Solutions to

Exam: Calculus II, 22 June 2022

done by Sonia (I just decided to learn LaTeX this way, okay??)

Ex. 1. Let $f(x) = x^3$ on $[0,\ a]$, $a > 0$.

(a) Calculate lower and upper Riemann sums for corresponding to the partition of $[0,\ a]$ into n equal subintervals.

(b) What is the difference between upper and lower Riemann sums?

Hint. Use the formula

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Solution:

$$U(f,\ P) = \sum_{k=1}^{n} f(u_k)\Delta x_k.$$

$$L(f,\ P) = \sum_{k=1}^{n} f(l_k)\Delta x_k.$$

$$P_n : 0 < \frac{a}{n} < ... < \frac{na}{n} = a$$

$$x_k = \frac{ka}{n} \qquad x_{k-1} = \frac{(k-1)a}{n}$$

$$\Delta x_k = \frac{ka}{n} - \frac{(k-1)a}{n} = \frac{a}{n}$$

(a) Riemann sums:

$$U(f,\ P) = \sum_{k=1}^{n} [\frac{ka}{n}]^3\ \frac{a}{n}\ = \sum_{k=1}^{n} \frac{k^3a^3}{n^3}\ \frac{a}{n}\ = \sum_{k=1}^{n} k^3\ \frac{a^4}{n^4}\ = \frac{a^4}{n^4}\ \sum_{k=1}^{n} k^3\ = \frac{a^4}{n^4}\ \frac{n^2(n+1)^2}{4}$$

$$= \frac{a^4}{4}\ \frac{(n+1)^2}{n^2}.$$

$$L(f,\ P) = \sum_{k=1}^{n}[\frac{(k-1)a}{n}]^3\ \frac{a}{n} = \sum_{k=1}^{n}\frac{(k-1)^3a^3}{n^3}\ \frac{a}{n} = \sum_{k=1}^{n}(k-1)^3\ \frac{a^4}{n^4} = \frac{a^4}{n^4}\ \sum_{k=1}^{n}(k-1)^3$$

$$= \frac{a^4}{n^4}\ \frac{(n-1)^2((n-1)+1)^2}{4} = \frac{a^4}{n^4}\ \frac{(n-1)^2n^2}{4} = \frac{a^4}{4}\ \frac{(n-1)^2}{n^2}.$$

(b) Difference:

$$U(f,\ P) - L(f,\ P) = \frac{a^4}{4}\ \frac{(n+1)^2}{n^2} - \frac{a^4}{4}\ \frac{(n-1)^2}{n^2} = \frac{a^4}{4}\ \frac{(n+1)^2-(n-1)^2}{n^2}$$

$$= \frac{a^4}{4}\ \frac{n^2+2n+1-n^2+2n-1}{n^2} = \frac{a^4}{4}\ \frac{4n}{n^2} = \frac{a^4}{4}\ \frac{4}{n} = \frac{a^4}{n}.$$

P.S.: Mr Maligranda mentioned that he wanted us to find this difference (b) in a different way, but it is <u>not</u> specified in the task itself. But we can try to show limits:

$$\lim_{n\to\infty} U(f,P) = \lim_{n\to\infty}\frac{a^4}{4}\ \frac{(n+1)^2}{n^2} = \frac{a^4}{4}\ \lim_{n\to\infty}\frac{n^2+2n+1}{n^2}$$

$$= \frac{a^4}{4}\ \lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1} = \frac{a^4}{4}.$$

$$\lim_{n\to\infty} L(f,P) = \lim_{n\to\infty}\frac{a^4}{4}\ \frac{(n-1)^2}{n^2} = \frac{a^4}{4}\ \lim_{n\to\infty}\frac{n^2-2n+1}{n^2}$$

$$= \frac{a^4}{4}\ \lim_{n\to\infty}\frac{1-\frac{2}{n}+\frac{1}{n^2}}{1} = \frac{a^4}{4}.$$

Ex. 2.

(a) Find the function $F(x) = \int_0^x f(t)dt$ for $x \in [0,3]$ if

$$f(x) = \begin{cases} x-1 & \text{if } 0 \le x \le 1, \\ -2x+4 & \text{if } 1 < x \le 2, \\ 1 & \text{if } 2 < x \le 3. \end{cases}$$

(b) Draw the graphs of f and F.
(c) Find left-derivatives and right-derivatives at $x = 1$ and at $x = 2$, i.e.,

$$F'(1^-) = \lim_{h\to 0^-} \frac{F(1+h) - F(1)}{h}, \ F'(2^-) = \lim_{h\to 0^-} \frac{F(2+h) - F(2)}{h},$$

and

$$F'(1^+) = \lim_{h\to 0^+} \frac{F(1+h) - F(1)}{h}, \ F'(2^+) = \lim_{h\to 0^+} \frac{F(2+h) - F(2)}{h}.$$

Is F differentiable at 1 and/or at 2?

Solution:

(a)

$0 \le x \le 1$:

$$F(x) = \int_0^x (t-1)\ dt \ = \ (\frac{1}{2}t^2 - t)\Big|_0^x \ = \ \frac{1}{2}x^2 - x.$$

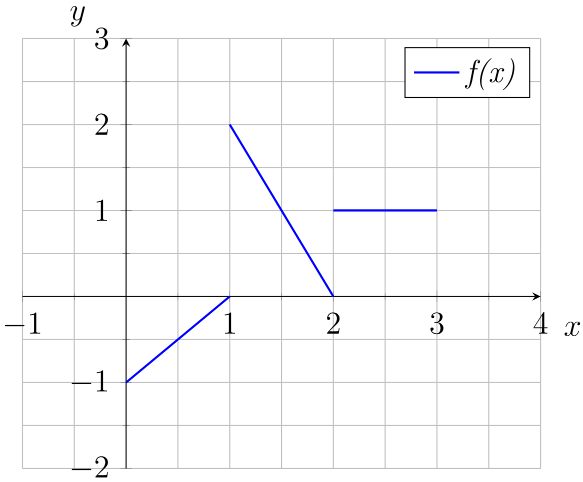
$1 < x \le 2$:

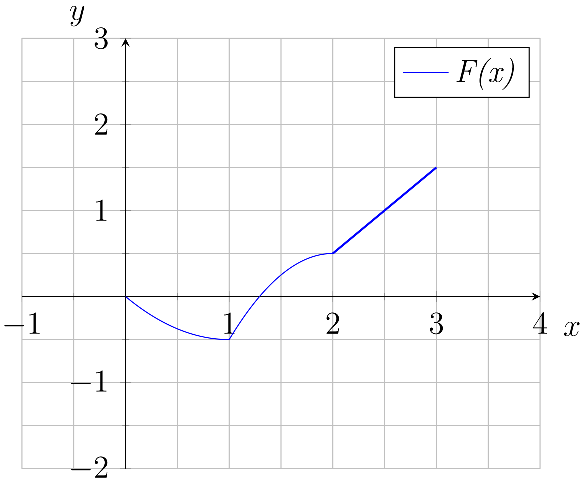
$$F(x) = \int_0^1 (t-1)\ dt + \int_1^x (-2t+4)\ dt \ = \ (\frac{1}{2}t^2 - t)\Big|_0^1 + (-t^2+4t)\Big|_1^x \ = \ -x^2 + 4x - \frac{7}{2}.$$

$2 < x \le 3$:

$$F(x) = \int_0^1 (t-1)\ dt + \int_1^2 (-2t+4)\ dt + \int_2^x 1\ dt \ = \ (\frac{1}{2}t^2 - t)\Big|_0^1 + (-t^2+4t)\Big|_1^2 + t\Big|_2^x$$
$$= \ x - \frac{3}{2}.$$

(b)





(c)

$$F'(1^-) = \lim_{h\to 0^-} \frac{F(1+h)-F(1)}{h} = \lim_{h\to 0^-} \frac{\frac{1}{2}(1+h)^2-(1+h)+\frac{1}{2}}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{1}{2}+h+\frac{1}{2}h^2-1-h+\frac{1}{2}}{h} = \lim_{h\to 0^-} \frac{\frac{1}{2}h^2}{h} = \lim_{h\to 0^-} \frac{1}{2}h = 0.$$

$$F'(1^+) = \lim_{h\to 0^+} \frac{F(1+h)-F(1)}{h} = \lim_{h\to 0^+} \frac{-(1+h)^2+4(1+h)-\frac{7}{2}+\frac{1}{2}}{h}$$

$$= \lim_{h\to 0^+} \frac{-1-2h-h^2+4+4h-3}{h} = \lim_{h\to 0^+} \frac{-h^2+2h}{h} = \lim_{h\to 0^+} (-h+2) = 2.$$

Since $0 = F'(1^-) \neq F'(1^+) = 2$, F is <u>not</u> differentiable at 1.

$$F'(2^-) = \lim_{h\to 0^-} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0^-} \frac{-(2+h)^2+4(2+h)-\frac{7}{2}-\frac{1}{2}}{}$$

$$= \lim_{h\to 0^-} \frac{-4-4h-h^2+8+4h-4}{h} = \lim_{h\to 0^-} (-h) = 0.$$

$$F'(2^+) = \lim_{h\to 0^+} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0^+} \frac{2+h-\frac{3}{2}-\frac{1}{2}}{h}$$

$$= \lim_{h\to 0^+} 1 = 1.$$

Since $0 = F'(2^-) \neq F'(2^+) = 1$, F is <u>not</u> differentiable at 2.

Ex. 3. (a) Find the partial sums

$$G_n = \sum_{k=3}^{n} (\frac{2}{3})^k, \quad T_n = \sum_{k=3}^{n} \frac{1}{k(k+2)} \quad \text{for} \quad n \geq 3.$$

(b) Find sum of the series $\displaystyle\sum_{k=3}^{\infty} \left[(\frac{2}{3})^k + \frac{1}{k(k+2)} \right]$.

Solution:

Preamble

The formula used for Geometric n-series:

$$\sum_{i=m}^{n} r^i = \frac{r^m - r^{n+1}}{1-r}.$$

(a)

G_n :

$$S_n = \frac{\left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}} = \frac{\frac{8}{27} - \frac{2^{n+1}}{3^{n+1}}}{\frac{1}{3}} = \frac{8}{9} - \frac{2^{n+1}}{3^n}.$$

T_n :

$$S_n = \sum_{k=3}^{n} \frac{1}{k(k+2)} = \frac{1}{2} \sum_{k=3}^{n} \frac{1}{k} - \frac{1}{k+2} = \frac{1}{2} [\frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + ... + \frac{1}{n-4} - \frac{1}{n-2}$$

$$+\frac{1}{n-3} - \frac{1}{n-1} + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}] = |cancelling\ out|$$

$$= \frac{1}{2} [\frac{1}{3} + \frac{1}{4} - \frac{1}{n+1} - \frac{1}{n+2}] = \frac{7}{24} + \frac{-2n-3}{2(n+2)(n+1)} = \frac{7(n+2)(n+1) - 12(2n+3)}{24(n+2)(n+1)}$$

$$= \frac{7n^2 + 21n + 14 - 24n - 36}{24(n+2)(n+1)} = \frac{7n^2 - 3n - 22}{24(n+2)(n+1)}.$$

(b)

$$\lim_{n \to \infty} \left[\frac{8}{9} - \frac{2^{n+1}}{3^n} \right] = \frac{8}{9}.$$

$$\lim_{n \to \infty} \left[\frac{7n^2 - 3n - 22}{24(n+2)(n+1)} \right] = \lim_{n \to \infty} \left[\frac{7n^2 - 3n - 22}{24(n^2 + 3n + 2)} \right] = \frac{1}{24} \lim_{n \to \infty} \frac{7 - \frac{3}{n} - \frac{22}{n^2}}{1 + \frac{3}{n} + \frac{2}{n^2}} = \frac{7}{24}.$$

Since both of the series converge:

$$\sum_{k=3}^{\infty} \left[(\frac{2}{3})^k + \frac{1}{k(k+2)} \right] = \sum_{k=3}^{\infty} (\frac{2}{3})^k + \sum_{k=3}^{\infty} \frac{1}{k(k+2)} = \frac{8}{9} + \frac{7}{24} = \frac{85}{72}.$$

Ex. 4. Find global maximum and global minimum of the function

$$f(x, y) = x^2y^2 + x^3y - 4x^2y$$

over triangle with vertices (0, 0), (0, 6), (6, 0).

Solution:

$$\begin{aligned} f_x &= xy(3x + 2y - 8) \\ f_y &= x^2(x + 2y - 4) \end{aligned}$$

$$\begin{cases} xy(3x + 2y - 8) = 0; \\ x^2(x + 2y - 4) = 0; \end{cases}$$

$$\begin{cases} 3(4 - 2y) + 2y - 8 = 0; \\ x = 4 - 2y; \end{cases}$$

$$\begin{cases} 12 - 4y - 8 = 0; \\ x = 4 - 2y; \end{cases}$$

$$\begin{cases} y = 1; \\ x = 2. \end{cases}$$

Which gives us *P(2, 1)*.

(0, 0) and (0, 6):

$f(0, y) = 0.$

(0, 0) and (6, 0):

$f(x, 0) = 0.$

(0, 6) and (6, 0):

$$y = -x + 6$$

$$\begin{aligned} f(x, -x+6) &= x^2(-x+6)^2 + x^3(-x+6) - 4x^2(-x+6) \\ &= (6-x)(6x^2 - x^3 + x^3 - 4x^2) = 2x^2(6-x). \end{aligned}$$

$$\begin{gathered} f' = (12x^2 - 2x^3)' = 24x - 6x^2. \\ 24x - 6x^2 = 0 \\ 6x(4-x) = 0 \\ x = 0 \ \text{ or } \ x = 4 \end{gathered}$$

Therefore:

$$\begin{cases} y = -x + 6; \\ x = 4; \end{cases}$$

$$\begin{cases} y = 2; \\ x = 4; \end{cases}$$

Which gives us $\underline{P(4,\ 2)}$.

Therefore, global extrema are:

$$\begin{gathered} f(2, 1) = -4. \\ f(4, 2) = 64. \end{gathered}$$

Ex. 5. Solve the differential equation

$$y'' - 2y' = 6\,e^{2x}.$$

(b) Solve the initial value problem

$$\begin{cases} y'' - 2y' = 6\,e^{2x}, \\ y(0) = 1, \\ y'(0) = 2. \end{cases}$$

Solution:

All solutions of differential equation: $$y = y_H + y_P.$$

(a)

$$y'' - 2y' = 6\,e^{2x}.$$

1) <u>homo</u>geneous:

$$\begin{gathered} y'' - 2y' = 0 \\ \lambda^2 - 2\lambda = 0 \\ \lambda(\lambda - 2) = 0 \\ \lambda_1 = 0 \ \text{ or } \ \lambda_2 = 2 \end{gathered}$$

Since $\lambda_1 \neq \lambda_2$ real:

$$\underline{y_H = C_1 e^{0x} + C_2 e^{2x} = C_1 + C_2 e^{2x}}.$$

2) particular:

$$y_P = Axe^{2x}.$$
$$y'_P = Ae^{2x} + 2Axe^{2x} = Ae^{2x}(1 + 2x).$$
$$y''_P = 2Ae^{2x}(1 + 2x) + 2Ae^{2x} = 2Ae^{2x}(2x + 2).$$

$$2Ae^{2x}(2x + 2) - 2Ae^{2x}(1 + 2x) = 6e^{2x}$$
$$2Ae^{2x}(2x + 2 - 1 - 2x) = 6e^{2x}$$
$$2A = 6$$
$$A = 3$$

Therefore: $\underline{y_P = 3xe^{2x}}$.

Then, all solutions of the differential equation:
$\underline{y = C_1 + C_2e^{2x} + 3xe^{2x}}$.

(b)

$$y = C_1 + C_2 e^{2x} \quad \textbf{from homogeneous part of point (a)}$$

From the initially given conditions:

$$\begin{cases} y(0) = 1, \\ y'(0) = 2. \end{cases}$$

$$y(0) = C_1 + C_2 e^0 = C_1 + C_2. \quad \text{and} \quad y(0) = 1$$

Which gives us: $\underline{C_1 + C_2 = 1}$.

$$y' = 2C_2 e^{2x}$$
$$y'(0) = 2C_2 \quad \text{and} \quad y'(0) = 2$$

Which gives us: $\underline{C_2 = 1}$.

Therefore:

$$\begin{cases} C_1 + C_2 = 1; \\ C_2 = 1; \end{cases}$$
$$\begin{cases} C_1 = 0; \\ C_2 = 1; \end{cases}$$

Therefore, the solution to IVP:

$$\underline{y = e^{2x}}.$$

P.S.: I am <u>not</u> sure in the solution to IVP, but hopefully it is correct.

Good luck with retake preparation and may the odds be even in your favor!