Lech M.

Lecture 10

A) Partial derivatives and tangent plane

Let $z = f(x,y)$ be a function of two variables.

(*) If y is held constant, $f(x,y)$ becomes a function of x alone, and its derivative (if it exists) is called the <u>partial derivative of $f(x,y)$ with respect to x</u>.

(**) Similarly, if x is held constant, then we have <u>partial derivative of $f(x,y)$ with respect to y</u>.

These derivatives, variously denoted by

$f_x(x,y)$, $\frac{\partial f}{\partial x}$, z_x, $\frac{\partial z}{\partial x}$ and

$f_y(x,y)$, $\frac{\partial f}{\partial y}$, z_y, $\frac{\partial z}{\partial y}$

~~$f_1(x,y)$~~
~~$D_1 f(x,y)$~~
No!

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Ex. 1 Find

$f_x(0,1)$, $f_y(0,1)$

if

$$f(x,y) = 2x^3y^2 + 3xy + 4x + 5y.$$

$$f_x(x,y) = 6x^2y^2 + 3y + 4$$

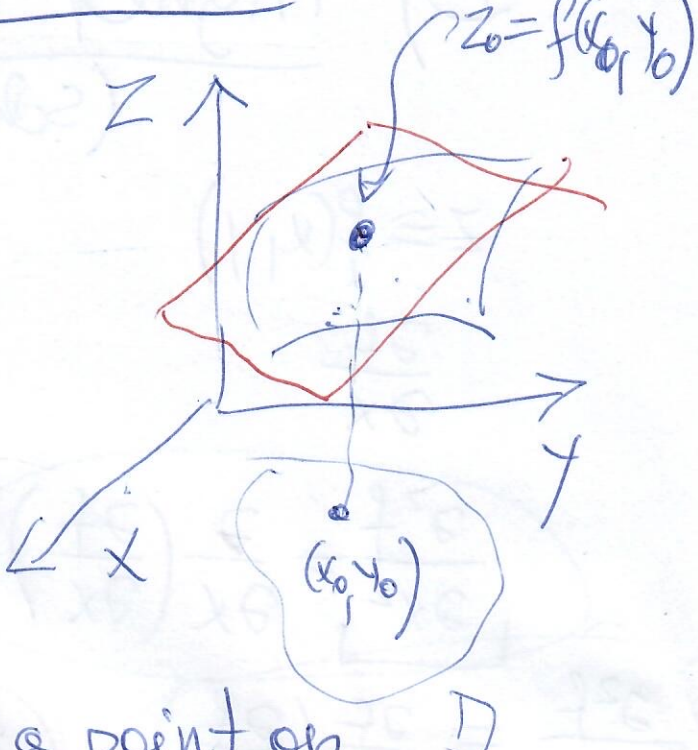
$$f_x(0,1) = 3 + 4 = 7$$

$$f_y(x,y) = 4x^3y + 3x + 5$$

$$f_y(0,1) = 5$$

Tangent plane

$z = f(x,y)$



Let $P_0(x_0, y_0, z_0)$ be a point on the surface $z = f(x,y)$.

Thm 1 If $f(x,y)$ is differentiable at (x_0, y_0), then the surface has a tangent plane at P_0, and this plane has the equation

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0) - (z-z_0) = 0}$$

Remark
Normal vector

$\vec{n} = [f_x(x_0,y_0), f_y(x_0,y_0), -1]$ is a normal vector to the surface $z = f(x,y)$ at $P_0(x_0, y_0, z_0)$.

<u>Proof</u>: Tangent plane intersects the vertical plane $y=y_0$ in a stright line that is tangent at P_0 to the curve of intersection of the surface $z=f(x,y)$ and the plane $y=y_0$.

Thus line has slope $f_x(x_0,y_0)$ and is parallel to the vector

$$\vec{v_1}=\vec{i}+f_x(x_0,y_0)\vec{k}.$$

Similarly, the tangent plane intersects the vertical plane $x=x_0$ in a stright line having slope $f_y(x_0,y_0)$.

This line is parallel to the vector

$$\vec{v_2}=\vec{j}+f_y(x_0,y_0)\vec{k}.$$

The surface $z=f(x,y)$ has normal vector

$$\vec{n}=\vec{v_2}\times\vec{v_1}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & f_y(x_0,y_0)\\ 1 & 0 & f_x(x_0,y_0)\end{vmatrix}$$

$$=f_x(x_0,y_0)\vec{i}+f_y(x_0,y_0)\vec{j}-\vec{k}$$

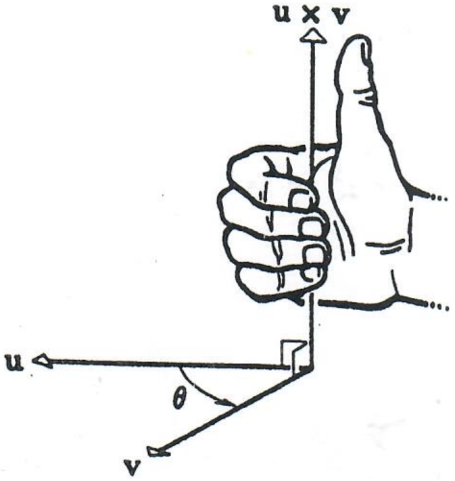
Equation of tangent plane goingt threag $P_0=(x_0,y_0,z_0)$

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

Vector product $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

1) $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

2) the direction of $\vec{u} \times \vec{v}$ is determined by the "right-hand rule"



fingers of the right hand are cupped in the direction of rotation

$\Downarrow$

the <u>thumb</u> indicates the direction of $\vec{u} \times \vec{v}$

3) $\|\vec{u} \times \vec{v}\| = \underbrace{\|\vec{u}\| \|\vec{v}\| \sin\Theta}$

= area of the parallelogram determined by $\vec{u}$ and $\vec{v}$

<u>EX. 2</u> Find the equation of the tangent plane to the surface

$$z = x^2 y$$

at $P_0 = (2,1,4)$.

1° Point is on the surface since

$$2^2 \cdot 1 = 4$$

$f_x = 2xy$, $f_y = x^2$

$f_x(2,1) = 4$, $f_y(2,1) = 4$

$\vec{n} = [4, 4, -1]$

Tangent plane

$$4(x-2) + 4(y-1) - (z-4) = 0$$

$$\underline{4x + 4y - z = 8}$$

B) Higher order derivatives

(Schwarz theorem)

$z = f(x, y)$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Sometimes we can write

$$f_{xx}, f_{yy}, f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

We can then have higher-order partial derivatives, for example

$$f_{yyx} = \frac{\partial^3 f}{\partial x \partial y^2} \text{ or } f_{xxyy} = \frac{\partial^4 f}{\partial y^2 \partial x^2}$$

Ex. 3 Find the second-order partial derivatives of the function

$$f(x,y)=x^2y^3+x^4y$$

$f_x=2xy^3+4x^3y$, $f_y=3x^2y^2+x^4$

$f_{xx}=2y^3+12x^2y$, $f_{yy}=6x^2y$

$f_{xy}=6xy^2+4x^3$, $f_{yx}=6xy^2+4x^3$

The same!

Ex. 4 Let $f(x,y)=y^2e^x+y$. Find f_{xy}, f_{yx} and f_{xyyx}.

$f_x=y^2e^x$, $f_y=2ye^x+1$

$f_{xy}=2ye^x$, $f_{yx}=2ye^x$

mixed second-order derivatives are the same!

$$f_{xyyx}=(2ye^x)_{yx}=(2e^x)_x=2e^x$$

EX. 5 Find f_x, f_y, f_z and f_{xy}, f_{yx}, f_{xz}, f_{zx} and f_{yz}, f_{zy} when

$$f(x,y,z) = z\ln(x^2 y\cos z)$$

$$f_x = \frac{z}{\cancel{x^2 y\cos z}} \cdot 2\cancel{xy\cos z} = \frac{2z}{x}$$

$$f_y = \frac{z}{\cancel{x^2} y \cancel{\cos z}} \cdot \cancel{x^2}\cancel{\cos z} = \frac{z}{y}$$

$$f_z = \ln(x^2 y\cos z) + \frac{z}{\cancel{x^2}\cancel{y}\cos z} \cdot \cancel{x^2}\cancel{y}(-\sin z)$$

$$= \ln(x^2 y\cos z) - z\cot z$$

$$f_{xy} = \left(\frac{2z}{x}\right)_y = 0 \qquad f_{yx} = \left(\frac{z}{y}\right)_x = 0$$

$$f_{xz} = \left(\frac{2z}{x}\right)_z = \frac{2}{x} \qquad f_{zx} = \left[\ln(x^2 y\cos z) - z\cot z\right]_x$$

$$= \frac{1}{x^2 \cancel{y\cos z}} \cdot 2x\cancel{y\cos z} = \frac{2}{x}$$

$$f_{yz} = \left(\frac{z}{y}\right)_z = \frac{1}{y} \qquad f_{zy} = \left[\ln(x^2 y\cos z) - z\cot z\right]_y$$

$$= \frac{1}{\cancel{x^2} y\cancel{\cos z}} \cdot \cancel{x^2}\cancel{\cos z} = \frac{1}{y}$$

10

Thm 2 (Schwarz theorem on equality of mixed second-order partial derivatives, 1873).

Let $f(x,y)$ be a function of two variables. If f_x, f_y, f_{xy} and f_{yx} are <u>continuous</u> on some set (open set), then

$\boxed{f_{xy} = f_{yx}}$ at each point of the set

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$$

Proof. (x_0, y_0) point in the set.

For
$$F(h,k) = f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)$$
if we denote by
$$\varphi(x) = f(x, y_0+k) - f(x, y_0), \quad \psi(y) = f(x_0+h, y) - f(x_0, y)$$
then
$$F(h,k) = \varphi(x_0+h) - \varphi(x_0) \text{ and } F(h,k) = \psi(y_0+k) - \psi(y_0).$$
By the Mean-Value Theorem
$$\varphi(x_0+h) - \varphi(x_0) \overset{f_x \text{ continous}}{=} \varphi'(x_0+\theta h)\cdot h, \quad 0<\theta<1$$
$$\psi(y_0+k) - \psi(y_0) \overset{f_y \text{ continous}}{=} \psi'(y_0+\eta k)\cdot k, \quad 0<\eta<1$$
Thus
$$F(h,k) = \begin{cases} \varphi'(x_0+\theta h)h \\ \psi'(y_0+\eta k)k \end{cases} = \begin{cases} [f_x(x_0+\theta h, y_0+k) - f_x(x_0+\theta h, y_0)]h \\ [f_y(x_0+h, y_0+\eta k) - f_y(x_0, y_0+\eta k)]k \end{cases}$$
$$\overset{MVT}{=} \quad \begin{matrix} f_{xy} \text{ continous} \\ f_{yx} \text{ continous} \end{matrix} \quad \begin{cases} f_{xy}(x_0+\theta h, y_0+\mu k)hk, & 0<\theta,\mu<1 \\ f_{yx}(x_0+\nu h, y_0+\eta k)hk, & 0<\eta,\nu<1 \end{cases}$$
Hence
$$f_{xy}(x_0+\theta h, y_0+\mu k) = f_{yx}(x_0+\nu h, y_0+\eta k)$$
When $h \to 0$, $k \to 0$ then continuity of f_{xy} and f_{yx} gives
$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

c) Chain Rule

Thm 3 (one independent variable)

$x = x(t)$, $y = y(t)$ differentiable

$z = f(x,y)$ differentiable $\Longrightarrow z(t) = f(x(t), y(t))$ differentiable of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

$$\boxed{z_t = z_x \cdot x_t + z_y \cdot y_t}$$

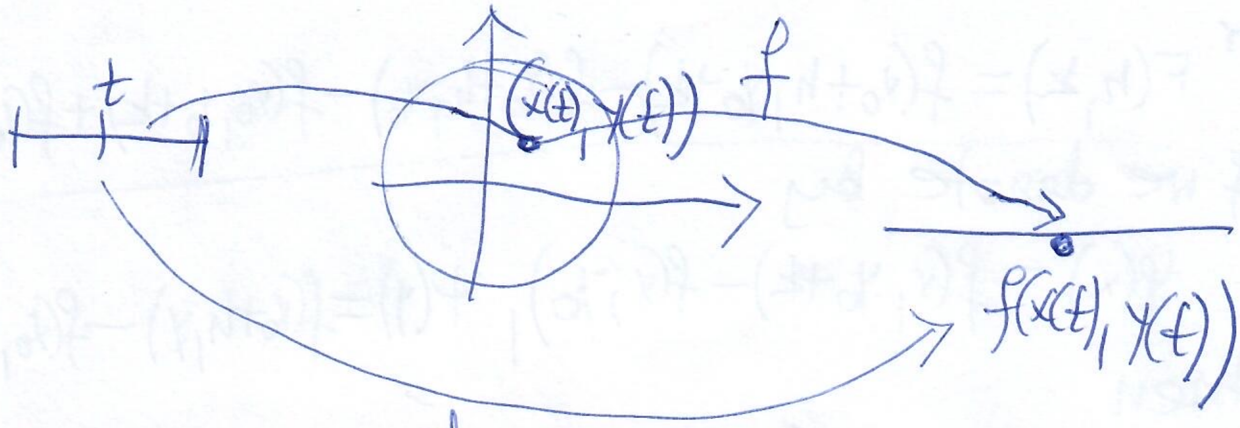
Thm 4 (two independent variables)

$x = x(u,v)$, $y = y(u,v)$ have first-order partial derivatives

$z = f(x,y)$ differentiable $\longrightarrow z(u,v) = f(x(u,v), y(u,v))$ differentiable of (u,v) and

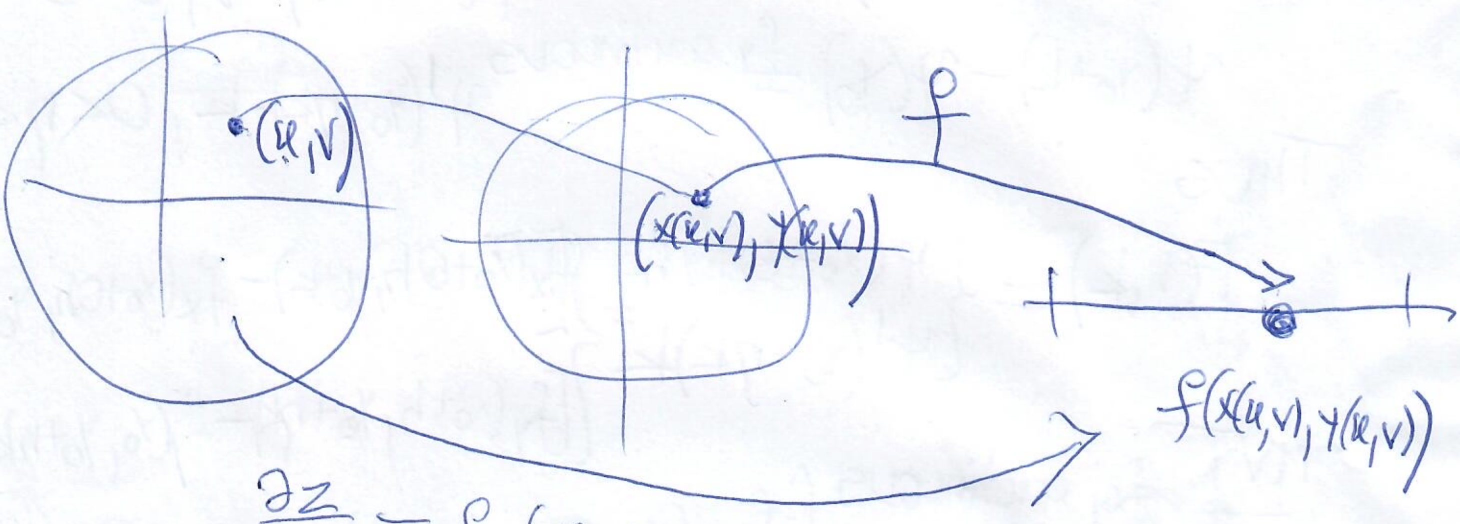
$$\boxed{\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}}$$

Thm 3



$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

Thm 4



$$\frac{\partial z}{\partial u} = f_x(x(u,v), y(u,v)) \frac{\partial x}{\partial u}(u,v) + f_y(x(u,v), y(u,v)) \frac{\partial y}{\partial u}(u,v)$$

$$\frac{\partial z}{\partial v} = f_x(\quad) \frac{\partial x}{\partial v}(\,) + f_y(\quad) \cdot \frac{\partial y}{\partial v}(\cdot)$$

Ex. 6 Let

$$z = e^{3x+2y}, \text{ where } x = \cos t,\ y = t^2$$

Find $\dfrac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{3x+2y}\cdot 3\cdot(-\sin t) + e^{3x+2y}\cdot 2\cdot 2t$$

$$= e^{3x+2y}(4t - 3\sin t)$$

$$= e^{3\cos t + 2t^2}(4t - 3\sin t)$$

Ex. 7 Let $z = \dfrac{x}{y}$, where $x = e^{uv}$, $y = \ln(uv)$

Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ using the chain rule.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{1}{y}\cdot e^{uv}\cdot v + \left(-\frac{x}{y^2}\right)\cdot\frac{1}{u\cancel{v}}\cdot\cancel{v} =$$

$$= \frac{1}{y}e^{uv}\cdot v - \frac{x}{y^2}\cdot\frac{1}{u} = \frac{v}{\ln(uv)}e^{uv} - \frac{e^{uv}}{[\ln(uv)]^2}\cdot\frac{1}{u}$$

verte!

15

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{1}{y} \cdot e^{uv} \cdot u + \left(-\frac{x}{y^2}\right) \cdot \frac{1}{uv} \cdot u$$

$$= \frac{u}{\ln(u,v)} e^{uv} - \frac{e^{uv}}{[\ln(u,v)]^2 \cdot v}$$