- This is set of review problem for the final. It could give you an idea of what types of questions you will get at the final (length has not been calibrated here).
- We will discuss solutions in class next week.
- The review questions from 2019 should also be a good training to revise the concepts (links in last year's website).

Problem 0.1 (points)**.** *Manipulations of norms and inner products.*

(a) *Show that for any $x, y \in \mathbb{R}^n$ we have $x^T y = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$.*

(b) *Prove the following converse to the Pythagorean theorem: For any x, y in $\mathbb{R}^n$ if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ then $\langle x, y\rangle = 0$.*

Problem 0.2 (points)**.** *Correct/Incorrect. Justify.*
Consider $A \in \mathbb{R}^{n\times m}$ with $n \geq m$.

(a) *If A has linearly independent columns, then $A^\dagger = (A^\top A)^{-1}A^\top$.*

(b) *Suppose $m = n$ and v_1 and v_2 are eigenvectors with the same eigenvalue, then any vector in $\mathrm{Span}(v_1, v_2)$ is also an eigenvector of A.*

(c) *The set of symmetric matrices is convex.*

(d) *The map $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times m} & \to & \mathbb{R} \\ A & \to & \|A\|_F \end{array} \right.$ (Frobenius norm) is a linear map.*

Problem 0.3 (points)**.** *Correct/Incorrect. Justify or give a counter example.*
Consider f and g two convex, twice differentiable, functions from $\mathbb{R}^n \to \mathbb{R}$ and

(a) *The function $h(x_1, x_2) = x_1 x_2$ defined on $\mathbb{R}^2$ is convex.*

(b) *The function $h(x) = f(x) + \alpha g(x)$ is convex for any $\alpha \in \mathbb{R}$.*

(c) *For $n = 1$, the function $h(x) = -f(x) + \log(x)$ is concave.*

Problem 0.4 (points)**.** *Compute the singular value decomposition of $A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$.*

Problem 0.5. *Solve the following constrained minimization problems*

(a) $x + y + z$ *subject to* $e^{-x} + e^{-y} \leq 1$ *and* $z + 2y = 0$

(b) $x^2 + y^2 - 2y + z^2$ *subject to* $x + y + z \leq 1$

Problem 0.6. *Let $x \leq -1$ a real number.*

(a) *Show that for $0 < \alpha < 1$, $(1+x)^\alpha \leq 1 + \alpha x$.*

(b) *Show that for $\alpha > 1$, $(1+x)^\alpha \geq 1 + \alpha x$.*

(c) *Show that $(1+x)^\alpha = 1 + \alpha x$ if and only if $x = 0$.*

Problem 0.7. *Let $A \in \mathbb{R}^{n\times d}$, assume that the columns of A are linearly independent. How many steps of Newton's method do you need to minimize $\|Ax - y\|^2$, for y a fixed vector in $\mathbb{R}^n$? Justify your answer.*