Lech M.

Lecture 8

(A) Power series. Radius of convergence

Def. 1 A series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \dots$$

is called a power series in x.

Here $c_0, c_1, \dots$ are numbers, x is a variable

Ex. 1

a) $1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k$

b) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\Big\{ \, 0! = 1$

c) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{k+1}}{k+1}$

d) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

Thm 1 (convergence of power series)

For any power series in x

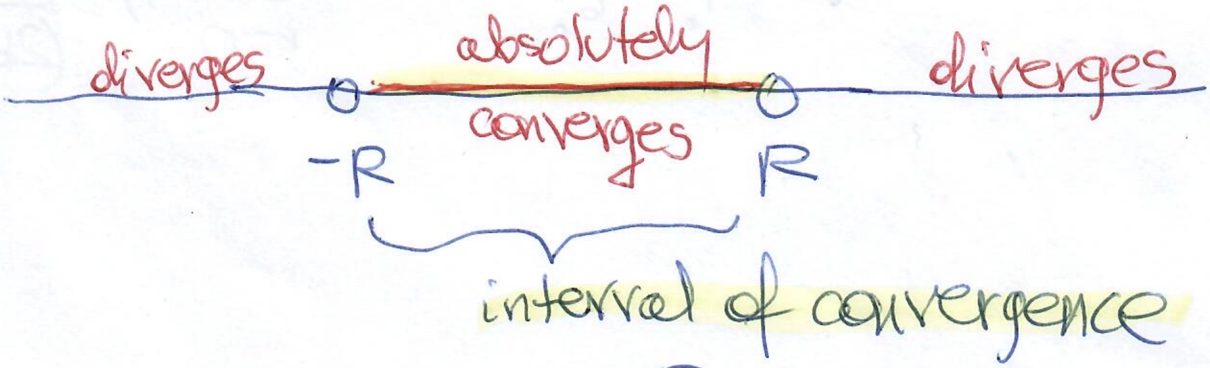
$$\sum_{k=0}^{\infty} c_k x^k$$

exactly one of the following is true:

1° The series converges only for $x=0$.

2° The series converges absolutely for all real x.

3° The series converges absolutely for all x in some open interval $(-R, R)$, diverges if $x<-R$ or $x>R$. At the points $x=R$ and $x=-R$ the series may converge absolutely, converge conditionally or diverge, depending on particular series.



$\underbrace{|x|<R}$ → radius of convergence

Ratio test (d'Alembert test, 1768)

$a_k > 0$, $\sum_{k=0}^{\infty} a_k$ converges if $\rho_A = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$

diverges if $\rho_A > 1$

Root test (Cauchy test, 1821)

$a_k \geq 0$, $\sum_{k=0}^{\infty} a_k$ converges if $\rho_C = \lim_{k\to\infty} \sqrt[k]{a_k} < 1$

diverges if $\rho_C > 1$

Thm 2 Power series

$$\sum_{k=0}^{\infty} a_k x^k \quad \boxed{= \sum_{k=0}^{\infty} c_k, \; c_k = a_k x^k}$$

converges absolutely for $|x| < R$ and diverges for $|x| > R$, where $R = \frac{1}{\rho}$

Proof

Ratio test

$$\underline{1} > \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty} \left|\frac{a_{k+1} x^{k+1}}{a_k x^k}\right| = |x| \underline{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$$

$$= |x| \rho_A$$

Power series converges if

$$|x| \rho_A < 1 \iff |x| < \frac{1}{\rho_A} = R$$

Root test

$$1 > \lim_{k\to\infty} \sqrt[k]{|c_k|} = \lim_{k\to\infty} \left[|a_k x^k|\right]^{1/k} = |x| \lim_{k\to\infty} \sqrt[k]{|a_k|}$$

$$= |x| \rho_C$$

$$|x| < \frac{1}{\rho_C} = R$$

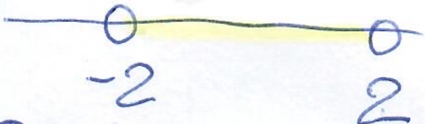
EX. 2 Find the interval of convergence and radius of convergence of power series:

a) $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$ geometric series converges $\Longleftrightarrow |x| < 1$

$(-1, 1)$, $R = 1$

b) $\sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \dots$

converges $\Longleftrightarrow \left|\frac{x}{2}\right| < 1 \Longleftrightarrow |x| < 2$



$(-2, 2)$, $R = 2$

c) $\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ $= \underline{e^x}$!

$$\rho = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)!} \frac{k!}{x^k} \right|$$

$$= \lim_{k \to \infty} |x| \frac{k!}{k!(k+1)} = |x| \lim_{k \to \infty} \frac{1}{k+1} = 0$$

$R = \infty$

$(-\infty, \infty)$

EX. 2

d) $\sum_{k=0}^{\infty} \underbrace{\frac{k!}{2^k}}_{a_k} x^k = 1 + \frac{1}{2}x + \frac{2}{4}x^2 + \frac{3!}{2^3}x^3 + \ldots$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)!}{2^{k+1}} \cdot \frac{2^k}{k!}$$

$$= \lim_{k\to\infty} \frac{\cancel{k!}(k+1)}{2\cdot \cancel{2^k}} \cdot \frac{\cancel{2^k}}{\cancel{k!}} = \lim_{k\to\infty} \frac{k+1}{2} = \infty$$

$\underline{R = 0}$

series converges only if $x = 0$

e) $\sum_{k=0}^{\infty} \frac{x^k}{k+1}$, $a_k = \frac{1}{k+1}$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

series absolutely convergent for $|x| < R = \frac{1}{\rho} = 1$

$x = 1$

$\sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ harmonic series diverges

$x = -1$

$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ alternating harmonic series is convergent

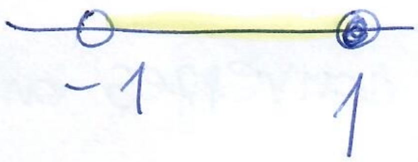
$\underline{[-1, 1)}$ interval of convergence

(f) $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k+1}$, $a_k = \frac{(-1)^k}{k+1}$

$$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

$$|x| < \frac{1}{\rho} = 1$$

Series converges absolutely for $|x|<1$



$x=1$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges

$x=-1$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \dots$$

diverges

$(-1, 1]$ interval of convergence

Power series in x

$$\sum_{k=0}^{\infty} c_k x^k.$$

Def. 2 A series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is called power series in $x-a$.

a, $c_0, c_1, \dots$ are numbers

x variable

Ex. 3

a) $$\sum_{k=0}^{\infty} \frac{(x-2)^k}{k+1} = 1 + \frac{x-2}{2} + \frac{(x-2)^2}{3} + \dots$$

b) $$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)^k}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \dots$$

Interval of convergence

Consider $x-a = X$

Then $$\underbrace{\sum_{k=0}^{\infty} c_k (x-a)^k} \longleftrightarrow \underbrace{\sum_{k=0}^{\infty} c_k X^k}$$

$$-R < X < R \longleftrightarrow |X| < R$$

$$-R < x-a < R$$

$$-R+a < x < R+a$$

EX. 4 Find the radius of convergence and the interval of convergence

$$\sum_{k=0}^{\infty} \underbrace{(-1)^k \frac{(x-2)^k}{k+1}}_{u_k}$$

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(x-2)^{k+1}}{k+2} \cdot \frac{k+1}{(-1)^k (x-2)^k} \right|$$

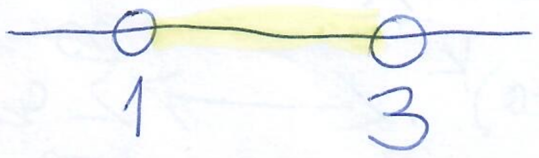
$$= \lim_{k\to\infty} \left[\left| \frac{\cancel{(x-2)^k}(x-2)}{\cancel{(x-2)^k}} \right| \cdot \frac{k+1}{k+2} \right] =$$

$$= |x-2| \lim_{k\to\infty} \frac{k+1}{k+2} = |x-2|$$

series converges if $|x-2| < 1$ $\qquad R = \frac{1}{\rho} = 1$

$$-1 < x-2 < 1$$

$$1 < x < 3$$



$x = 1$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} \text{ diverges}$$

$x = 3$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \text{ alternating harmonic} \rightarrow \text{converges}$$

$(1, 3]$ interval of convergence

Lesson 8B

Taylor's formula; Taylor and Maclaurin series

THM 3 (Taylor's theorem, 1715)

If $f^{(n+1)}(t)$ exists for all t in an interval containing a and x, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \underline{R_n(f,a,x)},$$

where

$$R_n(f,a,x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some c between a and x

Lagrange remainder (1813)

$n=0$

$$f(x) = f(a) + f'(c)(x-a)$$

$$\frac{f(x)-f(a)}{x-a} = f'(c) \quad \text{MVT}$$

$n=1$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2!}(x-a)^2$$

When $a=0$ then we have Mclaurin formula

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+R_n$$

$$R_n=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

Taylor's formula

$$f(x)=\underbrace{\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(f,a,x)}$$

THM 4 If

$\lim_{n\to\infty} R_n(f,a,x)=0$ for all x in some interval I,

then

$$f(x)=\lim_{n\to\infty}P_n(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

for $x\in I$

$a=0 \implies$ Mclaurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \quad \text{for } x\in I$$

Ex. 5 Write the Maclaurin series of functions:

a) $f(x) = \frac{1}{1+x}$ for $-1 < x < 1$

$f'(x) = -\frac{1}{(1+x)^2}$, $f''(x) = \frac{2}{(1+x)^3}$,

$f'''(x) = \frac{-2 \cdot 3}{(1+x)^4}$ ....., $\boxed{f^{(n)}(x) = \frac{(-1)^n \cdot n!}{(1+x)^{n+1}}}$,

$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ $\qquad f^{(n)}(0) = (-1)^n n!$

$= \sum_{k=0}^{n} \frac{(-1)^k \cancel{k!}}{\cancel{k!}} x^k = \sum_{k=0}^{n} (-1)^k x^k$

$= 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$

$R_n(f, 0, x) = \frac{(-1)^{n+1} \cancel{(n+1)!}}{\cancel{(n+1)!} (1+c)^{n+1}} x^{n+1}$

For $0 < x < 1$. [for some $c \in (0, x)$]

$0 \leq |R_n(f, 0, x)| \leq \frac{|x|^{n+1}}{(1+c)^{n+1}} \leq |x|^{n+1} \xrightarrow[\text{as } n \to \infty]{} 0$

From Thm 4

$\boxed{\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k} = 1 - x + x^2 - x^3 + \ldots$ for $|x| < 1$

Then

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} (-1)^k (-x)^k = \sum_{k=0}^{\infty} (-1)^{2k} x^k$$

$$= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$$

b) $f(x) = \ln(1+x)$ for $-1 < x < 1$.

$$\ln(1+x) = \underbrace{\int_0^x \frac{1}{1+t}\, dt}_{\ln(1+t)\big|_0^x = \ln(1+x) - \ln 1 = \ln(1+x)} = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^k \, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^x t^k \, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{t^{k+1}}{k+1} \bigg|_0^x \right)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Maclaurin series of some functions

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\dots \qquad -1<x<1.$$

$$\ln(1+x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots, \qquad -1<x\leq 1$$

$$\arctan x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots, \qquad -1\leq x\leq 1$$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots, \quad x\in\mathbb{R}$$

$$\sinh x=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots, \quad x\in\mathbb{R}$$

$$\cosh x=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots, \quad x\in\mathbb{R}$$

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots, \quad x\in\mathbb{R}$$

$$\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots, \quad x\in\mathbb{R}$$