambda} k_\lambda/m^6 \ll 1$ [35–37]; region (3) corresponds to the strong-field and high-energy limit $1 \ll e^2 k_\mu f^\mu{}_\nu f^{\nu\lambda} k_\lambda/m^6 \ll (k_0/m)^6 \times \alpha^{-3}$ studied in Ref. [36], where $\alpha = e^2/4\pi$ is the fine structure constant; region (4) is the region that satisfies $(f/f_c) \lesssim 1$ and $(f/f_c) \times (k_0/m) \lesssim 1$ explored in Ref. [38]. The orange lines indicate the regions in which the refractive indices are computed numerically in this paper. Note that our method can treat the whole region in this figure in principle.
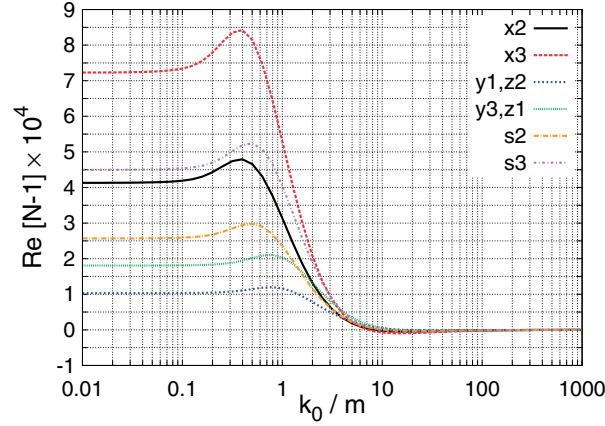
Fig. 3. Plot of Re[$N-1$] as a function of the probe photon energy in the crossed field. Here, N is a refractive index. We set $f/f_\mathrm{c}=1$. The colors specify different modes.

ranges we adopt in this paper are displayed in orange in the same figure: we first calculate the refractive index for the external field of the critical value to validate our formulation by comparing our results with those in the previous studies; then we vary the strength of the external field.

The polarization tensor $\Pi_\mu{}^\nu$ in the crossed field is obtained by simply taking the limit of $\Omega \to 0$ in Eq. (50). Setting the strength of the external field to the critical value $f/f_\mathrm{c} = 1$, we compute the refractive indices for the range of $0.01 \leq k_0/m \leq 1000$.[5] Note that the low-energy regime ($k_0/m \lesssim 1$) has already been investigated, as shown in Fig. 2. The real part Re[$N-1$] is shown in Fig. 3 with colors indicating different modes of the probe photon.

It is found that the deviation of the refractive index from unity is of the order of 10^{-4}. As k_0/m gets smaller, the refractive index approaches the values in the weak-field or low-energy limit [region (2) in Fig. 2], which are written as

$$N_{x2} \simeq 1 + \frac{2\alpha}{45\pi} \frac{\kappa^2 m^2}{k_0^2}, \tag{56}$$

$$N_{x3} \simeq 1 + \frac{7\alpha}{90\pi} \frac{\kappa^2 m^2}{k_0^2} \tag{57}$$

for the x2 and x3 modes, respectively, where $\kappa^2 = e^2 k_\mu f^\mu{}_\nu f^{\nu\lambda} k_\lambda/m^6 = e^2 f^2 (k_0 + k_1)^2/m^6$ is the product of the probe photon energy and the field strength normalized by the critical value. Then the typical value of $N_{x2} - 1$ can be estimated as

$$N_{x2} - 1 \simeq \frac{8\alpha}{45\pi} \left(\frac{f}{f_\mathrm{c}}\right)^2 \sim 4.1 \times 10^{-4} \left(\frac{I}{4.6 \times 10^{29}\,\mathrm{W\,cm^{-2}}}\right), \tag{58}$$

where $I = f^2/4\pi$ is the intensity of the plane wave. The results are hence in agreement with what was already published in Refs. [35–38]. The refractive indices depend on the propagation direction of the probe photon: the modulus $|\mathrm{Re}[N-1]|$ is larger for a photon propagating in the opposite direction to the background plane wave (the x mode) than those going perpendicularly (the y/z modes); the s mode that propagates obliquely normally lies in between, although the modulus is greater for the s3

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[5] Shore studied the refractive index of super-critical magnetic fields for a wider range of the photon energy [24]. The results are similar to ours for the crossed field.
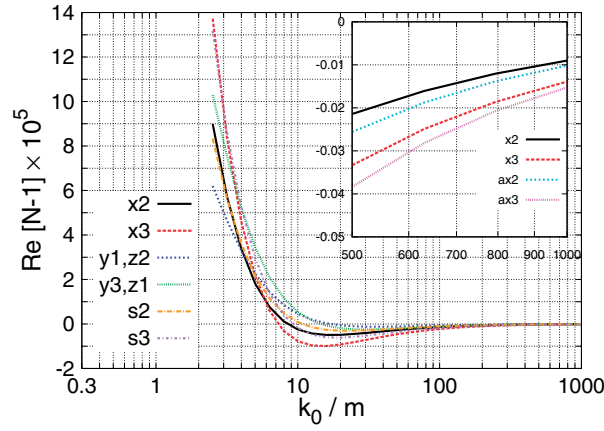
Fig. 4. As Fig. 3, but for high energies alone on a different vertical scale. The inset zooms into the high-energy range of $500 \leq k_0/m \leq 1000$ and asymptotic formulae are also shown as ax2 and ax3 for the x2 and x3 modes, respectively.

mode than for the x2 mode. The photons polarized in the *z*-direction have larger moduli in general, except that the z mode, which propagates in this direction, has a greater modulus when it is polarized in the *x*-direction. These trends are also true for other results obtained below in this paper.

As k_0/m becomes larger than ~ 10, all the refractive indices for different propagation directions appear to converge to unity, which is consistent with Refs. [17,38]. This is more apparent in Fig. 4, which zooms in to the region of $3 \lesssim k_0/m \leq 1000$. It can also be seen in the same figure that $\text{Re}[N-1]$ is negative and the modulus $|\text{Re}[N-1]|$ decreases for $k_0/m \gtrsim 10$. This trend is consistent with the high-energy limits given in Ref. [36] [region (3) in Fig. 2], which are written as

$$N_{\text{x2}} \simeq 1 - \frac{\sqrt{3}\alpha m^2}{14\pi^2 k_0^2}(3\kappa)^{2/3}\Gamma^4\left(\frac{2}{3}\right)\left(1 - i\sqrt{3}\right),\tag{59}$$

$$N_{\text{x3}} \simeq 1 - \frac{3\sqrt{3}\alpha m^2}{28\pi^2 k_0^2}(3\kappa)^{2/3}\Gamma^4\left(\frac{2}{3}\right)\left(1 - i\sqrt{3}\right)\tag{60}$$

for the x2 and x3 modes, respectively. In our formulation, these results are reproduced by putting $e^{-im^2 s}$ to unity and setting $(kk) = k_0^2 - k_1^2$ equal to zero in Eqs. (52)–(54) for the polarization tensor $\Pi_\mu{}^\nu$ or Eqs. (E.57) and (E.59) for the induced electromagnetic current $\langle j_\mu \rangle$. Note, however, that our numerical results for $\text{Re}[N-1]$ are not yet settled to the asymptotic limits, with deviations of $\sim 10\%$ still remaining at $k_0/m \sim 1000$. In this figure, the high-energy limits for the x2 and x3 modes are displayed as the lines labeled as ax2 and ax3, respectively. The imaginary parts, on the other hand, have already reached the asymptotic limits at $k_0/m \sim 1000$ (see below).

Toll [17] pointed out that unless the Poynting vectors of the probe photon and the external field are parallel to each other, an appropriate Lorentz transformation makes them anti-parallel and, as a result, the refractive index depends only on the reduced field strength f_{rd},

$$f_{\text{rd}} = f \sin^2\left(\frac{\theta}{2}\right),\tag{61}$$

as long as the field strength is not much larger than the critical value. Here, θ is the angle between the Poynting vectors of the probe photon and the external field. We hence redraw Fig. 3 as Fig. 5 in the range of $0.1 \leq k_0/m \leq 10$ after adjusting the external field strength so that $f_{\text{rd}} = f_c$ for all
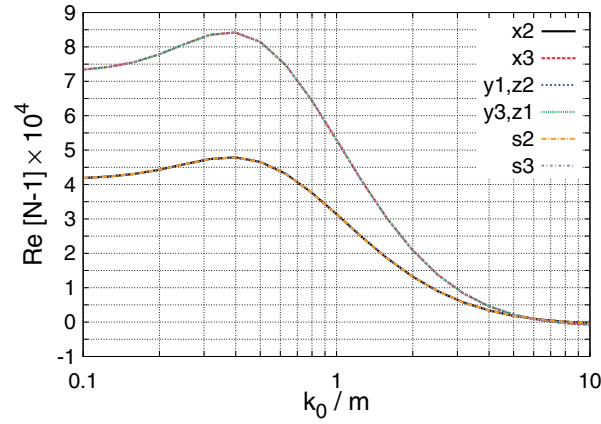
**Fig. 5.** As Fig. 3, but for $f_{\rm rd} = f_{\rm c}$ in the energy range of $0.1 \leq k_0/m \leq 10$ for all the modes.
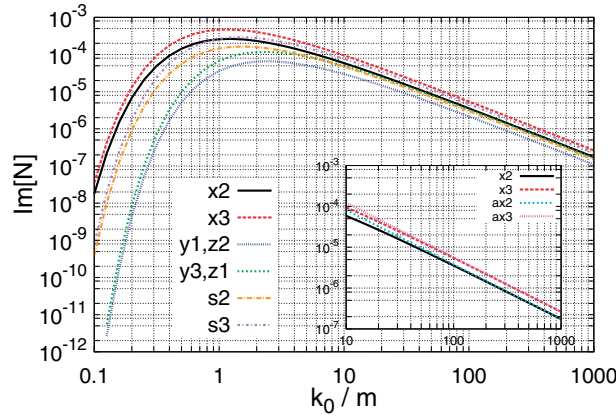
Fig. 6. As Fig. 3, but for the imaginary part of the refractive index Im[N]. The inset shows the behavior in the high-energy regime as in Fig. 4. The lines labeled as ax2 and ax3 show the high-energy limit expressed as Eqs. (59) and (60), respectively.

the modes. As expected, the x3, s3, y3, and z1 modes become identical, which is also true for the x2, s2, y1, and z2 modes. The relation also holds for the imaginary part. It is important that these relations are obtained as a result of separate calculations for different propagation directions in our formulation, the fact that guarantees the correctness of our calculations.

The imaginary part of the refractive index Im[N] is shown in Fig. 6 for the same case. It is found that the imaginary part is non-vanishing down to $k_0 = 0$, although it diminishes very rapidly for $k_0/m \lesssim 0.1$. It is also seen that Im[N] for each photon mode reaches its maximum at $k_0/m \sim 1$ and decreases monotonically for higher energies. These behaviors are also consistent with the known limits [36]. In fact, as mentioned above, they are already settled to the asymptotic values at $k_0/m \sim 1000$, as shown in the inset of the figure. The imaginary parts Im[N] for different modes follow the general trend mentioned earlier for |Re[$N - 1$]|, with the x3 mode being the largest and the y1/z2 being the smallest except around $k_0/m \sim 1$, where some crossings occur.

The imaginary part of the refractive index in the weak-field or low-energy region [region (2) in Fig. 2] was considered in Ref. [37,38]. Although we cannot obtain the analytic expression, we try to compute the imaginary part numerically in this regime. The results are displayed in Fig. 7 for the x2 and x3 modes in the range $0.03 \lesssim k_0/m \leq 1$. The lines labeled as ax2 and ax3 are the results
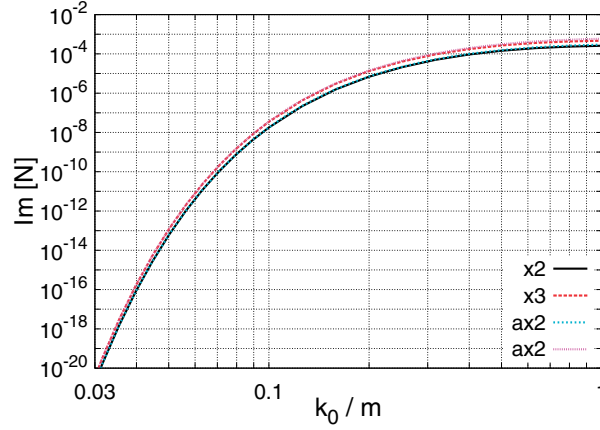
Fig. 7. As Fig. 6, but for the low-energy range. The lines labeled as ax2 and ax3 show the weak-field or low-energy limits expressed as Eqs. (62) and (63), respectively.

obtained in Ref. [37], which are expressed as

$$\mathrm{Im}[N_{x2}] \simeq \frac{1}{8}\sqrt{\frac{3}{2}}\frac{\alpha\epsilon}{\nu}e^{-\frac{4}{3\epsilon\nu}}, \tag{62}$$

$$\mathrm{Im}[N_{x3}] \simeq \frac{1}{4}\sqrt{\frac{3}{2}}\frac{\alpha\epsilon}{\nu}e^{-\frac{4}{3\epsilon\nu}}, \tag{63}$$

where $\epsilon = f/f_c$ and $\nu = k_0/m$. It is found that the imaginary parts $\mathrm{Im}[N]$ are better approximated in this regime by Eqs. (62) and (63) than by

$$\mathrm{Im}[N_{x2}] \simeq \frac{4\alpha\epsilon^2}{45}\frac{4}{3\epsilon\nu}e^{-\frac{4}{3\epsilon\nu}}, \tag{64}$$

$$\mathrm{Im}[N_{x3}] \simeq \frac{7\alpha\epsilon^2}{45}\frac{4}{3\epsilon\nu}e^{-\frac{4}{3\epsilon\nu}} \tag{65}$$

as obtained in Ref. [38].

Next, we show the dependence of the refractive index on the external field strength, setting $k_0/m = 1$. This has never been published in the literature before. In Fig. 8, $\mathrm{Re}[N - 1]$ is shown as a function of f/f_c in the range $0.01 \leq f/f_c \leq 1000$. Figure 9 zooms in to the range $0.01 \leq f/f_c \leq 3$, setting the vertical axis in the logarithmic scale. The quadratic behavior observed for $0.01 \leq f/f_c \lesssim 0.5$ is in accord with the weak-field or low-energy limits [36], which are given as ax2 and ax3 for the x2 and x3 modes in the inset of this figure, respectively. $\mathrm{Re}[N - 1]$ is negative at $f/f_c \gtrsim 10$, which is consistent with the earlier findings. The modulus $|\mathrm{Re}[N - 1]|$ is an increasing function of f at $f/f_c \gtrsim 10$.

The imaginary part $\mathrm{Im}[N]$ is shown in Fig. 10. It increases monotonically with the external field strength. The slopes are steeper at $f/f_c \lesssim 0.5$, which is consistent with the analytic expression in the weak-field or low-energy limit of $\mathrm{Im}[N]$ [37,38]. The inset of this figure shows the comparison of our numerical results with the asymptotic limits, Eqs. (62) and (63), labeled as ax2 and ax3 for the x2 and x3 modes, respectively. They almost coincide with each other at $f/f_c \lesssim 0.5$. Note, on the other hand, that the behavior of the imaginary part at high field strengths has not been reported in the literature.
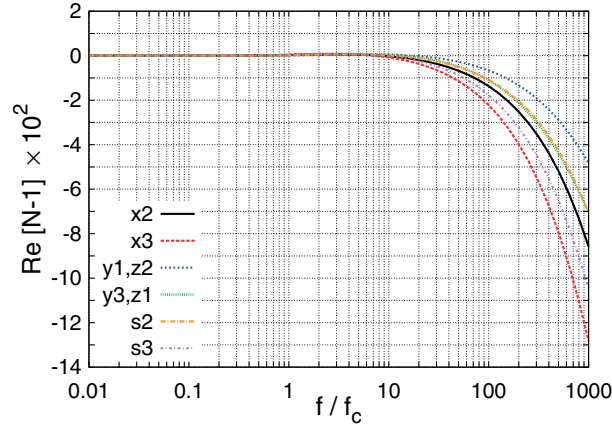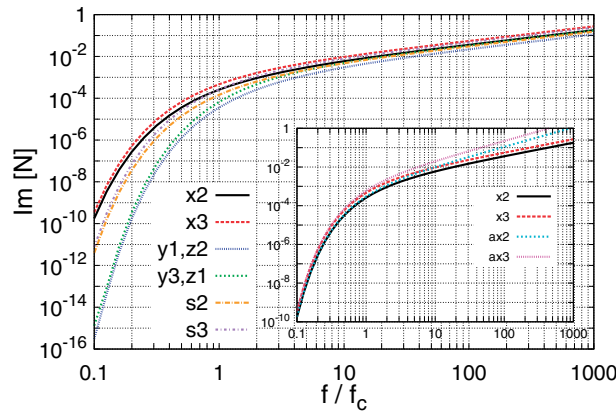
Fig. 8. Plot of $\mathrm{Re}[N-1]$ as a function of the field strength. We assume $k_0/m = 1$ this time.
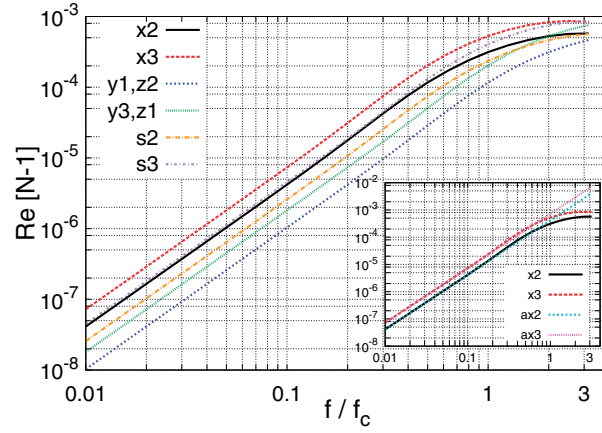
Fig. 9. As Fig. 8, but for weak fields. The inset shows the comparison between our numerical results and asymptotic expressions, Eqs. (56) and (57), labeled as ax2 and ax3 for the x2 and x3 modes, respectively, in the weak-field or low-energy limits.

Fig. 10. As Fig. 8, but for the imaginary part of the refractive index $\mathrm{Im}[N]$. The inset shows the comparison between our numerical results and the asymptotic expressions in the weak-field or low-energy limits, Eqs. (62) and (63), labeled as ax2 and ax3 for the x2 and x3 modes, respectively.

Table 2. Proportionality coefficients in the correction δN from temporal and spatial non-uniformities.*

Mode	C_{Re}	C_{Im}
x2	-1.30×10^{-3}	3.16×10^{-3}
x3	-3.08×10^{-3}	5.17×10^{-3}
y1,z2	1.42×10^{-4}	4.35×10^{-4}
y3,z1	1.83×10^{-4}	8.28×10^{-4}
s2	-3.10×10^{-4}	1.79×10^{-3}
s3	-9.69×10^{-4}	3.11×10^{-3}

* $k_0/m = 1$ and $f/f_{\mathrm{c}} = 1$.

4.2. Plane wave

We next consider the "local" refractive index for the plane wave field, which is also original in this paper. We evaluate numerically the polarization tensor is given in Eqs. (50) and (52)–(54) and solve the Maxwell equation, Eq. (55), obtained in the gradient expansion. Since our formulation is based on perturbation theory, it is natural to express the refractive index in the plane wave as $N + \delta N$, where N is the refractive index for the crossed field and δN is the correction from the temporal and spatial non-uniformities. As mentioned for the crossed field, the refractive indices for the y1 and z2 modes are identical to each other. In fact, the relevant components of the Maxwell equations, Eq. (55), are the same for these modes. This is also true for the y3 and z1 modes.

It is found that the correction δN indeed starts with the linear order of Ω/m for both the real and imaginary parts. It is then written as

$$\delta N = (C_{\mathrm{Re}} + iC_{\mathrm{Im}}) \times \Omega/m + O((\Omega/m)^2), \tag{66}$$

and the numerical values of the coefficients C_{Re} and C_{Im} are given for $k_0/m = 1$ and $f/f_{\mathrm{c}} = 1$ in Table 2. The temporal and spatial variations are found to mainly affect the imaginary part: $|\mathrm{Im}[\delta N]| > |\mathrm{Re}[\delta N]|$ from these results. It is also seen that $\mathrm{Im}[\delta N]$ is larger for the photons propagating in the opposite direction to the external plane wave (x-direction), as in the crossed field limit. The real parts $\mathrm{Re}[\delta N]$ are negative for photons other than those propagating perpendicularly to the external plane wave. The modulus $|\mathrm{Re}[N + \delta N]|$ is hence reduced for these modes by the field variation.

We next present the dependence on k_0/m of δN for $f/f_{\mathrm{c}} = 1$, $\Omega/m = 10^{-3}$ in Figs. 11 and 12. The real part $\mathrm{Re}[\delta N]$ is exhibited in Fig. 11. It can be seen that the real part can be both positive and negative: it tends to be negative at higher values of k_0/m, although the range depends on the mode; in fact, the values of the photon energy, above which δN gets positive, are smaller for the photons propagating opposite to the external plane wave. $\mathrm{Re}[\delta N]$ is much smaller than $\mathrm{Re}[N - 1]$ for the crossed field at $0.1 \leq k_0/m \leq 1$ and decreases very rapidly like $\mathrm{Im}[N]$ for the crossed field.

The imaginary part $\mathrm{Im}[\delta N]$ is shown in Fig. 12 for probe photon energies of $0.03 \leq k_0/m \leq 1$. The inset indicates the comparison for the x2 and x3 modes between the numerically computed results of $\mathrm{Im}[\delta N]$ and the asymptotic values in the weak-field or low-energy limits. The expressions of $\mathrm{Im}[\delta N]$ in this regime are given from Eqs. (52) and (53) as

$$\mathrm{Im}[\delta N_{\mathrm{x2}}] \simeq \frac{263\alpha}{3780\pi} \left(\frac{\Omega k_\mu n^\mu}{m^2} \right) \frac{\kappa^2 m^2}{k_0^2}, \tag{67}$$

$$\mathrm{Im}[\delta N_{\mathrm{x3}}] \simeq \frac{71\alpha}{540\pi} \left(\frac{\Omega k_\mu n^\mu}{m^2} \right) \frac{\kappa^2 m^2}{k_0^2}, \tag{68}$$
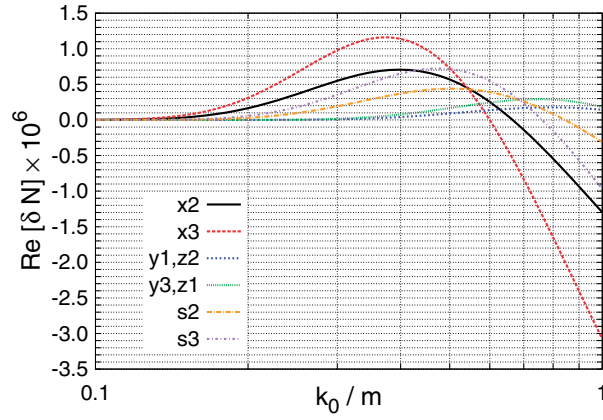
Fig. 11. Plot of the correction to the real part of the refractive index for the crossed field from the temporal and spatial variations in the plane wave field. The field strength is set to the critical value, i.e., $f/f_c = 1$, and the frequency of the external wave field is chosen as $\Omega/m = 10^{-3}$.
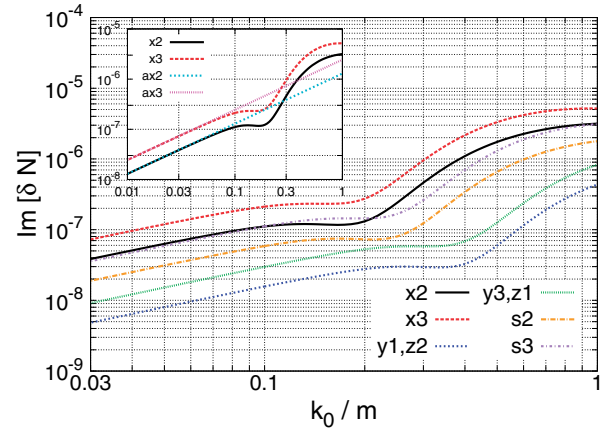

Fig. 12. As Fig. 11, but for $\mathrm{Im}[\delta N]$. The inset shows the comparison for the x2 and x3 modes between the asymptotic limits, Eqs. (67) and (68), labeled as ax2 and ax3, and the numerically computed results.

where $\Omega k_\mu n^\mu/m^2 = \Omega(k_0 + k_1)/m^2$ is the product of the momentum of the external plane wave and that of the probe photon normalized by the electron mass, and is a representative term in the gradient expansion $F_{\mu\nu} \sim f_{\mu\nu}(1 + \Omega\xi)$, being proportional to Ω with the proportional factor $k_\mu n^\mu$ originating from the commutation relation of ξ that accompanies Ω. $\kappa^2 = e^2 k_\mu f^\mu{}_\nu f^{\nu\lambda} k_\lambda/m^6 = e^2 f^2 (k_0 + k_1)^2/m^6$, as previously defined in Eq. (56). Equations (67) and (68) are convenient for the evaluation of the typical value of $\mathrm{Im}[\delta N]$:

$$\mathrm{Im}[\delta N] \sim \alpha \left(\frac{\Omega}{m}\right)\left(\frac{k_0}{m}\right)\left(\frac{f}{f_c}\right)^2$$

$$\sim 7 \times 10^{-6} \left(\frac{\Omega}{0.5\,\mathrm{keV}}\right)\left(\frac{k_0}{510\,\mathrm{keV}}\right)\left(\frac{I}{4.6 \times 10^{29}\,\mathrm{W\,cm^{-2}}}\right), \tag{69}$$

where I is the intensity of the external electromagnetic wave. It is found from the inset that $\mathrm{Im}[\delta N]$ is well approximated for $\Omega/m = 10^{-3}$ by the asymptotic expressions at $k_0/m \leq 0.03$ for $f/f_c = 1$. There occurs a dent at $k_0/m \simeq 0.2$ and $\mathrm{Im}[\delta N]$ rises more rapidly with k_0/m at larger energies, where $\mathrm{Im}[N]$ of the crossed field also becomes substantial. The location of the dent depends on the
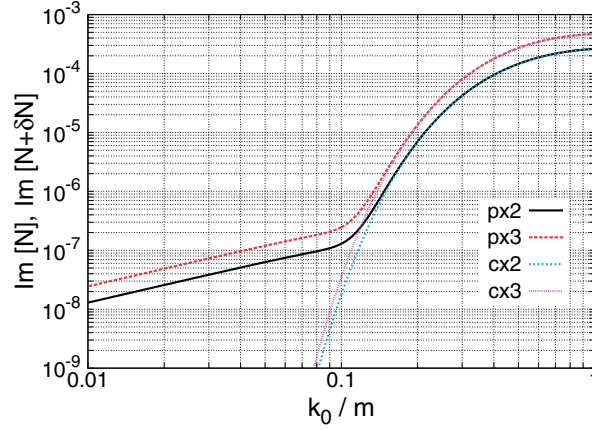
Fig. 13. Comparison for the x2 and x3 modes of $\text{Im}[N + \delta N]$ labeled as px2 and px3, respectively, and $\text{Im}[N]$ labeled as cx2 and cx3, respectively, as a function of the probe photon energy, k_0/m. Here we set $f/f_c = 1$ as in the previous two figures.
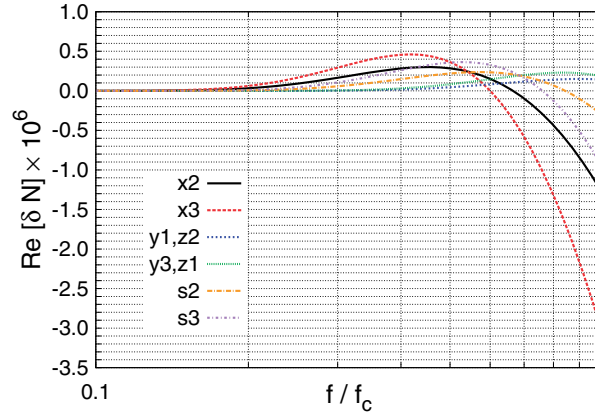

Fig. 14. $\text{Re}[\delta N]$ for the external plane wave field of frequency $\Omega/m = 10^{-3}$. The probe photon energy is chosen as $k_0/m = 1$.

propagation direction of the probe photon, with the x (y/z) mode having the smallest (largest) value of k_0/m at the dent, respectively.

Since the imaginary part of the refractive index declines rapidly below these energies for the crossed field, it is dominated by the first-order correction $\text{Im}[\delta N]$ from the temporal and spatial variations in the plane wave at these low energies. In fact, the latter is commonly more than ten times larger than the former: $\text{Im}[\delta N] \gtrsim 10 \times \text{Im}[N]$ at $k_0/m \lesssim 0.1$. See also Fig. 13, where we plot $\text{Im}[N + \delta N]$ and $\text{Im}[N]$ as a function of k_0/m. This is especially the case for probe photons propagating transversally to the background plane wave. In accordance with the trend for the crossed field, the x (y/z) modes have largest (smallest) moduli $|\text{Im}[\delta N]|$, and the s modes come in between in general.

Finally, we look into the dependence of δN on the field strength in the range $f/f_c \leq 1$. The real and imaginary parts of δN are shown in Figs. 14 and 15, respectively. The probe photon energy is set to $k_0/m = 1$, and the frequency of the external field is assumed to be $\Omega/m = 10^{-3}$ again, though the results scale linearly with the latter. It is evident that the results are quite similar to those shown in Figs. 11 and 12: the real part, $\text{Re}[\delta N]$, has a hump at $f/f_c \sim 0.5$, whereas the imaginary part, $\text{Im}[\delta N]$, is quadratic in f/f_c at weak fields and becomes dominant over the crossed field contribution, $\text{Im}[N]$,
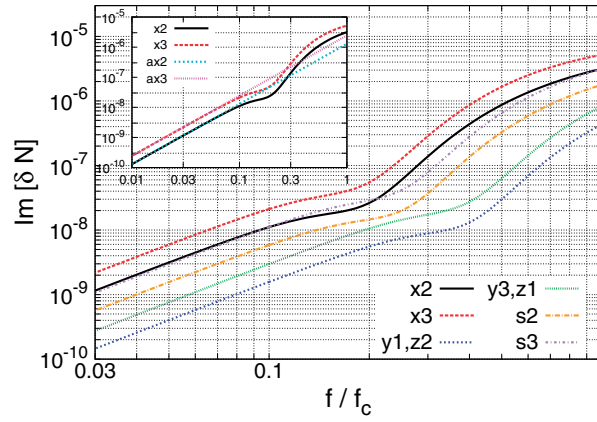
Fig. 15. As Fig. 14, but for Im[δN]. The inset shows the comparison for the x2 and x3 modes between the asymptotic limits, Eqs. (67) and (68), labeled as ax2 and ax3, respectively, and the numerically computed results for the x2 and x3 modes, labeled as x2 and x3, respectively.
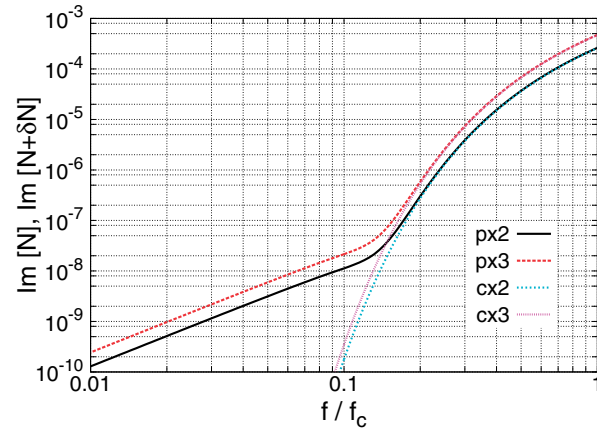
Fig. 16. As Fig. 13, but as a function of field strength, f/f_c. Here, k_0/m is set to 1 as in the previous two figures.

at $f/f_c \lesssim 0.1$; the order in the magnitudes of Im[δN] for different modes is the same as that in Fig. 11; Re[δN] is negative for a certain range of f/f_c, which depends on the mode, occurring for stronger fields for the mode propagating transversally to the background plane field. The reason for these behaviors is the following: although δN depends not only on the product of k_0/m and $(f/f_c)^2$ but also on $k^\mu k_\mu/m^2$, the latter dependence is minuscule in the regime we consider here. As a result, the dependence of the refractive index on k_0/m can be translated into that of f/f_c. In fact, the numerical results for Im[δN] are well approximated by the same asymptotic formulae, Eqs. (67) and (68), in the weak-field regime $f/f_c \lesssim 0.1$, which can be seen in the inset of Fig. 15; Im[δN] has a dent at $f/f_c \sim 0.2$ and changes its behavior at larger field strengths, where the crossed field contribution, Im[N], becomes large, overwhelming Im[δN]. See also Fig. 16, where we plot Im[$N + \delta N$] and Im[N] as a function of f/f_c.

5. Summary and discussion

In this paper we have developed a perturbation theory adapted to Schwinger's proper time method to calculate the induced electromagnetic current, which should be plugged into the Maxwell equations to obtain the refractive indices, for external, linearly polarized plane waves, considering them as

the unperturbed states and regarding a probe photon as the perturbation to them. Although this is nothing new, and indeed has been employed previously [20], our formulation is based on the interaction picture, a familiar tool in quantum mechanics and also referred to as the Furry picture in strong-field QED, rather than utilizing the properties of particular electromagnetic fields from the beginning. Moreover, assuming that the wavelength of the external plane wave is much longer than the Compton wavelength of the electron, and employing the gradient expansion, we have evaluated locally the polarization tensor via the induced electromagnetic current to the lowest order of the spatial and temporal variations of the external fields, which is the main achievement in this paper. It has been shown that the vacuum polarization is given locally by the field strength and its gradient of the external plane waves at each point. We have then considered the dispersion relations for the probe photons propagating in various directions and derived the local refractive indices.

We have first evaluated them for the crossed fields, which are the long-wavelength limit of the plane waves. In so doing, the field strength and the energy of the probe photon are not limited but are allowed to take any values. Note that even for the crossed field not all the parameter regime has been investigated, and we have explored those portions unconsidered so far. We have shown that the refractive index is larger for the photons propagating opposite to the external field than for those propagating perpendicularly. We have also confirmed some limiting cases that were already known in the literature analytically or numerically [35–38], particularly the behavior in weak external fields demonstrated in Ref. [17]. Note, however, that the assumption of a fixed classical background field becomes rather questionable at field strengths near or, in particular, above the critical field strength, since the back reactions to the background field from pair creations should then be taken into account. This issue is certainly much beyond the scope of this paper, and in spite of this conceptual problem we think that the results in such very strong fields are still useful to understand the scale and qualitative behavior of the corrections from the field gradient.

We have then proceeded to the evaluation of the refractive index for the plane wave to the lowest order of the temporal and spatial variations of the background field. The local correction δN to the refractive index for the crossed field N has been numerically evaluated for the first time. We have demonstrated that the modulus of its imaginary part is larger than that of the real part, i.e., the field variations mainly affect the imaginary part of the refractive index. Note that the refractive index we have obtained in this study is local, depending on the local field strength and its gradient, and is meaningful in the sense of the WKB approximation. This is in contrast to the refractive index averaged over the photon path in Ref. [13].

In optical laser experiments ($\Omega/m \sim 10^{-6}$), the refractive index may be approximated very well by that for the crossed field. The correction from the field variations is typically $|\delta N| \sim 10^{-5} \times \text{Re}[N-1]$. The weak-field limit may be also justified, since the current maximum laser intensity, $\sim 2.0 \times 10^{22}$ W cm^{-2}, is still much lower than the critical value, 4.6×10^{29} W cm^{-2}. Then the numerical results given in Fig. 9 are applicable: $\text{Re}[N-1] \sim 10^{-4} \times (f/f_c)^2$ for the probe photon with $k_0/m = 1$, which corresponds to $\sim 10^{-8}$ at $\sim 10^{25}$ W cm^{-2}, the power expected for future laser facilities such as ELI. Note that how to observe the local refractive index in the electromagnetic wave is a different issue, and the averaged one will be better suited for experiments [13].

Unlike for the optical laser, the field variations may not be ignored for X-ray lasers with $\Omega/m \simeq 10^{-2}$. We find from Fig. 9 and Table 2 that the refractive index for the crossed field and the first-order correction to it are $|N-1| \sim 10^{-4}$ and $|\delta N| \sim 10^{-5}$, respectively, for a probe photon with $k_0/m = 1$ propagating opposite to external fields with the critical field strength. It may be

more interesting that the imaginary part of the first-order correction, $\text{Im}[\delta N]$, becomes larger than that for the crossed field $\text{Im}[N]$ at $f/f_c \lesssim 0.1$ for $k_0/m = 1$ or at $k_0/m \lesssim 0.1$ for $f/f_c = 1$. It should be noted, however, that the suppression is much relaxed by the presence of the temporal and spatial variations in the background plane field. This is because the imaginary part of the refractive index is exponentially suppressed for the crossed field while it is suppressed only by powers for the plane wave.

Very strong electromagnetic fields and their temporal and/or spatial variations may also be important for some astronomical phenomena. For example, burst activities called giant flares and short bursts have been observed in magnetars, i.e., strongly magnetized neutron stars [3]. Although the energy source of these activities is thought to be the magnetic fields of magnetars, the mechanism of bursts is not understood yet. In the analysis of the properties of the emissions from these bursts, the results obtained in this paper may be useful.

As for the burst mechanism, one interesting model related to the strong field variation was proposed by some authors [50–52], in which they considered shock formations in electromagnetic waves propagating in strong magnetic fields around the magnetar. The shock dissipation may produce a fireball of electrons and positrons via pair creation. Their discussion is based on the Rankine–Hugoniot-type jump condition and the Euler–Heisenberg Lagrangian, which is certainly not able to treat the close vicinity of the shock wave, since the shock is essentially a discontinuity. Note, however, that our result in this paper is not very helpful for this problem, either, since the field variation is very rapid and has quite short wavelengths and, moreover, finite amplitudes of waves are essential for shock formation while our method is limited to the linear level. It is hence necessary to extend the formulation to accommodate these non-linear effects somehow, which will be a future task.

Appendix A.   Detailed derivations

We begin with the following transformation amplitudes: $\langle x^{(0)}(s)|\hat{\Pi}^\mu(s)U(s)|x(0)\rangle$, $\langle x^{(0)}(s)|U(s)\hat{\Pi}^\mu(0)|x(0)\rangle$. They are written as

$$
\langle x(s)|\hat{\Pi}^\mu(s)U(s)|x(0)\rangle
$$

$$
= \langle x(s)|\hat{\Pi}^\mu(s)\left[1 - i\int_0^s du\Big\{e\hat{\Pi}^\alpha(u)b_\alpha \exp\left[-ik_\delta\hat{x}^\delta(u)\right]\right.
$$

$$
\left. + eb_\alpha \exp\left[-ik_\delta\hat{x}^\delta(u)\right]\hat{\Pi}^\alpha(u) + \frac{1}{2}e\sigma^{\alpha\beta}(u)g_{\alpha\beta}(u)\Big\}\right]|x(0)\rangle, \tag{A.1}
$$

$$\langle x(s)|U(s)\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$= \langle x(s)| \left[1 - i \int_0^s du \left\{ e\hat{\Pi}^{\alpha}(u)b_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] \right. \right.$$

$$\left. \left. + eb_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\alpha}(u) + \frac{1}{2}e\sigma^{\alpha\beta}(u)g_{\alpha\beta}(u) \right\} \right] \hat{\Pi}^{\mu}(0)|x(0)\rangle, \qquad (A.2)$$

with the proper time evolution operator given in Eq. (40). In this expression, $g_{\alpha\beta}(u) = g_{\alpha\beta} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$. We rearrange the first two terms in the integrand as

$$\hat{\Pi}^{\mu}(s) \left(e\hat{\Pi}^{\alpha}(u)b_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] + eb_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\alpha}(u) \right)$$

$$= 2eb_{\alpha}\hat{\Pi}^{\mu}(s)\hat{\Pi}^{\alpha}(u) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] - eb_{\alpha}k^{\alpha}\hat{\Pi}^{\mu}(s) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right], \qquad (A.3)$$

$$\left(e\hat{\Pi}^{\alpha}(u)b_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] + eb_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\alpha}(u) \right) \hat{\Pi}^{\mu}(0)$$

$$= 2eb_{\alpha}\hat{\Pi}^{\alpha}(u) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0) - eb_{\alpha}k^{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0), \qquad (A.4)$$

using the relation

$$\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}_{\alpha}(u) = \left\{ \hat{\Pi}_{\alpha}(u) + \left[-ik_{\delta}\hat{x}^{\delta}(u), \hat{\Pi}_{\alpha}(u)\right] \right\} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$$

$$= \hat{\Pi}_{\alpha}(u) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] - k_{\alpha} \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right], \qquad (A.5)$$

which is obtained from Eqs. (A.29) and (B.12). The calculations of the remaining terms in the integrand, $\langle x(s)|\hat{\Pi}^{\mu}(s)\left(-i\int_0^s du \frac{1}{2}e\sigma^{\alpha\beta}(u)g_{\alpha\beta}(u)\right)|x(0)\rangle$ and $\langle x(s)|\left(-i\int_0^s du \frac{1}{2}e\sigma^{\alpha\beta}(u)g_{\alpha\beta}(u)\right)\hat{\Pi}^{\mu}(0)|x(0)\rangle$, proceed as follows:

$$\langle x(s)|\hat{\Pi}^{\mu}(s) \int_0^s du \left(-\frac{ie}{2}\right) \sigma^{\alpha\beta}(u)g_{\alpha\beta}(u)|x(0)\rangle$$

$$\simeq \langle x(s)| \int_0^s du\,\hat{\Pi}^{\mu}(s) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$$

$$\times \left(-\frac{ie}{2}\right) \left[(\sigma g) + \frac{ieu}{2}\left\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\right\} + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f) \right] |x(0)\rangle$$

$$+ \langle x(s)| \int_0^s du\,\hat{\Pi}^{\mu}(s) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right] \left(-\frac{ie}{2}\right)$$

$$\times \left[\frac{ieu}{2}\left\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\right\}(\Omega\xi(0)) + \frac{e^2 u^2}{2}(\sigma f)(\sigma g)(\sigma f)(\Omega\xi(0)) \right] |x(0)\rangle, \quad (A.6)$$

$$\langle x(s)| \int_0^s du \left(-\frac{ie}{2}\right) g_{\alpha\beta}(u)\sigma^{\alpha\beta}(u)\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$\simeq \langle x(s)| \int_0^s du \left(-\frac{ie}{2}\right) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)$$

$$\times \left[(\sigma g) + \frac{ieu}{2}\left\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\right\} + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f) \right] |x(0)\rangle$$

$$+ \langle x(s)| \int_0^s du \left(-\frac{ie}{2}\right) \exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)$$

$$\times \left[\frac{ieu}{2}\left\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\right\}(\Omega\xi(0)) + \frac{e^2 u^2}{2}(\sigma f)(\sigma g)(\sigma f)(\Omega\xi(0)) \right] |x(0)\rangle$$

$$+ \langle x(s)| \int_0^s du \left(-\frac{ie}{2}\right) \exp\left[-ik_\delta \hat{x}^\delta(u)\right](-in^\mu)$$

$$\times \left[\frac{ieu}{2}\{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}\Omega + \frac{e^2 u^2}{2}(\sigma f)(\sigma g)(\sigma f)\Omega\right]|x(0)\rangle. \tag{A.7}$$

On the second lines in the above equations, we employed the expansion of $\sigma^{\alpha\beta}(u)$ given in Eq. (42). The resultant expressions with Eqs. (A.3) and (A.4) give Eqs. (43) and (44). Note that all operators in these expressions, i.e., $\hat{\Pi}^\mu(s)$, $\hat{\Pi}^\mu(0)$, $\hat{\Pi}^\mu(u)$, and $\hat{x}^\mu(u)$, are defined in the interaction picture.

Remaining are the evaluations of the transformation amplitudes, such as

$$\langle x(s)|\hat{\Pi}^\mu(s)|x(0)\rangle, \tag{A.8}$$

$$\langle x(s)|\hat{\Pi}^\mu(0)|x(0)\rangle, \tag{A.9}$$

$$\langle x(s)|\exp\left[-ik_\delta \hat{x}^\delta(u)\right]|x(0)\rangle, \tag{A.10}$$

$$\langle x(s)|\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta \hat{x}^\delta(u)\right]|x(0)\rangle, \tag{A.11}$$

$$\langle x(s)|\hat{\Pi}^\mu(s)\exp\left[-ik_\delta \hat{x}^\delta(u)\right]|x(0)\rangle, \tag{A.12}$$

$$\langle x(s)|\exp\left[-ik_\delta \hat{x}^\delta(u)\right]\hat{\Pi}^\mu(0)|x(0)\rangle, \tag{A.13}$$

$$\langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta \hat{x}^\delta(u)\right]|x(0)\rangle, \tag{A.14}$$

$$\langle x(s)|\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta \hat{x}^\delta(u)\right]\hat{\Pi}^\mu(0)|x(0)\rangle. \tag{A.15}$$

Each operator in these amplitudes can be represented with $\hat{x}^\mu(s)$ and $\hat{x}^\mu(0)$. For example, $\hat{\Pi}^\mu(s)$ and $\hat{\Pi}^\mu(0)$ are derived from Eqs. (27) and (28) to the lowest order of Ω as

$$\hat{\Pi}^\mu(s) = \frac{\hat{x}^\mu(s) - \hat{x}^\mu(0)}{2s} + \frac{e}{2}f^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0)) + \Omega\frac{e}{2}f^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0))\left(\frac{2}{3}\xi(s) + \frac{1}{3}\xi(0)\right)$$

$$+ n^\mu e^2 f^2 s\left(\frac{1}{6}\xi(s) - \frac{1}{6}\xi(0)\right) + \Omega n^\mu e^2 f^2 s\left(\frac{1}{4}\xi^2(s) - \frac{1}{6}\xi(s)\xi(0) - \frac{1}{12}\xi^2(0)\right)$$

$$+ \frac{1}{4}\Omega e s n^\mu(\sigma f), \tag{A.16}$$

$$\hat{\Pi}^\mu(0) = \frac{\hat{x}^\mu(s) - \hat{x}^\mu(0)}{2s} - \frac{e}{2}f^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0)) + \Omega\frac{e}{2}f^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0))\left(-\frac{1}{3}\xi(s) - \frac{2}{3}\xi(0)\right)$$

$$+ n^\mu e^2 f^2 s\left(\frac{1}{6}\xi(s) - \frac{1}{6}\xi(0)\right) + \Omega n^\mu e^2 f^2 s\left(\frac{1}{12}\xi^2(s) + \frac{1}{6}\xi(s)\xi(0) - \frac{1}{4}\xi^2(0)\right)$$

$$- \frac{1}{4}\Omega e s n^\mu(\sigma f). \tag{A.17}$$

Using the fact that the left-hand side (and hence the right-hand side also) of Eq. (A.17) is independent of s, we obtain the operator $\hat{x}^\mu(u)$ in terms of $\hat{x}^\mu(s)$ and $\hat{x}^\mu(0)$ as

$$\hat{x}^\mu(u)$$

$$= \hat{x}^\mu(0) + \frac{u}{s}(\hat{x}^\mu(s) - \hat{x}^\mu(0))$$

$$+ ef^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0))\left[-u + \frac{u^2}{s} + \Omega\left\{\left(-\frac{u}{3} + \frac{1}{3}\frac{u^3}{s^2}\right)\xi(s) + \left(-\frac{2}{3}u + \frac{u^2}{s} - \frac{1}{3}\frac{u^3}{s^2}\right)\xi(0)\right\}\right]$$

$$+ n^\mu e^2 f^2 \left\{ \left(\frac{su}{3} - u^2 + \frac{2}{3}\frac{u^3}{s} \right) \xi(s) + \left(-\frac{su}{3} + u^2 - \frac{2}{3}\frac{u^3}{s} \right) \xi(0) \right\}$$

$$+ \Omega n^\mu e^2 f^2 \left\{ \left(\frac{su}{6} - \frac{1}{3}u^2 - \frac{1}{3}\frac{u^3}{s} + \frac{1}{2}\frac{u^4}{s^2} \right) \xi^2(s) \right.$$

$$\left. + \left(\frac{su}{3} - \frac{4}{3}u^2 + 2\frac{u^3}{s} - \frac{u^4}{s^2} \right) \xi(s)\xi(0) + \left(-\frac{su}{2} + \frac{5}{3}u^2 - \frac{5}{3}\frac{u^3}{s} + \frac{1}{2}\frac{u^4}{s^2} \right) \xi^2(0) \right\}$$

$$+ \frac{1}{2}\Omega e \sigma^{\nu\lambda} f_{\nu\lambda} n^\mu \left(u^2 - su \right). \tag{A.18}$$

Replacing s with u in Eq. (A.16) and plugging Eq. (A.18) into Eq. (A.16), we can express $\hat{\Pi}^\mu(u)$ as

$$\hat{\Pi}^\mu(u)$$
$$= \frac{\hat{x}^\mu(s) - \hat{x}^\mu(0)}{2s}$$

$$+ ef^\mu{}_\nu(\hat{x}^\nu(s) - \hat{x}^\nu(0)) \left[-\frac{1}{2} + \frac{u}{s} + \Omega \left\{ \left(\frac{1}{2}\left(\frac{u}{s}\right)^2 - \frac{1}{6} \right) \xi(s) + \left(-\frac{1}{2}\left(\frac{u}{s}\right)^2 + \frac{u}{s} - \frac{1}{3} \right) \xi(0) \right\} \right]$$

$$+ n^\mu e^2 f^2 s \left[\left\{ \frac{1}{6} - \frac{u}{s} + \left(\frac{u}{s}\right)^2 \right\} \xi(s) + \left\{ -\frac{1}{6} + \frac{u}{s} - \left(\frac{u}{s}\right)^2 \right\} \xi(0) \right]$$

$$+ \Omega n^\mu e^2 f^2 s \left[\left\{ \frac{1}{12} - \frac{1}{3}\left(\frac{u}{s}\right) - \frac{1}{2}\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 \right\} \xi^2(s) \right.$$

$$\left. + \left\{ \frac{1}{6} - \frac{4}{3}\frac{u}{s} + 3\left(\frac{u}{s}\right)^2 - 2\left(\frac{u}{s}\right)^3 \right\} \xi(s)\xi(0) + \left\{ -\frac{1}{4} + \frac{5}{3}\frac{u}{s} - \frac{5}{2}\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 \right\} \xi^2(0) \right]$$

$$+ \Omega e \sigma^{\nu\lambda} f_{\nu\lambda} n^\mu s \left(-\frac{1}{4} + \frac{1}{2}\frac{u}{s} \right). \tag{A.19}$$

It is now easy to evaluate the amplitudes in Eqs. (A.8) and (A.9), which appear in the induced electromagnetic current as $\mathrm{tr}\left(\langle x(s)|\hat{\Pi}^\mu(s) + \hat{\Pi}^\mu(0)|x(0)\rangle \right)$ and $\mathrm{tr}\left(\sigma^{\mu\nu} \langle x(s)|\hat{\Pi}^\mu(s) - \hat{\Pi}^\mu(0)|x(0)\rangle \right)$. They are given as

$$\mathrm{tr}\left(\langle x(s)|\hat{\Pi}^\mu(s) + \hat{\Pi}^\mu(0)|x(0)\rangle \right) \simeq \mathrm{tr}\left(\langle x(s)|0|x(0)\rangle \right) = 0, \tag{A.20}$$

$$\mathrm{tr}\left(\sigma^{\mu\nu} \langle x(s)|\hat{\Pi}_\nu(s) - \hat{\Pi}_\nu(0)|x(0)\rangle \right) \simeq \mathrm{tr}\left[\sigma^{\mu\nu} n_\nu (\sigma f) \right] \frac{1}{i(4\pi)^2 s} \frac{e}{2} \Omega = 0, \tag{A.21}$$

where we used the relation

$$\langle x(s)|x(0)\rangle = \frac{1}{i(4\pi)^2 s^2} \left(1 - \frac{ies}{2}(\sigma f)(1 + \Omega\xi) \right), \tag{A.22}$$

which is derived from Eq. (31). There is hence no contribution to the induced electromagnetic current from $\langle x(s)|\hat{\Pi}^\mu(s)|x(0)\rangle$ and $\langle x(s)|\hat{\Pi}^\mu(0)|x(0)\rangle$.

The amplitude given in Eq. (A.10) is calculated to the linear order of Ω by using the Zassenhaus formula:

$$e^{X+\Omega Y} \simeq e^X e^{\Omega Y} e^{-\frac{1}{2}[X,\Omega Y]} e^{\frac{1}{6}(2[\Omega Y,[X,\Omega Y]]+[X,[X,\Omega Y]])}$$

$$\simeq e^X + e^X \Omega Y + e^X \left(-\frac{1}{2}[X,\Omega Y]\right) + e^X \frac{1}{6}[X,[X,\Omega Y]]. \tag{A.23}$$

In this expression, X stands collectively for the terms that do not include Ω in the argument of the exponential function in Eq. (A.10), whereas ΩY represents those terms that depend on Ω. The commutation relations in this equation are evaluated as follows:

$$[X,\Omega Y] = i\Omega(k\cdot n)^2 e^2 f^2 \left(-\frac{4}{3}su^2 + \frac{10}{3}u^3 - 2\frac{u^4}{s}\right)\xi(s)$$

$$+ i\Omega(k\cdot n)^2 e^2 f^2 \left(-\frac{2}{3}s^2 u + \frac{10}{3}su^2 - \frac{14}{3}u^3 + 2\frac{u^4}{s}\right)\xi(0)$$

$$+ i\Omega(k\cdot n)ek_\beta f^\beta{}_\nu\left[\hat{x}^\nu(s) - \hat{x}^\nu(0)\right]\left(-\frac{2}{3}su + 2u^2 - \frac{4}{3}\frac{u^3}{s}\right), \tag{A.24}$$

$$[X,[X,\Omega Y]] = i\Omega(k\cdot n)^3 e^2 f^2 \left(\frac{4}{3}s^3 u - \frac{16}{3}s^2 u^2 + 8su^3 - 4u^4\right). \tag{A.25}$$

Putting these results together, we obtain the explicit expression of the exponential operator suited for the calculation of the amplitude as

$$e^{-ik_\alpha \hat{x}^\alpha(u)} = \exp\left\{-ik_\alpha\left[\frac{u}{s}\hat{x}^\alpha(s) + ef^\alpha{}_\beta \hat{x}^\beta(s)\left(-u+\frac{u^2}{s}\right) + n^\alpha e^2 f^2\left(\frac{su}{3} - u^2 + \frac{2}{3}\frac{u^3}{s}\right)\xi(s)\right]\right\}$$

$$\times \left(1 + \Omega\left\{iek_\alpha f^\alpha{}_\beta\left[\hat{x}^\beta(s) - \hat{x}^\beta(0)\right]\left[\left(\frac{u}{3} - \frac{1}{3}\frac{u^3}{s^2}\right)\xi(s) + \left(\frac{2}{3}u - \frac{u^2}{s} + \frac{1}{3}\frac{u^3}{s^2}\right)\xi(0)\right]\right.$$

$$+ i(k\cdot n)e^2 f^2\left[\left(-\frac{su}{6} + \frac{1}{3}u^2 + \frac{1}{3}\frac{u^3}{s} - \frac{1}{2}\frac{u^4}{s^2}\right)\xi^2(s)\right.$$

$$+ \left(-\frac{su}{3} + \frac{4}{3}u^2 - 2\frac{u^3}{s} + \frac{u^4}{s^2}\right)\xi(s)\xi(0) + \left.\left(\frac{su}{2} - \frac{5}{3}u^2 + \frac{5}{3}\frac{u^3}{s} - \frac{1}{2}\frac{u^4}{s^2}\right)\xi^2(0)\right]$$

$$+ ie(k\cdot n)k_\alpha f^\alpha{}_\beta\left[\hat{x}^\beta(s) - \hat{x}^\beta(0)\right]\left(-\frac{2}{3}u^2 + \frac{4}{3}\frac{u^3}{s} - \frac{2}{3}\frac{u^4}{s^2}\right)$$

$$+ i(k\cdot n)^2 e^2 f^2 \xi(s)\left(\frac{1}{3}su^2 - 2u^3 + 3\frac{u^4}{s} - \frac{4}{3}\frac{u^5}{s^2}\right)$$

$$+ i(k\cdot n)^2 e^2 f^2 \xi(0)\left(-su^2 + \frac{10}{3}u^3 - \frac{11}{3}\frac{u^4}{s} + \frac{4}{3}\frac{u^5}{s^2}\right)$$

$$+ i(k\cdot n)^3 e^2 f^2\left(-\frac{2}{3}s^3 u + \frac{14}{9}s^2 u^2 - \frac{4}{9}su^3 - \frac{10}{9}u^4 + \frac{2}{3}\frac{u^5}{s}\right)$$

$$+ \left.ie(\sigma f)(k\cdot n)\left(\frac{1}{2}su - \frac{1}{2}u^2\right)\right\}\right)$$

$$\times \exp\left\{-ik_\mu\left[\left(1-\frac{u}{s}\right)\hat{x}^\mu(0) + ef^\mu{}_\nu \hat{x}^\nu(0)\left(u - \frac{u^2}{s}\right)\right.\right.$$

$$+ n^\mu e^2 f^2 \left(-\frac{su}{3} + u^2 - \frac{2}{3}\frac{u^3}{s} \right) \xi(0) \Bigg] \Bigg\}$$

$$\times \exp\left[i(k)^2 \left(u - \frac{u^2}{s} \right) + i(k \cdot n)^2 e^2 f^2 \left(-\frac{1}{3}su^2 + \frac{2}{3}u^3 - \frac{1}{3}\frac{u^4}{s} \right) \right]. \tag{A.26}$$

The transformation amplitude is then given as

$$\langle x(s) | \exp\left[-ik_\mu \hat{x}^\mu(u) \right] | x(0) \rangle$$

$$= \langle x(s) | x(0) \rangle \times \exp\left(-ik_\mu x^\mu \right) \exp\left[i(k)^2 \left(u - \frac{u^2}{s} \right) + i(k \cdot n)^2 e^2 f^2 \left(-\frac{1}{3}su^2 + \frac{2}{3}u^3 - \frac{1}{3}\frac{u^4}{s} \right) \right]$$

$$\times \Bigg\{ 1 + \Omega \Bigg[i(k \cdot n)^2 e^2 f^2 \xi \left(-\frac{2}{3}su^2 + \frac{4}{3}u^3 - \frac{2}{3}\frac{u^4}{s} \right)$$

$$+ i(k \cdot n)^3 e^2 f^2 \left(-\frac{2}{3}s^3 u + \frac{14}{9}s^2 u^2 - \frac{4}{9}su^3 - \frac{10}{9}u^4 + \frac{2}{3}\frac{u^5}{s} \right) + ie(\sigma f)(k \cdot n) \left(\frac{1}{2}su - \frac{1}{2}u^2 \right) \Bigg] \Bigg\}. \tag{A.27}$$

We next calculate the amplitudes in Eqs. (A.11)–(A.13). The operators $\hat{\Pi}^\mu(u)$, $\hat{\Pi}^\mu(s)$, and $\hat{\Pi}^\mu(0)$ are written in terms of $\hat{x}^\mu(s)$ and $\hat{x}^\mu(0)$, and the amplitudes can be calculated after rearranging the order of operators. We first consider the rearrangement of $\hat{x}^\alpha(0) \exp\left[-ik_\mu \hat{x}^\mu(u) \right]$. Using the relations

$$B e^{-A} = e^{-A} B + e^{-A} [A, B] + \frac{1}{2} e^{-A} [A, [A, B]], \tag{A.28}$$

$$e^A B = B e^A + [A, B] e^A + \frac{1}{2} [A, [A, B]] e^A, \tag{A.29}$$

which are derived from Hadamard's lemma

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]], \tag{A.30}$$

one can obtain

$$\hat{x}^\alpha(0) \exp\left[-ik_\mu \hat{x}^\mu(u) \right]$$

$$= \exp\left[-ik_\mu \hat{x}^\mu(u) \right]$$

$$\times \Bigg[\hat{x}^\alpha(0) - 2uk^\alpha + 2e(f^{\alpha\mu}k_\mu)u^2 - \frac{4}{3}u^3 n^\alpha (k \cdot n) e^2 f^2$$

$$+ \Omega e(f^{\alpha\mu}k_\mu) \left\{ \frac{2}{3}\frac{u^3}{s}\xi(s) + \left(2u^2 - \frac{2}{3}\frac{u^3}{s} \right) \xi(0) \right\} + \Omega e n^\alpha k_\mu f^\mu{}_\nu \left(\hat{x}^\nu(s) - \hat{x}^\nu(0) \right) \left(-\frac{2}{3}\frac{u^3}{s} \right)$$

$$+ \Omega n^\alpha (k \cdot n) e^2 f^2 \left\{ \left(\frac{2}{3}u^3 - 2\frac{u^4}{s} \right) \xi(s) + \left(-\frac{10}{3}u^3 + 2\frac{u^4}{s} \right) \xi(0) \right\}$$

$$+ \Omega e(f^{\alpha\mu}k_\mu)(k \cdot n) \left(-\frac{4}{3}u^3 \right) + \Omega n^\alpha (k \cdot n)^2 e^2 f^2 (2u^4) \Bigg], \tag{A.31}$$

which is still inappropriate for the calculation of the amplitudes because some $\hat{x}^\mu(s)$ are sitting to the right of $\exp\left[-ik_\delta \hat{x}^\delta(u) \right]$, which contains $\hat{x}^\mu(0)$. We hence have to rearrange further the terms

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that contain $\hat{x}^\mu(s)$ to obtain

$$\hat{x}^\alpha(0)\exp\left[-ik_\mu\hat{x}^\mu(u)\right]$$

$$= \exp\left[-ik_\mu\hat{x}^\mu(u)\right]\left[\hat{x}^\alpha(0) + \Omega e(f^{\alpha\mu}k_\mu)\left(2u^2 - \frac{2}{3}\frac{u^3}{s}\right)\xi(0)\right.$$

$$\left. + \Omega e n^\alpha k_\mu f^\mu{}_\nu\hat{x}^\nu(0)\frac{2}{3}\frac{u^3}{s} + \Omega n^\alpha(k\cdot n)e^2f^2\left(-\frac{10}{3}u^3 + 2\frac{u^4}{s}\right)\xi(0)\right]$$

$$+ \left[\Omega e(f^{\alpha\mu}k_\mu)\frac{2}{3}\frac{u^3}{s}\xi(s) + \Omega e n^\alpha k_\mu f^\mu{}_\nu\hat{x}^\nu(s)\left(-\frac{2}{3}\frac{u^3}{s}\right)\right.$$

$$\left. + \Omega n^\alpha(k\cdot n)e^2f^2\left(\frac{2}{3}u^3 - 2\frac{u^4}{s}\right)\xi(s)\right]\exp\left[-ik_\mu\hat{x}^\mu(u)\right]$$

$$+ \exp\left[-ik_\mu\hat{x}^\mu(u)\right]\left[-2uk^\alpha + 2e(f^{\alpha\mu}k_\mu)u^2 - \frac{4}{3}u^3n^\alpha(k\cdot n)e^2f^2\right.$$

$$\left. + \Omega e(f^{\alpha\mu}k_\mu)(k\cdot n)\left(\frac{4}{3}\frac{u^4}{s} - \frac{8}{3}u^3\right) + \Omega n^\alpha(k\cdot n)^2e^2f^2\left(-\frac{8}{3}\frac{u^5}{s} + \frac{14}{3}u^4\right)\right]. \quad\text{(A.32)}$$

This is the expression suitable for the calculation of the transformation amplitudes.

The rearrangement of $\exp\left[-ik_\mu\hat{x}^\mu(u)\right]\hat{x}^\alpha(s)$ goes similarly. The amplitudes of these operators are then written as follows:

$$\langle x(s)|\hat{x}^\mu(0)\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times\left[x^\mu + \Omega ef^{\mu\nu}k_\nu\xi 2u^2 + \Omega n^\mu(k\cdot n)e^2f^2\xi\left(-\frac{8}{3}u^3\right) - 2uk^\alpha + 2ef^{\mu\nu}k_\nu u^2\right.$$

$$\left. - \frac{4}{3}u^3n^\mu(k\cdot n)e^2f^2 + \Omega ef^{\mu\nu}k_\nu(k\cdot n)\left(\frac{4}{3}\frac{u^4}{s} - \frac{8}{3}u^3\right)\right.$$

$$\left. + \Omega n^\mu(k\cdot n)^2e^2f^2\left(-\frac{8}{3}\frac{u^5}{s} + \frac{14}{3}u^4\right)\right], \quad\text{(A.33)}$$

$$\langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]\hat{x}^\mu(s)|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times\left[x^\mu + \Omega ef^{\mu\nu}k_\nu\xi(-2s^2 + 4su - 2u^2) + \Omega n^\mu(k\cdot n)e^2f^2\left(-\frac{8}{3}s^3 + 8s^2u - 8su^2 + \frac{8}{3}u^3\right)\xi\right.$$

$$\left. + 2(u-s)k^\mu + 2ef^{\mu\nu}k_\nu(-s^2 + 2su - u^2) + n^\mu(k\cdot n)e^2f^2\left(-\frac{4}{3}s^3 + 4s^2u - 4su^2 + \frac{4}{3}u^3\right)\right.$$

$$\left. + \Omega e(k\cdot n)f^{\mu\nu}k_\nu\left(\frac{4}{3}s^3 - \frac{8}{3}s^2u + \frac{8}{3}u^3 - \frac{4}{3}\frac{u^4}{s}\right)\right.$$

$$\left. + \Omega n^\mu(k\cdot n)^2e^2f^2\left(2s^4 - \frac{16}{3}s^3u + \frac{4}{3}s^2u^2 + 8su^3 - \frac{26}{3}u^4 + \frac{8}{3}\frac{u^5}{s}\right)\right]. \quad\text{(A.34)}$$

The quadratic terms in x, e.g., $\langle x(s)|f^{\mu}{}_{\nu}\hat{x}^{\nu}(0)\xi(0)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$, can be calculated by successive commutations. All results combined, the amplitude of $\hat{\Pi}^{\mu}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$ is given as

$$\langle x(s)|\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\mu}\hat{x}^{\mu}(u)\right]|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_{\mu}\hat{x}^{\mu}(u)\right]|x(0)\rangle$$

$$\times\left[\frac{u}{s}k^{\alpha} + ef^{\alpha\beta}k_{\beta}\left(\frac{u^2}{s}-u\right) + n^{\alpha}(k\cdot n)e^2f^2\left(\frac{1}{3}su-u^2+\frac{2}{3}\frac{u^3}{s}\right) + \Omega ef^{\alpha\beta}k_{\beta}\xi\left(\frac{u^2}{s}-u\right)\right.$$

$$+ \Omega n^{\alpha}(k\cdot n)e^2f^2\xi\left(\frac{2}{3}su-2u^2+\frac{4}{3}\frac{u^3}{s}\right) + \Omega ef^{\alpha\beta}k_{\beta}(k\cdot n)\left(\frac{4}{3}u^2-\frac{8}{3}\frac{u^3}{s}+\frac{4}{3}\frac{u^4}{s^2}\right)$$

$$\left.+ \Omega n^{\alpha}(k\cdot n)^2e^2f^2\left(-su^2+4u^3-5\frac{u^4}{s}+2\frac{u^5}{s^2}\right) + \Omega e(\sigma f)n^{\alpha}\left(-\frac{1}{4}s+\frac{1}{2}u\right)\right]. \quad (A.35)$$

Similar expressions are obtained for the amplitudes of $\hat{\Pi}^{\mu}(s)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$ and $\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)$, which are shown, respectively, as follows:

$$\langle x(s)|\hat{\Pi}^{\mu}(s)\exp\left[-ik_{\alpha}\hat{x}^{\alpha}(u)\right]|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_{\alpha}\hat{x}^{\alpha}(u)\right]|x(0)\rangle$$

$$\times\left[\Omega ef^{\mu\nu}k_{\nu}\xi\left(u-\frac{u^2}{s}\right) + \Omega n^{\mu}(k\cdot n)e^2f^2\xi\left(\frac{2}{3}su-2u^2+\frac{4}{3}\frac{u^3}{s}\right) + \left(u-\frac{u^2}{s}\right)ef^{\mu\nu}k_{\nu}\right.$$

$$+ \left(\frac{1}{3}su-u^2+\frac{2}{3}\frac{u^3}{s}\right)n^{\mu}(k\cdot n)e^2f^2 + \frac{u}{s}k^{\mu} + \left(-\frac{2}{3}u^2+\frac{4}{3}\frac{u^3}{s}-\frac{2}{3}\frac{u^4}{s^2}\right)\Omega ef^{\mu\nu}k_{\nu}(k\cdot n)$$

$$\left.+ \left(-\frac{1}{3}su^2+2u^3-3\frac{u^4}{s}+\frac{4}{3}\frac{u^5}{s^2}\right)\Omega n^{\mu}(k\cdot n)^2e^2f^2 + \frac{1}{4}\Omega esn^{\mu}(\sigma f)\right], \quad (A.36)$$

$$\langle x(s)|\exp\left[-ik_{\alpha}\hat{x}^{\alpha}(u)\right]\hat{\Pi}^{\mu}(0)|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_{\alpha}\hat{x}^{\alpha}(u)\right]|x(0)\rangle$$

$$\times\left[\Omega ef^{\mu\nu}k_{\nu}\xi\left(u-\frac{u^2}{s}\right) + \Omega n^{\mu}(k\cdot n)e^2f^2\xi\left(\frac{2}{3}su-2u^2+\frac{4}{3}\frac{u^3}{s}\right) + \left(\frac{u}{s}-1\right)k^{\mu}\right.$$

$$+ \left(u-\frac{u^2}{s}\right)ef^{\mu\nu}k_{\nu} + \left(\frac{1}{3}su-u^2+\frac{2}{3}\frac{u^3}{s}\right)n^{\mu}(k\cdot n)e^2f^2$$

$$+ \Omega\left(-\frac{2}{3}u^2+\frac{4}{3}\frac{u^3}{s}-\frac{2}{3}\frac{u^4}{s^2}\right)ef^{\mu\nu}k_{\nu}(k\cdot n)$$

$$\left.+ \Omega\left(-su^2+\frac{10}{3}u^3-\frac{11}{3}\frac{u^4}{s}+\frac{4}{3}\frac{u^5}{s^2}\right)n^{\mu}(k\cdot n)^2e^2f^2 + \left(-\frac{1}{4}\right)\Omega esn^{\mu}(\sigma f)\right]. \quad (A.37)$$

Finally, Eqs. (A.14) and (A.15) are calculated. We rewrite them in terms of $\langle x(s)|\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$ and $\langle x(s)|\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]|x(0)\rangle$, which have already been evaluated. In so doing, the products of the operators such as $\hat{x}^{\mu}(0)\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$ in $\hat{\Pi}^{\mu}(s)\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]$ and $\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{x}^{\mu}(s)$ in $\hat{\Pi}^{\alpha}(u)\exp\left[-ik_{\delta}\hat{x}^{\delta}(u)\right]\hat{\Pi}^{\mu}(0)$ have to be rearranged. To accomplish this, we need the following commutation relations for $\hat{\Pi}^{\mu}(u)$,

which are obtained from the results given in Appendix B:

$$\left[\hat{x}^\mu(0), \hat{\Pi}^\alpha(u)\right] = -i\eta^{\mu\alpha} + 2uief^{\mu\alpha} - 2u^2 in^\mu n^\alpha e^2 f^2 + \Omega ief^{\mu\alpha}\xi(s)\left(\frac{u^2}{s}\right)$$

$$+ \Omega ief^{\mu\alpha}\xi(0)\left(2u - \frac{u^2}{s}\right) + \Omega in^\mu n^\alpha e^2 f^2 \xi(s)\left(u^2 - 4\frac{u^3}{s}\right)$$

$$+ \Omega in^\mu n^\alpha e^2 f^2 \xi(0)\left(-5u^2 + 4\frac{u^3}{s}\right) + \Omega ien^\mu f^\alpha{}_\beta\left(\hat{x}^\beta(s) - \hat{x}^\beta(0)\right)\left(-\frac{u^2}{s}\right),$$

(A.38)

$$\left[\hat{\Pi}^\alpha(u), \hat{x}^\mu(s)\right] = i\eta^{\alpha\mu} + (-2s + 2u)ief^{\alpha\mu} + (2s^2 - 4su + 2u^2)in^\alpha n^\mu e^2 f^2$$

$$+ \Omega ief^{\alpha\mu}\xi(s)\left(\frac{u^2}{s} - s\right) + \Omega ief^{\alpha\mu}\xi(0)\left(-\frac{u^2}{s} + 2u - s\right)$$

$$+ \Omega in^\alpha n^\mu e^2 f^2 \xi(s)\left(s^2 + 2su - 7u^2 + 4\frac{u^3}{s}\right)$$

$$+ \Omega in^\alpha n^\mu e^2 f^2 \xi(0)\left(3s^2 - 10su + 11u^2 - 4\frac{u^3}{s}\right)$$

$$+ \Omega ien^\mu f^\alpha{}_\beta\left(\hat{x}^\beta(s) - \hat{x}^\beta(0)\right)\left(\frac{u^2}{s} - 2u + s\right).$$

(A.39)

The employment of these relations produces the following results:

$$\langle x(s)|\hat{x}^\mu(0)\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times\left[-i\eta^{\mu\alpha} + 2uief^{\mu\alpha} - 2u^2 in^\mu n^\alpha e^2 f^2 + i\Omega en^\alpha f^{\mu\nu}k_\nu\frac{2}{3}\frac{u^3}{s} + i\Omega en^\mu f^{\alpha\beta}k_\beta\left(-\frac{4}{3}\frac{u^3}{s}\right)\right.$$

$$+ i\Omega e(k\cdot n)f^{\mu\alpha}\left(-4u^2 + 2\frac{u^3}{s}\right) + i\Omega n^\mu n^\alpha(k\cdot n)e^2 f^2\left(\frac{28}{3}u^3 - \frac{20}{3}\frac{u^4}{s}\right)$$

$$\left.+ i\Omega n^\mu n^\alpha e^2 f^2 \xi(-4u^2) + i\Omega ef^{\mu\alpha}\xi 2u\right]$$

$$+ \langle x(s)|\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times\left[x^\mu - 2uk^\mu + 2ef^{\mu\nu}k_\nu u^2 - \frac{4}{3}u^3 n^\mu(k\cdot n)e^2 f^2 + \Omega ef^{\mu\nu}k_\nu(k\cdot n)\left(\frac{4}{3}\frac{u^4}{s} - \frac{8}{3}u^3\right)\right.$$

$$+ \Omega n^\mu(k\cdot n)^2 e^2 f^2\left(-\frac{8}{3}\frac{u^5}{s} + \frac{14}{3}u^4\right)$$

$$\left.+ \Omega ef^{\mu\nu}k_\nu(2u^2)\xi + \Omega n^\mu(k\cdot n)e^2 f^2\left(-\frac{8}{3}u^3\right)\xi\right],$$

(A.40)

$$\langle x(s)|\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta\hat{x}^\delta(u)\right]\hat{x}^\mu(s)|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times\left[i\Omega ef^{\alpha\mu}\xi(2u - 2s) + i\Omega n^\alpha n^\mu e^2 f^2 \xi(4s^2 - 8su + 4u^2) + i\eta^{\alpha\mu} + (2u - 2s)ief^{\alpha\mu}\right.$$

$$+ (2s^2 - 4su + 2u^2)in^\alpha n^\mu e^2 f^2 + i\Omega e n^\alpha f^{\mu\nu} k_\nu \left(-\frac{4}{3}s^2 + 2su - \frac{2}{3}\frac{u^3}{s} \right)$$

$$+ i\Omega e n^\mu f^{\alpha\beta} k_\beta \left(\frac{2}{3}s^2 - 2u^2 + \frac{4}{3}\frac{u^3}{s} \right) + i\Omega e (k \cdot n) f^{\alpha\mu} \left(2\frac{u^3}{s} - 4u^2 + 2su \right)$$

$$+ i\Omega n^\mu n^\alpha (k \cdot n) e^2 f^2 \left(-\frac{8}{3}s^3 + \frac{4}{3}s^2 u + 12su^2 - \frac{52}{3}u^3 + \frac{20}{3}\frac{u^4}{s} \right) \Bigg]$$

$$+ \langle x(s)|\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] |x(0)\rangle$$

$$\times \Bigg[x^\mu + \Omega e f^{\mu\nu} k_\nu \xi(-2s^2 + 4su - 2u^2) + \Omega n^\mu (k \cdot n) e^2 f^2 \xi \left(-\frac{8}{3}s^3 + 8s^2 u - 8su^2 + \frac{8}{3}u^3 \right)$$

$$+ 2(u - s)k^\mu + 2e f^{\mu\nu} k_\nu (-s^2 + 2su - u^2) + n^\mu (k \cdot n) e^2 f^2 \left(-\frac{4}{3}s^3 + 4s^2 u - 4su^2 + \frac{4}{3}u^3 \right)$$

$$+ \Omega e (k \cdot n) f^{\mu\nu} k_\nu \left(\frac{4}{3}s^3 - \frac{8}{3}s^2 u + \frac{8}{3}u^3 - \frac{4}{3}\frac{u^4}{s} \right)$$

$$+ \Omega n^\mu (k \cdot n)^2 e^2 f^2 \left(2s^4 - \frac{16}{3}s^3 u + \frac{4}{3}s^2 u^2 + 8su^3 - \frac{26}{3}u^4 + \frac{8}{3}\frac{u^5}{s} \right) \Bigg]. \tag{A.41}$$

We are now ready to write down the amplitudes of the addition $\langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] + \hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] \hat{\Pi}^\mu(0)|x(0)\rangle$ and the subtraction $\langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] - \hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] \hat{\Pi}^\mu(0)|x(0)\rangle$, which appear in the induced electromagnetic current. The results are as follows:

$$\langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] + \hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u) \right] \hat{\Pi}^\mu(0)|x(0)\rangle$$

$$= \langle x(s)| \exp\left[-ik_\delta \hat{x}^\delta(u) \right] |x(0)\rangle$$

$$\times \Bigg[ie f^{\mu\alpha} \left(1 - 2\frac{u}{s} \right) + k^\mu k^\alpha \left(-\frac{u}{s} + 2\frac{u^2}{s^2} \right) + e k^\mu f^{\alpha\beta} k_\beta \left(u - 3\frac{u^2}{s} + 2\frac{u^3}{s^2} \right)$$

$$+ e f^{\mu\nu} k_\nu k^\alpha \left(2\frac{u^2}{s} - 2\frac{u^3}{s^2} \right) + e^2 f^{\mu\nu} k_\nu f^{\alpha\beta} k_\beta \left(-2u^2 + 4\frac{u^3}{s} - 2\frac{u^4}{s^2} \right)$$

$$+ k^\mu n^\alpha (k \cdot n) e^2 f^2 \left(-\frac{1}{3}su + \frac{5}{3}u^2 - \frac{8}{3}\frac{u^3}{s} + \frac{4}{3}\frac{u^4}{s^2} \right)$$

$$+ f^{\mu\nu} k_\nu n^\alpha (k \cdot n) e^3 f^2 \left(\frac{2}{3}su^2 - \frac{8}{3}u^3 + \frac{10}{3}\frac{u^4}{s} - \frac{4}{3}\frac{u^5}{s^2} \right)$$

$$+ n^\mu k^\alpha (k \cdot n) e^2 f^2 \left(\frac{2}{3}u^2 - 2\frac{u^3}{s} + \frac{4}{3}\frac{u^4}{s^2} \right)$$

$$+ n^\mu f^{\alpha\beta} k_\beta (k \cdot n) e^3 f^2 \left(-\frac{2}{3}su^2 + \frac{8}{3}u^3 - \frac{10}{3}\frac{u^4}{s} + \frac{4}{3}\frac{u^5}{s^2} \right) + in^\mu n^\alpha e^2 f^2 \left(\frac{1}{3}s - 2u + 2\frac{u^2}{s} \right)$$

$$+ n^\mu n^\alpha (k \cdot n)^2 e^4 f^4 \left(\frac{2}{9}s^2 u^2 - \frac{4}{3}su^3 + \frac{26}{9}u^4 - \frac{8}{3}\frac{u^5}{s} + \frac{8}{9}\frac{u^6}{s^2} \right) + i\eta^{\mu\alpha}\frac{1}{s} \Bigg]$$

$$+ i\Omega \langle x(s)| \exp\left[-ik_\delta \hat{x}^\delta(u) \right] |x(0)\rangle$$

$$\times \Bigg[e f^{\mu\alpha} (k \cdot n) \left(\frac{1}{3}s - \frac{5}{3}u + 4\frac{u^2}{s} - 2\frac{u^3}{s^2} \right) + e f^{\mu\alpha} \xi \left(1 - 2\frac{u}{s} \right)$$

$$
+ ie(k \cdot n)k^\mu f^{\alpha\beta}k_\beta \left(\frac{4}{3}u^2 - \frac{16}{3}\frac{u^3}{s} + \frac{20}{3}\frac{u^4}{s^2} - \frac{8}{3}\frac{u^5}{s^3} \right) + ie\xi k^\mu f^{\alpha\beta}k_\beta \left(-u + 3\frac{u^2}{s} - 2\frac{u^3}{s^2} \right)
$$

$$
+ ie(k \cdot n)f^{\mu\nu}k_\nu k^\alpha \left(\frac{4}{3}\frac{u^3}{s} - \frac{8}{3}\frac{u^4}{s^2} + \frac{4}{3}\frac{u^5}{s^2} \right) + ie\xi f^{\mu\nu}k_\nu k^\alpha \left(-2\frac{u^2}{s} + 2\frac{u^3}{s^2} \right)
$$

$$
+ ie^2(k \cdot n)f^{\mu\nu}k_\nu f^{\alpha\beta}k_\beta \left(-4u^3 + 12\frac{u^4}{s} - 12\frac{u^5}{s^2} + 4\frac{u^6}{s^3} \right)
$$

$$
+ ie^2\xi f^{\mu\nu}k_\nu f^{\alpha\beta}k_\beta \left(4u^2 - 8\frac{u^3}{s} + 4\frac{u^4}{s^2} \right)
$$

$$
+ ik^\mu n^\alpha(k \cdot n)^2 e^2 f^2 \left(-su^2 + 6u^3 - 13\frac{u^4}{s} + 12\frac{u^5}{s^2} - 4\frac{u^6}{s^3} \right)
$$

$$
+ ik^\mu n^\alpha \xi(k \cdot n)e^2 f^2 \left(\frac{2}{3}su - \frac{10}{3}u^2 + \frac{16}{3}\frac{u^3}{s} - \frac{8}{3}\frac{u^4}{s^2} \right)
$$

$$
+ iek^\mu n^\alpha(\sigma f) \left(-\frac{1}{4}s + u - \frac{u^2}{s} \right) + ef^{\mu\nu}k_\nu n^\alpha \left(-\frac{1}{3}s + \frac{1}{3}u - \frac{2}{3}\frac{u^3}{s^2} \right)
$$

$$
+ if^{\mu\nu}k_\nu n^\alpha(k \cdot n)^2 e^3 f^2 \left(\frac{22}{9}su^3 - \frac{110}{9}u^4 + 22\frac{u^5}{s} - \frac{154}{9}\frac{u^6}{s^2} + \frac{44}{9}\frac{u^7}{s^3} \right)
$$

$$
+ if^{\mu\nu}k_\nu n^\alpha \xi(k \cdot n)e^3 f^2 \left(-2su^2 + 8u^3 - 10\frac{u^4}{s} + 4\frac{u^5}{s^2} \right)
$$

$$
+ ie^2 f^{\mu\nu}k_\nu n^\alpha(\sigma f) \left(\frac{1}{2}su - \frac{3}{2}u^2 + \frac{u^3}{s} \right)
$$

$$
+ in^\mu k^\alpha(k \cdot n)^2 e^2 f^2 \left(\frac{4}{3}u^3 - \frac{16}{3}\frac{u^4}{s} + \frac{20}{3}\frac{u^5}{s^2} - \frac{8}{3}\frac{u^6}{s^3} \right)
$$

$$
+ in^\mu k^\alpha \xi(k \cdot n)e^2 f^2 \left(-\frac{4}{3}u^2 + 4\frac{u^3}{s} - \frac{8}{3}\frac{u^4}{s^2} \right)
$$

$$
+ en^\mu f^{\alpha\beta}k_\beta \left(\frac{1}{3}s - \frac{u^2}{s} + \frac{4}{3}\frac{u^3}{s^2} \right)
$$

$$
+ in^\mu f^{\alpha\beta}k_\beta(k \cdot n)^2 e^3 f^2 \left(-\frac{20}{9}su^3 + \frac{100}{9}u^4 - 20\frac{u^5}{s} + \frac{140}{9}\frac{u^6}{s^2} - \frac{40}{9}\frac{u^7}{s^3} \right)
$$

$$
+ in^\mu f^{\alpha\beta}k_\beta \xi(k \cdot n)e^3 f^2 \left(2su^2 - 8u^3 + 10\frac{u^4}{s} - 4\frac{u^5}{s^2} \right)
$$

$$
+ n^\mu n^\alpha(k \cdot n)e^2 f^2 \left(-\frac{4}{3}su + 8u^2 - \frac{40}{3}\frac{u^3}{s} + \frac{20}{3}\frac{u^4}{s^2} \right)
$$

$$
+ in^\mu n^\alpha(k \cdot n)^3 e^4 f^4 \left(\frac{10}{9}s^2 u^3 - \frac{70}{9}su^4 + \frac{190}{9}u^5 - \frac{250}{9}\frac{u^6}{s} + \frac{160}{9}\frac{u^7}{s^2} - \frac{40}{9}\frac{u^8}{s^3} \right)
$$

$$
+ n^\mu n^\alpha \xi e^2 f^2 \left(\frac{2}{3}s - 4u + 4\frac{u^2}{s} \right)
$$

$$
+ in^\mu n^\alpha \xi(k \cdot n)^2 e^4 f^4 \left(-\frac{8}{9}s^2 u^2 + \frac{16}{3}su^3 - \frac{104}{9}u^4 + \frac{32}{3}\frac{u^5}{s} - \frac{32}{9}\frac{u^6}{s^2} \right)
$$

$$
+ in^\mu n^\alpha(k \cdot n)(\sigma f)e^3 f^2 \left(\frac{1}{6}s^2 u - \frac{5}{6}su^2 + \frac{4}{3}u^3 - \frac{2}{3}\frac{u^4}{s} \right) \bigg] , \tag{A.42}
$$

$$\langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u)\exp\left[-ik_\delta\hat{x}^\delta(u)\right] - \hat{\Pi}^\alpha(u)\exp\left[-ik_\delta\hat{x}^\delta(u)\right]\hat{\Pi}^\mu(0)|x(0)\rangle$$

$$= \langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times \left[k^\mu k^\alpha \frac{u}{s} + ek^\mu f^{\alpha\beta}k_\beta\left(-u+\frac{u^2}{s}\right) + k^\mu n^\alpha(k\cdot n)e^2f^2\left(\frac{1}{3}su - u^2 + \frac{2}{3}\frac{u^3}{s}\right)\right]$$

$$+ i\Omega\langle x(s)|\exp\left[-ik_\delta\hat{x}^\delta(u)\right]|x(0)\rangle$$

$$\times \left[ef^{\mu\alpha}(k\cdot n)\left(-\frac{1}{3}s+u\right) + iek^\mu f^{\alpha\beta}k_\beta(k\cdot n)\left(-\frac{4}{3}u^2 + \frac{8}{3}\frac{u^3}{s} - \frac{4}{3}\frac{u^4}{s^2}\right)\right.$$

$$+ ie\xi k^\mu f^{\alpha\beta}k_\beta\left(u - \frac{u^2}{s}\right)$$

$$+ ik^\mu n^\alpha(k\cdot n)^2e^2f^2\left(su^2 - 4u^3 + 5\frac{u^4}{s} - 2\frac{u^5}{s^2}\right)$$

$$+ ik^\mu n^\alpha\xi(k\cdot n)e^2f^2\left(-\frac{2}{3}su + 2u^2 - \frac{4}{3}\frac{u^3}{s}\right)$$

$$+ ik^\mu n^\alpha e(\sigma f)\left(\frac{1}{4}s - \frac{1}{2}u\right) + ef^{\mu\nu}k_\nu n^\alpha\left(\frac{1}{3}s - u\right)$$

$$+ in^\mu k^\alpha(k\cdot n)^2e^2f^2\left(-\frac{2}{3}u^3 + \frac{4}{3}\frac{u^4}{s} - \frac{2}{3}\frac{u^5}{s^2}\right)$$

$$+ in^\mu k^\alpha e(\sigma f)\left(-\frac{1}{2}u\right) + en^\mu f^{\alpha\beta}k_\beta\left(-\frac{1}{3}s + \frac{u^2}{s}\right)$$

$$+ in^\mu f^{\alpha\beta}k_\beta(k\cdot n)^2e^3f^2\left(\frac{2}{3}su^3 - 2u^4 + 2\frac{u^5}{s} - \frac{2}{3}\frac{u^6}{s^2}\right)$$

$$+ in^\mu f^{\alpha\beta}k_\beta e^2(\sigma f)\left(\frac{1}{2}su - \frac{1}{2}u^2\right) + n^\mu n^\alpha(k\cdot n)e^2f^2\left(\frac{2}{3}su - 2u^2 + \frac{4}{3}\frac{u^3}{s}\right)$$

$$+ in^\mu n^\alpha(k\cdot n)^3e^4f^4\left(-\frac{2}{9}s^2u^3 + \frac{10}{9}su^4 - 2u^5 + \frac{14}{9}\frac{u^6}{s} - \frac{4}{9}\frac{u^7}{s^2}\right)$$

$$\left.+ in^\mu n^\alpha e^3f^2(\sigma f)(k\cdot n)\left(-\frac{1}{6}s^2u + \frac{1}{2}su^2 - \frac{1}{3}u^3\right)\right]. \tag{A.43}$$

Appendix B.   Permutations of operators

We give some technical details relevant for permutations of operators in this section. The basic commutation relations are those among $\hat{x}^\mu(s)$, $\hat{x}^\mu(u)$, and $\hat{x}^\mu(0)$. It is written as

$$\left[\hat{x}^\mu(0), \hat{x}^\alpha(s)\right]$$

$$= -2is\eta^{\mu\alpha} + 2ies^2f^{\mu\alpha} - \frac{4}{3}in^\mu n^\alpha e^2f^2s^3 + i\Omega es^2f^{\mu\alpha}\left(\frac{4}{3}\xi(s) + \frac{2}{3}\xi(0)\right)$$

$$- \frac{2}{3}i\Omega es^2n^\alpha f^\mu{}_\nu[\hat{x}^\nu(s) - \hat{x}^\nu(0)] + i\Omega n^\mu n^\alpha e^2f^2s^3\left(-\frac{4}{3}\xi(s) - \frac{4}{3}\xi(0)\right) \tag{B.1}$$

for $\hat{x}^\mu(s)$ and $\hat{x}^\alpha(0)$. Its derivation is as follows. The canonical commutation relation is written as

$$\left[\hat{x}^\mu(0), \hat{\Pi}^\nu(0)\right] = \left[\hat{x}^\mu(s), \hat{\Pi}^\nu(s)\right] = -i\eta^{\mu\nu}, \tag{B.2}$$

and the expression of $\hat{x}^{\mu}(0)$ in terms of $\hat{\Pi}^{\mu}(s)$ is obtained from Eq. (A.16) as

$$\hat{x}^{\mu}(0) = -2s\hat{\Pi}^{\mu}(s) + \hat{x}^{\mu}(s) + 2es^2 f^{\mu}{}_{\nu}\hat{\Pi}^{\nu}(s) + n^{\mu}e^2f^2s^2\left(-\frac{2}{3}\xi(s) + \frac{2}{3}\xi(0)\right)$$

$$+ \Omega es^2 f^{\mu}{}_{\nu}\hat{\Pi}^{\nu}(s)\left(\frac{4}{3}\xi(s) + \frac{2}{3}\xi(0)\right) + \Omega n^{\mu}e^2f^2s^2\left(-\frac{5}{6}\xi^2(s) + \frac{1}{3}\xi(s)\xi(0) + \frac{1}{2}\xi^2(0)\right)$$

$$+ \frac{1}{2}\Omega es^2 n^{\mu}(\sigma f). \tag{B.3}$$

Let us first consider the commutation relation $\left[\xi(0), \hat{x}^{\alpha}(s)\right]$. From Eq. (B.3), we obtain

$$\xi(0) = -2sn_{\mu}\hat{\Pi}^{\mu}(s) + \xi(s). \tag{B.4}$$

We then easily derive the following relation:

$$\left[\xi(0), \hat{x}^{\alpha}(s)\right] = \left[-2sn_{\mu}\hat{\Pi}^{\mu}(s), \hat{x}^{\alpha}(s)\right]$$

$$= -2sn_{\mu}i\eta^{\mu\alpha}$$

$$= -2isn^{\alpha}. \tag{B.5}$$

Combining Eqs. (B.2)–(B.5), we obtain Eq. (B.1) easily.

The following commutation relations, which are frequently used, also follow immediately:

$$\left[\hat{x}^{\alpha}(0), f^{\mu}{}_{\nu}\hat{x}^{\nu}(s)\right] = 2isf^{\alpha\mu} - 2in^{\alpha}n^{\mu}ef^2s^2 + i\Omega n^{\alpha}n^{\mu}ef^2s^2\left(-\frac{4}{3}\xi(s) - \frac{2}{3}\xi(0)\right), \tag{B.6}$$

$$\left[\hat{x}^{\alpha}(s), f^{\mu}{}_{\nu}\hat{x}^{\nu}(0)\right] = -2isf^{\alpha\mu} - 2in^{\alpha}n^{\mu}ef^2s^2 + i\Omega n^{\alpha}n^{\mu}ef^2s^2\left(-\frac{2}{3}\xi(s) - \frac{4}{3}\xi(0)\right), \tag{B.7}$$

$$\left[f^{\alpha}{}_{\beta}\hat{x}^{\beta}(s), f^{\mu}{}_{\nu}\hat{x}^{\nu}(0)\right] = -2in^{\alpha}n^{\mu}f^2s, \tag{B.8}$$

$$\left[\xi(s), \xi(0)\right] = 0, \tag{B.9}$$

$$\left[\hat{x}^{\mu}(0), \xi(s)\right] = -2isn^{\mu}, \tag{B.10}$$

$$\left[\xi(0), \hat{x}^{\mu}(s)\right] = -2isn^{\mu}, \tag{B.11}$$

$$\left[\hat{x}^{\mu}(s), \hat{\Pi}^{\nu}(s)\right] = \left[\hat{x}^{\mu}(0), \hat{\Pi}^{\nu}(0)\right] = \left[\hat{x}^{\mu}(u), \hat{\Pi}^{\nu}(u)\right] = -i\eta^{\mu\nu}. \tag{B.12}$$

The commutation relations between $\hat{x}^{\alpha}(u)$ and $\hat{x}^{\beta}(0)$ or $\hat{x}^{\beta}(s)$ are derived by Eqs. (A.18) and (B.1) as

$$\left[\hat{x}^{\alpha}(u), \hat{x}^{\beta}(0)\right]$$

$$= 2iu\eta^{\alpha\beta} + 2ieu^2 f^{\alpha\beta} + \frac{4}{3}in^{\alpha}n^{\beta}e^2f^2u^3 + i\Omega ef^{\alpha\beta}\left[\frac{2}{3}\frac{u^3}{s}\xi(s) + \left(2u^2 - \frac{2}{3}\frac{u^3}{s}\right)\xi(0)\right]$$

$$+ i\Omega en^{\beta}f^{\alpha}{}_{\nu}[\hat{x}^{\nu}(s) - \hat{x}^{\nu}(0)]\left(\frac{2}{3}\frac{u^3}{s}\right)$$

$$+ i\Omega n^{\alpha}n^{\beta}e^2f^2\left[\left(-\frac{2}{3}u^3 + 2\frac{u^4}{s}\right)\xi(s) + \left(\frac{10}{3}u^3 - 2\frac{u^4}{s}\right)\xi(0)\right], \tag{B.13}$$

$$\left[\hat{x}^{\alpha}(s), \hat{x}^{\beta}(u)\right]$$

$$= 2i(s-u)\eta^{\alpha\beta} + 2ie(s^2 - 2su + u^2)f^{\alpha\beta} + in^{\alpha}n^{\beta}e^2 f^2 \left(\frac{4}{3}s^3 - 4s^2 u + 4su^2 - \frac{4}{3}u^3\right)$$

$$+ i\Omega e f^{\alpha\beta}\left[\left(\frac{4}{3}s^2 - 2su + \frac{2}{3}\frac{u^3}{s}\right)\xi(s) + \left(\frac{2}{3}s^2 - 2su + 2u^2 - \frac{2}{3}\frac{u^3}{s}\right)\xi(0)\right]$$

$$+ i\Omega e n^{\alpha} f^{\beta}{}_{\nu}[\hat{x}^{\nu}(s) - \hat{x}^{\nu}(0)]\left(\frac{2}{3}s^2 - 2su + 2u^2 - \frac{2}{3}\frac{u^3}{s}\right)$$

$$+ i\Omega n^{\alpha} n^{\beta} e^2 f^2 \left[\left(\frac{4}{3}s^3 - 2s^2 u - 2su^2 + \frac{14}{3}u^3 - 2\frac{u^4}{s}\right)\xi(s)\right.$$

$$+ \left.\left(\frac{4}{3}s^3 - 6s^2 u + 10su^2 - \frac{22}{3}u^3 + 2\frac{u^4}{s}\right)\xi(0)\right]. \tag{B.14}$$

Appendix C.   *x*-dependence of transformation amplitudes

Here we discuss the *x*-dependence of the results. Note that the calculations of the amplitudes in Eqs. (A.8)–(A.15) are calculated for x^{μ} in the neighborhood of each point under the assumption that the wavelength of the external wave field is much longer than the Compton wavelength of the electron. Then *x* appears explicitly only in the form of $\xi = n_{\mu}x^{\mu}$ and it turns out in addition that ξ occurs only as a combination of $f(0)(1 + \Omega\xi)$. For example, the amplitude in Eq. (A.10) is written as

$$\langle x(s)| \exp\left[-ik_{\mu}\hat{x}^{\mu}(u)\right] |x(0)\rangle$$

$$\simeq \langle x(s)|x(0)\rangle \times \exp\left(-ik_{\mu}x^{\mu}\right) \exp\left[i(k)^2\left(u - \frac{u^2}{s}\right)\right]$$

$$\times \exp\left[i(k \cdot n)^2 e^2 f^2(0)(1 + \Omega\xi)^2 \left(-\frac{1}{3}su^2 + \frac{2}{3}u^3 - \frac{1}{3}\frac{u^4}{s}\right)\right]$$

$$\times \left\{1 + \Omega\left[i(k \cdot n)^3 e^2 f^2(0)\left(-\frac{2}{3}s^3 u + \frac{14}{9}s^2 u^2 - \frac{4}{9}su^3 - \frac{10}{9}u^4 + \frac{2}{3}\frac{u^5}{s}\right)\right.\right.$$

$$+ \left.\left. ie(\sigma f(0))(k \cdot n)\left(\frac{1}{2}su - \frac{1}{2}u^2\right)\right]\right\}, \tag{C.1}$$

and that in Eq. (A.11) is given as

$$\langle x(s)| \hat{\Pi}^{\alpha}(u) \exp\left[-ik_{\mu}\hat{x}^{\mu}(u)\right] |x(0)\rangle$$

$$\simeq \langle x(s)| \exp\left[-ik_{\mu}\hat{x}^{\mu}(u)\right] |x(0)\rangle$$

$$\times \left[\frac{u}{s}k^{\alpha} + ef^{\alpha\beta}(0)k_{\beta}(1 + \Omega\xi)\left(\frac{u^2}{s} - u\right) + n^{\alpha}(k \cdot n)e^2 f^2(0)(1 + \Omega\xi)^2 \left(\frac{1}{3}su - u^2 + \frac{2}{3}\frac{u^3}{s}\right)\right.$$

$$+ \Omega e f^{\alpha\beta}(0)k_{\beta}(k \cdot n)\left(\frac{4}{3}u^2 - \frac{8}{3}\frac{u^3}{s} + \frac{4}{3}\frac{u^4}{s^2}\right)$$

$$+ \Omega n^{\alpha}(k \cdot n)^2 e^2 f^2(0)\left(-su^2 + 4u^3 - 5\frac{u^4}{s} + 2\frac{u^5}{s^2}\right)$$

$$+ \left.\Omega e(\sigma f(0))n^{\alpha}\left(-\frac{1}{4}s + \frac{1}{2}u\right)\right]. \tag{C.2}$$
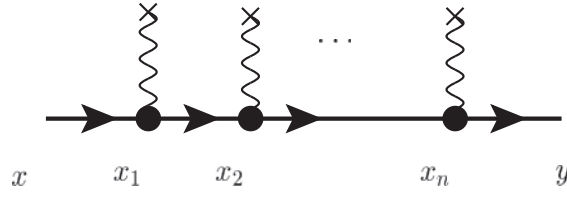
Fig. D.1. Electron propagator with external electromagnetic fields.

Note that the terms proportional to Ω in these equations are of higher order, and that $f(0)$ in these terms can be replaced with $f(0)(1 + \Omega\xi)$. Considering $f(0)(1 + \Omega\xi) \approx f(x)$ in the same approximation, we may conclude that all the explicit x-dependence can be included in the amplitude of the external field, and hence that the current term depends on the field strength f and its gradient Ω at each point. We can then assume that $x^\mu = 0$ at any points and the terms that contain ξ disappear in our results.

Appendix D.　Furry's theorem in the proper time method

It is well known as Furry's theorem in QED that all loop diagrams with an odd number of vertices vanish. The same reasoning applies to our theory, and we find that the terms in the induced electromagnetic current that include odd numbers of external electromagnetic fields should be dropped in our case. To understand this, we consider the charge conjugation of the electron propagator with the external electromagnetic fields.

The propagator with n external fields, $S_{n,A}$, is represented as

$$S_{n,A}(y - x) = S(y - x_1)[-e\gamma^\mu A_\mu(x_1)]S(x_1 - x_2) \cdots [-e\gamma^\mu A_\mu(x_n)]S(x_n - x), \quad \text{(D.1)}$$

where $S(y - x)$ is the electron free propagator. Because the charge conjugation of the free propagator is

$$
\begin{aligned}
S^c(y - x) &= \mathcal{C}S(y - x)\mathcal{C}^\dagger \\
&= CS^T(x - y)C^{-1},
\end{aligned}
\quad \text{(D.2)}
$$

where C is the matrix and $C = i\gamma^2\gamma^0$ for the Dirac representation, and the charge conjugation of the electromagnetic field A_μ is

$$A_\mu^c = \mathcal{C}A_\mu\mathcal{C}^\dagger = -A_\mu, \quad \text{(D.3)}$$

the charge conjugation of $S_{n,A}(y - x)$ is

$$
\begin{aligned}
S_{n,A}^c(y - x) &= \mathcal{C}S_{n,A}(y - x)\mathcal{C}^\dagger \\
&= \mathcal{C}S(y - x_1)\mathcal{C}^\dagger\mathcal{C}[-e\gamma^\mu A_\mu(x_1)]\mathcal{C}^\dagger \cdots \mathcal{C}^\dagger\mathcal{C}S(x_n - x)\mathcal{C}^\dagger \\
&= CS^T(x_1 - y)C^{-1}\left\{-e\gamma^\mu[-A_\mu(x_1)]\right\}C \cdots CS^T(x - x_n)\mathcal{C}^\dagger C^{-1} \\
&= CS^T(x_1 - y)(-e)(-\gamma^{\mu T})[-A_\mu(x_1)] \cdots S^T(x - x_n)C^{-1} \\
&= C\left\{S(x - x_n)[-e\gamma^\mu A_\mu(x_1)] \cdots S(x_1 - y)\right\}^T C^{-1} \\
&= CS_A^T(x - y)C^{-1}.
\end{aligned}
\quad \text{(D.4)}
$$

Another expression of charge conjugation is

$$
\begin{aligned}
S^c_{n,A}(y - x) &= \mathcal{C} S_{n,A}(y - x) \mathcal{C}^\dagger \\
&= \mathcal{C} S(y - x_1) \mathcal{C}^\dagger \mathcal{C}(-e\gamma^\mu A_\mu(x_1)) \cdots \mathcal{C}^\dagger \mathcal{C} S(x_n - x) \mathcal{C}^\dagger \\
&= S(y - x_1)(-e\gamma^\mu)(-A_\mu(x_1)) \cdots S(x_n - x) \\
&= S_{n,-A}(y - x),
\end{aligned}
\tag{D.5}
$$

because of the property of the electron propagator that it does not change by the charge conjugation $\mathcal{C} S \mathcal{C}^\dagger = S$. From Eq. (D.5), we conclude that the sign of the propagator changes when the propagator contains odd numbers of electromagnetic fields,

$$
S^c_{\text{odd},A}(y - x) = -S_{\text{odd},A}(y - x),
\tag{D.6}
$$

and that the sign of the propagator does not change when the propagator contains even numbers of electromagnetic fields,

$$
S^c_{\text{even},A}(y - x) = S_{\text{even},A}(y - x).
\tag{D.7}
$$

As shown in Eq. (2), the induced electromagnetic current is represented by the propagator

$$
j^\mu(x) = \frac{\partial \mathcal{L}[A, a](x)}{\partial a_\mu} = ie\text{tr}\left[\gamma^\mu G(x, x)\right].
\tag{D.8}
$$

There are two ways to obtain the charge conjugation of the propagator. One is to extract the matrix C:

$$
\begin{aligned}
j^{c\mu}_A(x) &= ie\text{tr}\left[\gamma^\mu G^c_A(x, x)\right] = ie\text{tr}\left[\gamma^\mu C G^T_A(x, x) C^{-1}\right] \\
&= ie\text{tr}\left[-\gamma^{\mu T} G^T_A(x, x)\right] = -ie\text{tr}\left[\gamma^{\mu T} G^T_A(x, x)\right] \\
&= -ie\text{tr}\left[\gamma^\mu G_A(x, x)\right] = -j^\mu_A(x).
\end{aligned}
\tag{D.9}
$$

The other is to change the sign of the electromagnetic field:

$$
\begin{aligned}
j^{c\mu}_A(x) &= ie\text{tr}\left[\gamma^\mu G^c_A(x, x)\right] = ie\text{tr}\left[\gamma^\mu G_{-A}(x, x)\right] \\
&= j^\mu_{-A}(x).
\end{aligned}
\tag{D.10}
$$

Comparing these two expression, we obtain

$$
-j^\mu_A(x) = j^\mu_{-A}(x).
\tag{D.11}
$$

Thus, the induced electromagnetic current should contain only those terms with odd numbers of external electromagnetic fields. Since it is represented as $\langle j_\mu \rangle = \Pi_\mu{}^\nu b_\nu$ with the probe photon b_ν and the polarization tensor $\Pi_\mu{}^\nu$, the number of external fields in $\Pi_\mu{}^\nu$ should be even.

Appendix E.   Expression for the induced electromagnetic current

The induced electromagnetic current $\langle j^\mu \rangle$ as given in Eq. (39) is given as follows:

$$
\langle j^\mu \rangle \simeq \frac{e}{2} \int_0^\infty ds \int_0^s du\, e^{-im^2 s}\left[A^\mu + B^\mu + C^\mu + D^\mu + E^\mu + F^\mu + G^\mu + H^\mu\right],
\tag{E.1}
$$

in which the terms $A^\mu, B^\mu, \ldots, H^\mu$ are expressed as follows:

$$A^\mu = \mathrm{tr}\left[ieb_\alpha k^\alpha \langle x(s)|\hat{\Pi}^\mu(s) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] + \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}^\mu(0)|x(0)\rangle\right], \tag{E.2}$$

$$B^\mu = \mathrm{tr}\left[\left(-\frac{ie}{2}\right) \langle x(s)|\hat{\Pi}^\mu(s) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] + \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}^\mu(0)|x(0)\rangle(\sigma g)\right], \tag{E.3}$$

$$C^\mu = \mathrm{tr}\left[(-2ie)b_\alpha \langle x(s)|\hat{\Pi}^\mu(s)\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u)\right]\right.$$
$$\left. + \hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}^\mu(0)|x(0)\rangle\right], \tag{E.4}$$

$$D^\mu = \mathrm{tr}\left[(-2eb_\alpha)\sigma^{\mu\nu}\langle x(s)|\hat{\Pi}_\nu(s)\hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u)\right]\right.$$
$$\left. - \hat{\Pi}^\alpha(u) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}_\nu(0)|x(0)\rangle\right], \tag{E.5}$$

$$E^\mu = \mathrm{tr}\left[\left(-\frac{e}{2}\right) \sigma^{\mu\nu}\langle x(s)|\hat{\Pi}_\nu(s) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] - \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}_\nu(0)|x(0)\rangle\right.$$
$$\left. \times \left\{(\sigma g) + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f)\right\}\right], \tag{E.6}$$

$$F^\mu = \mathrm{tr}\left[\left(-\frac{ie^2 u}{4}\right) \sigma^{\mu\nu}\langle x(s)|\hat{\Pi}_\nu(s) \exp\left[-ik_\delta \hat{x}^\delta(u)\right] - \exp\left[-ik_\delta \hat{x}^\delta(u)\right] \hat{\Pi}_\nu(0)|x(0)\rangle\right.$$
$$\left. \times \{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}\right], \tag{E.7}$$

$$G^\mu = \mathrm{tr}\left[\left(\frac{\Omega e^2 u}{4}\right) \sigma^{\mu\nu}n_\nu \langle x(s)| \exp\left[-ik_\delta \hat{x}^\delta(u)\right] |x(0)\rangle \{(\sigma f)(\sigma g) - (\sigma g)(\sigma f)\}\right], \tag{E.8}$$

$$H^\mu = \mathrm{tr}\left[\left(-\frac{i\Omega e^3 u^2}{4}\right) \sigma^{\mu\nu}n_\nu \langle x(s)| \exp\left[-ik_\delta \hat{x}^\delta(u)\right] |x(0)\rangle(\sigma f)(\sigma g)(\sigma f)\right]. \tag{E.9}$$

They are further decomposed: e.g., A^μ is written as the sum of A_i^μ as $A^\mu = \sum_{i=1}^{5} A_i^\mu$. The same notation is used for $B^\mu, C^\mu, \ldots, F^\mu$. All these components are explicitly written as follows:

$$A_1^\mu = K(u)ie(b \cdot k)k^\mu \left(2\frac{u}{s} - 1\right) - c.t., \tag{E.10}$$

$$A_2^\mu = K(u)ie^3 f^2(b \cdot k)(k \cdot n)n^\mu \left(\frac{2}{3}su - 2u^2 + \frac{4}{3}\frac{u^3}{s}\right), \tag{E.11}$$

$$A_3^\mu = K(u)i\Omega e^3 f^2(b \cdot k)(k \cdot n)^2 n^\mu \left(-\frac{4}{3}su^2 + \frac{16}{3}u^3 - \frac{20}{3}\frac{u^4}{s} + \frac{8}{3}\frac{u^5}{s^2}\right), \tag{E.12}$$

$$A_4^\mu = K(u)\Omega e^3 f^2(b \cdot k)(k \cdot n)^3 k^\mu \left(-\frac{2}{3}s^3 u + \frac{26}{9}s^2 u^2 - \frac{32}{9}su^3 - \frac{2}{9}u^4 + \frac{26}{9}\frac{u^5}{s} - \frac{4}{3}\frac{u^6}{s^2}\right), \tag{E.13}$$

$$A_5^\mu = K(u)\Omega e^5 f^4(b \cdot k)(k \cdot n)^4 n^\mu$$
$$\left(\frac{4}{9}s^4 u^2 - \frac{64}{27}s^3 u^3 + \frac{116}{27}s^2 u^4 - \frac{20}{9}su^5 - \frac{56}{27}u^6 + \frac{76}{27}\frac{u^7}{s} - \frac{8}{9}\frac{u^8}{s^2}\right), \tag{E.14}$$

$$B_1^\mu = \mathrm{tr}\left[(\sigma f)(\sigma g)\right] L(u)e^3 f^{\mu\nu}k_\nu \left(-\frac{su}{2} + \frac{u^2}{2}\right), \tag{E.15}$$

$$B_2^\mu = \mathrm{tr}\left[(\sigma f)(\sigma g)\right] L(u)i\Omega e^5 f^2(k \cdot n)^3 f^{\mu\nu}k_\nu$$

$$\left(\frac{1}{3}s^4u^2 - \frac{10}{9}s^3u^3 + s^2u^4 + \frac{1}{3}su^5 - \frac{8}{9}u^6 + \frac{1}{3}\frac{u^7}{s} \right), \tag{E.16}$$

$$B_3^\mu = \text{tr}\left[(\sigma f)(\sigma g) \right] L(u)\Omega e^3 (k\cdot n)f^{\mu\nu}k_\nu \left(\frac{1}{2}su^2 - u^3 + \frac{1}{2}\frac{u^4}{s} \right), \tag{E.17}$$

$$B_4^\mu = \text{tr}\left[(\sigma f)(\sigma g) \right] L(u)\Omega e^3 (k\cdot n)f^{\mu\nu}k_\nu \left(\frac{1}{3}su^2 - \frac{2}{3}u^3 + \frac{1}{3}\frac{u^4}{s} \right), \tag{E.18}$$

$$C_1^\mu = K(u)ie(b\cdot k)k^\mu \left(2\frac{u}{s} - 4\frac{u^2}{s^2} \right) - c.t., \tag{E.19}$$

$$C_2^\mu = K(u)ie^3(bfk)f^{\mu\nu}k_\nu \left(4u^2 - 8\frac{u^3}{s} + 4\frac{u^4}{s^2} \right), \tag{E.20}$$

$$C_3^\mu = K(u)ie^3f^2(b\cdot n)(k\cdot n)k^\mu \left(\frac{2}{3}su - \frac{10}{3}u^2 + \frac{16}{3}\frac{u^3}{s} - \frac{8}{3}\frac{u^4}{s^2} \right), \tag{E.21}$$

$$C_4^\mu = K(u)ie^3f^2(b\cdot k)(k\cdot n)n^\mu \left(-\frac{4}{3}u^2 + 4\frac{u^3}{s} - \frac{8}{3}\frac{u^4}{s^2} \right), \tag{E.22}$$

$$C_5^\mu = K(u)e^3f^2(b\cdot n)n^\mu \left(\frac{2}{3}s - 4u + 4\frac{u^2}{s} \right), \tag{E.23}$$

$$C_6^\mu = K(u)ie^5f^4(b\cdot n)(k\cdot n)^2 n^\mu \left(-\frac{4}{9}s^2u^2 + \frac{8}{3}su^3 - \frac{52}{9}u^4 + \frac{16}{3}\frac{u^5}{s} - \frac{16}{9}\frac{u^6}{s^2} \right), \tag{E.24}$$

$$C_7^\mu = K(u)eb^\mu \frac{2}{s} - c.t., \tag{E.25}$$

$$C_8^\mu = K(u)\Omega e^3f^2(b\cdot k)(k\cdot n)^3 k^\mu \left(\frac{4}{3}s^2u^2 - \frac{52}{9}su^3 + \frac{64}{9}u^4 + \frac{4}{9}\frac{u^5}{s} - \frac{52}{9}\frac{u^6}{s^2} + \frac{8}{3}\frac{u^7}{s^3} \right), \tag{E.26}$$

$$C_9^\mu = K(u)\Omega e^5f^2(bfk)(k\cdot n)^3 f^{\mu\nu}k_\nu$$
$$\times \left(\frac{8}{3}s^3u^3 - \frac{104}{9}s^2u^4 + \frac{152}{9}su^5 - \frac{16}{3}u^6 - \frac{88}{9}\frac{u^7}{s} + \frac{88}{9}\frac{u^8}{s^2} - \frac{8}{3}\frac{u^9}{s^3} \right), \tag{E.27}$$

$$C_{10}^\mu = K(u)\Omega e^5f^4(b\cdot n)(k\cdot n)^4 k^\mu$$
$$\times \left(\frac{4}{9}s^4u^2 - \frac{88}{27}s^3u^3 + \frac{244}{27}s^2u^4 - \frac{292}{27}su^5 + \frac{64}{27}u^6 + \frac{188}{27}\frac{u^7}{s} - \frac{176}{27}\frac{u^8}{s^2} + \frac{16}{9}\frac{u^9}{s^3} \right), \tag{E.28}$$

$$C_{11}^\mu = K(u)\Omega e^5f^4(b\cdot k)(k\cdot n)^4 n^\mu$$
$$\times \left(-\frac{8}{9}s^3u^3 + \frac{128}{27}s^2u^4 - \frac{232}{27}su^5 + \frac{40}{9}u^6 + \frac{112}{27}\frac{u^7}{s} - \frac{152}{27}\frac{u^8}{s^2} + \frac{16}{9}\frac{u^9}{s^3} \right), \tag{E.29}$$

$$C_{12}^\mu = K(u)i\Omega e^5f^4(b\cdot n)(k\cdot n)^3 n^\mu$$
$$\times \left(-\frac{4}{9}s^4u + \frac{100}{27}s^3u^2 - \frac{248}{27}s^2u^3 + \frac{196}{27}su^4 + \frac{28}{9}u^5 - \frac{64}{9}\frac{u^6}{s} + \frac{8}{3}\frac{u^7}{s^2} \right), \tag{E.30}$$

$$C_{13}^\mu = K(u)\Omega e^7f^6(b\cdot n)(k\cdot n)^5 n^\mu \left(-\frac{8}{27}s^5u^3 + \frac{200}{81}s^4u^4 - \frac{664}{81}s^3u^5 \right.$$
$$\left. +\frac{1072}{81}s^2u^6 - \frac{712}{81}su^7 - \frac{248}{81}u^8 + \frac{728}{81}\frac{u^9}{s} - \frac{448}{81}\frac{u^{10}}{s^2} + \frac{32}{27}\frac{u^{11}}{s^3} \right), \tag{E.31}$$

$$C_{14}^{\mu} = K(u)i\Omega e^3 f^2 (k \cdot n)^3 b^{\mu} \left(-\frac{4}{3}s^2 u + \frac{28}{9}su^2 - \frac{8}{9}u^3 - \frac{20}{9}\frac{u^4}{s} + \frac{4}{3}\frac{u^5}{s^2} \right), \tag{E.32}$$

$$C_{15}^{\mu} = K(u)i\Omega e^3 (bfk)(k \cdot n)f^{\mu\nu}k_{\nu} \left(-8u^3 + 24\frac{u^4}{s} - 24\frac{u^5}{s^2} + 8\frac{u^6}{s^3} \right), \tag{E.33}$$

$$C_{16}^{\mu} = K(u)i\Omega e^3 f^2 (b \cdot n)(k \cdot n)^2 k^{\mu} \left(-2su^2 + 12u^3 - 26\frac{u^4}{s} + 24\frac{u^5}{s^2} - 8\frac{u^6}{s^3} \right), \tag{E.34}$$

$$C_{17}^{\mu} = K(u)i\Omega e^3 f^2 (b \cdot k)(k \cdot n)^2 n^{\mu} \left(\frac{8}{3}u^3 - \frac{32}{3}\frac{u^4}{s} + \frac{40}{3}\frac{u^5}{s^2} - \frac{16}{3}\frac{u^6}{s^3} \right), \tag{E.35}$$

$$C_{18}^{\mu} = K(u)\Omega e^3 f^2 (b \cdot n)(k \cdot n)n^{\mu} \left(-\frac{8}{3}su + 16u^2 - \frac{80}{3}\frac{u^3}{s} + \frac{40}{3}\frac{u^4}{s^2} \right), \tag{E.36}$$

$$C_{19}^{\mu} = K(u)i\Omega e^5 f^4 (b \cdot n)(k \cdot n)^3 n^{\mu}$$
$$\left(\frac{20}{9}s^2 u^3 - \frac{140}{9}su^4 + \frac{380}{9}u^5 - \frac{500}{9}\frac{u^6}{s} + \frac{320}{9}\frac{u^7}{s^2} - \frac{80}{9}\frac{u^8}{s^3} \right), \tag{E.37}$$

$$D_1^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)ie^3 (bfk)k_{\nu}(-su + u^2), \tag{E.38}$$

$$D_2^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)\Omega e^5 f^2 (bfk)(k \cdot n)^3 k_{\nu}$$
$$\times \left(-\frac{2}{3}s^4 u^2 + \frac{20}{9}s^3 u^3 - 2s^2 u^4 - \frac{2}{3}su^5 + \frac{16}{9}u^6 - \frac{2}{3}\frac{u^7}{s} \right), \tag{E.39}$$

$$D_3^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)\Omega e^3 (k \cdot n)f_{\nu}^{\ \alpha}b_{\alpha} \left(\frac{1}{3}s^2 - su \right), \tag{E.40}$$

$$D_4^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)i\Omega e^3 (bfk)(k \cdot n)k_{\nu} \left(\frac{4}{3}su^2 - \frac{8}{3}u^3 + \frac{4}{3}\frac{u^4}{s} \right), \tag{E.41}$$

$$D_5^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)\Omega e^3 (b \cdot n)f_{\nu}^{\ \lambda}k_{\lambda} \left(-\frac{1}{3}s^2 + su \right), \tag{E.42}$$

$$D_6^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)\Omega e^3 (bfk)n_{\nu} \left(\frac{1}{3}s^2 - u^2 \right), \tag{E.43}$$

$$D_7^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)i\Omega e^5 f^2 (bfk)(k \cdot n)^2 n_{\nu} \left(-\frac{2}{3}s^2 u^3 + 2su^4 - 2u^5 + \frac{2}{3}\frac{u^6}{s} \right), \tag{E.44}$$

$$D_8^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)i\Omega e^3 (bfk)(k \cdot n)k_{\nu} \left(su^2 - 2u^3 + \frac{u^4}{s} \right), \tag{E.45}$$

$$D_9^{\mu} = \text{tr}\left[\sigma^{\mu\nu}(\sigma f) \right] L(u)\Omega e^3 (bfk)n_{\nu}(su - u^2), \tag{E.46}$$

$$E_1^{\mu} = \text{tr}\left[\sigma^{\mu\nu} \left((\sigma g) + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f) \right) \right] L(u) \left(-\frac{e}{2}k_{\nu} \right) - c.t., \tag{E.47}$$

$$E_2^{\mu} = \text{tr}\left[\sigma^{\mu\nu} \left((\sigma g) + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f) \right) \right]$$
$$L(u)\Omega e^3 f^2 (k \cdot n)^2 n_{\nu} \left(-\frac{1}{3}su^2 + \frac{2}{3}u^3 - \frac{1}{3}\frac{u^4}{s} \right), \tag{E.48}$$

$$E_3^{\mu} = \text{tr}\left[\sigma^{\mu\nu} \left((\sigma g) + \frac{e^2 u^2}{4}(\sigma f)(\sigma g)(\sigma f) \right) \right] L(u)i\Omega e^3 f^2 (k \cdot n)^3 k_{\nu}$$

$$\times \left(\frac{1}{3} s^3 u - \frac{7}{9} s^2 u^2 + \frac{2}{9} s u^3 + \frac{5}{9} u^4 - \frac{1}{3} \frac{u^5}{s} \right), \tag{E.49}$$

$$F_1^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)e^3 k_\nu \frac{su}{8}, \tag{E.50}$$

$$F_2^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)i\Omega e^5 f^2 (k \cdot n)^3 n_\nu$$
$$\times \left(-\frac{1}{12} s^4 u^2 + \frac{7}{36} s^3 u^3 - \frac{1}{18} s^2 u^4 - \frac{5}{36} s u^5 + \frac{1}{12} u^6 \right), \tag{E.51}$$

$$F_3^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)\Omega e^5 f^2 (k \cdot n)^2 k_\nu \left(\frac{1}{12} s^2 u^3 - \frac{1}{6} s u^4 + \frac{1}{12} u^5 \right), \tag{E.52}$$

$$F_4^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)\Omega e^3 (k \cdot n) k_\nu \left(-\frac{1}{8} s u^2 + \frac{1}{8} u^3 \right), \tag{E.53}$$

$$F_5^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)i\Omega e^3 n_\nu \frac{su}{8}, \tag{E.54}$$

$$G^\mu = F_5^\mu, \tag{E.55}$$

$$H^\mu = \mathrm{tr}[\sigma^{\mu\nu}(\sigma f)(\sigma g)(\sigma f)]L(u)\left(-i\Omega e^3 n_\nu \frac{u^2}{4} \right), \tag{E.56}$$

where we have employed the following abbreviations: $k \cdot n = k_\mu n^\mu$ and $bfk = b_\alpha f^{\alpha\beta} k_\beta$. In the above equations, $K(u)$ and $L(u)$ are defined as

$$K(u) = \frac{1}{4i\pi^2 s^2} \exp(-ikx) \exp\left\{ i(k)^2 \left(u - \frac{u^2}{s} \right) + i(k \cdot n)^2 e^2 f^2 \left(-\frac{1}{3} s u^2 + \frac{2}{3} u^3 - \frac{1}{3} \frac{u^4}{s} \right) \right\}, \tag{E.57}$$

$$L(u) = K(u)/4, \tag{E.58}$$

where $\exp(-ikx) = \exp(-ik_\mu x^\mu)$. The counter terms that originate from renormalization are denoted by $c.t.$ in some equations. For the crossed field, i.e., the long-wavelength limit ($\Omega \to 0$) of the external plane wave, the above expression is reduced to

$$\langle j^\mu \rangle |_{\Omega=0} \simeq \frac{e}{2} \int_0^\infty ds \int_0^s du \, e^{-im^2 s} \left[\sum_{i=1}^2 A_i^\mu + B_1^\mu + \sum_{i=1}^7 C_i^\mu + D_1^\mu + E_1^\mu + F_1^\mu \right]. \tag{E.59}$$

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