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Lecture 6

Improper Integrals

$I = \int_a^b f(x)\,dx$

f continuous and closed interval $[a,b]$

$\Downarrow$

f bounded on $[a,b]$
and integral I is a finite number.

$\boxed{\text{Proper integrals}}$

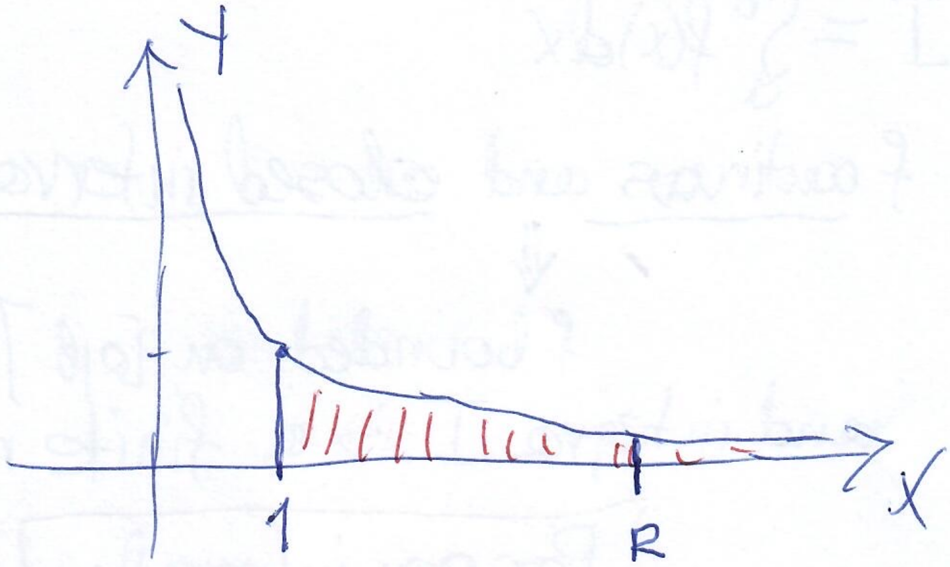
We can generalize the definite integral for two possibilities:

I) we may have $a = -\infty$ or $b = \infty$ or both

II) f may be become unbounded as $x \to a$ or $x \to b$ or both, i.e.
$\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to b^-} f(x) = \pm\infty$ or both.

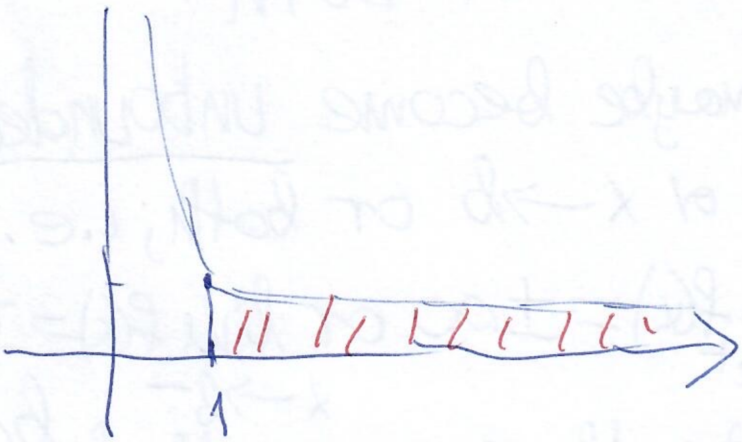
Integrals then are called improper integrals of type I or improper integrals of type II.

Ex. 1 Find the area A of the region lying under the curve $y=\frac{1}{x^2}$ and above the x-axis to the right of $x=1$



$$A=\int_1^\infty \frac{1}{x^2}dx := \lim_{R\to\infty}\int_1^R \frac{1}{x^2}dx = \lim_{R\to\infty}\left(-\frac{1}{x}\Big|_1^R\right) = \lim_{R\to\infty}\left(-\frac{1}{R}+1\right) = 1.$$

Ex. 2 $y=\frac{1}{x}$



$$A=\int_1^\infty \frac{1}{x}dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}dx = \lim_{R\to\infty}\left(\ln x\Big|_1^R\right) = \lim_{R\to\infty}(\ln R) = \infty$$

integral diverges to infinity

Improper Integrals of Type I

(I) If f is continuous on $[a,\infty)$, we define the improper integral as the limit of proper integrals

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.}$$

(II) Similarly, if f is continuous on $(-\infty, b]$, then we define

$$\boxed{\int_{-\infty}^{b} f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.}$$

If the limit

1° is a finite number we say that improper integral converges

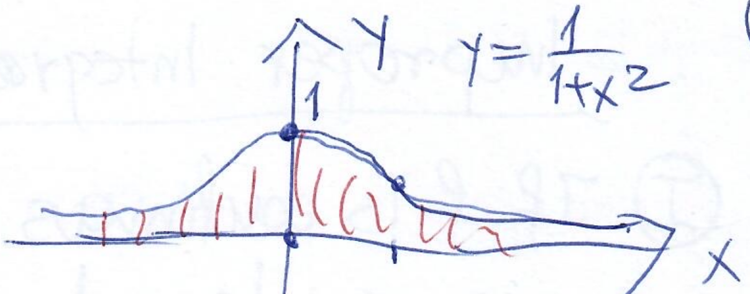
2° does not exist we say that improper integral diverges

3° is ∞ or $-\infty$ we say that improper integral diverges to infinity or $-$infinity

(III) f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx.$$

Ex. 3



a) $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

$$= \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx = \lim_{A\to-\infty}\int_{A}^{0}\frac{1}{1+x^2}dx$$

$$+ \lim_{R\to\infty}\int_{0}^{R}\frac{1}{1+x^2}dx = \lim_{A\to-\infty}\left(\arctan x\Big|_{A}^{0}\right)$$

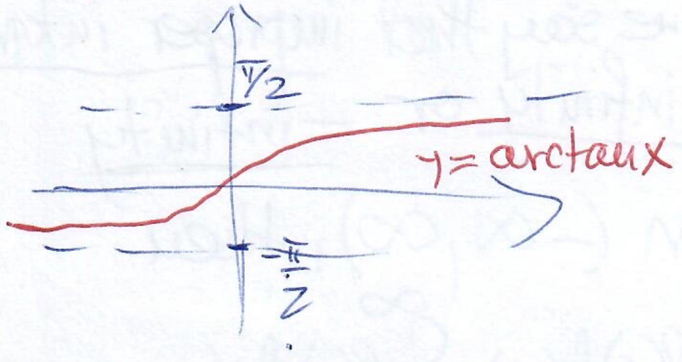
$$+ \lim_{R\to\infty}\left(\arctan x\Big|_{0}^{R}\right) =$$

$$= \lim_{A\to-\infty}\left(\arctan 0 - \arctan A\right)$$

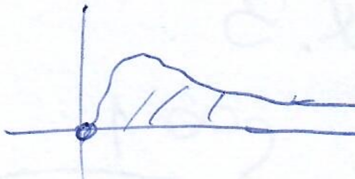
$$+ \lim_{R\to\infty}\left(\arctan R - \arctan 0\right)$$

$$= \lim_{A\to-\infty} -\arctan A + \lim_{R\to\infty}\arctan R$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



EX. 3



b) $\int_0^\infty x e^{-x}\,dx = \lim_{R\to\infty} \int_0^R x e^{-x}\,dx$

$$\int_0^R x e^{-x}\,dx = \left\{\begin{matrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{matrix}\right\} = -x e^{-x}\Big|_0^R + \int_0^R e^{-x}\,dx$$

$$= -Re^{-R} - \left(e^{-x}\Big|_0^R\right) = \frac{-R}{e^R} - \left(e^{-R} - 1\right)$$

$$= -\frac{R}{e^R} - \frac{1}{e^R} + 1$$

$$= \lim_{R\to\infty}\left[-\frac{R}{e^R} - \frac{1}{e^R} + 1\right] = 0 + 0 + 1 = 1$$

c) $\int_0^\infty \cos x\,dx = \lim_{R\to\infty} \int_0^R \cos x\,dx = \lim_{R\to\infty}\left(\sin x\Big|_0^R\right)$

$= \lim_{R\to\infty}(\sin R - \sin 0) = \lim_{R\to\infty} \sin R$ does not exist

and integral diverges

Ex. 3

d) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^r}\,dx, \quad r>1$

$$= \lim_{R\to\infty} \int_e^{R} \frac{1}{x(\ln x)^r}\,dx$$

$$\int_e^{R} \frac{1}{x(\ln x)^r}\,dx = \left\{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \\ x=e \to u=1 \\ x=R \to u=\ln R \end{array}\right\} = \int_1^{\ln R} \frac{1}{u^r}\,du$$

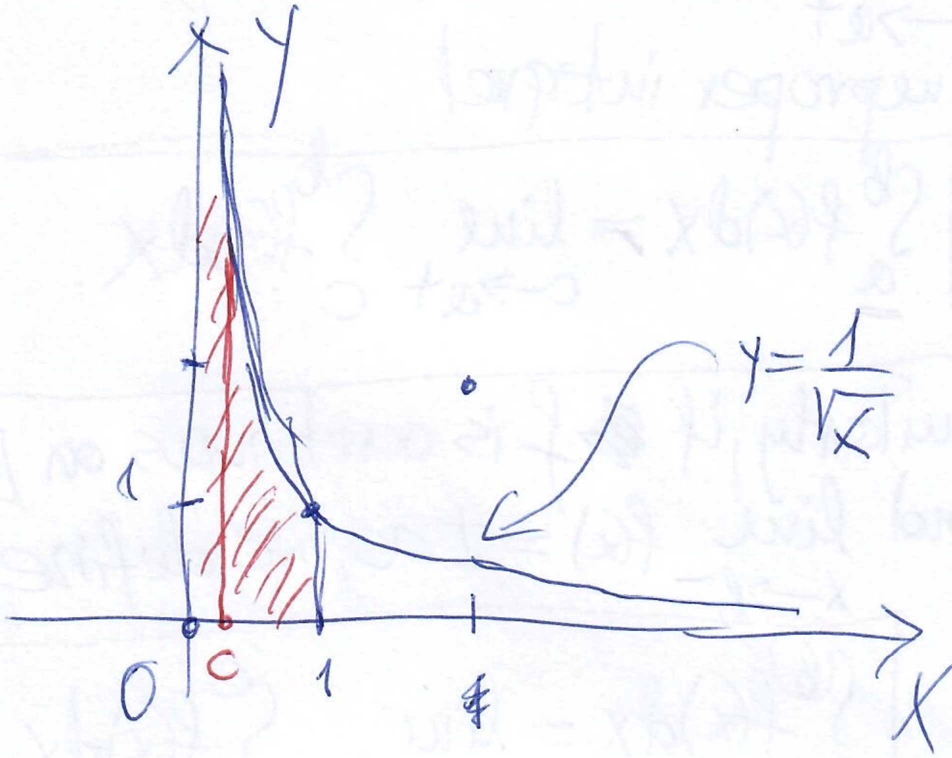
$$= \int_1^{\ln R} u^{-r}\,du = \left.\frac{u^{-r+1}}{-r+1}\right|_1^{\ln R} =$$

$$= \frac{(\ln R)^{-r+1}-1}{-r+1} = \frac{1}{r-1} - \frac{1}{r-1}\cdot\frac{1}{(\ln R)^{r-1}}$$

$$= \lim_{R\to\infty}\left[\frac{1}{r-1} - \frac{1}{r-1}\,\frac{1}{(\ln R)^{r-1}}\right] = \frac{1}{r-1} - 0 = \frac{1}{r-1}$$

integral converges

EX. 4 Find the area of the region S lying under $y=\frac{1}{\sqrt{x}}$ above the x-axis, between $x=0$ and $x=1$.



$f(x)=\frac{1}{\sqrt{x}}$ is not continuous on $[0,1]$ since $\lim_{x\to 0+}\frac{1}{\sqrt{x}}=+\infty$

$A=\int_0^1 \frac{1}{\sqrt{x}}\,dx$ improper integral since

$$A=\lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx=\lim_{c\to 0+}\left(2\sqrt{x}\Big|_c^1\right)$$

$$=\lim_{c\to 0+}\left(2-2\sqrt{c}\right)=2$$

integral converges, and S has a finite area 2.

Improper Integrals of Type II

If f is continuous on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$, we define the improper integral

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.}$$

Similarly, if f is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, we define

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.}$$

Again the possibilities are convergence, divergence, divergence to infinity and / divergence to minus infinity.

Ex. 5

a) $\int_0^1 \frac{1}{x}dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}dx = \lim_{c\to 0^+} \left(\ln x\Big|_c^1\right)$

$= \lim_{c\to 0^+} (\ln 1 - \ln c) = \lim_{c\to 0^+} \ln\frac{1}{c} = +\infty$

integral diverges to infinity.

b) $\int_0^1 \frac{1}{(1-x)^{1/4}}dx = \lim_{c\to 1^-} \int_0^c \frac{1}{(1-x)^{1/4}}dx =$

$= \lim_{c\to 1^-} \left[-\frac{(1-x)^{3/4}}{3/4}\Big|_0^c\right] = \lim_{c\to 1^-} \left[-\frac{4}{3}(1-c)^{3/4} + \frac{4}{3}\right]$

$= 0 + \frac{4}{3} = \frac{4}{3}$

c) $\int_0^1 \ln x\,dx = \lim_{c\to 0^+} \int_c^1 \ln x\,dx$

$\int \ln x\,dx = \left| \begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix} \right| = x\ln x - \int dx = x\ln x - x$

$\lim_{c\to 0^+} \left(x\ln x - x\Big|_c^1\right) = \lim_{c\to 0^+} (-1 - c\ln c + c)$

$\lim_{c\to 0^+} c\ln c$

$= -1$

Thm 1 If $0 < a < \infty$, then

i) $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges to } \dfrac{1}{(p-1)a^{p-1}} & \text{if } p>1 \\ \text{diverges to } \infty & \text{if } p \le 1 \end{cases}$

ii) $\displaystyle\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{converges to } \dfrac{a^{1-p}}{1-p} & \text{if } p<1 \\ \text{diverges to } \infty & \text{if } p \geq 1 \end{cases}$

Proof.

i) $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_a^b x^{-p}\,dx =$

$$= \lim_{b\to\infty} \begin{cases} \dfrac{x^{-p+1}}{-p+1}\Big|_a^b & \longleftrightarrow \quad p \neq 1 \\ \ln|x|\,\Big|_a^b & \longleftrightarrow \quad p = 1 \end{cases}$$

$$= \lim_{b\to\infty} \begin{cases} \dfrac{b^{-p+1}}{-p+1} - \dfrac{a^{-p+1}}{-p+1} & p \neq 1 \\ \ln b - \ln a = \ln\dfrac{b}{a} & p = 1 \end{cases}$$

$$= \lim_{b\to\infty} \begin{cases} \dfrac{a^{-p+1}}{p-1} - \dfrac{1}{(p-1)} \cdot \dfrac{1}{b^{p-1}} & p \neq 1 \\ \ln\dfrac{b}{a} & p = 1 \end{cases}$$

$$= \lim_{b\to\infty} \begin{cases} \dfrac{a^{-p+1}}{p-1} & p>1 \\ \infty & p=1 \\ \infty & p<1 \end{cases}$$

ii)

$$\int_0^a \frac{1}{x^p}\,dx = \lim_{c\to 0+} \int_c^a x^{-p}\,dx$$

$$= \lim_{c\to 0+} \begin{cases} \dfrac{x^{-p+1}}{-p+1}\Big|_c^a & p\neq 1 \\ \ln|x|\Big|_c^a & p=1 \end{cases}$$

$$= \lim_{c\to 0+} \begin{cases} \dfrac{a^{-p+1}}{-p+1} - \dfrac{c^{-p+1}}{-p+1} & p\neq 1 \\ \ln a - \ln c = \ln\dfrac{a}{c} & p=1 \end{cases}$$

$$= \lim_{c\to 0+} \begin{cases} \dfrac{a^{-p+1}}{-p+1} & p<1 \\ \infty & p=1 \\ \infty & p>1 \end{cases}$$