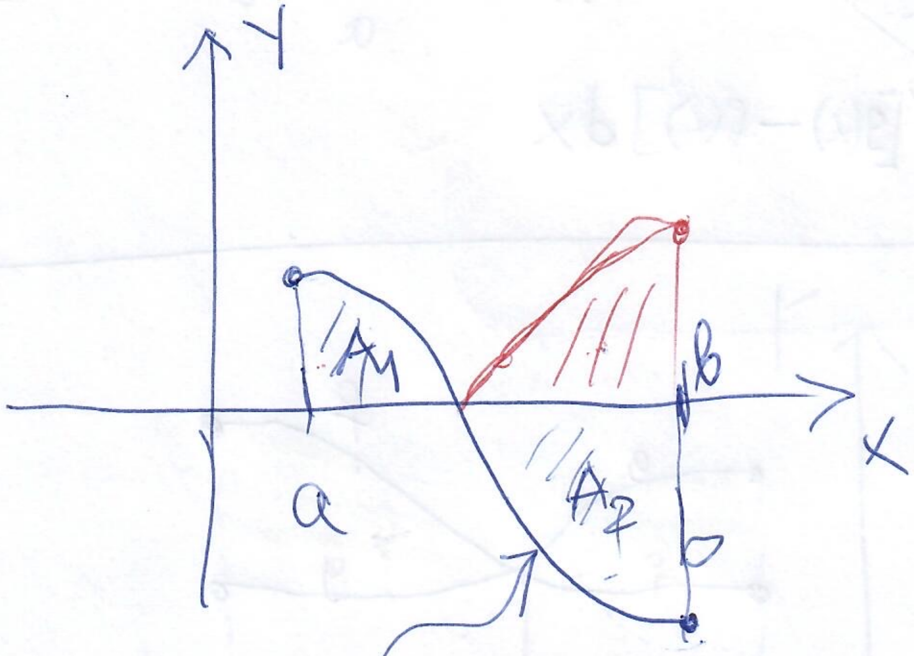
Lech M.

Lecture 5

Applications of integration

A) Areas of plane regions (bounded)

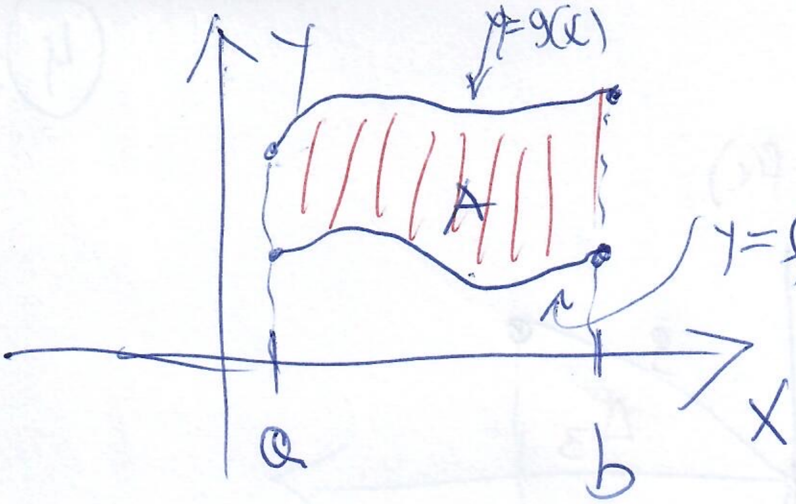
① Area under the curve



$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

② Area between two curves

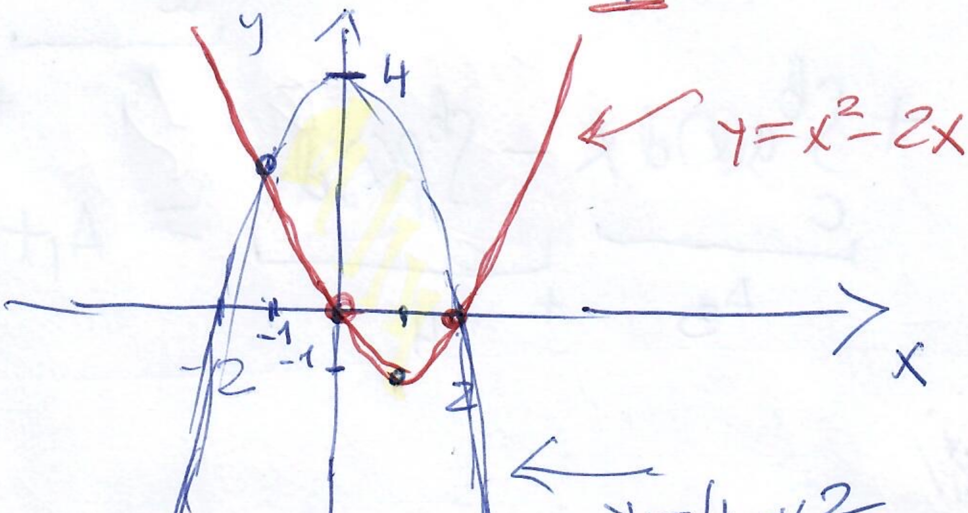


$f(x) \le g(x) \ \forall x \in [a,b]$

$$A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b [g(x)-f(x)]\,dx$$

Remark f or g can be also negative.

Ex. 1 Find the area between the curves $y = 4-x^2$ and $y = x^2-2x$.

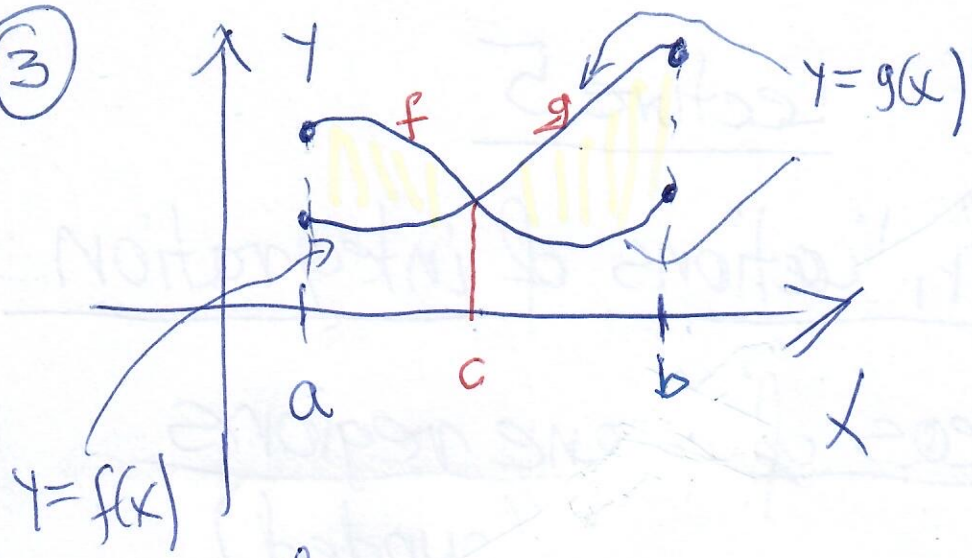


$x^2-2x \le 4-x^2 \ \forall x \in [-1,2]$

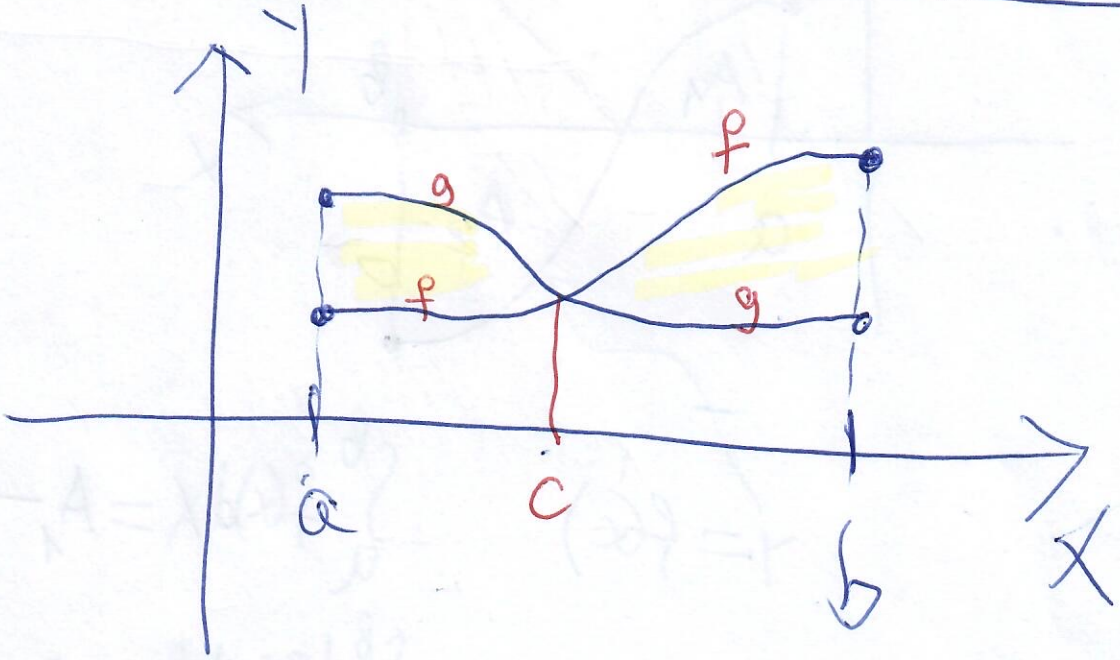
$$A = \int_{-1}^{2} [4-x^2-(x^2-2x)]\,dx = \int_{-1}^{2} (4-2x^2+2x)\,dx$$

$$= 4x - \frac{2}{3}x^3 + x^2 \Big|_{-1}^{2} = 8 - \frac{2}{3}\cdot 8 + 4 - \left(-4 + \frac{2}{3} + 1\right)$$

$$= 16 - 1 - \frac{18}{3} = 9$$

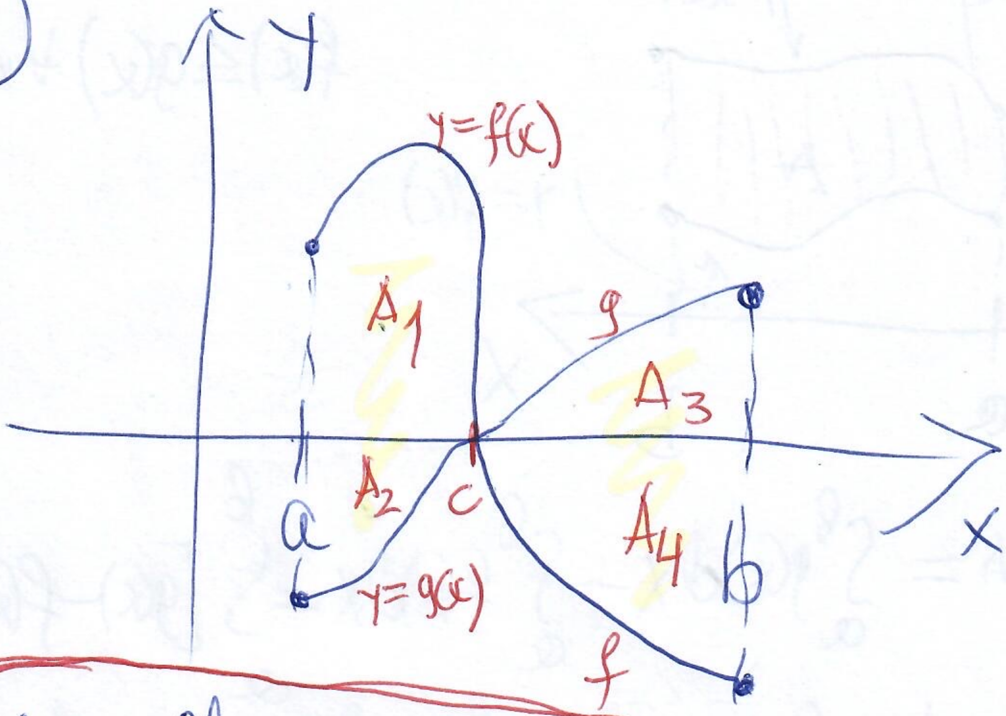


$$A=\int_a^b |f(x)-g(x)|\,dx=\int_a^c [f(x)-g(x)]\,dx$$
$$+\int_c^b [g(x)-f(x)]\,dx$$



$$A=\int_a^b |f(x)-g(x)|\,dx=\int_a^c [g(x)-f(x)]\,dx$$
$$+\int_c^b [f(x)-g(x)]\,dx$$

④ ④

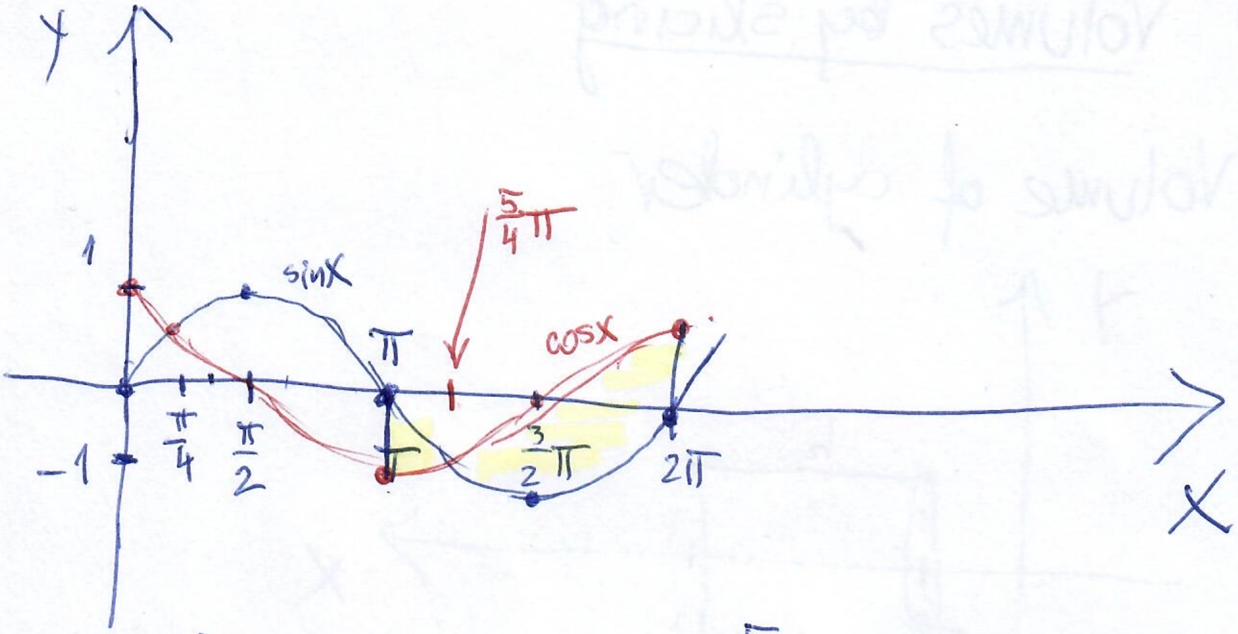


$$\boxed{A = \int_a^b |f(x)-g(x)|\,dx} = \int_a^c [f(x)-g(x)]\,dx$$

$$+\int_c^b -[f(x)-g(x)]\,dx = \underbrace{\int_a^c f(x)\,dx}_{A_1} - \underbrace{\int_a^c g(x)\,dx}_{+A_2}$$

$$+\underbrace{\int_c^b g(x)\,dx}_{A_3} - \underbrace{\int_c^b f(x)\,dx}_{+A_4} = A_1+A_2+A_3+A_4$$

EX. 2 Find the area between the curves $y=\sin x$ and $y=\cos x$ from $x=\pi$ to $x=2\pi$.



$$A=\int_{\pi}^{2\pi}|\sin x-\cos x|\,dx=\int_{\pi}^{\frac{5}{4}\pi}(\sin x-\cos x)\,dx$$

$$+\int_{\frac{5}{4}\pi}^{2\pi}(\cos x-\sin x)\,dx=-\cos x-\sin x\Big|_{\pi}^{\frac{5}{4}\pi}$$

$$+\sin x+\cos x\Big|_{\frac{5}{4}\pi}^{2\pi}=-\cos\tfrac{5}{4}\pi-\sin\tfrac{5}{4}\pi+\cos\pi+\sin\pi$$

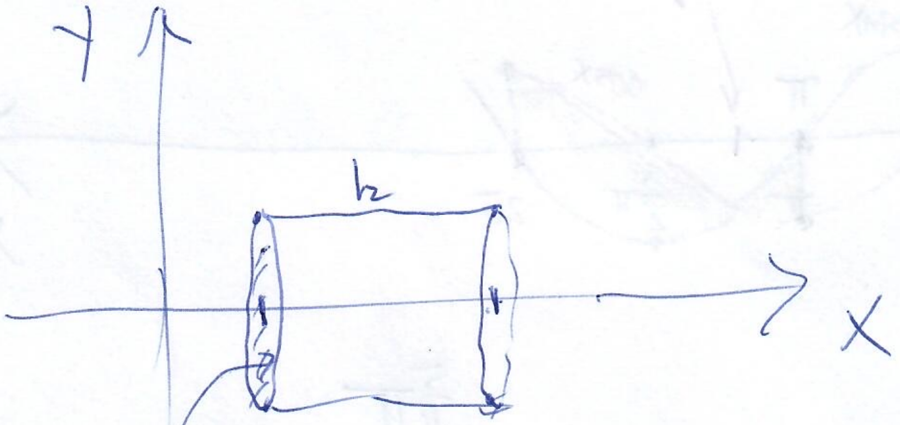
$$+\sin 2\pi+\cos 2\pi-\sin\tfrac{5}{4}\pi-\cos\tfrac{5}{4}\pi=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-1+0$$

$$+0+1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=4\,\frac{\sqrt{2}}{2}=2\sqrt{2}.$$

B) Volumes of solids of revolution

1 Volumes by slicing

Volume of cylinder

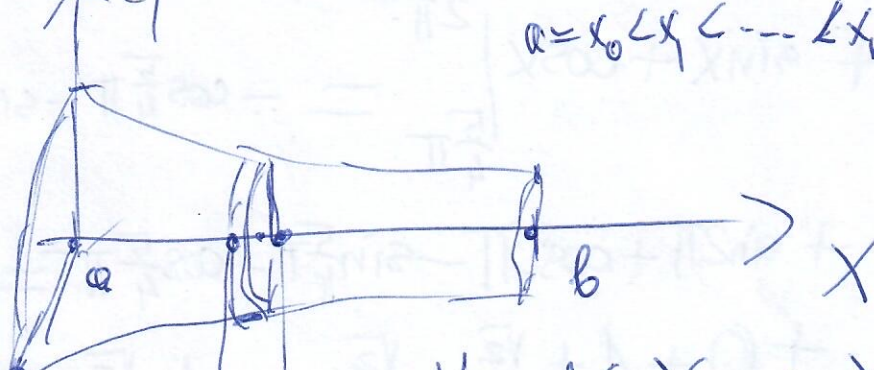


A = area of slice

$V = A \cdot h$

More general slides

$a = x_0 < x_1 < \dots < x_n = b$



$V_k = A(c_k)(x_k - x_{k-1})$

$A(x)$ – slice of the solid at point x (assume that this is continuous functions)

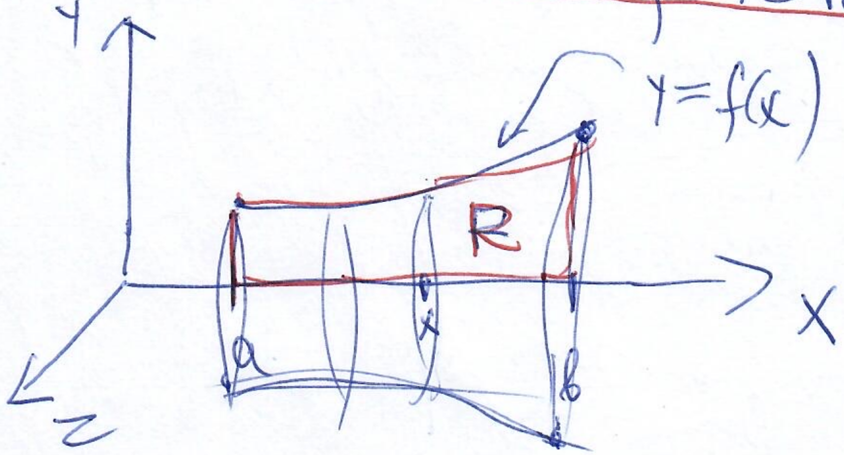
$$V=\sum_{k=1}^{n} V_k=\sum_{k=1}^{n} A(c_k)(x_k-x_{k-1})=$$

$$=\sum_{k=1}^{n} A(c_k)\Delta x_k \xrightarrow[\max \Delta x_k \to 0]{} \int_a^b A(x)\,dx$$

Thm 1 Suppose a solid lies between the planes $x=a$ and $x=b$ and has cross-sectional area perpendicular to x-axis given by the continuous function $A(x)$, $x\in[a,b]$. Then the volume of the solid is

$$V=\int_a^b A(x)\,dx.$$

② Volumes of solids of revolution



If the region R bounded by $y=f(x)$, $y=0$, $x=a$ and $x=b$ is rotated about the x-axis, then the volume of such a solid of revolution is

$$\boxed{V=\pi\int_a^b f(x)^2\,dx.}$$

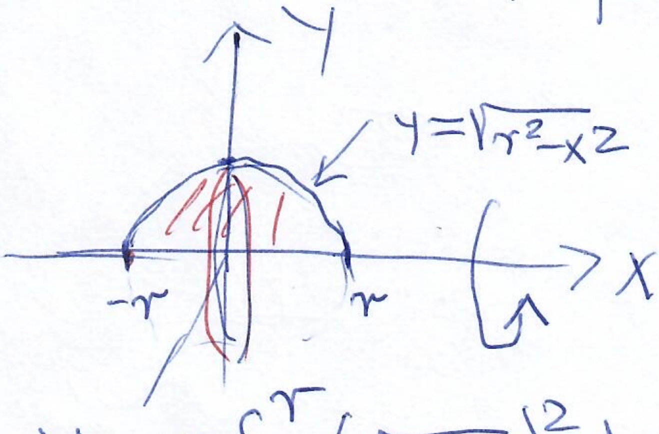
Why?

cross-section has radius $r=f(x)$

$$A(x)=\pi\,[f(x)]^2$$

$$V=\int_a^b A(x)\,dx=\pi\int_a^b f(x)^2\,dx.$$

Ex. 3 (volume of sphere of radius r)



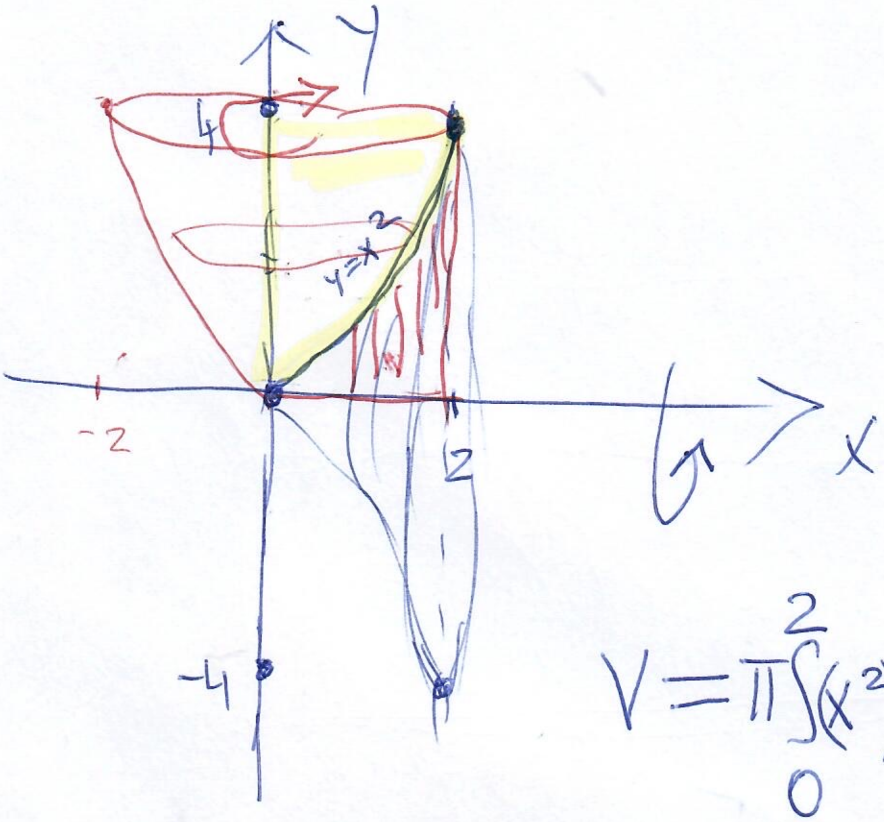
$$V=\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2dx=\pi\int_{-r}^{r}(r^2-x^2)\,dx$$

$$=\pi\left(r^2x-\frac{x^3}{3}\Big|_{-r}^{r}\right)=\pi\left[r^3-\frac{r^3}{3}-\left(-r^3+\frac{r^3}{3}\right)\right]$$

$$=\pi\left(2r^3-\frac{2}{3}r^3\right)=\frac{4}{3}\pi r^3$$

EX. 4 Find the volumes of the solids obtained if the region R under $y=x^2$ over the interval $[0,2]$ is rotated about

(a) the x-axis (b) the y-axis.



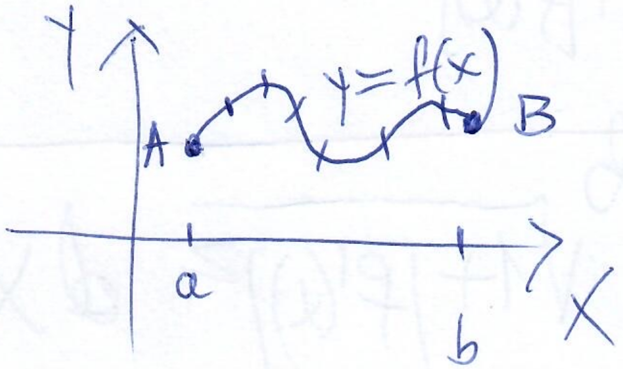
$$V=\pi\int_0^2 (x^2)^2\,dx$$

$$=\pi\left(\frac{x^5}{5}\Big|_0^2\right)=\frac{32}{5}\pi$$

$$V=\pi\int_0^4\left[2^2-(\sqrt{y})^2\right]dy=\pi\int_0^4(4-y)\,dy$$

$$=\pi\left(4y-\frac{y^2}{2}\Big|_0^4\right)=\pi\left(16-\frac{16}{2}\right)=8\pi$$

c) Arc length of the curve



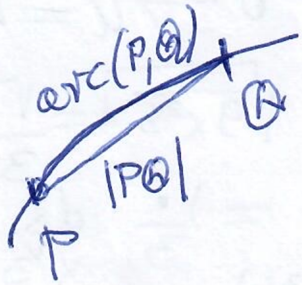
Let C be a curve joining two points A and B.

We can form a polygonal line $P_0 P_1 P_2 \dots P_n$ which length is

$$L_n = |P_0 P_1| + |P_1 P_2| + \dots + |P_{n-1} P_n| = \sum_{k=1}^{n} |P_{k-1} P_k|$$

Such polygonal line approximate the length of the curve C.

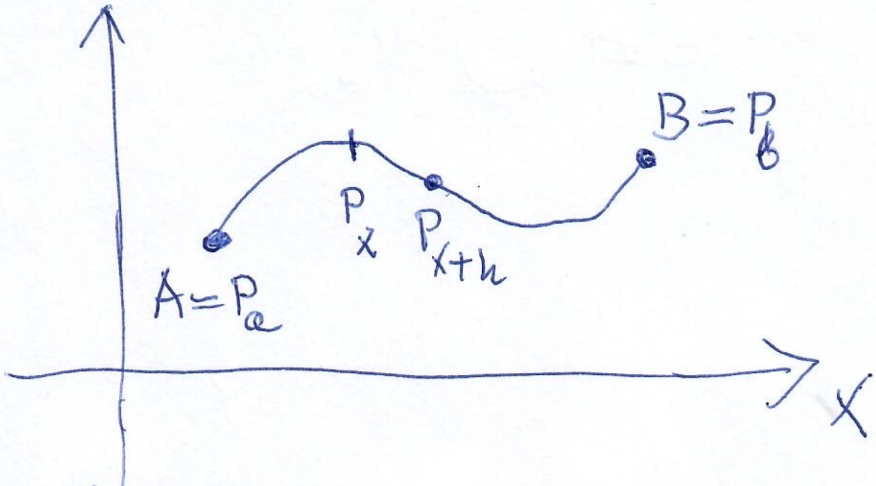
We assume that oure curves are smooth - i.e. they have continuously turning tangent. Then is true that



$$\lim_{Q \to P} \frac{\mathrm{arc}(P,Q)}{|PQ|} = 1.$$

"very short" pices of a smooth curve are "almost straight"

Now let C be the graph of $y=f(x)$, $x\in[a,b]$ and f' be continuous on $[a,b]$.



$P_x=(x,f(x))$

Let $s(x)=$ arc length along C from $A=P_a$ to P_x. Then

$$\frac{s(x+h)-s(x)}{h}=\frac{arc(P_{x+h},P_x)}{h}=\frac{arc(P_{x+h},P_x)}{|P_{x+h}P_x|}\cdot\frac{|P_{x+h}P_x|}{h}$$

$$=\frac{arc(P_{x+h},P_x)}{|P_{x+h}P_x|}\,\frac{\sqrt{h^2+[f(x+h)-f(x)]^2}}{h}$$

$$|P_{x+h}P_x|=\sqrt{(x+h-x)^2+[f(x+h)-f(x)]^2}$$

$P_x=(x,f(x))$

$P_{x+h}=(x+h,f(x+h))$

$$=\frac{arc(P_{x+h},P_x)}{|P_{x+h}P_x|}\sqrt{1+\left(\frac{f(x+h)-f(x)}{h}\right)^2}$$

$$\xrightarrow{h\to 0}\sqrt{1+[f'(x)]^2}$$

$$\frac{ds}{dx} = \sqrt{1+[f'(x)]^2}$$

$$\Downarrow$$

$$\boxed{S = \int_a^b \sqrt{1+[f'(x)]^2}\, dX}$$

arclength of $y = f(x)$ from $x = a$ to $x = b$

Ex. 5 Find the length of the curve $y = x^{2/3}$ from $x = 0$ to $x = 8$

$$s = \int_0^8 \sqrt{1+\left(\frac{2}{3}x^{-1/3}\right)^2}\, dx = \int_0^8 \sqrt{1+\frac{4}{9}x^{-2/3}}\, dx$$

$$= \int_0^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx = \left\{ \begin{array}{l} u = 9x^{2/3}+4 \\ du = 6x^{-1/3}dx \\ x=0 \to u=4 \\ x=8 \to u=40 \end{array} \right\}$$

$$\sqrt{\frac{\cdot}{9x^{2/3}}}\, dx = \frac{\sqrt{\cdot}}{3x^{1/3}}\, dx = \frac{\sqrt{\cdot}}{3}\, \frac{dx}{x^{1/3}}$$

$$= \int_4^{40} \frac{u^{1/2}}{18}\, du = $$

$$= \frac{1}{18}\cdot \frac{u^{3/2}}{3/2}\Big|_4^{40} = \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(40\cdot\sqrt{40} - 4\cdot\sqrt{4}\right)$$

$$= \frac{1}{27}\left[80\sqrt{10} - 8\right]$$