ORCS 4529: Reinforcement Learning

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Parameterizing the Policy space

$\pi_\theta : S \to \Delta^A$ with parameter vector $\theta \in \mathbb{R}^d$:

- $\pi_\theta(s)$ denotes a probability vector of dimension A with each component being the probability of taking the corresponding action.

Or, $\pi_\theta : S \times A \to \mathbb{R}$,

- $\pi_\theta(s, a)$ being the probability of taking action a, and $\sum_a \pi_\theta(s, a) = 1$.

For a scalable formulation, we want $d << |S|$.

Examples

Let x_s denote the features of state s.

- Softmax policy (N discrete actions): Parameters $\theta = (\theta_1, \ldots, \theta_N)$ for N actions. Probability of playing action a in state s

$$\pi_\theta(s, a) = \frac{e^{f_{\theta_a}(x_s)}}{\sum_{a' \in A} e^{f_{\theta_{a'}}(x_s)}}$$

  where $f_\theta(x)$ is a function e.g., the outcome of a neural network

- Gaussian policy (continuous unrestricted action space): Distribution over actions given state s

$$\pi_\theta(s) = \mathcal{N}(f_\theta(s), \sigma^2)$$

  (Single mode distribution centered at one action)

Policy Optimization

Let Π_θ is a collection of all policies in a given parameteric class with parameter θ. Then, among all policies in the given parametric class, the policy that optimizes infinite horizon discounted value:

$$\max_\theta V^{\pi_\theta}(s_1)$$

Similarly for the average reward objective:

$$\max_\theta \rho^{\pi_\theta}(s_1)$$

Similarly for the Finite reward objective:

$$\max_\theta V_H^{\pi_\theta}(s_1)$$

Can we compute/estimate the gradient $\nabla_\theta V^{\pi_\theta}(s_1)$? (Or $\nabla_\theta V_H^{\pi_\theta}(s_1)$, $\nabla_\theta \rho^{\pi_\theta}(s_1)$ etc.)

Finite horizon MDP

Theorem

For finite horizon MDP (S, A, s_1, P, R, H)*, for a policy* π_θ*,*

$$\nabla_\theta V_H^{\pi_\theta}(s_1) = \mathbb{E}_\tau \left[R(\tau) \sum_{t=1}^{H} \nabla_\theta \log(\pi_\theta(s_t, a_t)) | s_1 \right]$$

where τ *denotes a random sample trajectory of states-actions*

$$\tau = (s_1, a_1, s_2, a_2, \ldots, s_H, a_H)$$

on starting from state s_1 *and following policy* π_θ *and* $R(\tau)$ *is the total expected discounted reward* $R(\tau) = \sum_{t=1}^{T} \gamma^{t-1} R(s_t, a_t)$ *on the given trajectory.*

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$$\begin{aligned}
\nabla_\theta V_H^{\pi_\theta}(s) &= \nabla_\theta \mathbb{E}[\sum_{t=1}^{H} \gamma^{t-1} r_t | s_1 = s] \\
&= \nabla_\theta \mathbb{E}[\sum_{t=1}^{H} \gamma^{t-1} \mathbb{E}_\tau [r_t | s_1 = s] \\
&= \nabla_\theta \mathbb{E}[\sum_{t=1}^{H} \gamma^{t-1} R(s_t, a_t) | s_1 = s] \\
&= \nabla_\theta \mathbb{E}_{\tau \sim D^{\pi_\theta}} [R(\tau) | s_1 = s]
\end{aligned}$$

Let $D^{\pi_\theta}(\tau)$ be probablity of sampling a trajectory τ from policy π.

$$D^\pi(\tau) := \prod_{t=1}^{H} \pi(s_t, a_t) P(s_t, a_t, s_{t+1})$$

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For finite horizon MDP (S, A, s_1, P, R, H), for a policy π_θ,

$$\nabla_\theta V_H^{\pi_\theta}(s_1) = \mathbb{E}_\tau \left[R(\tau) \sum_{t=1}^{H} \nabla_\theta \log(\pi_\theta(s_t, a_t)) | s_1 \right]$$

where τ denotes a random sample trajectory of states-actions

$$\tau = (s_1, a_1, s_2, a_2, \ldots, s_H, a_H)$$

on starting from state s_1 and following policy π_θ and $R(\tau)$ is the total expected discounted reward $R(\tau) = \sum_{t=1}^{T} \gamma^{t-1} R(s_t, a_t)$ on the given trajectory.

Then,

$$\begin{aligned}
\nabla_\theta V_H^{\pi_\theta}(s) &= \nabla_\theta \mathbb{E}_{\tau \sim D^{\pi_\theta}} [R(\tau) | s_1 = s] \\
&= \nabla_\theta \sum_{\tau : D^{\pi_\theta}(\tau) > 0} D^{\pi_\theta}(\tau | s_1 = s) R(\tau) \\
&= \sum_{\tau : D^{\pi_\theta}(\tau) > 0} D^{\pi_\theta}(\tau | s_1 = s) \nabla_\theta \log(D^{\pi_\theta}(\tau | s_1 = s)) R(\tau) \\
&= \mathbb{E}_{\tau \sim D^{\pi_\theta}} [\nabla_\theta \log(D^{\pi_\theta}(\tau)) R(\tau) | s_1 = s]
\end{aligned}$$

Further, for a given sample trajectory τ^i.

$$\begin{aligned}
\nabla_\theta \log(D^{\pi_\theta}(\tau^i)) &= \sum_{t=1}^{H} \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i)) + \nabla_\theta \log P(s_t^i, a_t^i, s_{t+1}^i) \\
&= \sum_{t=1}^{H} \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i))
\end{aligned}$$

Vanilla Policy gradient algorithm

aka REINFORCE [Williams 1988, Williams 1992

Initialize policy parameter θ_1, and baseline.

In each iteration $k = 1, 2, \ldots,$

- Execute current policy π^θ to obtain several sample trajectories τ^i, $i = 1, \ldots, m$ where

  $\tau^i = (s_1, a_1^i, s_2^i, \ldots, s_H^i, a_H^i)$, $\hat{R}(\tau^i) = r_1^i + \gamma r_2^i + \cdots + \gamma^{H-1} r_H^i$
- Use these sample trajectories and chosen baseline to compute an unbiased gradient estimator $\hat{\mathbf{g}}$ using Policy gradient theorem

$$\hat{\mathbf{g}}_k = \frac{1}{m} \sum_{i=1}^{m} \hat{R}(\tau^i) \sum_{t=1}^{H} \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i))$$

$$\hat{\mathbf{g}}_k = \frac{1}{m} \sum_{i=1}^{m} \sum_{t=1}^{H} (\hat{R}(\tau^i) - b_t(s_t^i)) \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i))$$

- Update $\theta_{k+1} \leftarrow \theta_k + \alpha_k \, \hat{\mathbf{g}}_k$
- Update baseline as required.

Baseline

Introducing a baseline does not change the expectation of gradient, but may improve variance if selected carefully.

An example of a good state-dependent baseline

$$b_t(s) = V^{\pi_\theta}_{H-t}(s)$$

i.e., the (Estimated) value of policy π_θ, starting from state s at time t. (Some more insights into this later)

Baseline

Introducing a baseline does not change the expectation of gradient, but may improve variance if selected carefully.

An example of a good state-dependent baseline

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i.e., the (Estimated) value of policy π_θ, starting from state s at time t. (Some more insights into this later)

Below we show this is unbiased. The expectations below are over trajectories $(s_1, a_1, \ldots, a_{H-1}, s_H, a_H)$, where given state s_t, the action $a_t \sim \pi(s_t, \cdot)$. For any fixed θ, t, the baseline $b_t(s_t)|s_t$ needs to be deterministic or independent of $a_t|s_t$. For simplicity, we assume it is deterministic.

$$\begin{aligned}
\mathbb{E}_\tau[\sum_{t=1}^{H} b_t(s_t)\frac{\partial}{\partial\theta_j}\log(\pi_\theta(s_t,a_t))|\theta,s_1] &= \mathbb{E}[\sum_{t=1}^{H}\mathbb{E}[b_t(s_t)\frac{\partial}{\partial\theta_j}\log(\pi_\theta(s_t,a_t))|s_t]|\theta,s_1] \\
&= \mathbb{E}[\sum_{t=1}^{H-1} b_t(s_t)\mathbb{E}[\frac{\partial}{\partial\theta_j}\log(\pi_\theta(s_t,a_t))|s_t]|\theta,s_1] \\
&= \mathbb{E}[\sum_{t=1}^{H} b_t(s_t)\sum_a \pi_\theta(s_t,a)\frac{\partial}{\partial\theta_j}\log(\pi_\theta(s_t,a))|\theta,s_1] \\
&= \mathbb{E}[\sum_{t=1}^{H} b_t(s_t)\sum_a \frac{\partial}{\partial\theta_j}\pi_\theta(s_t,a)|\theta,s_1] \\
&= \mathbb{E}[\sum_{t=1}^{H} b_t(s_t)\frac{\partial}{\partial\theta_j}\sum_a \pi_\theta(s_t,a)|\theta,s_1]
\end{aligned}$$

Infinite horizon discounted rewards

The Policy optimization problem

$$\max_{\theta} V_{\gamma}^{\pi_\theta}(s_1)$$

where

$$V_{\gamma}^{\pi_\theta}(s_1) = \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \gamma^{t-1} r_t | s_1; a_t \sim \pi_\theta(s_t)]$$

Equivalently

$$V_{\gamma}^{\pi_\theta}(s_1) = \sum_{s} d^{\pi_\theta}(s) \sum_{a} \pi_\theta(s,a) R(s,a)$$

where $d^{\pi}(s) = \lim_{T\to\infty} \sum_{t=1}^{T} \gamma^{t-1} \Pr(s_t = s | s_1, \pi)$, the total discounted probability of being in state s under policy π.

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Infinite horizon discounted rewards

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where $d^{\pi}(s) = \lim_{T\to\infty} \sum_{t=1}^{T} \gamma^{t-1} \Pr(s_t = s | s_1, \pi)$, the total discounted probability of being in state s under policy π.

Proof is by simple conditional expectations.

Policy Gradient Theorem [Sutton, 1999]

Theorem

For infinite horizon MDP discounted reward case,

$$\nabla_\theta V_\gamma^{\pi_\theta}(s_1) = \sum_s d^{\pi_\theta}(s) \sum_a Q_\gamma^{\pi_\theta}(s,a) \nabla_\theta \pi_\theta(s,a)$$
$$= \frac{1}{(1-\gamma)} \mathbb{E}_{s \sim (1-\gamma) d^{\pi_\theta}, a \sim \pi_\theta(s)} \left[Q_\gamma^{\pi_\theta}(s,a) \nabla_\theta \log(\pi_\theta(s,a)) \right]$$

That is gradient of gain with respect to θ can be expressed in terms of gradient of (log of) policy function with respect to θ.

Remark: $(1-\gamma)d^\pi$ is a distribution over states, in particular $\sum_s (1-\gamma) d^\pi(s) = 1$

Remarks on Policy Gradient Theorem

- The key aspect of the expression for the policy gradient is that there are no terms of the form $\nabla_\theta d^{\pi_\theta}(s)$: the effect of policy changes on the (unknown) distribution over states does not appear.
- The distribution over actions given a state s is known, and its gradient $\nabla_\theta \log \pi_\theta(s, a)$ can be conveniently calculated e.g., by autodiff.
- The expectation is over the trajectories collected from the current policy which is convenient for approximating the gradient by sampling (on-policy).
- A difficulty compared to the finite horizon case is that current policy's Q-value $Q^{\pi_\theta}(s, a)$ is also not normally known, but it can be estimated e.g., by Monte Carlo or TD-learning.

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Regarding last point, note that we only need the Q-value estimates, not the gradient of Q-values.

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We abbreviate π_θ as π in below. We have:

$$
\begin{aligned}
Q^\pi(s,a) &= R(s,a) + \gamma \sum_{s'} P(s,a,s') V^\pi(s') \\
\nabla_\theta Q^\pi(s,a) &= \gamma \sum_{s'} P(s,a,s') \nabla_\theta V^\pi(s')
\end{aligned}
$$

$$
\begin{aligned}
V^\pi(s) &= \sum_a \pi(s,a) Q^\pi(s,a) \\
\nabla_\theta V^\pi(s) &= \sum_a Q^\pi(s,a) \nabla_\theta \pi(s,a) + \sum_a \pi(s,a) \nabla_\theta Q^\pi(s,a)
\end{aligned}
$$

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Therefore,

$$
\begin{aligned}
&\sum_s d^\pi(s)\nabla_\theta V^\pi(s) \\
&= \sum_s d^\pi(s)\sum_a Q^\pi(s,a)\nabla_\theta \pi(s,a) + \sum_s d^\pi(s)\sum_a \pi(s,a)\nabla_\theta Q^\pi(s,a) \\
&= \sum_s d^\pi(s)\sum_a Q^\pi(s,a)\nabla_\theta \pi(s,a) + \sum_s d^\pi(s)\sum_a \pi(s,a)\left(\gamma\sum_{s'} P(s,a,s')\nabla_\theta V^\pi(s')\right) \\
&= \sum_s d^\pi(s)\sum_a Q^\pi(s,a)\nabla_\theta \pi(s,a) + \sum_{s'} d^\pi(s')\nabla_\theta V^\pi(s') - \nabla_\theta V^\pi(s_1) \qquad (1)
\end{aligned}
$$

where we obtained the last equation using the following derivation for $d^\pi(s')$. Let $\Pr(s \to x, k, \pi)$ is the probability of going from state s to state x in k steps under policy π. Moving the terms around in (1):

$$\nabla_\theta V^\pi(s_1) = \sum_s d^\pi(s)\sum_a Q^\pi(s,a)\nabla_\theta \pi(s,a)$$

Remarks on Policy Gradient Theorem

- The key aspect of the expression for the policy gradient is that there are no terms of the form $\nabla_\theta d^{\pi_\theta}(s)$: the effect of policy changes on the (unknown) distribution over states does not appear.
- The distribution over actions given a state s is known, and its gradient $\nabla_\theta \log \pi_\theta(s, a)$ can be conveniently calculated e.g., by autodiff.
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- A difficulty compared to the finite horizon case is that current policy's Q-value $Q^{\pi_\theta}(s, a)$ is also not normally known, but it can be estimated e.g., by Monte Carlo or TD-learning.

$$
\begin{aligned}
d^\pi(s') &= \sum_{t=1}^{\infty} \gamma^{t-1} \Pr(s_t = s' | s_1, \pi) \\
&= \sum_{t=1}^{\infty} \gamma^{t-1} \Pr(s_1 \to s', t-1, \pi) \\
&= \sum_{t=2}^{\infty} \gamma^{t-1} \Pr(s_1 \to s', t-1) + \mathbf{1}(s' = s_1) \\
&= \sum_{t=2}^{\infty} \gamma^{t-1} \left(\sum_{s,a} \Pr(s_1 \to s, t-2, \pi) \pi(s,a) P(s,a,s') \right) + \mathbf{1}(s' = s_1) \\
&= \sum_{t=1}^{\infty} \gamma^{t} \left(\sum_{s,a} \Pr(s_1 \to s, t-1, \pi) \pi(s,a) P(s,a,s') \right) + \mathbf{1}(s' = s_1) \\
&= \gamma \sum_{s,a} \left(\sum_{t=1}^{\infty} \gamma^{t-1} \Pr(s_t = s | s_1, \pi) \right) \pi(s,a) P(s,a,s') + \mathbf{1}(s' = s_1) \\
&= \gamma \sum_{s,a} d^\pi(s) \pi(s,a) P(s,a,s') + \mathbf{1}(s' = s_1)
\end{aligned}
$$

Policy gradient estimation

Run policy π several times starting from s_1 to observe sample trajectories $\{\tau^i\}$ of length T for some large T (small γ^T)

- Q-value estimation using Monte Carlo method: at each time step t in a trajectory τ, set

$$\hat{Q}_t := \sum_{t'=t}^{T} \gamma^{t'-t} r_{t'}$$

- Estimate of policy gradient from single trajectory (using Policy Gradient Theorem)

$$\hat{\mathbf{g}} = \sum_{t=1}^{T} \gamma^{t-1} \hat{Q}_t \nabla_\theta \log(\pi_\theta(s_t, a_t))$$

- Can add a baseline without introducing bias

$$\hat{\mathbf{g}} = \sum_{t=1}^{T} \gamma^{t-1} (\hat{Q}_t - b_t(s_t)) \nabla_\theta \log(\pi_\theta(s_t, a_t))$$

REINFORCE algorithm

Vanilla Policy gradient algorithm

Initialize policy parameter θ, and baseline function $b_t(s), \forall s$.
In each iteration k,

1. Policy (π_θ) evaluation Execute current policy π^θ to obtain several sample trajectories τ^i, $i = 1, \ldots, m$.
   For any given sample trajectory i, use observed rewards $r_1, r_2, \ldots$, to compute $\hat{Q}_t^i := \sum_{t'=t}^{T-1} \gamma^{t'-t} r_{t'}$.
2. Policy Improvement Use $\hat{Q}_t^i$ and baseline function $b_t(s)$ to compute a gradient estimator $\hat{\mathbf{g}}_k$ using Policy gradient theorem.

$$\hat{\mathbf{g}} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \gamma^{t-1} \hat{Q}_t^i \, \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i)) \tag{2}$$

$$\hat{\mathbf{g}} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \gamma^{t-1} (\hat{Q}_t^i - b_t(s_t^i)) \nabla_\theta \log(\pi_\theta(s_t^i, a_t^i)) \tag{3}$$

Update $\theta_{k+1} \leftarrow \theta_k + \alpha \hat{\mathbf{g}}_k$.
Re-optimize baseline.

Variance reduction using baseline

Theorem

Let

$$A_t = (\hat{Q}_t - b_t(s_t))\nabla_\theta \log(\pi_\theta(s_t, a_t)).$$

the estimator of policy gradient Then, $Var(A_t|s_t)$ is mimimized by base line:

$$b_t(s_t) = \frac{\mathbb{E}\left(Q^{\pi_\theta}(s_t, a_t)\nabla_\theta \log(\pi_\theta(s_t, a_t))^2|s_t\right)}{\mathbb{E}\left(\log(\pi_\theta(s_t, a_t))^2|s_t\right)}$$

The value function $V^{\pi_\theta}(s) = \mathbb{E}\left(Q^{\pi_\theta}(s_t, a_t)|s_t\right)$ is an approximation of the above optimal baseline.

Baseline optimization

We use the following lemma:

Lemma

For any two random variables A, B*, such that for some filtration* $\mathcal{F}$*,* $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$*, almost surely,*

$$Var(A) = \mathbb{E}[(A - B)^2] + Var(B)$$

Baseline optimization

We use the following lemma:

Lemma
For any two random variables A, B, such that for some filtration $\mathcal{F}$, $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$, almost surely,

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Lemma

For any two random variables A, B, such that for some filtration $\mathcal{F}$, $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$, almost surely,

$$Var(A) = \mathbb{E}[(A - B)^2] + Var(B)$$

Proof.

For such r.v., $\mathbb{E}[A|B] = \mathbb{E}[\mathbb{E}[A|\mathcal{F}, B]|B] = \mathbb{E}[B|B] = B$ Therefore, firstly, $\mathbb{E}[A - B] = \mathbb{E}[\mathbb{E}[A - B|B]] = \mathbb{E}[B - B] = 0$, so that $\mathbb{E}[A] = \mathbb{E}[B]$. And, $\mathbb{E}[B(A - B)] = \mathbb{E}[\mathbb{E}[B(A - B)|B]] = \mathbb{E}[B^2 - B^2] = 0$. Then,

$$\begin{aligned}
\mathrm{Var}(A) &= \mathbb{E}[(A - \mathbb{E}[A])^2] \\
&= \mathbb{E}[(A - \mathbb{E}[B])^2] \\
&= \mathbb{E}[((B - \mathbb{E}[B]) + (A - B))^2] \\
&= \mathbb{E}[(B - \mathbb{E}[B])^2 + 2(B - \mathbb{E}[B])(A - B) + (A - B)^2] \\
&= \mathrm{Var}(B) + \mathbb{E}[(A - B)^2]
\end{aligned}$$

Baseline optimization

We use the following lemma:

Lemma

For any two random variables A, B, such that for some filtration $\mathcal{F}$, $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$, almost surely,

$$Var(A) = \mathbb{E}[(A-B)^2] + Var(B)$$

Now, consider the expression for policy gradient estimate at $\hat{\theta}$ (for infinite horizon discounted case) in (**??**). For simplicity of notation, let's consider a single sample and omit the superscript i.

$$\hat{\mathbf{g}} = \sum_{t=1}^{T} \gamma^{t-1}(\hat{Q}_t - b_t(s_t)) \ \nabla_\theta \log(\pi_\theta(s_t, a_t; \hat{\theta}))$$

Let

$$A_t = F'_t = (\hat{Q}_t - b_t(s_t))\nabla_\theta \log(\pi_\theta(s_t, a_t)).$$

And,

$$B_t = \mathbb{E}[A_t | s_1, a_1, \ldots, s_t, a_t] = (Q_t(s_t, a_t) - b_t(s_t))\nabla_\theta \log(\pi_\theta(s_t, a_t))$$

Note that $\mathbb{E}[B_t|s_t] = \sum_a Q^{\pi_\theta}(s_t, a)\nabla_\theta \pi_\theta(s_t, a)$.

Also, the conditions in the above lemma are satisfied by A_t, B_t for $\mathcal{F} = \{s_1, a_1, \ldots, s_t, a_t\}$

Baseline optimization

We use the following lemma:

Lemma

For any two random variables A, B, such that for some filtration $\mathcal{F}$, $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$, almost surely,

$$Var(A) = \mathbb{E}[(A-B)^2] + Var(B)$$

Therefore, using the lemma above:

$$\begin{aligned}
\mathrm{Var}(F_t'|s_t) = \mathrm{Var}(A_t|s_t) &= \mathbb{E}[(A_t - B_t)^2|s_t] + \mathrm{Var}(B_t|s_t) \\
&= \mathbb{E}\left[\left((\hat{Q}_t - \mathbb{E}[\hat{Q}_t|s_t, a_t])\nabla_\theta \log(\pi(s_t, a_t))\right)^2 |s_t\right] \\
&\quad + \mathbb{E}[B_t^2|s_t] - \mathbb{E}[B_t|s_t]^2 \\
&= \mathbb{E}\left[\left((\hat{Q}_t - Q^{\pi_\theta}(s_t, a_t))\nabla_\theta \log(\pi(s_t, a_t))\right)^2 |s_t\right] \\
&\quad + \mathbb{E}\left[((Q^{\pi_\theta}(s_t, a_t) - b_t(s_t))\nabla_\theta \log(\pi(s_t, a_t)))^2 |s_t\right] \\
&\quad - (\sum_a Q^{\pi_\theta}(s_t, a)\nabla_\theta \pi_\theta(s_t, a))^2
\end{aligned}$$

Baseline optimization

We use the following lemma:

Lemma

For any two random variables A, B, such that for some filtration $\mathcal{F}$, $\mathbb{E}[B|\mathcal{F}] = B, \mathbb{E}[A|\mathcal{F}] = B$, almost surely,

$$Var(A) = \mathbb{E}[(A-B)^2] + Var(B)$$

This is minimized by baseline:

$$b_t(s_t) = \frac{\mathbb{E}\left(Q^{\pi_\theta}(s_t, a_t)\nabla_\theta \log(\pi(s_t, a_t))^2 | s_t\right)}{\mathbb{E}\left(\nabla_\theta \log(\pi(s_t, a_t))^2 | s_t\right)}$$

The value function $V^{\pi_\theta}(s) = \mathbb{E}\left(Q^{\pi_\theta}(s_t, a_t)|s_t\right)$ is an approximation of the above optimal baseline.

The difference $Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s)$ is also referred to as **Advantage** of action *a* in state *s*. This terminology appears in algorithms like 'Asynchronous Advantage Actor Critic Algorithm (A3C)' and 'Generalized Advantage Estimation (GAE)'

However, even this baseline can only be estimated, and needs to be updated every time the policy changes (i.e., as θ changes). It seems natural to use Q-function/value function approximation methods as a subroutine to make these estimations, thus combining the two categories of methods – policy-gradient and value-function based. We will discuss further motivations for combining the two when studying actor-critic methods.

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Actor critic methods

- Actor-only methods (REINFORCE vanilla policy gradient) work with a parameterized family of policies. The gradientof the performance, with respect to the actor parameters, is directly estimated by simulation, and the parameters are updated in a direction of improvement. Maintain Policy Network.
- Critic-only methods (e.g., Q-learning, TD-learning) use TD-updates with function approximation to estimate optimal Q-values Q^* or Q-value of a given policy Q^π. Maintain Deep Q-network

Actor-critic algorithm

Maintain two networks: a policy network π_θ, and Q-network f_ω that estimates Q-value Q^{π_θ} of the current policy.

In iteration $k = 1, 2, 3, \ldots$,

- **Policy evaluation (train a critic network):** Use TD-learning (Q-learning with fixed policy) or Monte-Carlo to fit parameters ω such that $f_\omega(s, a) \approx Q^{\pi_{\theta_k}}(s, a)$
- **Policy improvement (Update the actor network):**

$$\theta_{k+1} \leftarrow \theta_k + \alpha_k \, \hat{\mathbb{E}}_{s \sim (1-\gamma) d^{\pi_k}, a \sim \pi_k(s)} \left[f_\omega(s, a) \nabla_\theta \log(\pi_{\theta_k}(s, a)) \right]$$

Q-network can also be used to estimate the current baseline (value function).

Contents I

Example: Tabular case

- $\theta \in \mathbb{R}^{S\times A}$ (Scores) Arg max policy:

$$\pi_\theta(s,a) = \frac{e^{\theta_{s,a}}}{\sum_{a'} e^{\theta_{s,a'}}}$$

$$\frac{\partial}{\partial \theta_{s',a'}} \log(\pi_\theta(s,a)) = \begin{cases} 1-\pi_\theta(s,a), & s,a = s',a', \\ -\pi_\theta(s,a'), & s = s', a \neq a' \\ 0 & otherwise \end{cases}$$

- Policy evaluation step in REINFORCE: Estimate $\hat{Q}^{\pi_\theta}(s,a)$ using Monte Carlo method
- Policy improvement step in REINFORCE:

$$\theta_{k+1} \leftarrow \theta_k + \alpha_k \hat{\mathbb{E}}_{s,a\sim\pi_\theta}\left[\hat{Q}^{\pi_\theta}(s,a)\nabla_\theta \log(\pi_\theta(s,a))\right]$$

Which is same as (approximate greedy policy improvement)

$$\theta_{k+1}(s,a) \approx \theta_k(s,a) + \alpha_k d^{\pi_\theta}(s)\pi_\theta(s,a)(Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s))$$

Gradient is already normalized, no baseline needed

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Gradient is already normalized, no baseline needed

$$\mathbf{g}(s, a) = d^{\pi_\theta}(s)\pi_\theta(s, a)(Q^{\pi_\theta}(s, a) - \sum_{a'} Q^{\pi_\theta}(s, a')\pi_\theta(s, a'))$$

$$\mathbf{g}(s, a) = d^{\pi_\theta}(s)\pi_\theta(s, a)(Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s))$$

Connection to tabular policy iteration method

$$\theta_{k+1}(s,a) \approx \theta_k(s,a) + \alpha_k d^{\pi_\theta}(s)\pi_\theta(s,a)(Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s))$$

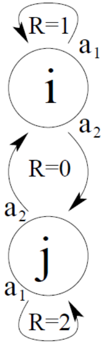
Moving towards but not jumping to

$$\arg\max_a Q^{\pi_\theta}(s,a)$$

Why not jump? Note that we only have estimates of $Q^{\pi_\theta}(s,a)$ for states and actions visited often under current policy.

Example illustrating difficulty in convergence

Kakade and Langford 2002



Assume initial policy is

$$\pi(i, a_1) = 0.8, \pi(i, a_2) = 0.2,\ \pi(j, a_1) = 0.2, \pi(j, a_2) = 0.8.$$

with stationary distribution $p(i) = 0.8, p(j) = 0.2$.
Optimal policy $\pi^*(i, a_2) = 1, \pi^*(j, a_1) = 1$.

Example cont.

Figure from Kakade and Langford, 2002

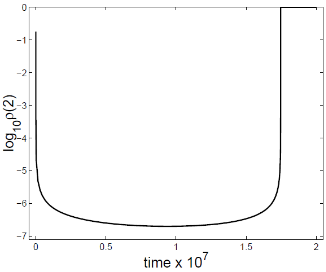


Figure: Stationary probability of state j under policy gradient algorithm

What happens under policy gradient?

$$\theta_{k+1}(s,a) \approx \theta_k(s,a) + \alpha_k d^{\pi_\theta}(s)\pi_\theta(s,a)(Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s))$$

- Suppose the current policy favors a_1 on state i and a_2 on state j (like the initial policy),
- For both states $Q^{\pi_\theta}(s,a_1)$ is more than $Q^{\pi_\theta}(s,a_2)$ (because immediate reward is 0 for a_2).
- For state i, $\pi_\theta(i,a_1)$ is also more than a_2, so $\theta(i,a_1)$ increases more than a_2 (goes farther from optimal)
- For state j, even though $Q^{\pi_\theta}(s,a_1)$ favors a_1, $\pi_\theta(j,a_2)$ is more. So there might be no or small improvement in favor of a_1.
- Also $d^\pi(j) < d^\pi(i)$, so any improvement for state j may be overshadowed by the decline for state i.

What happens under policy gradient?

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- For state j, even though $Q^{\pi_\theta}(s,a_1)$ favors a_1, $\pi_\theta(j,a_2)$ is more. So there might be no or small improvement in favor of a_1.
- Also $d^\pi(j) < d^\pi(i)$, so any improvement for state j may be overshadowed by the decline for state i.

Note that the gradient depends on $d^\pi(s), s = i, j$, where $(1-\gamma)d^\pi(s)$ is roughly (for γ close to 1) the stationary distribution of the current policy π, and $Q^\pi(s,a)$. For state i, $Q^\pi(i,a_1)$ is higher that $Q^\pi(i,a_2)$ To see this, note that $Q^\pi(i,a_1) = 1 + \gamma V^\pi(i) = 1 + \gamma 0.8 + 0.8\gamma^2 V^\pi(i)) + \gamma^2 0.2 V^\pi(j)$, and $Q^\pi(i,a_2) = \gamma V^\pi(j) = \gamma^2 0.8 V^\pi(i) + 0.2 \times 2\gamma + 0.2\gamma^2 V^\pi(j) = Q^\pi(i,a_1) - 1 - 0.4\gamma$. Therefore, θ_{i,a_1} increases compared to θ_{i,a_2}, which means the updated policy will favor looping on i even more rather than transitioning to j. For j, again θ_{j,a_1} increases more, but since the stationary probability $d^\pi(i) \approx \frac{1}{1-\gamma}p(i)$ is higher for state i than $d^\pi(j) \approx \frac{1}{1-\gamma}p(j)$, the state i gets an update of higher magnitude - moves more aggressively towards the new θ. The next policy (updated θ) is likely to be even worse, i.e., the stationary probability is even higher for state i because of the increased probability of taking action a_1 in state i. Due to these reasons, initially the stationary probability of j decreases, and becomes exponentially small before it comes back to the correct policy in exponential time

How does it compare to the greedy policy improvement?

Greedy policy improvement

$$\pi(s) = \arg\max_a Q^{\pi_\theta}(s, a)$$

(Not necessarily a good idea if we don't have good estimates of $Q^{\pi_\theta}(s, a)$ for s, a not visited)

- In the given example, $\pi(j, a_1) = 1, \pi(j, a_2) = 0$ after the first iteration.
- And, for the new policy $Q^{\pi}(i, a_2) > Q^{\pi}(i, a_1)$, so that $\pi(i, a_2) = 1, \pi(i, a_1) = 0$ in second iteration.

Optimal policy in two iterations.

Approximately Optimal Approximate RL [Kakade and Langford 2002]

Can we design an algorithm that is guaranteed to improve some performance measure at every step?

Conservative Greedy policy improvement algorithm

Main idea: Move to the greedy policy but not fully: New policy and old policy are same with probability α.

If current policy is π_k, conservative policy improvement:

$$\pi^{k+1} \leftarrow (1-\alpha)\pi^k + \alpha\pi_*^{k+1}$$

where π_*^{k+1} is the greedy policy

$$\pi_*^{k+1}(s) = \arg\max_a Q^{\pi_k}(s,a) = \arg\max_a \underbrace{Q^{\pi_k}(s,a) - V^{\pi_k}(s)}_{Advantage\ \ function\ \ A^{\pi_k}(s,a)}$$

Regular policy iteration has $\alpha = 1$.

Generalizing this idea to large state space

If current policy parameter is θ_k, conservative greedy policy improvement:

$$\pi^{k+1} \leftarrow (1-\alpha)\pi^k + \alpha\pi_{\theta^*_{k+1}}$$

where

$$\theta^*_{k+1}(s) = \arg\max_{\theta} \mathbb{E}_{s\sim(1-\gamma)d^{\pi_k}}[\sum_a \pi_{\theta}(s,a)A^{\pi_k}(s,a)]$$

That is, $\pi_{\theta^*_{k+1}}$ is an approximate greedy policy – close to greedy in expectation over states.

After k iterations, the policy π^k is a randomized policy which plays policy π^i with probability $(1-\alpha)^{i-1}\alpha$.

Trust Region Policy Optimization (TRPO)

(To be disucssed later)

Fit a θ so that π_θ is close to greedy in expectation over states, *and* is not too far from the current policy.

$$\theta_{k+1} = \begin{array}{ll} \arg\max_{\theta} & \mathbb{E}_{s\sim(1-\gamma)d^{\pi_{\theta_k}}}\left[\sum_a \pi_\theta(s,a) A^{\pi_{\theta_k}}(s,a)\right] \\ s.t. & \max_s KL(\pi_\theta(s) \| \pi_{\theta_k}(s)) \leq \alpha \end{array}$$

This is the popular TRPO (Trust Region Policy Optimization algorithm) [Schulman et al. 2015]

Conservative Greedy Policy Improvement

Lemma 4.1 of Kakade and Langford, 2002

Let π be the current policy, π' be the greedy policy,

$$\pi' = \arg\max_{\hat{\pi}\in\Pi} A_\pi(\hat{\pi}) := \mathbb{E}_{s\sim(1-\gamma)d^\pi}[\sum_a \hat{\pi}(s,a)A^\pi(s,a)]$$

and π^{new} is conservative greedy policy which is same as π with probability $1-\alpha$.

Theorem (Policy improvement theorem)

$$V^{\pi^{new}}(s_1) - V^\pi(s_1) \geq \frac{\alpha}{1-\gamma}A_\pi(\pi') - \frac{\alpha^2}{(1-\gamma)^2}2\gamma A_\pi^{\max}$$

where $A_\pi^{\max}$ is the maximum advantage over all states:

$$A_\pi^{\max} = \max_s \left|\sum_a \pi'(s,a)A^\pi(s,a)\right|$$

Intuitive proof sketch for policy improvement theorem

using policy gradient theorem

Given π, π', consider the set of policies

$$\pi_\alpha = (1 - \alpha)\pi + \alpha\pi'$$

for all $\alpha \in (0, 1)$.

Note that $\pi^{new} = \pi_\alpha, \pi = \pi_0, \pi' = \pi_1$.

How much does the value of policy V^{π_α} change if we change the policy parameter from 0 to α? Policy gradient theorem!!

Intuitive proof through policy gradient

By policy gradient theorem, policy gradient at $\alpha = 0$

$$\nabla_\alpha V^{\pi_\alpha}(s_1)\Big|_{\alpha=0} = \sum_s d^{\pi_\alpha}(s) \sum_a \left(\nabla_\alpha \pi_\alpha(s,a)\right) A^{\pi_\alpha}(s,a)\Big|_{\alpha=0}$$

where $\pi_0 = \pi$, and

$$\nabla_\alpha \pi_\alpha(s,a) = \pi'(s,a) - \pi(s,a)$$

$$\begin{aligned}\nabla_\alpha V^{\pi_\alpha}(s_1)\Big|_{\alpha=0} &= \sum_s d^{\pi}(s) \sum_a \left(\pi'(s,a) - \pi(s,a)\right) A^{\pi}(s,a) \\ &= \sum_s d^{\pi}(s) \sum_a \pi'(s,a) A^{\pi}(s,a) =: \frac{A_\pi(\pi')}{1-\gamma}\end{aligned}$$

Then, for small enough α, the lemma we want to prove follows (roughly) from Taylor approximation ($\pi_\alpha = \pi_{new}, \pi_0 = \pi$).

$$V^{\pi_{new}}(s_1) - V^{\pi}(s_1) \geq \alpha \, \frac{A_\pi(\pi')}{1-\gamma} - O(\alpha^2)$$

Intuitive proof through policy gradient

By policy gradient theorem, policy gradient at $\alpha = 0$

$$\nabla_\alpha V^{\pi_\alpha}(s_1)\Big|_{\alpha=0} = \sum_s d^{\pi_\alpha}(s) \sum_a (\nabla_\alpha \pi_\alpha(s,a))\, A^{\pi_\alpha}(s,a)\Big|_{\alpha=0}$$

where $\pi_0 = \pi$, and

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$$\begin{aligned}\nabla_\alpha V^{\pi_\alpha}(s_1)\Big|_{\alpha=0} &= \sum_s d^{\pi}(s) \sum_a (\pi'(s,a) - \pi(s,a))\, A^{\pi}(s,a) \\ &= \sum_s d^{\pi}(s) \sum_a \pi'(s,a) A^{\pi}(s,a) =: \frac{A_\pi(\pi')}{1-\gamma}\end{aligned}$$

Then, for small enough α, the lemma we want to prove follows (roughly) from Taylor approximation ($\pi_\alpha = \pi_{new}, \pi_0 = \pi$).

$$V^{\pi_{new}}(s_1) - V^{\pi}(s_1) \geq \alpha\, \frac{A_\pi(\pi')}{1-\gamma} - O(\alpha^2)$$

$$\begin{aligned}\nabla_\alpha V(\pi_\alpha^{new})\Big|_{\alpha=0} &= \sum_s d^{\pi_\alpha^{new}}(s) \sum_a (\nabla_\alpha \pi_\alpha^{new}(s,a))\, A^{\pi_\alpha^{new}}(s,a)\Big|_{\alpha=0} \\ &= \sum_s d^{\pi_\alpha^{new}}(s) \sum_a (\pi'(s,a) - \pi(s,a)) A^{\pi_\alpha^{new}}(s,a)\Big|_{\alpha=0} \\ &= \sum_s d^{\pi}(s) \sum_a (\pi'(s,a) - \pi(s,a)) A^{\pi}(s,a) \\ &= \sum_s d^{\pi}(s) \sum_a \pi'(s,a) A^{\pi}(s,a) - \sum_s d^{\pi}(s) \pi(s,a) A^{\pi}(s,a) \\ &= \sum_s d^{\pi}(s) \sum_a \pi'(s,a) A^{\pi}(s,a) \\ &= \frac{1}{(1-\gamma)} A_\pi(\pi')\end{aligned}$$

The second last step follows because $\sum_s d^\pi(s)\pi(s,a)A^\pi(s,a) = \sum_s d^\pi(s)\pi(s,a)(Q^\pi(s,a) - V^\pi(s)) = \sum_s d^\pi(s)(V^\pi(s) - V^\pi(s)) = 0$. In fact, by the same insight, $A_\pi(\pi^{new}) = \alpha A_\pi(\pi')$. Therefore, using Taylor expression, a lower bound on the improvement is given by

Proof of Policy improvement lemma

Lemma (Policy improvement lemma)

$$V^{\pi^{new}}(s_1) - V^{\pi}(s_1) \geq \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - \frac{\alpha^2}{(1-\gamma)^2} 2\gamma A_\pi^{\max}$$

$$A_\pi(\pi') = \mathbb{E}_{s\sim(1-\gamma)d^\pi}[\sum_a \pi'(s,a) A^\pi(s,a)]$$

Proof Outline:

- A "Performance Difference Lemma" characterizes the exact difference in in the two value functions, but involves new state distribution $d^{\pi_{new}}$.
- Proof of "policy improvement lemma" is by showing that new state distribution $d^{\pi_{new}}$ is close to the old state distribution d^π.

Performance difference Lemma

Lemma 6.1 of Kakade and Langford 2002

Following characterizes the exact change in value of policy

Lemma (Performance difference Lemma)

For any two policies π^{new}, π*,*

$$
\begin{aligned}
V^{\pi^{new}}(s_1) - V^{\pi}(s_1) &= \sum_s d^{\pi^{new}}(s) \sum_a \pi^{new}(s,a) A^{\pi}(s,a) \\
&= \frac{\alpha}{(1-\gamma)} \mathbb{E}_{s \sim (1-\gamma) d^{\pi^{new}}} \left[\sum_a \pi'(s,a) A^{\pi}(s,a) \right]
\end{aligned}
$$

Policy improvement lemma proof is by comparing the distribution over states under the new and old policies: $d^{\pi^{new}}$ and d^{π}.

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Policy improvement lemma proof is by comparing the distribution over states under the new and old policies: $d^{\pi^{new}}$ and d^{π}.

Proof of Performance Difference Lemma:

$$\begin{aligned}
V^{\tilde{\pi}}(s_1) &= \mathbb{E}_{s_1,a_1,s_2,a_2,\ldots\sim\tilde{\pi}}[\sum_{t=1}^{\infty} \gamma^{t-1} R(s_t,a_t)|s_1] \\
&= \sum_{t=1}^{\infty} \gamma^{t-1} \mathbb{E}_{s_t,a_t\sim\tilde{\pi}}[R(s_t,a_t) + V^{\pi}(s_t) - V^{\pi}(s_t)|s_1] \\
&= \sum_{t=1}^{\infty} \gamma^{t-1} \mathbb{E}_{s_t,a_t\sim\tilde{\pi},s_{t+1}\sim P_{s_t,a_t}}[R(s_t,a_t) + \gamma V^{\pi}(s_{t+1}) - V^{\pi}(s_t)|s_1] + V^{\pi} \\
&= \sum_{t=1}^{\infty} \gamma^{t-1} \mathbb{E}_{s_t,a_t}[Q^{\pi}(s_t,a_t) - V^{\pi}(s_t)|s_1] + V^{\pi}(s_1) \\
&= \sum_{t=1}^{\infty} \gamma^{t-1} \mathbb{E}_{s_t,a_t}[A^{\pi}(s_t,a_t)|s_1] + V^{\pi}(s_1) \\
&= \sum d^{\tilde{\pi}}(s) \sum \tilde{\pi}(s,a) A^{\pi}(s,a) + V^{\pi}(s_1)
\end{aligned}$$

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$$\begin{aligned} V^{\pi^{new}}(s_1) - V^{\pi}(s_1) &= \sum_s d^{\pi^{new}}(s) \sum_a \pi^{new}(s,a) A^{\pi}(s,a) \\ &= \frac{\alpha}{(1-\gamma)} \mathbb{E}_{s \sim (1-\gamma) d^{\pi^{new}}} \left[\sum_a \pi'(s,a) A^{\pi}(s,a) \right] \end{aligned}$$

Policy improvement lemma proof is by comparing the distribution over states under the new and old policies: $d^{\pi^{new}}$ and d^{π}.

Proof of Performance Improvement Lemma:

To compare the state distributions under the two policies, a coupling argument is used. In any given state s, π^{new} picks actions according to π' with probability α and according to π with probability $1-\alpha$. Now, for any fixed time t, let η_t be the number of steps before time t where π^{new} did not take action according to π, i.e., η_t is the number of mismatches in the actions suggested by π^{new} and π. Then, conditional on event $\eta_t = 0$, the distribution of states before time t is same for trajectories generated from π^{new} and π. More precisely,

$$\Pr_{\tau \sim \pi}(s_t = s | \eta_t = 0) = \Pr_{\tau \sim \pi}(s_t = s)$$

where random variable $\tau = (s_1, s_2, \ldots, s_t, \ldots)$ denotes a trajectory . Further, the probability that there was some mismatch before t is given by $p_t := \Pr(\eta_t > 0) = 1 - \Pr(\eta_t = 0) = 1 - (1-\alpha)^{t-1}$.

Performance difference Lemma

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\end{aligned}
$$

Policy improvement lemma proof is by comparing the distribution over states under the new and old policies: $d^{\pi^{new}}$ and d^{π}.

$$
\begin{aligned}
\rho(\pi^{new}) - \rho(\pi) &= \sum_s d^{\pi^{new}}(s) \sum_a \pi^{new}(s,a) A^{\pi}(s,a) \\
&= \mathbb{E}_{\tau=(s_1,s_2,\ldots,)\sim\pi^{new}}[\sum_t \gamma^{t-1} \sum_a \pi^{new}(s_t,a) A^{\pi}(s_t,a)] \\
&= \mathbb{E}_{\tau=(s_1,s_2,\ldots,)\sim\pi^{new}}[\sum_t \gamma^{t-1} \sum_a \alpha\pi'(s_t,a) A^{\pi}(s_t,a)] \\
&= \alpha \sum_t (1-p_t)\gamma^{t-1} \mathbb{E}_{\tau=(s_1,s_2,\ldots,s_t)\sim\pi^{new}}[\sum_a \pi'(s_t,a) A^{\pi}(s_t,a) | \eta_t = 0] \\
&\quad + \alpha \sum_t p_t \gamma^{t-1} \mathbb{E}_{\tau=(s_1,s_2,\ldots,)\sim\pi^{new}}[\sum_a \pi'(s_t,a) A^{\pi}(s_t,a) | \eta_t > 0] \\
&= \alpha \sum_t (1-p_t)\gamma^{t-1} \mathbb{E}_{(s_1,s_2,\ldots,s_t)\sim\pi}[\sum_a \pi'(s_t,a) A^{\pi}(s_t,a)] \\
&\quad + \alpha \sum_t p_t \gamma^{t-1} \mathbb{E}_{\tau=(s_1,s_2,\ldots,)\sim\pi^{new}}[\sum_a \pi'(s_t,a) A^{\pi}(s_t,a) | \eta_t > 0]
\end{aligned}
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For any two policies π^{new}, π,

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V^{\pi^{new}}(s_1) - V^{\pi}(s_1) &= \sum_s d^{\pi^{new}}(s) \sum_a \pi^{new}(s,a) A^{\pi}(s,a) \\
&= \frac{\alpha}{(1-\gamma)} \mathbb{E}_{s\sim(1-\gamma)d^{\pi^{new}}} \left[\sum_a \pi'(s,a) A^{\pi}(s,a) \right]
\end{aligned}
$$

Policy improvement lemma proof is by comparing the distribution over states under the new and old policies: $d^{\pi^{new}}$ and d^{π}.

$$
\begin{aligned}
&= \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - \alpha \sum_t p_t \gamma^{t-1} \mathbb{E}_{(s_1,s_2,\ldots,s_t)\sim\pi} [\sum_a \pi'(s_t,a) A^{\pi}(s_t,a)] \\
&\quad + \alpha \sum_t \gamma^{t-1} p_t \mathbb{E}_{\tau=(s_1,s_2,\ldots,)\sim\pi^{new}} [\sum_a \pi'(s_t,a) A^{\pi}(s_t,a) | \eta_t > 0] \\
&\geq \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - 2\alpha \sum_t (1-(1-\alpha)^{t-1}) \gamma^{t-1} \left(\max_s \left| \sum_a \pi'(s,a) A^{\pi}(s,a) \right| \right) \\
&= \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - 2\alpha A^{\pi}_{\max} \left(\frac{1}{1-\gamma} - \frac{1}{1-(1-\alpha)\gamma} \right) \\
&= \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - 2\alpha A^{\pi}_{\max} \frac{\alpha\gamma}{(1-\gamma)(1-(1-\alpha)\gamma)} \\
&\geq \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - \frac{\alpha^2}{(1-\gamma)^2} \cdot 2 A^{\pi}_{\max} \gamma
\end{aligned}
$$

How is the policy improvement lemma useful?

Algorithm Design

Important: $A_\pi(\pi')$ can be estimated by simulation/sampling from current policy!

We can select a step size to always have a positive improvement as long as $A_\pi(\pi') > 0$. Let R be an upper bound on rewards, so that $A_\pi(\pi') \leq A_\pi^{\max} \leq \frac{R}{1-\gamma}$. Then, setting

$$\alpha = \frac{A_\pi(\pi')(1-\gamma)^2}{4R},$$

and substituting in policy improvement lemma 7, we get

$$V(\pi^{new}) - V(\pi) \geq \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - \frac{2\alpha^2 A_\pi^{\max}}{(1-\gamma)^2} \geq \frac{A_\pi(\pi')^2(1-\gamma)}{8R} \tag{4}$$

Conservative policy improvement algorithm

Initialize π. Repeat:

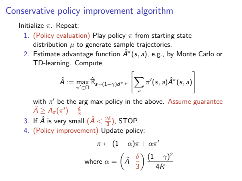
1. (Policy evaluation) Play policy π from starting state distribution μ to generate sample trajectories.
2. Estimate advantage function $\hat{A}^{\pi}(s, a)$, e.g., by Monte Carlo or TD-learning. Compute

$$\hat{A} := \max_{\pi' \in \Pi} \hat{\mathbb{E}}_{s \sim (1-\gamma) d^{\pi,\mu}} \left[\sum_{a} \pi'(s, a) \hat{A}^{\pi}(s, a) \right]$$

   with π' be the arg max policy in the above. Assume guarantee $\hat{A} \geq A_{\pi}(\pi') - \frac{\delta}{3}$
3. If $\hat{A}$ is very small $(\hat{A} < \frac{2\delta}{3})$, STOP.
4. (Policy improvement) Update policy:

$$\pi \leftarrow (1 - \alpha)\pi + \alpha\pi'$$

$$\text{where } \alpha = \left(\hat{A} - \frac{\delta}{3} \right) \frac{(1-\gamma)^2}{4R}$$



The estimation can be done for example by estimating $A^\pi(s,a)$ as function approximation $f_\omega(s,a)$ where parameter ω is set through sample estimation with loss function

$$\min_{\omega} \max_{\tilde{\pi} \in \Pi} \mathbb{E}_{s \sim (1-\gamma)d^\pi(s)} [\sum_a \tilde{\pi}(s,a) |\hat{A}^\pi(s,a) - f_\omega(s,a)|]$$

Since this is an expected error over state distribution under the current policy π, this loss can be approximated using trajectory samples from the current policy. Since $|\max_a |A^\pi(s,a)| \leq \frac{R}{(1-\gamma)}$, roughly $\frac{R^2}{(1-\gamma)^2\delta^2} \log \frac{1}{\delta}$ samples are required to ensure a δ error with probability $1-\delta$.

Conservative policy improvement algorithm

Lemma

The conservative greedy policy improvement algorithm terminates in at most $\frac{72R^2}{\delta^2(1-\gamma)^3}$ *iterations to find a policy* π *such that*

$$\max_{\pi'} A_\pi(\pi') \leq \delta$$

Conservative policy improvement algorithm

Lemma

The conservative greedy policy improvement algorithm terminates in at most $\frac{72R^2}{\delta^2(1-\gamma)^3}$ iterations to find a policy π such that

$$\max_{\pi'} A_\pi(\pi') \leq \delta$$

Why? In every iteration $\hat{A} - \frac{\delta}{3} \geq \frac{2\delta}{3} - \frac{\delta}{3} = \frac{\delta}{3}$, therefore, from (4), the increase in value function is at least

$$\left(\hat{A} - \frac{\delta}{3}\right)^2 \frac{(1-\gamma)^2}{8R} \geq \frac{\delta^2(1-\gamma)^2}{72R}$$

Since the total improvement to be made is at the most maximum value, i.e., $R/(1-\gamma)$, the procedure terminates in at most $\frac{R}{1-\gamma}\frac{72R}{\delta^2(1-\gamma)^2} = \frac{72R^2}{\delta^2(1-\gamma)^3}$ steps.

How good is the policy? local improvement vs. optimality

Theorem (Theorem 6.2 of Kakade and Langford 2002)

Let $(1-\gamma)d^{\pi,\mu}$ denote the discounted state distribution for policy π when starting state distribution is μ. We run our algorithm using starting state distribution μ and obtain a policy π with

$$\max_{\pi'} A_{\pi,\mu}(\pi') \leq \delta$$

where $A_\pi(\pi') = \mathbb{E}_{s\sim(1-\gamma)d^{\pi,\mu}(s)}[\sum_a \pi'(s,a)A^\pi(s,a)]$. Then, for any policy π^* and starting state distribution μ^*,

$$\begin{aligned}
\mathbb{E}_{s\sim\mu^*}[V^{\pi^*}(s) - V^\pi(s)] &\leq \frac{\delta}{1-\gamma}\left\|\frac{d^{\pi^*,\mu^*}}{d^{\pi,\mu}}\right\|_\infty \\
&\leq \frac{\delta}{(1-\gamma)}\left\|\frac{d^{\pi^*,\mu^*}}{\mu}\right\|_\infty
\end{aligned}$$

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As demonstrated in the last section, the conservative greedy algorithm (with the right choice of π' and step size α) is guaranteed to terminate. It terminates at the policy π such that $\max_{\pi'} A_\pi(\pi') \leq \delta$. This can be interpreted as the condition that there is no (or very little) advantage increase on changing the policy under the state distribution of the current policy. In other words, it cannot be improved locally. But, how does this policy compare to the "optimal policy", which may have a completely different state distribution?

The following theorem shows that the gap can be large if the stationary distribution over states for the chosen policy is very different from the stationary distribution over states for the optimal policy. This can happen if there isn't enough exploration over states. The following theorem also provides a way to ensure exploration. It states that one could start from a different starting state distribution (e.g. uniform) than the target starting state distribution, and then the gap depends only on how the stationary distribution of optimal policy differs from the uniform distribution.

How good is the policy? local improvement vs. optimality

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Let $(1-\gamma)d^{\pi,\mu}$ denote the discounted state distribution for policy π when starting state distribution is μ. We run our algorithm using starting state distribution μ and obtain a policy π with

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By Performance difference lemma

$$\begin{aligned}\mathbb{E}_{s_1\sim\mu^*}[V^{\pi^*}(s_1) - V^\pi(s_1)] &= \sum_s d^{\pi^*,\mu^*}(s)\pi^*(s,a)A^\pi(s,a) \\ &= \sum_s \frac{d^{\pi^*,\mu^*}(s)}{d^{\pi,\mu}(s)} d^{\pi,\mu}(s)\sum_a \pi^*(s,a)A^\pi(s,a) \\ &\le \left\|\frac{d^{\pi^*,\mu^*}}{d^{\pi,\mu}}\right\|_\infty \sum_s d^{\pi,\mu}(s)\sum_a \pi^*(s,a)A^\pi(s,a) \\ &= \left\|\frac{d^{\pi^*,\mu^*}}{d^{\pi,\mu}}\right\|_\infty \frac{1}{(1-\gamma)}A_\pi(\pi^*) \\ &\le \left\|\frac{d^{\pi^*,\mu^*}}{\mu}\right\|_\infty \frac{\delta}{(1-\gamma)}\end{aligned}$$

The last step follows from the observation that $d^{\pi,\mu}(s) \ge \Pr(s_1 = s; \pi, \mu) = \mu(s)$.

Therefore, if we can choose the starting state distribution to be uniform distribution, we can bound the gap from optimal policy by $\frac{n\delta}{(1-\gamma)^2}$, where n is the number of states. (Some applications may not have the flexibility of choosing the starting state distribu-

TRPO [Schulman et al., ICML 2015]

- Provides policy improvement guarantees similar to the Conservative greedy method
- Gets rid of unwieldy mixture policies
- In every iteration moves to a new policy within α distance of the old policy.

Some definitions

Distance between two distributions p, q:

$$D_{TV}(p||q) = \frac{1}{2}\sum_i |p_i - q_i|$$

$$D_{KL}(p||q) = \sum_i p_i \log \frac{p_i}{q_i}$$

Known result:

$$D_{TV}(p||q)^2 \leq D_{KL}(p||q)$$

Distance between two policies π^{new}, π^{old}:

$$D_{TV}^{\max}(\pi^{old}, \pi^{new}) = \max_s D_{TV}(\pi^{old}(\cdot|s), \pi^{new}(\cdot|s))$$

$$D_{KL}^{\max}(\pi^{old}, \pi^{new}) = \max_s D_{KL}(\pi^{old}(\cdot|s), \pi^{new}(\cdot|s))$$

New theorem for bounding policy improvement

[Schulman eta al. 2015]

Theorem

Let $\alpha = D_{TV}^{\max}(\pi, \tilde{\pi})$. *Then, the following bound holds for any starting state* s_1*:*

$$V^{\tilde{\pi}}(s_1) - V^{\pi}(s_1) \geq \frac{1}{(1-\gamma)} A_{\pi}(\tilde{\pi}) - \frac{\alpha^2}{(1-\gamma)^2} 4\gamma\epsilon$$

where

$$A_{\pi}(\tilde{\pi}) = \mathbb{E}_{s\sim(1-\gamma)d^{\pi}}[\sum_a \tilde{\pi}(s,a) A^{\pi}(s,a)]$$

$$\epsilon = \max_{s,a} |A^{\pi}(s,a)|$$

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$$\epsilon = \max_{s,a} |A^\pi(s,a)|$$

Essentially same as Conservative greedy policy improvement guarantee but no mixture policy required.
Compare to Kakade and Langford 2002:

$$V^{\pi^{new}}(s_1) - V^{\pi}(s_1) \geq \frac{\alpha}{(1-\gamma)} A_\pi(\pi') - \frac{\alpha^2}{(1-\gamma)^2} 2\gamma A_\pi^{\max}$$

$$A_\pi(\pi') = \mathbb{E}_{s\sim(1-\gamma)d^\pi}[\sum_a \pi'(s,a) A^\pi(s,a)]$$

where

$$A_\pi^{\max} = \max_s |\sum_a \pi'(s,a) A^\pi(s,a)|$$

TRPO algorithm design

In each iteration $k+1$:

$$\theta_{k+1} = \begin{array}{cc} \arg\max_{\theta} & \mathbb{E}_{s\sim\pi_{\theta_k}}\left[\sum_a \pi_{\theta}(s,a)A^{\pi_{\theta_k}}(s,a)\right] \\ s.t. & \mathbb{E}_{s\sim\pi_{\theta_k}}\left[D_{KL}(\pi_{\theta}(\cdot|s),\pi_{\theta_k}(\cdot|s))\right] \leq \delta \end{array}$$

- Relaxes $D_{KL}^{\max}$ to expected KL divergence over states sampled from old policy.

Some implementation details

- Monte Carlo estimates of $A^{\pi_{\theta_k}}(s, a)$ or $Q^{\pi_{\theta_k}}(s, a)]$, and expectation terms from sample trajectories generated from π_θ.
- 'Vine" simulation and Importance sampling
  - multiple alternate actions can be tried from the observed states in simulation settings.
  - objective replaced by

$$\mathbb{E}_{s\sim\pi_{\theta_k}, a\sim q}\left[\frac{\pi_{\theta}(s, a)}{q(s, a)} A^{\pi_{\theta_k}}(s, a)\right]$$

- Conjugate gradient method for constrained optimization
  - Requires computing Hessian of the KL divergence term.
  - Several approximations proposed for making it efficient.

Proximal Policy Optimization (PPO)

Schulman et al 2017

Recall TRPO:

$$\theta_{k+1} = \begin{array}{rl} \arg\max_{\theta} & \mathbb{E}_{s\sim\pi_{\theta_k}}[\sum_a \pi_{\theta}(s,a)A^{\pi_{\theta_k}}(s,a)] \\ s.t. & \mathbb{E}_{s\sim\pi_{\theta_k}}[D_{KL}(\pi_{\theta}(\cdot|s),\pi_{\theta_k}(\cdot|s))] \leq \delta \end{array}$$

Equivalently

$$\theta_{k+1} = \begin{array}{rl} \arg\max_{\theta} & \mathbb{E}_{s,a\sim\pi_{\theta_k}}[\frac{\pi_{\theta}(s,a)}{\pi_{\theta_k}(s,a)}A^{\pi_{\theta_k}}(s,a)] \\ s.t. & \mathbb{E}_{s,a\sim\pi_{\theta_k}}\left[\frac{\pi_{\theta}(s,a)}{\pi_{\theta_k}(s,a)}\log(\frac{\pi_{\theta}(s,a)}{\pi_{\theta_k}(s,a)})\right] \leq \delta \end{array}$$

PPO algorithm design

For a sample state action pair s^i, a^i, let ratio

$$r_i(\theta) := \frac{\pi_\theta(s^i, a^i)}{\pi_{\theta_k}(s^i, a^i)}$$

PPO replaces the trust region constrained by clipped ratio in the objective and solve the unconstrained problem:

$$\theta_{k+1} = \max_\theta \hat{E}_i \left[\min\left(r_i(\theta)\hat{A}_i, \text{clip}(r_i(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_i\right)\right]$$

where $\hat{E}$ is expectation over state and action pairs s^i, a^i generated from policy π_{θ_k}, and $\hat{A}^i$ is a Monte Carlo estimate of $A^{\pi_{\theta_k}}(s_i, a_i)$.

PPO algorithm

Initialize θ^1. In iteration k

- Generate sample trajectories from π_{θ_k}.
- For each sample s^i, a^i in the trajectories, construct Monte Carlo estimate $\hat{A}^i$ for $A^{\pi_{\theta_k}}(s_i, a_i)$.
- Let $r_i(\theta) := \frac{\pi_\theta(s^i,a^i)}{\pi_{\theta_k}(s^i,a^i)}$.
- Solve

$$\theta_{k+1} = \max_\theta \hat{E}_i \left[\min \left(r_i(\theta)\hat{A}_i, \mathrm{clip}(r_i(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_i \right) \right]$$

Comparisons on MuJoCo

Figure from [Schulman et al. 2017]

PPO used with $\epsilon = 0.2$

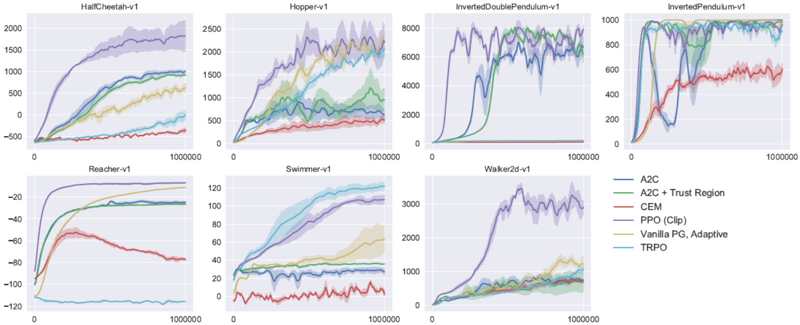

Figure 3: Comparison of several algorithms on several MuJoCo environments, training for one million timesteps.

Other related methods

- Soft policy iteration: softmax instead of greedy policy update. Project back to policy space using KL divergence

$$\pi^{soft-greedy}(\cdot|s^i) \propto \exp(\hat{Q}^i)$$

$\hat{Q}^i$ is a Monte Carlo estimate of $Q^{\pi_{\text{old}}}(s^i, a^i)$

$$\pi^{new} = \arg\min_{\pi' \in \Pi} \hat{\mathbb{E}}_{s^i \sim \pi^{old}} \left[D_{KL}(\pi'(\cdot|s^i) \parallel \pi^{soft-greedy}(\cdot|s^i)) \right]$$

- Soft actor-critic [Harnooja et al. 2018]: (Deep) Q-learning/TD learning to estimate Q-values

$$\pi^{new} = \arg\min_{\pi' \in \Pi} \hat{\mathbb{E}}_{s^i \sim \pi^{old}} \left[D_{KL}(\pi'(\cdot|s^i) \parallel \frac{\exp(Q_\theta(s^i, a^i))}{Z_\theta}) \right]$$

Gradient descent methods for optimizing the above objective.
Removes the need to estimate normalization constant Z_θ.