Introduction to probability

1. Classical and geometric probability

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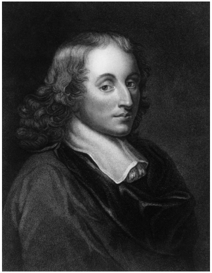
Historical overview

- France, XVII century: gambling is popular and is becoming more complicated
- 1654: known gambler Chevalier de Méré consults Blaise Pascal about the chance of winning in a certain game of dice
- Pascal starts to correspond with Pierre de Fermat and they jointly formulate the mathematical basis of probability
- The ideas of Pascal and Fermat are developed in the following centuries (e.g., de Moivre, Bernoulli)



Blaise Pascal (1623-1662)

Pierre de Fermat (1601-1665)

Historical overview

- 1814: Pierre Laplace in his book *Théorie analytique des probabilités* formulates the mathematical theory of probability
- Laplace's theory is now known as classical probability
- Based on a principle of indifference:

  „Having n mutually-exclusive and collectively exhausting possible outcomes, in the absence of any prior information, assign equal probability to every outcome."

Pierre Simon de Laplace (1749-1827)

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- Every possible result of a random experiment (trial) is called an outcome or an elementary event and is denoted with ω

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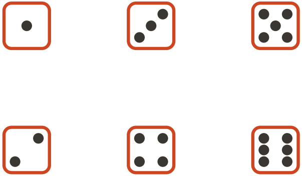

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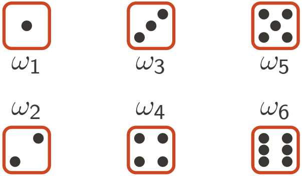

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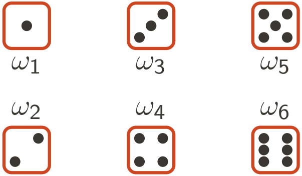

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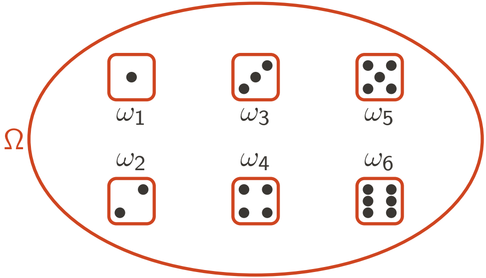

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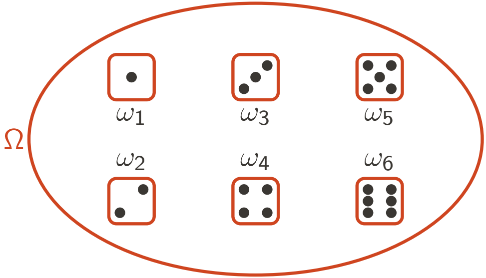

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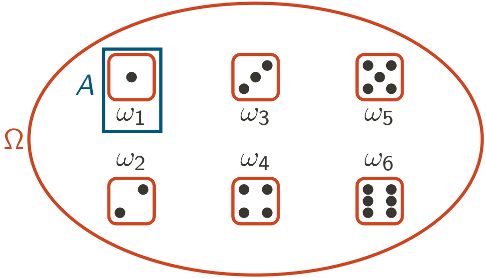

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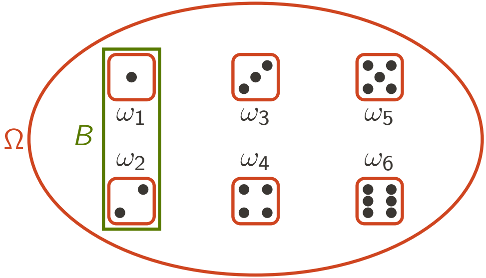

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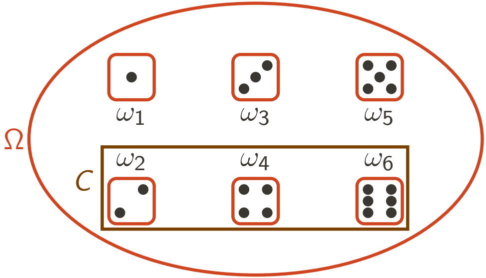

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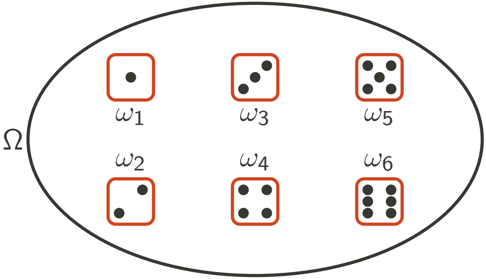
total number of outcomes

- Principle of indifference: „every outcome is equally likely"
- It holds:

$$P(\emptyset) = 0$$
$$P(\Omega) = 1$$
$$P(A) \in [0, 1]$$

Example – rolling a die



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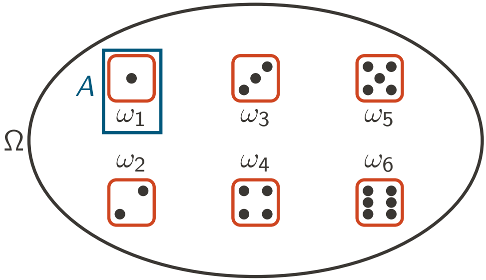

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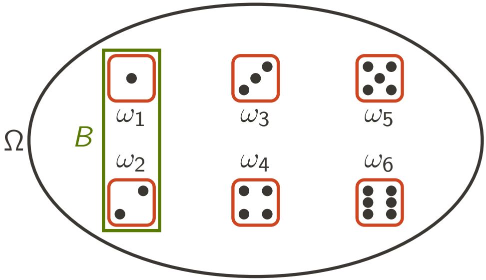

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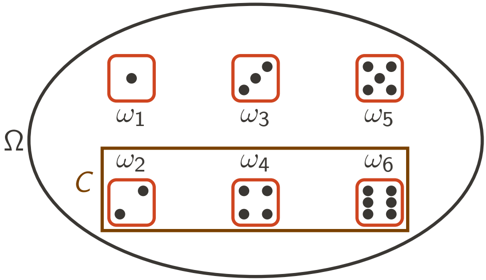

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- Event „roll even number":
  $C = \{\omega_2, \omega_4, \omega_6\}, \qquad |C| = 3, \qquad P(A) = \frac{3}{6} = \frac{1}{2}$

Example – tossing three coins

 Head  Tail

- Sample space:

- Event „getting three heads":

- Event „getting (exactly) two heads":

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Tail

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$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$$|\Omega| = 8$$

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Example – rolling two dice

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- Sample space ($|\Omega| = 36$):

$$\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(6,5),(6,6)\}$$

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- Event: „the sum of both dice is S"

S	Event	Probability
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

Example – rolling two dice

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$$\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(6,5),(6,6)\}$$

- Event: „the sum of both dice is S"

S	Event	Probability
2	$A_2 = \{(1,1)\}$	
3	$A_3 = \{(1,2),(2,1)\}$	
4	$A_4 = \{(1,3),(2,2),(3,1)\}$	
5	$A_5 = \{(1,4),(2,3),(3,2),(4,1)\}$	
6	$A_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$	
7	$A_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$	
8	$A_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$	
9	$A_9 = \{(3,6),(4,5),(5,4),(6,3)\}$	
10	$A_{10} = \{(4,6,(5,5),(6,4)\}$	
11	$A_{11} = \{(5,6),(6,5)\}$	
12	$A_{12} = \{(6,6)\}$	

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- Sample space ($|\Omega| = 36$):

$$\Omega = \{(1,1),(1,2),\dots,(1,6),(2,1),(2,2),\dots,(6,5),(6,6)\}$$

- Event: „the sum of both dice is S"

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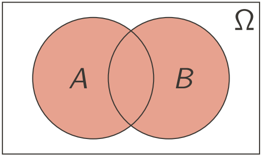
2	$A_2 = \{(1,1)\}$	$P(A_2) = 1/36$
3	$A_3 = \{(1,2),(2,1)\}$	$P(A_3) = 2/36$
4	$A_4 = \{(1,3),(2,2),(3,1)\}$	$P(A_4) = 3/36$
5	$A_5 = \{(1,4),(2,3),(3,2),(4,1)\}$	$P(A_5) = 4/36$
6	$A_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$	$P(A_6) = 5/36$
7	$A_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$	$P(A_7) = 6/36$
8	$A_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$	$P(A_8) = 5/36$
9	$A_9 = \{(3,6),(4,5),(5,4),(6,3)\}$	$P(A_9) = 4/36$
10	$A_{10} = \{(4,6,(5,5),(6,4)\}$	$P(A_{10}) = 3/36$
11	$A_{11} = \{(5,6),(6,5)\}$	$P(A_{11}) = 2/36$
12	$A_{12} = \{(6,6)\}$	$P(A_{12}) = 1/36$

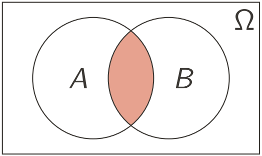
Operations with events

Events are sets!



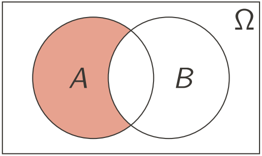
sum $A \cup B$

„A or B occurred"



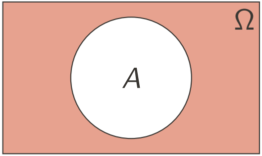
intersection $A \cap B$

„A and B occurred"



difference $A \setminus B$

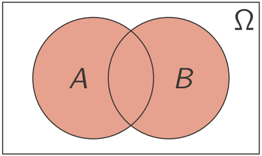
„A occurred, but not B"



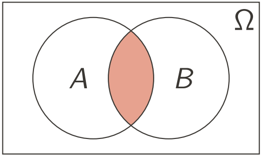
complement A'

„A did not occur"

Operations with events



$A \cup B$



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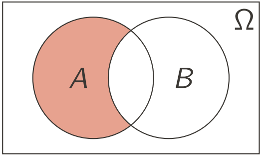

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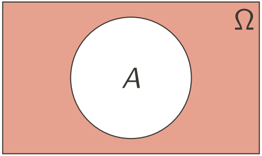
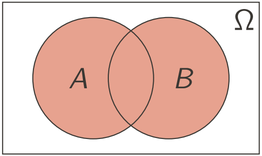


A'

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |A \setminus B| &= |A| - |A \cap B| \\ |A'| &= |\Omega| - |A| \end{aligned}$$

Operations with events



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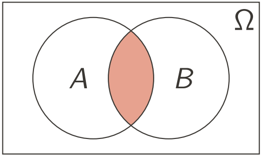

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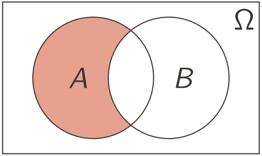

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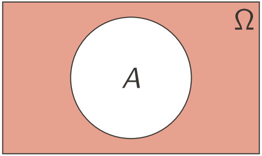
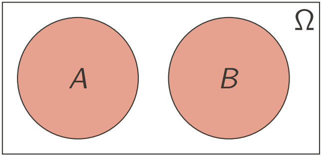


A'

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \setminus B) &= P(A) - P(A \cap B) \\
P(A') &= 1 - P(A)
\end{aligned}
$$

Mutually exclusive (disjoint) events



- If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

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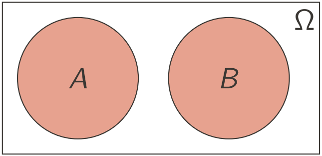

- If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$
- More generally: if $A_1, \ldots, A_n$ are mutually exclusive (disjoint), $A_i \cap A_j = \emptyset$ for $i \neq j$, then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Counting: variation with repetition

If an experiment consists of k independent trials, and there are n possible outcomes in each trial, then the total number of outcomes is $\underbrace{n \cdot n \cdot \ldots \cdot n}_{k} = n^k$.

This is the number of ways one can choose k elements (with repetitions allowed) from a set of n elements (the order of elements matters)

- Number of possible outcomes from rolling 4 dice? $6 \cdot 6 \cdot 6 \cdot 6 = 6^4$
- Number of possible outcomes from tossing 10 coints? $2^{10} = 1024$
- Number of binary sequences of length n? $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$
- Number of 5-letter words formed from a 26-letter Latin alphabet? 26^5

Counting: variation without repetition

The number of ways one can choose k distinct (without repetition) elements from a set of n elements (the order of elements matters):
$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$.

- Number of ways one can draw 5 cards (without replacement) from a deck of 52 cards (the order of cards matters)? $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$
- Number of possible outcomes from rolling 3 dice when each die gives a different number? $6 \cdot 5 \cdot 4$
- Number of 5-letter words formed from a 26-letter alphabet in which every letter occurs at most once? $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

Counting: permutations

The number of ways one can order a set of n elements:
$n! = 1 \cdot 2 \cdot \ldots \cdot n$

- Number of ways 5 people can be arranged in a queue? $5! = 120$
- Number of all possible results of shuffling a deck of 52 cards? $52!$

Counting: combination

The number of ways one can choose a k-element subset (the order does not matter) from a set of n elements:
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

- The number of ways one can form a group of size 3 from a set of 5 people? $\binom{5}{3} = \frac{5!}{3!2!} = 10$
- 10 teams play with each other. How many matches are going to be played? $\binom{10}{2} = \frac{10!}{8!2!} = 45$
- The number of binary sequences of length 8 with exactly 3 ones? $\binom{8}{3} = \frac{8!}{5!3!} = 56$ (hint: with each binary sequence associate a subset of an n-element set)

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We draw 5 cards from a deck. What is the probability of getting "four of a kind" (4 cards of the same rank)

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$$|A| = 13 \times 48$$

$$P(A) = \frac{13 \cdot 48}{\binom{52}{5}} \simeq 0.00024$$

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$$|A| = \binom{20}{10}$$

Exercise

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$$|\Omega| = 2^{20}$$

Event A – „we got exactly 10 heads"

$$|A| = \binom{20}{10}$$

$$P(A) = \frac{\binom{20}{10}}{2^{20}} \simeq 0.176$$

The problem of Chevalier de Méré

What is more likely?

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This reasoning is actually wrong!!!

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Experiment 2:

- Ω: all possible outcomes of rolling a pair of dice 24 times:
  $|\Omega| = 36^{24}$
- A: "at least one double 1"
- A': "no double 1 rolled"
  $|A'| = 35^{24}, \qquad P(A') = \frac{35^{24}}{36^{24}}$
- Therefore:

$$P(A) \;=\; 1 - P(A') \;=\; 1 - \frac{35^{24}}{36^{24}} \;\simeq\; 0.4914$$

Lotto

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$$|A| = 1 \qquad \Longrightarrow \qquad P(A) = \frac{1}{\binom{49}{6}} = \frac{6!43!}{49!} = \frac{1}{13\,983\,816}$$

Not much
(thus, playing Lotto is sometimes called a "tax on dreams")

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- Therefore:

$$P(A') \; = \; \frac{365 \cdot 364 \cdot \ldots 343}{365^{23}} \simeq 0.493 \qquad P(A) \; = \; 1 - P(A') \simeq 0.507$$

- The probability is surprisingly large (above 50%)!
- Direct applications to calculating the probability of collision for hash functions

Geometric probability

- Application of the principle of indifference to sample space $\mathbb{R}^n$
- In complete analogy to classical probability, but:
  - The outcomes of an experiment are points in $\mathbb{R}^n$,
  - The events are sets in $\mathbb{R}^n$
  - The "size" of a set is its n-dimensional measure: length ($n = 1$), area ($n = 2$), volume ($n = 3$), etc.
- The concept of geometric probability already considered by Newton in 1666-1668

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- Interpretation:
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  - Geometric probability: „each point in Ω equally likely"
- Geometric probability has identical properties as its classical counterpart, e.g. $P(\emptyset) = 0$, $P(A') = 1 - P(A)$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, itp.

Geometric probability: example

We random draw a point from interval $[-1, 1]$. What is the probability that the point is contained within the interval $[0.1, 0.5]$?

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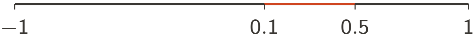

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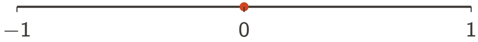
What is the probability that we draw a point exactly equal to zero?



$A = \{0\}, \quad P(A) = \frac{0}{2} = 0$

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We draw two points x_1 and x_2 from a unit interval $[0, 1]$. Calculate the probability that $x_1 > x_2$?

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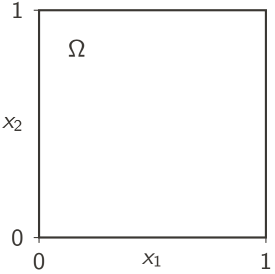

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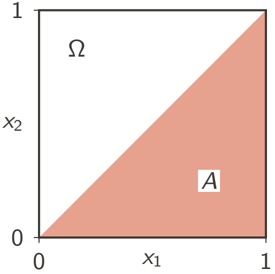

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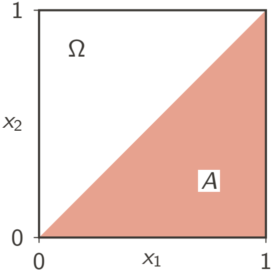

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Geometric probability: example

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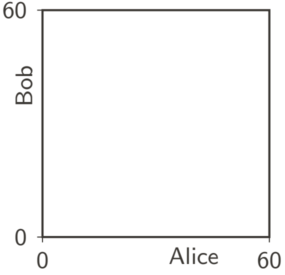

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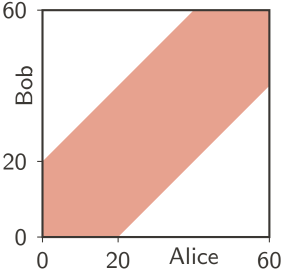

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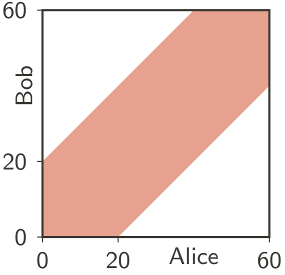

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$$P(A) \;=\; 1 - P(A') \;=\; 1 - \frac{|A'|}{|\Omega|} \;=\; 1 - \frac{40 \times 40}{60 \times 60} \;=\; \frac{5}{9}.$$

Geometric probability: Buffon's needle

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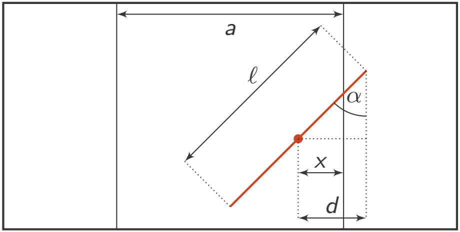

Outcomes are pairs (x, α):

x – distance from the center of the needle to the closest line, $x \in [0, a/2]$

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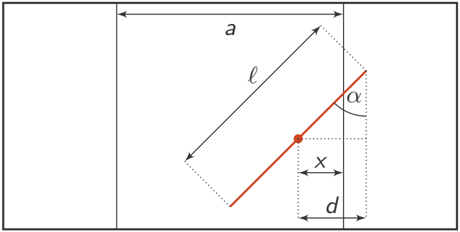

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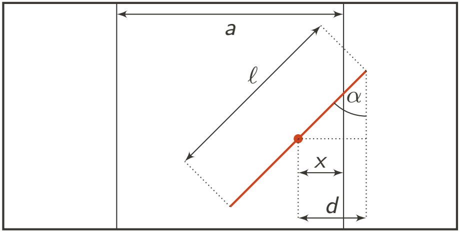

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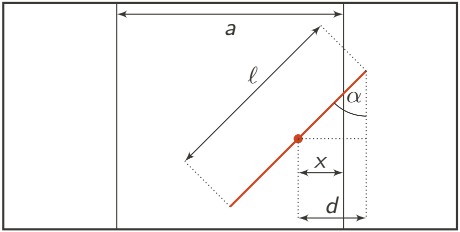

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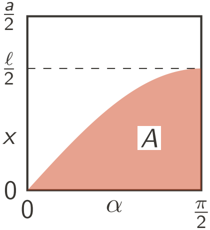

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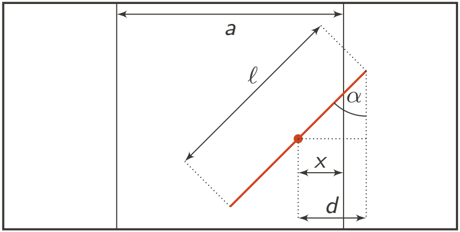

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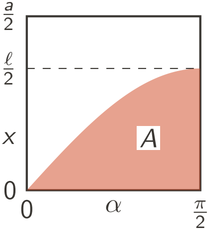

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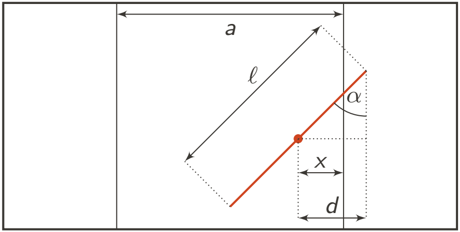

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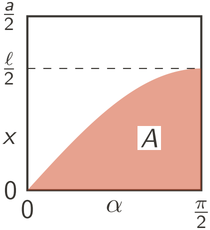

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If $\ell = \frac{a}{2}$, $P(A) = \frac{1}{\pi}$. Can be used to experimentally estimate the number π!