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Test from exercises.
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$\boxed{1}$ ex. 4.

Find the length of the curve:

$$y = 2 \cdot \cosh \frac{x}{2}$$

from $x=0$ to $x=4$.

$$\text{length} = \int_0^4 \sqrt{1 + \left(\left(2 \cdot \cosh \frac{x}{2}\right)'\right)^2}\, dx =$$

$$\cancel{= \int_0^4 \sqrt{1 + 4\left(\cosh \frac{x}{2}\right)^2}\, dx =}$$

$$= \int_0^4 \sqrt{1 + \sinh^2 \frac{x}{2}}\, dx =$$

$$= \left\{ \begin{array}{l} u = \frac{x}{2} \\ du = \frac{1}{2}\, dx \end{array} \right\} = 2 \cdot \int_0^2 \sqrt{1 + \sinh^2 u}\, du =$$

$$= 2 \int_0^2 \cosh u \, du \overset{FTC}{=} 2 \cdot \sinh u \Big|_0^2 =$$

$$= 2 \sinh(2) - 2 \sinh(0) = \underline{2 \sinh(2)}.$$

If $2 \sinh(2)$ bigger than 9?

$$2 \cdot \sinh(2) = 2 \cdot \frac{e^2 - e^{-2}}{2} = e^2 - e^{-2} =$$

$$= e^2 - \frac{1}{e^2} = \frac{e^4 - 1}{e^2}$$

$$e \approx 2{,}72 \Rightarrow \frac{e^4 - 1}{e^2} \approx \frac{54}{7} \approx 7 < 9.$$

=> No, length is smaller than 9.

$\boxed{2}$ ex. 3.

a) evaluate $I_b = \int_{e^2}^{b} \frac{1}{x \ln x (\ln \ln x)}$

for $b > e^2$.

$\boxed{3}$ ex. 4.

Find the radius of convergence and the interval of convergence of power series:

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{k}{2^k} (x-3)^k.$$

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{k}{2^k}(x-3)^k = 0 - \frac{1}{2}(x-3) + \frac{1}{2}(x-3)^2 - \frac{3}{8}(x-3)^3 + \frac{1}{4}(x-3)^4 \ldots$$

$$\rho = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^k (k+1)(x-3)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{(-1)^k k (x-3)^k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^k(-1)(k+1)(x-3)(x-3)^k}{2^k \cdot 2} \cdot \frac{2^k}{(-1)^k (x-3)^k \cdot k} \right| = \lim_{k\to\infty} \left| \frac{-(k+1)(x-3)}{2k} \right| =$$

$$= \lim_{k\to\infty} \left| -\frac{(1+1/k)(x-3)}{2} \right| = \left| -\frac{1}{2}(x-3) \right| = \frac{|x-3|}{2} < 1 \Rightarrow \text{power series - convergent.}$$

$$-1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2$$

$$1 < x < 5.$$

• If $x = 1$:

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{k}{2^k}(x-3)^k = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{k}{2^k} \cdot (-2)^k \Rightarrow \text{diverges.}$$

ex.4 $\boxed{3}$

• If $x = 5$:

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{k}{2^k} \cdot 2^k \Rightarrow \text{diverges.}$$

$\Rightarrow$ Interval of convergence: $(1, 5)$

$\Rightarrow$ Radius of convergence: $R = 2$

$\boxed{4}$ ex. 2.

Locate all local extrema and saddle points of:

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y$$

$$\begin{cases} f_x = 3x^2 + 3y^2 - 15 = 0 \\ f_y = 6xy - 12 = 0. \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3(x^2+y^2) = 15. \\ xy = 2. \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{4}{y^2} + y^2 = 5 \\ x = \frac{2}{y}. \end{cases} \Rightarrow \begin{cases} \frac{4+y^4}{y^2} = 5. \\ x = \frac{2}{y} \end{cases}$$

$$5y^2 = y^4 + 4$$

$$y^4 - 5y^2 + 4 = 0.$$

Put $y^2 = t$:

$$t^2 - 5t + 4 = 0.$$

$$D = \cancel{25 - 4 \cdot 1 \cdot 4} = 9.$$

$$t_1 = \frac{5+3}{2} = 4.$$

$$t_2 = 1.$$

$\Rightarrow y_1 = \pm 1$, $y_2 = \pm 2$.

$\Rightarrow x_1 = \pm 2$, $x_2 = \pm 1$.

$\Rightarrow$ Critical points: $(2,1)$, $(-2,1)$, $(1,2)$, $(-1,-2)$.

$f_{xx} = 6x$. $f_{xy} = 6y$. $f_{yy} = 6x$.

① $(2, 1)$:

$f_{xx}(2,1) = 12 > 0$. $f_{xy} = 6$. $f_{yy} = 12$.

$$D = \begin{vmatrix} 12 & 6 \\ 6 & 12 \end{vmatrix} = 144 - 36 > 0.$$

$\Rightarrow (2,1)$ – local min.

② $(-2, 1)$:

$f_{xx}(-2,1) = -12 < 0$.

$f_{xy} = 6$.

$f_{yy} = -12$.

$$D = \begin{vmatrix} -12 & 6 \\ 6 & -12 \end{vmatrix} = 144 - 36 > 0$$

$\Rightarrow (-2,1)$. – local max.

③ $(1, 2)$:

$f_{xx} = 6 > 0$.

$f_{xy} = 12$.

$f_{yy} = 6$.

$$D = \begin{vmatrix} 6 & 12 \\ 12 & 6 \end{vmatrix} = 36 - 144 < 0$$

$\Rightarrow (1,2)$ – saddle point.

④ $(-1, -2)$:

$f_{xx}(-1,-2) = -6 < 0$.

$f_{xy} = -12$. $f_{yy} = -6$.

$$D = \begin{vmatrix} -6 & -12 \\ -12 & -6 \end{vmatrix} = 36 - 144 < 0$$

$\Rightarrow (-1, -2)$ - saddle point.

$\boxed{5}$ ex. 1.

Solve IVP:

$$\begin{cases} y'(x) + y(x) \cdot \tan x = \sin 2x, \\ y(0) = 1 \end{cases}$$

$$y'(x) + y(x) \cdot \tan x = \sin 2x.$$

1) homogeneous equation:

$$y'(x) + y(x) \cdot \tan x = 0.$$

$$y' = -\tan x \cdot y.$$

$$\frac{y'}{y} = -\tan x.$$

$$(\ln|y|)' = -\tan x.$$

$$\ln|y| = -\int \tan x \, dx + C_1 =$$

$$= -\int \frac{\sin x}{\cos x} dx + C_1 = \left\{ \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right\} =$$

$$= + \int \frac{(-1) \sin x \, dx}{\cos x} + C_1 = \int \frac{du}{u} + C_1 =$$

$$= \ln|u| + C_1 = \ln|\cos x| + C_1.$$

$$|y| = e^{-\ln|\cos x|} + C_1.$$

$$|y| = e^{C_1} \cdot e^{-\ln|\cos x|}.$$

$$y = C_1 \cdot e^{-\ln|\cos x|} \quad \text{for } \forall C_1 \in \mathbb{R}$$

$$y\left(\frac{\pi}{2}\right) = C_1 \cdot e^{-1} = 0.$$

2) $y = C(x) \cdot e^{-\ln|\cos x|}$

$y' = C'(x) \cdot e^{-\ln|\cos x|} +$

$+ C(x) \cdot e^{-\ln|\cos x|} \cdot (-\tan x)$.

$y' + \tan x \, y = C'(x) \cdot e^{-\ln|\cos x|}$

$- \tan x \cdot C(x) \cdot e^{-\ln|\cos x|} +$

$+ \tan x \cdot C(x) \cdot e^{-\ln|\cos x|} =$

$= C'(x) \cdot e^{-\ln|\cos x|}$.

$C'(x) \cdot e^{-\ln|\cos x|} = \sin 2x$.

$C'(x) = \sin 2x \cdot e^{\ln|\cos x|}$.

$C(x) = \int \sin 2x \cdot e^{\ln|\cos x|} dx + D$

=> solution:

$y = \left[\int \sin 2x \cdot e^{\ln|\cos x|} dx + D \right] \cdot$

$\cdot e^{-\ln|\cos x|}$.