Lech M.

Lecture 13

Ordinary differential equations of the second order

General form

$$F(x, y, y', y'') = 0$$

We are looking on $y = y(x)$ which satisfying this equation.

Simpler form

$$\boxed{(1) \quad y'' = f(x, y, y')}$$

Thm 1 If f is continuous in rectangular parallelepiped $P \subset \mathbb{R}^3$ containing x_0, y_0, y_0', then equation (1) has solution $y = y(x)$ such that

$$y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

Moreover, if $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are continuous in P, then solution is unique.

Particular cases of (1)

I $y'' = f(x)$

$$\Downarrow$$

$$y' = \int f(x)\,dx + C_1 = F(x) + C_1$$

$$\Downarrow$$

$$y = \int F(x)\,dx + C_1 x + C_2$$

II $y'' = f(x, y')$

Put $z = y'$. Then

$z' = f(x, z)$ d.e. of the first order

III **Linear differential equation of the second order**

$$\boxed{(2) \quad y'' + p(x)y' + q(x)y = r(x)}$$

If $r(x) = 0$, then (2) is said to be homogeneous

if $r(x) \neq 0$, then (2) is said to be nonhomogeneous

Homogeneous case

$$(2_H) \quad y'' + p(x)y' + q(x)y = 0, \quad x \in D$$

Remarks 1. $y=0$ is a solution of homogeneous equation.

2. If y_1 and y_2 are solutions of homogeneous equation, then

$$y = c_1 y_1 + c_2 y_2 \quad \forall c_1, c_2 \in \mathbb{R}$$

are also solutions of homogeneous equation.

Def. 1 Two solutions y_1, y_2 of (2_H) are fundamental if the Wronski determinant (Wronskian)

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0 \text{ at some } x \in D$$

y_1, y_2 are linearly independent in D

THM 2 (fundamental) If y_1, y_2 are fundamental solutions of

(2_H) $y'' + p(x)y' + q(x)y = 0$,

then

$$y = c_1 y_1 + c_2 y_2 \quad \forall c_1, c_2 \in \mathbb{R}$$

are all solutions of (2_H).

THM 3 If we know one solution y_1 of (2_H), then by variation of constant we can find the second one y_2 such that y_1, y_2 are fundamental (and so all solutions).

Proof of Thm 3

Solutions $y_1(x)$ and $c y_1(x)$ are not fundamental

$$\begin{vmatrix} y_1 & c y_1 \\ y_1' & c y_1' \end{vmatrix} = c y_1 y_1' - c y_1 y_1' = 0.$$

Let us do variation of constant

$$y_1(x), \quad \underbrace{c(x) y_1(x)}_{y_2(x)}$$

Then

$$y_2''(x) = (c'y_1 + cy_1')' = c''y_1 + c'y_1' + c'y_1' + cy_1''$$

$$y_2'' + p(x)y_2' + q(x)y_2 = c''y_1 + 2c'y_1' + cy_1'' +$$
$$p(x)\left[c'y_1 + cy_1'\right] + q(x)\,cy_1 =$$
$$= c''y_1 + 2c'y_1' + p(x)c'y_1 + c\underbrace{\left[y_1'' + p(x)y_1' + q(x)y_1\right]}_{=0}$$

since y_1 is the solution of (2_H)

$$\Downarrow$$

$$c''y_1 + 2c'y_1' + p(x)c'y_1 = 0$$

$$c''y_1 = -2c'y_1' - c'p(x)y_1$$

$$\frac{c''}{c'} = -\frac{2y_1'}{y_1} - p(x)$$

$$(\ln|c'(x)|)' = -2(\ln|y_1(x)|)' - p(x)$$

$$\ln|c'(x)| = -2\ln|y_1(x)| - \int p(x)\,dx$$

$$\Downarrow$$

we can get $C(x)$.

Ex.1 Equation

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0, \quad x \neq 0$$

has solution $y_1(x) = x$. Find all solutions.

Solution

$x, \quad C(x) \cdot x$

$$(C \cdot x)'' = (C' \cdot x + C)' = C'' \cdot x + C' + C' = C'' \cdot x + 2C'$$

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = C'' \cdot x + 2C' - \frac{1}{x}(C'x + C) + \frac{1}{x^2} \cdot C \cdot x$$

$$= \underbrace{C'' \cdot x + 2C' - C' - \cancel{\frac{C}{x}} + \cancel{\frac{C}{x}}}_{} = 0,$$

$$C''x + C' = 0 \longrightarrow C''x = -C'$$

$$\frac{C''}{C'} = -\frac{1}{x}$$

$$(\ln|C'|)' = -\frac{1}{x}$$

$$\ln|C'| = -\ln|x|$$

$$C'(x) = \frac{1}{x}$$

$$C(x) = \ln|x|$$

$x, \quad x\ln|x|$ fundamental

$x, x\ln|x|$ is fundamental

$$\begin{vmatrix} x & x\ln|x| \\ 1 & \ln|x|+1 \end{vmatrix} = x(\ln|x|+1) - x\ln|x| = $$
$$= x \neq 0$$

All solutions

$$y_H = c_1 x + c_2 x\ln|x|$$

Nonhomogeneous case

(2) $\quad y'' + p(x)y' + q(x)y = r(x) \quad \forall x \in D$

THM 4 All solutions of (2) are

$$y(x) = y_H(x) + y_p(x),$$

where y_H are all solutions of homogeneous equation and y_p is a particular solution of nonhomogeneous equation.

Why?

$$y'' + p(x)y' + q(x)y = y_H'' + y_p'' + p(x)\left[y_H' + y_p'\right] + q(x)\left[y_H + y_p\right]$$
$$= \underbrace{y_H'' + p(x)y_H' + q(x)y_H}_{=0} + \underbrace{y_p'' + p(x)y_p' + q(x)y_p}_{r(x)} = r(x).$$

Ex. 2 Solve d. e.

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = 1, \quad x \neq 0$$

1° Homogeneous

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = 0$$

From Ex. 1 $\quad y_H = c_1 x + c_2 x \ln|x|$

2° $y_p(x) = x^2$. We control this

$$y_p'' - \frac{1}{x} y_p' + \frac{1}{x^2} y_p = \cancel{2} - \cancel{\frac{1}{x} \cdot 2x} + \frac{1}{x^2} \cdot x^2 = 1$$

OK

All solutions are

$$y = y_H(x) + y_p(x) = c_1 x + c_2 x \ln|x| + x^2$$

IV Linear differential equations of the second order with constant coefficients

$$(3) \quad \boxed{y'' + py' + qy = r(x)}, \quad p, q \in \mathbb{R}$$

First homogeneous equation

$$(3_H) \quad \boxed{y'' + py' + qy = 0} \quad p, q \in \mathbb{R}$$

Particular solution has form $y = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$

How to find λ?

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$y'' + py' + qy = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x}$$

$$= \underbrace{e^{\lambda x}}_{\neq 0}[\lambda^2 + p\lambda + q] = 0$$

$$(4) \quad \boxed{\lambda^2 + p\lambda + q = 0}$$

characteristic equation of (3_H).

There are three cases:

1° Solutions are real numbers and different

2° Solutions are real numbers and equal

3° Solutions are complex numbers

(3_H) $y'' + py' + qy = 0$, $p, q \in \mathbb{R}$

(4) $\boxed{\lambda^2 + p\lambda + q = 0}$

1° $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = \lambda_2 e^{(\lambda_1+\lambda_2)x} - \lambda_1 e^{(\lambda_1+\lambda_2)x}$$

$$= e^{(\lambda_1+\lambda_2)x} \cdot (\lambda_2 - \lambda_1) \neq 0$$

linearly independent

$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$ all solutions of (3_H)

2° $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

$y_1 = e^{\lambda x}$, $y_2(x) = C(x) e^{\lambda x}$

$\downarrow$ calculate !

$C(x) = x$

$y_1 = e^{\lambda x}$, $y_2(x) = x e^{\lambda x}$

All solutions

$y_H = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$

$$y_1 = e^{\lambda x}$$

$$\underbrace{\lambda^2 + p\lambda + q}_{(\lambda - a)^2} = 0 \quad \text{and} \quad \lambda_1 = \lambda_2 = \lambda = -\frac{p}{2}$$

$$(\lambda - a)^2 = 0$$

$$(\lambda - a)^2 = \lambda^2 \underbrace{- 2a}\lambda + a^2$$

$$-2a = p \rightarrow a = -\frac{p}{2}$$

$$\underline{y_2 = c(x) e^{\lambda x}}$$

$$y_2'' = \left(c' e^{\lambda x} + c\lambda e^{\lambda x}\right)' = c'' e^{\lambda x} + c'\lambda e^{\lambda x} + \lambda\left(c' e^{\lambda x} + c\lambda e^{\lambda x}\right)$$

$$y_2'' + p y_2' + q y_2 = \ldots$$

$$c'' + c' \underbrace{(2\lambda + p)}_{=0} = 0$$

$$\Downarrow$$

$$c'' = 0 \Rightarrow c' = \text{const}$$

$$\Downarrow$$

$$\underline{c(x) = x}$$

3° λ_1, λ_2 complex numbers

$$\lambda^2 + p\lambda + q = 0, \quad p, q \in \mathbb{R}$$

$$\lambda_1 = \alpha + \beta i, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0$$

$$\lambda_2 = \overline{\lambda_1} = \overline{\alpha + \beta i} = \alpha - \beta i$$

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

$$\left\{ e^{iv} = \cos v + i \sin v \right.$$

$$y = C y_1 + D y_2 =$$

$$C, D \in \mathbb{C}$$

$$= C e^{(\alpha + \beta i)x} + D e^{(\alpha - \beta i)x}$$

$$= C e^{\alpha} \left[\cos \beta x + i \sin \beta x\right] + D e^{\alpha x} \left[\cos \beta x - i \sin \beta x\right]$$

$$= e^{\alpha x} \left[(C + D) \cos \beta x + i (C - D) \sin \beta x\right]$$

* $C = D = \frac{1}{2}$

- $y_1 = e^{\alpha x} \cos \beta x$

** $C = -D = -\frac{i}{2} \Rightarrow i(C - D) = i\left(-\frac{i}{2} - \frac{i}{2}\right) = i(-i) = -i^2 = 1$

- $y_2 = e^{\alpha x} \sin \beta x$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ (e^{\alpha x} \cos \beta x)' & (e^{\alpha x} \sin \beta x)' \end{vmatrix} = \ldots = e^{2\alpha x} \cdot \beta \neq 0$$

Homework

$$y_H = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin \beta x$$

$$\forall c_1, c_2 \in \mathbb{R}$$

3°

$$\lambda_1 = \alpha + \beta i, \ \lambda_2 = \alpha - \beta i, \ \alpha, \beta \in \mathbb{R}$$

$$\beta \neq 0$$

$$y_H = e^{\alpha x} [c_1 \cos\beta x + c_2 \sin \beta x]$$

Ex. 3

$$y'' - y' - 2y = 0 \quad \Big\} \ y = e^{\lambda x}$$

$$\lambda^2 - \lambda - 2 = 0 \longrightarrow \lambda_{1,2} = \frac{1 \mp \sqrt{1+8}}{2} \begin{cases} -1 \\ 2 \end{cases}$$

$$y_H = c_1 e^{-x} + c_2 e^{2x}$$

Ex. 4

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \longrightarrow \lambda_1 = \lambda_2 = 1$$

$$y_H = c_1 e^{x} + c_2 x e^{x}$$

Ex. 5

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0 \longrightarrow \lambda^2 = -1$$

$$\lambda_{1,2} = \pm i$$

$$y_H = c_1 \cos x + c_2 \sin x$$

Nonhomogeneous case

(3) $y'' + py' + qy = r(x)$, $p, q \in \mathbb{R}$

All solutions

$$y = \underbrace{y_H}_{\text{we know}} + y_p$$

How to find particular solution of (3)

Equation	Initial GUESS for y_p
$r(x) = ke^{\alpha x}$	$y_p(x) = A e^{\alpha x}$
$r(x) = a_0 + a_1 x + \dots + a_n x^n$	$y_p(x) = A_0 + A_1 x + \dots + A_n x^n$
$r(x) = a_1 \cos bx + a_2 \sin bx$	$y_p(x) = A_1 \cos bx + A_2 \sin bx$

Remark (brothers or sisters)

If $r(x) = 2\cos 3x$, then $y_p(x) = A_1 \cos 3x + A_2 \sin 3x$

or

$r(x) = 3\sin 4x$, then $y_p(x) = A_1 \cos 4x + A_2 \sin 4x$.