Let $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$. Define the following operations on $\mathbb{R}^2$:

(1) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, (called the pintwise addition)
(2) $\alpha \cdot (x, y) = (\alpha x, \alpha y)$, for all $\alpha \in \mathbb{R}, (x, y) \in \mathbb{R}^2$, (called the scalar multiplication)

The triple $(\mathbb{R}^2, +, \cdot)$ forms a nice algebraic structure, which is called a vector space over $\mathbb{R}$.

Any non empty subset $W \subset \mathbb{R}^2$ which is closed under the addition and scalar multiplication on $\mathbb{R}^2$, is called a subspace of $\mathbb{R}^2$

Example: $W = \{(x, y) \in \mathbb{R}^2|x = y\}$.

Informally, a "linear transformation" preserves algebraic operations. Thus sum of two vectors is mapped to the sum of their images and the scalar multiple of a vector is mapped to the same scalar multiple of its image.

A linear transformation (or linear map) T from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a map which satisfies

(1) $T(0) = 0$
(2) $T(v_1 + v_2) = T(v_1) + T(v_2)$
(3) $T(\alpha v) = \alpha v$

Geometrically, a linear map sends lines passing through the origin to line passing through the origin or onto the origin.

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, y)$.

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (2x, y)$, Stretching along e_1 (x direction).

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, 0)$, Stretching along e_1 (x direction).

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (2x, y)$, projection

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (0, x)$, projection + rotation

A non empty set V is said to be a vector space over $\mathbb{R}$ (o real vector space) if there exist maps $+ : V \times V \to V$, $(x, y) \mapsto x + y$, and $\cdot : \mathbb{R} \times V \to V$, $(\alpha, x) \mapsto \alpha \cdot x$, called scalar multiplication, satisfying the following properties:

(1) $x + y = y + x$
(2) $(x + y) + z = x + (y + z)$
(3) There exists $0 \in V$ such that $x + 0 = x = 0 + x$.
(4) For every $x \in V$, there exists $y \in V$ such that $x + y = 0 = y + x$.
(5) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.
(6) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
(7) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
(8) $1 \cdot x = x$.