Answer 1 -------------

1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Computing Eigen values & Eigen vectors

- if 'e' is an eigen vector of the matrix A, then

$$Ae = \lambda e$$

- Some scalar 'λ' is eigen value of A.

$$Ae = \lambda I e$$

$$(A - \lambda I)e = 0$$

$$\det(A - \lambda I) = 0$$

For given matrix 'A',

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \lambda I\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)\left[(5-\lambda)(9-\lambda) - 48\right] - 2\left[4(9-\lambda) - 42\right] + 3\left[32 - 7(5-\lambda)\right] = 0$$

$$-\lambda^3 + 15\lambda^2 + 18\lambda = 0 \quad - (1)$$

$$-\lambda(\lambda^2 - 15\lambda - 18) = 0$$

solving for roots of quadratic eq., $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$-\lambda\left(\lambda + \frac{3\sqrt{33} - 15}{2}\right)\left(\lambda - \frac{3\sqrt{33} + 15}{2}\right) = 0$$

$$\lambda = 0, \quad \frac{3\sqrt{33}+15}{2}, \quad -\frac{3\sqrt{33}+15}{2}$$

For $\lambda_3 = -\frac{3\sqrt{33}+15}{2}$

$$(A - \lambda_3 I) = 0$$

$$\left[\begin{array}{ccc|c} \frac{3\sqrt{33}-13}{2} & 2 & 3 & 0 \\ 4 & \frac{3\sqrt{33}-5}{2} & 6 & 0 \\ 7 & 8 & \frac{3\sqrt{33}+3}{2} & 0 \end{array}\right] \quad R_1 \leftarrow \frac{R_1}{\left(\frac{3\sqrt{33}+13}{64}\right)}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3\sqrt{33}+13}{32} & \frac{9\sqrt{33}+39}{64} & 0 \\ 4 & \frac{3\sqrt{33}-5}{2} & 6 & 0 \\ 7 & 8 & \frac{3\sqrt{33}+3}{2} & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3\sqrt{33}+13}{32} & \frac{9\sqrt{33}+39}{64} & 0 \\ 0 & \frac{9\sqrt{33}-33}{8} & (-9\sqrt{33}+57)/16 & 0 \\ 7 & 8 & \frac{3\sqrt{33}+3}{2} & 0 \end{array}\right] \quad R_3 \leftarrow R_3 - 7R_1$$

$$\left[\begin{array}{ccc|c} 1 & (3\sqrt{33}+13)/13 & (9\sqrt{33}+39)/64 & 0 \\ 0 & (9\sqrt{33}-33)/8 & (-9\sqrt{33}+57)/16 & 0 \\ 0 & (-21\sqrt{33}+165)/32 & (33\sqrt{33}-177)/64 & 0 \end{array}\right] \quad R_2 \leftarrow \frac{R_2}{\left(\frac{9\sqrt{33}-33}{8}\right)}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3\sqrt{33}+13}{32} & (9\sqrt{33}+39)/64 & 0 \\ 0 & 1 & (3\sqrt{33}-11)/44 & 0 \\ 0 & \frac{(-21\sqrt{33}+165)}{32} & (33\sqrt{33}-177)/64 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3\sqrt{33}+13}{32} & \frac{9\sqrt{33}+39}{64} & 0 \\ 0 & 1 & \frac{3\sqrt{33}-11}{44} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2\left(-\frac{21\sqrt{33}+165}{32}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3\sqrt{33}+11}{22} & 0 \\ 0 & 1 & \frac{3\sqrt{33}-11}{44} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \leftarrow R_1 - \left(\frac{3\sqrt{33}+13}{32}\right)R2$$

$$x_1 + \frac{3\sqrt{33}+11}{22} = 0 \Rightarrow x_1 = \frac{-(3\sqrt{33}+11)}{22}x_3$$

$$x_2 + \frac{3\sqrt{33}-11}{44} = 0 \Rightarrow x_2 = -\left(\frac{3\sqrt{33}-11}{44}\right)x_3$$

Assuming $x_3 = 1$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(3\sqrt{33}+11)/22 \\ -(3\sqrt{33}-11)/44 \\ 1 \end{bmatrix} \dashrightarrow \text{Vector } V_3$$

Similarly we can find for $\lambda_1 = 0$ & $\lambda_2 = \frac{3\sqrt{33}+15}{2}$

For $\lambda_1 = 0$, $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$x_1 = \frac{-1}{2}x_2$

$2x_1 + x_2 = 0$

If $x_2 = -2$,

For λ_2, $x = \begin{bmatrix} \frac{3\sqrt{33}-11}{22} \\ \frac{3\sqrt{33}+11}{44} \\ 1 \end{bmatrix}$

$x_1 = \frac{3\sqrt{33}-11}{22}x_3$

$x_2 = \frac{3\sqrt{33}+11}{44}x_3$

If x_3 is 1,

> Trace of matrix = sum of Eigen values

$$15 = 0 + \frac{3\sqrt{33}+15}{2} + \left(\frac{-3\sqrt{33}+15}{2}\right)$$

$$15 = 15$$

> Rank of matrix = No. of Eigen values

$$2 = 2$$

> Product of Eigen values is determinant.

Answer 2 -----------

2) $$\underset{p \times q}{\overset{A}{\left[\quad\right]}} \; \underset{q \times 1}{\overset{x}{\left[\quad\right]}} = \underset{p \times 1}{\overset{y}{\left[\quad\right]}}$$

1) Matrix 'A' has dimensions $p \times q$.

2) 'A' such that,

$$\|y_2 - y_1\| = \|x_2 - x_1\|$$

{condition to satisfy} Euclidean distance

$$\|Ax_2 - Ax_1\| = \|x_2 - x_1\|$$

$$\|A\| \, \|x_2 - x_1\| = \|x_2 - x_1\|$$

For this equation to be satisfied,

$$x_2 - x_1 = 0 \Rightarrow \boxed{x_1 = x_2}$$

OR

$$\boxed{\det(A) = \pm 1}$$

$$\|x_2 - x_1\| = \|y_2 - y_1\|$$

$$(x_2 - x_1)^T . (x_2 - x_1) = (Ax_2 - Ax_1)^T . (Ax_2 - Ax_1)$$

$$= (A(x_2 - x_1))^T . (A(x_2 - x_1))$$

$$= (x_2 - x_1)^T A^T A (x_2 - x_1)$$

Hence, $A^T A = I$, which means 'A' is an orthogonal matrix.

2) Feasibility of unchanged Euclidean distance

3. a) $q = 2$; $p = 2$ 'A' is orthogonal

Answer 3 ------

3) 1. $l : w_1 x_1 + w_2 x_2 + w_3 = 0$

$\mu = [\mu_1, \mu_2]$

The subtraction of 'mean' can be assumed as translation of axis & the new coordinate system will have line 'l' pass through origin (new

Assume, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x^T = [x_2 \ \ x_1]$

$xx^T = \begin{bmatrix} x_1 \cdot x_2 & x_1 \cdot x_1 \\ x_2 \cdot x_2 & x_2 \cdot x_1 \end{bmatrix}$

Here, we see that $\boxed{R_2 = \frac{x_2}{x_1} R_1}$ -①

Similarly we can establish that a matrix formed has linearly dependent rows, with only one independent row.

Hence, Rank of such a matrix is 1.

We know that number of non-zero Eigen vectors is equal to the rank.

$\boxed{\text{For } A', \ \text{1 eigen-} \cancel{\text{vector}} \text{ value (non-zero)}}$

For A, rank is still 1, only the scalar component will differ from eq①.

$\boxed{\text{For } A, \ \text{1 non-zero Eigen Value}}$

2. The answers will be same as above,

For B', $x = [x_1' - \mu_1 ; x_2' - \mu_2]$

xx^T will still give linearly dependent rows.

$\boxed{\text{Number of Eigen values, only 1}}$

For B, $\boxed{\text{Eigen values only 1 in number}}$

3.

~~Eigen vector is $\|x_i - \mu\|$~~

Eigen value is $\|\Sigma\|$

Eigen vector is any row in Σ.

For xx^T, $\|x\|^2$ is the eigen value with eigen vector 'x'.

Proof:- $(x x^T) x = x (x^T x) = x \|x\|^2 = \|x\|^2 x = \lambda x$.