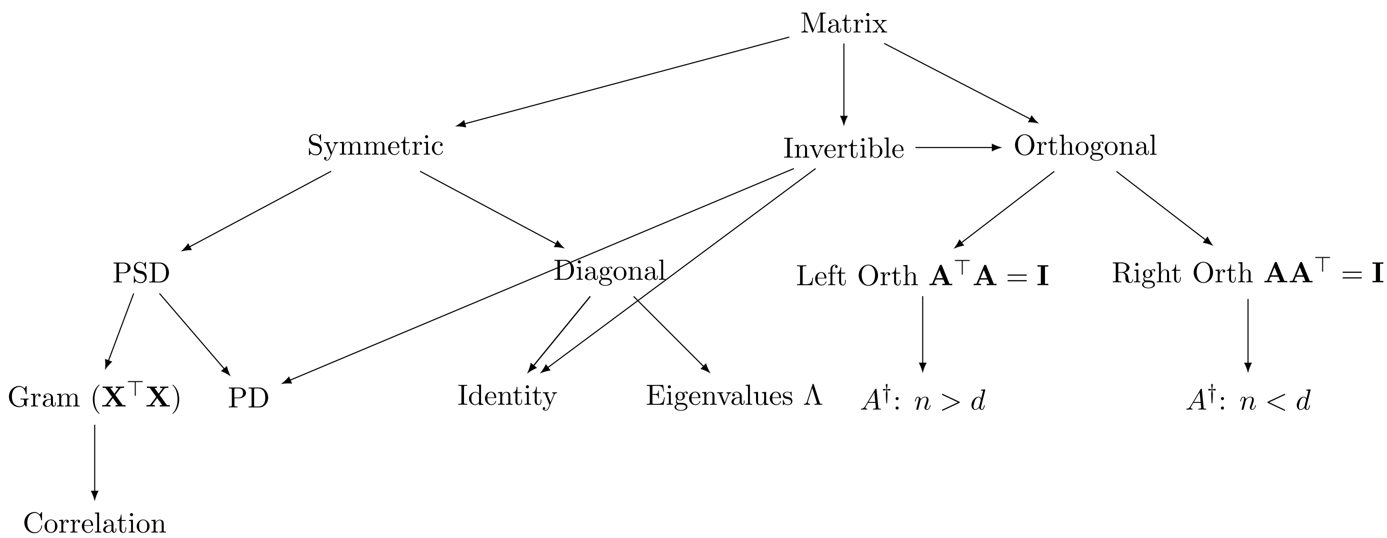
Special Matrices and their Properties



Term	**Definition**	**Unique Attributes / Key Properties**
Diagonal	$a_{ij} = 0$ for $i \neq j$	- Determinant is product of diagonal entries - Eigenvalues are diagonal entries - $\mathbf{A}^k$ is diagonal with entries raised to k
Symmetric	$\mathbf{A} = \mathbf{A}^T$, i.e., $a_{ij} = a_{ji}$	- $\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{x}$ for all $\mathbf{x}$ if $\mathbf{A}$ is symmetric - $\mathbf{v}^\top\mathbf{A}\mathbf{v} \leq \lambda_{\max}(\mathbf{A})\|\mathbf{v}\|^2$ - All eigenvalues are real - Eigenvectors for distinct eigenvalues are orthogonal - Diagonalizable via orthogonal matrices
Identity (I)	$\mathbf{I}_{ij} = \delta_{ij}$	- $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ - $\mathbf{I}^{-1} = \mathbf{I}$ - All eigenvalues are 1
Invertible (Non-singular)	Exists $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	- $\det(\mathbf{A}) \neq 0$ - Linearly independent rows and columns - No zero eigenvalues
Singular	A square matrix with $\det(\mathbf{A}) = 0$	- Not invertible - Linearly dependent rows or columns - At least one zero eigenvalue
Orthogonal	$\mathbf{A}^{-1} = \mathbf{A}^T$, so $\mathbf{A}^T\mathbf{A} = \mathbf{I}$	- Rows and columns form an orthonormal set - Preserves lengths and angles - $\det(\mathbf{A}) = \pm 1$
Left Orthogonal	$\mathbf{A}^T\mathbf{A} = \mathbf{I}$ (tall matrix, $n \geq d$)	- Columns are orthonormal - $\mathbf{A}^\dagger = \mathbf{A}^T$ (left pseudoinverse) - Used when $\mathbf{A}$ has full column rank - Condition: $\text{rank}(\mathbf{A}) = d$ where $\mathbf{A} \in \mathbb{R}^{n\times d}$
Right Orthogonal	$\mathbf{A}\mathbf{A}^T = \mathbf{I}$ (wide matrix, $n \leq d$)	- Rows are orthonormal - $\mathbf{A}^\dagger = \mathbf{A}^T$ (right pseudoinverse) - Used when $\mathbf{A}$ has full row rank - Condition: $\text{rank}(\mathbf{A}) = n$ where $\mathbf{A} \in \mathbb{R}^{n\times d}$

Pseudoinverse (Moore–Penrose)	Generalized inverse $\mathbf{A}^\dagger$ satisfying the four Penrose conditions	- Always exists and is unique for any matrix - **Full column rank** ($n \geq d$, $\text{rank}(\mathbf{A}) = d$): $\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ (left pseudoinverse) - **Full row rank** ($n \leq d$, $\text{rank}(\mathbf{A}) = n$): $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$ (right pseudoinverse) - **Square and invertible**: $\mathbf{A}^\dagger = \mathbf{A}^{-1}$ - **Rank deficient**: Computed via SVD decomposition
Positive Definite	$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all non-zero $\mathbf{x}$	- All eigenvalues are positive - $\det(\mathbf{A}) > 0$ - Always defines an inner product: for any PD matrix, there exists a unique inner product - For any inner product, there exists a PD matrix $\mathbf{A}$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{A}\mathbf{y}$ - Invertible
Positive Semi-Definite	$\mathbf{A} = \mathbf{A}^T$, $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$	- All eigenvalues are non-negative - $\det(\mathbf{A}) \geq 0$ - Always symmetric - Can always be written as $\mathbf{X}^T\mathbf{X}$ for some $\mathbf{X}$ (Gram matrix) - Defines a semi-inner product
Triangular (Upper / Lower)	Upper: $a_{ij} = 0$ for $i > j$; Lower: $a_{ij} = 0$ for $i < j$	- Determinant is product of diagonal entries - Eigenvalues are diagonal entries - Solves systems efficiently via substitution
Skew-Symmetric	$\mathbf{A} = -\mathbf{A}^T$, so $a_{ii} = 0$	- Eigenvalues are 0 or purely imaginary - $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ for all $\mathbf{x}$
Idempotent	$\mathbf{A}^2 = \mathbf{A}$	- Used in projection operations - Eigenvalues are 0 or 1
Nilpotent	$\mathbf{A}^k = 0$ for some $k \in \mathbb{N}$	- All eigenvalues are 0
Orthogonal Projection	Linear transformation $\mathbf{p}$ where $\mathbf{p}^2 = \mathbf{p}$ and $\mathbf{p} = \mathbf{p}^T$	- Projects onto a subspace along its orthogonal complement - Minimizes distance to the subspace (best approximation) - $\text{Im}(\mathbf{p})$ is a subspace; $\text{Ker}(\mathbf{p}) = \text{Im}(\mathbf{I} - \mathbf{p})$
Linear Map (Transformation)	Function $T : V \to W$ satisfying $T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y})$	- Represented by a matrix in finite dimensions - Preserves linear structure - Kernel and image define fundamental subspaces
Fundamental Subspaces	Column space, row space, null space, left null space	- Column space: span of columns (range of $\mathbf{A}$) - Row space: span of rows = col space of $\mathbf{A}^T$ - Null space: solutions to $\mathbf{A}\mathbf{x} = 0$ - Left null space: null space of $\mathbf{A}^T$
Unitary	$\mathbf{U}^\dagger = \mathbf{U}^{-1}$, where $\mathbf{U}^\dagger$ is the conjugate transpose of $\mathbf{U}$	- $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$ - Preserves inner products: $\langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ - All eigenvalues lie on the unit circle: $\lvert\lambda\rvert = 1$ - Generalization of orthogonal matrices to complex space
Gram Matrix	$\mathbf{G} = \mathbf{X}^T\mathbf{X}$ for some matrix $\mathbf{X}$	- Always positive semi-definite - Symmetric by construction - Entries are inner products: $G_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$ - Used in least squares and quadratic forms
Eigenvalue Matrix	Matrix with known eigenvalues $\lambda_1, \ldots, \lambda_n$	- Can be diagonalized as $\mathbf{A} = \mathbf{p}\boldsymbol{\Lambda}\mathbf{p}^{-1}$ - $\boldsymbol{\Lambda}$ is diagonal matrix of eigenvalues - $\mathbf{p}$ contains corresponding eigenvectors as columns - Powers: $\mathbf{A}^k = \mathbf{p}\boldsymbol{\Lambda}^k\mathbf{p}^{-1}$

Eigenvalue Matrix ($\mathbf{\Lambda}$)	Diagonal matrix $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_n)$	- Contains eigenvalues on the diagonal - Result of eigendecomposition/PCA - $\mathbf{\Lambda}_{ii} = \lambda_i$, $\mathbf{\Lambda}_{ij} = 0$ for $i \neq j$ - Used in spectral analysis and dimensionality reduction
Covariance Matrix	$\text{Cov}(\mathbf{X}) = \frac{1}{n-1}(\mathbf{X}-\mu)^T(\mathbf{X}-\mu)$ (unbiased)	- Always positive semi-definite - Symmetric by construction - Diagonal entries are variances - Off-diagonal entries are covariances
Correlation Matrix	Normalized covariance matrix with unit diagonal	- Positive semi-definite - Symmetric with $\rho_{ii} = 1$ - Entries satisfy $-1 \leq \rho_{ij} \leq 1$ - Used in statistics and data analysis
Householder Matrix	$H = I - 2\frac{vv^T}{v^Tv}$ for vector v	- Symmetric and orthogonal - $H^2 = I$ (involutory) - Reflects vectors across hyperplane - Used in QR decomposition
Permutation Matrix	Matrix with exactly one 1 in each row and column	- Orthogonal matrix - Determinant is ± 1 - Reorders rows/columns when multiplied - $P^TP = I$ and $P^{-1} = P^T$
Circulant Matrix	Each row is cyclic shift of previous row	- Diagonalized by Discrete Fourier Transform - Eigenvalues given by DFT of first row - Used in signal processing - Fast matrix-vector multiplication via FFT
Toeplitz Matrix	Constant along each diagonal: $a_{ij} = a_{i-j}$	- Determined by $2n-1$ parameters - Includes circulant matrices as special case - Used in time series and signal processing - Fast algorithms available for solving systems
Hankel Matrix	Constant along each anti-diagonal: $a_{ij} = a_{i+j}$	- Symmetric if square - Related to Toeplitz matrices - Used in system identification - Connection to moment problems