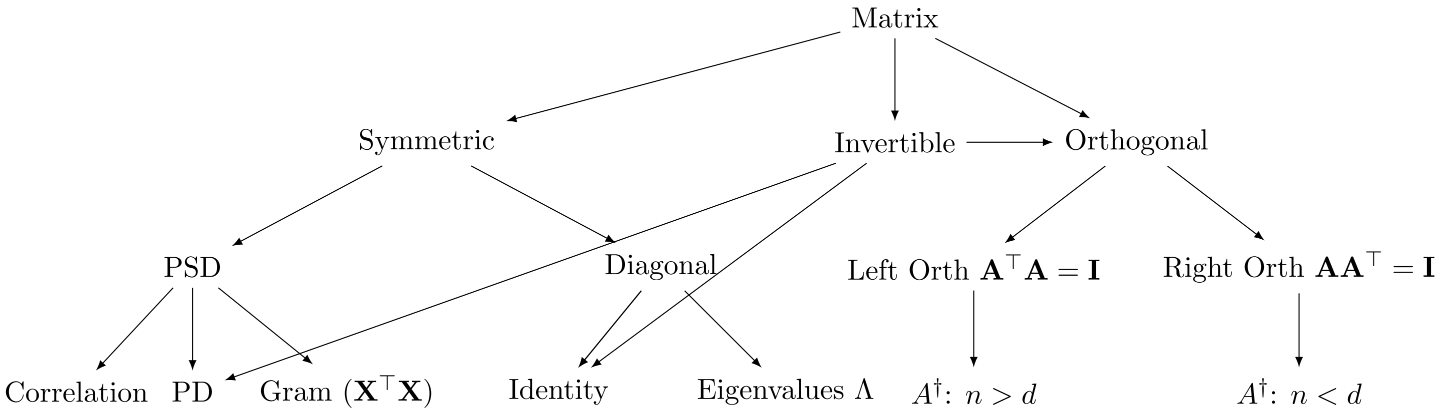
Special Matrices and their Properties



Term	**Definition**	**Unique Attributes / Key Properties**
Diagonal	$a_{ij} = 0$ for $i \neq j$	- Determinant is product of diagonal entries - Eigenvalues are diagonal entries - A^k is diagonal with entries raised to k
Symmetric	$A = A^T$, i.e., $a_{ij} = a_{ji}$	- $x^T Ax = (Ax)^T x$ for all x if A is symmetric - **v** - All eigenvalues are real - Eigenvectors for distinct eigenvalues are orthogonal - Diagonalizable via orthogonal matrices
Identity (I)	$I_{ij} = \delta_{ij}$	- $AI = IA = A$ - $I^{-1} = I$ - All eigenvalues are 1
Invertible (Non-singular)	Exists A^{-1} such that $AA^{-1} = A^{-1}A = I$	- $\det(A) \neq 0$ - Linearly independent rows and columns - No zero eigenvalues
Singular	A square matrix with $\det(A) = 0$	- Not invertible - May have linearly dependent rows or columns - At least one zero eigenvalue
Orthogonal	$A^{-1} = A^T$, so $A^T A = I$	- Rows and columns form an orthonormal set - Preserves lengths and angles - $\det(A) = \pm 1$
Left Orthogonal	$A^T A = I$ (tall matrix, $n \geq d$)	- Columns are orthonormal - $A^\dagger = A^T$ (left pseudoinverse) - Used when A has full column rank - Condition: $\text{rank}(A) = d$ where $A \in \mathbb{R}^{n\times d}$
Right Orthogonal	$AA^T = I$ (wide matrix, $n \leq d$)	- Rows are orthonormal - $A^\dagger = A^T$ (right pseudoinverse) - Used when A has full row rank - Condition: $\text{rank}(A) = n$ where $A \in \mathbb{R}^{n\times d}$
Pseudoinverse (Moore–Penrose)	Generalized inverse $A^\dagger$ satisfying the four Penrose conditions	- Always exists and is unique for any matrix - **Full column rank** ($n \geq d$, $\text{rank}(A) = d$): $A^\dagger = (A^T A)^{-1} A^T$ (left pseudoinverse) - **Full row rank** ($n \leq d$, $\text{rank}(A) = n$): $A^\dagger = A^T (AA^T)^{-1}$ (right pseudoinverse) - **Square and invertible**: $A^\dagger = A^{-1}$ - **Rank deficient**: Computed via SVD decomposition

Positive Definite	$\vec{x}^T A\vec{x} > 0$ for all non-zero $\vec{x}$	- All eigenvalues are positive - $\det(A) > 0$ - Invertible
Positive Semi-Definite	$A = A^T$, $\vec{x}^T A\vec{x} \geq 0$ for all $\vec{x}$	- All eigenvalues are non-negative - $\det(A) \geq 0$ - Always symmetric - Can always be written as $X^T X$ for some X (Gram matrix) - Defines a semi-inner product
Triangular (Upper / Lower)	Upper: $a_{ij} = 0$ for $i > j$; Lower: $a_{ij} = 0$ for $i < j$	- Determinant is product of diagonal entries - Eigenvalues are diagonal entries - Solves systems efficiently via substitution
Skew-Symmetric	$A = -A^T$, so $a_{ii} = 0$	- Eigenvalues are 0 or purely imaginary - $\vec{x}^T A\vec{x} = 0$ for all $\vec{x}$
Idempotent	$A^2 = A$	- Used in projection operations - Eigenvalues are 0 or 1
Nilpotent	$A^k = 0$ for some $k \in \mathbb{N}$	- All eigenvalues are 0
Orthogonal Projection	Linear transformation P where $P^2 = P$ and $P = P^T$	- Projects onto a subspace along its orthogonal complement - Minimizes distance to the subspace (best approximation) - $\mathrm{Im}(P)$ is a subspace; $\mathrm{Ker}(P) = \mathrm{Im}(I - P)$
Linear Map (Transformation)	Function $T : V \to W$ satisfying $T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$	- Represented by a matrix in finite dimensions - Preserves linear structure - Kernel and image define fundamental subspaces
Fundamental Subspaces	Column space, row space, null space, left null space	- Column space: span of columns (range of A) - Row space: span of rows = col space of A^T - Null space: solutions to $A\vec{x} = 0$ - Left null space: null space of A^T
Unitary	$U^\dagger = U^{-1}$, where $U^\dagger$ is the conjugate transpose of U	- $U^\dagger U = UU^\dagger = I$ - Preserves inner products: $\langle U\vec{x}, U\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ - All eigenvalues lie on the unit circle: $\lvert\lambda\rvert = 1$ - Generalization of orthogonal matrices to complex space
Gram Matrix	$G = X^T X$ for some matrix X	- Always positive semi-definite - Symmetric by construction - Entries are inner products: $G_{ij} = \langle x_i, x_j\rangle$ - Used in least squares and quadratic forms
Eigenvalue Matrix	Matrix with known eigenvalues $\lambda_1, \ldots, \lambda_n$	- Can be diagonalized as $A = P\Lambda P^{-1}$ - Λ is diagonal matrix of eigenvalues - P contains corresponding eigenvectors as columns - Powers: $A^k = P\Lambda^k P^{-1}$
Eigenvalue Matrix (Λ)	Diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$	- Contains eigenvalues on the diagonal - Result of eigendecomposition/PCA - $\Lambda_{ii} = \lambda_i$, $\Lambda_{ij} = 0$ for $i \neq j$ - Used in spectral analysis and dimensionality reduction
Covariance Matrix	$\mathrm{Cov}(X) = \frac{1}{n-1}(X-\mu)^T(X-\mu)$	- Always positive semi-definite - Symmetric by construction - Diagonal entries are variances - Off-diagonal entries are covariances
Correlation Matrix	Normalized covariance matrix with unit diagonal	- Positive semi-definite - Symmetric with $\rho_{ii} = 1$ - Entries satisfy $-1 \leq \rho_{ij} \leq 1$ - Used in statistics and data analysis

Householder Matrix	$H = I - 2\frac{vv^T}{v^Tv}$ for vector v	- Symmetric and orthogonal - $H^2 = I$ (involutory) - Reflects vectors across hyperplane - Used in QR decomposition
Permutation Matrix	Matrix with exactly one 1 in each row and column	- Orthogonal matrix - Determinant is ± 1 - Reorders rows/columns when multiplied - $P^TP = I$ and $P^{-1} = P^T$
Circulant Matrix	Each row is cyclic shift of previous row	- Diagonalized by Discrete Fourier Transform - Eigenvalues given by DFT of first row - Used in signal processing - Fast matrix-vector multiplication via FFT
Toeplitz Matrix	Constant along each diagonal: $a_{ij} = a_{i-j}$	- Determined by $2n - 1$ parameters - Includes circulant matrices as special case - Used in time series and signal processing - Fast algorithms available for solving systems
Hankel Matrix	Constant along each anti-diagonal: $a_{ij} = a_{i+j}$	- Symmetric if square - Related to Toeplitz matrices - Used in system identification - Connection to moment problems