Calculus Cheat Sheet

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ABSTRACT

This cheat sheet lists all appliances of various mathematical techniques as shown by prof. Lech Maligranda during his Calculus classes and lectures. All examples are taken directly from the recorded footage.

Every example contains a link to the source material, where you can view the entire problem solving process; this cheat sheet's purpose is only to help *recognize* some of the patterns in a dense, consolidated form. For that reason, many steps of the solving processes were skipped, to showcase just the end result.

The cheat sheet started as an attempt to encapsulate all tricky integration techniques (unintuitive substitutions and other aspects of exercises that are easier to simply memorize). This primary goal is still firmly kept in mind, with some extra material added (e.g. sequences). I try to include as much as I can, but I exclude some trivial examples that just don't make sense to note down. This is a cheat sheet, not a coursebook; it expects you already grasp the theory and came to practice.

Note that there's **no warranty** that this is a complete database — below is a list of all material that's already been pieced into the document.

Errors can (and should!) be reported to the author of the document — Arthur Intel.

COVERED SO FAR

TABLE OF CONTENTS

(sorry if the displayed page numbers don't exactly match)

Part A - Integration

1. Substitution

1.1. L3 — The general case

source

$$\int f'(g(x))\, g'(x)\, dx = \begin{Bmatrix} u = g(x) \\ du = g'(x)\, dx \end{Bmatrix} = \int f'(u)\, du = f(u) + C$$

1.2. L3 — The general case (definite integral)

source

1. g differentiable on $[a, b]$, $\quad g(a) = A, \quad g(b) = B$
2. f continuous in the range of g

$$\int\limits_a^b f(g(x))\, g'(x)\, dx = \int\limits_A^B f(u)\, du$$

1.3. L3E1a

source

$$\frac{1}{a^2} \int \frac{1}{1 + \frac{x^2}{a^2}}\, dx = \begin{Bmatrix} u = \frac{x}{a} \\ du = \frac{1}{a}\, dx \end{Bmatrix} = \cdots = \frac{1}{a} \arctan \frac{x}{a} + C$$

1.4. L3E1b

source

$$\int \frac{x}{x^2 + 1}\, dx = \begin{Bmatrix} u = x^2 + 1 \\ du = 2x\, dx \end{Bmatrix} = \int \frac{1}{u} \frac{du}{2} = \cdots = \frac{1}{2} \ln(x^2 + 1) + C$$

1.5. L3E1c

source

$$\int e^x \sqrt{1 + e^x}\, dx = \begin{Bmatrix} u = 1 + e^x \\ du = e^x dx \end{Bmatrix} = \int \sqrt{u}\, du = \cdots = \frac{2}{3} (1 + e^x)^{\frac{3}{2}} + C$$

1.6. L3E1d

source

$$\int \frac{1}{(x+2)^2+1}\,dx = \left\{\begin{matrix} u = x+2 \\ du = dx \end{matrix}\right\} = \cdots = \arctan(x+2) + C$$

1.7. L3E2a

source

$$\int_0^1 \sqrt{3x+1}\,dx = \left\{\begin{matrix} u = 3x+1 \\ du = 3\,dx \\ x = 0 \rightarrow u = 1 \\ x = 1 \rightarrow u = 4 \end{matrix}\right\} = \int_1^4 \sqrt{u}\,\frac{1}{3}\,du = \cdots = \frac{14}{9}$$

1.8. L3E2b

source

$$\int_0^2 \frac{2x}{2x^2+1}\,dx = \left\{\begin{matrix} u = 2x^2+1 \\ du = 4x\,dx \\ x = 0 \rightarrow u = 1 \\ x = 2 \rightarrow u = 9 \end{matrix}\right\} = \int_1^9 \frac{1}{u}\,\frac{du}{2} = \cdots = \ln 3$$

1.9. L3E2c

source

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx = \left\{\begin{matrix} u = \sqrt{x+1} \\ du = \dfrac{1}{2\sqrt{x+1}}\,dx \\ x = 0 \rightarrow u = 1 \\ x = 8 \rightarrow u = 3 \end{matrix}\right\} = \int_1^3 \cos u \cdot 2\,du = \cdots = 2(\sin 3 - \sin 1)$$

1.10. L3E3 — 1st method

source

$$\int \sin 2x\,dx = \left\{\begin{matrix} u = 2x \\ du = 2\,dx \end{matrix}\right\} = \int \sin u\,\frac{du}{2} = \cdots = -\frac{\cos 2x}{2} + C$$

1.11. L3E3 — 2nd method

source

$$\int \sin 2x\,dx = 2\int \sin x\cos x\,dx = \begin{Bmatrix} u = \sin x \\ du = \cos x\,dx \end{Bmatrix} = 2\int u\,du = \sin^2 x + D$$

1.12. L3E4

source

$$\int \frac{g'(x)}{g(x)}\,dx = \begin{Bmatrix} u = g(x) \\ du = g'(x)\,dx \end{Bmatrix} = \int \frac{1}{u}\,du = \cdots = \ln|g(x)| + C$$

In particular,

$$\int cotx\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$$

1.13. L3E4

source

$$2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx = \begin{Bmatrix} x = a\sin t \\ dx = a\cos t\,dt \\ x = -a \to t = -\frac{\pi}{2} \\ x = a \to t = \frac{\pi}{2} \end{Bmatrix} = \cdots = \pi ab$$

1.14. C4E1

source

$$\int \sin^3 x\cos^5 x\,dx = \begin{Bmatrix} u = \cos x \\ du = -\sin x\,dx \end{Bmatrix} = \cdots = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$$

1.15. C4E2i

source

$$\frac{1}{a}\int \frac{1}{1-\frac{x^2}{a^2}}\,dx = \begin{Bmatrix} t = \frac{x}{a} \\ dt = \frac{1}{a}\,dx \end{Bmatrix} = \cdots = \arcsin\frac{x}{a} + C$$

1.16. C4E2ii — Euler substitution

source

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \left\{\begin{aligned} & u = x+\sqrt{a^2+x^2} \\ & du = \left(1+\frac{2x}{2\sqrt{a^2+x^2}}\right)dx = \\ & = \frac{u}{\sqrt{a^2+x^2}}\,dx \end{aligned}\right\} = \cdots = \ln\left|x+\sqrt{a^2+x^2}\right| + C$$

1.17. C4E3a

source

$$\int\limits_{-a}^{a} f(x)\,dx = \int\limits_{-a}^{0} f(x)\,dx + \int\limits_{0}^{a} f(x)\,dx = \left\{\begin{aligned} & x = -t \\ & dx = -dt \\ & x = -a \to t = a \\ & x = 0 \to t = 0 \end{aligned}\right\} = \cdots = 2\int\limits_{0}^{a} f(x)\,dx$$

1.18. C4E3c

source

The substitution in this exercise is very circumstantial — be sure to watch the source material.

$$\int\limits_{a}^{0} f(x)\,dx = \left\{\begin{aligned} & x = u-T \\ & dx = du \\ & x = a \to u = a+T \\ & x = 0 \to u = T \end{aligned}\right\} = \cdots = -\int\limits_{T}^{a+T} f(u)\,du$$

1.19. C4E4 — 1st method

source

$$\int \frac{x^3}{\sqrt{1-x^2}} = \left\{\begin{aligned} & u = 1-x^2 \\ & du = -2x\,dx \end{aligned}\right\} = \cdots = -\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

1.20. C4E4 — 2nd method

source

$$\int \frac{x^3}{\sqrt{1-x^2}} = \left\{\begin{aligned} & 1-x^2 = t^2 \\ & -2x\,dx = 2t\,dt \end{aligned}\right\} = \cdots = -\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

1.21. C4E4 — 3rd method

source

$$\int \frac{x^3}{\sqrt{1-x^2}} = \left\{ \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \right\} = \frac{1}{2} \int \frac{t}{\sqrt{1-t}}\,dt =^{(by\ parts)}_{\cdots} = -\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

1.22. C4E4 — 4th method

source

$$\int \left(1-\cos^2 t\right)\sin t\,dt = \left\{ \begin{matrix} u = \cos t \\ du = -\sin t\,dt \end{matrix} \right\} = \cdots = -\sqrt{1-x^2} + \frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + C$$

1.23. C4E5a

source

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \left\{ \begin{matrix} u = 1+x^2 \\ du = 2x\,dx \end{matrix} \right\} = \cdots = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + C$$

1.24. C4E6a

source

$$\int \frac{x}{x^4+16}\,dx = \left\{ \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \right\} = \cdots =$$

$$= \frac{1}{32} \int \left[\frac{1}{1+\frac{u^2}{16}} \right] du = \left\{ \begin{matrix} \frac{u}{4} = t \\ \frac{1}{4}\,du = dt \end{matrix} \right\} = \cdots = \frac{1}{8} \arctan \frac{x^2}{4} + C$$

1.25. L4E6

source

$$\int \frac{1}{\cos x}\,dx = \left\{ \begin{matrix} u = \dfrac{1+\sin x}{\cos x} \\ du = \dfrac{\cos^2 x + \sin x(1+\sin x)}{\cos^2 x}\,dx = \dfrac{1+\sin x}{\cos^2 x}\,dx \end{matrix} \right\} = \cdots = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

1.26. C5E6 1°

source

$$\int \frac{1}{\cos x}\,dx = \left\{\begin{aligned} u &= \frac{1+\sin x}{\cos x} \\ du &= \frac{\cos^2 x + (1+\sin x)\sin x}{\cos^2 x}\,dx = \cdots = \frac{1+\sin x}{\cos x}\cdot\frac{dx}{\cos x}\end{aligned}\right\} = \cdots = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

1.27. C5E7b

source

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \left\{\begin{aligned} t &= x+\frac{1}{2} \\ dt &= dx\end{aligned}\right\} = \cdots = \frac{4}{3}\int \frac{1}{1+\left(\frac{2t}{\sqrt{3}}\right)^2}\,dt$$

$$\frac{4}{3}\int \frac{1}{1+\left(\frac{2t}{\sqrt{3}}\right)^2}\,dt = \left\{\begin{aligned} u &= \frac{2}{\sqrt{3}}\,t \\ du &= \frac{2}{\sqrt{3}}\,dt\end{aligned}\right\} = \cdots = \frac{2\sqrt{3}}{3}\arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C$$

1.28. C5E8

source

$$\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx = \left\{\begin{aligned} t &= \frac{x}{\sqrt{3}} \\ dt &= \frac{1}{\sqrt{3}}\,dx\end{aligned}\right\} = \cdots = \frac{2\sqrt{3}}{3}\arctan\frac{x}{\sqrt{3}} + C$$

1.29. C5E9 — 2nd method

source

$$\int \frac{1}{\sin x}\,dx = \left\{\begin{aligned} t &= \frac{1+\cos x}{\sin x} \\ dt &= \cdots = -\frac{1+\cos x}{\sin x}\cdot\frac{dx}{\sin x} \\ \frac{dt}{t} &= -\frac{dx}{\sin x}\end{aligned}\right\} = -\int \frac{dt}{t} = \cdots = \ln\left|\frac{\sin x}{1+\cos x}\right| + C$$

1.30. L5E5

source

$$\int_0^8 \sqrt{\frac{9x^{\frac{2}{3}}+4}{9x^{\frac{2}{3}}}} = \left\{\begin{array}{l} u = 9x^{\frac{2}{3}}+4 \\ du = 6x^{-\frac{1}{3}}dx \\ x = 0 \to u = 4 \\ x = 8 \to u = 40 \end{array}\right\} = \cdots = \frac{1}{27}\left(80\sqrt{10}-8\right)$$

1.31. C6E1

source

$$\int_0^b \frac{e^x}{1+e^x}\,dx \left\{\begin{array}{l} 1+e^x = t \\ e^x dx = dt \\ x = 0 \to t = 2 \\ x = b \to t = 1+e^b \end{array}\right\} = \cdots = \ln\left(1+e^b\right) - \ln 2$$

1.32. C6E7

source A.4.6

$$\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x^2}\,dx = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}\cdot x\,dx}{x^2} = \left\{\begin{array}{l} x^2+1 = t^2 \\ x\,dx = t\,dt \\ x = \sqrt{3} \to t = 2 \\ x = \sqrt{8} \to t = 3 \end{array}\right\} =^{(A.4.6)} \cdots = 1 + \frac{1}{2}\ln\frac{3}{2}$$

1.33. L6E3d

source

$$\int_e^R \frac{1}{x(\ln x)^r}\,dx = \left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \\ x = e \to u = 1 \\ x = R \to u = \ln R \end{array}\right\} = \cdots = \frac{1}{r-1} - \frac{1}{r-1}\cdot\frac{1}{(\ln R)^{r-1}}$$

1.34. L6E6

source

$$\int \frac{1}{x^2 e^{\frac{1}{x}}}\, dx = \left\{ \begin{aligned} u &= -\frac{1}{x} \\ du &= \frac{1}{x^2} \end{aligned} \right\} = \int e^u du = e^{-\frac{1}{x}}$$

1.35. C7E2

source

$$\int \frac{x}{x^2+3}\, dx = \left\{ \begin{aligned} t &= x^2+3 \\ dt &= 2x\, dx \end{aligned} \right\} = \cdots = -\frac{1}{2} \cdot \frac{1}{x^2+3}$$

1.36. C7E3

source

$$\int \frac{a}{1+\frac{b^2}{a^2}x^2}\, dx = \left\{ \begin{aligned} \frac{b}{a}x &= t \\ \frac{b}{a}dx &= dt \end{aligned} \right\} = \cdots = \frac{a}{b}\arctan\left(\frac{b}{a}x\right)$$

1.37. C7E4

source

$$\int \frac{1}{\sqrt{1-(2x-1)^2}}\, dx = \left\{ \begin{aligned} 2x-1 &= t \\ 2dx &= dt \end{aligned} \right\} = \frac{\arcsin(2x-1)}{2} + C$$

1.38. C7E5

source

$$\int \frac{-\sin x}{\cos x}\, dx = \left\{ \begin{aligned} \cos x &= t \\ -\sin x\, dx &= dt \end{aligned} \right\} = \cdots = -\ln|\cos x|$$

1.39. C7E10

source

$$\int \frac{1}{x(\ln x)^2}\, dx = \left\{ \begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\, dx \end{aligned} \right\} = \cdots = -\frac{1}{\ln x} + C$$

2. Integration by parts

2.1. L3 — The general case

source

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$$

2.2. L3 — The general case (for definite integral)

source

$$\int_a^b u(x)\,v'(x)\,dx = u(x)\,v(x)\Big|_a^b - \int_a^b u'(x)\,v(x)\,dx$$

2.3. L3E1a

source

$$\int xe^x\,dx = \begin{Bmatrix} u = x & dv = e^x\,dx \\ du = dx & v = e^x \\ u(x) = x & v'(x) = e^x \\ u'(x) = 1 & v(x) = e^x \end{Bmatrix} = \cdots = xe^x - e^x + C$$

2.4. L3E1b

source

$$\int x\cos x\,dx = \begin{Bmatrix} u(x) & v'(x) = \cos x \\ u' = 1 & v = \sin x \end{Bmatrix} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

2.5. L3E1c

source

$$\int x^2\cos x\,dx = \begin{Bmatrix} u = x^2 & v' = \cos x \\ u' = 2x & v = \sin x \end{Bmatrix} =$$

$$= x^2\sin x - 2\int x\sin x\,dx = \begin{Bmatrix} u_1 = x & v_1' = \sin x \\ u_1' = 1 & v_1 = -\cos x \end{Bmatrix} =$$

$$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right) = \cdots = x^2\sin x + 2x\cos x - 2\sin x + C$$

2.6. L3E1d

source

$$\int x\arctan x\,dx = \begin{Bmatrix} u = \arctan x & v' = x \\ u' = \dfrac{1}{1+x^2} & v = \dfrac{x^2}{2} \end{Bmatrix} = \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C$$

2.7. L3E2

source

$$\int_e^{e^2} \ln x\,dx = \begin{Bmatrix} u = \ln x & v' = 1 \\ u' = \dfrac{1}{x} & v = x \end{Bmatrix} = x\ln x\Big|_e^{e^2} - \int_e^{e^2} dx = \cdots = e^2$$

2.8. L3E3

source

$$\int_e^{e^2} \frac{\ln(\ln x)}{x}\,dx = \begin{Bmatrix} u = \ln(\ln x) & v' = \dfrac{1}{x} \\ u' = \dfrac{1}{\ln x}\dfrac{1}{x} & v = \ln x \end{Bmatrix} = \cdots = \ln 4 - 1$$

2.9. L3E5

source

$$\int_0^{\pi} e^{-x}\sin x\,dx = \begin{Bmatrix} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{Bmatrix} = \cdots = e^{-\pi} + 1 - \int_0^{\pi} e^{-x}\cos x\,dx$$

$$\int_0^{\pi} e^{-x}\cos x\,dx = \begin{Bmatrix} u_1 = e^{-x} & v_1' = \cos x \\ u_1' = -e^{-x} & v_1 = \sin x \end{Bmatrix} = \cdots = \int_0^{\pi} e^{-x}\sin x\,dx$$

2.10. C4E4 — 2nd method

source

$$\frac{1}{2}\int \frac{t}{\sqrt{1-t}}\,dt = \begin{Bmatrix} u = t & v' = \dfrac{1}{\sqrt{1-t}} \\ u' = 1 & v = -2\sqrt{1-t} \end{Bmatrix} = \cdots = -x^2\sqrt{1-x^2} - \frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

2.11. C4E7

source

$$\int e^{2x}\sin 3x\,dx = \left\{\begin{matrix} u = \sin 3x & v' = e^{2x} \\ u' = 3\cos^3 x & v = \dfrac{e^{2x}}{2} \end{matrix}\right\} = \cdots =$$

$$\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx = \left\{\begin{matrix} u_1 = \cos 3x & v_1' = e^{2x} \\ u_1' = -3\sin 3x & v_1 = \dfrac{e^{2x}}{2} \end{matrix}\right\} = \cdots = \frac{\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x}{\frac{13}{4}} + C$$

2.12. C4E8

source

$$I_n = \int x^n e^{-x}dx = \left\{\begin{matrix} u = x^n & v' = e^{-x} \\ u' = nx^{n-1} & v = -e^{-x} \end{matrix}\right\} = \cdots = -x^n e^{-x} + n\,I_{n-1}$$

2.13. C4E9

source

$$\int \ln x dx = \left\{\begin{matrix} u = \ln x & v' = 1 \\ u' = \dfrac{1}{x} & v = x \end{matrix}\right\} = x\ln x - \int dx = x\ln x - x$$

$$\int (\ln x)^n dx = \left\{\begin{matrix} u = (\ln x)^n & v' = 1 \\ u' = n(\ln x)^{n-1} & v = x \end{matrix}\right\} = x(\ln x)^n - n\int (\ln x)^{n-1}dx$$

2.14. L6E3b

source

$$\int_0^R xe^{-x}dx = \left\{\begin{matrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{matrix}\right\} = \cdots = -\frac{R}{e^R} - \frac{1}{e^R} + 1$$

2.15. L6E5c

source

$$\int \ln x \, dx = \left\{ \begin{matrix} u = \ln x & v' = 1 \\ u' = \dfrac{1}{x} & v = x \end{matrix} \right\} = x\ln x - x$$

2.16. C7E7

source

$$\int (\ln x)^2 dx = \left\{ \begin{matrix} u = (\ln x)^2 & v' = 1 \\ u' = \dfrac{2\ln x}{2} & v = x \end{matrix} \right\} = x\ln(x)^2 - 2 \int \ln x dx$$

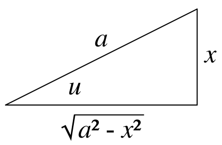
$$x\ln(x)^2 - 2 \int \ln x dx = \left\{ \begin{matrix} u_1 = \ln x & v_1' = 1 \\ u_1' = \dfrac{1}{x} & v_1 = x \end{matrix} \right\} = \cdots = x(\ln x)^2 - 2x\ln x + 2x$$

3. Inverse substitution

3.1. L4 — The inverse sine substitution

source

Integrals involving $\sqrt{a^2 - x^2}$, $(a > 0)$ can be reduced to a simpler form by the substitution



$$u = \arcsin \frac{x}{a} \longleftrightarrow x = a \sin u$$

$$\left(-\frac{\pi}{2} \le u \le \frac{\pi}{2} \right)$$

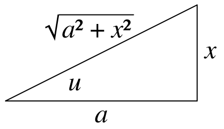
Then

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 u} = \sqrt{a^2 \left(1 - \sin^2 u\right)}$$

3.2. L4 — The inverse tangent substitution

source

Integrals involving $\sqrt{a^2+x^2}$ or $\left(\dfrac{1}{a^2+x^2}\right)$, $(a>0)$ are often simplified by the substitution



$$u = \arctan\frac{x}{a} \longleftrightarrow x = a\tan u$$

$$\left(-\frac{\pi}{2} \le u \le \frac{\pi}{2}\right)$$

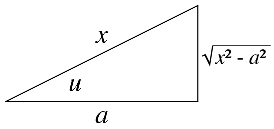
Then

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 u} = a\sqrt{1+\frac{\sin^2 u}{\cos^2 u}} = a\sqrt{\frac{\cos^2 u+\sin^2 u}{\cos^2 u}} = \frac{a}{\cos u}$$

3.3. L4 — The inverse cosine substitution

source

Integrals involving $\sqrt{x^2-a^2}$, $(a>0)$ can be simplified by using substitution



$$u = \arccos\frac{a}{x} \longleftrightarrow x = \frac{a}{\cos u}$$

$$(0 \le u \le \pi)$$

Then

$$\sqrt{x^2-a^2} = \sqrt{\frac{a^2}{\cos^2 u}-a^2} = a\sqrt{\frac{1-\cos^2 u}{\cos^2 u}} = a\sqrt{\frac{\sin^2 u}{\cos^2 u}} = a\,|\tan u|$$

3.4. L4 — Substitution $x = \tan \frac{u}{2}$

source

$$\int R(\sin u, \cos u)\, du, \qquad R - \textit{rational function}$$

$$u = 2\arctan x \longleftrightarrow x = \tan\frac{u}{2}$$

Then

$$\begin{cases} \cos u = \dfrac{1-x^2}{1+x^2} \\ \sin u = \dfrac{2x}{1+x^2} \\ du = \dfrac{2}{1+x^2}\, dx \end{cases}$$

3.5. C4E4 — 4th method

source A.1.22

$$\int \frac{x^3}{\sqrt{1-x^2}}\, dx = \begin{Bmatrix} x = \sin t \\ dx = \cos t\, dt \end{Bmatrix} = \int \left(1-\cos^2 t\right)\sin t\, dt \overset{(A.1.22)}{=} \cdots = -\sqrt{1-x^2} + \frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + C$$

3.6. L4E4

source (prof. Maligranda has a mistake in the last step of his solution)

$$\int \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\, dx = \begin{Bmatrix} x = 2\sin u \\ dx = 2\cos u\, du \end{Bmatrix} = \cdots = \frac{x}{4\sqrt{4-x^2}} + C$$

3.7. L4E5

source

$$\int \frac{1}{\left(4+x^2\right)^{\frac{3}{2}}}\, dx = \begin{Bmatrix} x = 2\tan u \\ dx = \dfrac{2}{\cos^2 u}\, du \end{Bmatrix} = \cdots = \frac{1}{4}\,\frac{x}{\sqrt{4+x^2}} + C$$

3.8. L4E6

source A.1.25

$$\int \frac{1}{\sqrt{x^2-4}} = \left\{ \begin{aligned} x &= \frac{2}{\cos u} \\ dx &= 2\,\frac{\sin u}{\cos^2 u}\,du \end{aligned} \right\} = \cdots = \int \frac{1}{\cos u}\,du \overset{(A.1.25)}{\underset{\cdots}{=}} = \ln\left|x+\sqrt{x^2-4}\right| + C$$

3.9. L4E7

source

$$\int \frac{1}{\cos u}\,du = \left\{ \begin{aligned} x &= \tan\frac{u}{2} \\ \cos u &= \frac{1-x^2}{1+x^2} \\ du &= \frac{2}{1+x^2}\,dx \end{aligned} \right\} = 2\int \frac{1}{1-x^2}\,dx \overset{(PF)}{=} \int \left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \cdots = \ln\left|\frac{1+\sin u}{\cos u}\right| + C$$

3.10. C5E4

source A.5.1

$$\int \sqrt{a^2-x^2}dx = \left\{ \begin{aligned} x &= a\sin u \\ dx &= a\cos u\,du \end{aligned} \right\} = \cdots = a|a| \int \cos^2 u\,du = \qquad (a \neq 0)$$

$$\overset{(A.5.1)}{\underset{\cdots}{=}} = \frac{a|a|}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2-x^2}\right) + C$$

3.11. C5E5

source A.5.1

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx = \left\{ \begin{aligned} x &= \tan u \\ dx &= \frac{1}{\cos^2 u}\,du \end{aligned} \right\} \overset{(A.5.1)}{\underset{\cdots}{=}} = \frac{\arctan x}{2} + \frac{1}{2}\cdot\frac{x}{1+x^2} + C$$

3.12. C5E6

source A.1.26

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \left\{\begin{array}{l} x = \dfrac{a}{\cos u} \\ dx = a\,\dfrac{\sin u}{\cos^2 u}\,du \end{array}\right\} =$$

1° if $(x \geq a)$, then

$$= \int \left(\frac{1}{\sqrt{\dfrac{a^2}{\cos^2 u} - a^2}} \cdot a\,\frac{\sin u}{\cos^2 u} \right) du = \cdots = \int \frac{1}{\cos u}\,du \overset{(A.1.26)}{=\cdots} = \ln\left| x + \sqrt{x^2-a^2} \right| + C$$

2° if $(x \leq -a)$, then for $y = -x$ we have $y \geq a$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \left\{\begin{array}{l} x = -y \\ dx = -dy \end{array}\right\} = -\int \frac{1}{\sqrt{y^2-a^2}}\,dy \overset{(1°)}{=} -\ln\left| y + \sqrt{y^2-a^2} \right| + D = \cdots = \ln\left| x + \sqrt{x^2-a^2} \right| + C$$

3.13. C5E8

source A.1.28

$$\int \frac{1}{2+\cos u}\,du = \left\{\begin{array}{l} x = \tan\dfrac{u}{2} \\ \cos u = \dfrac{1-x^2}{1+x^2} \\ du = \dfrac{2}{1+x^2}\,dx \end{array}\right\} = \cdots = \frac{2}{3}\int \frac{1}{1+\left(\dfrac{x}{\sqrt{3}}\right)^2}\,dx \overset{(A.1.28)}{=\cdots} = \frac{2\sqrt{3}}{3}\arctan \cdot \frac{\tan\dfrac{u}{2}}{\sqrt{3}} + C$$

3.14. C5E9 — 1st method

source

$$\int \frac{1}{\sin x} = \left\{ \begin{aligned} u &= \tan\frac{x}{2} \\ \sin x &= \frac{2u}{1+u^2} \\ dx &= \frac{2}{1+u^2}\,du \end{aligned} \right\} = \cdots = \ln\left|\tan\frac{x}{2}\right| + C$$

4. Method of partial fractions

4.1. L4E2

source

$$\int \frac{1}{x^2-a^2} = \frac{1}{2a}\left(\int \frac{1}{x-a}\,dx - \int \frac{1}{x+a}\,dx\right) = \cdots = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\uparrow$$

$$\left[\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)\right]$$

4.2. L4E3

source

$$\int \frac{x+4}{x^2-5x+6}\,dx = \cdots = -6\ln|x-2| + 7\ln|x-3| + C$$

$$\uparrow$$

$$\left[\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{-6}{x-2} + \frac{7}{x-3}\right]$$

4.3. C5E1

source

$$\int \frac{x^2+3x+2}{x\left(x^2+1\right)}\,dx = \cdots = 2\ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + 3\arctan x + C$$

$$\uparrow$$

$$\left[\frac{x^2+3x+2}{x\left(x^2+1\right)} = \cdots = \frac{2}{x} + \frac{3-x}{x^2+1}\right]$$

4.4. C5E2

source

$$\int \frac{1}{x(x-1)^2}\,dx = \cdots = \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C$$

$$\uparrow$$

$$\left[\frac{1}{x(x-1)^2} = \cdots = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right]$$

4.5. C5E3

source

$$\int \frac{1}{x^3(x-3)}\,dx = \cdots = -\frac{1}{27}\ln|x| + \frac{1}{9x} - \frac{1}{3}\cdot\frac{x^{-2}}{-2} + \frac{1}{27}\ln|x-3| + C$$

$$\uparrow$$

$$\left[\frac{1}{x^3(x-3)} = \cdots = -\frac{1}{27}\cdot\frac{1}{x} - \frac{1}{9}\cdot\frac{1}{x^2} - \frac{1}{3}\cdot\frac{1}{x^3} + \frac{1}{27}\cdot\frac{1}{x-3}\right]$$

4.6. C6E7 | C7E1

source1 source2

$$\int \frac{1}{x^2-1}\,dx = \cdots = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$

$$\uparrow$$

$$\left[\frac{1}{x^2-1} = \cdots = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)\right]$$

4.7. L6E7

source

$$\lim_{R\to\infty}\int_1^R \frac{6}{(2x+1)(x+2)}\,dx = \cdots = 2\ln 2$$

$$\uparrow$$

$$\left[\frac{6}{(2x+1)(x+2)} = \cdots = \frac{4}{2x+1} - \frac{2}{x+2}\right]$$

5. Miscellaneous

5.1. Trigonometric Identities

source

$$\begin{array}{rl} & \cos^2 x + \sin^2 x = 1 \\ \pm & \cos^2 x - \sin^2 x = \cos 2x \\ \hline & \begin{cases} 2\cos^2 x = 1 + \cos 2x \\ 2\sin^2 x = 1 - \cos 2x \end{cases} \end{array}$$

5.2. Impossible integral

source

This integral is impossible to calculate.

$$\int e^{-x^2}\,dx$$

5.3. Euler Formula (complex numbers)

source

$$e^{i\gamma} = \cos\gamma + i\sin\gamma$$

Part B - Infinite Sequences

1. Theorems

1.1. L7T1 — Geometric series

source

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}, \quad (|r| < 1)$$

1.2. L7T2

source

$$\textit{If } \sum_{k=1}^{\infty} a_k \textit{ converges, then } \lim_{k\to\infty} a_k = 0.$$

$$\textit{Conversely, if } \lim_{k\to\infty} a_k \neq 0, \textit{ then } \sum_{k=1}^{\infty} a_k \textit{ diverges.}$$

REMARK

$$\rho = \lim_{k\to\infty} a_k = 0 \textit{ alone is } \textbf{NOT} \textit{ sufficient to guarantee that } \sum_{k=1}^{\infty} a_k \textit{ converges.}$$

remark source

1.3. L7T3 — Harmonic series

source

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \underset{\textit{diverges}}{+\infty}$$

1.4. L7T4 — Comparison test

source

$$\textit{Let } 0 \leq a_k \leq b_k, \quad k = 1, 2, \ldots$$

$$a) \quad \textit{If } \sum_{k=1}^{\infty} b_k < \infty, \textit{ then } \sum_{k=1}^{\infty} a_k < \infty$$

$$b) \quad \textit{If } \sum_{k=1}^{\infty} a_k = \infty, \textit{ then } \sum_{k=1}^{\infty} b_k = \infty$$

1.5. L7T5 — Integral test

source

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms and let $f(x)$ be the function obtained by replacing k by x in the formula for a_k, i.e. $f(k) = a_k$.

If f is *decreasing and continuous* on $[1, \infty]$, then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\, dx$ both converge or both diverge.

1.6. L7T6 — Convergence of p-series

source

Let $p \in \mathbb{R},\ p > 0$. Series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges if $(p > 1)$ and diverges if $(0 < p \le 1)$.

1.7. L7T7 — Ratio test (d'Alambert test, 1768)

source

If the terms $a_k > 0,\ \lim\limits_{n\to\infty} \frac{a_{k+1}}{a_k} = R$

$$a) \quad \text{if} \quad R < 1\ , \ \ then \ \ \sum_{k=1}^{\infty} a_k \quad converges$$

$$b) \quad \text{if} \quad R > 1\ , \ \ then \ \ \sum_{k=1}^{\infty} a_k \quad diverges$$

1.8. L7T8 — Root test (Cauchy test, 1821)

source

If $(a_k > 0)$ and $R = \lim\limits_{k\to\infty} \sqrt[k]{a_k}$ exists, then

$$a) \quad \text{if} \quad R < 1\ , \ \ then \ \ \sum_{k=1}^{\infty} a_k \quad converges$$

$$b) \quad \text{if} \quad R > 1\ , \ \ then \ \ \sum_{k=1}^{\infty} a_k \quad diverges$$

1.9. L8 — Power series

source

A series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots$$

is called a **power series in x**.

For any power series in x exactly one of the following is true:

1. The series converges only for $x = 0$
2. The series converges absolutely for all real x.
3. The series converges absolutely for all x in some open interval $(-R, R)$, diverges if $(x < -R)$ or $(x > R)$. At the points $x = R$ and $x = -R$ the series may converge absolutely, converge conditionally or diverge, depending on particular series.

1.10. L8 — Taylor's formula

source

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

1.11. L8 — Maclaurin's formula

source

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

1.12. L8T4

source

$$\begin{cases} f(x) = P_n(x) + R_n(f, a, c) \\ P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k \\ R_n(f, a, c) = \frac{f^{(n+1)}(c)}{(c+1)!} (c-a)^{n+1} \end{cases}$$

If

$$\lim_{n\to\infty} R_n(f, a, x) = 0 \qquad \text{for all } x \text{ in some interval } I,$$

then

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \qquad \text{for } x \in I$$

2. Selected Examples

2.1. L7E1

source

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^{n-1}} + \frac{3}{10^n}$$

$$10S_n = 3 + \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^{n-1}}$$

$$9S_n = 3 - \frac{3}{10^n}$$

$$S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

converges to $\frac{1}{3}$

2.2. L7E2

source

$$S_n = 1 - 1 + 1 - 1 + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1}$$

diverges, i.e. has no sum

2.3. L7E4 — Telescoping series

source

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = ?$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

2.4. L7E6

source

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}} = \sum_{k=1}^{n} \left(\sqrt{k+1} - \sqrt{k} \right) = \sqrt{k+1} - 1$$

2.5. C8E1a

source

$$\sum_{k=1}^{\infty} \left(\frac{5}{4^k} + \frac{2}{5^{k-1}} \right) \overset{(geometric)}{=} \cdots = \frac{25}{6}$$

2.6. C8E1b

source

$$\sum_{k=3}^{\infty} \left[\frac{5}{2^k} + \frac{3}{k(k+1)} \right] \overset{(geometric/telescoping)}{=} \cdots = \frac{9}{4}$$

2.7. C8E2a

source

$$\sum_{k=1}^{\infty} \frac{1}{4^k + 21} \leq \sum_{k=1}^{\infty} \frac{1}{4^k} \overset{(geometric)}{=} \cdots = \frac{1}{3}$$

series converges

2.8. C8E2b

source

$$\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{3}} \geq \sum_{k=1}^{\infty} \frac{1}{k}$$

series diverges

2.9. C8E3a

source

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \overset{(integral\ test)}{\rightarrow} \int_{1}^{\infty} \frac{1}{x^2}\, dx = \cdots = 1$$

series converges

2.10. C8E3b — method 1

source

$$\sum_{k=1}^{\infty} \frac{k}{e^{k^2}} \overset{(integral\ test)}{\rightarrow} = \int_{1}^{\infty} \frac{x}{e^{x^2}}\, dx = \cdots = \frac{1}{2e}$$

series converges

2.11. C8E3b — method 2

source

$$\sum_{k=1}^{\infty} \frac{k}{e^{k^2}} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{\infty} \frac{1}{(k-1)k} \overset{(telescoping)}{=} \cdots = 2$$

series converges

2.12. C8E4a

source

$$\sum_{k=1}^{\infty} \frac{1}{k!} \overset{(ratio\ test)}{\rightarrow} \rho = \lim_{k \to \infty} \frac{k!}{k!\,(k+1)} = \cdots = 0$$

series converges

2.13. C8E4b

source

$$\sum_{k=1}^{\infty} \frac{k}{4^k} \overset{(ratio\ test)}{\rightarrow} \rho = \lim_{k\to\infty} \left(\frac{\frac{k+1}{4^{k+1}}}{\frac{k}{4^k}} \right) = \cdots = \frac{1}{4}$$

series converges

2.14. C8E4c

source

$$\sum_{k=1}^{\infty} \frac{k!}{k^k} \overset{(ratio\ test)}{\rightarrow} \rho = \lim_{k\to\infty} \left(\frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} \right) = \cdots = \frac{1}{e}$$

series converges

2.15. C8E4d

source

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} \overset{(ratio\ test)}{\rightarrow} \rho = \lim_{k\to\infty} \left(\frac{\frac{\left[(k+1)!\right]^2}{\left[2(k+1)\right]!}}{\frac{(k!)^2}{(2k)!}} \right) = \cdots = \frac{1}{4}$$

series converges

2.16. C8E5a

source

$$\sum_{k=1}^{\infty} \left(\frac{2k+1}{4k+7} \right)^k \overset{(root\ test)}{\rightarrow} \rho = \lim_{k\to\infty} \left[\left(\frac{2k+1}{4k+7} \right)^k \right]^{\frac{1}{k}} = \cdots = \frac{1}{2}$$

series converges

2.17. C8E5b

source

$$\sum_{k=1}^{\infty} \frac{1}{\left[\ln(k+1)\right]^{k}} \overset{\textit{(root test)}}{\rightarrow} \rho = \lim_{k\to\infty} \frac{1}{\left[\ln(k+1)\right]^{k\cdot\frac{1}{k}}} = \cdots = 0$$

series converges

2.18. C8E5c

source

$$\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2} \overset{\textit{(root test)}}{\rightarrow} \lim_{k\to\infty} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{\frac{1}{k}} = \cdots = \frac{1}{e}$$

series converges

2.19. L8E2c

source

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \overset{\textit{(ratio test)}}{\rightarrow} \rho = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \cdots = 0$$

$$R = \infty$$

SIDENOTE This sum is equal to e^x.

2.20. L8E2d

source

$$\sum_{k=0}^{\infty} \frac{k!}{2^k} x^k \overset{\textit{(ratio test)}}{\rightarrow} \lim_{k\to\infty} \left(\frac{(k+1)!}{2^k+1} \cdot \frac{2^k}{k!} \right) = \cdots = \infty$$

$$R = 0$$

Series converges only if $x = 0$.

2.21. L8E2e

source

$$\sum_{k=0}^{\infty} \frac{x^k}{k+1}, \quad c_k = \frac{1}{k+1} \overset{\text{(ratio test)}}{\rightarrow} \rho = \lim_{k\to\infty}\left(\frac{k}{k+1}\right) = 1$$

$$I = [-1, 1)$$
$$R = 1$$

Series converges only if $x = 0$.

2.22. C9E1

source

$$\sum_{k=2}^{\infty} \frac{1}{k(k+2)} = \cdots = \frac{1}{2}\left[\sum_{k=2}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \sum_{k=2}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\right] =^{\text{(telescoping)}} \cdots = \frac{5}{12}$$

2.23. C9E2

source

$$A = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \overset{\left(a=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots>0\right)}{\rightarrow} A = -\frac{a}{\sqrt{2}-1} < 0$$

A diverges

2.24. C9E3a

source

$$\sum_{k=1}^{\infty} \frac{4^k}{k^2} x^k \overset{\text{(ratio test)}}{\rightarrow} \rho = \cdots = |x| \cdot 4$$

$\left[-\frac{1}{4}, \frac{1}{4}\right]$ interval of convergence

2.25. C9E3b

source

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}} \overset{\text{(ratio test)}}{\rightarrow} \rho = \cdots = |x|$$

$(-1, 1]$ interval of convergence

2.26. C9E4

source

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k+1}}{(2k+1)!} \overset{\text{(ratio test)}}{\rightarrow} \rho = \cdots = 0$$

$(-\infty, \infty)$ interval of convergence

2.27. C9E5

source

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+1)^k}{k} \overset{\text{(ratio test)}}{\rightarrow} \rho = \cdots = |x+1|$$

$(-2, 0]$ interval of convergence

2.28. C9E6

source

$$\sum_{k=1}^{\infty} (-1)^k \frac{k!}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)} x^{2k+1} \overset{\text{(ratio test)}}{\rightarrow} \rho = \cdots = \frac{x^2}{2}$$

$\left(-\sqrt{2}, \sqrt{2}\right)$ interval of convergence

2.29. C9E6

source

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

2.30. C9E7

source

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

2.31. C9E9b

source

$$\lim_{x \to 0} \frac{(e^{2x} - 1)\ln(1 + x^3)}{(1 - \cos 3x)^2} \overset{\text{(Maclaurin)}}{=} \cdots = \frac{8}{81}$$

3. Known series of some functions

3.1. L8

source

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\ln(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Part C - Differential Equations

1. Theorems

1.1. L12T1 (Peano, 1896)

source

If f is continuous in $[a, b] \times [c, d]$, then there exists y with continuous derivative which satisfy Initial Value Problem (Cauchy Problem).

1.2. L12T2 (Picard-Lindelöf, 1890, 1894)

source

If f and $\dfrac{\partial f}{\partial y}$ are continuous in $[a, b] \times [c, d]$, then there exists a unique solution to Cauchy Problem.

1.3. L12T3 — Linear superposition principle

source

If y_1 and y_2 are solutions of the homogeneous linear differential equation of degree n, then

$$y = C_1 y_1 + C_2 y_2$$

are also solutions for any $C_1, C_2 \in \mathbb{R}$.

1.4. L13T1

source

If f is continuous in rectangular parallelepiped $\mathrm{P} \subset \mathbb{R}^3$ containing x_0, y_0, y_0' , then equation $y'' = f(x, y, y')$ **has solution** $y = y(x)$ such that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'$$

Moreover, if $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y'}$ are continuous in P, then solution is **unique**.

1.5. L13T2 (fundamental)

source

If y_1, y_2 are fundamental solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2h}$$

then

$$y = C_1 y_1 + C_2 y_2 \quad , \quad \forall C_1, C_2 \in R$$

are **all solutions** of **(2h)**.

1.6. L13T3

source

If we know one solution y_1 of **(2h)**, then by variation of constant we can find the second one y_2 such that y_1, y_2 are fundamental (and so all solutions).
See proof and example here.

1.7. L13T4 (nonhomogeneous case)

source

$$y'' + p(x)y' + q(x)y = r(x) \qquad \textbf{(2)}$$

All solutions of **(2)** are

$$y(x) = y_H(x) + y_p(x)$$

...where y_H are **all solutions** of homogeneous equation and y_p is a **particular solution** of non-homogeneous equation.

<u>How to find particular solution?</u>

Method #1: GUESS

source

Equation	Initial GUESS for y_p
$r(x) = ke^{ax}$	$y_p(x) = \underline{A}e^{ax}$
$r(x) = a_0 + a_1x + \cdots + a_nx^n$	$y_p(x) = \underline{A_0} + \underline{A_1}x + \cdots + \underline{A_n}x^n$
$r(x) = a_1\cos bx + a_2\sin bx$	$y_p(x) = \underline{A_1}\cos bx + \underline{A_2}\sin bx$ *

(*) REMARK

Even if $a_1 = 0$ or $a_2 = 0$, you should still write both sin and cos in particular solution:

$r(x) = 2\cos 3x \quad \rightarrow \quad y_p(x) = A_1\cos 3x + A_2\sin 3x$

$r(x) = 3\sin 4x \quad \rightarrow \quad y_p(x) = A_1\cos 4x + A_2\sin 4x$

2. Linear Differential Equations of order 1

2.1. L12 — Method of variation of constant

source

We want to find all solutions of

$$y' + p(x)y = r(x)$$

First - homogeneous equation

$$y' + p(x)y = 0 \tag{1}$$

$y=0$ or $y \neq 0$, assume $y \neq 0$:

$$y' = -p(x)y$$

$$\frac{y'}{y} = -p(x)$$

$$\left(\ln|y|\right)' = -p(x) \quad \to \quad \ln|y| = -\int p(x)\,dx + C_1$$

Let

$$P(x) = \int p(x)\,dx$$

Then

$$|y| = e^{-P(x)+C_1} \quad \to \quad \mathbf{y = Ce^{-P(x)}}$$

Second - variation of constant

$$y = C(x)e^{-P(x)}$$

$$y' = \cdots = C'(x)e^{-P(x)} - p(x)C(x)e^{-P(x)}$$

$$y' + p(x)y = \cdots = C'(x)e^{-P(x)}$$

$$C(x) = \int r(x)e^{P(x)}dx + A$$

solution of **(1)** is

$$y = \left[\int r(x)e^{P(x)}dx + A\right]e^{-P(x)}$$

2.2. L12 — Method of integrating factor

source

$$y' + p(x)y = r(x) \tag{1}$$

Multiply **(1)** by $e^{P(x)}$, where $P(x)$ is a primitive function of $p(x)$.

$$\mathbf{y' \cdot e^{P(x)} + p(x)y \cdot e^{P(x)}} = r(x) \cdot e^{P(x)}$$

$$\frac{d}{dx}\left(ye^{P(x)}\right) = \mathbf{y' \cdot e^{P(x)} + ye^{P(x)} \cdot p(x)} \qquad \text{nice trick!}$$

$$\frac{d}{dx}\left(ye^{P(x)}\right) = r(x)e^{P(x)}$$

solution of **(1)** is

$$y = e^{-P(x)}\left[\int r(x)e^{P(x)}dx + A\right]$$

2.3. L12E6a

source

$$y' = -2y \xrightarrow{(assume\ y \neq 0)} \frac{\mathbf{y'}}{\mathbf{y}} = -2 \rightarrow \left(\ln|y|\right)' = -2 \rightarrow \ln|y| = -2x + A$$

$$\uparrow$$

$$\left[\left(\ln|y|\right)' = \cdots = \frac{\mathbf{y'}}{\mathbf{y}}\right]$$

All solutions:

$$y = Ce^{-2x} \qquad \forall c \in \mathrm{R}$$

2.4. L12E6b

source

$$y' = -xy \xrightarrow{(assume\ y \neq 0)} \frac{\mathbf{y'}}{\mathbf{y}} = -x \rightarrow \left(\ln|y|\right)' = -x \rightarrow \ln|y| = -\frac{x^2}{2} + A$$

$$\uparrow$$

$$\left[\left(\ln|y|\right)' = \cdots = \frac{\mathbf{y'}}{\mathbf{y}}\right]$$

All solutions:

$$y = Ce^{\frac{-x^2}{2}} \qquad \forall c \in \mathrm{R}$$

3. Linear Differential Equations of order 2

3.1. Particular cases

source

I. $\mathbf{y'' = f(x)}$ – Just integrate the right side twice.
II. $\mathbf{y'' = f(x, y')}$ – Let $z = y'$. Then you have d.e. of the first order.
III. Linear differential equation of the 2nd order – see C.3.2.
IV. Linear differential equation of the 2nd order with constant coefficients (special case of **III**, [source]) – see C.3.4.

3.2. Linear differential equation of the 2nd order

source

$$y'' + p(x)y' + q(x)y = r(x) \tag{2}$$

If $r(x) = 0$, then **(2)** is said to be **homogeneous**.
Otherwise, it is said to be **nonhomogeneous**.

Homogeneous case

$$y'' + p(x)y' + q(x)y = 0 \tag{2h}$$

REMARKS

1. $y = 0$ is a solution
2. If y_1 and y_2 are solutions, then

$$y = C_1 y_1 + C_2 y_2 \quad , \quad \forall C_1, C_2 \in R$$

are also solutions of homogeneous equation.

Two solutions y_1, y_2 of **(2h)** are **fundamental** ($\Leftrightarrow$ linearly independent in D) if the **Wroński determinant** (**Wronskian**)

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0 \qquad \textit{at some } x \in D$$

Nonhomogeneous case

See Theorem 4 (C.1.7)

3.3. Proof of THM3 — how to find y_2 based on y_1 (homogeneous case)

source

Let

$$y_2(x) = C(x) y_1(x)$$

Then, calculate the 1st and 2nd derivative of y_2:

$$y_2''(x) = (C'y_1 + Cy_1')' = C''y_1 + C'y_1' + Cy_1''$$

Then, plug those into the formula and simplify:

$$y_2''(x) + p(x)y_2' + q(x)y_2 =$$

$$= C''y_1 + 2C'y_1' + Cy_1'' + p(x)\left[C'y_1 + Cy_1'\right] + q(x)Cy_1 =$$

$$= C''y_1 + 2C'y_1' + p(x)C'y_1 + C\left[y_1'' + p(x)y_1' + q(x)y_1\right]$$

(the part in square brackets is equal to 0, since y_1 is a solution of **(2h)**)
Next, write the equation for y_2 and solve for C:

$$C''y_1 + 2C'y_1' + p(x)C'y_1 = 0$$

$$C''y_1 = -2C'y_1' - C'p(x)y_1$$

$$\frac{C''}{C'} = -\frac{2y_1'}{y_1} - p(x)$$

$$\left(\ln|C'(x)|\right)' = -2\left(\ln|y_1(x)|\right)' - p(x)$$

$$\ln|C'(x)| = -2\ln|y_1(x)| - \int p(x)\,dx$$

From this we can get $C(x)$.

3.4. Linear differential equations of the 2nd order with constant coefficients

source

$$y'' + py' + qy = r(x) \quad , \quad p, q \in R \tag{3}$$

First homogeneous equation

$$y'' + py' + qy = 0 \quad , \quad p, q \in R \tag{3H}$$

Particular solution of **(3H)** has form $y = e^{\lambda x}$ for some $\lambda \in C$ (complex number).

How to find λ?

First, calculate the 1st and 2nd derivative of y:

$$y = e^{\lambda x} \quad , \quad y' = \lambda e^{\lambda x} \quad , \quad y'' = \lambda^2 e^{\lambda x}$$

Then, plug those into the **(3H)** equation:

$$y'' + py' + qy = \lambda^2 e^{\lambda x} + pxe^{\lambda x} + qe^{\lambda x} =$$
$$= e^{\lambda x}\left[\lambda^2 + p\lambda + q\right] = 0$$

Since $e^{\lambda x}$ is never 0, we can drop it, which leaves only the part in the brackets:

$$\lambda^2 + p\lambda + q = 0 \tag{4}$$

This form is called the **characteristic equation for (3H)**.
Now there are 3 cases:

1° Solutions are real numbers and different.
2° Solutions are real numbers and equal.
3° Solutions are complex numbers.

Ad.1°

$$y_1(x) = e^{\lambda_1 x} \quad , \quad y_2(x) = e^{\lambda_2 x}$$

All solutions:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Ad.2°

$$y_1(x) = e^{\lambda x} \quad , \quad y_2(x) = x e^{\lambda x}$$

All solutions:

$$y_H = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

Ad.3°

$$\alpha, \beta \in R \ , \quad \beta \neq 0$$
$$\lambda_1 = \alpha + \beta i \ , \quad \lambda_2 = \alpha - \beta i$$

All solutions:

$$y_H = e^{\alpha x}\left[C_1 \cos\beta x + C_2 \sin\beta x\right]$$

4. Linear Differential Equations of order n

4.1. The general case

source

$$g_n(x)y^{(n)} + g_{n-1}(x)y^{(n-1)} + \ \cdots \ + g_2(x)y'' + g_1(x)y' + g_0(x)y = h(x) \qquad \textbf{(1)}$$

If $h(x) = 0$, then equation **(1)** is called **homogeneous**.

Note that each coefficient of $y^{(k)}$ is independent of y and only ever dependent on x. The same must apply to $h(x)$.

Examples [source]:

1. **linear:**
   - $y'' + 3xy' - (x-2)y = e^x$
2. **nonlinear:**
   - $y \cdot y'' + y' = 2x$
   - $y'' = 2x\cos(3y)$
   - $(y')^2 = y$