6

NULL DOUBLE INJECTION (NDI)

AND THE

EXTRA ELEMENT THEOREM (EET)

Null Double Injection

Consider a system with a single injected (input) signal u_{i1}, and two dependent (output) signals u_{o1}, u_{o2}:

Consider a second injected signal u_{i2} which alone would give:

If the system is linear, in general each output is a linear sum of the outputs due to each input:

$$\circ\!\xrightarrow{\;u_{i1}\;} \qquad\qquad \xrightarrow{\;u_{o1} = A_1 u_{i1} + A_2 u_{i2}\;}\!\circ$$

$$\circ\!\xrightarrow{\;u_{i2}\;} \qquad\qquad \xrightarrow{\;u_{o2} = B_1 u_{i1} + B_2 u_{i2}\;}\!\circ$$

Note definition of "gains:"

$$\left.\frac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0} = A_1 \qquad \left.\frac{u_{o1}}{u_{i2}}\right|_{u_{i1}=0} = A_2$$

$$\left.\frac{u_{o2}}{u_{i1}}\right|_{u_{i2}=0} = B_1 \qquad \left.\frac{u_{o2}}{u_{i2}}\right|_{u_{i1}=0} = B_2$$

By adjustment of u_{i2} relative to u_{i1} (in magnitude and phase), either output can be nulled:

For example, if u_{o1} is nulled:

$$0 = A_1 u_{i1} + A_2 u_{i2}\Big|_{u_{o1}=0}$$

$$u_{o2} = B_1 u_{i1} + B_2 u_{i2}\Big|_{u_{o1}=0}$$

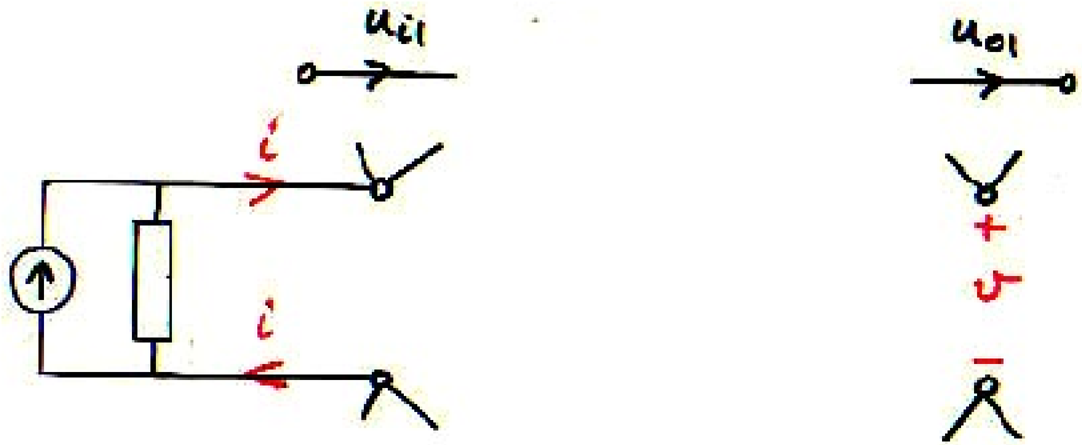
By elimination of u_{i1}:

$$\left.\frac{u_{o2}}{u_{i2}}\right|_{u_{o1}=0} = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Notice the difference from

$$\left.\frac{u_{o2}}{u_{i2}}\right|_{u_{i1}=0} = B_2$$

Particular example: $u_{i2} = i$, $u_{o2} = v$:



$$u_{o1} = A_1 u_{i1} + A_2 i$$

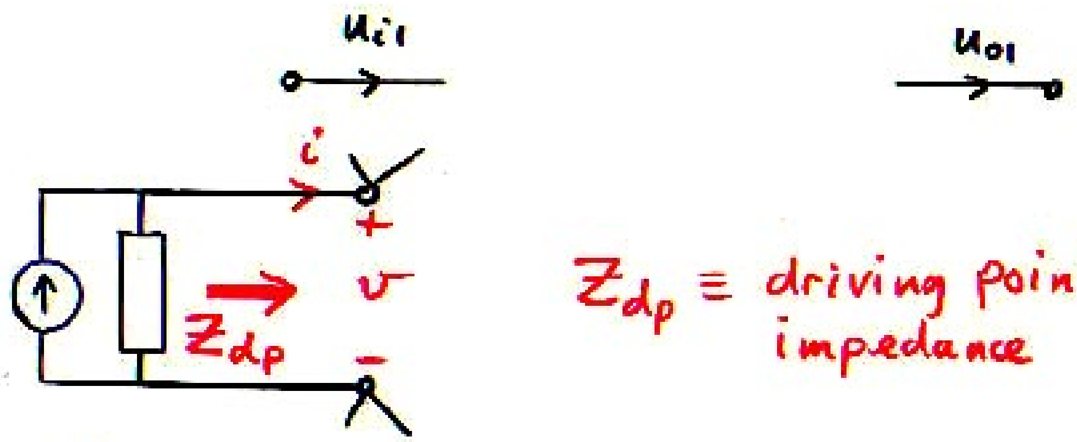
$$v = B_1 u_{i1} + B_2 i$$

So $$\left.\frac{v}{i}\right|_{u_{i1}=0} = B_2$$

$$\left.\frac{v}{i}\right|_{u_{o1}=0} = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Special case: the Extra Element Theorem

The second output v is across the second input terminals:



$Z_{dp} \equiv$ driving point impedance

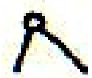
Then

$$\left.\frac{v}{i}\right|_{u_{i1}=0} = \left.Z_{dp}\right|_{u_{i1}=0} \equiv Z_d = B_2$$

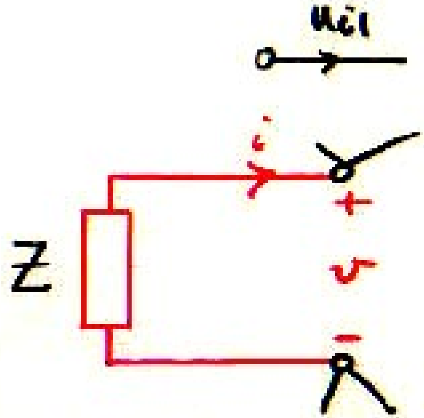
$$\left.\frac{v}{i}\right|_{u_{o1}=0} = \left.Z_{dp}\right|_{u_{o1}=0} \equiv Z_n = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Consider the original circuit without $u_{i2} = i$:

$$\text{Original gain} = A_1 = \left.\frac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0}$$



Consider the original circuit without $u_{i2} = i$:



Original gain $= A_1 = \left.\dfrac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0}$

Now consider the circuit with an <u>extra element</u> Z:

$$i = -\frac{v}{Z}$$

So

$$u_{o1} = A_1 u_{i1} - \frac{A_2}{Z} v$$

$$v = B_1 u_{i1} - \frac{B_2}{Z} v$$

Eliminate v:

$$u_{o1} = A_1 \frac{1 + \dfrac{1}{Z}\dfrac{A_1 B_2 - A_2 B_1}{A_1}}{1 + \dfrac{1}{Z} B_2} u_{i1}$$

Hence

$$u_{o1} = A_1 \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} u_{i1}$$

or

$$\text{gain}\Big|_Z = \text{gain}\Big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}}$$

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

Hence

$$u_{o1} = A_1 \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} u_{i1}$$

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This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

The Theorem also proves that any transfer function (e.g. gain) of a linear system is a bilinear function of any single element (e.g. Z).

forming Δ_{11} the terms by which z is multiplied must be the minor Δ_{11jj} obtained by omitting both the first and jth rows and columns. If we let Δ^0 and Δ_{11}^0 represent, respectively, Δ and Δ_{11} when $z = 0$, therefore, we have

$$Z = \frac{\Delta^0 + z\Delta_{jj}}{\Delta_{11}^0 + z\Delta_{11jj}}. \tag{1-11}$$

Since Δ_{jj} and Δ_{11jj} are evidently independent of z they can equally well be written as Δ_{jj}^0 and Δ_{11jj}^0. This will occasionally be done in later analysis in order to facilitate further transformations.

The relation between Z_T and z can be found in similar fashion. It is given by

$$Z_T = \frac{\Delta^0 + z\Delta_{jj}}{\Delta_{12}^0 + z\Delta_{12jj}}. \tag{1-12}$$

If z represents a unilateral coupling term, instead of a bilateral element, the expansion is essentially the same. Thus, if we suppose that z is a part of Z_{ij} in the original determinant, we readily find

$$Z = \frac{\Delta^0 + z\Delta_{ij}}{\Delta_{11}^0 + z\Delta_{11ij}} \tag{1-13}$$

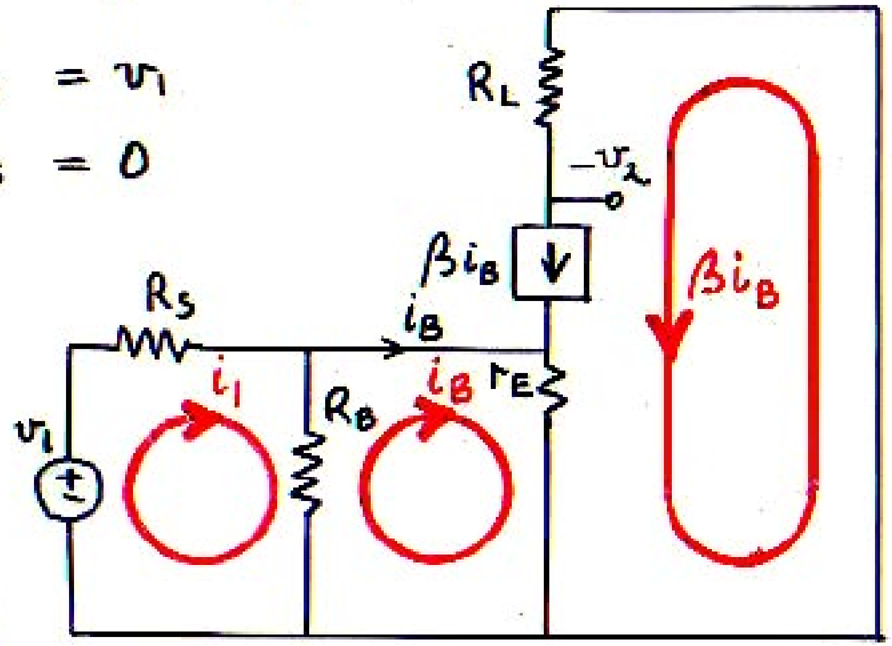
and

$$Z_T = \frac{\Delta^0 + z\Delta_{ij}}{\Delta_{12}^0 + z\Delta_{12ij}}. \tag{1-14}$$

The "brute-force" method: loop analysis

$$(R_s + R_B)\,i_1 \qquad - R_B\, i_B = v_1$$

$$-R_B\, i_1 + \left[R_B + (1+\beta)r_E\right] i_B = 0$$



$$i_B = \frac{\begin{vmatrix} R_s+R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s+R_B & -R_B \\ -R_B & R_B+(1+\beta)r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s+R_B)\left[R_B+(1+\beta)r_E\right] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta)r_E R_s + R_B^2 + (1+\beta)r_E R_B - R_B^2}$$

Finally, $\quad v_2 = R_L \beta i_B$

which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta)r_E R_s + (1+\beta)r_E R_B + R_s R_B}$$

Implementation:

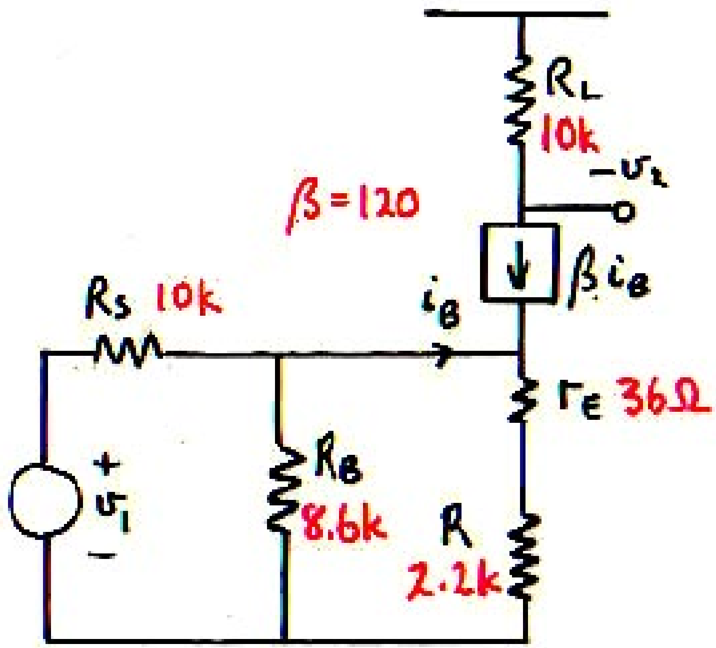
All that is needed is to calculate the driving point impedance across the terminals to which the extra element is to be added, under two conditions:

$$Z_d \equiv Z_{dp}\Big|_{u_{i1}=0} \quad \text{(original input zero)}$$

$$Z_n \equiv Z_{dp}\Big|_{u_{o1}=0} \quad \text{(original output nulled)}$$

Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \to 1/sC_2$ is desired.

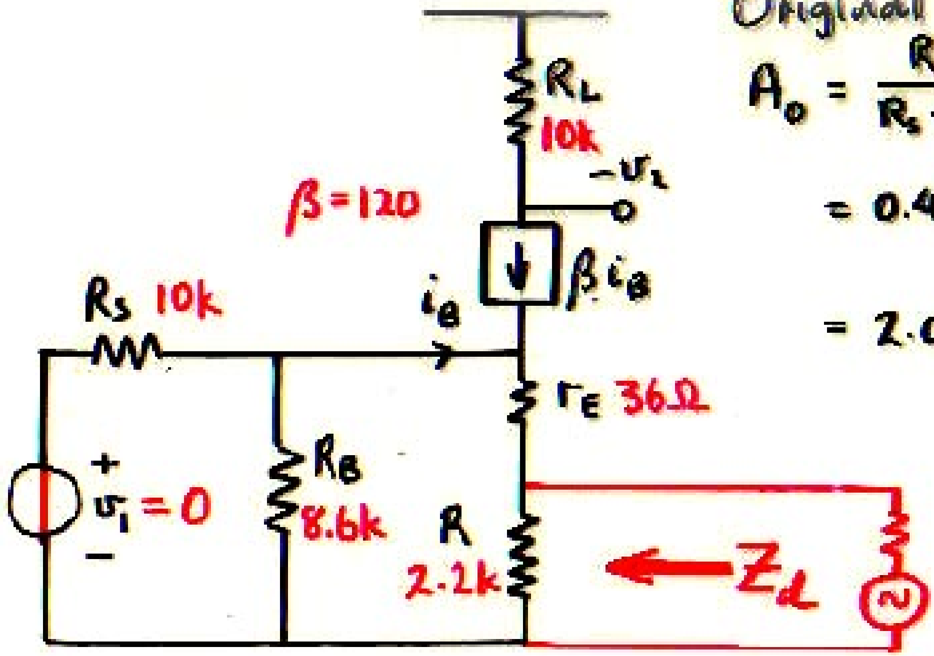


Original gain:

$$A_0 = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 0.46 \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2.0 \Rightarrow 6\,\mathrm{dB}$$



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$$= 0.46\;\frac{10}{2.2 + 0.036 + 0.039}$$

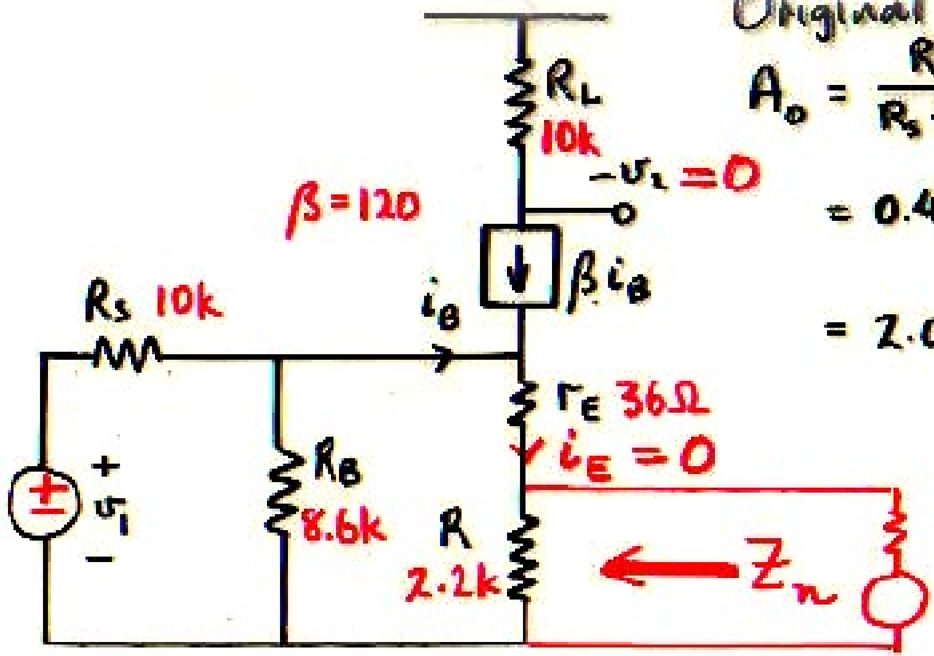
$$= 2.0 \Rightarrow 6\,dB$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_1$, and applying a second injected signal across R:

$$Z_d = R_d = R\|[r_E + (R_s\|R_B)/(1+\beta)]$$

$$= 2.2\,\|\,[0.036 + \underbrace{(10\|8.6)/120}_{0.039}]$$

$$= 75\,\Omega$$



Original gain:

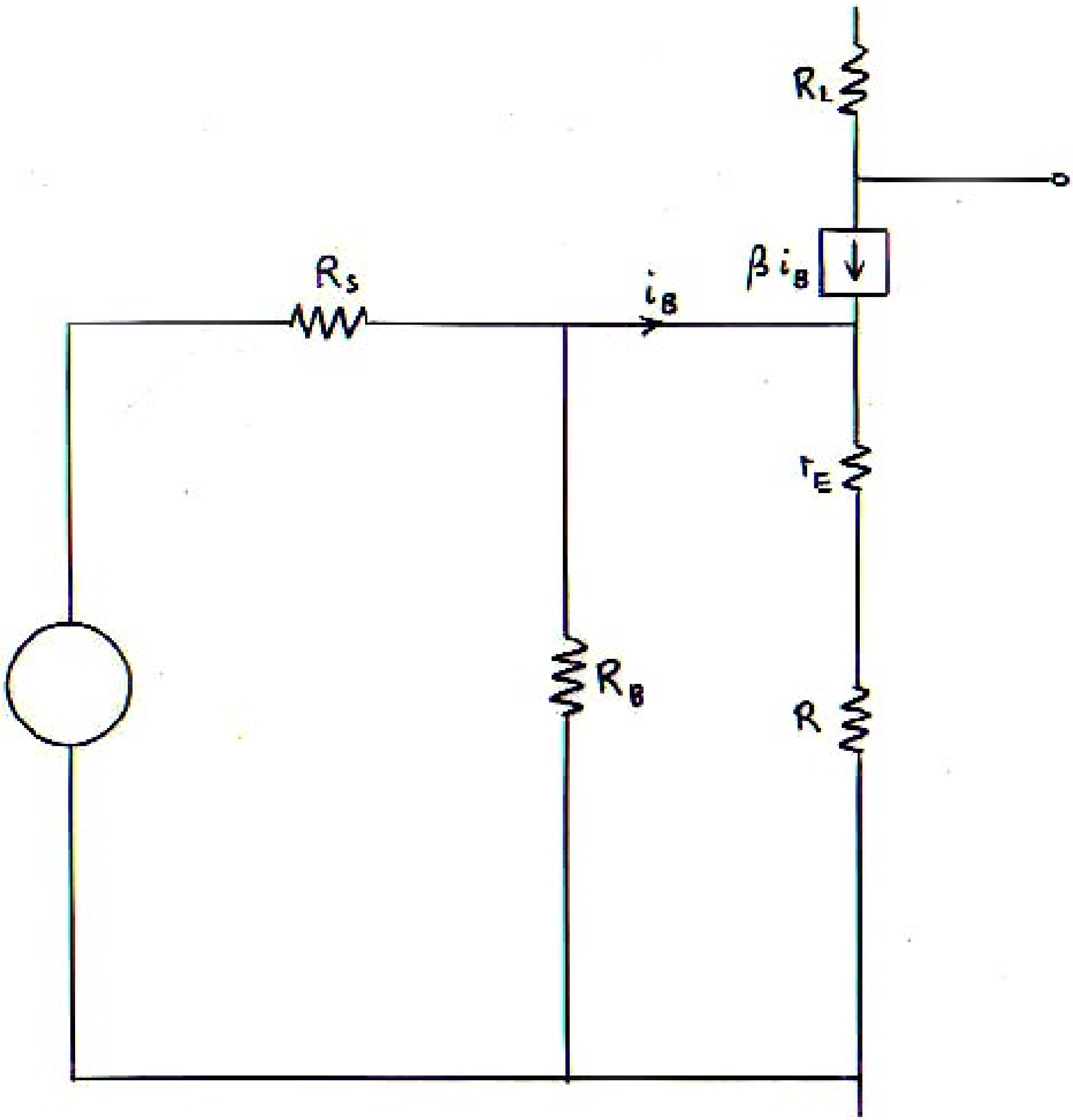
$$A_o = \frac{R_B}{R_s + R_B}\,\frac{\alpha R_L}{R + r_e + (R_s \| R_B)/(1+\beta)}$$

$$= 0.46\,\frac{10}{2.2 + 0.036 + 0.039}$$

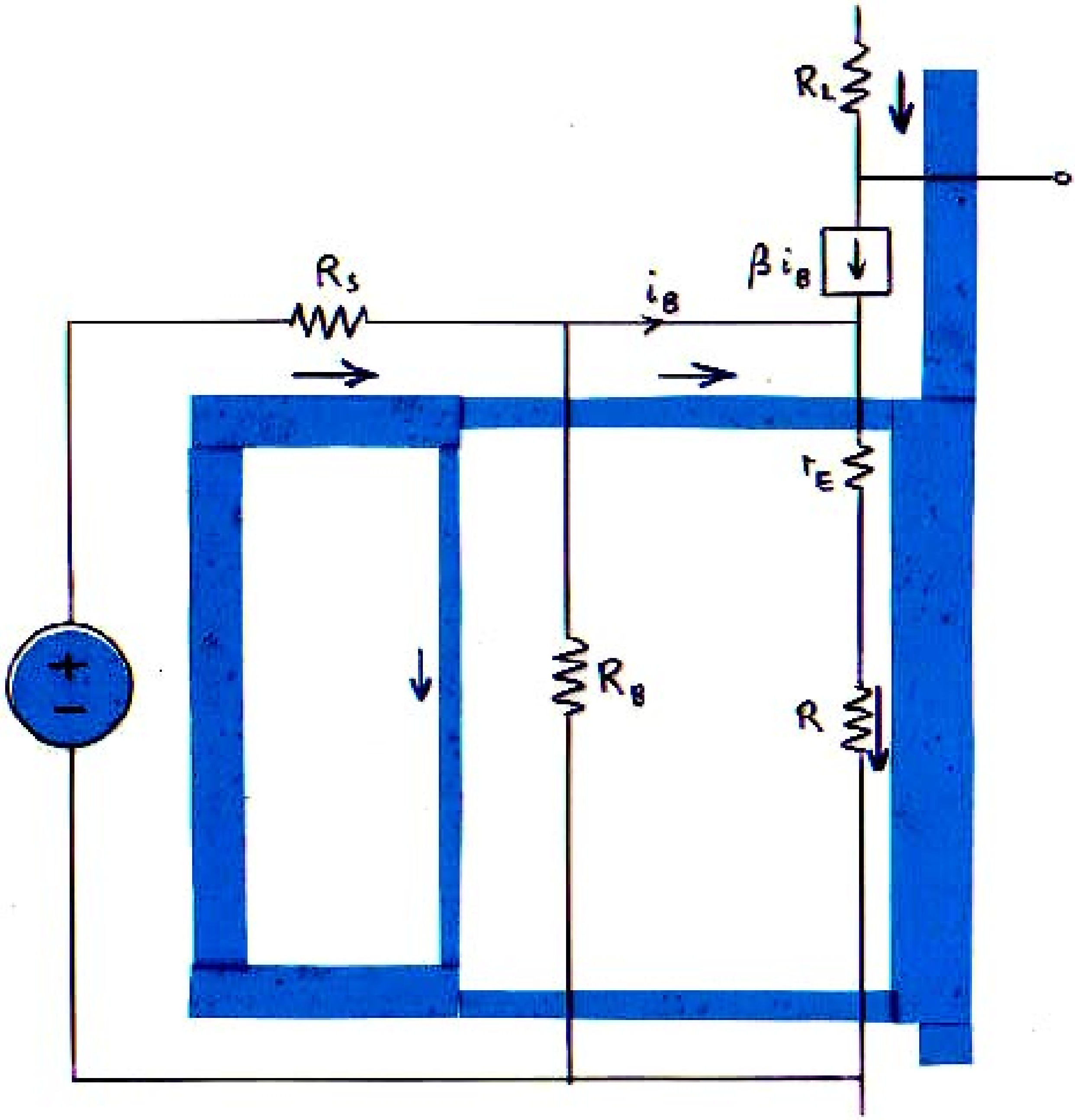
$$= 2.0 \Rightarrow 6\,\mathrm{dB}$$

Step 2. Calculate Z_n by applying a second injected signal across R, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence:

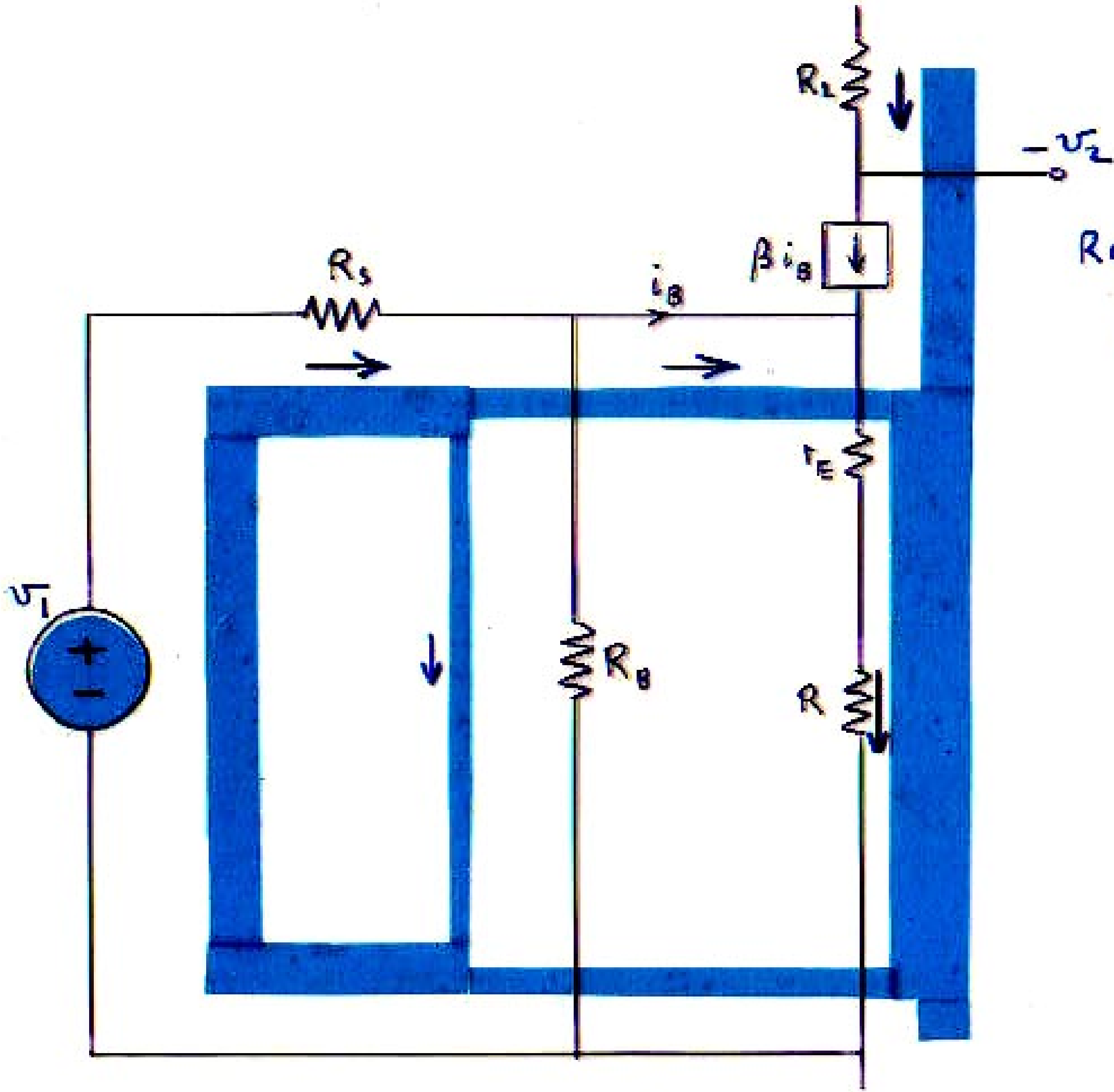
$$Z_n = R_n = R = 2.2\text{k}$$



R_L
βi_B
R_S
i_B
r_E
R_B
$R
$

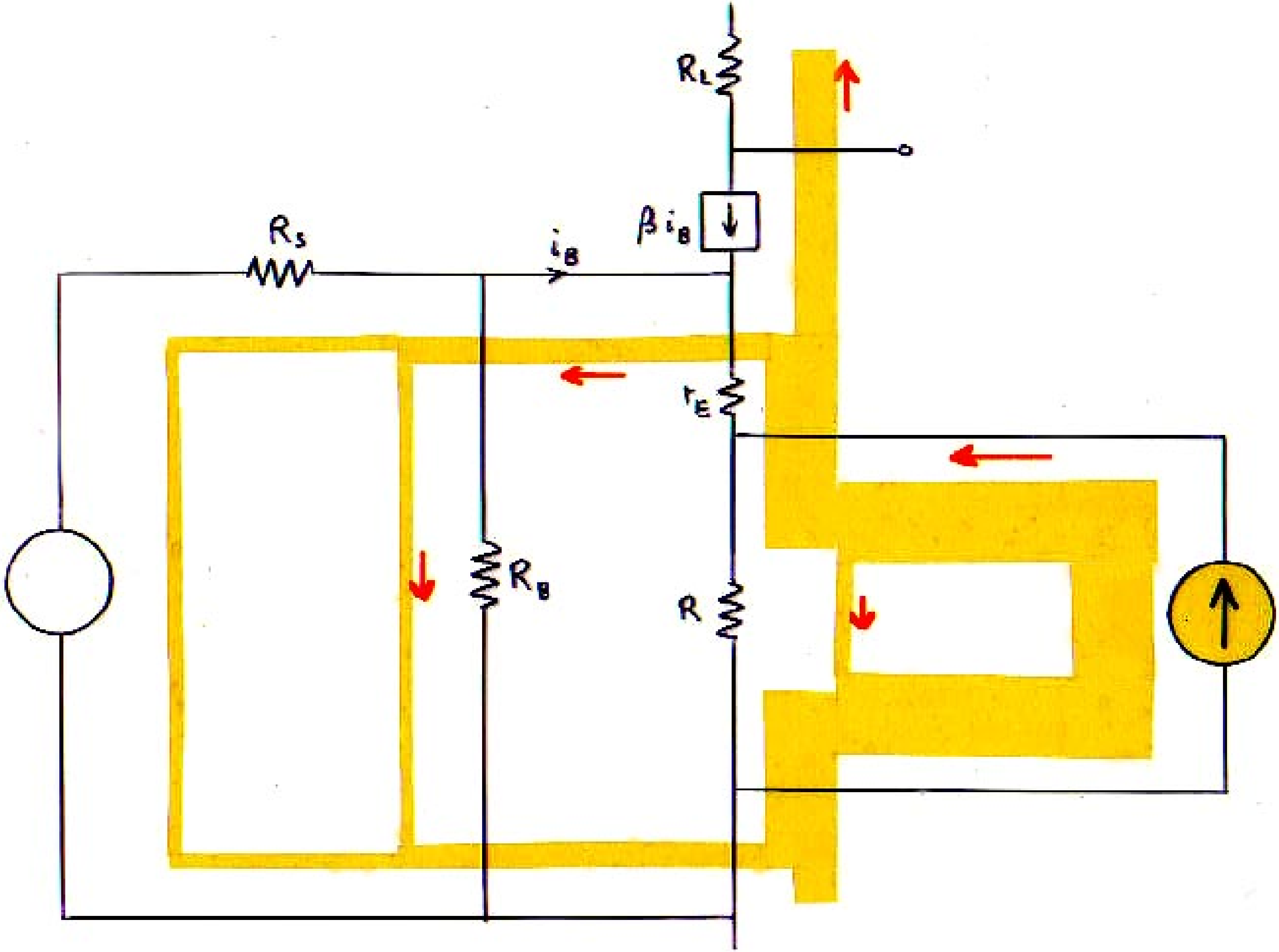


R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−

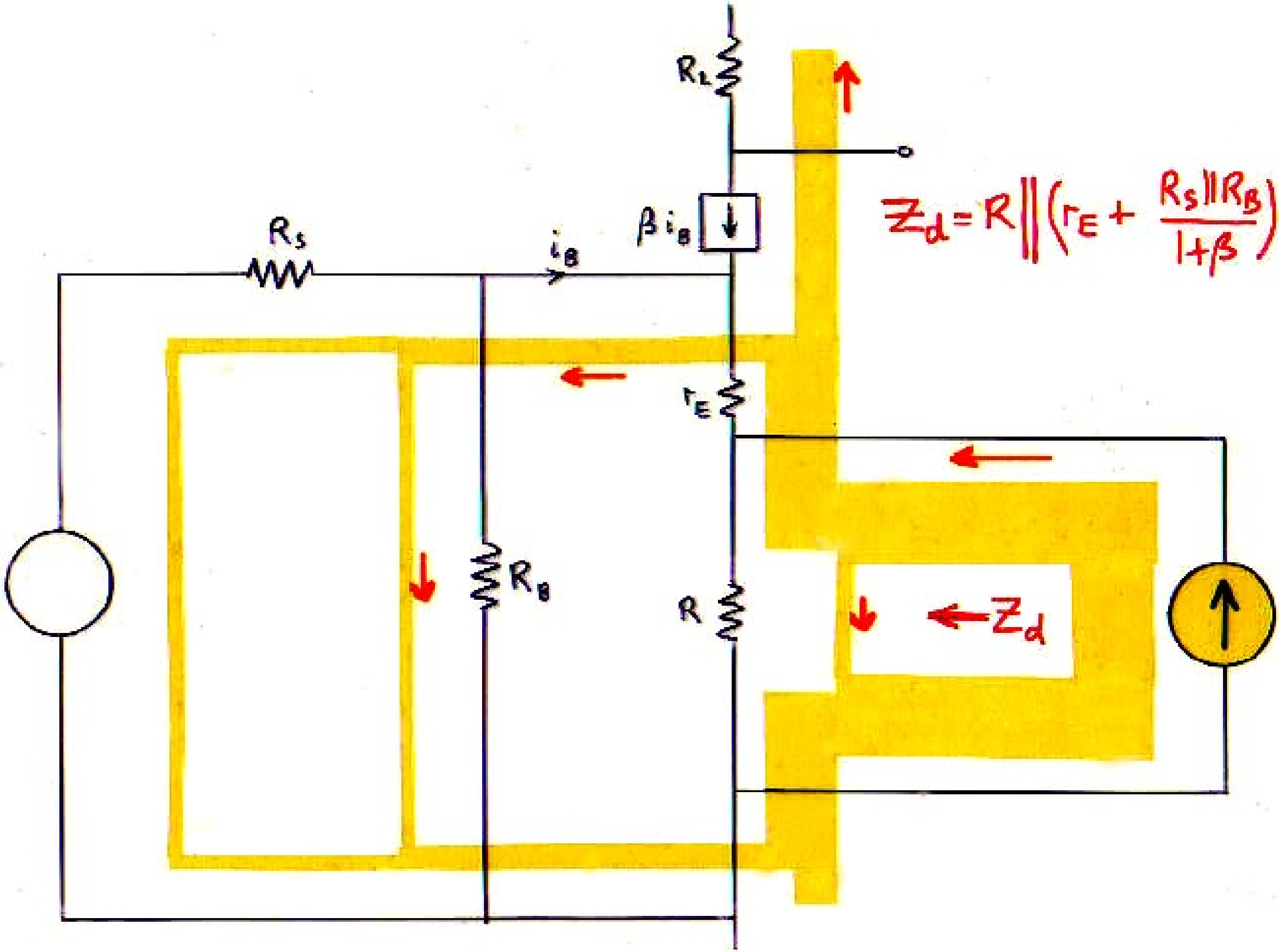


Reference gain:

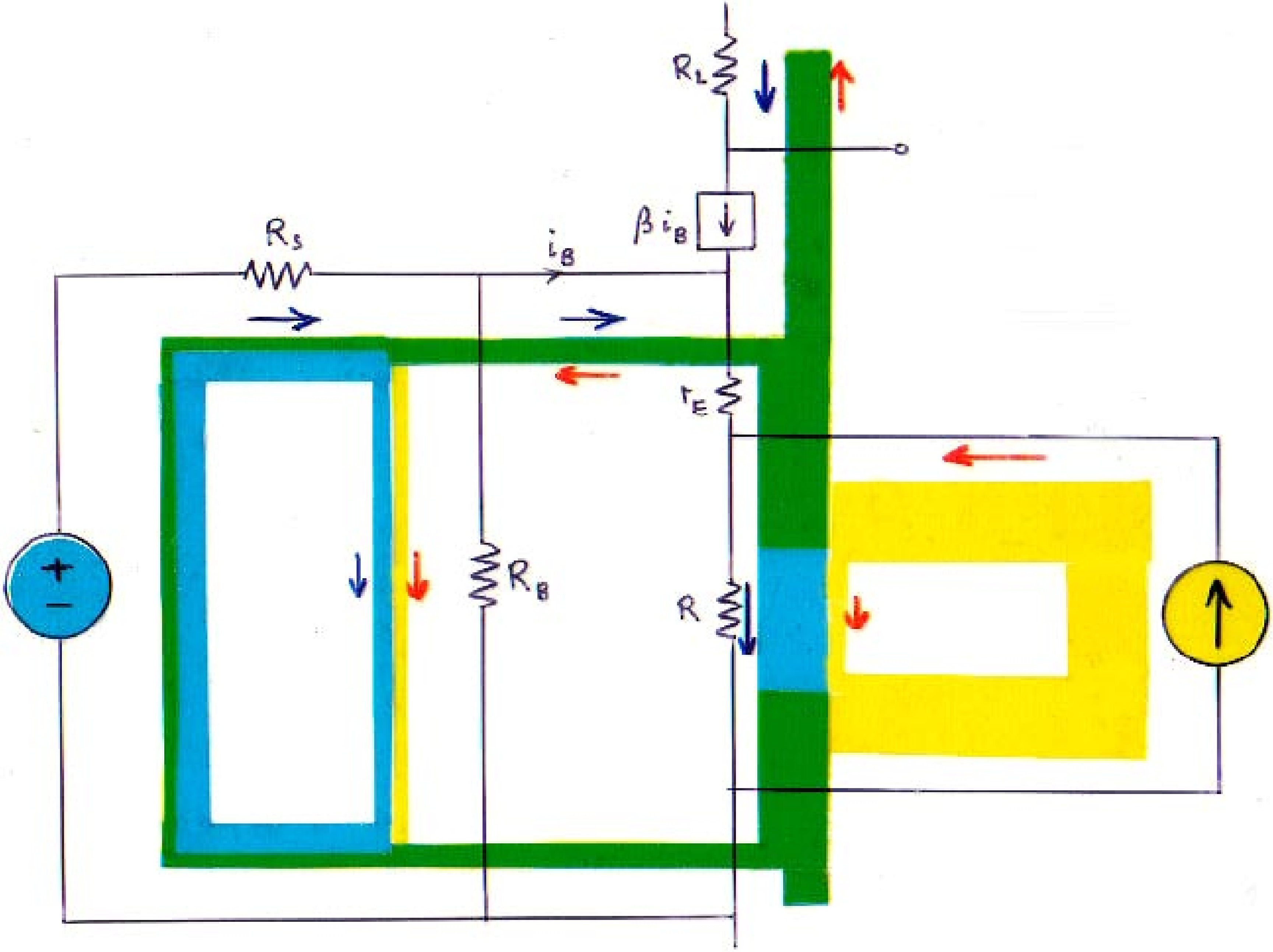
$$A_o = \frac{v_2}{v_1}$$



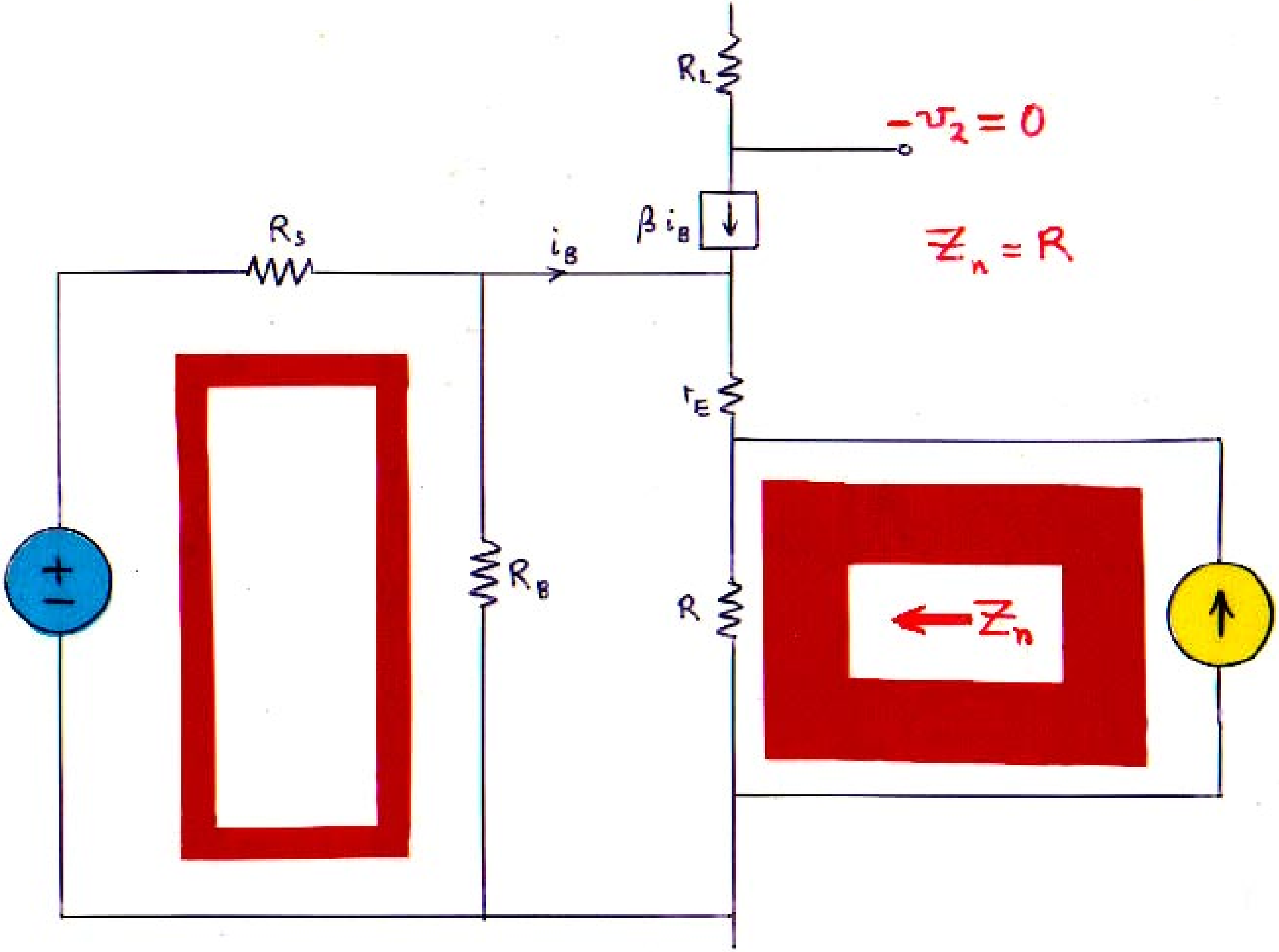
RL
βiB
RS
iB
rE
RB
R



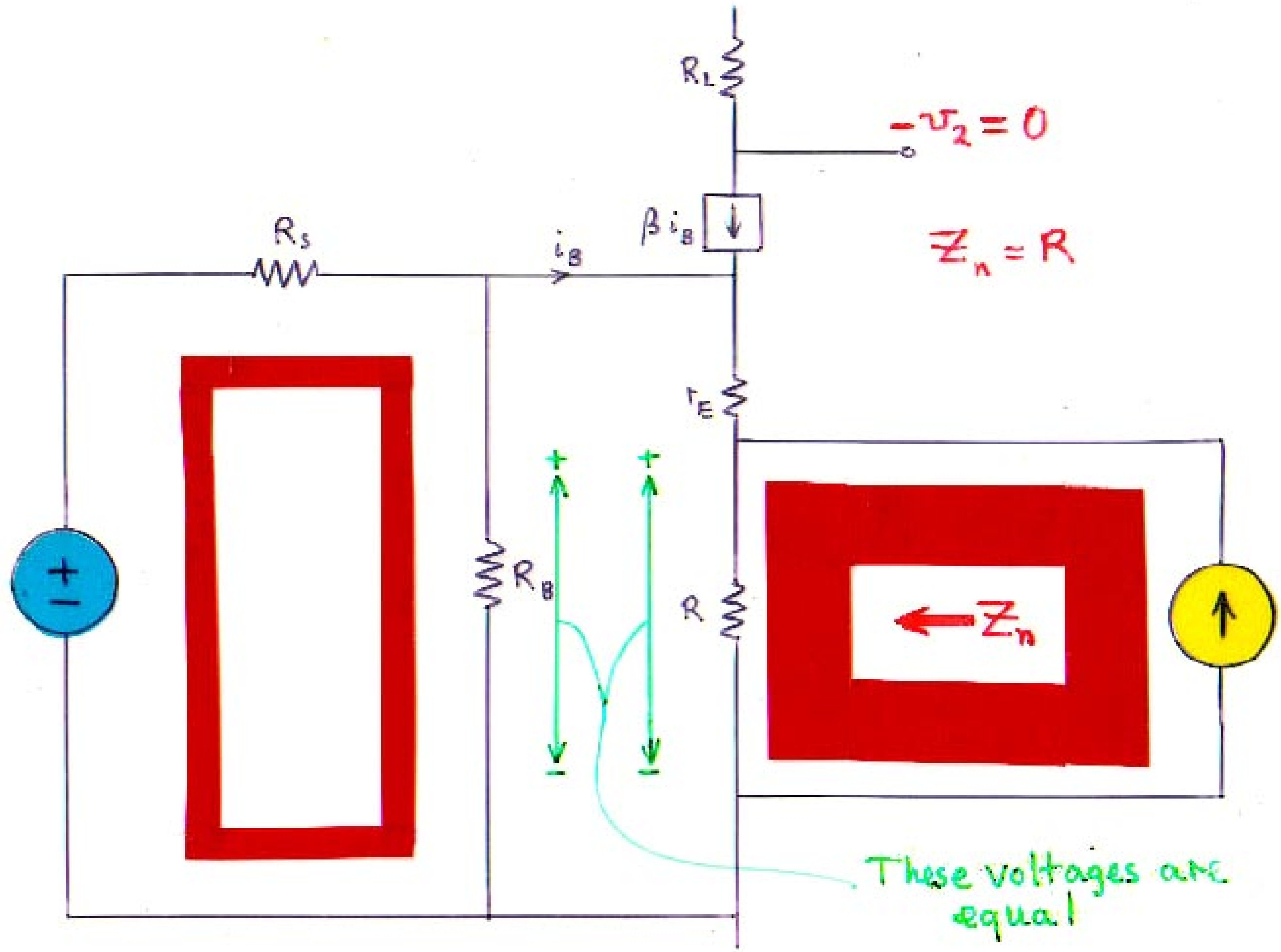
R_L
βi_B
R_S
i_B
r_E
R_B
R
$Z_d = R \| \left(r_E + \frac{R_S \| R_B}{1+\beta} \right)$
Z_d



R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−



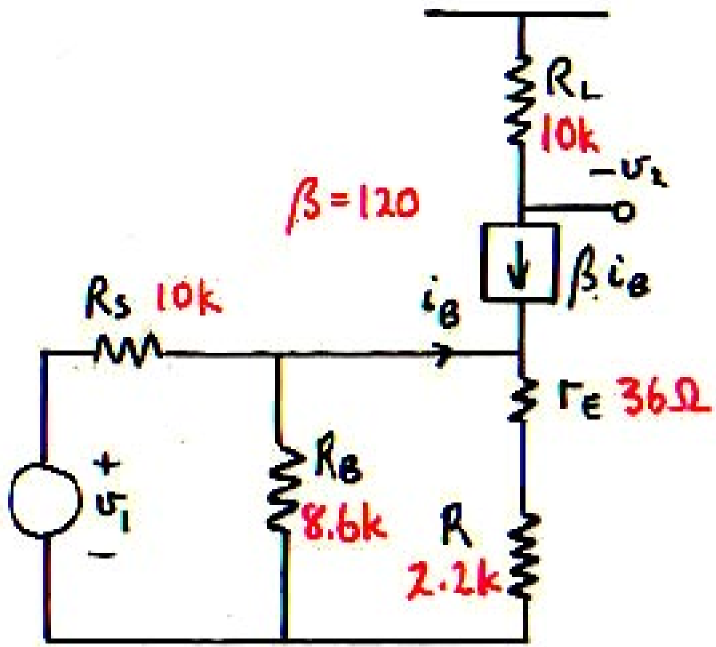
R_L
$-v_2 = 0$
βi_B
R_S
i_B
$Z_n = R$
r_E
R_B
R
Z_n



R_L
$-v_2 = 0$
βi_B
R_S
i_B
$Z_n = R$
r_E
R_B
R
Z_n
These voltages are equal

Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \rightarrow 1/sC_2$ is desired.

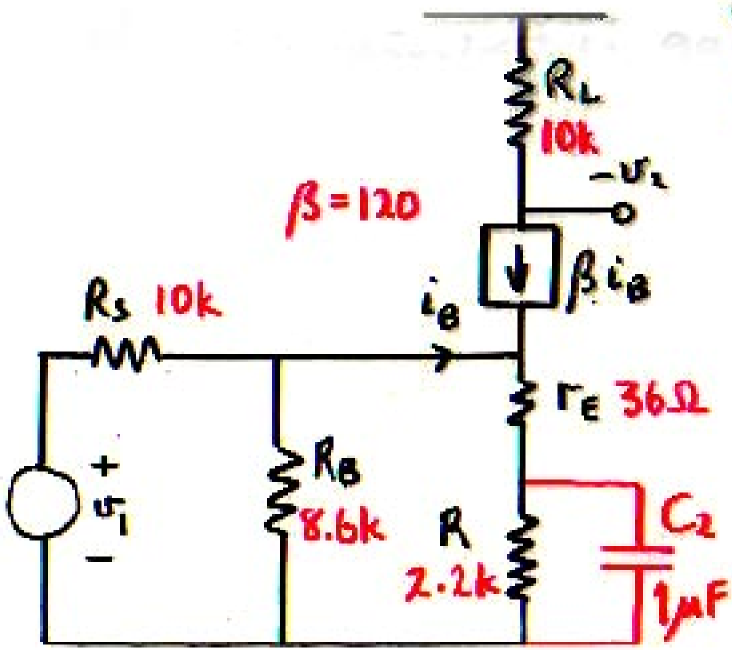


Original gain:

$$A_0 = \frac{R_B}{R_S + R_B} \, \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 0.46 \, \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2.0 \Rightarrow 6\,dB$$



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Hence, corrected gain in presence of $C_2 = 1\mu F$ bypass capacitance is:

$$A = A_o \frac{1 + \frac{R_n}{Z}}{1 + \frac{R_d}{Z}} = A_o \frac{1 + sC_2R_n}{1 + sC_2R_d} = A_o \frac{1 + \frac{s}{\omega_1}}{1 + \frac{s}{\omega_2}} = A_m \frac{1 + \frac{\omega_1}{s}}{1 + \frac{\omega_2}{s}}$$

where

$$\omega_1 \equiv \frac{1}{C_2 R_n} \quad f_1 = \frac{159}{1 \times 2.2} = 72\,Hz \quad \omega_2 \equiv \frac{1}{C_2 R_d} = \frac{159}{1 \times 0.075} = 2.1\,kHz$$

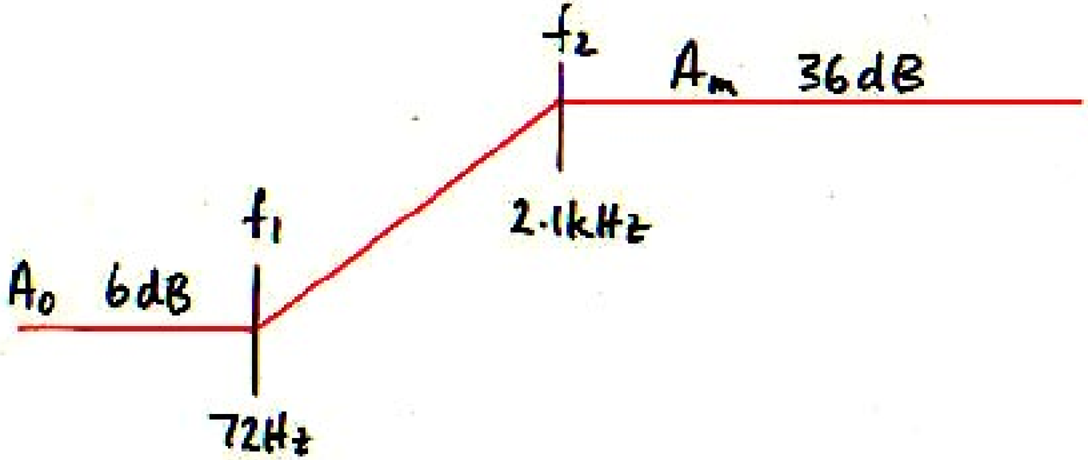
$$A_m = A_o \frac{\omega_2}{\omega_1} = A_o \frac{R_n}{R_d}$$

$$= \frac{R_B}{R_s + R_B} \frac{\alpha R_L}{R + r_E + (R_s \| R_B)/(1+\beta)} \frac{R[R + r_E + (R_s \| R_B)/(1+\beta)]}{R[r_E + (R_s \| R_B)/(1+\beta)]}$$

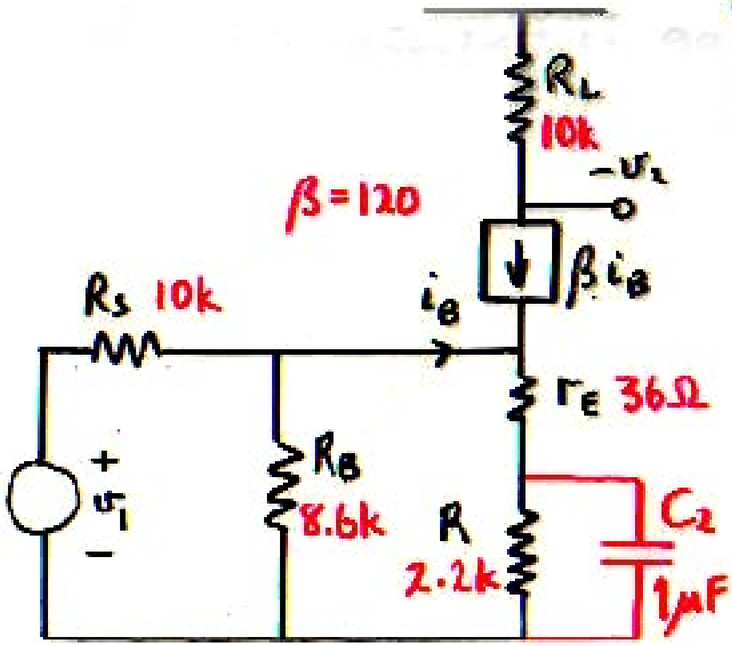
$$= \frac{R_B}{R_s + R_B} \frac{\alpha R_L}{r_E + (R_s \| R_B)/(1+\beta)} = 62 \Rightarrow 36\,dB$$

NOTE: <u>Nulling</u> a voltage is <u>not</u> the same as <u>shorting</u> it!

NOTE: the null double injection <u>calculation</u> is easier than the single injection <u>calculation</u>!



f_2
A_m 36dB
f_1
2.1kHz
A_0 6dB
72Hz



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Hence, corrected gain in presence of $C_2 = 1\mu F$ bypass capacitance is:

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$$A_m = A_0\,\frac{\omega_2}{\omega_1} = A_0\,\frac{R_n}{R_d}$$

$$= \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{R+r_E+(R_s\|R_B)/(1+\beta)}\,\frac{R[R+r_E+(R_s\|R_B)/(1+\beta)]}{R[r_E+(R_s\|R_B)/(1+\beta)]}$$

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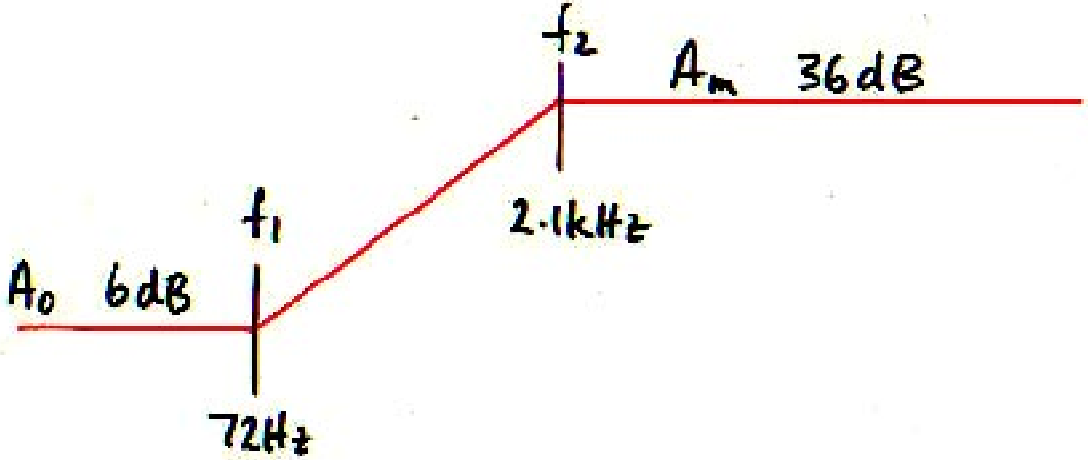
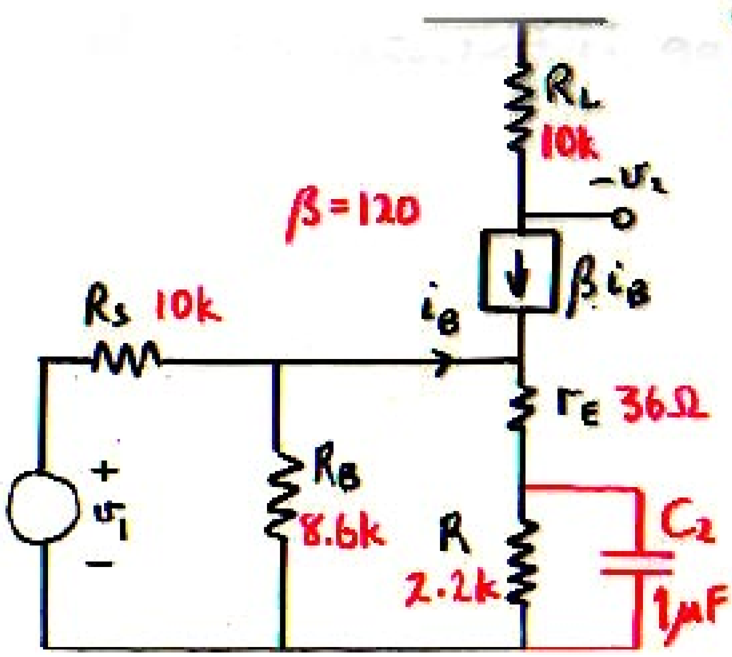


f2
Am 36dB
f1
2.1kHz
A0 6dB
72Hz



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Hence, corrected gain in presence of $C_2 = 1\mu F$ bypass capacitance is:

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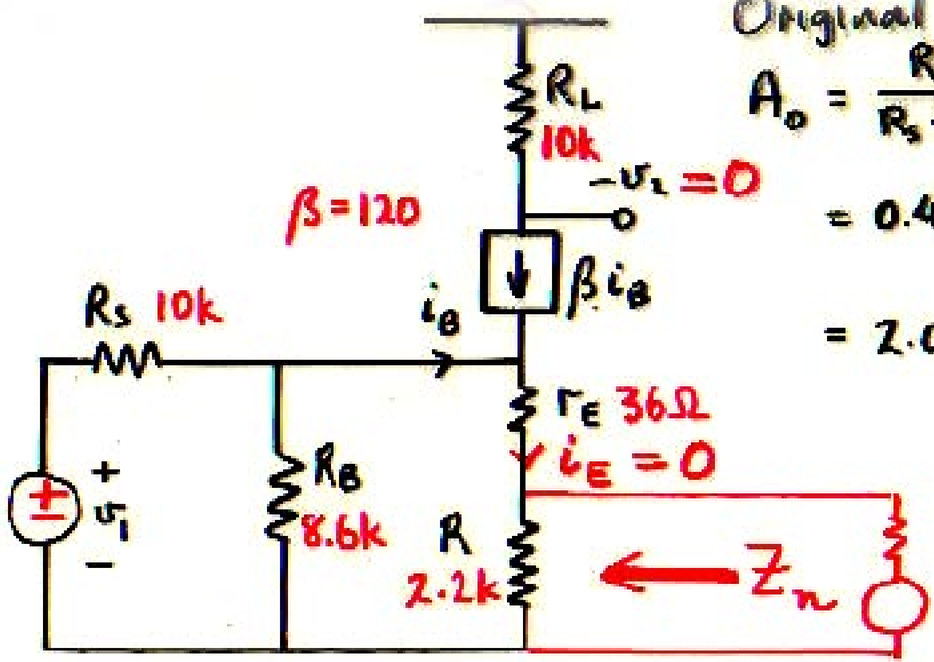

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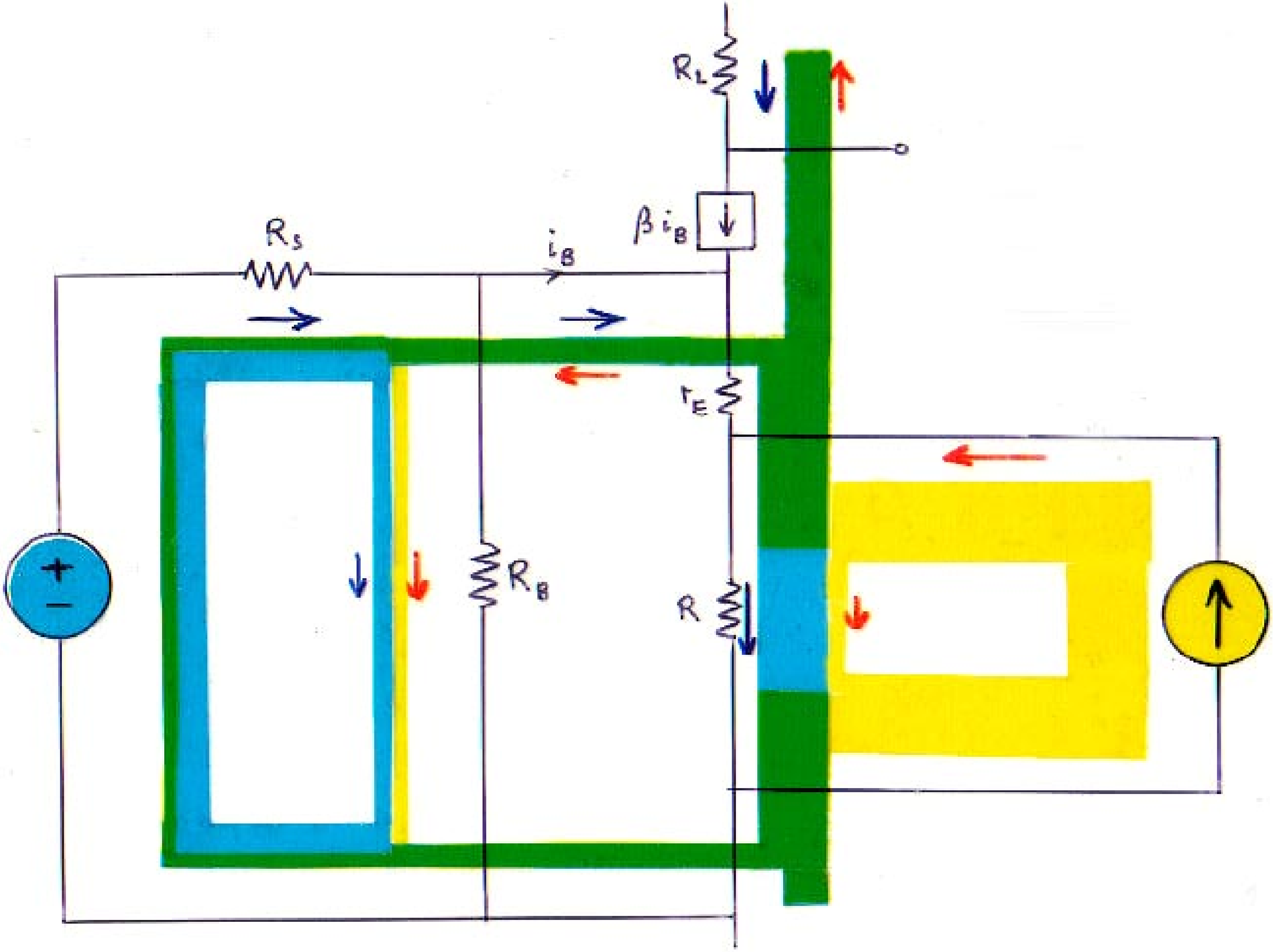
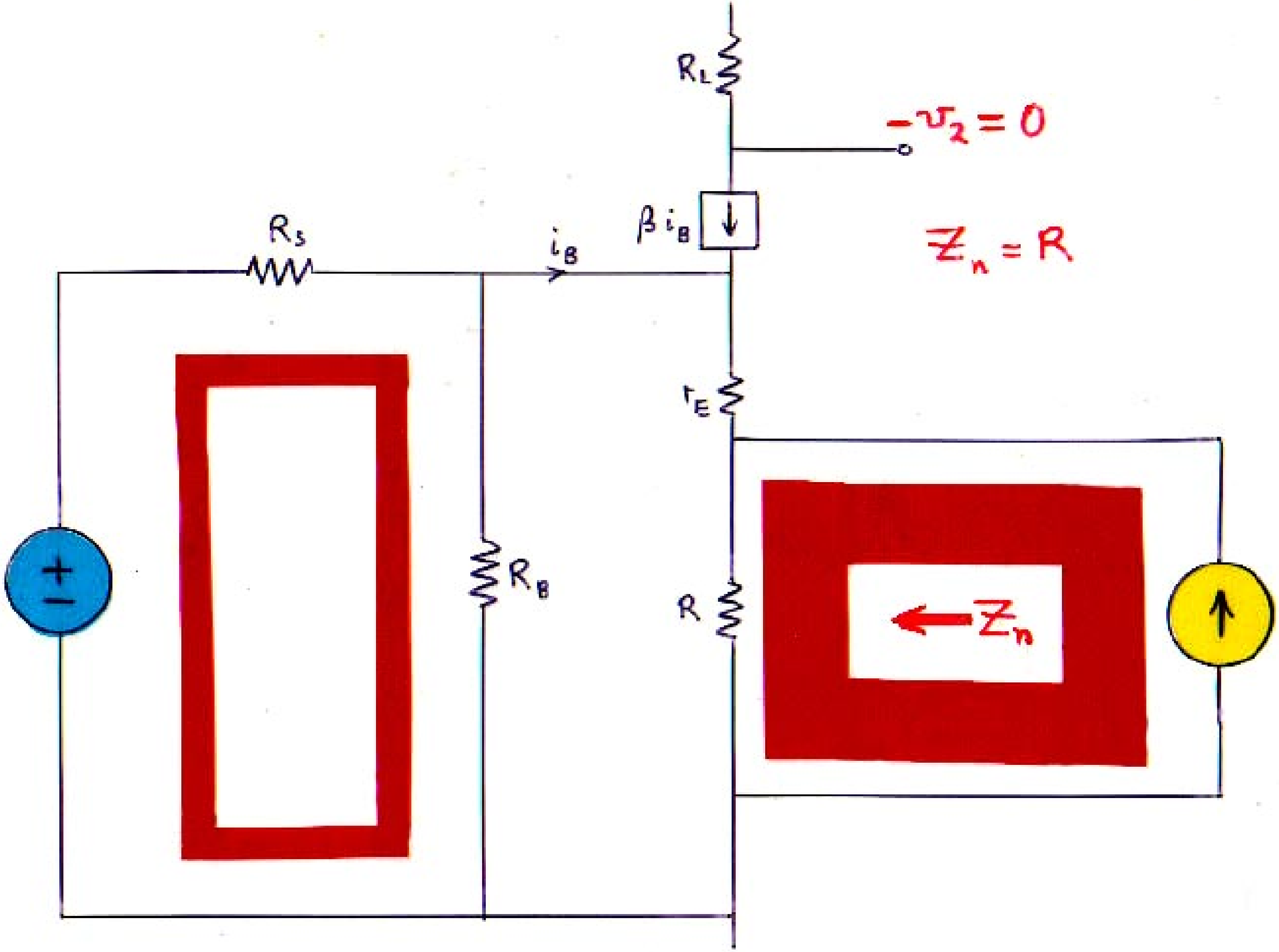
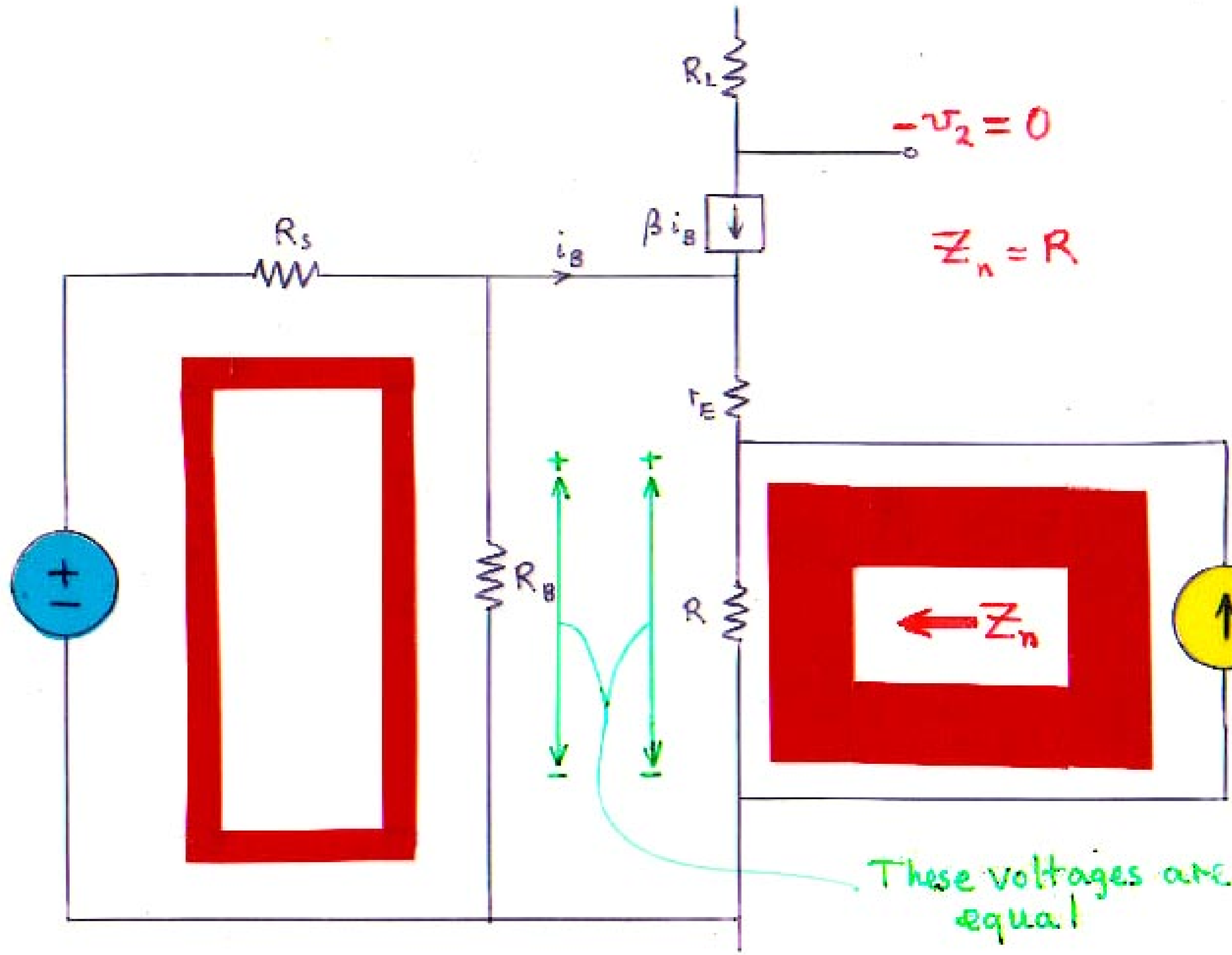
$$Z_n = R_n = R = 2.2k$$



R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−



R_L
$v_2 = 0$
βi_B
$Z_n = R$
R_S
i_B
r_E
R_B
R
Z_n



R_L
$-v_2 = 0$
βi_B
R_s
i_B
$Z_n = R$
r_E
+
+
R_B
R
Z_n
−
−
These voltages are equal

The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

reference gain

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\Big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

$$= \frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

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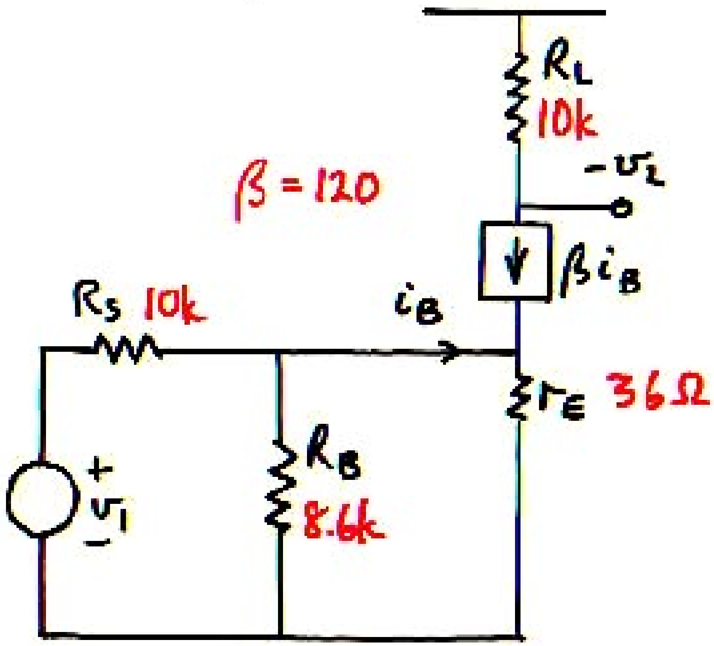
$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

reference gain

$$= \text{gain}\big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} = \frac{Z_n}{Z_d}\,\text{gain}\big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

This must be the gain when $Z=0$

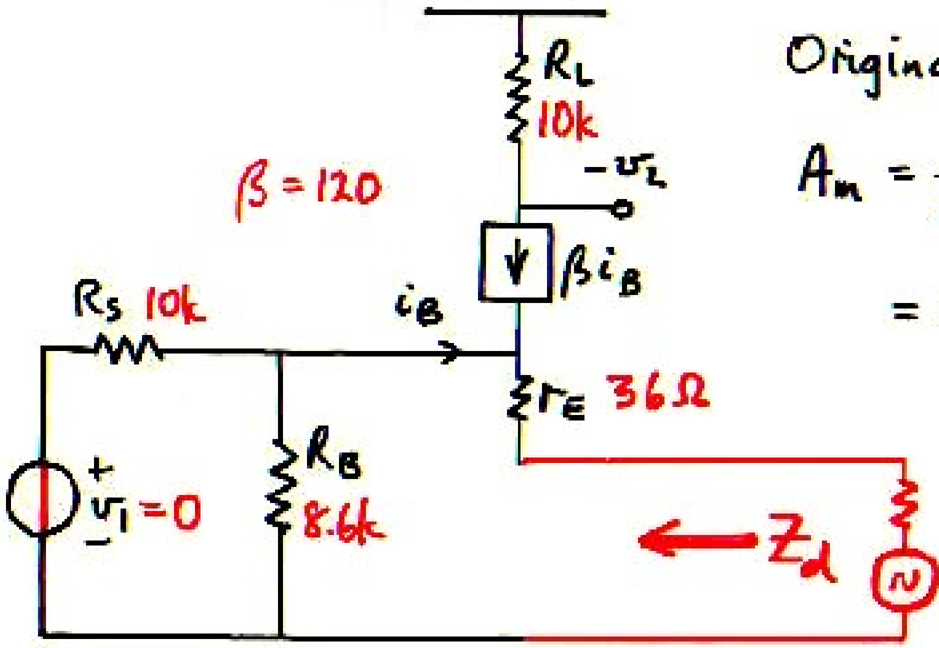
Example: An alternative to the method of the previous example is to find the correction factor to the midband gain A_m resulting from addition of the series "extra element" $Z \to R \| 1/sC$.



Original gain:

$$A_m = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,dB$$



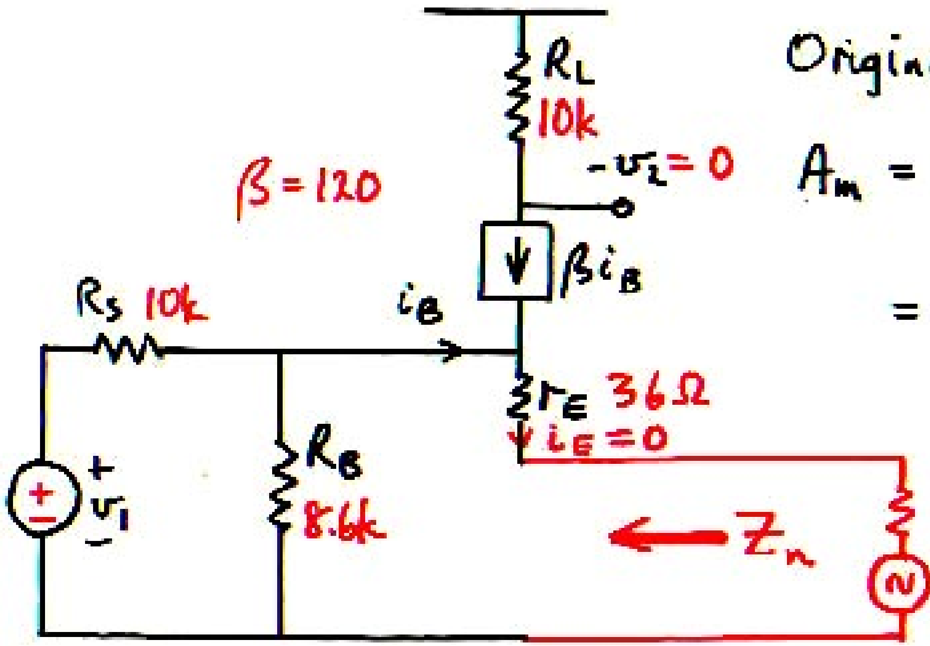
Original gain:

$$A_m = \frac{R_B}{R_S + R_B} \, \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,\text{dB}$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_i$ and applying a second injected signal in series with r_E:

$$Z_d \equiv R_d' = r_E + (R_S \| R_B)/(1+\beta)$$



Original gain:

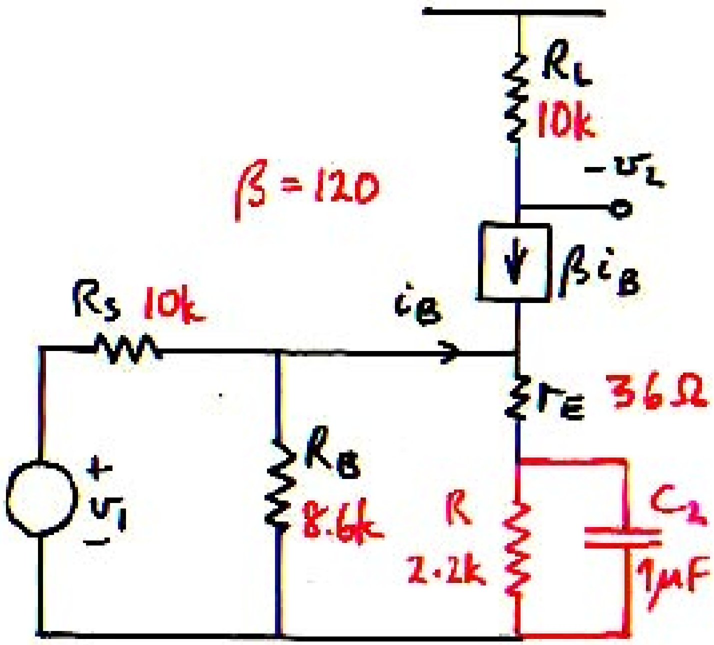
$$A_m = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,\text{dB}$$

Step 2. Calculate Z_n by applying a second injected signal in series with r_E, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$.

Then, since $v_2 = 0$, $i_E = 0$, hence

$$Z_n \equiv R_n' = \infty$$



Original gain:

$$A_m = \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{r_E + (R_s \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,\text{dB}$$

Hence, corrected gain in presence of $R \| 1/sC_2$ is:

$$A = A_m \frac{1+\frac{Z}{R_n'}}{1+\frac{Z}{R_d'}} = A_m \frac{1}{1+\frac{R}{R_d'}\,\frac{1}{1+sC_2R}} = A_m \frac{1+1/sC_2R}{1+1/sC_2(R \| R_d')}$$

However, $R \| R_d' = R_d$, so

$$A = A_m \frac{1+1/sC_2R}{1+1/sC_2R_d} \longrightarrow \text{same result as before}$$

Generalization: Extra Element Theorem — #1

There are two forms of Extra Element Theorem:

1.
$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}}$$

Provides a correction factor for an extra element added in shunt across a node pair.

2.
$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=0} \frac{1 + \frac{Z}{Z_n}}{1 + \frac{Z}{Z_d}}$$

Provides a correction factor for an extra element added in series with a branch.

The "extra element" Z can be any two-terminal combination of impedances.

Note that in all cases the null double injection calculation is easier than the single injection calculation.

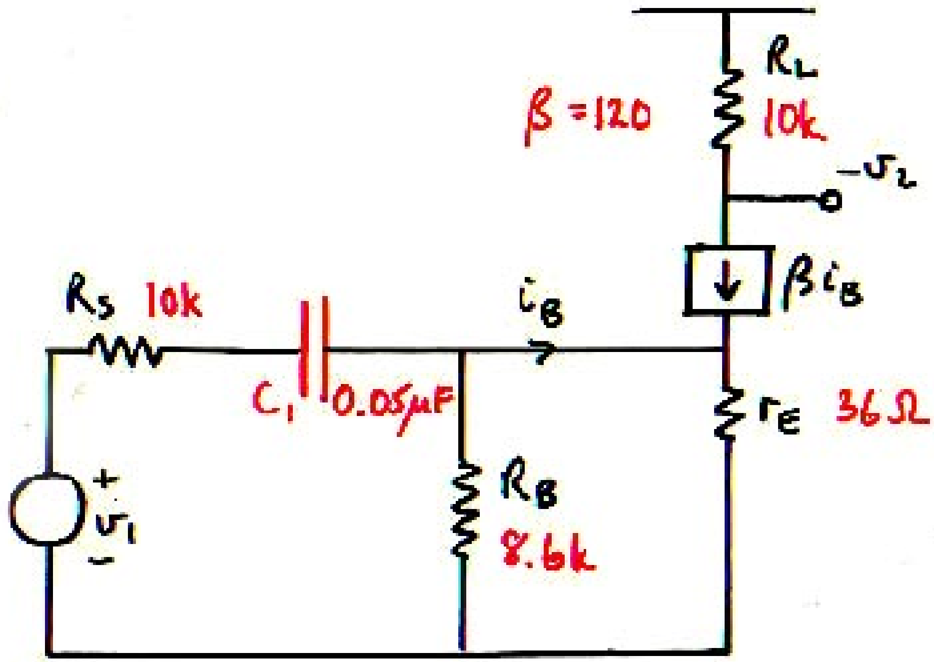
This results from <u>use</u> of the null condition (which makes several <u>other</u> quantities zero);

<u>and</u>

because the <u>relation</u> between u_{i1} and u_{i2} to <u>produce</u> the null is <u>never needed</u> — only the <u>null itself</u> is used.

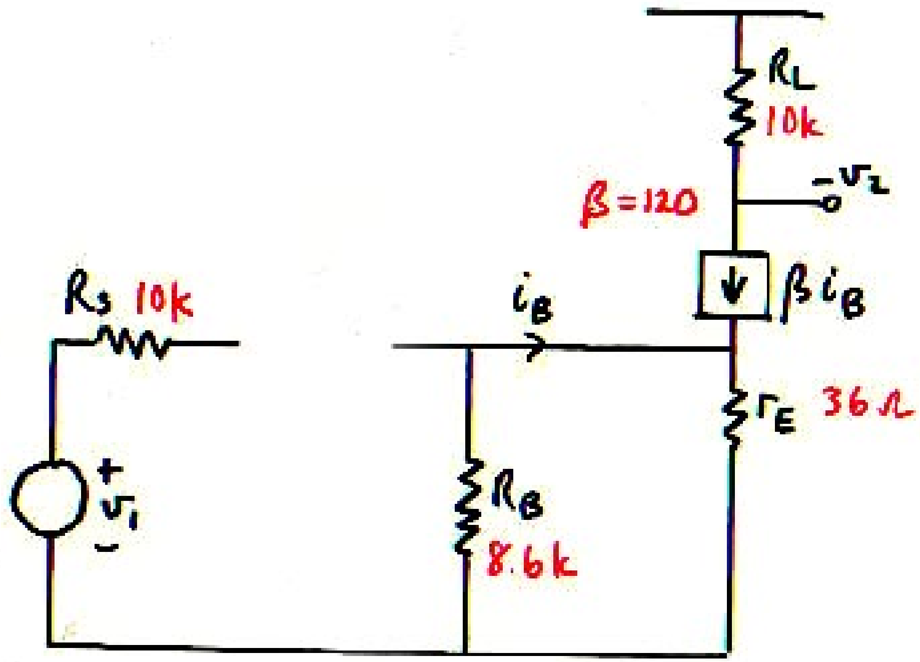
Exercise

In the CE amplifier stage, find the correction factor to the midband gain A_m resulting from inclusion of the coupling capacitance $C_1 = 0.05\mu F$:



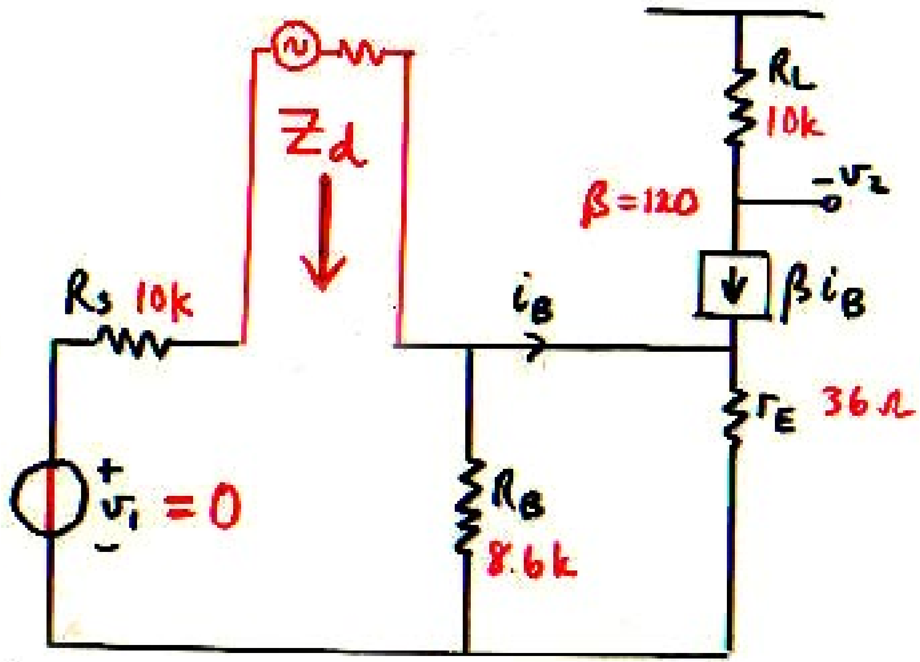
$A_m = 36dB$

Exercise Solution



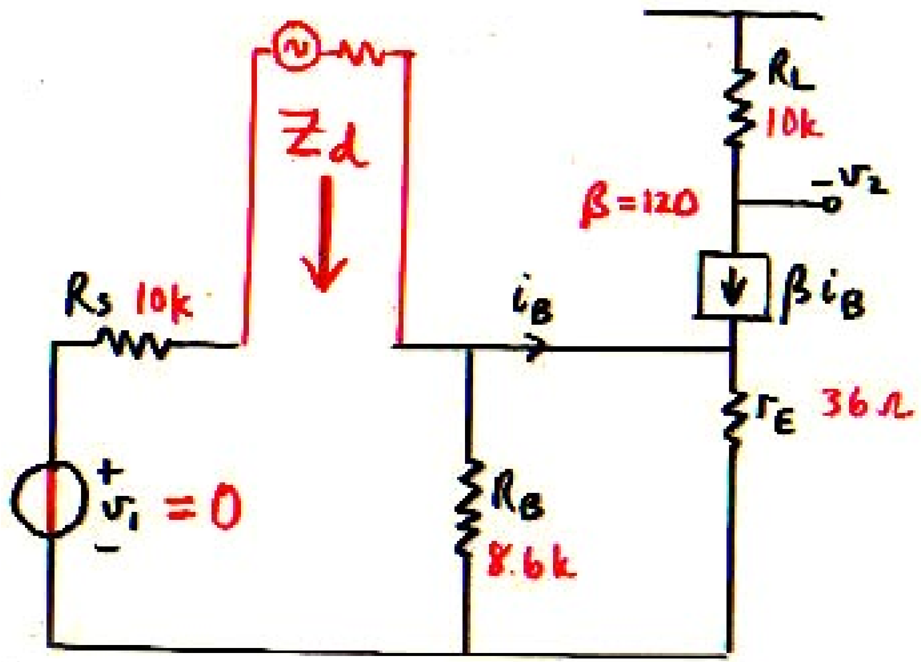
$A_m = 36\text{dB}$

Exercise Solution



$A_m = 36\text{dB}$

Exercise Solution

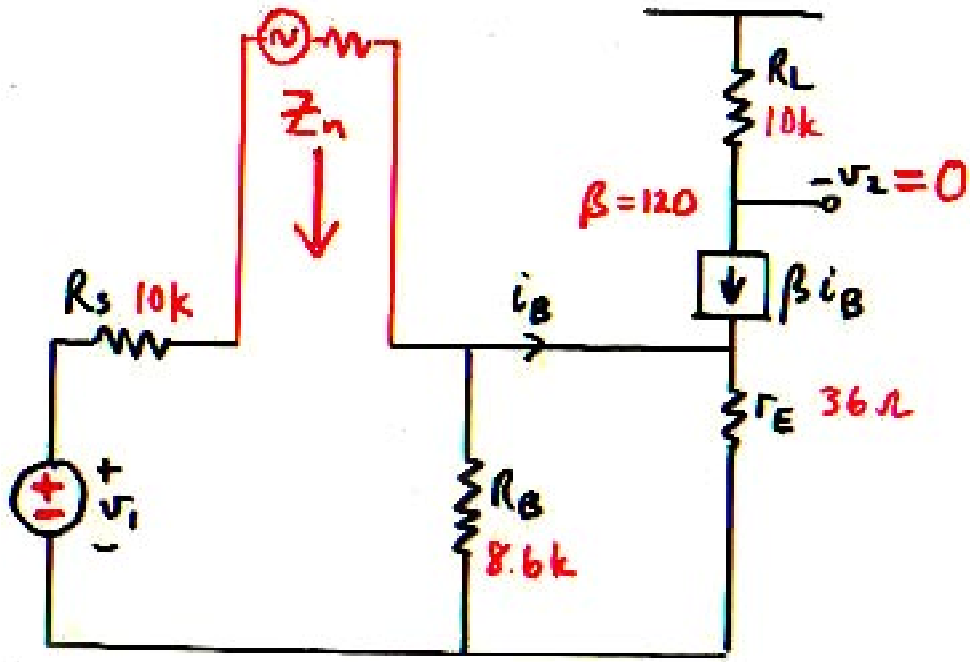


$A_m = 36dB$

Step 1.

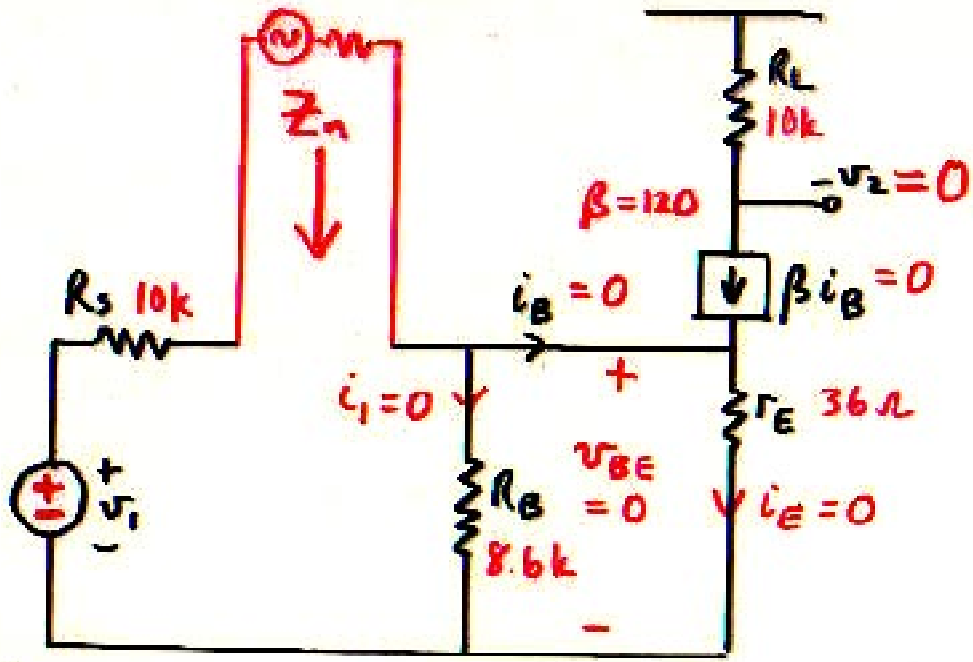
$$Z_d = R_d = R_s + R_B \| (1+\beta) r_E$$
$$= 10 + 8.6 \| (120 \times 0.036)$$
$$= 10 + 8.6 \| 4.3$$
$$= 13k$$

Exercise Solution



$A_m = 36\,dB$

Exercise Solution

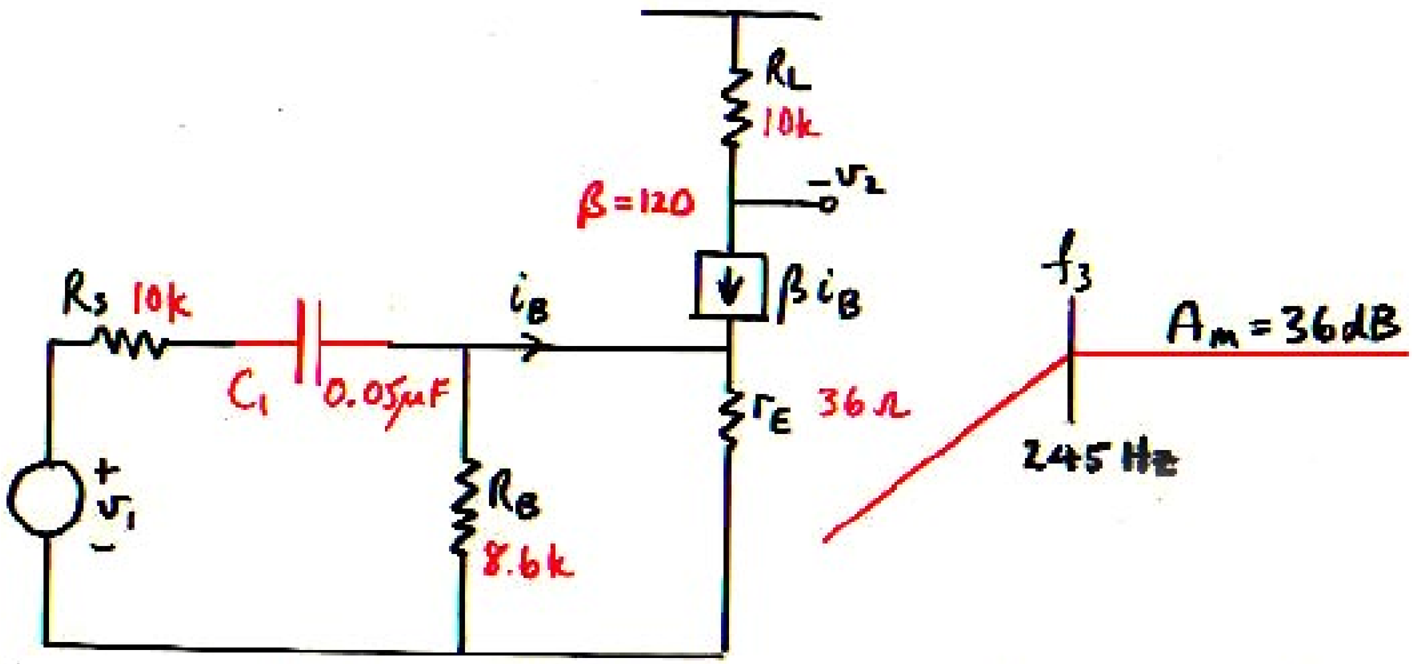


$A_m = 36dB$

Step 2.

$$Z_n = R_n = \infty$$

Exercise Solution



Hence corrected gain in the presence of $Z \to 1/sC_1$ is

$$A = A_m \frac{1 + \frac{Z}{Z_n}}{1 + \frac{Z}{Z_d}} = A_m \frac{1}{1 + \frac{1}{sC_1R_d}} = A_m \frac{1}{1 + \frac{\omega_3}{s}}$$

where

$$\omega_3 \equiv \frac{1}{C_1R_d} \qquad f_3 = \frac{159}{0.05 \times 13} = 245\text{ Hz}$$

Generalization: Extra Element Theorem — #2

If the reference circuit is purely resistive,
$Z_d = R_d$ and $Z_n = R_n$ are pure resistances.

If, also, the extra element is a pure reactance,
the Extra Element Theorem correction factor gives the corner frequencies directly.

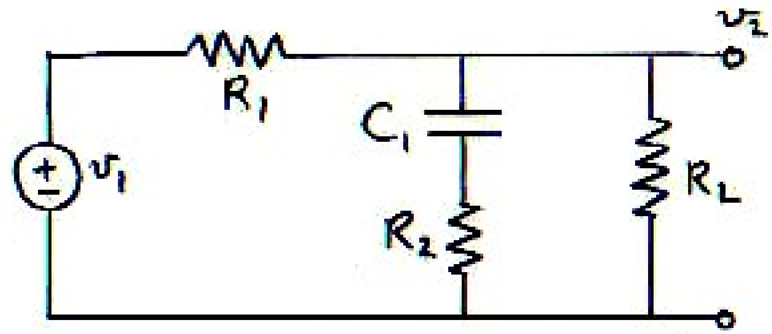
Generalization: Extra Element Theorem — #3

The Extra Element Theorem can profitably be used to divide the analysis of a complicated circuit into successive simpler steps:

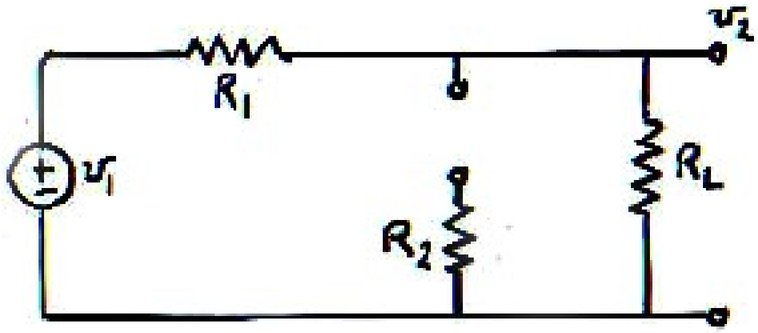
Designate one element as "extra," and the circuit without the element as the "reference circuit." Calculate the gain of the (simpler) reference circuit, then restore the omitted element by the Extra Element Theorem correction factor.

This is a particularly useful approach when the designated "extra" element is a reactance and the reference circuit is purely resistive.

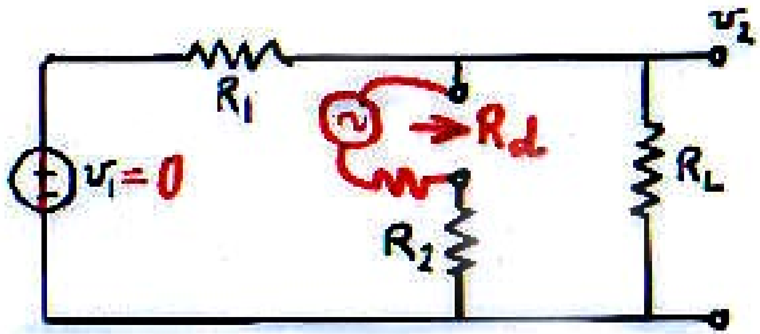
Exercise: Lag-lead network



Find the transfer function $A \equiv v_2/v_1$ by designating C_1 as an "extra" element.

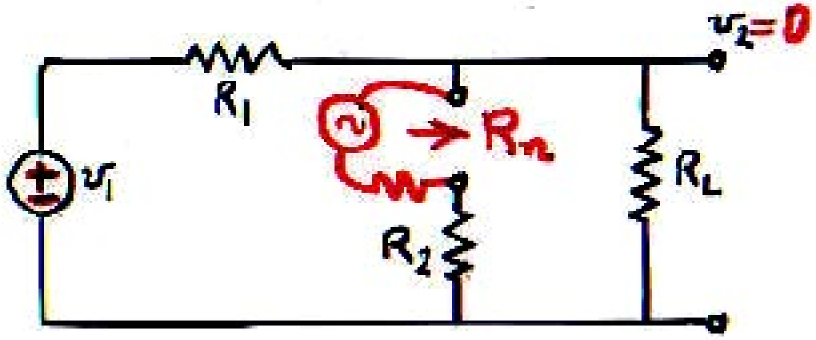


$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

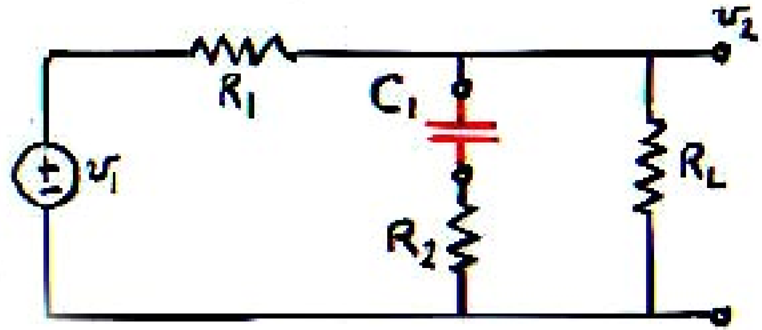
$$R_d = R_2 + R_1 \| R_L$$



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$

$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1 R_2}{1 + sC_1 (R_2 + R_1 \| R_L)}$$

The Extra Element Theorem may be used to find an extra element correction factor for any transfer function of a linear circuit. It is necessary merely to identify the "input" and "output" signals; Z_d and Z_n are then calculated as the driving point impedance seen by the extra element with the "input" zero and with the "output" nulled, respectively.

Examples of transfer functions:

"output" → current drawn from power supply / "input" → input voltage (a transadmittance)

$$\frac{\text{current drawn from power supply}}{\text{input voltage}}$$

"output" → output voltage ripple component / "input" → power supply ripple voltage (a voltage gain; audio susceptibility of a power supply)

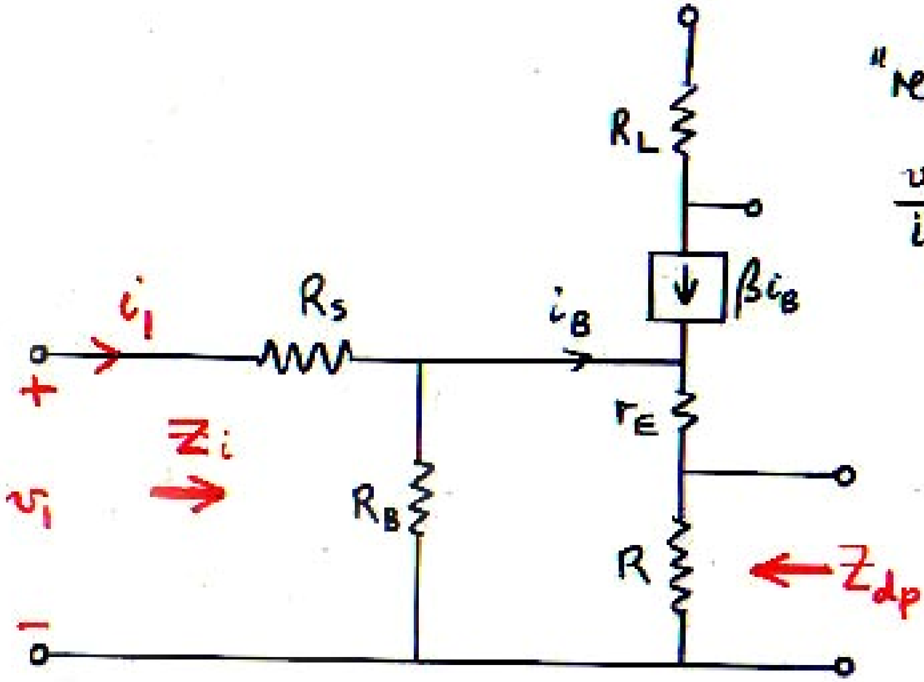
$$\frac{\text{output voltage ripple component}}{\text{power supply ripple voltage}}$$

"output" → corresponding driving voltage / "input" → any driving current (a self-impedance, e.g. input or output impedance)

$$\frac{\text{corresponding driving voltage}}{\text{any driving current}}$$

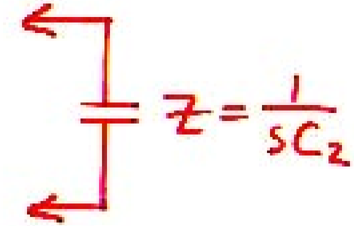
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E+R)$$



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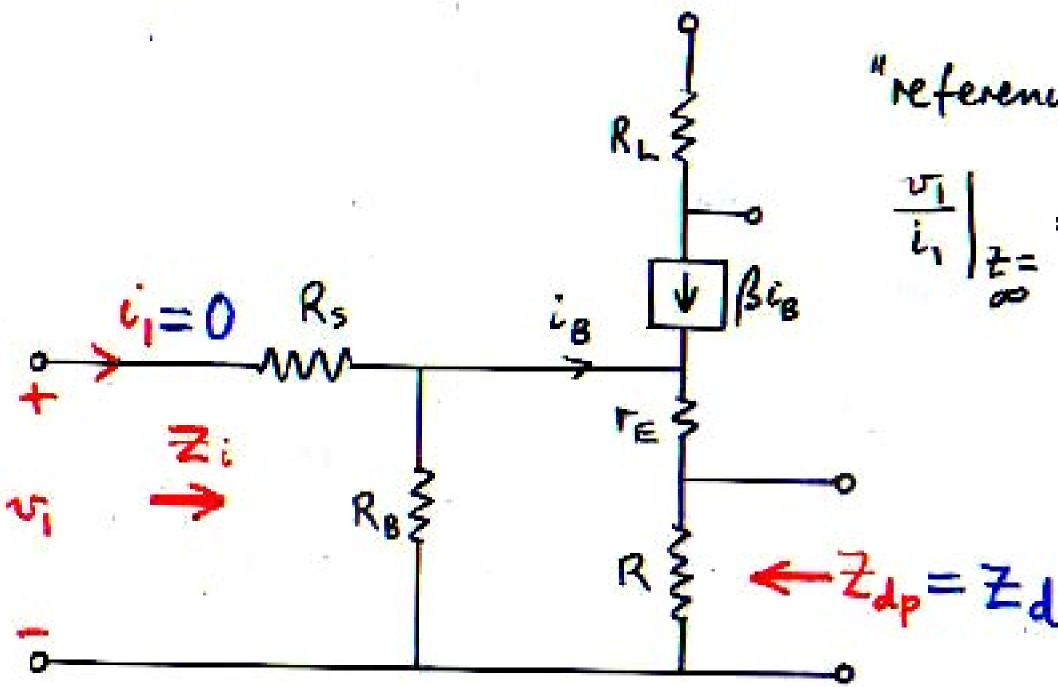

"reference transfer function":

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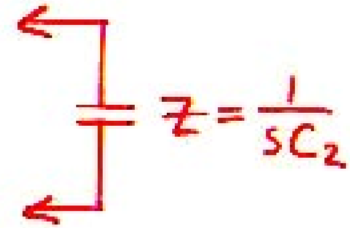
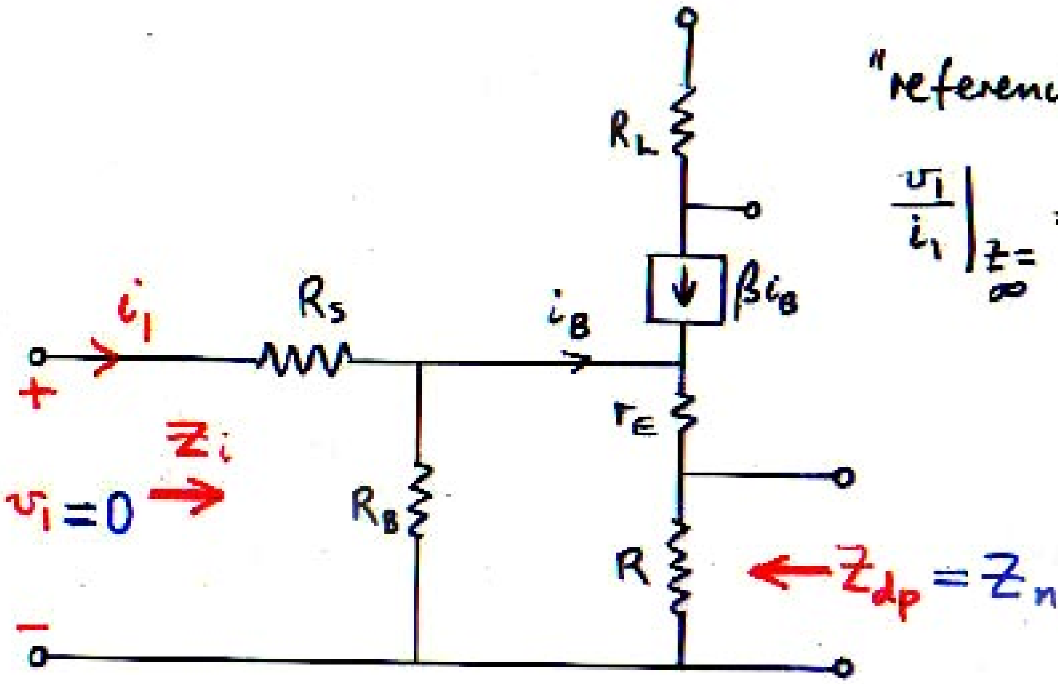


$$Z_d = \left.Z_{dp}\right|_{\text{"input" zero}} = \left.Z_{dp}\right|_{i_1=0} = R_d = R \| \left(r_E + \frac{R_B}{1+\beta}\right)$$

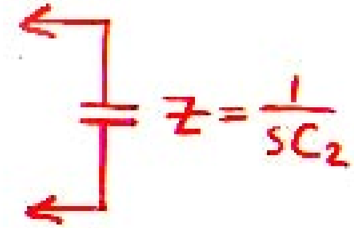
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

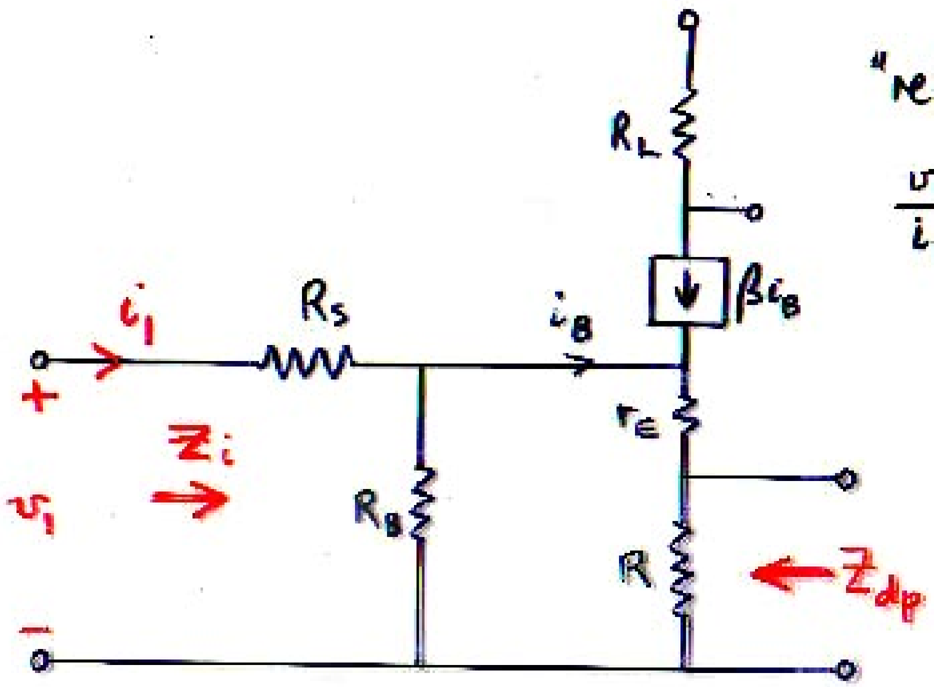
$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E + R)$$



$$Z_n = \left.Z_{dp}\right|_{\substack{\text{"output"}\\ \text{nulled}}} = \left.Z_{dp}\right|_{v_1=0} = R_n = R \| \left(r_E + \frac{R_S \| R_B}{1+\beta}\right)$$

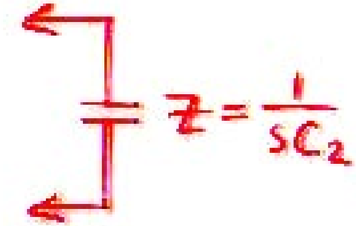
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

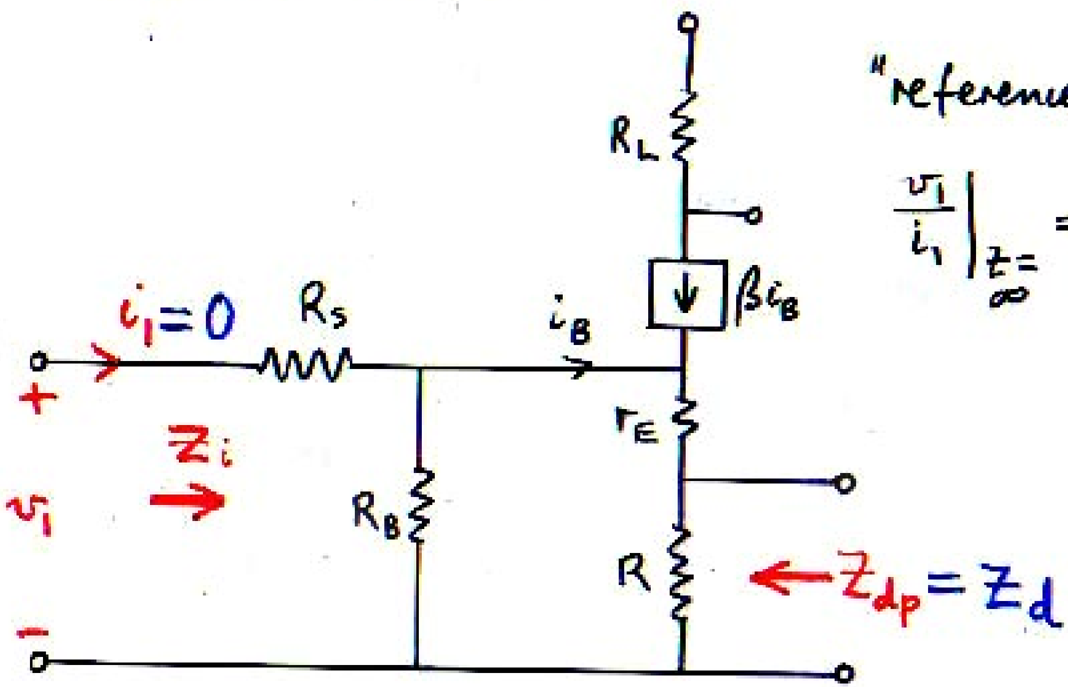
$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_s + R_B \| (1+\beta)(r_E+R)$$



Hence, $Z_i = \left[R_s + R_B \| (1+\beta)(r_E+R)\right] \dfrac{1 + sC_2 R_n}{1 + sC_2 R_d}$

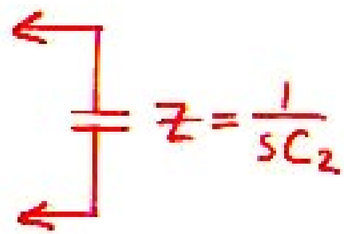
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



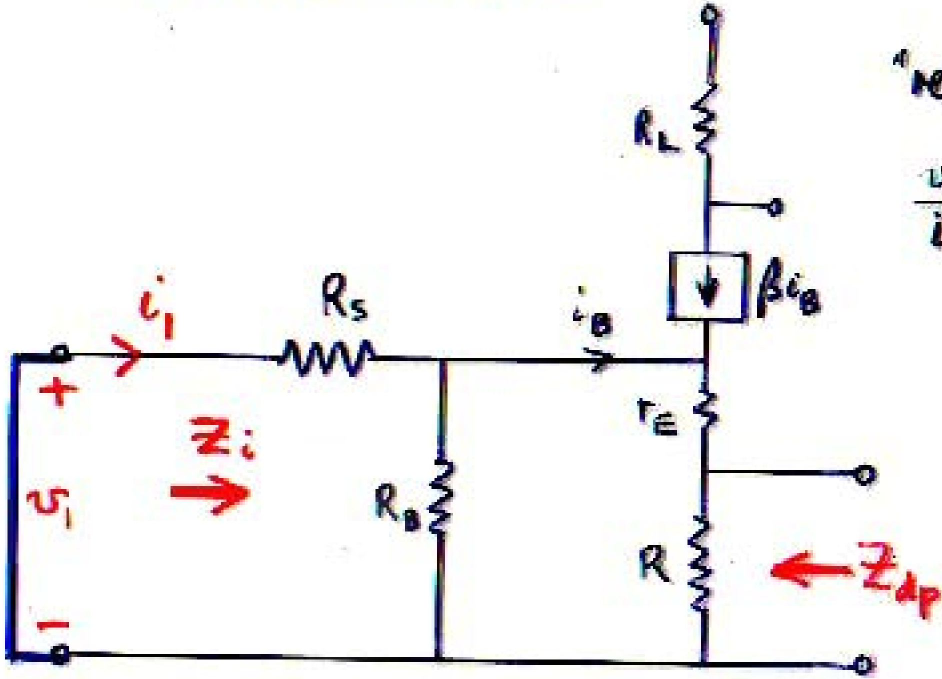
"reference transfer function":

$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E + R)$$



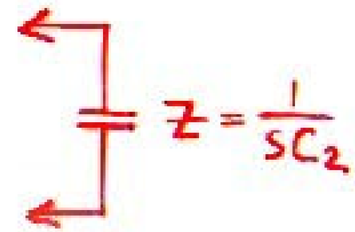
$$Z_d = \left.Z_{dp}\right|_{\text{"input" zero}} = \left.Z_{dp}\right|_{i_1=0} = R_d = R \| \left(r_E + \frac{R_B}{1+\beta}\right)$$

"Reference circuit":



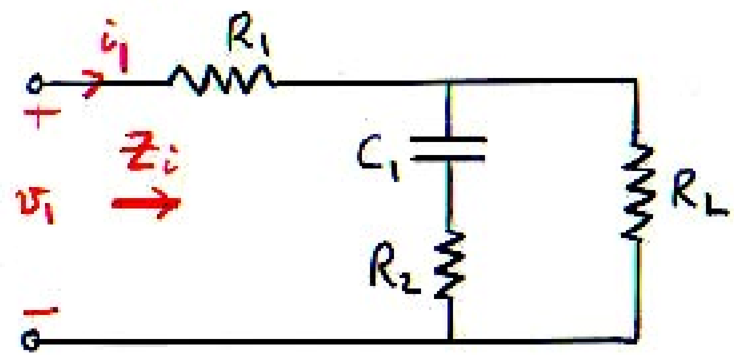
"reference transfer function":

$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E+R)$$



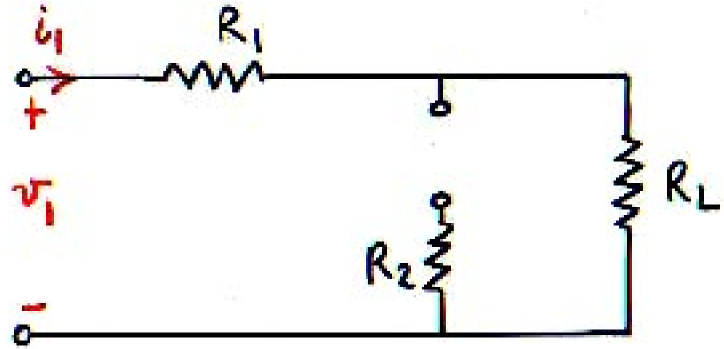
NOTE: In the special case of a self-impedance, nulling the "output" voltage is the same as shorting the "input" current, because the "output" and "input" are at the same node pair.

Example: Lag-lead network



Find the input impedance $Z_i = v_1 / i_1$ by designating C_1 as an "extra" element.

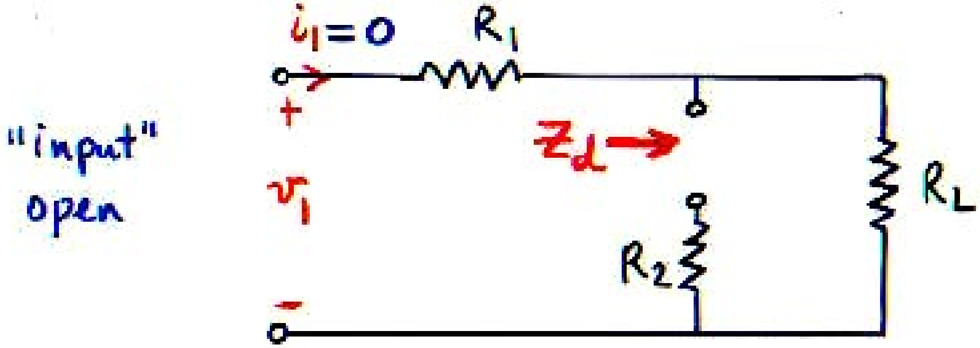
"Reference circuit:"



"Reference" input impedance:

$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1 + R_L$$

"Reference circuit:"

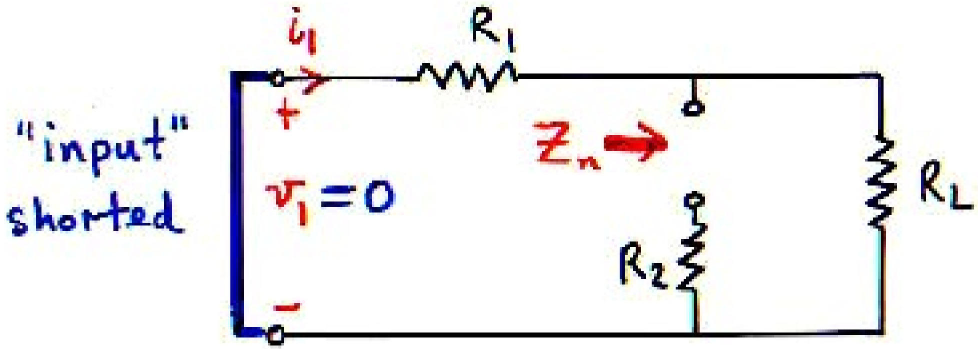


"Reference" input impedance:

$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1 + R_L$$

$$Z_d = R_d = R_2 + R_L$$

"Reference circuit:"



"Reference" input impedance:

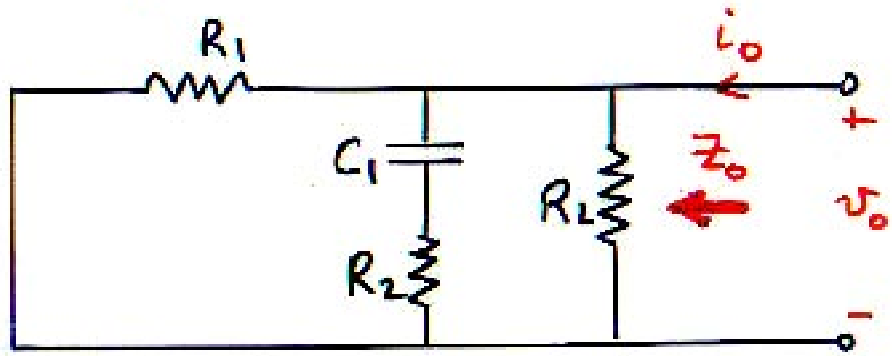
$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1 + R_L$$

$$Z_n = R_n = R_2 + R_1 \| R_L$$

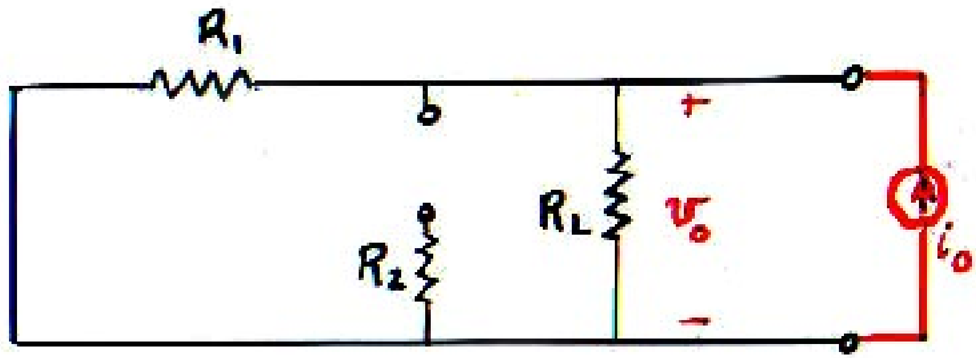
Hence:

$$Z_i = (R_1 + R_L)\,\frac{1 + sC_1R_n}{1 + sC_1R_d}$$

Exercise: Lag-lead network

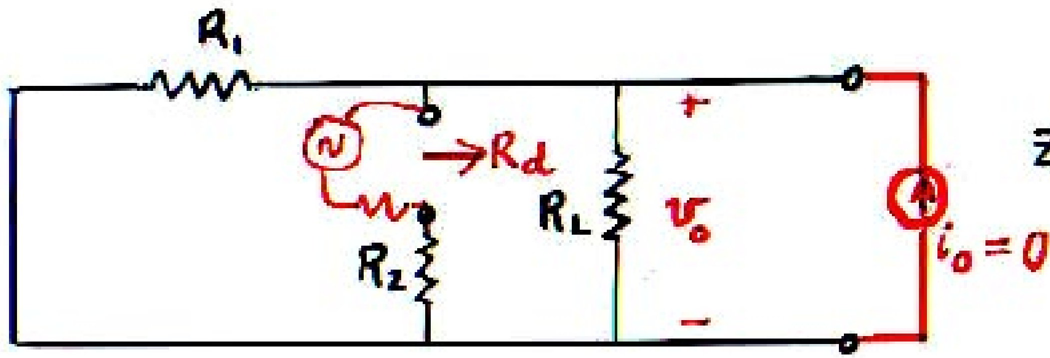


Find the output impedance $Z_o = v_o / i_o$ by designating C_1 as an "extra" element.



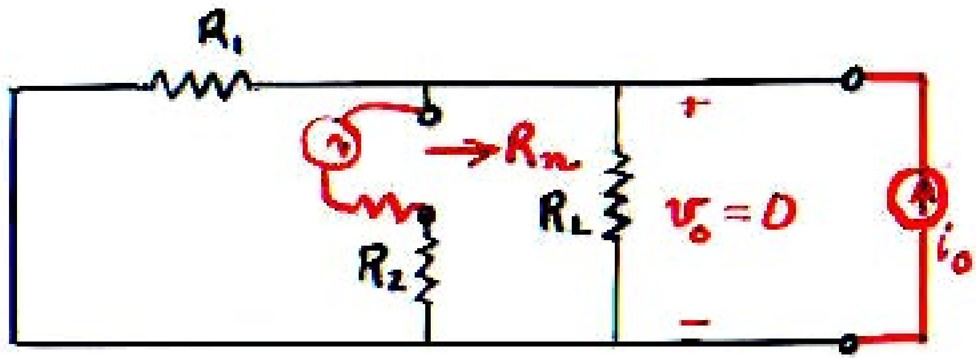
$$Z_o\Big|_{Z=\infty} \equiv \frac{v_o}{i_o}\Big|_{Z=\infty}$$

$$= R_1 \| R_L$$



$$Z_o\Big|_{Z=\infty} \equiv \frac{v_o}{i_o}\Big|_{Z=\infty}$$
$$= R_1 \| R_L$$

$$R_d = R_2 + R_1 \| R_L$$



$$Z_o\Big|_{Z=\infty} \triangleq \frac{v_o}{i_o}\Big|_{Z=\infty}$$
$$= R_1 \| R_L$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$

With C_1 replaced:

$$Z_o = R_1 \| R_L \, \frac{1 + sC_1 R_2}{1 + sC_1 (R_2 + R_1 \| R_2)}$$

Generalization: Extra Element Theorem - #4

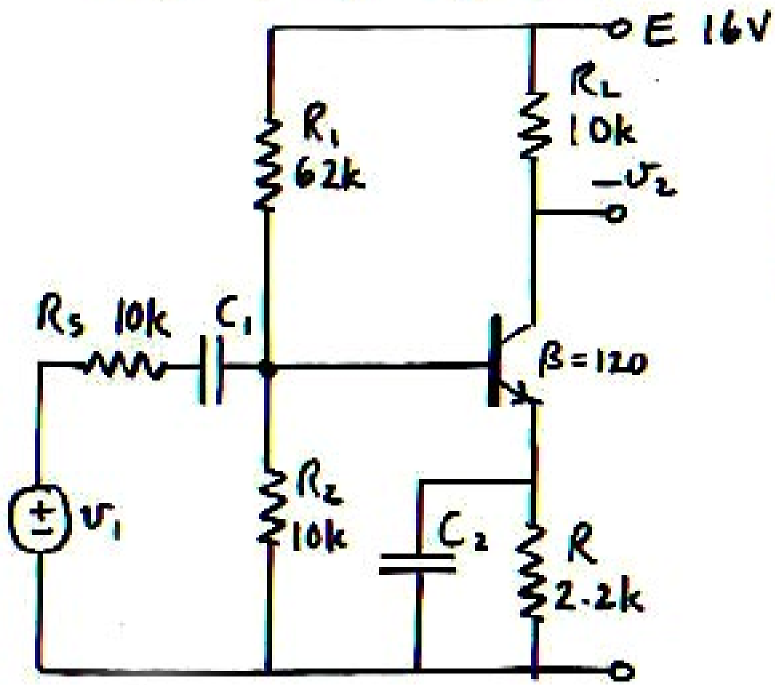
The Extra Element theorem can be used to find an extra element correction factor for any transfer function; Z_d and Z_n are then the driving point impedances seen by the extra element with the "input" zero and with the "output" nulled, respectively.

When the transfer function is a self-impedance, such as the input impedance Z_i or the output impedance Z_o, nulling the "output" is the same as shorting the "input," hence

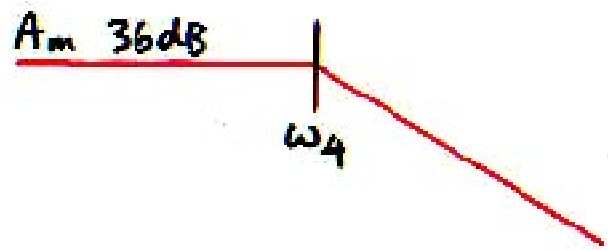
$$Z_d = Z_{dp}\Big|_{\substack{\text{"input"}\\ \text{zero}}} = Z_{dp}\Big|_{\substack{\text{"input"}\\ \text{open}}}$$

$$Z_n = Z_{dp}\Big|_{\substack{\text{"output"}\\ \text{nulled}}} = Z_{dp}\Big|_{\substack{\text{"input"}\\ \text{shorted}}}$$

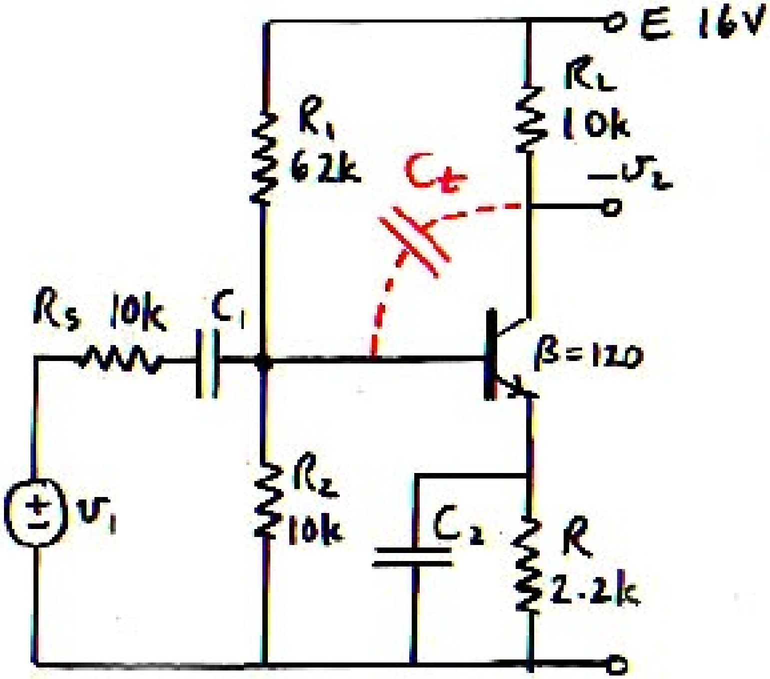
High-frequency properties of CE amplifier



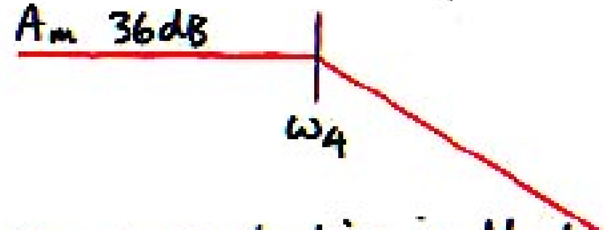
Measurement indicates that there is a high-frequency pole ω_4:



High-frequency properties of CE amplifier



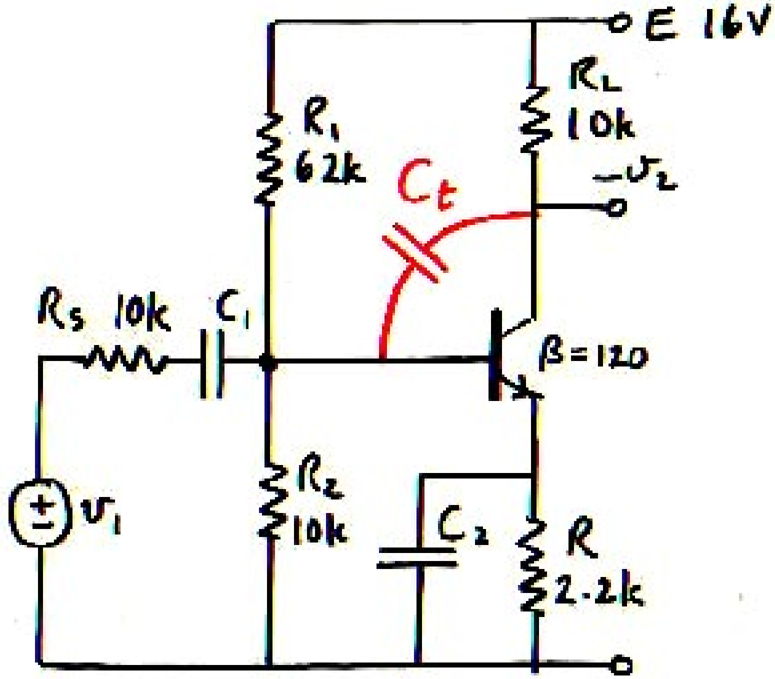
Measurement indicates that there is a high-frequency pole ω_4:



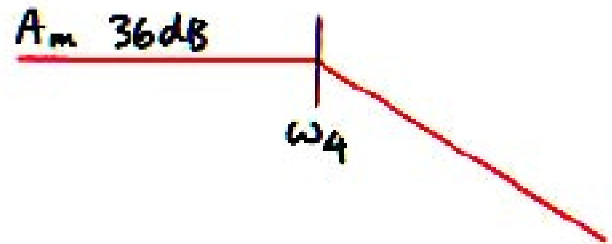
The expectation is that this is caused by the collector-base transition-layer capacitance C_t.

A typical value is $C_t = 5\text{pF}$. The resulting corner frequency with $R_L = 10\text{k}$ is $159/5 \times 10^{-6} \times 10 = 3.2\,\text{MHz}$. Since the actual corner frequency is much lower, there must be a multiplying effect on C_t resulting from its connection to the transistor base instead of to ground.

High-frequency properties of CE amplifier

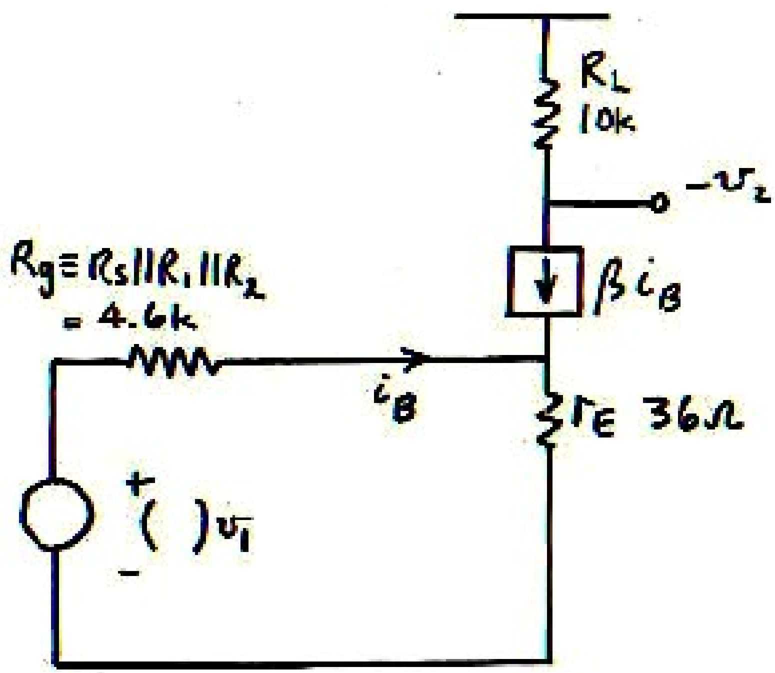


Measurement indicates that there is a high-frequency pole ω_4:

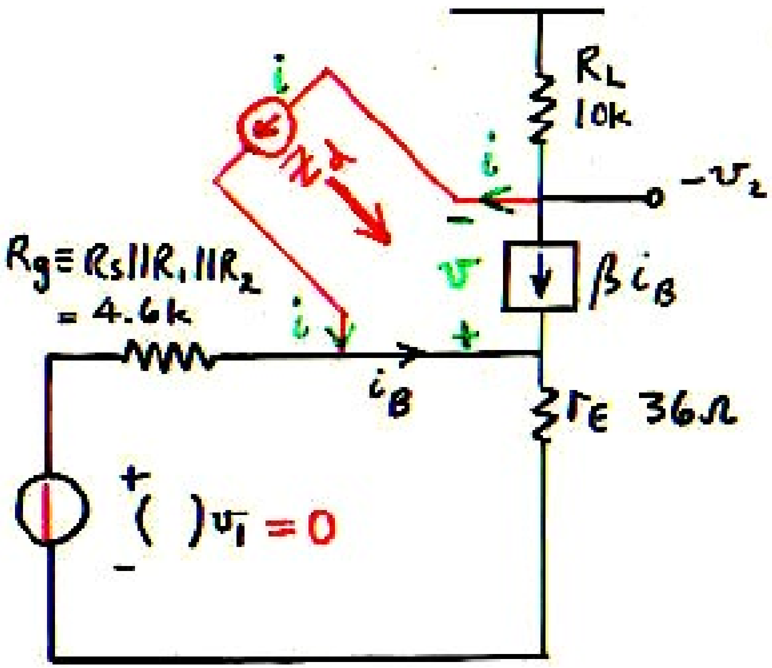


Since the midband gain $A_m = 36\text{dB}$ has already been determined, use the Extra Element Theorem to find the correction factor resulting from inclusion of $Z \to 1/sC_t$.

Midband model after Thevenin reduction of R_S, R_1, R_2:

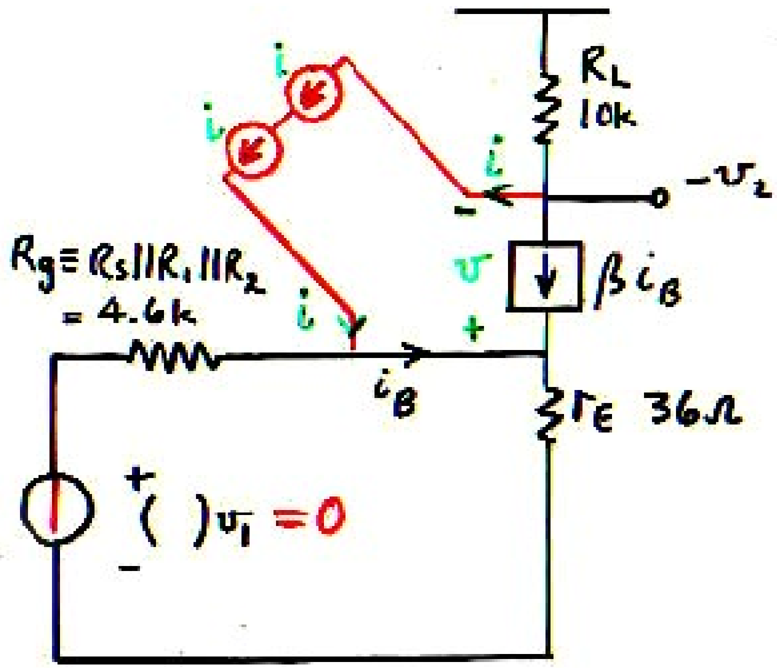


Midband model after Thevenin reduction of R_S, R_1, R_2:



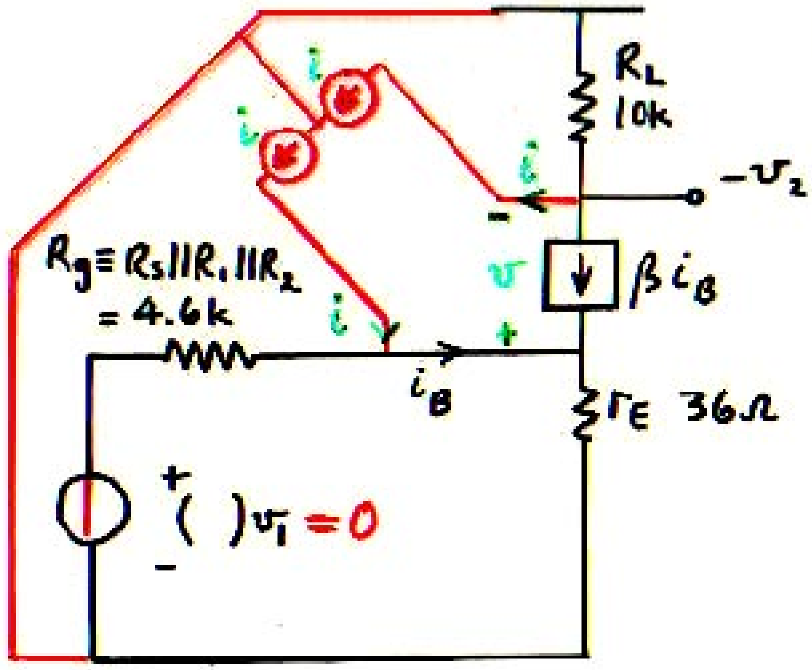
$$Z_d = \left.\frac{v}{i}\right|_{v_1 = 0}$$

Midband model after Thevenin reduction of R_S, R_1, R_2:



The current generator i can be divided into two equal current generators in series.

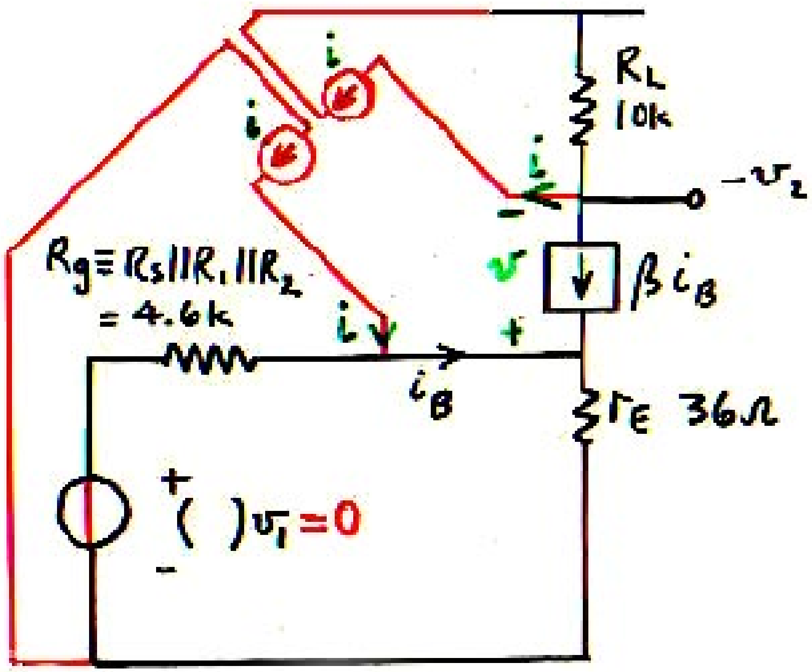
Midband model after Thevenin reduction of R_s, R_1, R_2:



The current generator i can be divided into two equal current generators in series.

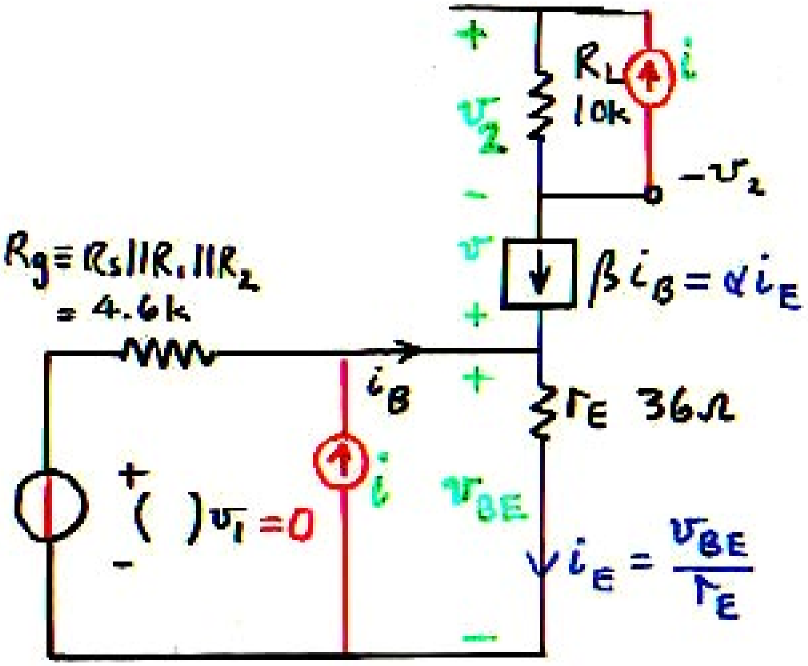
Since the voltage at the junction of the two current generators i is immaterial, the junction can be grounded.

Midband model after Thevenin reduction of R_S, R_1, R_2:



A separate ground can be identified for each current generator i.

Midband model after Thevenin reduction of R_s, R_1, R_2:



Rearranged diagram.

$$Z_d = R_d = \frac{v}{i} = \frac{v_{BE}}{i} + \frac{v_2}{i}$$

$$v_{BE} = [R_g \| (1+\beta) r_E]\, i$$

$$v_2 = R_L(\alpha i_E + i) = R_L\left(\frac{\alpha}{r_E} v_{BE} + i\right)$$

$$R_d = \frac{v_{BE}}{i} + R_L\left(\frac{\alpha}{r_E}\frac{v_{BE}}{i} + 1\right)$$

$$R_d = \left(1 + \frac{\alpha R_L}{r_E}\right)[R_g \| (1+\beta) r_E] + R_L = R_L [R_g \| (1+\beta) r_E]\left[\frac{1}{R_L} + \frac{\alpha}{r_E} + \frac{1}{R_g \| (1+\beta) r_E}\right]$$

$$= R_L [R_g \| (1+\beta) r_E]\left[\frac{1}{R_L} + \frac{\beta}{(1+\beta) r_E} + \frac{1}{R_g} + \frac{1}{(1+\beta) r_E}\right]$$

$$= \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} R_L = \frac{4.6 \| 4.3}{4.6 \| 0.036 \| 10} R_L = 62 R_L$$

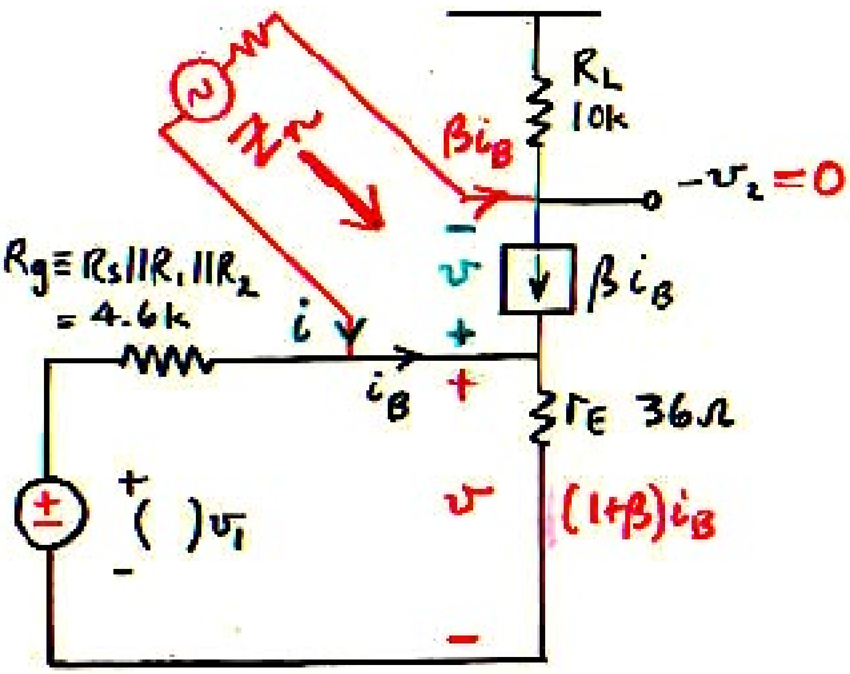
$$= 620k$$

Generalization: Floating Current Generator

A floating current generator can be replaced by two separate, equal, grounded current generators.

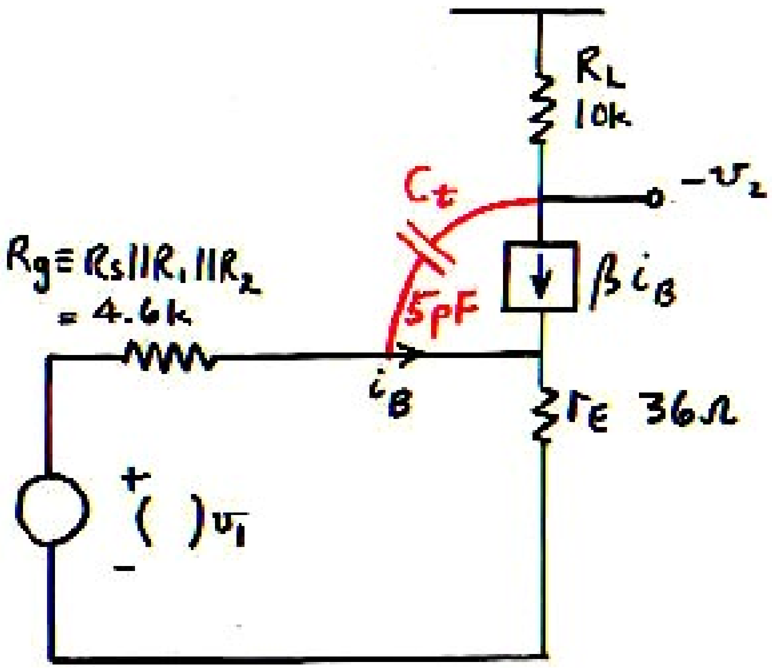
This is a useful technique in "doing the algebra on the circuit diagram."

Midband model after Thevenin reduction of R_s, R_1, R_2:



$$Z_n = R_n = \frac{v}{i} = \frac{(1+\beta) r_E i_B}{-\beta i_B} = -\frac{r_E}{\alpha} = -36\Omega$$

Midband model after Thevenin reduction of R_s, R_1, R_2:



Hence corrected gain after inclusion of C_t is

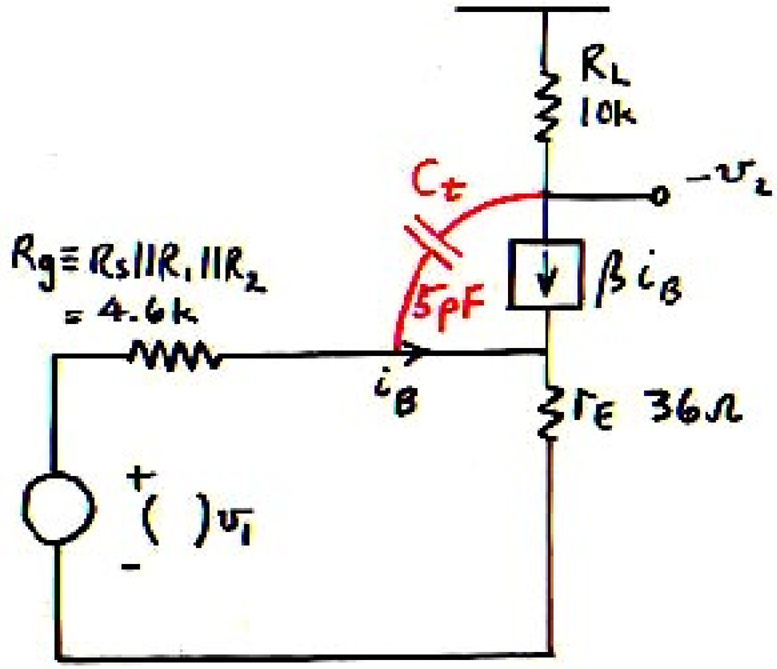
$$A = A_m \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} = A_m \frac{1 + sC_t R_n}{1 + sC_t R_d}$$

$$= A_m \frac{1 - \frac{s}{\omega_5}}{1 + \frac{s}{\omega_4}} \quad \text{where}$$

$$\omega_4 \equiv \frac{1}{C_t R_d} \qquad f_4 = \frac{159}{5 \times 10^{-6} \times 620} = 51\,\text{kHz}$$

$$\omega_5 \equiv \frac{1}{C_t R_n} \qquad f_5 = \frac{159}{5 \times 10^{-6} \times 0.036} = 880\,\text{MHz}$$

Midband model after Thevenin reduction of R_s, R_1, R_2:



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$$\omega_4 \equiv \frac{1}{C_tR_d} \qquad f_4 = \frac{159}{5\times10^{-6}\times 620} = 51\text{kHz}$$

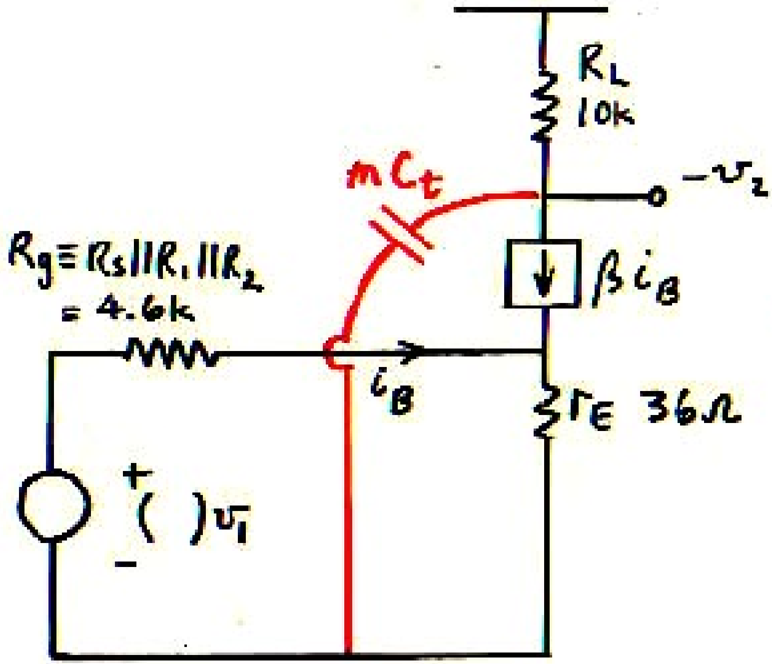
$$\omega_5 \equiv \frac{1}{C_tR_n} \qquad f_5 = \frac{159}{5\times10^{-6}\times 0.036} = 880\text{MHz}$$

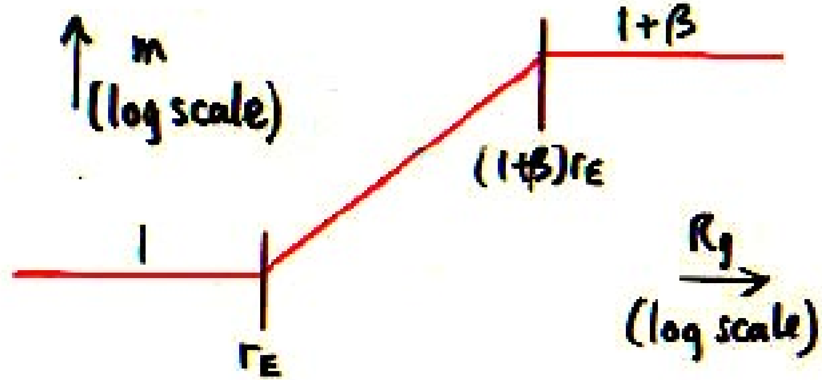
Note that the zero $\omega_5 = \frac{1}{C_tR_n} = \frac{\alpha}{C_tr_E}$ is negative (right half-plane), and is at a very high frequency unless there is substantial external emitter resistance and/or there is substantial external collector-base capacitance (as often exists).

Note that the pole $\omega_4 = \frac{1}{C_t R_d} = \frac{1}{C_t R_L}\frac{R_g \| r_E \| R_L}{R_g \| (1+\beta) r_E}$ is at a much lower frequency than $\omega_t \equiv \frac{1}{C_t R_L}$, and can be ascribed to an effective multiplication of C_t by a factor

$$m \equiv \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} = \frac{R_g \| (1+\beta) r_E}{R_g \| r_E}\left(1+\cancel{\frac{R_g \| r_E}{R_L}}\right) \begin{cases} \xrightarrow{R_g \gg (1+\beta) r_E} 1+\beta \\ \xrightarrow[R_g \ll r_E]{} 1 \end{cases}$$

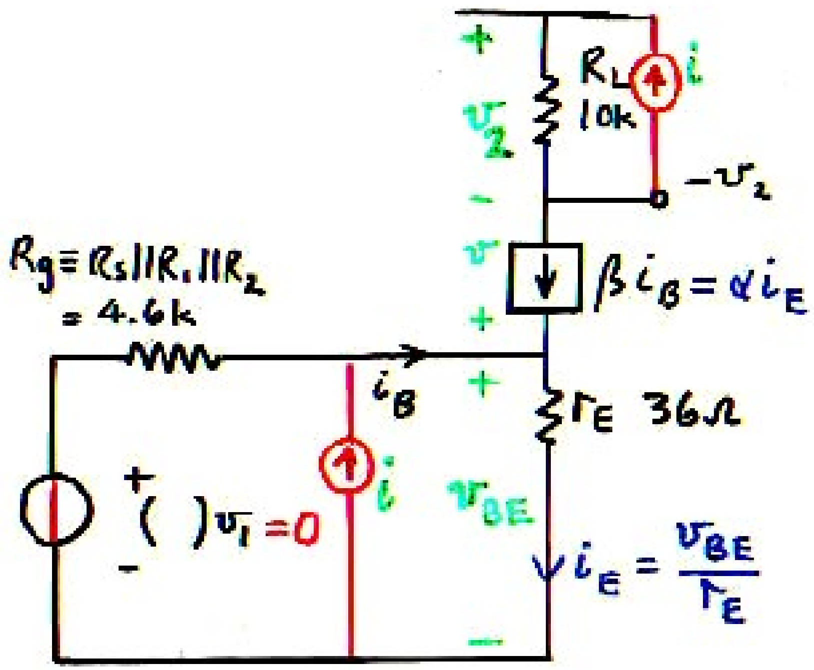
Midband model after Thevenin reduction of R_S, R_1, R_2:





This example: $m = 62$

Midband model after Thevenin reduction of R_s, R_1, R_2:



Rearranged diagram.

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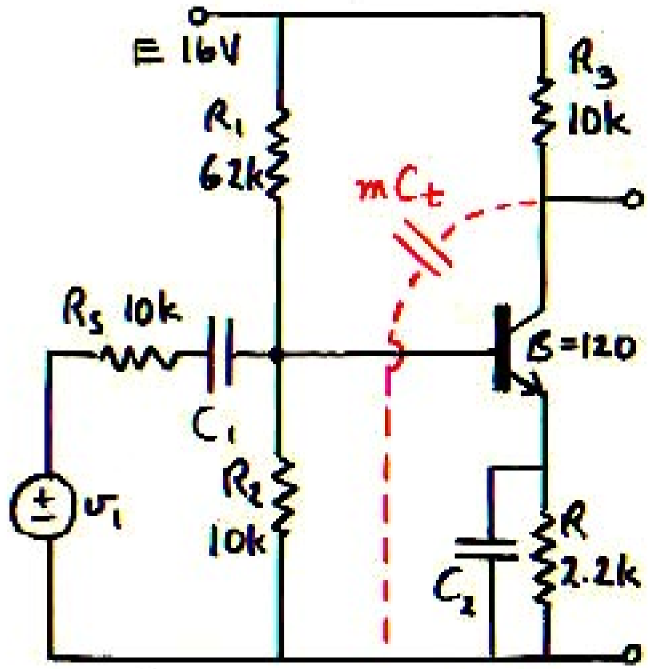
$$R_d = \frac{v_{BE}}{i} + R_L\left(\frac{\alpha}{r_E}\frac{v_{BE}}{i} + 1\right)$$

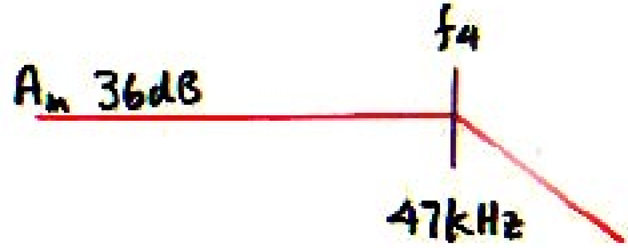
$$R_d = \left(1 + \frac{\alpha R_L}{r_E}\right)[R_g \| (1+\beta) r_E] + R_L = R_L[R_g \| (1+\beta) r_E]\left[\frac{1}{R_L} + \frac{\alpha}{r_E} + \frac{1}{R_g \| (1+\beta) r_E}\right]$$

$$= R_L[R_g \| (1+\beta) r_E]\left[\frac{1}{R_L} + \frac{\beta}{(1+\beta) r_E} + \frac{1}{R_g} + \frac{1}{(1+\beta) r_E}\right]$$

$$= \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} R_L = \frac{4.6 \| 4.3}{4.6 \| 0.036 \| 10} R_L = 62 R_L$$

$$= 620k$$





By measurement, $f_4 = 47\text{kHz}$; hence

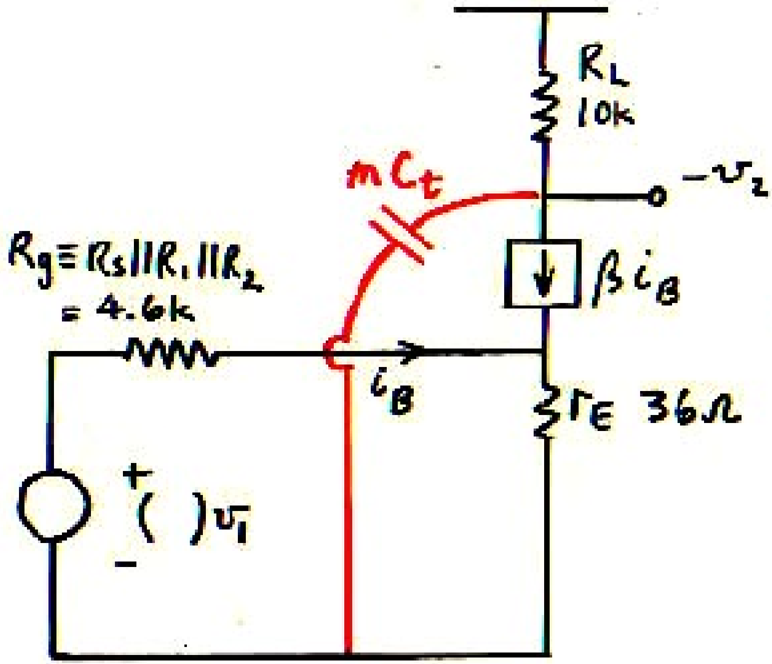
$$mC_t = \frac{159}{0.047 \times 10} = 340\text{pF}$$

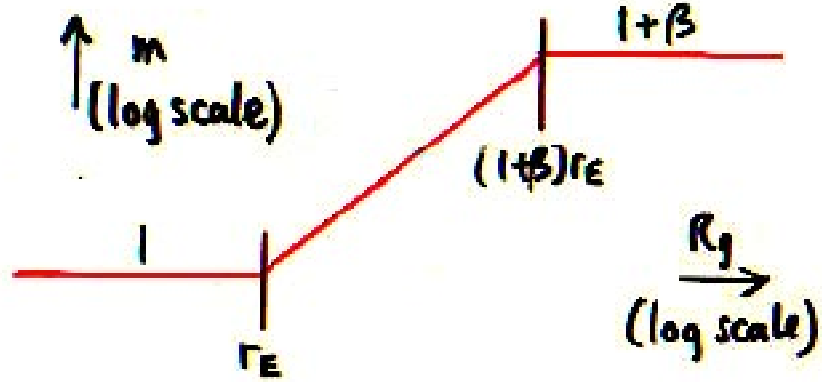
So:

$$C_t = 340/62 = 5.5\text{pF}$$

(Note: 13 pF of 10:1 scope probe is negligible)

Midband model after Thevenin reduction of R_S, R_1, R_2:





This example: m = 62

Alternative method for calculation of Z_d

There are two forms of the Extra Element theorem:

$$A = A\big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = A\big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

where
$$A\big|_{Z=0} = \frac{Z_n}{Z_d} A\big|_{Z=\infty}$$

Hence in general

$$\frac{A\big|_{Z=0}}{A\big|_{Z=\infty}} = \frac{Z_n}{Z_d}$$

It may be easier to find $A\big|_{Z=0}$, $A\big|_{Z=\infty}$, and Z_n than to find Z_d directly.

Example: Addition of collector-base capacitance C_t to the CE amplifier stage. $A\big|_{Z=\infty}$ and Z_n were easily found:
$$A\big|_{Z=\infty} = A_m = \frac{R_B}{R_S+R_B} \cdot \frac{\beta R_L}{R_g+(1+\beta) r_E} \qquad Z_n = R_n = -\frac{r_E}{\alpha}$$

The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

reference gain

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\Big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

reference gain

$$= \text{gain}\Big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} = \frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

This must be the gain when $Z=0$

Alternative method for calculation of Z_d

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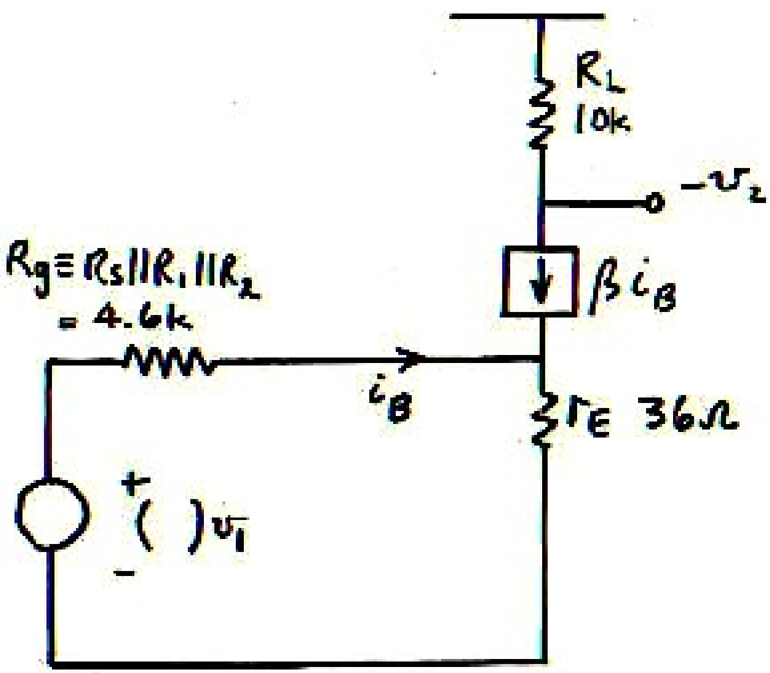
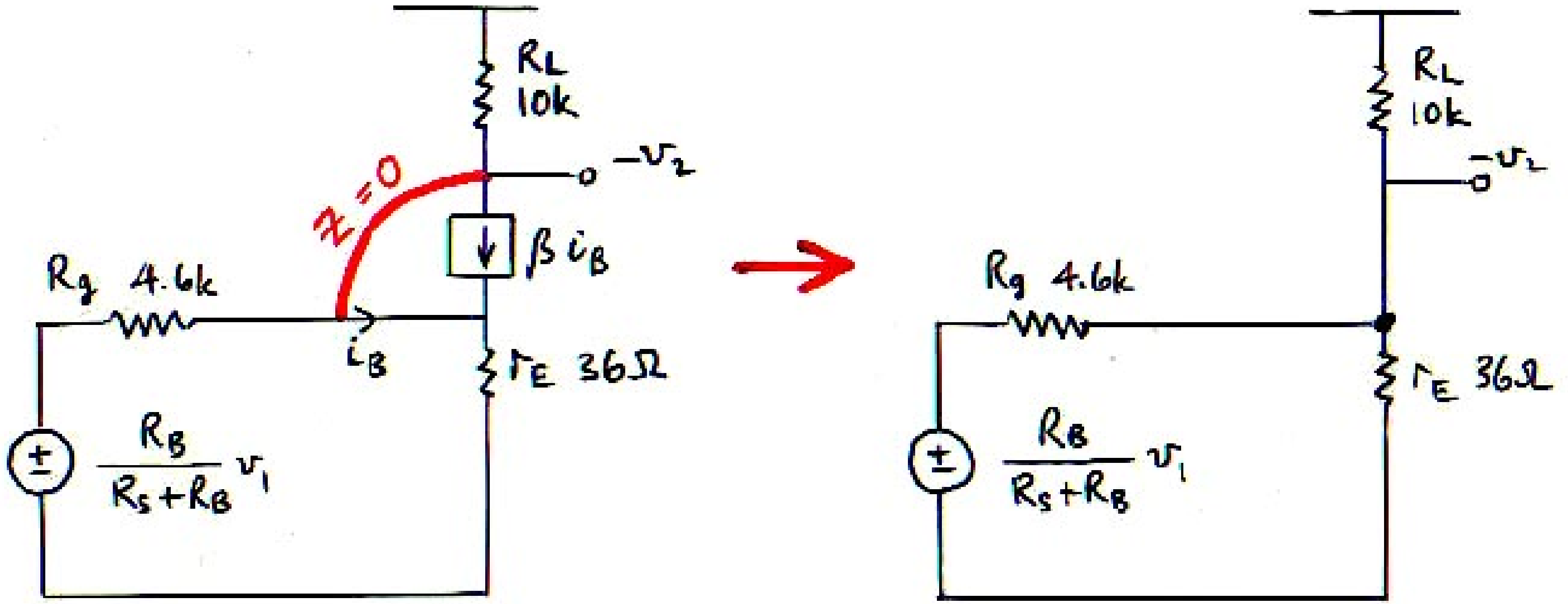
Midband model after Thevenin reduction of R_s, R_1, R_2:



Model for calculation of $A|_{z=0}$



$$A|_{z=0} = -\frac{R_B}{R_s+R_B}\,\frac{r_E\|R_L}{R_g + r_E\|R_L} = -\frac{R_B}{R_s+R_B}\,\frac{R_g\|r_E\|R_L}{R_g}$$

Hence: $$Z_d = R_d = R_n\,\frac{A|_{z=\infty}}{A|_{z=0}} = \frac{r_E}{\alpha}\,\frac{\beta R_L}{R_g+(1+\beta)r_E}\,\frac{R_g}{R_g\|r_E\|R_L} = \frac{R_g\|(1+\beta)r_E}{R_g\|r_E\|R_L}\,R_L$$

This is much easier than was the direct calculation of Z_d!

Generalization: Extra Element theorem — #5

The two reference gains and the two driving point impedances are related by:

$$\frac{A|_{Z=0}}{A|_{Z=\infty}} = \frac{Z_n}{Z_d}$$

One reference gain is always known or is easily found.
Z_n is always easier to find than Z_d.

Therefore:

It is often easier to find the other reference gain and to use the above ratio relation for Z_d, than to find Z_d directly.