A note on long-term distribution of wind induced load effects with applications to structures with high natural periods

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1 Introduction and background

Response and load effects for structures subjected to wind loading needs to be studied within the framework of random vibration since the wind loading is stochastic in nature. In this note we will denote the dynamic response of the system $X(t)$, which can be displacement response or a load effect.

The characteristic load effect used in design is often defined as the load effect with a yearly probability p of exceedence. A return period of 50 years $p = 1/50 = 0.02$ is often used. In wind engineering it is common practice to calculate the dynamic response of the system considering a mean wind velocity with 50 years return period and to further multiply the standard deviation of $X(t)$ with a peak factor to obtain the expected value of the extreme response during the averaging period considered. This will however in general not give a load effect with 50 years return period since this will require a long-term extreme value analysis of the extremes of the stochastic variable $X(t)$ and not of the mean wind velocity V

This note starts by giving a brief overview of the classical theory of short-term extremes to set the scene for more appropriate and accurate long-term extreme value analysis of the response. Only the main concepts are presented so [3] [6] and [2] are recommended reading for short-term extreme value analysis, while a comprehensive overview of long-term extreme value analysis is presented in [2].

The extreme response of a generalized simple one degree of freedom system is studied in detail to investigate how the long-term and short-term extreme values corresponds when considering 10 minutes and 1 hour short term averaging periods.

2 Basic short-term extreme value theory

2.1 Distribution of peaks

Step1: Assume that the process is stationary

Let us consider a stationary stochastic process $X(t)$. In design of structures it is often the distribution of peak values that are of interest. We thus want to calculate how often an arbitrary realization of the stochastic variable $X(t)$ can be expected to exceed a given level a. This can be assessed by considering the probability that the process reach the level a with a positive time derivative, resulting in the well known Rice formula.

$$\nu_X^+(a) = \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}) d\dot{x} \tag{1}$$

Here $\nu_X^+(a)$ is the expected upcrossing rate, or the expected number of upcrossings per unit time.

Step2: Assume that the process is narrow banded

We define the random variable X_p as the peaks of $X(t)$. The probability of X_p exceeding a $Prob\{X_p > a\}$ is equal to the number of peaks above the threshold divided by the total number of peaks. The zeros crossing rate will approximately be the same as the rate of peaks for a narrow banded process $N_{Peaks} = \nu_X^+(0)T_{ST}$.

$$Prob\{X_p > a\} = \frac{\nu_X^+(a)}{\nu_X^+(0)} \tag{2}$$

The cumulative distribution function of the peaks can thus be expressed as

$$Prob\{X_p < a\} = F_{X_p}(a) = 1 - \frac{\nu_X^+(a)}{\nu_X^+(0)} \tag{3}$$

and the probability density function can be obtained by taking the derivative with respect to a

$$f_{X_p}(a) = -\frac{1}{\nu_X^+(0)} \frac{d\nu_X^+(a)}{da} \tag{4}$$

Step3: Assume that the process is Gaussian

Assuming that the stochastic variable $X(t)$ is Gaussian renders the following joint distribution of $X(t)$ and $\dot{X}(t)$ since the variable and its time derivative are independent.

$$f_{X\dot{X}}(x, \dot{x}) = f_X(x) f_{\dot{X}}(\dot{x}) \tag{5}$$

$$f_{X\dot{X}}(x, \dot{x}) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left\{ -\frac{1}{2} \left[\left(\frac{x - m_X}{\sigma_X} \right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{X}}} \right)^2 \right] \right\} \tag{6}$$

Here σ_X and m_X denotes the standard deviation and mean value of X. Note that the variable $X(t)$ and its time derivative might be dependent for other distributions. The upcrossing rate is obtained using Rice's formula Eq.(1)

$$\nu_X^+(a) = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp\left\{-\frac{1}{2}\left(\frac{a-m_x}{\sigma_x}\right)^2\right\} \tag{7}$$

which for a zero mean value $m_x = 0$ renders

$$\nu_X^+(a) = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp\left\{-\frac{1}{2}\left(\frac{a}{\sigma_x}\right)^2\right\} \tag{8}$$

The probability density function of the peaks is then given by

$$f_{X_p}(a) = \frac{a}{\sigma_X^2}\exp\{\frac{1}{2}\left(\frac{a}{\sigma_X}\right)^2\} \tag{9}$$

This is the relatively well known Rayleigh distribution.

2.2 Extreme values

Let us denote the largest value that $X(t)$ assumes during the short-term period T_{ST} by $M(T_{ST})$. That is, $M(T_{ST}) = max\{X(t), 0 \leq t \leq T_{ST}\}$. Assuming that the peaks at high levels are statistically independent events, the random number of upcrossings in the interval T_{ST} is Poisson distributed.

$$Prob\{M(T_{ST}) \leq a\} = F_{M(T_{ST})}(a) = \exp\{-\nu_X^+(a)T_{ST}\} \tag{10}$$

The expected value of $M(T_{ST})$ is often used in design of structures. This can be obtained by

$$E\left[M(T_{ST})\right] = \int_0^\infty a\frac{dF_{M(T_{ST})}(a)}{da}da \tag{11}$$

An approximate analytical solution of the integral renders

$$E\left[M(T_{ST})\right] = \sigma_X\sqrt{2\ln(\nu_X^+(0)T_{ST})}\left\{1 + \frac{\gamma}{2\ln(\nu_X^+(0)T_{ST})} - \frac{\frac{\pi^2}{6}+\gamma^2}{8\Big(\ln(\nu_X^+(0)T_{ST})\Big)^2} + \ldots\right\} \tag{12}$$

The first two terms are often used in design codes, but more terms are neccesarry when $\ln(\nu_X^+(0)T_{ST})$ is low, for instance if the duration of the short-term period is $T_{ST} = 600s$ and the natural period is $T_n = 100s$

3 Long term extreme values

The stochastic variable $X(t)$ cannot be considered as stationary in the long-term period since its mean value and variance will change when the mean wind velocity varies. It is however fairly simple to generalize Eq.(10) for a nonstationary process [1]

$$Prob\{M(T_{LT}) \leq a\} = F_{M(T_{LT})}(a) = \exp\{-T_{LT}\frac{1}{T_{LT}}\int_0^{T_{LT}} \nu_X^+(a,t)dt\} \tag{13}$$

Here the short-term upcrossing rate has been replaced by its mean value for the entire period $\frac{1}{T_{LT}}\int_0^{T_{LT}} \nu_X^+(a,t)dt$. This can also be written as

$$F_{M(T_{LT})}(a) = \exp\{-T_{LT}\int_{\boldsymbol{w}} \nu_X^+(a|\boldsymbol{w})f_{\boldsymbol{W}}(\boldsymbol{w})d\boldsymbol{w}\} \tag{14}$$

Here $\nu_X^+(a|\boldsymbol{w})$ represents the mean upcrossing rate given the stochastic variables $\boldsymbol{w}$ (environmental parameters), while $f_{\boldsymbol{W}}(\boldsymbol{w})$ is the joint probability distribution of the enviromental parameters (mean wind velocity in our case)

$$F_{M(T_{LT})}(a) = \exp\{-T_{LT}\int_{v} \nu_X^+(a|v)f_V(v)dv\} \tag{15}$$

4 A simple example

It has been decided to use 1 hour as the averaging period instead of the more familiar 10 minutes for the bridges in the E39 project. It is therefore of great interest to study how this will influence the load effects estimated using the short-term approach and how the predictions compare to a full long-term approach.
Let us start by defining a very simple generalized dynamic system. We assume that the system is subjected to wind loading and that the standard deviation of the displacement response is given by

$$\sigma_X = \frac{2}{50^2}V^2 \tag{16}$$

Here V is the mean wind velocity in the short-term period with duration T_{ST}. We further assume that the process has a rectangular spectral density with bandwidth $\Delta\omega = 0.02(rad/s)$ as shown in Fig.(1). The distribution of the mean wind velocity for both short-term averaging periods is assumed given by the Weibull distribution displayed in Fig.(2). The distribution for the two averaging periods will in theory be slightly different, but this effect is neglected in this note. The distributions of the yearly maximum mean wind velocity considering $N = 8760$ and $N = 52560$ realizations in one year for the 1 hour and 10 minutes averaging period can be obtained by

$$F_{V,Year,600}(v) = (F_V(v))^{52560} \tag{17}$$

$$F_{V,Year,3600}(v) = (F_V(v))^{8760} \tag{18}$$

The corresponding return periods as function of mean wind velocity is given in Fig.(3). The mean wind velocity for a 50 years return period is 40.8 and 37.8 m/s for a 10 minutes and 1 hour averaging periods respectively. It needs me emphasized that these relations are approximate since it is in general not possible to get one dataset from the other without introducing assumptions similar to what has been assumed above, but the obtained values corresponds very well to the conversion factors presented in the literature [5] [4]. It is therefore concluded that the approach gives fair results when going from 1 hour to 10 minutes

A mean wind velocity of 40.8 m/s and 37.8 m/s gives according to Eq.(16) a standard deviation of the response of 1.33 and 1.14 meters respectively. The expected maximum response during the short-term periods can then be obtained by using the peak factor presented in Eq.(12). The peak factors for a range of natural periods and averaging periods for the mean wind velocity are presented in Table (1) considering only the two first terms which is normally done in design codes and considering three terms in Table (2). As can be seen from the results presented in the tables it is not sufficient to only use the two first terms in the expression for the peak factor when the averaging period is 600 seconds and the natural period of the system reach about 50 seconds.

The cumulative probability function of the extreme response using the short-term approach considering 10 minutes and 1 hour averaging periods and long-term periods of 1 and 50 years are shown in Fig.(4). The mean value of the short-term results are as discussed above commonly used in design, which can be obtained using Eq.(11). It should be noted that this will be slightly different then the median value (p=0.5). The load effect with 50 years return period can be picked from the long-term CDF for a period of $T_{LT} = 50$ years using

$$Prob\{M(T_{LT} < a)\} = (1 - \frac{1}{50})^{50} = 0.364 \tag{19}$$

Or more directly considering a long term period of $T_{LT} = 1$ year

$$Prob\{M(T_{LT} < a)\} = (1 - \frac{1}{50}) = 0.98 \tag{20}$$

As can be seen from the figure the two short-term approaches provides very similar predictions while the extreme value with 50 years return period is significantly higher then the short-term predictions. Short and long-term predictions considering the two averaging periods and a range of natural periods of the system are shown in Table (3) while the ratios of the predictions are shown in Table (4). It is seen that the short-term predictions considering 10 minutes and one hour averaging periods provides very similar predictions when the natural period is between 5 and 50 seconds, wile there is a discrepancy for $T_n = 100$ seconds. The similar results for the two averaging periods is due to that the mean wind velocity with 50 years return period is lower for a one hour averaging then 10 minutes. This compensates for that the peak factor is larger for the one hour period. It is however important to keep in mind that the mean value of the response will become lower when using a 1 hour averaging period. This effect is not studied further in this note.

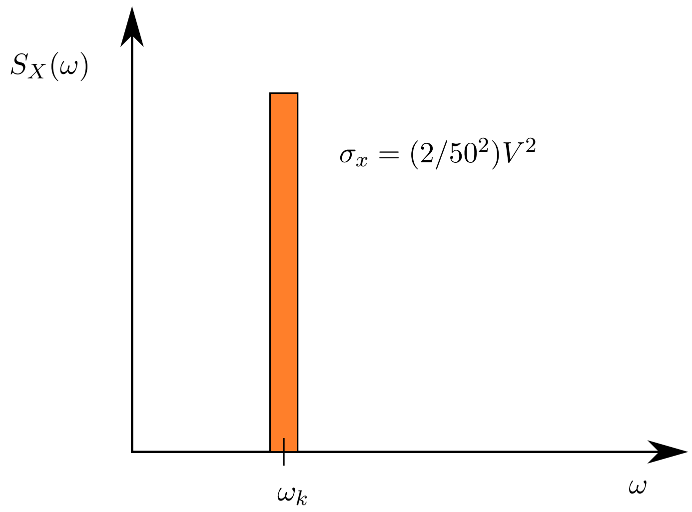

Figure 1: Auto spectral density of the narrow banded process

Table 1: Peak factor k_p (Two first terms in Eq.(12))

$T_{ST}(s)$	$T_n = 5(s)$	$T_n = 10(s)$	$T_n = 20(s)$	$T_n = 50(s)$	$T_n = 100(s)$
600	3.2809	3.0633	2.8294	2.4882	2.1979
3600	3.7866	3.5993	3.4018	3.1220	2.8927

The long-term prediction of the extreme value with 50 years return period is systematically about 10% higher then the short-term prediction considering a one hour averaging period. This is also the case when comparing the results for the periods from 5 to 20 seconds for the 10 minutes averaging period while the discrepancy increases for the periods of 50 and 100 seconds. This is probably because there are two few cycles within the 10 minutes period which make the velocities slightly below and above the selected mean velocity for the short-term analysis more important. This indicates that the short-term period is to short, which also violates that the dynamic system needs to have a steady state dynamic response in the short-term interval.

In marine engineering the expected value of the long-term extreme is commonly used. If this is selected in this case also, the long-term extreme values are about 20% higher then the short term predictions when using a one hour averaging period. This is also the case when comparing the results for the periods from 5 to 20 seconds for the 10 minutes averaging period while the discrepancy increases for the periods of 50 and 100 seconds du to the same reasons as discussed above.

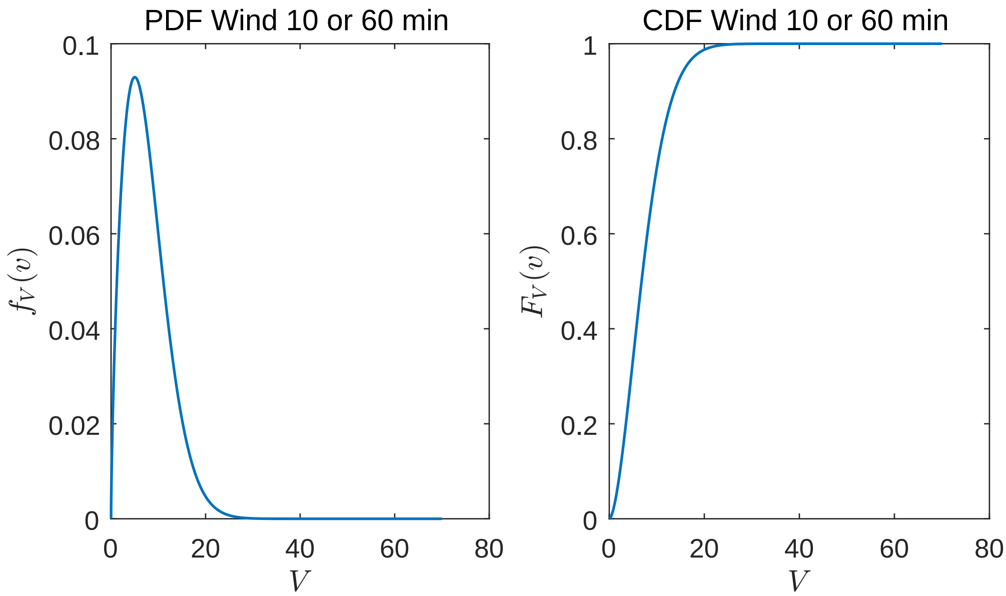

Figure 2: Weibull distribution of the mean wind velocity. The distribution is used for both 10 minutes and 1 hour averaging period, which is an approximation

Table 2: Peak factor k_p (All terms in Eq.(12))

$T_{LT}(s)$	$T_n = 5(s)$	$T_n = 10(s)$	$T_n = 20(s)$	$T_n = 50(s)$	$T_n = 100(s)$
600	3.2475	3.0211	2.7737	2.3990	2.0521
3600	3.7659	3.5748	3.3723	3.0824	2.8412

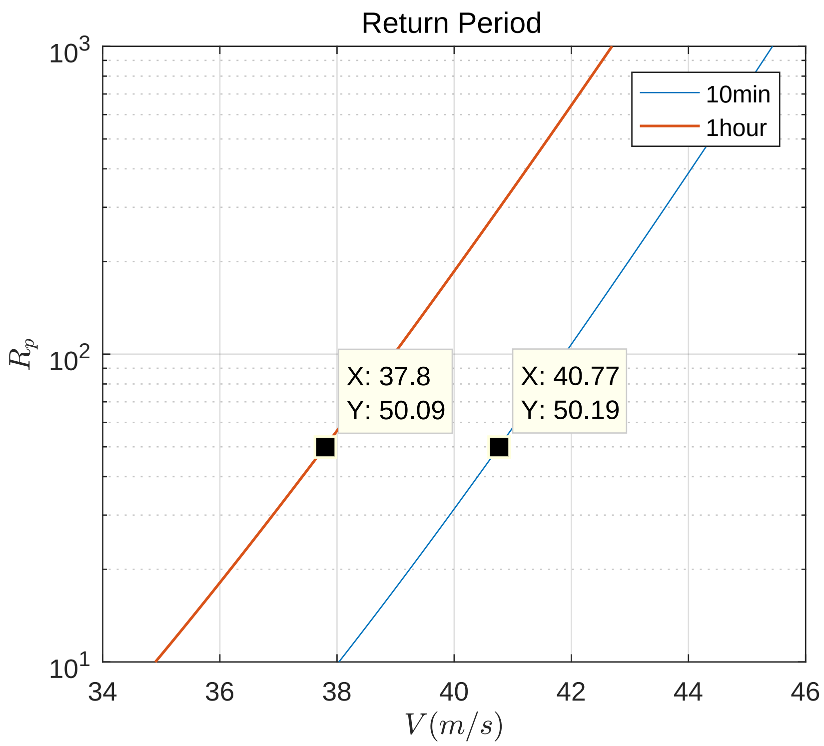

Figure 3: Return period for yearly maximum average mean wind velocity considering two averaging periods

Table 3: Extreme response: The first column is the natural period of the system. The second and third column is the short-term results using Eq.(11) for a 600s and 3600s averaging periods. This is the expected value of the extreme response considering a mean wind velocity with 50 years return period. The right column is the response with 50 years return period predicted using the long-term approach(Eq.(15)). The last column is the expected extreme value in the 50 years long-term period

$T_n(s)$	$E[M(T_{ST} = 600)]$	$E[M(T_{ST} = 3600)]$	$Prob[M(T_{LT} = 1) > a] = 0.02$	$E[M(T_{LT} = 50)]$
5	4.32	4.30	4.80	4.94
10	4.03	4.08	4.52	4.66
20	3.70	3.85	4.26	4.40
50	3.22	3.52	3.94	4.08
100	2.78	3.25	3.70	3.82

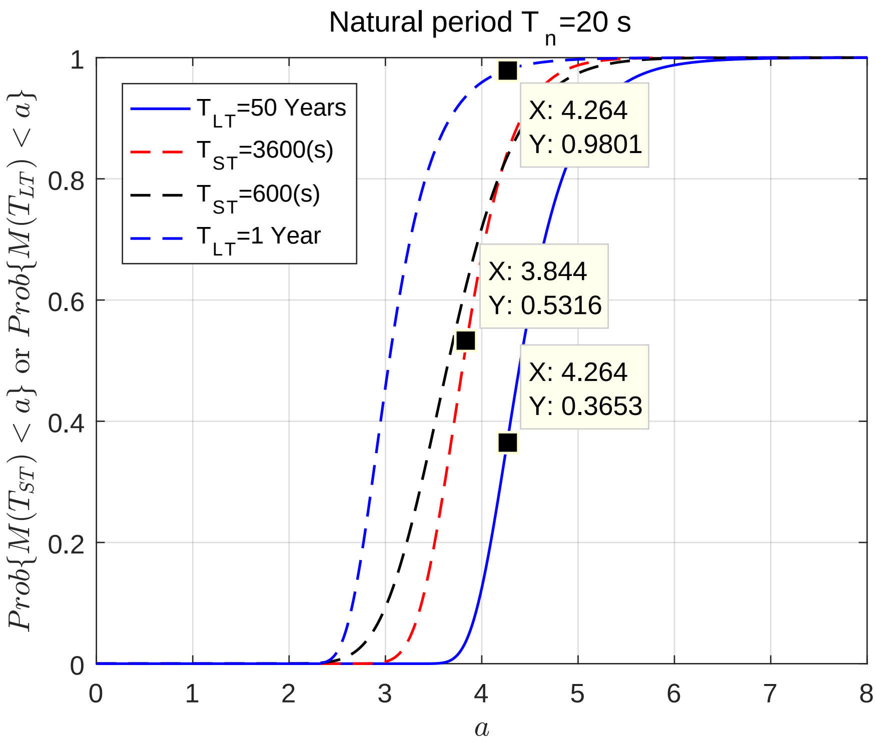

Figure 4: Predicted CDF using the short-term approach considering 10 minutes and 1 hour averaging period and the full long-term approach for a long-term period of 1 and 50 years

Table 4: Ratios of the extreme response predicted using the short-term approach considering 600 and 3600 seconds averaging periods and the long-term approach selecting values corresponding to the expected extreme value and the extreme value corresponding to 50 years return period

$T_n(s)$	$\frac{E[M(T_{ST}=3600)]}{E[M(T_{ST}=600)]}$	$\frac{Prob[M(T_{LT}=1)>a]=0.02}{E[M(T_{ST}=3600)]}$	$\frac{E[M(T_{LT}=50)]}{E[M(T_{ST}=3600)]}$	$\frac{Prob[M(T_{LT}=1)>a]=0.02}{E[M(T_{ST}=600)]}$	$\frac{E[M(T_{LT}=50)]}{E[M(T_{ST}=600)]}$
5	0.99	1.12	1.17	1.11	1.16
10	1.01	1.11	1.16	1.12	1.18
20	1.04	1.11	1.16	1.15	1.21
50	1.09	1.12	1.18	1.23	1.29
100	1.17	1.14	1.20	1.33	1.40

5 Concluding remarks

The basic principles of long and short-term extreme value analysis is discussed in this paper. The theory has been applied to a simple example to illustrate the basic principles.

- The expected extreme value of the load effect considering a mean wind velocity with 50 years return period will in general NOT give a load effect with 50 years return period.

- The extreme values predicted for the example in this note seems to be insensitive to the length of the assumed averaging period. This is because the mean wind with 50 years return period will decrease when the averaging period increases. Note that only the dynamic part of the response have been considered in the example. The mean value will be lower for an one hour averaging period which might result in a total response or load effect that is lower then the corresponding value considering a 10 min averaging period.

- The long-term predictions seems to be systematically 10% higher then the short-term predictions when long term values corresponding to a return period of 50 years is used, but higher values are seen when the short-term period is very short compared to the natural period of the system. It is therefore recommended to use one hour averaging periods for structures with very high natural periods, but problems non stationary behaviour of the wind field should be carefully considered. If the expected maximum response during the long term period is used, the long term predictions are about 20% higher then the corresponding short term estimates.

- Only the dynamic part of the response have been considered in the example discussed in this note. The mean value of the response will have an influence on the ratios of the short- and long-term predictions. In particular since a long averaging period will give a lower mean response in the short term predictions. This will probably make the difference between the short and long term predictions larger. Further studies are therefore required before one can conclude on the factor that should be included in design codes to get from the short-term estimate to a corresponding long-term value.

References

[1] Arvid Naess. Technical note: On the long-term statistics of extremes. *Applied Ocean Research*, 6(4):227–228, 1984.

[2] Arvid Naess and Torgeir Moan. *Stochastic dynamics of marine structures*. Cambridge University Press, 2012.

[3] David Edward Newland. *An introduction to random vibrations, spectral & wavelet analysis*. Courier Corporation, 2012.

[4] Emil Simiu and Toshio Miyata. *Design of buildings and bridges for wind: a practical guide for ASCE-7 standard users and designers of special structures*. 2006.

[5] Emil Simiu and Robert H Scanlan. *Wind effects on structures*. Wiley, 1996.

[6] Paul H Wirsching, Thomas L Paez, and Keith Ortiz. *Random vibrations: theory and practice*. Courier Corporation, 2006.