Report - Assignment 10

Akshay Anand (EE16B046)

April 28, 2018

Abstract

In this assignment, the fourier transform of some functions, with and without a windowing function is found out and their graphs plotted. Without windowing, for some functions, whose peak is not at any point in the discrete time interval taken, the peak appears flat, and not entirely accurate, whereas with windowing, for those function, the peaks become visible, though they appear broad as a multiplication in time domain, done by the Windowing function, results in a convolution in the frequency domain, which results in broader peaks. The Windowing function used is a Hamming window, given by the equation:

$$w[n] = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) & |\ n\ | \leq \frac{N-1}{2} \\ 0 & else \end{cases}$$

Libraries Used

```
import numpy as np
import numpy.fft as f
import matplotlib.pyplot as plt
import math
import mpl_toolkits.mplot3d.axes3d as p3
pi = math.pi
np.random.seed(5)
```

Plotting Function Defined

```
def plot_function(w,Y,x_lim, function):
    fig, axes = plt.subplots(2, 1, figsize=(15, 7), sharex = True)

    plt.suptitle("The DFT plots for " + function, fontsize=18)
    # The magnitude plot is plotted
    axes[0].plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
    axes[0].set_xlim([-x_lim,x_lim])
    axes[0].set_ylabel(r"$|Y|$",size=16)
    axes[0].set_title("Spectrum of " + function, fontsize=14)
    axes[0].grid(True)

    # The Phase plot is plotted
    ii=np.where(abs(Y)>1e-3)
    axes[1].plot(w[ii],np.angle(Y[ii]),'ro',lw=2)
    axes[1].set_xlim([-x_lim,x_lim])
    axes[1].set_ylim([-4,4])
    axes[1].set_ylabel(r"Phase of $Y$",size=16)
    axes[1].set_title("Phase Plot of " + function, fontsize=14)
    axes[1].set_xlabel(r"$\omega$",size=16)
    axes[1].grid(True)
    plt.show()
```

1 Example plots are plotted

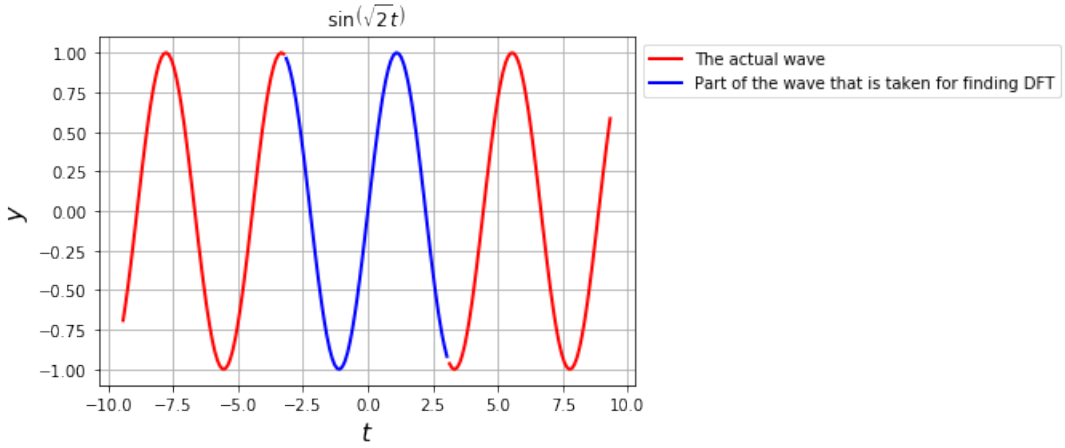
The example plots given in the question ($\sin(\sqrt{2}x)$) is plotted with Hamming window and without Hamming window to know the difference. First of all, a Hamming window is required for this particular function because, even though it

is periodic, its ω is $\sqrt{2}$, and hence if we plot the points that are sampled by the program, repeated periodically (whose fourier transform is actually found) the function is not the same as the continuous time $\sin(\sqrt{2}x)$, as shown below.

Actual $\sin(\sqrt{2}x)$ plot is as follows:

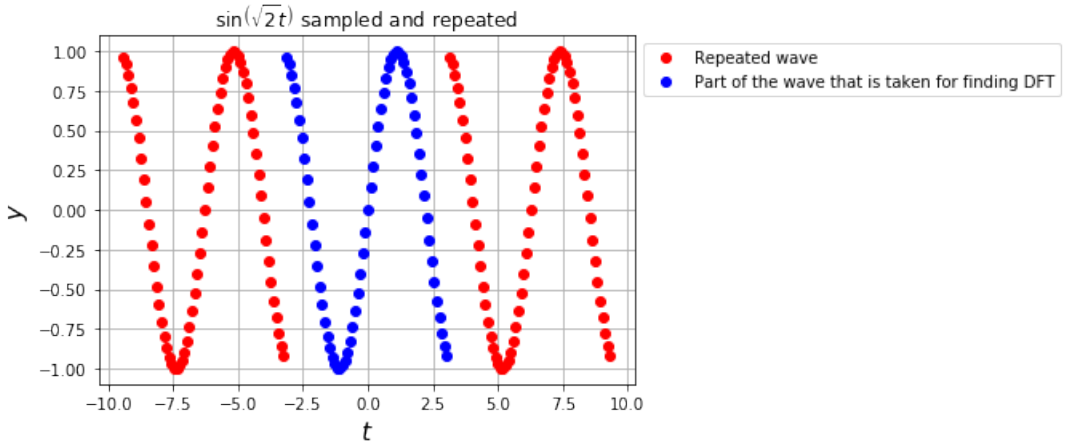
```
t1=np.linspace(-pi,pi,65);t1=t1[:-1]    # The interval used to find the DFT.
t2=np.linspace(-3*pi,-pi,65);t2=t2[:-1]
t3=np.linspace(pi,3*pi,65);t3=t3[:-1]

plt.plot(t2,np.sin(np.sqrt(2)*t2),'r',lw=2)
plt.plot(t1,np.sin(np.sqrt(2)*t1),'b',lw=2)    # The interval t1 is plotted in a
    different colour.
plt.plot(t3,np.sin(np.sqrt(2)*t3),'r',lw=2)

plt.legend(('The actual wave','Part of the wave that is taken for finding DFT'),
    bbox_to_anchor=(1, 1))
plt.ylabel(r"$y$",size=16)
plt.xlabel(r"$t$",size=16)
plt.title(r"$\sin\left(\sqrt{2}t\right)$")
plt.grid(True)
plt.show()
```

Plot obtained:



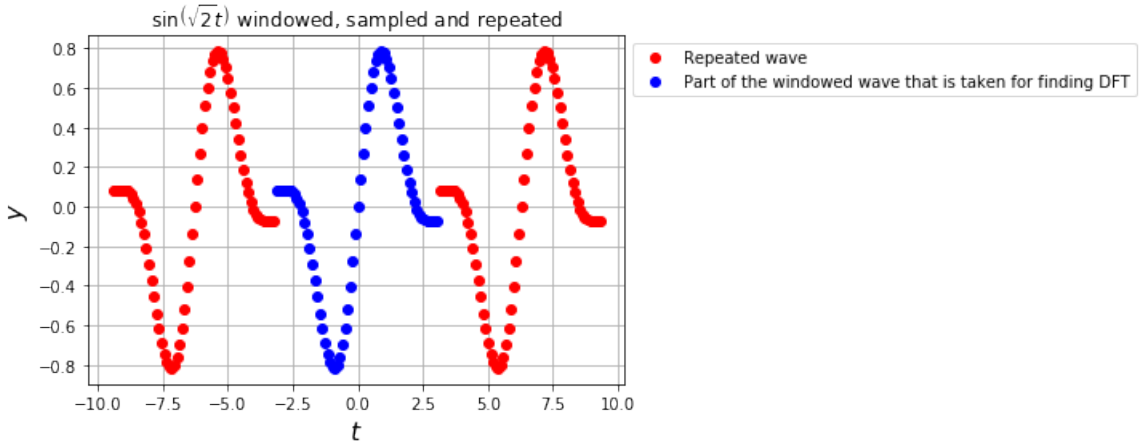
From the above plot, only the blue part is taken for DFT, and therefore, if we repeat just that part to form a periodic function, the plot obtained is:

```
t1=np.linspace(-pi,pi,65);t1=t1[:-1]    # The interval used to find the DFT.
t2=np.linspace(-3*pi,-pi,65);t2=t2[:-1]
t3=np.linspace(pi,3*pi,65);t3=t3[:-1]

y = np.sin(np.sqrt(2)*t1)

# The function on interval t1 is repeated periodically.
plt.plot(t2,y,'ro',lw=2)
plt.plot(t1,y,'bo',lw=2)
plt.plot(t3,y,'ro',lw=2)

plt.legend(('Repeated wave','Part of the wave that is taken for finding DFT'),
    bbox_to_anchor=(1, 1))
plt.ylabel(r"$y$",size=16)
plt.xlabel(r"$t$",size=16)
plt.title(r"$\sin\left(\sqrt{2}t\right)$ sampled and repeated")
plt.grid(True)
plt.show()
```



Thus, it can be seen that, in the actual plot that is used to find DFT, there is a huge discontinuity between successive periods, which is mainly due to the fact that $\sqrt{2}$ is an irrational number and therefore can't be plotted correctly in a discrete time waveform. It is this discontinuity that poses problems when calculating DFT, as Gibb's phenomenon occurs here. The main aim of windowing is to reduce this discontinuity without affecting the rest of the graphs very much, and as seen from the DFT plots below, it does work to a large extent. Before that, just to show, the same graph above, after windowing, the plot in time domain is as follows:

```
t1=np.linspace(-pi,pi,65);t1=t1[:-1]    # The interval used to find the DFT.
t2=np.linspace(-3*pi,-pi,65);t2=t2[:-1]
t3=np.linspace(pi,3*pi,65);t3=t3[:-1]

n=np.arange(64)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/63))
y=np.sin(np.sqrt(2)*t1)*wnd

# The function on interval t1 is windowed and repeated periodically.
plt.plot(t2,y,'ro',lw=2)
plt.plot(t1,y,'bo',lw=2)
plt.plot(t3,y,'ro',lw=2)
plt.legend(('Repeated wave','Part of the windowed wave that is taken for finding
    DFT'),bbox_to_anchor=(1, 1))
plt.ylabel(r"$y$",size=16)
plt.xlabel(r"$t$",size=16)
plt.title(r"$\sin\left(\sqrt{2}t\right)$ windowed, sampled and repeated")
plt.grid(True)
plt.show()
```

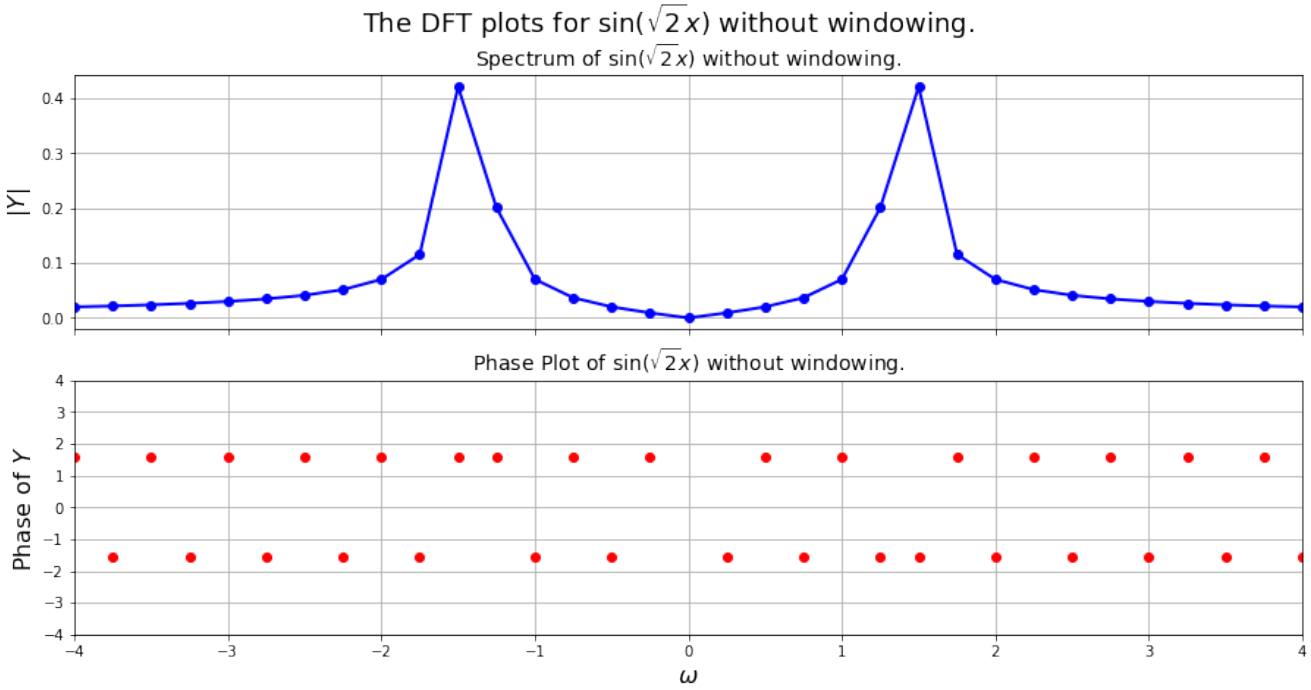


Thus, it is seen that, after windowing, the overall shape of the graph is preserved with a much reduced discontinuity. Now, the DFT of the function is calculated with and without windowing and observations are noted.

1.1 Plot without Hamming window

```
t = np.linspace(-4*pi,4*pi,257)    # Time interval declared
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
y = np.sin(np.sqrt(2)*t)           # Function defined
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/256.0
w=np.linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
plot_function(w,Y,4,'$\sin(\sqrt{2}x)$ without windowing.')
```

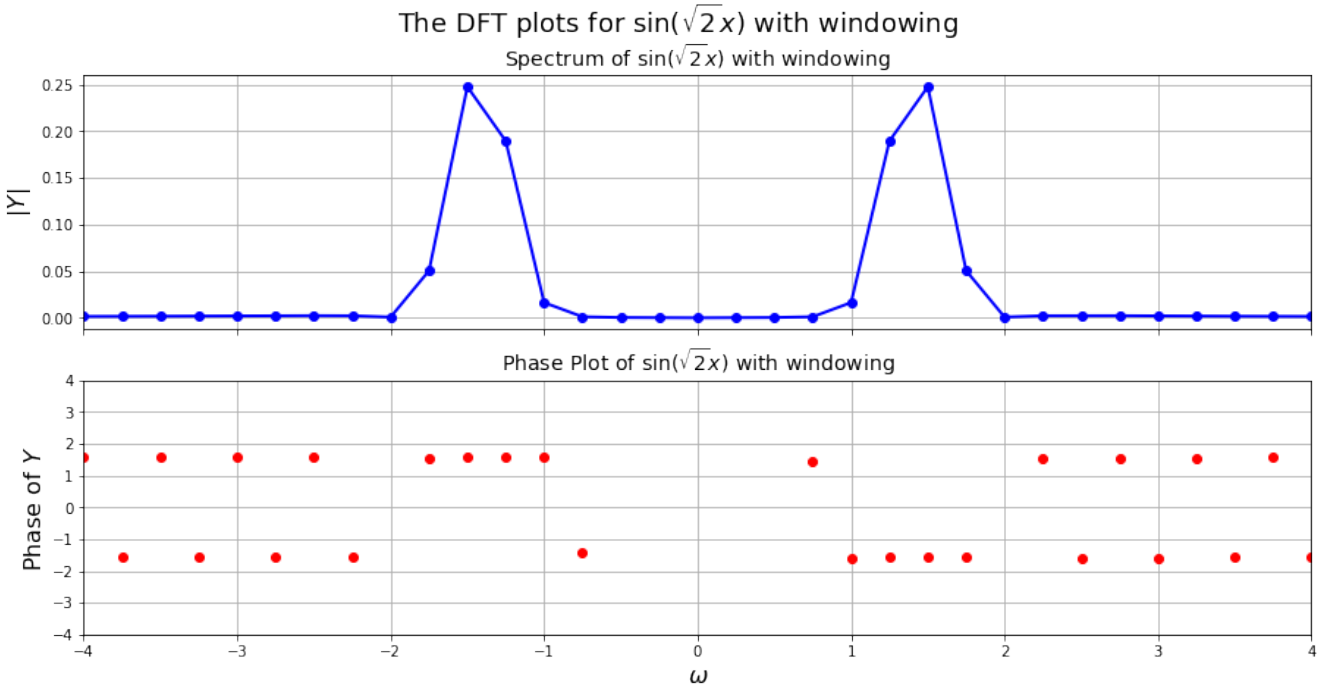
The graph obtained is:



1.2 Plot with Hamming window

```
t = np.linspace(-4*pi,4*pi,257)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
n = np.arange(256)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/255))     # Hamming window declared.
y = np.sin(np.sqrt(2)*t)
y = y * wnd                # Hamming window multipled in the time domain
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/256.0
w=np.linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
plot_function(w,Y,4,'$\sin(\sqrt{2}x)$ with windowing')
```

Graph obtained is:



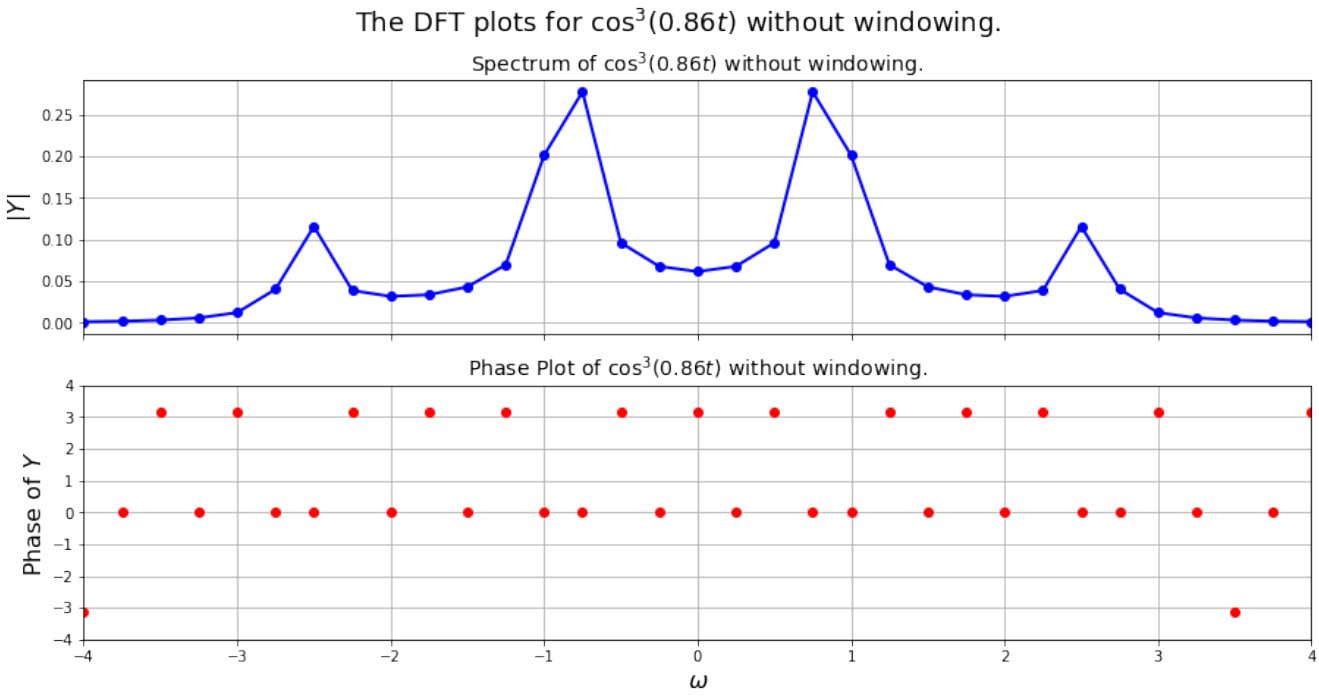
Thus, from the plots above, it can be seen that, without windowing, due to the high discontinuity, though the peaks exist, it doesn't reduce to 0 fast, and a phase exist for the wave even in between the peaks where its supposed to be 0. From the latter plot, with windowing, it is observed that, although the peak is broader (which is expected as multiplication in time domain by the windowing function is equivalent to convolution in frequency domain, which results in a broader peak), the magnitude does go to 0 fast, and there is no phase in between the peaks. Also, the obtained peak is in between 1 and 2, which is expected as ideally the peak is at $\sqrt{2} \approx 1.414$.

2 DFT of $\cos^3(0.86t)$.

Just like the previous example, 0.86 is also not in the sampled points and hence, it will also have a discontinuity, when the interval for which DFT is taken is repeated periodically, which is seen in its DFT as a wide peak, which doesn't go to 0 anywhere in between. This is because of the Gibb's phenomenon at the discontinuity, because of which, a lot of frequency components would be present in the DFT. This is very much reduced when the function is windowed as seen below.

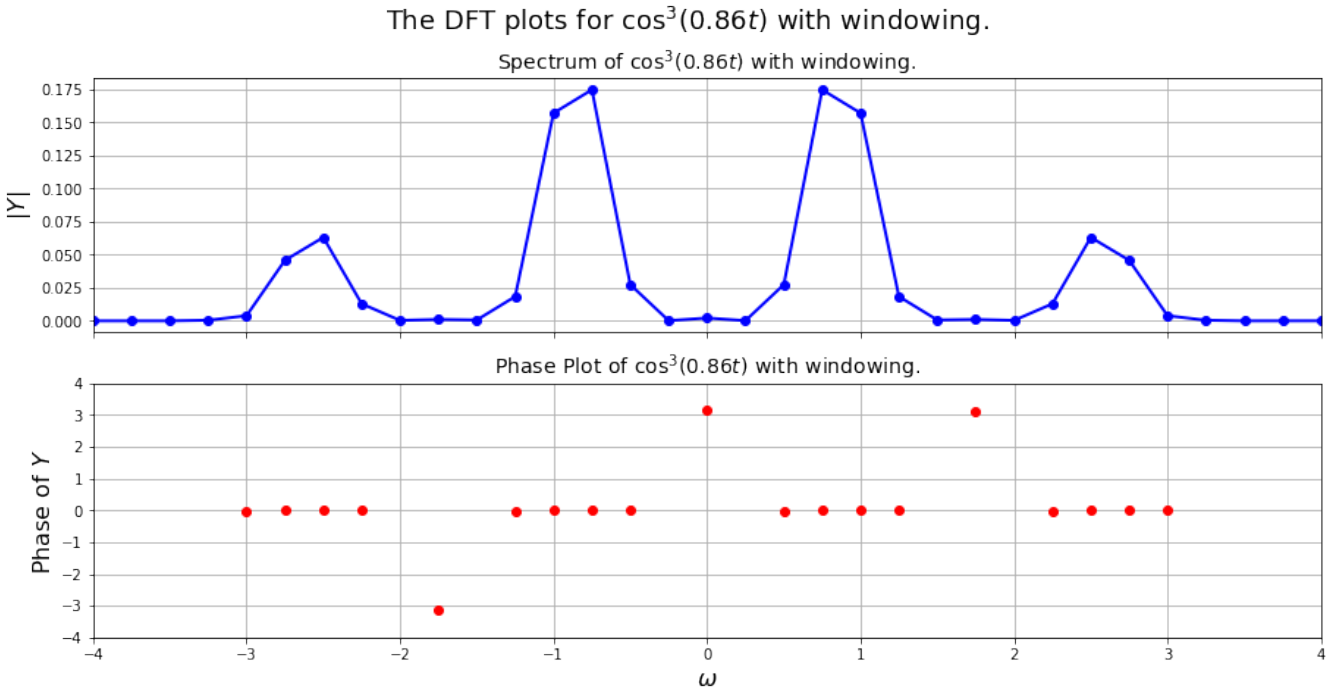
2.1 Without Windowing

```
t = np.linspace(-4*pi,4*pi,257)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
y = np.cos(0.86*t) ** 3
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/256.0
w=np.linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
plot_function(w,Y,4,'$\cos^3(0.86t)$ without windowing.')
```



2.2 With Windowing

```
t = np.linspace(-4*pi,4*pi,257)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
n = np.arange(256)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/255))
y = np.cos(0.86*t) ** 3
y = y * wnd
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/256.0
w=np.linspace(-pi*fmax,pi*fmax,257);w=w[:-1]
plot_function(w,Y,4,'$\cos^3(0.86t)$ with windowing.')
```



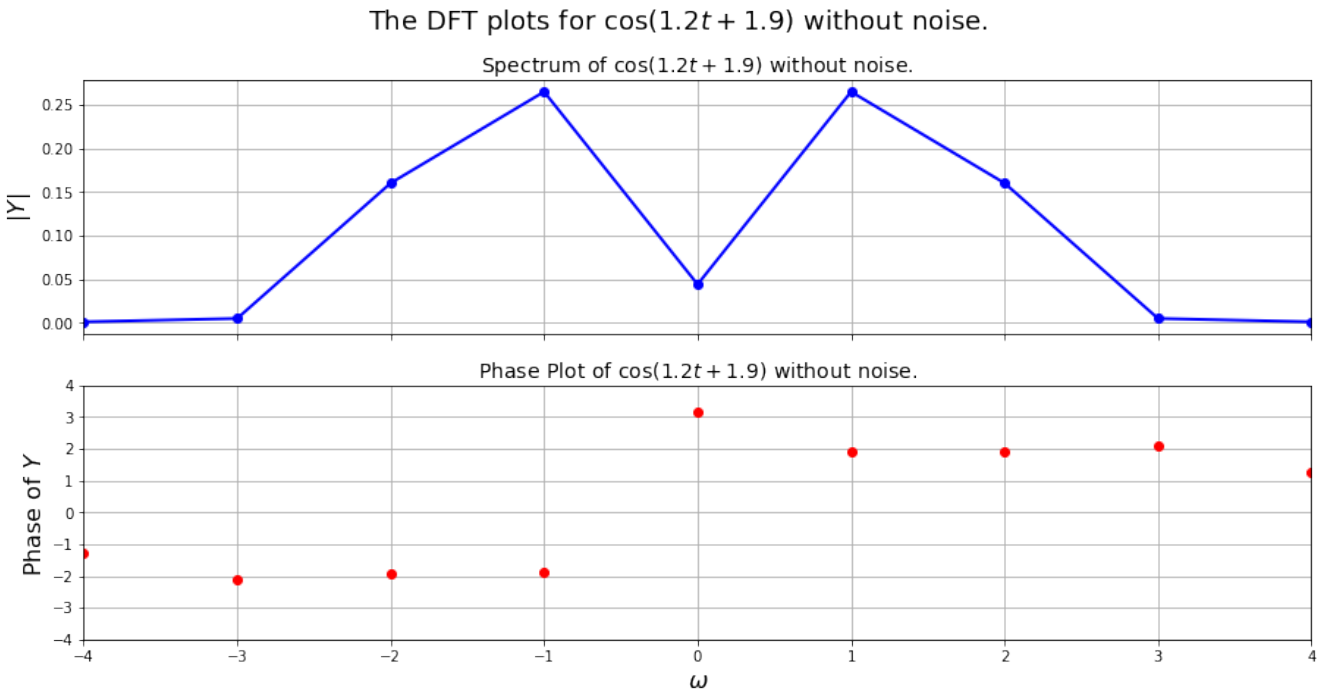
Thus, the observations stated above are verified.

3 Estimating ω_0 and δ in $\cos(\omega_0 t + \delta)$

To do this, first the DFT of the function (modelled as a 128 element vector) is plotted after windowing so as to get accurate peaks in the magnitude plot. Since, it is a cos function, the phase would be 0 normally. So δ would be the phase of the spectrum at its peaks and ω_0 would be approximately equal to the average of elements near the peak, as the peak is broadened due to windowing. Some examples are considered below:

```
# A random delta and omega declared to find error
delta = 1.9
omega = 1.2
t = np.linspace(-pi,pi,129)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
n = np.arange(128)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/127))
y = np.cos(omega*t + delta)
y = y * wnd
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/128.0
w=np.linspace(-pi*fmax,pi*fmax,129);w=w[:-1]
plot_function(w,Y,4,'$\cos(1.2t␣+␣1.9)$␣without␣noise.')
```

The graph obtained is:



Now, the values of δ and ω_0 are estimated from the calculated DFT:

```
ii = np.where(w>0)[0]     # All values of DFT above w=0 are taken (only one half of
    plot)
ii = np.where((abs(Y) == max(abs(Y[ii]))))    # The maximum in this region is taken
    as the peak.
est_delta = abs(np.angle(Y[ii])[0])    # The phase of the graph at the peak is the
    delta.

ii = np.where((abs(Y) > 3.5e-2) & (w >= 0))[0]     # The points greater than or
    equal to w=0 and with a significant magnitude are taken.
est_omega = abs(Y[ii]*w[ii])
est_omega = sum(est_omega)/(sum(abs(Y[ii])))  # As peak is spread out, omega is
    estimated as the weighted average (centre of mass) of the broad area near the
    peak (on one half of the plot).
print ('Without␣noise,␣the␣calculated␣delta␣is␣%.6f␣and␣the␣error␣in␣the␣calculated
    ␣delta␣is␣%.6f' %(est_delta, abs(est_delta - delta)))
print ('Without␣noise,␣the␣Calculated␣omega␣is␣%.6f␣and␣the␣error␣in␣the␣calculated
    ␣omega␣is␣%.6f' %(est_omega, abs(est_omega - omega)))
```

In this case, the output obtained was:

```
Without noise, the calculated delta is 1.890923 and the error in the calculated delta is 0.009077
Without noise, the Calculated omega is 1.247086 and the error in the calculated omega is 0.047086
```

Thus, with windowing, we are able to estimate the original values to a reasonable level of accuracy.

4 Estimating of ω_0 and δ with noise

Now, some white gaussian noise in the form of $0.1randn(128)$ is added to the above function, and its δ and ω_0 are estimated again. The actual δ and ω taken are same as before (1.9 and 1.2 respectively).

The only extra line, when compared to the previous code is:

```
y = y + 0.1*np.random.randn(128)
```

which is written before the windowing function is multiplied.

After which, the same code above is used to find and estimate the 2 parameters, for which the output obtained is:

```
With noise, the calculated delta is 1.884112 and the error in the calculated delta is 0.015888
With noise, the Calculated omega is 1.260343 and the error in the calculated omega is 0.060343
```

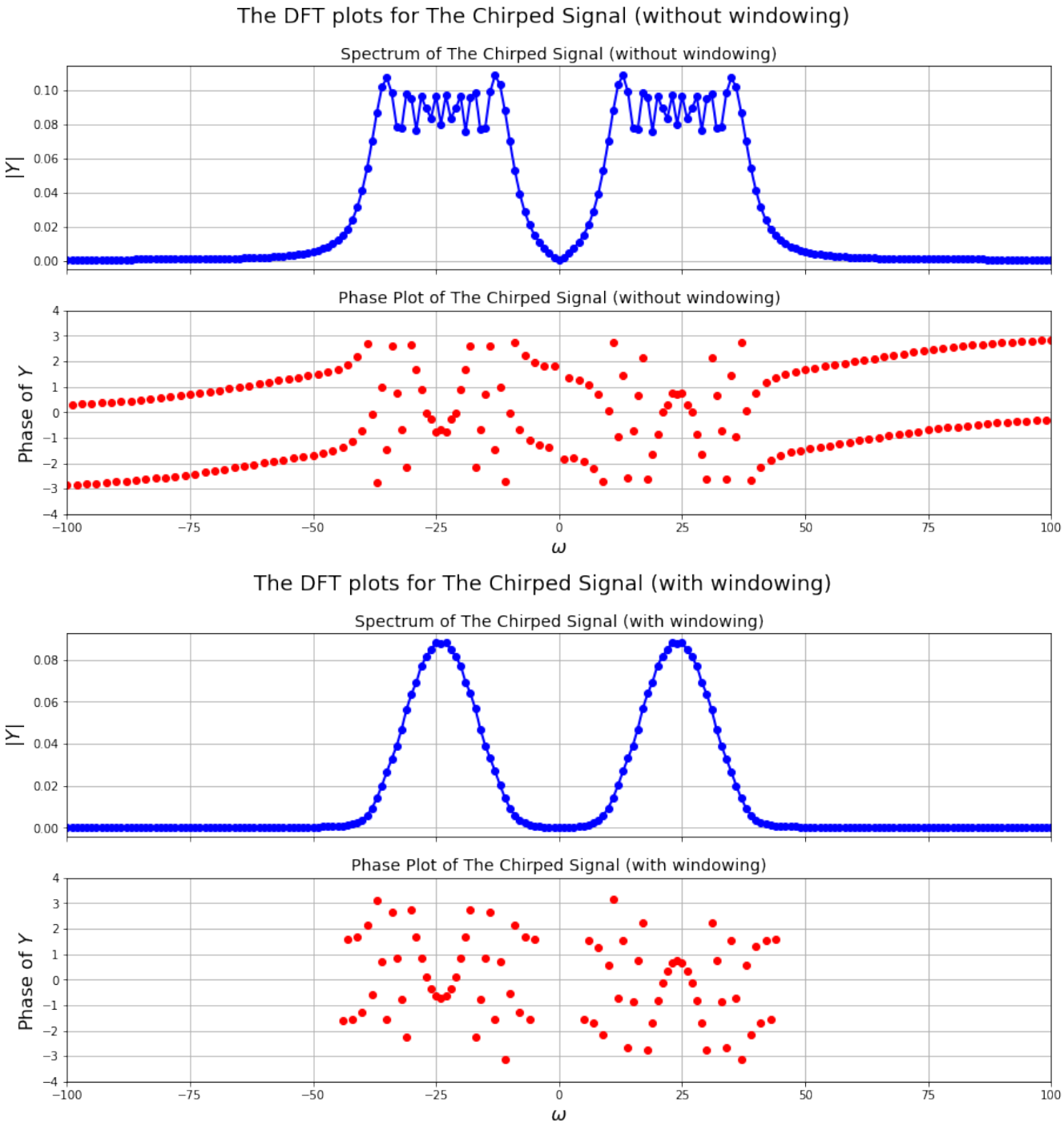
Thus, it can be seen that, when the noise was added, the error increased, though not by a lot.

5 Chirped Signal

In this section, the DFT plots of the chirped signal (defined below) is plotted, both with and without windowing. Though, windowing is not really required here, as the function would be continuous even after sampling it, as the beginning and ending values of the function in the interval $[-\pi, \pi]$ are the same.

$$x(t) = \cos\left(16\left(1.5 + \frac{t}{2\pi}\right)t\right)$$

```
# Plotting the chirped signal without windowing.
t = np.linspace(-pi,pi,1025)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
y = np.cos(16*t*(1.5 + (t/(2*pi))))
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/1024.0
w=np.linspace(-pi*fmax,pi*fmax,1025)
w=w[:-1]
plot_function(w,Y,100,'The Chirped Signal (without windowing)')

# Plotting the chirped signal with windowing.
t = np.linspace(-pi,pi,1025)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
n = np.arange(1024)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/1023))
y = np.cos(16*t*(1.5 + (t/(2*pi))))
y = y * wnd
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/1024.0
w=np.linspace(-pi*fmax,pi*fmax,1025)
w=w[:-1]
plot_function(w,Y,100,'The Chirped Signal (with windowing)')
```



Thus, again, it can be seen that when the signal is windowed the peaks are more clearer (distinct), even though they are broader, whereas without windowing the peaks are jagged and not clear at all. Also, the magnitude also decreases at a much slower rate in the one without windowing, as seen from the fact that a phase is present throughout, for the case without windowing, whereas, for the one with, it decreases faster.

6 Frequency - Time plot for Chirped Signal

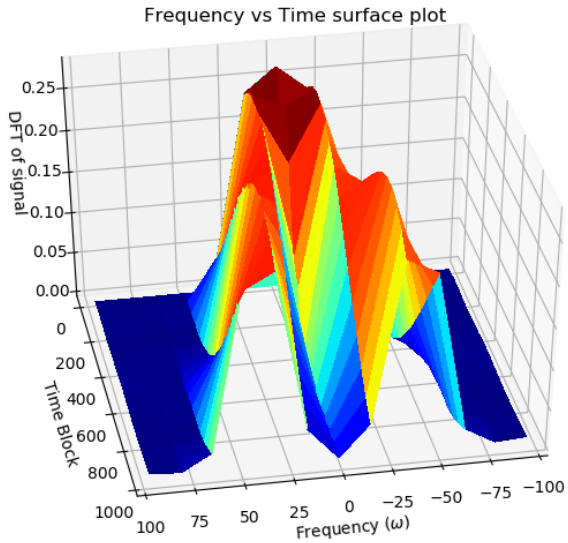
In this, the 1024 vector created earlier is divided into segments of 64 samples each. Then, the DFT of each sample is taken, and this is plotted in a surface plot with respect to time.

6.1 Disjoint samples taken

In this case, the 64 sample wide section taken is disjoint, and therefore a rough graph would be obtained, but this will be faster to plot, as number of elements in graph would be lesser.

```
t = np.linspace(-pi,pi,1025)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
t = np.array(np.split(t, 16))     # The entire 1024 elements are split into 16
    disjoint sets of 64 elements each.
n = np.arange(64)
wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/63))
y = np.cos(16*t*(1.5 + (t/(2*pi))))
y = y * wnd
y[0]=0
y=f.fftshift(y)
Y=f.fftshift(f.fft(y))/64.0
w=np.linspace(-pi*fmax,pi*fmax,65)
w=w[:-1]

n = np.arange(0,1024,64)
fig1 = plt.figure(4)
ax = p3.Axes3D(fig1)
plt.title('Frequency␣vs␣Time␣surface␣plot')
ax.set_xlabel('Frequency␣($\omega$)')
ax.set_ylabel('Time␣Block')
ax.set_xlim([-100,100])
ax.set_zlabel('DFT␣of␣signal')
x,y = np.meshgrid(w,n)
x[x>100]= np.nan       # Without this and the next line, the surface plot overflows
    due to the setting of xlim.
x[x<-100]= np.nan
surf = ax.plot_surface(x, y, abs(Y), rstride=1, cstride=1,cmap=plt.cm.jet,linewidth
    =0, antialiased=False)
plt.show()
```

The graph obtained is:

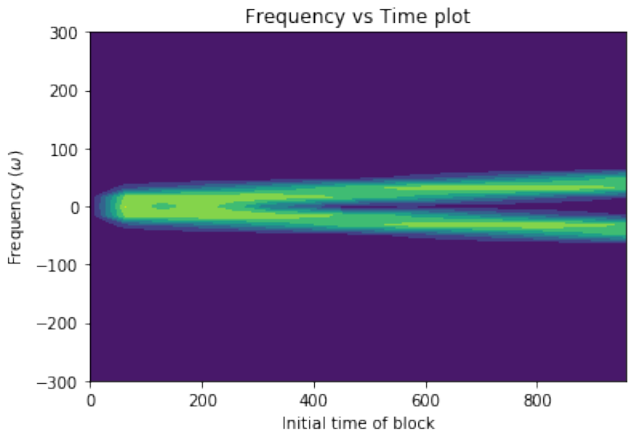


Thus, it is seen that at lesser time instants, the peaks of the DFT are closer to each other, while as the time instant (from which the 64 time points are taken) increases, the peaks become more wide apart. Because the graph was plotted with sets that are disjoint, this variation is not that clearly seen in the surface plot. It is more clearly seen in the one below.

The contour plot of the above surface plot is:

```
plt.contourf(n,w,abs(Y).transpose())
plt.ylabel('Frequency␣($\omega$)')
plt.xlabel('Initial␣time␣of␣block')
plt.title('Frequency␣vs␣Time␣plot')
```

```
plt.ylim(-300,300)
plt.show()
```

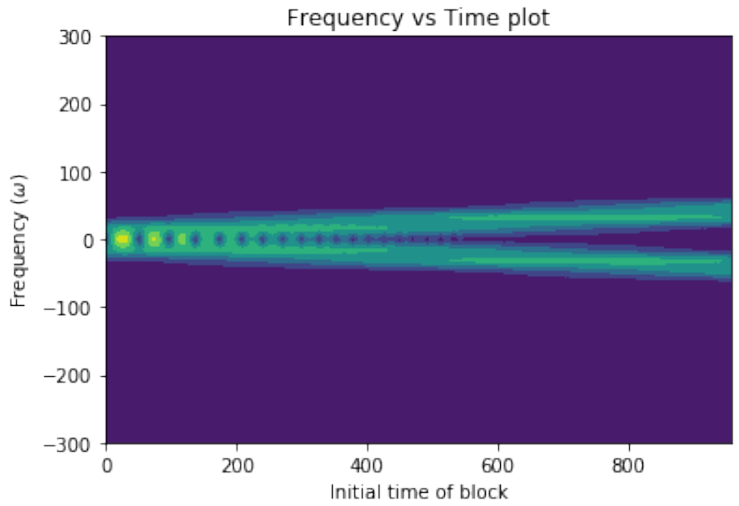


6.2 Continuous samples taken

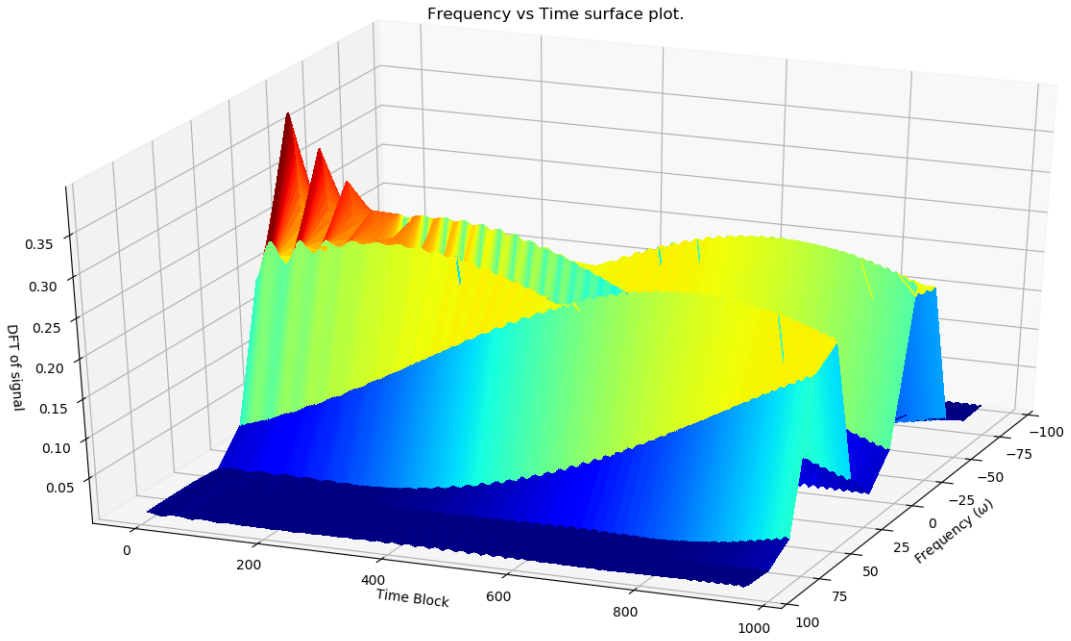
In this case, continous samples of 64 element wide sets are taken, i.e unlike the previous case where the sets were non-overlapping, in this case, the sets are overlapping, so as to provide a more clearer graph. Though the disadvantage of this method is that, plotting a 3D surface plot with these many elements will make the rendering of the graph very slow, and very difficult to interact with, hence a filled contour plot is plotted from which the same information can be extracted.

```
t = np.linspace(-pi,pi,1025)
t = t[:-1]
dt = t[1]-t[0]
fmax = 1/dt
Y = []
for i in range(0,960):              # A loop is run to take sample at each instant.
    x = np.array(t[i:i+64])         # 64 elements are taken starting at each time.
    y = np.cos(16*x*(1.5 + (x/(2*pi))))
    n = np.arange(64)
    wnd=f.fftshift(0.54+0.46*np.cos(2*pi*n/63))
    y = y * wnd
    y[0]=0
    y=f.fftshift(y)
    Y.append(f.fftshift(f.fft(y))/64.0)    # FFT taken for each 64 element set.
n = np.arange(960)
w=np.linspace(-pi*fmax,pi*fmax,65)
w=w[:-1]
Y = np.array(Y)

plt.contourf(n,w,abs(Y).transpose())
plt.ylabel('Frequency␣($\omega$)')
plt.xlabel('Initial␣time␣of␣block')
plt.title('Frequency␣vs␣Time␣plot')
plt.ylim(-300,300)
plt.show()

fig1 = plt.figure(4)
ax = p3.Axes3D(fig1)
plt.title('Frequency␣vs␣Time␣surface␣plot.')
ax.set_xlabel('Frequency␣($\omega$)')
ax.set_ylabel('Time␣Block')
ax.set_xlim([-100,100])
ax.set_zlabel('DFT␣of␣signal')
x,y = np.meshgrid(w,n)
x[x>100]= np.nan
```

```
x[x<-100]= np.nan
surf = ax.plot_surface(x, y, abs(Y), rstride=1, cstride=1,cmap=plt.cm.jet,linewidth
    =0, antialiased=False)
plt.show()
```



Thus, it can be seen that the two peaks do become more and more separate as time increases, which was observed before also, but this time, as continuous sets were taken, this transition is clearer. Just for reference, the surface plot obtained is shown below (although interacting with the graph when running the python file is very slow due to the large number of points that constitute the plot):



Thus, it is seen that, at lesser time instants, there is just a single peak in the middle which eventually splits into 2 peaks as tie increases, which was seen in simpler way in the contour plot plotted above. Also, compared to the disjoint samples, both the contour plot and the surface plot vary much more smoothly, which is expected, as continuous samples were taken.

It can also be seen that, initially there was just a single peak at a very high magnitude, which reduced and split into 2 peaks with reduced magnitudes, thus qualitatively, we can say that the energy of the DFT remains constant as time increases, which is expected as energy can't be created or destroyed.

7 Conclusion

The observations and conclusions for each section is written in the respective sections. Some general conclusions are explained below.

The normal DFT need not provide accurate peaks of the function, because of the fact that, it depends on the rate at which sampling is done on the continous time signal. If the rate of sampling is not a multiple of the signal frequency, a mismatch between signals occur after sampling. Because of this mismatch, periodic continuous time signals become discontinuous periodic signals after sampling, and this discontinuity is the one that causes the ambiguity in the fourier transform in the form of Gibb's Phenomenon.

One way to reduce this ambiguity is to reduce the discontinuity in the initial function, which is done by mulitplying the function with a windowing function, which, as seen above, reduces the discontinuity, and makes the peaks more visible in the DFT. Though these peaks do become broader, because of the fact that, multiplication in the time domain leads to convolution in the frequency domain, which leads to broadening of the peaks.

Thus, in general windowing helps in better distinguishing the function which becomes discontinuous after sampling.