Lech M.

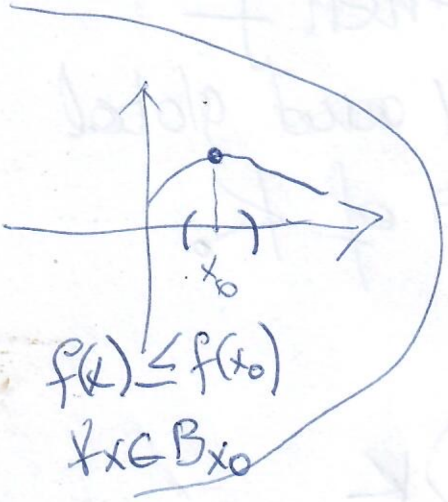
Lecture 11

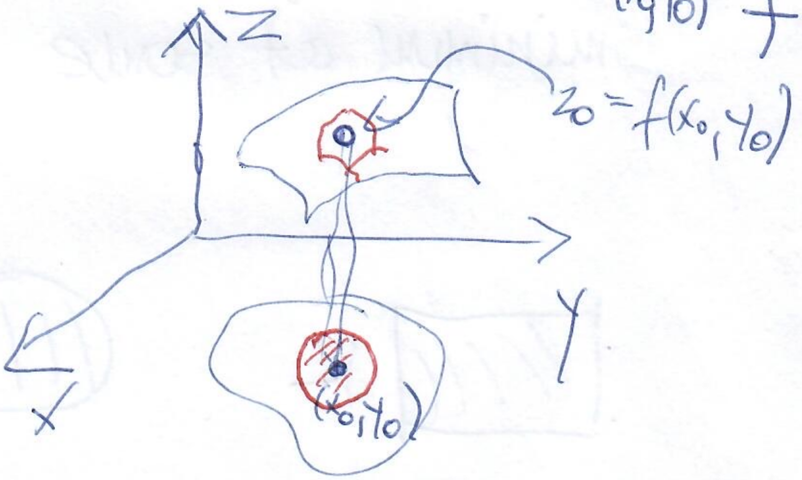
Local and global extreme values of functions of two variables

Def. $f: \mathbb{R}^2 \supset D \longrightarrow \mathbb{R}$

$f(x,y)$ has a <u>local maximum</u> at $(x_0,y_0) \in D$ if there is a circle $B_{(x_0,y_0)}$ centred at (x_0,y_0) such that

$$f(x,y) \le f(x_0,y_0) \quad \forall (x,y) \in B_{(x_0,y_0)} \cap D_f .$$





$f(x,y)$ has a <u>global maximum</u> (or <u>absolute maximum</u>) at (x_0,y_0) if

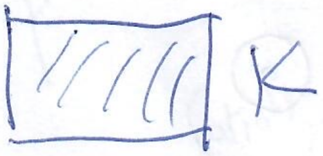
$$f(x,y) \le f(x_0,y_0) \quad \forall (x,y) \in D_f .$$

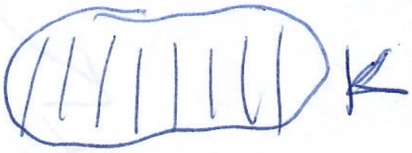
If we have reverse inequalities $\ge$ the $f(x,y)$ has <u>local minimum</u> or <u>global minimum</u> at (x_0,y_0)

$f(x,y)$ has <u>local extremum</u> at (x_0, y_0)
= local maximum or local minimum at (x_0, y_0)

$f(x,y)$ has <u>global extremum</u> at (x_0, y_0)
= global maximum or global minimum at (x_0, y_0).

<u>THM 1</u> (Weierstrass theorem):
If $f(x,y)$ is continuous on a closed and bounded set K, then f has both global maximum and global minimum at some points of K.





THM 2

1° $f(x,y)$ has local extreme at (x_0,y_0)
2° $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist

$$\left.\begin{array}{l} 1^\circ \\ 2^\circ \end{array}\right\} \Rightarrow \begin{cases} f_x(x_0,y_0)=0 \\ f_y(x_0,y_0)=0 \end{cases}$$

Proof

If f has local maximum at (x_0,y_0), then

$g(x)=f(x,y_0)$ has local maximum at x_0 $\Rightarrow g'(x_0)=0$

$h(y)=f(x_0,y)$ has local maximum at y_0 $\Rightarrow h'(y_0)=0$

Def. Critical point of $f(x,y)$ is a point (x_0,y_0) in the interior of the domain of f at which

$f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$

or at which one or both partial derivatives do not exist.

A critical point at which function does not have a local extreme is called a saddle point of the function.

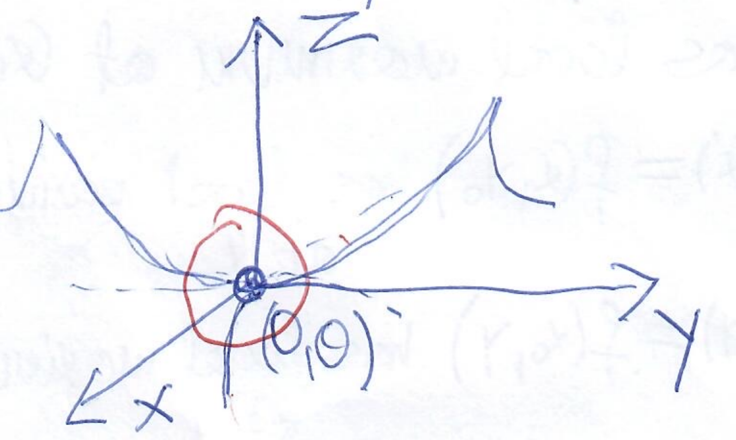
Ex. 1

$$f(x,y) = y^2 - x^2$$

$(0,0)$ is a saddle point

$f_x = -2x$, $f_x(0,0) = 0$

$f_y = 2y$, $f_y(0,0) = 0$



$$\lim_{x \to \pm\infty} f(x,y) = \lim_{x \to \pm\infty} (y^2 - x^2) = -\infty$$

$$\lim_{y \to \pm\infty} f(x,y) = \lim_{y \to \pm\infty} (y^2 - x^2) = +\infty$$

THM 3

1° $\begin{cases} f_x(x_0,y_0)=0 \\ f_y(x_0,y_0)=0 \end{cases} \Longrightarrow (x_0,y_0)$ critical point

2° $D(x_0,y_0)=\begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}$

a) Matrix D positive definite

$\Updownarrow$

$\underline{f_{xx}(x_0,y_0)>0}$, $D=\det D(x_0,y_0)>0 \Longrightarrow f$ has <u>local minimum</u> at (x_0,y_0)

b) $\underline{f_{xy}(x_0,y_0)<0}$, $D>0 \Longrightarrow f$ has local <u>maximum</u> at (x_0,y_0)

c) $D<0 \Longrightarrow$ at (x_0,y_0) is a saddle point of f

d) $D=0$ no conclusion.

Proof

1. Taylor expansion

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \frac{f_x(x_0,y_0)h + f_y(x_0,y_0)k}{1!} + \frac{f_{xx}(x_0,y_0)h^2 + 2f_{xy}(x_0,y_0)hk + f_{yy}(x_0,y_0)k^2}{2!} + \dots$$

2. (x_0,y_0) local extremum $\Longrightarrow$ $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$

Then

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \frac{f_{xx}(x_0,y_0)h^2 + 2f_{xy}(x_0,y_0)hk + f_{yy}(x_0,y_0)k^2}{2!} + \dots$$

$$\overset{t=\frac{h}{k},\, k\neq 0}{=} \frac{k^2}{2}\Big[\underbrace{\underbrace{f_{xx}}\, t^2 + 2\underbrace{f_{xy}}(x_0,y_0)\,t + \underbrace{f_{yy}}}_{At^2+2Bt+C}\Big] = Q(t)$$

Let

$$\Delta = 4\left(f_{xy}^2 - f_{xx}f_{yy}\right) = -4\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix} = -4D$$

a) If $f_{xx} > 0$ and $\Delta < 0$ ($D>0$) $\Longrightarrow$ $Q(t) > 0 \ \forall t \in \mathbb{R}$

at (x_0,y_0) is local minimum

b) If $f_{xx} < 0$ and $\Delta < 0$ ($D>0$) $\Longrightarrow$ $Q(t) < 0 \ \forall t \in \mathbb{R}$

at (x_0,y_0) is local maximum

c) If $\Delta > 0$ ($D<0$) the sign of $Q(t)$ is undefined (it has + and −)

i.e. at (x_0,y_0) is a saddle point.

Taylor series

One variable

$F \in C^{n+1}(\alpha, \beta)$ with $(\alpha, \beta) \supset [a, a+h]$

$$F(a+h) = F(a) + F'(a)h + \frac{F''(a)}{2!}h^2 + \dots + \frac{F^{(n)}(a)}{n!}h^n + \frac{F^{(n+1)}(c)}{(n+1)!}h^{n+1}$$

for some c between a and $a+h$.

Two variables

$f(x,y)$ has continuous partial derivatives up to order $n+1$ in square containing (a,b) and $(a+h, b+k)$

$\Downarrow$

Put $F(t) = f(a+th, b+tk)$, $0 \le t \le 1$

Then $F(0) = f(a,b)$, $F(1) = f(a+h, b+k)$

$$F'(t) = f_x(a+th, b+tk)h + f_y(a+th, b+tk)k$$

$$F''(t) = h^2 f_{xx}(a+th, b+tk) + 2hk f_{xy}(a+th, b+tk) + k^2 f_{yy}(a+th, b+tk)$$

$$F'''(t) = \dots$$

Therefore

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \dots$$

or

$$f(a+h, b+k) = f(a,b) + f_x(a,b)h + f_y(a,b)k + \frac{h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)}{2} + \dots$$

EX. 2 Locate all local extrema and saddle points of

$$f(x,y) = 3x^2 - 2xy + y^2 - 8y.$$

$$\begin{cases} f_x = 6x - 2y = 0 \longrightarrow y = 3x \\ f_y = -2x + 2y - 8 = 0 \end{cases}$$

$-2x + 6x - 8 = 0$

$4x = 8,\ x = 2$

$y = 6$

$(2,6)$ critical point.

$\boxed{f_{xx} = 6 > 0}$, $f_{xy} = -2$, $f_{yy} = 2$

$$D = \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} = 12 - 4 = \boxed{8 > 0}$$

local minimum

$$f_{min} = f(2,6) = \cancel{12} - 24 + \cancel{36} - \cancel{48} = -24$$

EX. 3

$$f(x,y) = 4xy - x^4 - y^4$$

$$\begin{cases} f_x = 4y - 4x^3 = 0 \longrightarrow y = x^3 \\ f_y = 4x - 4y^3 = 0 \end{cases}$$

$x = y^3 \longrightarrow x = x^9$

$x(x^8 - 1) = 0$

$\Downarrow$

$x = 0$ or $x = 1$ or $x = -1$

Critical points

$(0,0), (1,1), (-1,-1)$

$$f_{xx} = -12x^2, \quad f_{xy} = 4, \quad f_{yy} = -12y^2$$

1° $(0,0)$

$$f_{xx}(0,0) = 0, \quad D = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$$

saddle point

2° $(1,1)$

$$f_{xx}(1,1) = -12 < 0, \quad D = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$$

local maximum

3° $(-1,-1)$

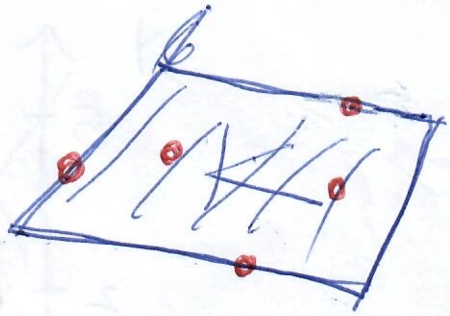
$$f_{xx}(-1,-1) = -12 < 0, \quad D = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$$

local maximum

How to find the absolute extrema of a continuous function on a closed and bounded set K

Step 1

Find the critical points of $f(x,y)$ that lie in the interior of K



Step 2

Find all boundary points at which the absolut extrema can occur.

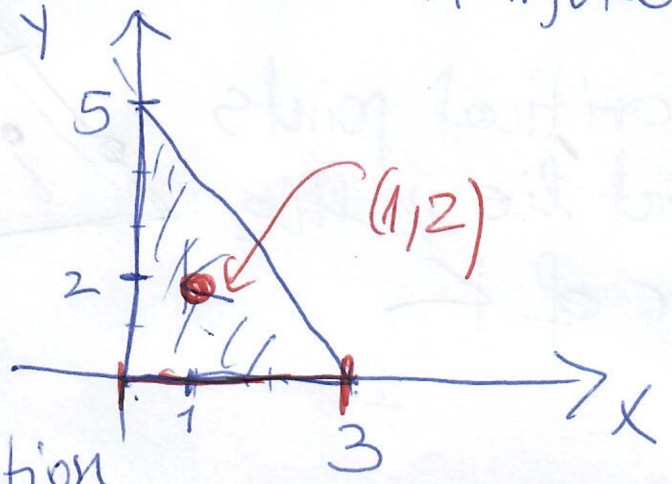
Step 3 Evaluate $f(x,y)$ at the points obtained in the previous steps.

The largest of these values is the absolute maximum and the smallest the absolute minimum.

Ex. 4 Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region K



Solution

Step 1
$$\begin{cases} f_x = 3y - 6 = 0 & \to y = 2 \\ f_y = 3x - 3 = 0 & \to x = 1 \end{cases}$$

$(1,2)$ critical point

Step 2 Boundary of K consists of three line segments

1^0 line segment $(0,0)$ and $(3,0)$, i.e. $y = 0$

$u(x) = f(x,0) = -6x + 7$ for $0 \leq x \leq 3$

$u'(x) = -6 \to u$ decreasing

$\min u(x) = u(3) = -11$

$\max u(x) = u(0) = 7$

2^0 line segment $(0,0)$ and $(0,5)$ i.e. $x=0$

$$v(y)=f(0,y)=-3y+7 \text{ for } 0\le y\le 5$$

$$v'(y)=-3 \longrightarrow v \text{ decreasing}$$

$$\min v(y)=v(5)=-8$$

$$\max v(y)=v(0)=7$$

3^0 line segment $(3,0)$ and $(0,5)$

$$y=-\frac{5}{3}x+5 \text{ for } 0\le x\le 3$$

$$w(x)=f\left(x,-\frac{5}{3}x+5\right)=\dots$$

$$=-5x^2+14x-8,\quad 0\le x\le 3$$

$$w'(x)=-10x+14=0$$

$$\Updownarrow$$

$x=\frac{7}{5}$ critical point of w

$$(0,0),\ (3,0),\ (0,5),\ \left(\frac{7}{5},\frac{8}{3}\right)\ \ (1,2)$$

$$\downarrow\quad\ \downarrow\quad\ \downarrow\quad\ \downarrow\quad\ \downarrow$$

$$7\quad -11\quad -8\quad \frac{9}{5}\quad 1$$

max min