Exam.
23.06.2021.

Aksenyuk Sofya, 150284.

$\boxed{1}$ ex.2.

Let $f(x)=\dfrac{2\cos x}{1+\sin^2 x}$ for $x\in[0,\frac{\pi}{2}]$

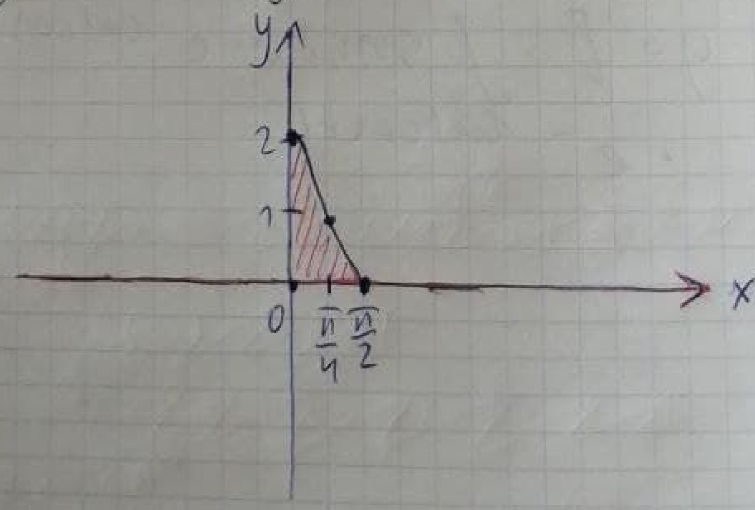
a) $f(0), f(\frac{\pi}{4}), f(\frac{\pi}{2})$ - ?

$$f(0)=\frac{2\cos 0}{1+\sin^2 0}=\frac{2\cdot 1}{1+0}=2.$$

$$f\left(\frac{\pi}{4}\right)=\frac{2\cos\frac{\pi}{4}}{1+\sin^2\frac{\pi}{4}}=\frac{2\cdot\frac{\sqrt{2}}{2}}{1+\frac{2}{4}}=\frac{\sqrt{2}}{\frac{3}{2}}=\frac{2\sqrt{2}}{3}$$

$$f\left(\frac{\pi}{2}\right)=\frac{2\cos\frac{\pi}{2}}{1+\sin^2\frac{\pi}{2}}=\frac{2\cdot 0}{2}=0.$$

b) area of the region R between: $y=f(x)$, $y=0$ and $x\in[0,\frac{\pi}{2}]$.



$$\frac{2\sqrt{2}}{3}\approx\frac{2\cdot 1{,}42}{3}=\frac{2{,}9}{3}\approx 1$$

$$R=\int_0^{\frac{\pi}{2}} f(x)\,dx=\int_0^{\frac{\pi}{2}}\frac{2\cos x}{1+\sin^2 x}\,dx=$$

$$= 2\cdot\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx = \left\{\begin{array}{l} u=\sin x \\ du=\cos x\,dx\end{array}\right\} =$$

$$= 2\cdot\int_0^1 \frac{1}{1+u^2}\,du \overset{FTC}{=} 2\cdot\arctan u \Big|_0^1 =$$

$$= 2\arctan 1 - 2\arctan 0 = 2\cdot\frac{\pi}{4} - 2\cdot 0 =$$

$$= \frac{\pi}{2}.$$

$\boxed{2}$ ex. 3.

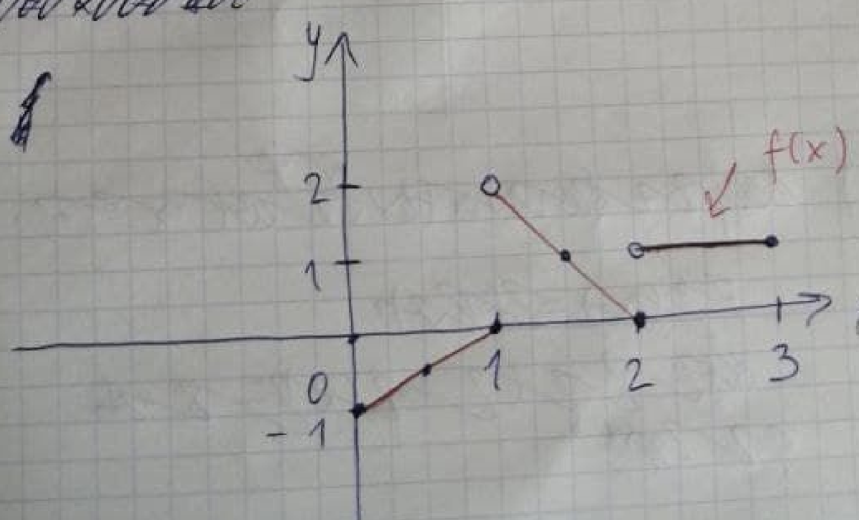
Find $F(x)=\int_0^x f(t)\,dt$ for $x\in[0,3]$ if:

$$f(x) = \begin{cases} x-1, & \text{if } 0\le x\le 1 \\ -2x+4, & \text{if } 1< x\le 2 \\ 1, & 2< x\le 3 \end{cases}$$

Using FTC:

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

and $F'(x) = f(x)$.



• $0\le x\le 1$:

$$F(x) = \int_0^x f(t)\,dt = \int_0^x (t-1)\,dt =$$

$$= \frac{(t-1)^2}{2}\Big|_0^x = \frac{(x-1)^2}{2} - \frac{1}{2} = \frac{(x-1)^2-1}{2}.$$

• $1 < x \leq 2$:

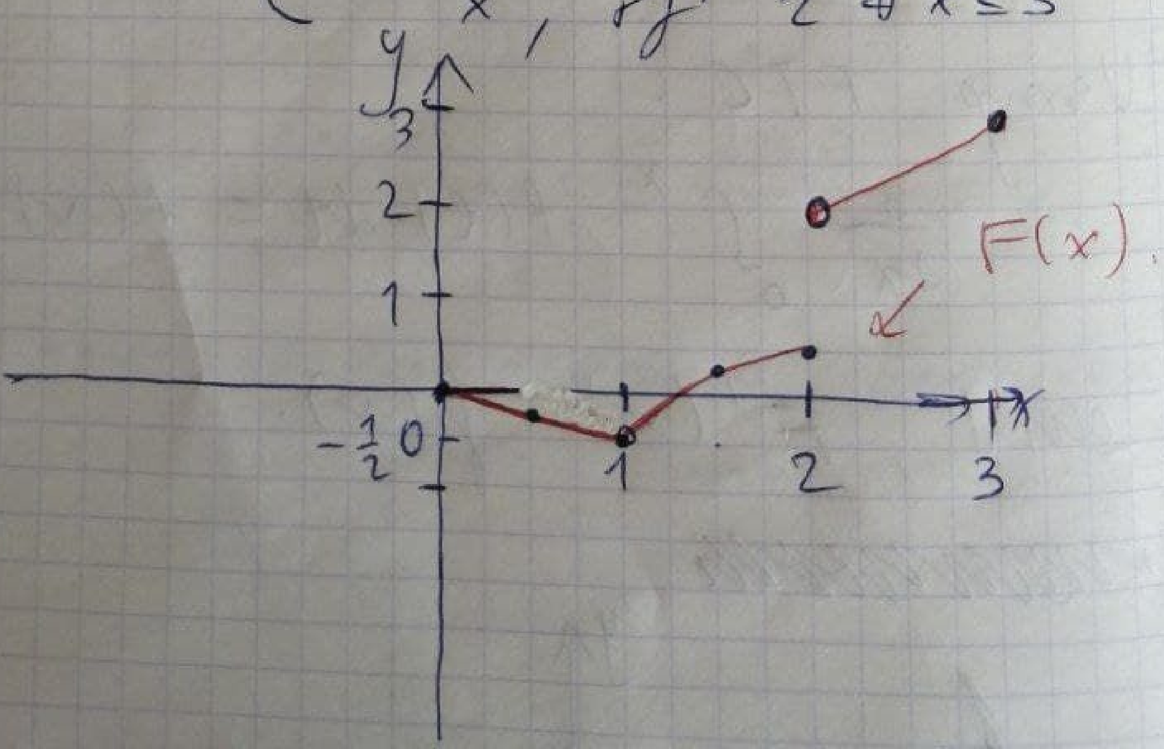
$$F(x) = \int_0^x f(t)\,dt = \int_0^x (-2t+4)\,dt =$$

$$= \int_0^1 (t-1)\,dt + \int_1^x (-2t+4)\,dt =$$

$$= \frac{(t-1)^2}{2}\Big|_0^1 + (4t-t^2)\Big|_1^x =$$

$$= 0 - \frac{1}{2} + (4x - x^2) - 3 = -x^2 + 4x - 3{,}5$$

• $$F(x) = \begin{cases} \frac{(x-1)^2}{2} - \frac{1}{2}, & \text{if } 0 \leq x \leq 1, \\ -x^2 + 4x - 3{,}5, & \text{if } 1 < x \leq 2, \\ x, & \text{if } 2 < x \leq 3 \end{cases}$$



ex. 2.

$$f(x,y) = xye^{-x^2-y^4}$$

$$\begin{cases} f_x = e^{-x^2-y^4}(-2x^2y+y)=0 \\ f_y = e^{-x^2-y^4}(-4xy^4+x)=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -2x^2y+y=0 \\ -4xy^4+x=0 \end{cases} \Rightarrow \begin{cases} y(-2x^2+1)=0 \\ x(-4y^4+1)=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -2x^2+1=0 \\ -4y^4+1=0 \end{cases} \Rightarrow \begin{cases} x=\pm\frac{\sqrt{2}}{2} \\ (-2y^2-1)(2y^2+1)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=\pm\frac{\sqrt{2}}{2} \\ y=\pm\frac{\sqrt{2}}{2} \end{cases}$$

$\Rightarrow$ critical points: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$f_{xx} = 2xy(2x^2-3)\cdot e^{-x^2-y^4}$$

$$f_{xy} = (2x^2-1)(4y^4-1)\cdot e^{-x^2-y^4}$$

$$f_{yy} = 4xy^3\cdot(4y^4-5)\cdot e^{-x^2-y^4}$$

1) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$f_{xx}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) =$$

5 ex. 3.

$$4y''(x) - 4y'(x) + 5y(x) = 16e^{x/2}.$$

1) homogeneous:

$$4y'' - 4y' + 5y = 0.$$

$$4\lambda^2 - 4\lambda + 5 = 0.$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4\cdot 4\cdot 5}}{8} \begin{cases} \frac{1}{2} + i \\ \frac{1}{2} - i \end{cases}$$

$$y_H = e^{\frac{1}{2}x}(C_1 \cdot \cos x + C_2 \sin x).$$

2) particular:

$$y_p = Ae^x + B\sin x + C\cos x.$$

$$y'_p = Ae^x + B\cos x - C\sin x.$$

$$y''_p = Ae^x - B\sin x - C\cos x.$$

$$4y_p'' - 4y_p' + 5y = 4(Ae^{x/2} - B\sin x - C\cos x) - 4(Ae^{x/2} + B\cos x - C\sin x) + 5(Ae^{x/2} + B\sin x + C\cos x)$$

$$= \cancel{4Ae^{x/2}} - 4B\sin x - 4C\cos x - \cancel{4Ae^{x/2}} - 4B\cos x + 4C\sin x + 5Ae^{x/2} + 5B\sin x + 5C\cos x =$$

$$= 5Ae^{x/2} + B\sin x + C\cos x - 4B\cos x + 4C\sin x =$$

$$= 5Ae^{x/2} + \sin x(B + 4C) + \cos x(C - 4B) = 16e^{x/2}.$$

$\longrightarrow$

$$\begin{cases} 5A = 16 \rightarrow A = \frac{16}{5} \\ B + 4C = 0 \rightarrow B + 16B = 0 \rightarrow B = 0 \\ C - 4B = 0 \rightarrow C = 4B \rightarrow C = 0 \end{cases}$$

$\Rightarrow$ All solutions:

$$y = y_H + y_P = e^{x/2}(C_1 \cos x + C_2 \sin x)$$

$$+ \frac{16}{5} e^{x/2}.$$