Introduction to probability: Solutions

1. Classical and geometric probability

3.03.2021

Exercise 1.

(a) $5!$

(b) 3^5

(c) $8 \cdot 10^6$

(d) $26 \cdot 25 \cdot 24 \cdot 23$

(e) $3! \cdot n_1! \cdot n_2! \cdot n_3!$

Exercise 3. BABA: $\frac{4!}{2!2!} = 6$; BARBARA: $\frac{7!}{3!2!2!} = 210$.

Exercise 4. Ω: set of all 6-element subsets of 49-element set, $|\Omega| = \binom{49}{6}$. If k numbers are matched ($k = 1, \ldots, 6$), the matched numbers can be chosen in $\binom{6}{k}$, while unmatched in $\binom{49-6}{6-k}$ ways. So, if A_k denotes an event that k numbers are matched, we have $|A_k| = \binom{6}{k}\binom{43}{6-k}$, and thus: $P(A_k) = \frac{\binom{6}{k}\binom{43}{6-k}}{\binom{49}{6}}$. In particular: $P(A_6) = 7.15{\cdot}10^{-8}$, $P(A_5) = 1.84{\cdot}10^{-5}$, $P(A_4) = 9.69{\cdot}10^{-4}$, $P(A_3) = 1.77{\cdot}10^{-2}$, $P(A_2) = 0.132$, $P(A_1) = 0.413$

Exercise 5. Ω: set of all words, $|\Omega| = 26^8$ (variation with repetition), A: set of words with distinct letters, $|A| = \frac{26!}{(26-8)!} = 26 \cdot 25 \cdot \ldots \cdot 19$, so $P(A) = \frac{26!}{26^8 18!}$

Exercise 6. Ω: set of all 20-element subsets of a 100-element set, $|\Omega| = \binom{100}{20}$. Let A_0 and A_1 denote the events that zero or one defective elements were selected, respectively. We have $|A_0| = \binom{95}{20}$ and $|A_1| = 5 \cdot \binom{95}{19}$. Since $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$, we have $P(A) = \frac{|A_0 \cup A_1|}{|\Omega|} = \frac{|A_0| + |A_1|}{|\Omega|} = \frac{\binom{95}{20} + 5 \cdot \binom{95}{19}}{\binom{100}{20}} \simeq 0.739$.

Exercise 7. $|\Omega| = 50!$, A_i – "the oldest at position i, the youngest at position $i+1$" ($i = 1, \ldots, 49$), $|A_i| = 48!$; similarly B_i – "the oldest at position $i+1$, the youngest at position i", $|B_i| = 48!$. Now the event we ask for is $A = A_1 \cup \ldots \cup A_{49} \cup B_1 \cup \ldots B_{49}$, all A_i and B_i are disjoint so $P(A) = \frac{2 \cdot 49 \cdot 48!}{50!} = \frac{2}{50}$

Exercise 8. There are 9 000 of these 4-digit numbers, out of each every third is divisible by 3 (1 000 in total), and every tenth is divisible by 10 (900 in total), so:

(a) $\frac{1}{3}$

(b) $\frac{1}{10}$

Exercise 9. $\binom{n}{k}2^{-n}$ (see the lecture no 1 for an explanation)

Exercise 10. One can represent a single outcome (permutation of dishes) as a permutation of a set $\{1, \ldots, 10\}$, where the i-th position represents the i-th customer, while a number of a dish represents who does this dish belong to. So, e.g., a permutation $\{4, 2, 1, 5, 6, 3, 10, 9, 7, 8\}$ means that the 1st customer got dish of the 4th customer, the 2nd customer got its own dish, the 3rd customer got a dish of the 1st customer, etc. Clearly, $|\Omega| = 10!$

(a) By symmetry, the answer is $\frac{1}{10}$, because a given (say, 1st) customer can get every dish equally likely. More formally, if A is the event "1st customer got his/her own dish", then $|A| = 9!$, because the first dish is fixed at the first position, while the remaining dishes can be permuted arbitrarily. So $P(A) = \frac{|A|}{|\Omega|} = \frac{9!}{10!} = \frac{1}{10}$

(b) If B denotes the even under question, then $|B| = 2!8!$. Why? Say the couple are the first two customers. Then, the permutation must have 1 and 2 and the first two positions (in any order $= 2!$), while the remaining 8 dishes can be permuted arbitrarily over the remaining 8 positions ($= 8!$). So $P(B) = \frac{2!8!}{10!} = \frac{1}{45}$.

(c) If C is the event in question then $|C| = 1$, because there is only one outcome (permutation) which belongs to C – the identity permutations. So $P(C) = \frac{1}{10!}$

Exercise 11.

(a) 13 possible ranks for a pair, and selection of 2 out of 4 cards of the same rank, then any other 3 accompanying cards which do not form a pair: $\frac{13 \cdot \binom{4}{2} \cdot 48 \cdot 44 \cdot 40}{3! \binom{52}{5}}$ (divided by 3! to account for different orders of the 3 accompanying cards)

(b) 13 possible ranks for tree of a kind, and selection of 3 out of 4 cards of the same rank, then any 2 other accompanying cards from the remaining 48: $\frac{13 \cdot \binom{48}{2} \cdot \binom{4}{3}}{\binom{52}{5}}$

(c) $\frac{\binom{13}{5}}{\binom{52}{5}}$

Represent a single outcome (assignment of pupils to desks) as a permutation of a set $\{1, \ldots, 20\}$, where the first two positions represent assignment to the first desk, the next two – to the second desk, etc. Numbers $1, \ldots, 10$ represent girls, while numbers $11, \ldots, 20$ represent boys. Clearly, $|\Omega| = 20!$

Exercise 12.

(a) At the first position there must be a girl (10 choices), at the second one a boy (another 10 choices), while the remaining pupils are assigned randomly (18! assignments). This is multiplied by 2, because we can have a boy at the first position and a boy at the second position. So if A is the event in question, $|A| = 2 \cdot 10 \cdot 10 \cdot 18!$ and $P(A) = \frac{|A|}{|\Omega|} = \frac{2 \cdot 10 \cdot 10 \cdot 18!}{20!} = \frac{10}{19}$

(b) Now, we have a girl at the first position (10 choices), a girl at the second position (9 choices, as one girl is already assigned to the first position), and 18! choices for the remaining pupils. So if B is the event in question, $|B| = 10 \cdot 10 \cdot 18!$ and $P(B) = \frac{9}{38}$

(c) If C is the even under question, then $|C| = 10!10!2^{10}$ and thus $P(C) = \frac{10!10!2^{10}}{20!}$ (all assignments of boys to desks – number of permutations of just boys ($= 10!$), similarly for girls ($= 10!$), now in each desk we need to consider 2 cases of assignments – boy and girl, or girl and boy, which gives 2^{10} cases in total)

Exercise 13. Represent a single outcome as a permutation of $\{1, \ldots, 20\}$ (the wives are numbered $1, \ldots, 10$, the corresponding husbands numbered $11, \ldots, 20$, where wife no i and husband no $i + 10$ are a marriage), so $|\Omega| = 20!$

(a) We fix a wife, say wife no 1 at any position $i = 1, \ldots, 20$. Her husband can sit at two positions – either $i+1$ (where 21 means position 1 as it is a round table), or $i-1$ (where 0 means position 20). So for each position of wife $i = 1, \ldots, 20$, there are two positions of her husband, while the remaining guests can be positioned at 18! ways. So after all denoting the event by A, we have $|A| = 20 \cdot 2 \cdot 18!$ and $P(A) = \frac{20 \cdot 2 \cdot 18!}{20!} = \frac{2}{19}$. One could also obtain this result by noting that if we fix a wife, we randomly select her two neighbors and there are $\binom{19}{2}$ equally probable ways to do that, while only 18 of these ways lead to having her husband as a neighbor, as this must be a selection containing her husband and any other of the remaining 18 guests; so we would get $P(A) = \frac{18}{\binom{19}{2}} = \frac{2}{19}$ as well.

(b) Fix a wife, say no 1, at any position $i = 1, \ldots, 20$, choose 2 out of 9 remaining wives as neighbors ($= 9 \cdot 8$, because this is an ordered selection – we distinguish the neighbor on the left from the neighbor on the right), and permute the remaining 17! guests randomly. Denoting the even by B, this gives $|B| = 20 \cdot 9 \cdot 8 \cdot 17!$ (where 20 is from considering all 20 positions of wife no 1), so $P(B) = \frac{20 \cdot 9 \cdot 8 \cdot 17!}{20!} = \frac{9 \cdot 8}{19 \cdot 18} = \frac{4}{19}$. You could also get to this answer by randomly selecting 2 out of 19 neighbors of wife no 1 (all (unordered) pairs equally likely, $\binom{19}{2}$ pairs in total) and noting that there are $\binom{9}{2}$ (unordered) pairs which consist only of wives. So $P(B) = \frac{\binom{9}{2}}{\binom{19}{2}} = \frac{9 \cdot 8}{19 \cdot 18} = \frac{4}{19}$ as well.

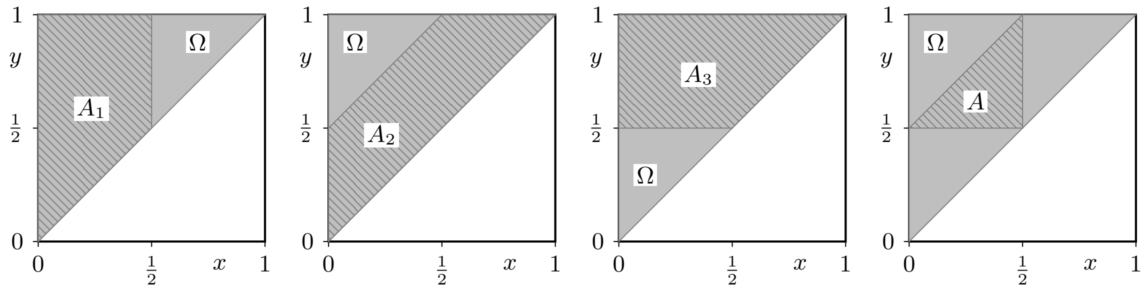
(c) That one seems a bit more tricky, if C is the event, then $|C| = 2 \cdot 2^{10} \cdot 10!$ and thus $P(C) = \frac{2 \cdot 2^{10} \cdot 10!}{20!}$. Why? First randomly permute the marriages (= 10! ways). Next, start sitting marriages either from position 1 or from position 2 (yes, we need to distinguish these two cases as the table is round, so marriage which is sitting at positions 20 and 1 is still sitting "together"). This exhaust all ways of positioning the marriages at the table ($= 2 \cdot 10!$ in total), but we can still swap a husband and a wife within a given marriage, which multiplies the number of cases by 2^{10}.

Exercise 14. If x_1, x_2, x_3 denote arrival times, then the event we ask for is $A = \{x_1 \leq 5 \vee x_2 \leq 5 \vee x_3 \leq 5\}$. Then $A' = \{x_1 > 5 \wedge x_2 > 5 \wedge x_3 > 5$. Since $|\Omega\| = 15^3$ and $|A'| = 10^3$, $P(A) = 1 - \frac{10^3}{15^3} = \frac{19}{27}$

Exercise 15. 0 – there are only countable number of rational numbers (in particular, in the unit interval), while there are uncountable number of real numbers in the unit interval.

Exercise 16. If x – breaking point, the area is $x(10-x)$. Solving $x(10-x) \leq 10$ gives $x \in (0, 5-\sqrt{15}) \cup (5+\sqrt{15}, 10)$, so that $P(A) = \frac{2(5-\sqrt{15})}{10}$

Exercise 17. The answer is $\frac{1}{4}$. Hint: this only happens when the longest part has length no more than $\frac{1}{2}$. Explanation: we define the sample space as $\Omega = \{(x,y) \in [0,1]^2 \colon x \leq y\}$, where an outcome $(x,y) \in \Omega$ denotes a pair of braking points with the convention that the first breaking point is the one closer to zero (this is not necessary but will make the future notation easier). This means that Ω is a right triangle with an area of $|\Omega| = 1/2$ (half of the unit square). Any (x,y) pair of breaking points will split the interval into three parts of lengths $x, y-x$ and $1-y$, respectively. Using the hint, if A denotes the event that one can form a triangle out these three parts, $A = \{(x,y) \in \Omega \colon x, y-x, 1-y \leq 1/2\}$. is the intersection of three sets $A_1 = \{(x,y) \in \Omega \colon x \leq 1/2\}$, $A_2 = \{(x,y) \in \Omega \colon y-x \leq 1/2\}$ and $A_3 = \{(x,y) \in \Omega \colon y \geq 1/2\}$. Let us draw these sets separately and together as A (Ω is always a gray area, while sets A_1, A_2, A_3, A are always shaded with diagonal lines):



Thus, A is also a right triangle of base and height equal to $\frac{1}{2}$, so its area is $|A| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, and thus $P(A) = \frac{|A|}{|\Omega|} = \frac{1/8}{1/2} = \frac{1}{4}$.

If you know any way of getting this answer easier, let me know!

Exercise 18. $\Omega = [0,60]^2$ is a square of side 60, where each outcome $(a,b) \in \Omega$ is pair of arrival times of, respectively, Alice and Bob. So $|\Omega| = 60 \cdot 60$. If C is the event under question, let C' be the complementary event ("Alice and Bob do not meet"). Split $C' = A \cup B$ it into two disjoint events: A – "Alice is later than 10 minutes after Bob" and B – "Bob is later than 20 minutes after Alice". Both events are right triangles, and A has base and height equal to 50 (so $|A| = \frac{1}{2} \cdot 50 \cdot 50$), while B has base and height 40 ($|B| = \frac{1}{2} \cdot 40 \cdot 40$). Thus $P(C') = P(A) + P(B) = \frac{\frac{50 \cdot 50}{2} + \frac{40 \cdot 40}{2}}{60 \cdot 60} = \frac{41}{72}$, and therefore $P(C) = 1 - \frac{41}{72} = \frac{31}{72}$.

Exercise 19. If A_z is the event under question, we have $|A_z'| = C_n z^n$ (the volume of an n-dimensional ball of radius z), so that $P(A_z) = 1 - P(A_z') = 1 - \frac{C_n z^n}{C_n} = 1 - z^n$, where C_n in the denominator is the size of Ω, which is the volume of an n-dimensional unit ball.