PROBLEM 0.1

(a) $$\|x+y\|^2 - \|x-y\|^2 = \|x\|^2 + 2x^Ty + \|y\|^2 - \left(\|x\|^2 + 2x^Ty + \|y\|^2\right)$$
$$= 4x^Ty.$$

(b) if $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|xy\|^2$

$\Rightarrow \quad 2 \; x^Ty = 0$

PROBLEM 0.2

(a) Write the SVD $A = U\Sigma V^T$, if linearly indep column

$$\Sigma = \begin{array}{c|c} & m \\ n & \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & 0 & \ddots & \\ & & & \sigma_m \\ & & 0 & \end{pmatrix} \end{array}$$

$$A^+ = V\Sigma' U^T = V \begin{pmatrix} 1/\sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1/\sigma_m & \end{pmatrix} U^T$$

then $A^TA \in \mathbb{R}^{m\times m}$ is invertible

$$A^TA = \left(V\Sigma^T U^T U \Sigma V^T\right)^2$$
$$= V \; \Sigma^T\Sigma \; V^T \qquad \Sigma^T\Sigma = \begin{pmatrix} \sigma_1^2 & & (0) \\ & \ddots & \\ (0) & & \sigma^2 \end{pmatrix}$$

$$= V \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{pmatrix} V^T$$

$$(A^TA)^{-1} = V \begin{pmatrix} 1/\sigma_1^2 & & \\ & \ddots & \\ & & 1/\sigma_m^2 \end{pmatrix} V^T$$

$$(A^TA)^{-1}A^T = V \begin{pmatrix} 1/\sigma_1^2 & & \\ & \ddots & \\ & & 1/\sigma_m^2 \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ 0 & & \sigma_m & \end{pmatrix} U^T$$

$$= V \begin{pmatrix} 1/\sigma_1 & & & \\ & \ddots & & 0 \\ & & 1/\sigma_m & \end{pmatrix} U^T = A^+$$

$\boxed{\text{TRUE}}$

(b) $\begin{cases} Av_1 = \lambda v_1 \\ Av_2 = \lambda v_2 \end{cases} \Rightarrow \forall v \in \text{Span}(v_1, v_2)$

$v = \alpha v_1 + \beta v_2$ for some $\alpha, \beta \in \mathbb{R}$.

$$Av = \alpha A v_1 + \beta A v_2$$
$$= \lambda(\alpha v_1 + \beta v_2)$$
$$= \lambda v \quad \checkmark$$

$\boxed{\text{TRUE}}$

(c) This a subspace $\Rightarrow$ convex. $\boxed{\text{TRUE}}$

(↳ showed it in midterm!)

(d) $\boxed{\text{FALSE}}$

Take $A \in \mathbb{R}^{n \times m}$ $\quad T(-A) = \|-A\|_F = \|A\|_F = T(A)$

PROBLEM 0.3

(a) False

$$\nabla f(x) = \begin{vmatrix} x_2 \\ x_1 \end{vmatrix} \qquad H_f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\mathrm{Tr}\left(H_f(x) \right) = 0 = \lambda_1 + \lambda_2$ but λ_1 and λ_2 are not both 0 since $H_f(x)$ is rank 2.

$\lambda_1 = -\lambda_2 \implies$ both signs.

(b) Take $g(x) = \frac{1}{2}x^2$ and $f(x) = x$ and $\alpha = -1$

$\implies h(x) = -\frac{1}{2}x^2 + x \rightarrow$ concave. $h''(x) = -1$.

not convex. $\boxed{\text{FALSE}}$

(c) True $\begin{cases} -f(x) \text{ concave} \\ \log(x) \text{ concave} \end{cases} \longrightarrow$

$g(x) = \log(x)$

$g'(x) = 1/x$

$g''(x) = -1/x^2 < 0$.

Sum of concave is concave.

PROBLEM 0.4.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$ matrix with $\begin{cases} 2 \text{ row} \\ 2 \text{ colums} \end{cases}$

Compute its SVD:

$$AA^T = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} = 5 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\begin{cases} Tr(AA^T) = 5 \times 5 = 25 \\ rank(AA^T) = 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 25 & \sigma_1 = 5 \\ \lambda_2 = 0 & \sigma_2 = 0 \end{cases}$$

<u>eigenvectors</u>: $AA^T v_2 = 0$ (will give us the left singular vectors).

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} x - 2y = 0 \\ 2x - 4y = 0 \end{cases} \Rightarrow u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

v_1 is necessarily orthogonal to that –

we are in dimension 2 there is not much more there is only one orthogonal direction

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Déduce the right singular vectors:

$$v_1 = \frac{1}{\sigma_1} A^T u_1 = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \times \frac{1}{\sqrt{3}}$$

$$= \frac{1}{3\sqrt{3}} \begin{pmatrix} 3 \\ -10 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/\sqrt{5} \\ 2\sqrt{5}/\sqrt{3} \end{pmatrix}$$

$v_2 \to$ orthogonal to $v_1 \to \begin{pmatrix} -2\sqrt{5}/\sqrt{3} \\ \sqrt{3}/\sqrt{5} \end{pmatrix}$.

$\hookrightarrow$ So we have it all:

$$A = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 5 & \\ & 0 \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^T.$$

PROBLEM

- min $e^{x} + e^{-y}$ subject to $x - 2y = 0$ $\longrightarrow$ convex problem.

$$\ell = e^{x} + e^{-2y} + \nu(x - 2y)$$

$$\nabla \ell = \begin{vmatrix} e^{x} + \nu \\ 2e^{-2y} - 2\nu \end{vmatrix} \Rightarrow x^*, y^* \text{ solution } \Leftrightarrow \begin{cases} (e^{x^*} + \nu) = 0 \\ 2e^{y^*} + 2\nu) = 0 \\ x^* + 2y^* = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^* = \log(-\nu) \\ y^* = \log(-\nu) \\ \log(-\nu) + 2\log(-\nu) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \nu = -1 \\ x^* = 0 \\ y^* = 0 \end{cases}$$ unique solution

- Minize $x^2 + y^2 - 2y + 2z^2$ suject to

$x + y + z \geqslant 1$

convex.

$$\ell = x^2 + y^2 - 2y + 2z^2 + \lambda(-x + y + z + 1)$$

$$\nabla \ell = \begin{vmatrix} 2x - \lambda \\ 2y - 2 - \lambda \\ 2z - \lambda \end{vmatrix}$$

$$\nabla \ell = 0 \longrightarrow \begin{cases} x = +\lambda/2 = z \\ y = \dfrac{+\lambda + 1}{2} \end{cases}$$

$$\lambda\left(+\frac{\lambda}{2} + \frac{+\lambda+1}{2} + \frac{\lambda}{2} - 1\right) = 0$$

$$\begin{cases} \lambda = 0 \\ \text{or} \\ +\lambda + \lambda + \lambda + \frac{1}{2} - 1 = 0 \end{cases} \Longrightarrow \begin{cases} \lambda = 0 \\ \text{or} \\ \lambda = +1/6 \end{cases}$$

if $\lambda = 0 \Rightarrow x + y + z = 0 + 1/2 + 0 \leq 1 \Rightarrow$ not in feasible set.

if $\lambda = 1/6$ $\begin{cases} x = z = 1/12 \\ y = \dfrac{1/6 + 1}{2} = \dfrac{7}{12} \end{cases}$ unique solution.

PROBLEM 0.6:

$x \leq -1$

$0 < \alpha < 1 \rightarrow (1+x)^\alpha \leq 1 + \alpha x$

Define $f(x) = (1+x)^\alpha \Rightarrow f'(x) = (\alpha \quad)(1+x)^{\alpha-1}$

$$\Rightarrow f''(x) = \underbrace{(\alpha \quad)(\alpha-1)}_{\leq 0.} \underbrace{(1+x)^{\alpha-2}}_{\geq 0}$$

f is concave $\rightarrow$ it will be below its tangent everywhere.

in particular at $x = 0$

$f(x) \simeq f(0) + f'(0)\, x$

$:= 1 + \alpha x.$

Same in the opposite direction $\rightarrow$ convex function

$(1+x)^\alpha - 1 - \alpha x = 0$

$f'(x) = 0 \longrightarrow \alpha(1+x)^{\alpha-1} - \alpha = 0 \qquad x=0$ solution

$f''(x) \longrightarrow \alpha(\alpha-1)(1+x)^{\alpha-2}$

will also be strictly positive or strictly negative depending on the values of α

strictly convex / concave.

one unique minimize / maximizer.

PROBLEM 0.7:

$f(x) = \| Ax - y \|^2$ $\longrightarrow$ start at some x_0

$$x_1 = x_0 - Hf^{-1}(x_0) \nabla f(x_0)$$

$Hf(x_0) = 2A^TA$ $\quad$ $\nabla f(x_0) = 2A^TAx_0 - 2A^Ty$

$$\Rightarrow x_1 = x_0 - (A^TA)^{-1}(A^TAx_0 - A^Ty)$$

$$\Rightarrow x_1 = (A^TA)^{-1}A^Ty = x_{LS}$$ 1 step!