Problem 5: Linear Program Duality

Lets first group constants and variables

$$\begin{aligned} v + 0.5x_1 - 0.2x_2 - 0.3x_3 &\geq 2 \\ v - 0.1x_1 + 0.1x_2 &\geq 3 \\ v - 0.2x_1 - 0.95x_2 + 0.95x_3 &\geq 5 \end{aligned}$$

Written in standard form, the coefficient vector c, coefficient matrix A, and constant vector b are:

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0.5 & -0.2 & -0.3 \\ 1 & -0.1 & 0.1 & 0 \\ 1 & -0.2 & -0.95 & 0.95 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

Dual LP Formulation

Here, the dual variable vector is $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

Our goal is to maximize $b^T y = 2y_1 + 3y_2 + 5y_3$.

The constraints are found by transposing A and multiplying by y, then setting it equal to c:

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 0.5 & -0.1 & -0.2 \\ -0.2 & 0.1 & -0.95 \\ -0.3 & 0 & 0.95 \end{pmatrix}$$

The constraints are:

$$\begin{aligned} y_1 + y_2 + y_3 &= 1 \\ 0.5y_1 - 0.1y_2 - 0.2y_3 &= 0 \\ -0.2y_1 + 0.1y_2 - 0.95y_3 &= 0 \\ -0.3y_1 + 0.95y_3 &= 0 \end{aligned}$$

Sign Constraints: Since the primal constraints are of the "$\geq$" type, the dual variables must be non-negative: $y_1, y_2, y_3 \geq 0$.

Final Dual Formulation: The complete dual LP is:

$$\begin{aligned} \text{Maximize} \quad & 2y_1 + 3y_2 + 5y_3 \\ \text{s.t.} \quad & y_1 + y_2 + y_3 = 1 \\ & 0.5y_1 - 0.1y_2 - 0.2y_3 = 0 \\ & -0.2y_1 + 0.1y_2 - 0.95y_3 = 0 \\ & -0.3y_1 + 0.95y_3 = 0 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$