Lech M.

Lecture 4

Techniques of Integration

A) Method of substitution

B) Integration by parts

C) Integration of rational functions, method of partial fractions

D) Inverse substitutions

$$\boxed{\int \frac{P(x)}{Q(x)}dx, \text{ where } P \text{ and } Q \text{ are polynomials}}$$

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_n \neq 0$

polynomial of degree n

$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$

We have basic case:

$$\boxed{\int \frac{P(x)}{Q(x)}dx, \quad \text{degree } P < \text{degree } Q}$$

Why?

If degree $P \geqslant$ degree Q, then

$$\frac{P(x)}{Q(x)} = \text{polynomial} + \frac{R(x)}{Q(x)}, \text{ where degree } R < \text{degree } Q$$

(2)

Ex. 1 Evaluate

$$\int \frac{x^3+2x^2}{x^2+1}\,dx$$

$$\begin{array}{r|l} x^3+2x^2 & x^2+1 \\ \cline{2-2} -(x^3+x) & x+2 \\ \hline 2x^2-x & \\ -(2x^2+2) & \\ \hline -x-2 & \end{array}$$

$$\frac{x^3+2x^2}{x^2+1} = x+2+\frac{-x-2}{x^2+1}$$

$$\int \frac{x^3+2x^2}{x^2+1}\,dx = \int (x+2)\,dx - \int \frac{x+2}{x^2+1}\,dx$$

$$= \frac{x^2}{2}+2x-\int\frac{x}{x^2+1}\,dx-\int\frac{2}{x^2+1}\,dx$$

$$= \frac{x^2}{2}+2x-\frac{1}{2}\ln(x^2+1)-2\arctan x+C$$

Ex. 2 $I = \int \frac{x}{2x-1}\,dx$

degree denominator = degree of numerator

$$I = \frac{1}{2}\int \frac{2x-1+1}{2x-1}\,dx = \frac{1}{2}\int \left(1+\frac{1}{2x-1}\right)dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\ln|2x-1|\right)+C =$$

$$= \frac{x}{2}+\frac{1}{4}\ln|2x-1|+C$$

The basic problem

$$\boxed{\int \frac{P(x)}{Q(x)}\,dx, \quad \text{degree } P < \text{degree } Q}$$

Case 1 Q is linear

$a \neq 0$, $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+C$

Case 2 Q quadratic

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)+C$$

$$\int \frac{x}{x^2-a^2}\,dx = \frac{1}{2}\ln|x^2-a^2|+C$$

$a \neq 0$, $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+C$

$a \neq 0$ $\int \frac{1}{x^2-a^2} dx =$

Method of partial fractions

= technique writing of a complicated fraction as a sum of simpler fractions

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$= \frac{A(x+a)+B(x-a)}{x^2-a^2} = \frac{x(A+B)+Aa-Ba}{x^2-a^2}$$

$$\begin{cases} A+B=0 \longrightarrow B=-A \\ Aa-Ba=1 \end{cases} \quad \forall x \in \mathbb{R}$$

$$\Downarrow$$

$$Aa+Aa=1$$

$$2Aa=1 \Longrightarrow A=\frac{1}{2a}, \; B=-\frac{1}{2a}$$

$$\frac{1}{x^2-a^2} = \frac{1/2a}{x-a} - \frac{1/2a}{x+a} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\left(\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx\right)$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

⑤ Ex. 3

$$I = \int \frac{x+4}{x^2-5x+6}\,dx$$

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$x^2-5x+6=0 \longrightarrow x_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} \begin{matrix} 3 \\ 2 \end{matrix}$$

$$\frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{x(A+B)-3A-2B}{(x-2)(x-3)}$$

$$\begin{cases} A+B=1 \longrightarrow B=1-A \\ -3A-2B=4 \end{cases}$$

$$-3A-2(1-A)=4$$

$$-3A+2A-2=4$$

$$-A=6 \longrightarrow A=-6, \quad B=7$$

$$\frac{x+4}{x^2-5x+6} = \frac{-6}{x-2} + \frac{7}{x-3}$$

$$\int \frac{x+4}{x^2-5x+6}\,dx = -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx$$

$$= -6\ln|x-2| + 7\ln|x-3| + C$$

Techniques of Integration

D) Inverse substitutions

1 The inverse sine substitution

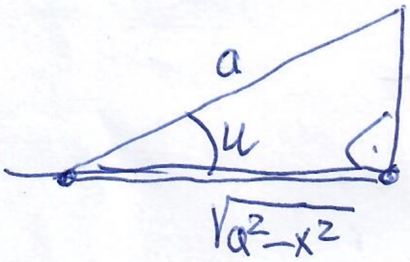
Integrals involving $\sqrt{a^2-x^2}$, $a>0$ can be reduced to a simpler form by the substitution

$$u = \arcsin\frac{x}{a} \iff \boxed{x = a\sin u} \text{ and } -\frac{\pi}{2} \le u \le \frac{\pi}{2}$$

Then

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2 u} = \sqrt{a^2(1-\sin^2 u)}$$

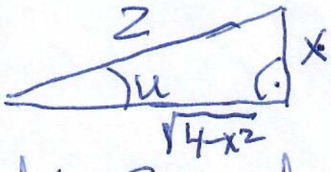
$$= \sqrt{a^2\cos^2 u} = a\cos u$$



$$\cos u = \frac{\sqrt{a^2-x^2}}{a}, \quad \tan u = \frac{x}{\sqrt{a^2-x^2}}$$

EX. 4

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \left\{ \begin{array}{l} x = 2\sin u \\ dx = 2\cos u\,du \end{array} \right\} \quad \tan u = \frac{x}{\sqrt{4-x^2}}$$



$$(4-x^2)^{3/2} = \left(\sqrt{4-x^2}\right)^3$$

$$(4-4\sin^2 u)^{3/2} = 8(\cos^2 u)^{3/2} = 8\cos^3 u$$

$$= \int \frac{1}{8\cos^3 u} \cdot 2\cos u\,du$$

$$= \frac{1}{4}\int \frac{1}{\cos^2 u}\,du = \frac{1}{4}\tan u + C$$

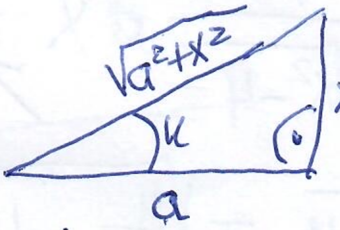
$$= \frac{1}{4}\tan\frac{x}{\sqrt{4-x^2}} + C$$

② The inverse tangent substitution

Integrals involving $\sqrt{a^2+x^2}$ or $\frac{1}{a^2+x^2}$, $a>0$ are often simplified by the substitution
$u = \arctan\frac{x}{a} \iff \boxed{x = a\tan u}$

and $-\frac{\pi}{2} < u < \frac{\pi}{2}$

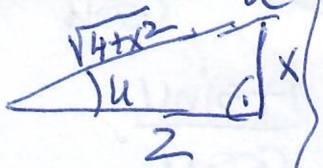
$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 u}$$
$$= a\sqrt{1+\frac{\sin^2 u}{\cos^2 u}} =$$
$$= a\sqrt{\frac{\cos^2 u+\sin^2 u}{\cos^2 u}} = \frac{a}{\cos u}$$



$$\cos u = \frac{a}{\sqrt{a^2+x^2}}$$
$$\sin u = \cancel{\pm}\frac{x}{\sqrt{a^2+x^2}}$$

Ex. 5

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx = \left\{ \begin{array}{l} x = 2\tan u \\ dx = \frac{2}{\cos^2 u}\,du \end{array} \right\}$$



$$= \int \frac{\cos^3 u}{8}\cdot\frac{2}{\cos^2 u}\,du$$
$$= \frac{1}{4}\int \cos u\,du$$
$$= \frac{1}{4}\sin u + C$$
$$= \frac{1}{4}\cdot\frac{x}{\sqrt{4+x^2}} + C$$

$$(4+x^2)^{3/2} = (4+4\tan^2 u)^{3/2} =$$
$$= 8\,(1+\tan^2 u)^{3/2}$$
$$= 8\left(\frac{\cos^2 u+\sin^2 u}{\cos^2 u}\right)^{3/2} = \frac{8}{\cos^3 u}$$

3 The inverse cosine substitution

Integrals involving $\sqrt{x^2-a^2}$, $a>0$ can be simplified by using substitution

$$u = \operatorname{arccos}\frac{a}{x} \iff \boxed{\frac{a}{x} = \cos u}, \quad 0 \le u \le \pi.$$

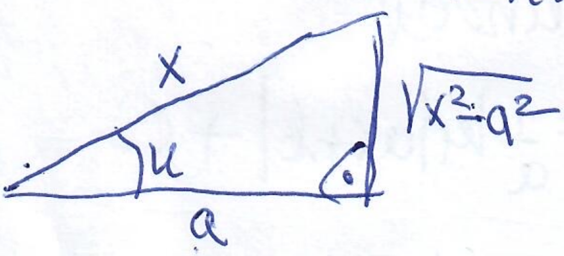
$$\Updownarrow$$

$$\boxed{x = \frac{a}{\cos u}}$$

$$\sqrt{x^2-a^2} = \sqrt{\frac{a^2}{\cos^2 u} - a^2} = \sqrt{a^2\left(\frac{1}{\cos^2 u} - 1\right)}$$

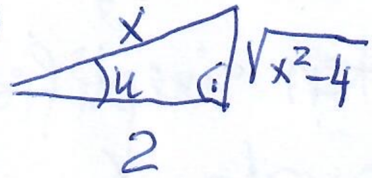
$$= a\sqrt{\frac{1-\cos^2 u}{\cos^2 u}} = a\sqrt{\frac{\sin^2 u}{\cos^2 u}} = a|\tan u|$$

$$= \begin{cases} a\tan u & 0 \le u < \frac{\pi}{2} \iff x \ge a \\ -a\tan u & \frac{\pi}{2} < u \le \pi \iff x \le -a \end{cases}$$



Ex. 6

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \left\{ \begin{array}{l} x = \dfrac{2}{\cos u} \end{array} \right.$$



$$dx = 2\,\frac{\pm \sin u}{\cos^2 u}\,du$$

$x \geq 2$

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{2\tan u}\cdot 2\,\frac{\sin u}{\cos^2 u}\,du =$$

$$\sqrt{x^2-4} = \sqrt{\frac{4}{\cos^2 u}-4} = 2\sqrt{\frac{1-\cos^2 u}{\cos^2 u}} = 2\tan u$$

$$= \int \frac{\cancel{\cos u}}{\cancel{\sin u}}\cdot\frac{\cancel{\sin u}}{\cos^2 u}\,du = \boxed{\int \frac{1}{\cos u}\,du}$$

$$= \left\{ \begin{array}{l} v = \dfrac{1+\sin u}{\cos u} \\ dv = \dfrac{\cos^2 u + \sin u(1+\sin u)}{\cos^2 u}\,du = \dfrac{1+\sin u}{\cos^2 u}\,du \\ \qquad = \dfrac{1+\sin u}{\cos u}\,\dfrac{du}{\cos u} = v\,\dfrac{du}{\cos u} \\ \dfrac{dv}{v} = \dfrac{du}{\cos u} \end{array} \right.$$

$$\int \frac{dv}{v} = \ln|v| + C = \ln\left|\frac{1+\sin u}{\cos u}\right| + C$$

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$$\int \frac{1}{\sqrt{x^2-4}}dx = \int \frac{dv}{v} = \ln|v| + C = \ln\left|\frac{1+\sin u}{\cos u}\right| + C$$

$$= \ln\left|\frac{1+\frac{\sqrt{x^2-4}}{x}}{\frac{2}{x}}\right| + C = \ln\left|\frac{x}{2} + \frac{x}{2}\frac{\sqrt{x^2-4}}{x}\right| + C$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C = \ln\left|x+\sqrt{x^2-4}\right| - \ln 2 + C$$

D

4 Substitution $x = \tan\frac{u}{2}$

universal

$$\int R(\sin u, \cos u)\,du, \quad R - \text{rational function}$$

$$u = 2\arctan x \Longleftrightarrow \boxed{x = \tan\frac{u}{2}}$$

$$1+\tan^2\frac{u}{2} = 1+\frac{\sin^2\frac{u}{2}}{\cos^2\frac{u}{2}} = \frac{\cos^2\frac{u}{2}+\sin^2\frac{u}{2}}{\cos^2\frac{u}{2}} = \frac{1}{\cos^2\frac{u}{2}}$$

$$\Downarrow$$

$$\cos^2\frac{u}{2} = \frac{1}{1+\tan^2\frac{u}{2}} = \frac{1}{1+x^2}$$

$$\cos u = \cos^2\frac{u}{2} - \sin^2\frac{u}{2} = 2\cos^2\frac{u}{2} - 1 = \frac{2}{1+x^2} - 1 = \frac{2-1-x^2}{1+x^2}$$

$$\boxed{\cos u = \frac{1-x^2}{1+x^2}}$$

$$\sin u = 2\sin\frac{u}{2}\cos\frac{u}{2} = 2\tan\frac{u}{2}\cos^2\frac{u}{2} = \frac{2x}{1+x^2}$$

$$\boxed{\sin u = \frac{2x}{1+x^2}}$$

$$dx = \frac{1}{\cos^2\frac{u}{2}}\cdot\frac{1}{2}\,du \Longleftrightarrow du = 2\cos^2\frac{u}{2}\,dx$$

$$\boxed{du = \frac{2}{1+x^2}\,dx}$$

EX. 7

$$\int \frac{1}{\cos u}\,du = \left\{ \begin{array}{l} x = \tan\frac{u}{2} \\ \cos u = \frac{1-x^2}{1+x^2} \\ du = \frac{2}{1+x^2}\,dx \end{array} \right\} = \int \frac{\cancel{1+x^2}}{1-x^2} \cdot \frac{2}{\cancel{1+x^2}}\,dx$$

$$= 2\int \frac{1}{1-x^2}\,dx = \int \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx =$$

$$= -\ln|1-x| + \ln|1+x| + C = \ln\left|\frac{1+x}{1-x}\right| + C$$

$$= \ln\left|\frac{1+\tan\frac{u}{2}}{1-\tan\frac{u}{2}}\right| + C = \ln\left|\frac{1+\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}}{1-\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}}\right| + C$$

$$= \ln\left|\frac{\cos\frac{u}{2}+\sin\frac{u}{2}}{\cos\frac{u}{2}-\sin\frac{u}{2}}\right| + C$$

$$= \ln\left|\frac{\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)}{\left(\cos\frac{u}{2}-\sin\frac{u}{2}\right)\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)}\right| + C$$

$$= \ln\left|\frac{\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)^2}{\cos^2\frac{u}{2}-\sin^2\frac{u}{2}}\right| + C$$

$$= \ln\left|\frac{1+2\sin\frac{u}{2}\cos\frac{u}{2}}{\cos u}\right| + C = \ln\left|\frac{1+\sin u}{\cos u}\right| + C$$