Recitation 3 Solution

1. (Exponential distribution, Gaussian distribution and the corresponding ML estimates)

2. (Estimating uniform parameters)

   (a) The pdf for uniform(a, b) one data value is $f(x_i|a,b) = \frac{1}{b-a}$ if x_i is in the interval[a, b] and 0 if it is not. So the likelihood function for our 5 data values is

   $$f(data|a,b) = \begin{cases} \frac{1}{(b-a)^5} & \text{if all data is in [a,b]} \\ 0 & \text{otherwise} \end{cases}$$

   This is maximized when $(b-a)$ is as small as possible. Since all the data has to be in the interval [a, b] we minimize $(b-a)$ by taking a = minimum of data and b = maximum of data. So a=1.2, b=10.5.

   (b) The same logic as in part (a) shows a = min$(x_1, ..., x_n)$ and b = max$(x_1, ..., x_n)$.

3. (Customer waiting time)

   (a)
   $$P(1 \le r \le 2) = \int_1^2 2e^{-2r} dr = -e^{-2r}|_1^2 = e^{-2} - e^{-4}$$

   (b)
   $$P(R \le r) = \int_0^r 2e^{-2u} du = -e^{-2u}|_0^r = 1 - e^{-2r}.$$

   So the cdf for R is

   $$F_R(r) = \begin{cases} 1 - e^{-2r} & \text{for } r \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

   (c) First, we find the cdf of T. T takes value in $(0, \infty)$
   for 0< t:

   $$F_T(t) = P(T \le t) = P(\frac{1}{R} < t) = P(\frac{1}{t} < R) = 1 - F_R(\frac{1}{t}) = e^{-2/t}$$

   We then differentiate to get

   $$f_T(t) = \frac{d}{dt}(e^{-2/t}) = \frac{2}{t^2} e^{-2/t}$$

4. (Bus waiting time)

   Let X1, X2, X3 be random variables denoting the number of minutes you have to wait for bus 1, 2, or 3 respectively. They have uniform distribution: X1, X2, X3 $\sim$ Uniform(0,10). T = min(X1, X2, X3). Let f_1, f_2, f_3 be the probability density functions for X1, X2, X3, and let F_1, F_2, F_3 be the cumulative distribution functions. Let g be the probability density function for T, and G be the cumulative distribution function for T. Then for x $\in$ [0, 10]

   $$f_i(x) = \frac{1}{10}$$

$$F_i(x) = \int_0^x \frac{1}{10} du = \frac{x}{10}$$

For t $\in$ [0, 10], the cdf is:

$$\begin{aligned}
G(t) = P(T \leq t) &= 1 - P(T > t) && (1)\\
&= 1 - P(min(X1, X2, X3) > t) && (2)\\
&= 1 - P((X1 > t) \cap (X2 > t) \cap (X3 > t)) && (3)\\
&= 1 - P(X1 > t) * P(X2 > t) * P(X3 > t) && (4)\\
&= 1 - (1 - \frac{t}{10})^3 && (5)
\end{aligned}$$

The pdf is the derivative of cdf:

$$g(t) = \frac{dG(t)}{dt} = \frac{3}{10}(1 - \frac{t}{10})^2$$

5. (Continuous Parametric Modeling)

(a) Similar to last week, the ML estimate of a and b for a uniform distribution are:

$$\begin{aligned}
a_{ML} &= \min(x_1, ...x_n) = 1 && (6)\\
b_{ML} &= \max(x_1, ...x_n) = 7 && (7)\\
&\Rightarrow U[1, 7] && (8)
\end{aligned}$$

(b) The ML estimate for λ in an exponential distribution is the inverse of the mean:

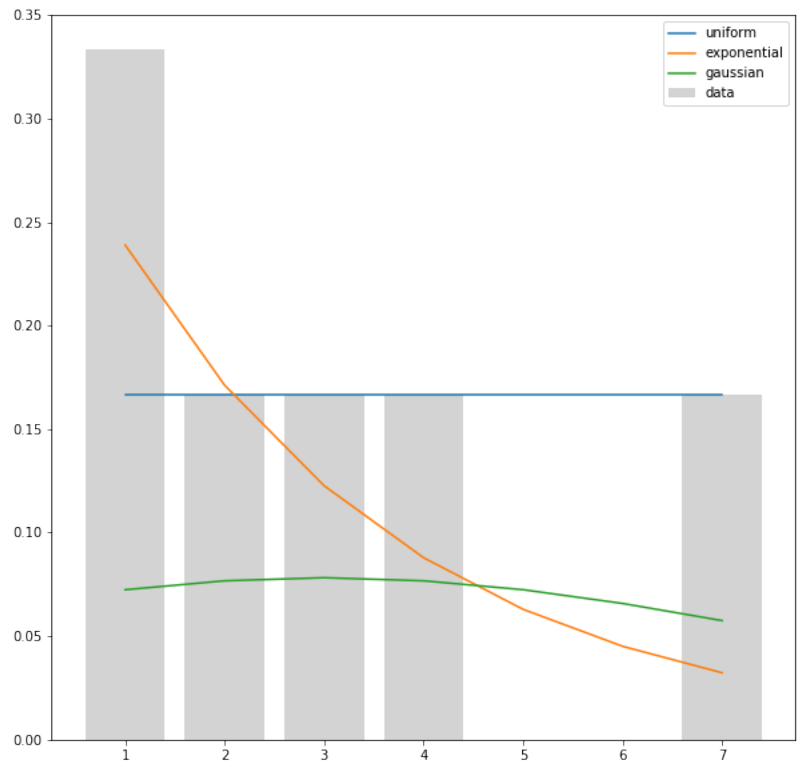
$$\lambda_{ML} = \frac{n}{\sum_{i=1}^n x_i} = \frac{6}{18} = \frac{1}{3} \Rightarrow Exp(\frac{1}{3}) \tag{9}$$

(c) The ML estimates for a gaussian distribution are:

$$\mu_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = 3 \tag{10}$$

$$\sigma_{ML} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \mu_{ML})^2} \approx 5.1 \tag{11}$$

(d) Plotting the data and 3 curves:



6. (Inverse sampling)

   (a) To transform each sample, we use Eq(152) from the notes:

   $$F_{\tilde{a}}^{-1}(u) = \frac{1}{\lambda}\log(\frac{1}{1-u}) \tag{12}$$

   For $u = 0.2$: $F_{\tilde{a}}^{-1}(u) = \log(\frac{1}{0.8}) = 0.2231$. The 5 samples are: 0.2231, 0.2877, 0.5108, 0.9163, 2.3026

   (b) Similar to part a: for $u = 0.2$: $F_{\tilde{a}}^{-1}(u) = \frac{1}{4}\log(\frac{1}{0.8}) = 0.0558$. The 5 samples are: 0.0558, 0.0719, 0.1277, 0.2291, 0.5756 . Note that these are much more tightly compressed than the samples in part a, since they are just divided by 4.

   (c) Trick question! 5 samples generated from the CDF do not indicate anything about the conditional likelihood of a given model, or set of model parameters. The point to take home from this question is that these generated samples don't tell us anything about which parameters fit best, maximize likelihood , etc.

   (d) Since $\mu = 2$ and $\sigma = 1$, we can use the lookup table after standardizing:

   $$z = \frac{u-2}{1} \tag{13}$$

   Once we look up each CDF in the table and find the corresponding z-value, we add back the mean to get our samples. For u=0.2:

   $$\Phi^{-1}(0.2) = -\Phi^{-1}(0.8) \approx -0.84 \Rightarrow x = -0.84 + 2 = 1.16 \tag{14}$$

   The 5 samples are (approximately): 1.16, 1.3, 1.75, 2.25, 3.28.