ORCS 4529: Reinforcement Learning

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Recall: Q-value iteration

1. Start with an arbitrary initialization $\mathbf{Q}^0 \in \mathbb{R}^{S \times A}$.
2. In every iteration k, improve the Q-value vector as:

$$\mathbf{Q}^k(s,a) = R(s,a) + \gamma \sum_{s'} P(s,a,s') \left(\max_{a'} Q^{k-1}(s',a') \right), \forall s, a$$

3. Stop if $\|Q^k - Q^{k-1}\|_\infty$ is small.
4. Output policy π^k defined as $\pi^k(s) = \arg\max_a Q^k(s,a)$

Recall: Policy iteration using Q-values

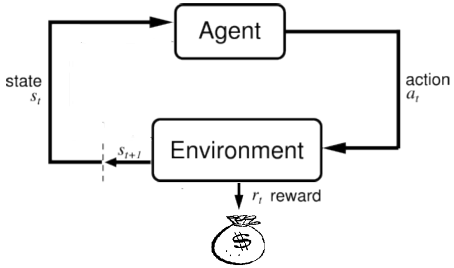
1. Initialize policy π^0.
2. In every iteration $k = 0, 1, 2, \ldots,$
   - (Policy evaluation) Compute value $Q^{\pi^k}(s, a)$, the Q-values of policy π^k for all s, a.
   - (Greedy Policy improvement) Compute new policy

$$\pi^{k+1}(s) := \arg\max_a Q^{\pi_k}(s, a), \forall s$$

3. Stop when $\pi^{k+1} = \pi^k$.

Relaxed stopping criteria: not much change in policy or its value.

Learning from observations



Unknown MDP model (R, P) of the environment, but can interact with the environment (take action a_t) to observe sample reward r_t and transition $s_t \to s_{t+1}$

$$\mathbb{E}[r_t | s_t = s, a_t = a] = R(s, a)$$

$$\Pr(s_{t+1} = s' | s_t = s, a_t = a) = P(s, a, s')$$

Reinforcement Learning: DP based Algorithms

How to implement Q-value iteration and policy iteration using samples?

How to interact with the environment to generate useful samples?

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First attempt: Implementing Q-VI with samples from environment

1. Start with an arbitrary initialization $\mathbf{Q}^0 \in \mathbb{R}^{S \times A}$.
2. For $t = 1, 2, \ldots,$,
   observe s_t, take action a_t, observe reward r_t, next state s_{t+1}
   improve the Q-value at s_t, a_t only as:

$$\mathbf{Q}^t(s_t, a_t) = r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a') \right)$$

3. Stop if $\|Q^t - Q^{t-1}\|_\infty$ is small.
4. Output policy π^t defined as $\pi^t(s) = \arg\max_a Q^t(s, a)$

Justification

For the state action pair s_t, a_t, we have an unbiased estimator of the DP update

$$\mathbb{E}[r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a') \right) | s_t = s, a_t = a]$$

$$= R(s,a) + \gamma \sum_{s'} P(s,a,s') \left(\max_{a'} Q^{t-1}(s', a') \right)$$

Concerns

- Error in update: Single sample used in every update.
  - We have an unbiased estimate of the Q-VI update but a large variance
  - (Why not take multiple samples before updating?)
  - Do not have unbiased estimate of Q-value
- Coverage: Update of only one component s_t, a_t: the visited state and action

Will it still converge? How to explore and update all states and actions? How to get more samples for important states and actions?

Q-learning

Mitigating the concerns

- Idea 1: Slow down the update

$$\mathbf{Q}^t(s_t, a_t) = (1 - \alpha_t)Q^{t-1}(s_t, a_t) + \alpha_t \left(r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a') \right) \right)$$

  where $\alpha_t \in [0, 1]$ acts as a "learning rate".

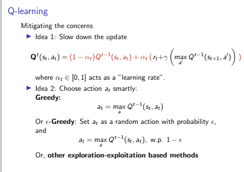
- Idea 2: Choose action a_t smartly:
  Greedy:

$$a_t = \max_a Q^{t-1}(s_t, a_t)$$

  Or ϵ-**Greedy**: Set a_t as a random action with probability ϵ, and

$$a_t = \max_a Q^{t-1}(s_t, a_t), \text{ w.p. } 1 - \epsilon$$

  Or, **other exploration-exploitation based methods**



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  Or, **other exploration-exploitation based methods**

1. α_t Acts as learning rate. Helps convergence
2. the choice of action is important because only the visited state and actions get samples and updates. Need to balance between visiting important (those taken and visited by optimal policy) actions and states vs. exploring all actions and states in order to learn the optimal policy

Q-learning algorithm (tabular setting)

1. Start with an arbitrary initialization $\mathbf{Q}^0 \in \mathbb{R}^{S \times A}$.
2. For $t = 1, 2, \ldots,$
   Observe s_t, take action a_t using a smart policy , observe reward r_t, next state s_{t+1}
   Improve the Q-value at s_t, a_t only as:
$$\mathbf{Q}^t(s_t, a_t) = (1-\alpha_t)Q^{t-1}(s_t, a_t) + \alpha_t(r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a')\right))$$
3. Stop if $\|Q^t - Q^{t-1}\|_\infty$ is small.
4. Output policy π^t defined as $\pi^t(s) = \arg\max_a Q^t(s, a)$

How to pick the actions? If ϵ-greedy that which ϵ? How to set the learning rate? Varies across implementations.

SGD interpretation of Q-learning update

Recall goal is to have $Q^t \rightarrow_{t\rightarrow\infty} Q^*$ where for all s, a

$$Q^*(s,a) = R(s,a) + \sum_{s'} P(s,a,s')(\max_{a'} Q^*(s',a'))$$

At the time t, we get an estimates z_t of rhs at $s = s_t, a = a_t$.

$$z_t := r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a') \right)$$

(z_ts are referred to as **target**).
We want to find $\hat{Q}$ such that

$$\hat{Q}(s_t, a_t) \approx z_t, \forall t$$

Supervised learning view: find best fit by minimizing the squared loss over all samples:

$$\min_{\hat{Q}} \sum_t (\hat{Q}(s_t, a_t) - z_t)^2$$

SGD interpretation

$$\min_{\hat{Q}} \sum_t (\hat{Q}(s_t, a_t) - z_t)^2$$

Gradient of t^{th} term in objective with respect to $\hat{Q}$

$$2(\hat{Q}(s_t, a_t) - z_t)\mathbf{1}_{s_t, a_t}$$

SGD or online gradient descent: At iteration t:

$$\begin{aligned}\hat{Q}(s_t, a_t) &\leftarrow \hat{Q}(s_t, a_t) - \alpha_t \underbrace{(\hat{Q}(s_t, a_t) - z_t)}_{\text{temporal difference}} \\ &= (1 - \alpha_t)\hat{Q}(s_t, a_t) + \alpha_t z_t\end{aligned}$$

Useful interpretation for studying convergence and extensions to deep learning.

Temporal difference (TD) in Q-learning

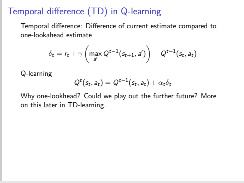
Temporal difference: Difference of current estimate compared to one-lookahead estimate

$$\delta_t = r_t + \gamma \left(\max_{a'} Q^{t-1}(s_{t+1}, a') \right) - Q^{t-1}(s_t, a_t)$$

Q-learning

$$Q^t(s_t, a_t) = Q^{t-1}(s_t, a_t) + \alpha_t \delta_t$$

Why one-lookhead? Could we play out the further future? More on this later in TD-learning.



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Why one-lookhead? Could we play out the further future? More on this later in TD-learning.

Difficult in this case since we don't know the optimal policy, but we could do that in policy iteration where policy is fixed in every iteration

Lab 1 Problem 1

Implement Tabular Q-learning for a simple OpenAI Gym (gymnasium) environment "Frozen-lake".

Q-learning with function approximation and deep learning

How to handle large state space?

Large-scale Q-learning

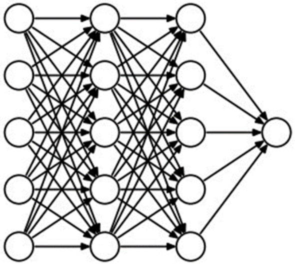
Use a parametric approximation $Q_\theta(s, a) = f_\theta(x_{s,a})$ for $Q(s, a)$.

- Example 1 Linear regression: Given feature embedding $x_{s,a}$ for state-action pair s, a, define
  $Q_\theta(s, a) = \theta_0 + \theta_1 x_1(s, a) + \ldots + \theta_n x_n(s, a)$
- Example 2 DNN: $Q_\theta(s, a) = f_\theta(x_{s,a})$ with θ being the parameters of the DNN

How do we find a good parameter θ?

Recall supervised learning basics

- Prediction Model (e.g. linear model or DNN)



$x \rightarrow$ $\rightarrow f_\theta(x)$

- Training: Given labeled dataset (x_i, y_i) a training algorithm (e.g., training in tensorflow, pytorch etc.) fits model parameters θ such that a loss function (e.g. square loss or cross-entropy loss) is minimized

$$\min_\theta \frac{1}{N} \sum_{i=1}^{N} L(f_\theta(x_i), y_i)$$

Q-learning via supervised learning

- Task: Find a θ such that for every s, a, the Bellman equation,

$$Q(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)}[R(s,a,s') + \gamma \max_{a'} Q(s',a')]$$

can be approximated well for all s, a, i.e.,

$$f_\theta(x_{s,a}) \approx \mathbb{E}_{s' \sim P(\cdot|s,a)}[R(s,a,s') + \gamma \max_{a'} f_\theta(x_{s',a'})]$$

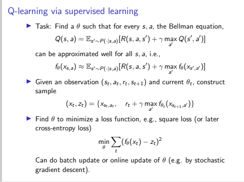
- Given an observation (s_t, a_t, r_t, s_{t+1}) and current θ_t, construct sample

$$(x_t, z_t) = \{x_{s_t,a_t}, \quad r_t + \gamma \max_{a'} f_{\theta_t}(x_{s_{t+1},a'})\}$$

- Find θ to minimize a loss function, e.g., square loss (or later cross-entropy loss)

$$\min_\theta \sum_t (f_\theta(x_t) - z_t)^2$$

Can do batch update or online update of θ (e.g. by stochastic gradient descent).



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$$\min_\theta \sum_t (f_\theta(x_t) - z_t)^2$$
Can do batch update or online update of θ (e.g. by stochastic gradient descent).

1. Recall SGD interpretation of Q learning: Q-learning update is a stochastic gradient step to minimize the squared loss function.

Deep Q-learning algorithm (DQN)

Let $L(x, y)$ be a loss function like squared loss function $L(x, y) = (x - y)^2$. Let $Q_\theta(s, a) = f_\theta(x_{s,a})$ be the function approximation, e.g., a DNN.

Repeat for $t = 1, 2, ..., T$,

- Observe state s_t, take an action a_t
  - ϵ-greedy policy $a_t = \arg\max_a Q_{\theta_t}(s_t, a) = \arg\max_a f_{\theta_t}(x_{s_t,a})$ with probability $1 - \epsilon$ and random action with probability ϵ
- Observe reward r_t, transition to state s_{t+1}.
- Construct feature embedding $x_t = x_{s_t,a_t}$. Use current θ_t to construct target

$$z_t = r_t + \gamma \max_{a'} Q_{\theta_t}(s_{t+1}, a') = r_t + \gamma \max_{a'} f_{\theta_t}(x_{s_{t+1},a'})$$

- Set learning rate α_t, and perform one stochastic gradient step

$$\theta_{t+1} \leftarrow \theta_t - \alpha_t \nabla_{\theta|_{\theta_t}} L(f_\theta(x_t), z_t)$$

Output parameter θ_T (learned policy $\pi(s) = \arg\max_a f_{\theta_T}(x_{s,a})$).

Remark on large action spaces

Can embed them using neural network just like states,

$$Q_\theta(s, a) = f_\theta(x_{s,a})$$

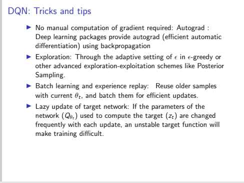
but in order to compute the policy and the target, we need to be able to compute

$$\max_a Q_\theta(s, a) \equiv \max_a f_\theta(x_{s,a})$$

Many practical scenarios: actions space is large but only few of them available in a given state, so above can be done by enumerating all feasible actions.

DQN: Tricks and tips

- No manual computation of gradient required: Autograd : Deep learning packages provide autograd (efficient automatic differentiation) using backpropagation
- Exploration: Through the adaptive setting of ϵ in ϵ-greedy or other advanced exploration-exploitation schemes like Posterior Sampling.
- Batch learning and experience replay: Reuse older samples with current θ_t, and batch them for efficient updates.
- Lazy update of target network: If the parameters of the network (Q_{θ_t}) used to compute the target (z_t) are changed frequently with each update, an unstable target function will make training difficult.



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1. These schemes use not only time but also the number of samples and hence uncertainty of the estimates to set the exploration parameter differently for different states and actions

Lab 3

Deep Q-learning and the OpenAI gym environment Cartpole

Contents I

Recall: Policy iteration using Q-values

1. Initialize policy π^0.
2. In every iteration $k = 0, 1, 2, \ldots,$
   - (Policy evaluation) Compute value $Q^{\pi^k}(s, a)$, the Q-values of policy π^k for all s, a.
   - (Greedy Policy improvement) Compute new policy

$$\pi^{k+1}(s) := \arg\max_a Q^{\pi_k}(s, a), \forall s$$

3. Stop when $\pi^{k+1} = \pi^k$.

Relaxed stopping criteria: not much change in policy or its value.

Computing Q-values of a fixed policy π

- Monte Carlo method:

$$Q^{\pi}(s,a) = \mathbb{E}[r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots | s_1 = s, a_1 = a; a_t = \pi(s)]$$

  Simulate the policy π starting from every state s and action a, and use the discounted sum of sample rewards to estimate the expected value.
  Easy extension to learning from observations

- Q-value-iteration : Based on Bellman equation (DP)

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s'} P^{\pi}(s,a,s') \mathbb{E}_{a' \sim \pi(s)}[Q^{\pi}(s',a')]$$

  TD-learning uses ideas similar to Q-learning

Q-value iteration for learning value of a policy π

A subroutine of policy iteration

1. Start with an arbitrary initialization $\mathbf{Q}^0 \in \mathbb{R}^{S \times A}$.
2. In every iteration k, improve the Q-value vector as:

$$\mathbf{Q}^k(s,a) = R(s,a) + \gamma \sum_{s'} P(s,a,s') \mathbb{E}_{a' \sim \pi(s)} \left[Q^{k-1}(s',a') \right], \forall s, a$$

3. Stop if $\|Q^k - Q^{k-1}\|_\infty$ is small.
4. Output policy π^k defined as $\pi^k(s) = \arg\max_a Q^k(s,a)$

TD (Temporal difference) learning algorithm for learning Q-values of policy π

A subroutine of policy iteration

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2. For $t = 1, 2, \ldots,$
   Observe s_t, take action a_t using a smart policy, observe reward r_t, next state s_{t+1}
   Update the Q-value at s_t, a_t only as:

   $$\mathbf{Q}^t(s_t, a_t) = (1 - \alpha_t) Q^{t-1}(s_t, a_t) + \alpha_t (r_t + \gamma \left(\mathbb{E}_{\mathbf{a'} \sim \pi(\mathbf{s_{t+1}})} [Q^{t-1}(s_{t+1}, a')] \right))$$

3. Stop if $\|Q^t - Q^{t-1}\|_\infty$ is small.

Question: What's the difference compared to Q-learning? Why is this learning Q^π and not Q^*?

Concerns and Choices

Concerns

- Sampling estimation error in update
- Coverage of all state-action pairs (Why?)

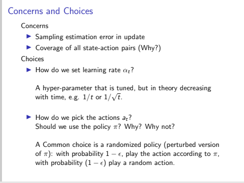
Choices

- How do we set learning rate α_t?

  A hyper-parameter that is tuned, but in theory decreasing with time, e.g. $1/t$ or $1/\sqrt{t}$.

- How do we pick the actions a_t?
  Should we use the policy π? Why? Why not?

  A Common choice is a randomized policy (perturbed version of π): with probability $1-\epsilon$, play the action according to π, with probability $(1-\epsilon)$ play a random action.



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1. For the learning rate, same explanation (and role in convergence) as Q-learning. We have a single sample - unbiased but high variance estimate of Q-VI update. Also not unbiased estimate of Q-value.
2. Why π? If we use π, we will have good estimates available for $Q(s', a')$ for $a' \sim \pi(s')$. This is the action we need for constructing the target.
3. Why not π? If we use only policy π to pick a_t, then we wont have any updates for $Q(s, a), a \notin \pi(s)$. This means we won't learn Q values of actions that are not taken by the current policy, this will make it impossible to improve the policy in the policy improvement step. Also, using policy π will mean that we will only visit learn Q-values on states that are visited by policy π. It is possible that the current policy does not go to the important states- e.g. states that are close to some winning state or are on path of the winning state. If we don't learn Q-values on those states we won't be able to learn to improve policy s on those states in the improvement step.

Monte Carlo Method for estimating Q-values of a fixed policy

Simulate policy π for T steps for a large enough T such that γ^T is very small.

- For $t = 1, 2, \ldots, T$
  Observe s_t, take action $a_t \sim \pi(s_t)$, observe reward r_t, next state s_{t+1}
- Output sample trajectory $\tau = (s_1, a_1, r_1, s_2, a_2, s_3, \ldots, s_T)$
- For each (s_t, a_t) in the trajectory τ,

$$\hat{Q}^{\pi}(s_t, a_t) \leftarrow r_t + \gamma r_{t+1} + \cdots + \gamma^{T-1} r_T$$

Comparison to TD-learning

- Sampling estimation error in certain cases can be better than TD-learning due to larger lookahead $\equiv$ less bias
- Coverage of state-action pairs: Perturbing the policy needed for exploration but will introduce bias in the estimation of Q^π.

Unlike TD-learning, in MC Sample collection policy has to be the same as the policy being evaluated.

Perturbing by ϵ_k is ok if we decrease ϵ_k over iterations k

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Examples: TD-learning vs. Monte Carlo

Comparison of estimation error

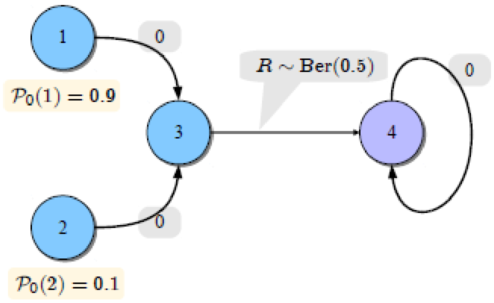

Figure: Source: Algorithms for Reinforcement Learning by Csaba Szepasvari

Examples: TD-learning vs. Monte Carlo

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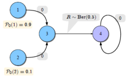
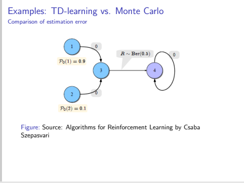


Figure: Source: Algorithms for Reinforcement Learning by Csaba Szepasvari

TD(0) converges faster: than Monte-carlo. The initial states are either 1 or 2. The process starts at state 1 with a higher probability (0.9), less frequently in state 2. Consider now how $TD(0)$ will behave at state 2. By the time state 2 is visited the k^{th} time, on the average state 3 would have already been visited $10k$ times. Assume that $\alpha_t = 1/(t+1)$ (the TD updates with this step size reduce to $\frac{1}{t+1}(t\hat{V}(s_t) + z_t)$, i.e., averaging of targets). At state 1 and 2, the target is $\hat{V}(3)$ (since immediate reward is 0 and transition probability to state 3 is 1). Therefore, whenever state 2 is visited the TD(0) sets its value as the average of estimates $\hat{V}^t(3)$ over the time steps t when state 1 was visited (similarly for state 2). At state 3 the TD(0) update reduces to averaging the Bernoulli rewards incurred upon leaving state 3. At the k^{th} visit of state 2, $\mathrm{Var}(\hat{V}(3)) \simeq 1/(10k)$ Clearly, $\mathbb{E}[\hat{V}(3)] = 0.5$. Thus, each update to the TD(0) prediction for state 2 is unbiased, and variance of the k^{th} target decreases as $1/(10k)$.

Now, consider the Monte-Carlo method. The Monte-Carlo method ignores the estimate of the value of state 3 and uses the Bernoulli rewards directly. In particular, $Var(\mathcal{R}_t|s_t = 2) = 0.25$, i.e., the variance of the target does not change with time. On this example, this makes the Monte-Carlo method slower to converge, showing that sometimes bootstrapping might indeed help.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Comparison of estimation error

Figure: Source: Algorithms for Reinforcement Learning by Csaba Szepasvari

To see an example when bootstrapping is not helpful, imagine that the problem is modified so that the reward associated with the transition from state 3 to state 4 is made deterministically equal to one. In this case, the Monte-Carlo method becomes faster since $\mathcal{R}_t = 1$ is the true target value. On the other hand, for the value of state 2 to get close to its true value, TD(0) has to wait until the estimate of the value at state 3 becomes close to its true value. This can slow down the convergence of TD(0). In fact, one can imagine a longer chain of states, where state $i+1$ follows state i, for $i \in 1, ..., N$ and the only time a nonzero reward is incurred is when transitioning from state $N-1$ to state N. In this example, the rate of convergence of the Monte-Carlo method is not impacted by the value of N, while TD(0) would get slower with N increasing (in TD(0), it will take N episodes for state 3's value to be updated to its correct value).

Lab 1 Problem 2

Implement Tabular Policy iteration with TD-learning on a simple OpenAI gym environment "Frozen-lake".

Policy iteration with function approximation

- Initialize θ^0, initialize policy $\pi^1 = \arg\max_a f_{\theta_0}(x_{s,a})$.
- In every iteration $k = 1, 2, \ldots$
  - (Policy evaluation): Use Deep TD-learning, train DNN $Q^{\pi_k}(s, a) := f_{\theta_k}(x_{s,a})$ to learn the value of policy π_k.
  - (Policy improvement): New policy π_{k+1} is defined by

$$\pi^{k+1}(s) := \arg\max Q^{\pi_k}(s, a) = \arg\max_a f_{\theta_k}(x_{s,a})$$

Note: Only the parameter θ_k needs to be stored.

Deep TD-learning subroutine

Learn the Q-value of given policy π using function approximation.

Repeat for $t = 1, 2, ..., T$,

- Observe state s_t, take an action a_t
  - ϵ-perturbed policy $a_t = \pi(s_t)$ w.p. $1 - \epsilon$, random action w.p. ϵ.
- Observe reward r_t, transition to state s_{t+1}.
- Use current θ_t to construct (1-lookahead) target

  $$z_t = r_t + \gamma f_{\theta_t}(x_{s_{t+1}, a_{t+1}})$$

  or use larger look-ahead up to $\tau + 1$ steps or Monte-Carlo

  $$z_t = \sum_{i=t}^{t+\tau} \gamma^{i-1} r_i + \gamma^{\tau+1} f_{\theta_t}(x_{s_{t+\tau+1}, a_{t+\tau+1}})$$

  where $a_{t+\tau+1} = \pi(s_{t+\tau+1})$.
- Set learning rate α_t, and perform one stochastic gradient step

  $$\theta_{t+1} \leftarrow \theta_t - \alpha_t \nabla_{\theta|_{\theta_t}} L(f_\theta(x_{s_t, a_t}), z_t)$$

Output θ_T, where $Q^\pi(s, a) \approx f_{\theta_T}(x_{s,a})$.

Contents I

Convergence of Tabular Q-learning, TD-learning

- We use the connections to the Stochastic Approximation method [Robbins and Monroe 1951]
- Asymptotic convergence proved in "Asynchronous Stochastic Approximation and Q-Learning" by John N. Tsitsiklis, 1994
- More recent works provide finite sample bounds:
  - Sheng Zhang, Zhe Zhang, Siva Theja Maguluri, Finite Sample Analysis of Average-Reward TD Learning and Q-Learning, NeurIPS 2021.
    (... and other more recent works by the last author)
  - Guannan Qu and Adam Wierman, Finite-time analysis of asynchronous stochastic approximation and Q-learning. COLT 2020.
  - Rayadurgam Srikant and Lei Ying. Finite-time error bounds for linear stochastic approximation and TD-learning. COLT 2019.

Stochastic Approximation (SA):

A method for finding a solution $x^* \in \mathbb{R}^n$ to

$$h(x) = 0$$

using noisy observations $\delta_t = h(x_t) + \omega_t$.

The SA method

- At time t on seeing sample δ_t, update

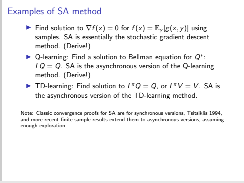
$$x_{t+1} \leftarrow x_t + \alpha_t \delta_t$$

Classic convergence theorems show $h(x_t) \to 0$ under certain conditions on $h(\cdot)$, the distribution of noise ω_t, and α_ts.

Examples of SA method

- Find solution to $\nabla f(x) = 0$ for $f(x) = \mathbb{E}_y[g(x, y)]$ using samples. SA is essentially the stochastic gradient descent method. (Derive!)
- Q-learning: Find a solution to Bellman equation for Q^*: $LQ = Q$. SA is the asynchronous version of the Q-learning method. (Derive!)
- TD-learning: Find solution to $L^\pi Q = Q$, or $L^\pi V = V$. SA is the asynchronous version of the TD-learning method.

Note: Classic convergence proofs for SA are for synchronous versions, Tsitsiklis 1994, and more recent finite sample results extend them to asynchronous versions, assuming enough exploration.



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1. $h(x) = -\nabla f(x)$, $\delta_t = -\nabla g(x_t, y_t)$, $x_{t+1} \leftarrow x_t - \alpha_t \nabla g(x_t, y_t)$
2. $h(x) = LQ - Q$,

$$\delta_t = \hat{L}Q_t - Q_t = r_t + \gamma \max_{a'} Q_t(s_{t+1}, a') - Q_t(s_t, a_t)$$

$$Q_{t+1}(s_t, a_t) \leftarrow Q_t(s_t, a_t) + \alpha_t \delta_t$$

SA method convergence

Essential Assumptions:

On step sizes:

$$\sum_{t=1}^{\infty} \alpha_t = \infty, \quad \sum_{t=1}^{\infty} \alpha_t^2 < \infty \tag{1}$$

On noise: martingale zero-mean and sub-Guassian:

$$\mathbb{E}[\omega_t|\mathcal{F}_{t-1}] = 0, \quad \mathbb{E}[\|\omega_t\|^2|\mathcal{F}_{t-1}] \leq A + B\|x_n\|^2 \tag{2}$$

for some constant A, B.

Here $\mathcal{F}_{n-1}$ is the σ-algebra defined by the history $\{x_1, \alpha_1, \omega_1, \ldots, x_{n-1}, \alpha_{n-1}, \omega_{n-1}, x_n, \alpha_n\}$. Noise ω_n is measurable with respect to $\mathcal{F}_n$. Note that Guassian noise $\omega_n \sim \mathcal{N}(\mathbf{0}, AI)$ satisfies the above assumption.

Classic SA convergence theorem

Theorem

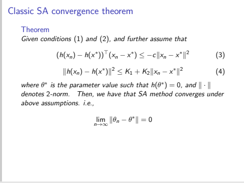
Given conditions (1) *and* (2), *and further assume that*

$$(h(x_n) - h(x^*))^\top (x_n - x^*) \leq -c\|x_n - x^*\|^2 \tag{3}$$

$$\|h(x_n) - h(x^*)\|^2 \leq K_1 + K_2\|x_n - x^*\|^2 \tag{4}$$

where θ^* *is the parameter value such that* $h(\theta^*) = 0$, *and* $\|\cdot\|$ *denotes* 2*-norm. Then, we have that SA method converges under above assumptions. i.e.,*

$$\lim_{n\to\infty} \|\theta_n - \theta^*\| = 0$$



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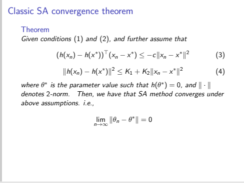
We can remove $h(\theta^*)$ from above expressions since it is 0 but the above form is more interpretable. Chech what this means for $h(x) = -\nabla f(x)$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Here is a proof sketch. Let θ be the target, i.e., $h(\theta) = 0$

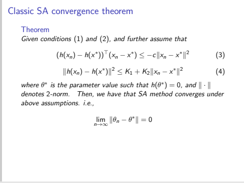
$$b_n = \mathbb{E}[(\theta_n - \theta)^2]$$

$$\begin{aligned}
b_{n+1} &= \mathbb{E}[\|\theta_{n+1} - \theta\|^2] \\
&= \mathbb{E}[\|\theta_{n+1} - \theta_n + \theta_n - \theta\|^2] \\
&= \mathbb{E}[\|\theta_{n+1} - \theta_n\|^2 + 2(\theta_{n+1} - \theta_n)^\top(\theta_n - \theta) + \|\theta_n - \theta\|^2] \\
b_{n+1} - b_n &= \mathbb{E}[\|\theta_{n+1} - \theta_n\|^2] + 2\mathbb{E}[(\theta_{n+1} - \theta_n)^\top(\theta_n - \theta)] \\
&= \mathbb{E}[\alpha_n^2\|Z_n\|^2] + 2\mathbb{E}[\mathbb{E}[\alpha_n(h(\theta_n) + \omega_n)^\top(\theta_n - \theta)|\mathcal{F}_{n-1}]] \\
&= \mathbb{E}[\alpha_n^2\|Z_n\|^2] + 2\alpha_n\mathbb{E}[h(\theta_n)^\top(\theta_n - \theta)] \\
&= \alpha_n^2\mathbb{E}[\mathbb{E}[\|h(\theta_n)\|^2 + \|\omega_n\|^2 + 2h(\theta_n)^\top\omega_n|\mathcal{F}_{n-1}]] + 2\alpha_n\mathbb{E}[h(\theta_n)^\top(\theta_n - \theta)] \\
&= \alpha_n^2\mathbb{E}[\|h(\theta_n)\|^2 + \|\omega_n\|^2] + 2\alpha_n\mathbb{E}[h(\theta_n)^\top(\theta_n - \theta)]
\end{aligned}$$

using $\mathbb{E}[\omega_n|\mathcal{F}_{n-1}] = 0$. Now, we use

$$(h(\theta_n))^\top(\theta_n - \theta) \le -c\|\theta_n - \theta\|^2 \tag{5}$$

$$\|h(\theta_n)\|^2 \le K_1 + K_2\|\theta_n - \theta\|^2 \tag{6}$$



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$$\lim_{n\to\infty} \|\theta_n - \theta^*\| = 0$$

Then,

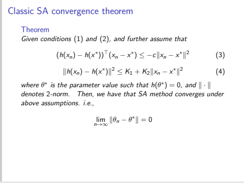
$$\begin{aligned} b_{n+1} - b_n &\le \alpha_n^2 K_1 + \alpha_n(K_2\alpha_n - 2c)\mathbb{E}[\|\theta_n - \theta\|^2] + \mathbb{E}[\alpha_n^2\omega_n^2] \\ &= \alpha_n^2 K_1 + \alpha_n(K_2\alpha_n - 2c)b_n + \mathbb{E}[\alpha_n^2\omega_n^2] \end{aligned} \tag{7}$$

Since $\sum_n \alpha_n^2 < \infty$, there exists some finite n after which $K_2\alpha_n < c$. Therefore, for that n and higher, the second term is negative. Therefore for some n_0,

$$\sum_{n \ge n_0} (b_{n+1} - b_n) \le \sum_{n \ge n_0} (\alpha_n^2 K_1 + \mathbb{E}[\alpha_n^2\omega_n^2]) < \infty$$

(using assumption of bounded variance of noise and $\sum_n \alpha_n^2 < \infty$)

This implies $b_{n+1} - b_n \to 0$, i.e., b_n converges.



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where θ^* is the parameter value such that $h(\theta^*) = 0$, and $\|\cdot\|$ denotes 2-norm. Then, we have that SA method converges under above assumptions. i.e.,

$$\lim_{n\to\infty} \|\theta_n - \theta^*\| = 0$$

Now, we want to show that b_n converges to 0. Suppose for contradiction that $\lim_{n\to\infty} b_n = \delta > 0$, then we would have $\sum_{n\geq n_0} \alpha_n c b_n \to \infty$ (since $\sum_n \alpha_n = \infty$). However, if we sum up (7) and move the terms around

$$\begin{aligned}
\sum_{n\geq n_0}^{N} \alpha_n c b_n &\leq b_{n_0} - b_{N+1} + \mathbb{E}[\sum_{n\geq n_0}^{N} \alpha_n^2(\omega_n^2 + K_1)] && (8)\\
\text{(using } b_{N+1} \geq 0) \quad &\leq b_{n_0} + \mathbb{E}[\sum_{n\geq n_0}^{N} \alpha_n^2(\omega_n^2 + K_1)] && (9)\\
&< \infty \text{ (using assumption of bounded variance of noise and } \sum_n \alpha_n^2 < \infty) && (10)
\end{aligned}$$

This is a contradiction. Therefore, it must hold that $\lim_{n\to\infty} b_n = 0$.

A corollary

Corollary (Corollary of Theorem 1)

Given conditions (1) *and* (2), *and further assume that* $h(V) = LV - V$*, where the L operator satisfied Euclidean norm contraction, ie.,*

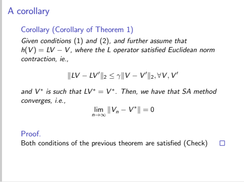
$$\|LV - LV'\|_2 \leq \gamma \|V - V'\|_2, \forall V, V'$$

and V^* *is such that* $LV^* = V^*$*. Then, we have that SA method converges, i.e.,*

$$\lim_{n \to \infty} \|V_n - V^*\| = 0$$

Proof.

Both conditions of the previous theorem are satisfied (Check) □



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and V^* is such that $LV^* = V^*$. Then, we have that SA method converges, i.e.,

$$\lim_{n\to\infty} \|V_n - V^*\| = 0$$

Proof.
Both conditions of the previous theorem are satisfied (Check) □

Suppose $h(\theta) = L\theta - \theta$, and L operator satisfied Euclidean norm contraction, ie., $\|LV - LV'\| \leq \gamma \|V - V'\|$. Then, using $L\theta^* = \theta^*$, and Cauchy-Schwartz,

$$(L\theta_n - \theta^*)^\top (\theta_n - \theta^*) \leq \|L\theta_n - L\theta^*\| \cdot \|\theta_n - \theta^*\| \leq \gamma \|\theta_n - \theta^*\|^2$$

Subtracting $(\theta_n - \theta^*)^\top (\theta_n - \theta^*)$ from both sides:

$$(L\theta_n - \theta_n)^\top (\theta_n - \theta^*) \leq -(1-\gamma)\|\theta_n - \theta^*\|^2$$

which is same as

$$h(\theta_n)^\top (\theta_n - \theta^*) \leq -c\|\theta_n - \theta^*\|^2 \tag{11}$$

That is, the first condition in Theorem 1 is satisfied. And, further

$$\begin{aligned}
\|h(\theta_n)\| &= \|L\theta_n - \theta_n\| \\
&= \|(L\theta_n - L\theta^*) + (L\theta^* - \theta_n)\| \\
&\leq \|L\theta_n - L\theta^*\| + \|\theta^* - \theta_n\| \\
&\leq (1+\gamma)\|\theta_n - \theta^*\| \\
&= (1+\gamma)\|\theta_n - \theta^*\|
\end{aligned}$$

Therefore, both the conditions of Theorem 1 are satisfied, and the convergence result

Convergence proofs relevant to Q-learning, TD-learning

Synchronous version

Theorem (Tsitsiklis 1994, Corollary of Proposition 4.5 in Bertsekas, 1996)

Given conditions (1) *and* (2), *and further assume that* $h(V) = LV - V$, *where* L *is a contraction mapping under infinity-norm, i.e., for all iterates* V_n

$$\|LV_n - LV^*\|_\infty \leq \gamma \|V_n - V^*\|_\infty$$

for some scalar $\gamma \in [0, 1)$, *then* $V_n \to V^*$ *as* $n \to \infty$ *with probability* 1.

Convergence counterexample under function approximation

Convergence: Counter-example The following example has been taken from [Bertsekas, 1995] Consider an MDP with states indexed by $i = 1, 2, \ldots, n$. Under the actions taken by a fixed policy π, the (deterministic) reward in state i is given by $g_i = 1, i \neq n$, and $g_n = -(n+1)$. Starting state is n, and from state i there is a deterministic transition to $i-1$ under this policy. State 1 is terminal state, the rewards are 0 after state 1. Therefore, (for $\gamma = 1$), $V(i) = i$ for all $i \neq n$, and $V(n) = -2$. Consider linear approximation (single parameter) $V_\theta(i) = \theta i$, where $\theta \in \mathbb{R}$. A desirable approximator in this family is $V_\theta(i) = i$, i.e., $\theta = 1$. For large n, $\theta = 1$ is in fact also close to the least square estimator $\arg\min_\theta \frac{1}{n}\sum_{i=1}^n (V(i) - i\theta)^2$ We show that $TD(0)$-learning converges to $\theta \approx -1$ instead.

For simplicity we consider lazy update of target in TD learning. We update the target only at the end of every episode. Let $\hat{\theta}$ is the parameter value at the beginning of an episode, then the update on observing reward-state transition $(i, g_i, i-1)$ is given by:

$$\theta \leftarrow \theta + \alpha(g_i + V_\theta(i-1) - V_\theta(i))\frac{\partial V_\theta(i)}{\partial \theta} = \theta + \alpha(g_i + \theta(i-1) - \theta i)i = \theta + \alpha(g_i - \theta)i$$

For simplicity suppose that the updates are applied in batch at the end of every episode: Then, at the end of an episode, θ would be updated as

$$\theta \leftarrow \theta + \alpha \sum_{i=1}^{n} (g_i - \theta) i$$

At convergence the update should converge to 0, i.e., θ would be solution of

$$\sum_{i=1}^{n} (g_i - \theta) i = 0$$

giving

$$\theta = \frac{\sum_{i=1}^{n} g_i i}{\sum_{i=1}^{n} i} = \frac{-(n+1)n + (n-1)n/2}{n(n+1)/2} = -1 - \frac{2}{(n+1)} \approx -1$$

In fact, if one uses TD(1) (full lookhead) instead in this case, the convergence is to the least square estimator: Per step update in that case is given by

$$\begin{aligned} \theta &\leftarrow \theta + \alpha (g_i + g_{i-1} + \cdots + g_1 - V_\theta(i)) \frac{\partial V_\theta(i)}{\partial \theta} \\ &= \theta + \alpha (g_i + g_{i-1} + \cdots + g_1 - \theta i) i \end{aligned}$$

So that the batch update at the end of the episode is given by:

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