Homework 1

September 12, 2021

Name: Aditya (Adi) Srikanth

Collaborators: Anna, Khevna, Andre, Tiffany, Steven, Soomin, Lev, Tanya, Changhyun

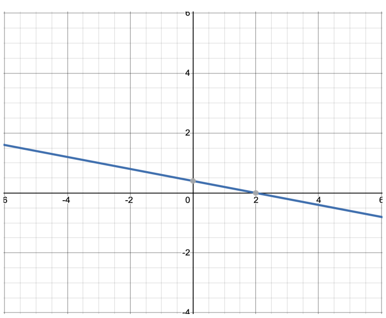
Problem 1.1

Part A

$E_1 = \{(x, y) \in R^2 \mid x + 5y = 2\}$

This is not a subspace as it violates both scalar multiplication and addition enclosure rules. To prove that this is not a subspace, we only show that scalar addition does not hold up as it is a sufficient counterexample. To illustrate, say that $e_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Both of these vectors are in E_1 and therefore $e_1 + e_2$ should also be in E_1. However, $e_1 + e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ which is not on the line $x + 5y = 2$. Therefore, E_1 is not a subspace.

A drawing of the line is below.



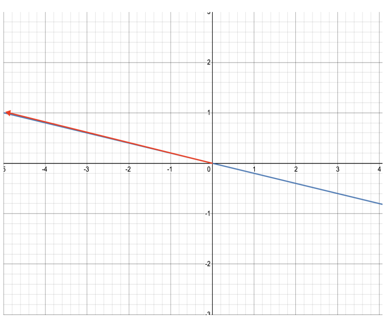
Part B

$E_1 = \{(x, y) \ \epsilon \ R^2 \mid x + 5y = 0\}$

Yes, this is a subspace. Justification is below:

1. There is at least one solution to the subspace (eg: (0,0)), therefore this space is a nonempty subset of R^2.
2. Addition Enclosed
   - Let e_1 be $x_1 + 5y_1 = 0$ and e_2 be $x_2 + 5y_2 = 0$. Both of these vectors are in E_2. If we add these together we should get an e_3 that is still in E_2 (which means it must be 0).
   - So, $e_1 + e_2 = e_3$
   - $(x_1 + 5y_1) + (x_2 + 5y_2) = e_3$
   - $(0) + (0) = e_3$
   - $e_3 = 0$
3. Scalar Multiplication Enclosed
   - Let e_1 be $x_1 + 5y_1 = 0$ and α be a scalar. e_1 is in E_2. If we multiply e_1 by α we should get a result e_2 that is still in E_2 (which means it must be 0).
   - So, $e_1 * \alpha = e_2$
   - $(x_1 + 5y_1) * \alpha = e_2$
   - $(0) * \alpha = e_2$
   - $e_2 = 0$

A drawing of the line is below.



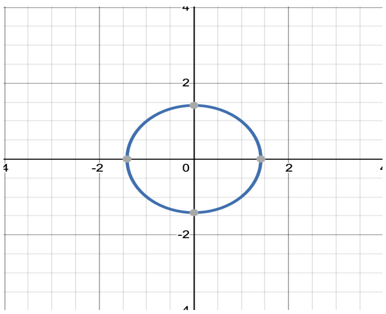
The dimension of this subspace is 1, $dim(E_2) = 1$. An example basis is $b_1 = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

Part C

$E_1 = \{(x, y) \ \epsilon \ R^2 \mid x^2 + y^2 = 2\}$

This is not a subspace as it violates both scalar multiplication and addition enclosure rules. To prove that this is not a subspace, we only show that scalar addition does not hold up as it is a sufficient counterexample. To illustrate, say that $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Both of these vectors are in E_3 and therefore $e_1 + e_2$ should also be in E_3. However, $e_1 + e_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ which is not on the line $x^2 + y^2 = 2$. Therefore, E_3 is not a subspace.

A drawing of the line is below.



Problem 1.2

Part A

$E_4 = \{(x, y, z) \in R^3 \mid x + y = 0\}$

Yes, this is a subspace. Justification is below:

1. There is at least one solution to the subspace (eg: (0,0,0)), therefore this space is a nonempty subset of R^3.
2. Addition Enclosed
   - Let e_1 be $x_1 + y_1 = 0$ and e_2 be $x_2 + y_2 = 0$. Both of these vectors are in E_4. If we add these together we should get an e_3 that is still in E_4 (which means it must be 0).
   - So, $e_1 + e_2 = e_3$
   - $(x_1 + y_1) + (x_2 + y_2) = e_3$
   - $(0) + (0) = e_3$
   - $e_3 = 0$
3. Scalar Multiplication Enclosed
   - Let e_1 be $x_1 + y_1 = 0$ and α be a scalar. e_1 is in E_4. If we multiply e_1 by α we should get a result e_2 that is still in E_4 (which means it must be 0).
   - So, $e_1 * \alpha = e_2$
   - $(x_1 + y_1) * \alpha = e_2$
   - $(0) * \alpha = e_2$
   - $e_2 = 0$

The dimension of this subspace is 2, $dim(E_4) = 2$. An example basis is $b_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 3 \\ -3 \\ 10 \end{bmatrix}$

Part B

$E_5 = \{(x, y, z) \ \epsilon \ R^3 \mid x + y = 0 \ \ \& \ \ -y + 3z = 0\}$

Yes, this is a subspace. Justification is below:

1. There is at least one solution to the subspace (eg: (0,0,0)), therefore this space is a nonempty subset of R^3.
2. Addition Enclosed
   - Let e_1 be $x_1 + y_1 = 0$ and $-y_1 + 3z_1 = 0$. Let e_2 be $x_2 + y_2 = 0$ and $-y_2 + 3z_2 = 0$. Both of these vectors are in E_5. If we add these together we should get an e_3 that is still in E_5 (which means both equations should solve to 0).
   - So, $e_1 + e_2 = e_3$
   - $(x_1 + y_1) + (x_2 + y_2) = e_3$ and $(y_1 + 3z_1) + (y_2 + 3z_2) = e_3$
   - $(0) + (0) = e_3$ and $(0) + (0) = e_3$
   - $e_3 = 0$ and $e_3 = 0$
3. Scalar Multiplication Enclosed
   - Let e_1 be $x_1 + y_1 = 0$ and $-y_1 + 3z_1 = 0$. Let α be a scalar. e_1 is in E_5. If we multiply these together we should get an e_2 that is still in E_5 (which means both equations should solve to 0).
   - So, $e_1 * \alpha = e_2$
   - $(x_1 + y_1) * \alpha = e_2$ and $(y_1 + 3z_1) * \alpha = e_2$
   - $(0) * \alpha = e_2$ and $(0) * \alpha = e_2$
   - $e_2 = 0$ and $e_2 = 0$

The dimension of this subspace is 1, $dim(E_5) = 1$. An example basis is $b_1 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$

Problem 1.3

Part A

In order to verify this claim, we must show that the canonical basis is both linearly independent and spans R^n.

1. For any canonical family (e_1 ... e_n), if we multiply each vector by some scalar α_i, we can represent the linear combination as $\begin{bmatrix} \alpha_1 \\ 0 \\ .. \\ 0 \end{bmatrix} + ... + \begin{bmatrix} 0 \\ 0 \\ .. \\ \alpha_n \end{bmatrix}$ which can be further simplified to be $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ .. \\ \alpha_n \end{bmatrix}$. The only way this can be set equal to a 0 vector is if all the α terms are 0,, which violates the rule for linear dependency. So, we conclude linear independence.

2. To show that the canonical family spans R^n, we first start with a simple case of R^2. Here, we know that $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and that $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For any x in R^2 where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we observe the following. If we multiply e_1 and e_2 by scalars α_1 and α_2, we get $x = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. So, we can construct any vector x in R^2 by setting $\alpha_1 = x_1$ and $\alpha_2 = x_2$.

We can generalize this to the case of R^n by noting that if we have $e_1 = \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}$ .. $e_n = \begin{bmatrix} 0 \\ 0 \\ .. \\ 1 \end{bmatrix}$ and scalars α_1 ... α_n we can create a linear combination and find $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ .. \\ \alpha_n \end{bmatrix}$. So, for any vector x in R^n, we can construct that vector using our canonical family simply by setting $x_i = \alpha_i$ for all i from 1 to n. Therefore, because we can create any vector x in R^n through a linear combination of the canonical family, we say that the canonical family spans R^n.

Because we show that the canonical family is linearly independent and spans R^n, we can show that the canonical family is indeed a basis of R^n.

The dimension of R^n is n, by rule.

Part B

- An example of a hyperplane in R^n can be represented as the Span(e_1, e_2, ..., e_{n-1}).
- An example of a line in R^n can be represented as the Span(e_1).

Problem 1.4

Part A

We show that q $\leqslant$ n with the following logic:

- We know that $dim(R^n) = n$, shown both by rule and also as a result of the canonical basis of R^n. Therefore, there must exist some linearly independent family $(x_1, ..., x_n)$ that Spans R^n (this family would be a basis of R^n).
- We are given that $(v_1, \ ... \ , v_q)$ are linearly independent and we know that the basis of R^n is also linearly independent.
- We now set up a proof by contradiction.
- We know by a given Lemma that for any family $(x_1, \ ... \ , x_n)$ and a family $(v_1, \ ... \ , v_q)$ ϵ Span$(x_1, \ ... \ , x_n)$, if q > n, then $(v_1, \ ... \ , v_q)$ are linearly dependent.
- We can say hypothetically that $(x_1, \ ... \ , x_n)$ is the basis of R^n, and $(v_1, \ ... \ , v_q)$ is of its span, R^n (this hypothetical is allowed by the information that is given to us). In this case, by the previous Lemma, if q > n, $(v_1, \ ... \ , v_q)$ would be linearly dependent.
- Because we know that $(v_1, \ ... \ , v_q)$ is not linearly dependent, we can also say that q > n cannot be true. Instead, we can conclude that q $\leqslant$ n

Part B

We show that p $\geqslant$ n with the following logic:

- We are given that $dim(Span(u_1, ..., u_p)) \leqslant p$
- We are also given that $Span(u_1, ..., u_p) = R^n$
- We can simplify to $dim(R^n) \leqslant p$
- We know by rule that $dim(R^n) = n$
- So we substitute back and conclude that $n \leqslant p$ or put differently, p $\geqslant$ n

Part C

Given the information we have, we begin our justification in cases to show that m = n.

Case 1: m < n

The second given statement says that $(v_1, ... , v_q, u_1, ... , u_{m-q}, u_i)$ is linearly dependent for any possible u_i. However, if m < n, this statement cannot be guaranteed. If m is less than n, the family of vectors $(v_1, ... , v_q, u_1, ... , u_{mq}, u_i)$ will not be greater than n total vectors. For any family of vectors in R^n we can only guarantee that the family will be linearly dependent if it has more than dim(R^n) (or, n) vectors (by given Lemma). Otherwise, it is possible that the family of vectors $(v_1, ... , v_q, u_1, ... , u_{mq}, u_i)$ is linearly independent, which violates the given statements (specifically statement c(ii)).

Case 2: m > n

If m > n then we have a clear violation. With m > n, the number of vectors in $(v_1, ... , v_q, u_1, ... , u_{mq})$ will be greater than n. We know that the $dim(R^n) = n$. By rule, every basis must be n vectors long. By the same logic laid out in 1.4(a), we also know that if any family of vectors in R^n is more than n vectors, it cannot be linearly independent. However, this contradicts our given statements (specifically part c(i)). Therefore, we know that m cannot be greater than n.

Case 3: m = n

Finally we reach the case of m = n. If m = n, then we know that $(v_1, ... , v_q, u_1, ... , u_{mq})$ is a total of n vectors long. We can ensure this by the following: $q + (m - q) = m = n$. Because we are given that $(v_1, ... , v_q, u_1, ... , u_{mq})$ is linearly independent and also has n vectors, we can conclude by rule that this family is a basis of R^n. This also means that if we add a term to the family, as is done in $(v_1, ... , v_q, u_1, ... , u_{mq}, u_i)$, this new family will have $n + 1$ terms in it. Also by rule, this must be a linearly dependent family. Therefore, when m = n (and only when m = n), we satisfy both conditions given to us.

Part D

A basis in R^4 can be the following: $(e_1,\, e_2,\, u_2,\, u_3)$, where:

- $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
- $u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- $u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

This can be justified by row reducing the concatenation of vectors $(e_1,\, e_2,\, u_2,\, u_3)$. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ reduces to the identity matrix, which by rule confirms that the family of vectors is linearly independent. Even without assuming any knowledge of matrices, we can show this to be true by setting the identity matrix equal to zero. After doing this, a linear combination will show us that each α must be zero for the linear combination to equal zero. By definition, this is linear independence. Again by rule, because this family of vectors is linearly independent and is of 4 vectors, we can confirm that this family is a basis of R^4.

Problem 1.5

The first step that we take is assuming that V is some subset of G. Therefore, we define some U such that V^c = U and U is of G. We can specifically define that $V + U = G$ and that U $\bigcap$ V = empty set.

From here, we follow the logic below
1. U has some basis of vectors (u_1, ... , u_n) and V has some basis of vectors (v_1, ... , v_k). The basis of V cannot map to the basis of U and the basis of U cannot map to the basis of V. We say this because basis vectors within a vector space can only transform into other vectors within the same vector space. Because U and V are distinct, the vector spaces cannot be the same.
2. From here, we can say $dim(G) = dim(u) + dim(v)$, because the family of vectors to form a basis for two distinct spaces requires the basis of both spaces. So the dimension reflects the total of all of these vectors.
3. We know that $dim(G) = dim(V)$, so we can simplify and get that $dim(U) = 0$
4. This implies that U is the empty set. However, this means that V is not a proper subset of G (because there is no valid complement of V within G). In other words, there is nothing in G that is not already in V.
5. Because V is a subset of G but not a proper subset, V is equal to G.

To show the equivalency from the other side, we start with the fact that V = G. This means that V and G admit the same basis as they are the same vector space. Because V and G have the same basis, $dim(V) = dim(G)$. Also, because V = G, we can say that V is a subset of G and that G is a subset of V. So, we can confirm the two conditions.

Because we can show the statements are equal reflexively, we say that we have found equivalency.