Assignment-1

Unless otherwise specified, R will denote a commutative ring with 1.

(1) Given families $\{M_i\}_{i\in I}$, $\{N_j\}_{j\in J}$ of R-modules, prove that there is a $\mathbb{Z}$-linear isomorphism:
$$\big(\oplus_{i\in I} M_i\big) \otimes \big(\oplus_{j\in J} N_{j\in J}\big) \cong \oplus_{i,j}\big(M_i \otimes N_j\big)$$

(2) Let M and N be finite free R-modules. Then prove that
$$M^* \otimes N \cong \mathrm{End}_R(M, N)$$

(3) If the R-modules M and N are not both finite free, is $M^* \otimes N$ isomorphic to $\mathrm{End}_R(M, N)$?

(4) Prove that tensor product commutes with the direct limits in the category of R-modules.

Assignment-2

(1) Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of sheaves on X. If $\mathcal{E}$ and $\mathcal{F}$ are flasque, then prove that $\mathcal{G}$ is flasque.

(2) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of sheaves X, and if $\mathcal{E}$ is flasque, then prove that
$$0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to 0$$
is exact.

(3) Prove that a scheme X is affine if and only if there exists a finite subset $\{f_1, \dots, f_r\}$ of $\Gamma(X, \mathcal{O}_X)$ such that the open sets X_{f_i} are affine, and $f_1, \dots, f_r$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$.

(4) Let X be a affine noetherian scheme. Show that
$$H^i(X, \mathcal{F}) = 0,$$
for all quasi-coherent sheaves $\mathcal{F}$ on X, and for all $i > 0$.

Assignment-3

(1) Prove that the cohomology groups commutes with the arbitrary direct sum.

(2) Let $\mathcal{O}_{\mathbb{P}^1_k}$ be the structure sheaf of $\mathbb{P}^1_k := \mathrm{Proj}(k[X_0, X_1])$ over a field k. Compute $H^i(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k})$.

(3) Let $\Omega_{\mathbb{P}^1_k}$ be the differential sheaf on $\mathbb{P}^1_k := \mathrm{Proj}(k[X_0, X_1])$ over a field k. Compute $H^i(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k})$.

(4) Let A be a ring, and $n \geq 1$. Prove that the set $\{X_1^{i_1}, \ldots, X_n^{i_n} | i_1, \ldots, i_n \in \mathbb{Z}\}$ forms a basis of $A[X_1, \ldots, X_n]_{X_1 \cdots X_n}$ as A-module. In particular, $A[X_1, \ldots, X_n]_{X_1 \cdots X_n}$ is a free A-module.

(5) Let X be a projective scheme of dimension n over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. The Euler characteristic of $\mathcal{F}$ is defined as

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathcal{F}).$$

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, prove that

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'').$$