Answer 2 :

Homework 6

$$p(x) = \left((2\pi)^p |V|\right)^{-1/2} \exp(-Q(x)/2) \quad -①$$

a) $P(BG) = P(FG)$ & $\sigma_1 = \sigma_2$

$p(x_1) = p(x_2)$ -② & $V_1 = V_2$ -③

optimal θ ~~x~~, i.e., x.

Using eq. ① & conditions ②, ③.

$$\exp(-Q(x_1)/2) = \exp(-Q(x_2)/2)$$

Here, $Q(x) = (x-m)' V^{-1} (x-m)$

$$\Rightarrow (x_1-\mu_1)' V^{-1} (x_1-\mu_1) = (x_2-\mu_2) V^{-1} (x_2-\mu_2)$$

Hence, $x_1 - \mu_1 = x_2 - \mu_2$

OR

$x_1 - \mu_1 = -(x_2 - \mu_2)$

At point of optimality $x_1 = x_2$,

So, $\mu_1 = \mu_2$

OR

$$x = \frac{\mu_1 + \mu_2}{2} = \theta$$

b) Since, we are taking one pixel at a time it is univariate.

$$\theta = \frac{\mu_1 + \mu_2}{2}$$

<u>case I</u>. $P(BG) = P(FG)$.

For given condition at θ,

$$\cancel{|\Sigma_1|^{-1/2} e^{-\left(\frac{\theta - \mu}{|\Sigma|}\right)}}$$

$$|\Sigma_1|^{-1/2} e^{-\left(\frac{(\theta - \mu_1)^2}{2|\Sigma_1|^2}\right)} = |\Sigma_2|^{-1/2} e^{-\left(\frac{(\theta - \mu_2)^2}{2|\Sigma_2|^2}\right)}$$

$$-\frac{1}{2} \log \left|\frac{\Sigma_1}{\Sigma_2}\right|^2 = \frac{(\theta - \mu_1)^2}{2|\Sigma_1|} - \frac{(\theta - \mu_2)^2}{2|\Sigma_2|}$$

$$\log \left|\frac{\Sigma_2}{\Sigma_1}\right|^2 = \frac{(\mu_2 - \mu_1)^2}{4|\Sigma_1|} - \frac{(\mu_1 - \mu_2)^2}{4|\Sigma_2|} \quad \text{— (b1)}$$

<u>case II</u>

In addition to assumptions in case I,

$$P(BG) = P(FG)$$

$$\mu_1 = \mu_2$$

substituting in eq. (b1),

$$\log \left|\frac{\Sigma_2}{\Sigma_1}\right|^2 = 0$$

$$\Rightarrow |\Sigma_1| = |\Sigma_2|$$

Here, $|\Sigma_1| = |\sigma_1|$

$|\Sigma_2| = |\sigma_2|$

since it is univariate.

case III

If nothing is assumed.

$$g_1(x_{ij}) = P\left(\frac{BG}{x_{ij}}\right) = P\left(\frac{x_{ij}}{BG}\right) P(BG) / P(x_{ij})$$

$$g_2(x_{ij}) = P\left(\frac{FG}{x_{ij}}\right) = P\left(\frac{x_{ij}}{FG}\right) P(FG) / P(x_{ij})$$

At the decision boundary $g_1(x_{ij}) = g_2(x_{ij})$.

Using pdf for each function

$$\frac{P(BG)}{\sqrt{2\pi}\,\sigma_1} e^{\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]} = \frac{P(FG)}{\sqrt{2\pi}\,\sigma_2} e^{\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right]} \quad - \text{(b3)}$$

c) If $P(BG) = 4 * P(FG)$

$\mu_1 = 100$; $\mu_2 = 200$

$\sigma_1 = \sigma_2$

Substituting in (b3),

$$4 * e^{\left[-\frac{1}{2}\left(\frac{x-100}{\sigma}\right)^2\right]} = e^{\left[-\frac{1}{2}\left(\frac{x-200}{\sigma}\right)^2\right]}$$

Taking log on both sides,

$$\log 4 - \frac{1}{2}\frac{(x-100)^2}{\sigma^2} = -\frac{1}{2}\frac{(x-200)^2}{\sigma^2}$$

$$\frac{(x-100)^2 - (x-200)^2}{2\sigma^2} = \log 4$$

For optimal boundary we can differentiate wrt σ,