Lech M.

Lecture 7

Infinite Series of Real Numbers

Let $\{a_k\}$ be a sequence of real numbers. An infinite series is an expression of the form

$$a_1 + a_2 + \dots = \sum_{k=1}^{\infty} a_k,$$

defined by the sequence $\{S_n\}$, where

$$S_n = \sum_{k=1}^{n} a_k.$$

The numbers a_k are called the terms of the series, and the numbers S_n are called the partial sums of the series.

Def. 1 The series $\sum_{k=1}^{\infty} a_k$ is said to converge to S if $\lim_{n\to\infty} S_n = S$.

If $\{S_n\}$ diverges, then we say that the series diverges.

$S_1 = a_1,\ S_2 = a_1 + a_2,\ S_3 = a_1 + a_2 + a_3, \dots$

$\{S_n\}$

EX. 1 $0.333\ldots$

$$= 0.3 + 0.03 + 0.003 + \ldots$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$ converges to $\frac{1}{3}$

$$-\; S_n = \frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^n}$$

$$10S_n = 3 + \frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^{n-1}}$$

$$10S_n - S_n = 3 - \frac{3}{10^n}$$

$$9S_n = 3 - \frac{3}{10^n}$$

$$S_n = \frac{3}{9} - \frac{3}{9}\cdot\frac{1}{10^n}$$

$$S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right) \xrightarrow[n\to\infty]{} \frac{1}{3}$$

EX. 2

$$\cancel{S} \; \underbrace{1-1}+\underbrace{1-1}+\underbrace{1-1}+\ldots = \sum_{k=1}^{\infty}(-1)^{k+1}$$

$S=0$?

$S=\frac{1}{2}$?

diverges, i.e. has no sum

$$S = 1 - \cancel{(1-1+1-1+\ldots)}$$

$$S = \cancel{1 - S} \Rightarrow 2S = 1 \rightarrow S = \frac{1}{2}$$

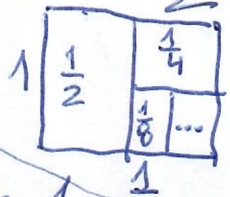
No sum

Both are wrong!

③

c) $1-1+1-1+1-1+\cdots$, $a=1,\ r=-1$

d) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$, $a=\frac{1}{2},\ r=\frac{1}{2}$



Thm 1 A geometric series

$$a+ar+ar^2+\cdots=\sum_{k=1}^{\infty} a_k r^{k-1},\ a\neq 0$$

converges if $|r|<1$ and diverges if $|r|\geq 1$.

If the series converges the sum is

$$\boxed{a+ar+ar^2+\cdots=\frac{a}{1-r}.}$$

Proof

① $|r|=1 \Rightarrow r=1$ or $r=-1$

$r=1$: $a+a+a+\cdots$

$S_n=na$

$\lim_{n\to\infty} S_n=\lim_{n\to\infty} na=\infty$

$r=-1$: $a-a+a-a+\cdots$

$=a(\underbrace{1-1+1-1+\cdots}_{\text{diverges}})$

$\lim_{n\to\infty} S_n$ does not exist

② $|r|\neq 1$

$$S_n=a+ar+ar^2+\cdots+ar^{n-1}$$

$$-rS_n=-ar-ar^2-ar^3-\cdots-ar^{n-1}-ar^n$$

$$S_n-rS_n=a-ar^n$$

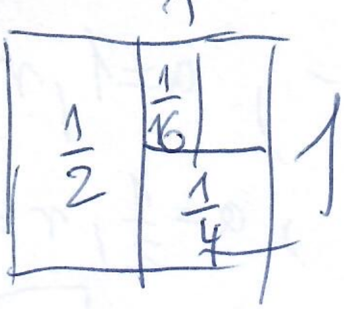
$$S_n(1-r)=a(1-r^n)$$

$$S_n=a\frac{1-r^n}{1-r}$$

$$\lim_{n\to\infty} S_n=\frac{a}{1-r}\lim_{n\to\infty}(1-r^n)=\begin{cases}\frac{a}{1-r} & \text{if } |r|<1\\ +\infty & \text{if } r>1\\ \text{not exists} & \text{if } r<-1\end{cases}$$

Ex. 3

a)



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\cdots$$

$$a=\frac{1}{2},\ r=\frac{1}{2}$$

$$S=\frac{1/2}{1-1/2}=\underline{1}$$

b) $$\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^{k+1}=\left(-\frac{2}{3}\right)^{2}+\left(-\frac{2}{3}\right)^{3}+\cdots$$

$$a=\frac{4}{9},\ r=-\frac{2}{3}$$

$$\left|-\frac{2}{3}\right|=\frac{2}{3}<1$$

converges

$$S=\frac{\frac{4}{9}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{4}{9}}{\frac{5}{3}}=\frac{4}{9}\cdot\frac{3}{5}=\underline{\frac{4}{15}}$$

<u>Ex. 4 (Telescoping series)</u>

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=1.\quad \text{Why?}$$

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cdots+\cancel{\frac{1}{n}}-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}\longrightarrow 1 \text{ as } n\to\infty$$

Thm 2 $\boxed{\text{If } \sum_{k=1}^{\infty} a_k \text{ converges, then } \lim_{k\to\infty} a_k = 0}$

Proof

$$a_k = S_k - S_{k-1} = (a_1 + a_2 + \dots + a_k) - (a_1 + a_2 + \dots + a_{k-1})$$

If $\lim_{n\to\infty} S_n = S$, then

$$a_k = S_k - S_{k-1}$$

$$\downarrow \quad \downarrow \quad k\to\infty$$

$$S - S = 0$$

$a_k \to 0$ as $k\to\infty$

Remark 1

$(P \Rightarrow Q) \Longleftrightarrow (\sim Q \Rightarrow \sim P)$

If series converges $\Rightarrow \lim_{k\to\infty} a_k = 0$

$\Updownarrow$

$\boxed{\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges}}$

Ex. 5 Series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges

since

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1$$

Remark 2

The condition $\lim_{k\to\infty} a_k = 0$ is NOT sufficient to guarantee convergence of the series $\sum_{k=1}^{\infty} a_k$.

EX. 6 For series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}}$

we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} = 0$$

but

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n} \left(\sqrt{k+1}-\sqrt{k}\right)$$

$$= \cancel{\sqrt{2}} - 1 + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \sqrt{4} - \cancel{\sqrt{3}} + \cdots + \sqrt{n+1} - \sqrt{n}$$

$$= \sqrt{n+1} - 1 \longrightarrow \infty \text{ as } n \to \infty$$

$$\frac{1}{\sqrt{k+1}+\sqrt{k}} = \sqrt{k+1} - \sqrt{k}$$

$$\Updownarrow$$

$$1 = \left(\sqrt{k+1}-\sqrt{k}\right)\left(\sqrt{k+1}+\sqrt{k}\right)$$

$$\cancel{k}+1-\cancel{k}$$

Thm 3 (Harmonic series) Harmonic series

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\dots = +\infty \quad \text{diverges}}$$

Proof. $S_n = \sum_{k=1}^{n} \frac{1}{k}$, $\{S_n\}$ is increasing

$$S_2 = 1+\frac{1}{2} > \frac{1}{2}+\frac{1}{2} = \frac{2}{2}$$

$$S_4 = S_2 + \frac{1}{3}+\frac{1}{4} > S_2 + \frac{1}{4}+\frac{1}{4} = S_2 + \frac{1}{2} > \frac{2}{2}+\frac{1}{2} = \frac{3}{2}$$

$$S_8 = S_4 + \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > S_4 + \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = S_4 + \frac{1}{2} > \frac{4}{2}$$

$$\vdots$$

$$S_{2^n} > \frac{n+1}{2} \longrightarrow \infty \text{ as } n \to \infty$$

$\{S_n\}$ is unbounded $\Rightarrow$ diverges

$$\Downarrow$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and $\lim_{k\to\infty} \frac{1}{k} = 0$

Remark 3

If $\sum_{k=1}^{\infty} a_k$ converges <u>and</u> $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty}(a_k \pm b_k)$ converges and

$$\sum_{k=1}^{\infty}(a_k \pm b_k) = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k.$$

Also

$$\sum_{k=1}^{\infty} c a_k = c \sum_{k=1}^{\infty} a_k \quad \forall c \in \mathbb{R}$$

Thm 4 (Comparison test) Let

$$0 \le a_k \le b_k, \quad k = 1, 2, \dots$$

a) If $\sum_{k=1}^{\infty} b_k < \infty \implies \sum_{k=1}^{\infty} a_k < \infty$

[if bigger series converges than smaller also converges]

b) If $\sum_{k=1}^{\infty} a_k = \infty \implies \sum_{k=1}^{\infty} b_k = \infty$

Proof

a) $\sum_{k=1}^{\infty} b_k = B < \infty \implies$

$$\underbrace{a_1 + a_2 + \dots + a_n}_{} \le b_1 + b_2 + \dots + b_n \le B$$

$\{s_n\}$ bounded above and increasing

$\Downarrow$

$\{s_n\}$ is convergent

$\Updownarrow$

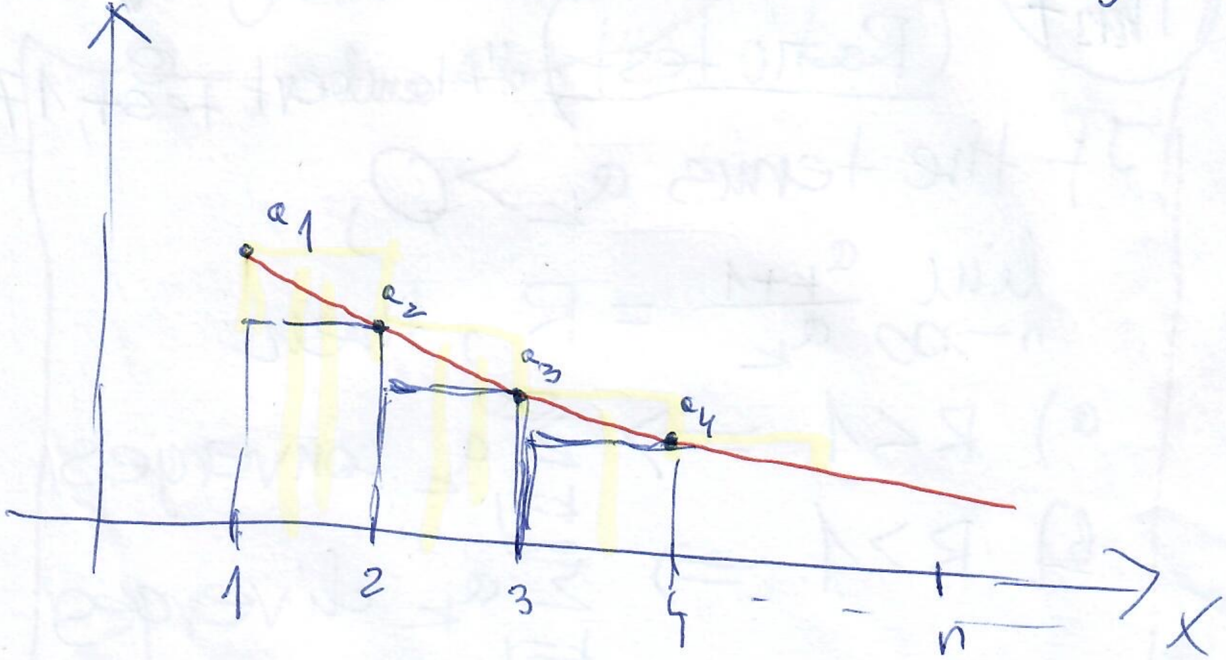
$\sum_{k=1}^{\infty} a_k$ convergent

b) Similar

Thm 5 (Integral test)

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms and let $f(x)$ be the function obtained by replacing k by x in the formula for a_k, i.e. $f(k)=a_k$.

If f is decreasing and continuous on $[1,\infty)$ then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)dx$ both converge or both diverge.



$$\underbrace{a_2+a_3+\dots+a_n}_{\text{are of rectangles}} \le \underline{\int_1^n f(x)dx}$$

area under the curve $y=f(x)$ from $x=1$ to $x=n$

$$S_n=\sum_{k=1}^{n} a_k = a_1+a_2+\dots+a_n \le a_1+\int_1^{n} f(x)dx$$

On the other hand

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \dots + a_n = S_n$$

Thus

$$\int_1^{n+1} f(x)\,dx \le S_n \le a_1 + \int_1^{n} f(x)\,dx$$

Thm 6 (Convergence of p-series)

Let $p \in \mathbb{R}$, $p > 0$. Series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges if $p > 1$ and diverges if $0 < p \le 1$

Proof $f(x) = \frac{1}{x^p}$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \begin{cases} \left(\ln|x| \Big|_1^b \right) & p = 1 \\ \left. \frac{x^{-p+1}}{-p+1} \right|_1^b & p \ne 1 \end{cases}$$

$$= \lim_{b\to\infty} \begin{cases} \ln b & p = 1 \\ \frac{b^{-p+1} - 1}{-p+1} & p \ne 1 \end{cases}$$

$$= \begin{cases} \infty & p = 1 \\ \frac{1}{p-1} & p > 1 \\ \infty & 0 < p < 1 \end{cases}$$

$p=2$ $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$ converges

Remark 4. Euler (1736) proved

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

but the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots = ?$$

not known from 1736 !

Thm 7 (Ratio test; d'Alembert test, 1768)

If the terms $a_k > 0$,

$$\lim_{n \to \infty} \frac{a_{k+1}}{a_k} = R, \text{ then}$$

a) $R < 1 \implies \sum_{k=1}^{\infty} a_k$ converges

b) $R > 1 \implies \sum_{k=1}^{\infty} a_k$ diverges

Proof

a) If $R<1$, then take $\varepsilon>0$ such that $R+\varepsilon<1$

$\Downarrow$

$$\frac{a_{k+1}}{a_k} < R+\varepsilon = r \quad \text{for } k \geq N$$

$\Downarrow$

$$a_{k+1} < r a_k \quad \forall k \geq N$$

$a_{N+1} < r a_N$, $a_{N+2} < r a_{N+1} < r^2 a_N$ etc

$$a_{N+1} + a_{N+2} + \dots < a_N (r + r^2 + \dots) = a_N \frac{r}{1-r}$$

and series

$$\underbrace{a_1 + a_2 + \dots + a_N}_{\text{finite}} + \underbrace{a_{N+1} + a_{N+2} + \dots}_{a_N \frac{r}{1-r}}$$

$< \infty$

series converges

b)

Ex. 7 Control if series

$$\sum_{k=1}^{\infty} \frac{k^{14}}{4^k} \text{ converges.}$$

Solution

$$R = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^{14}}{4^{k+1}}}{\frac{k^{14}}{4^k}} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{k}\right)^{14}}{4} = \frac{1}{4} < 1$$

series converges.

Thm 8 (Root test; Cauchy test, 1821)

If the terms $a_k > 0$,

$$\lim_{k\to\infty} \sqrt[k]{a_k} = R_0, \text{ then}$$

a) $R_0 < 1$, then series $\sum_{k=1}^{\infty} a_k$ converges.

b) $R_0 > 0$, then series —॥— diverges.

Ex. 8 Test for convergence

$$\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}.$$

Solution

$$R_0 = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{\frac{1}{k}} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k$$

$$= \lim_{k\to\infty} \left(\frac{1}{\frac{k+1}{k}}\right)^k = \frac{1}{\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k} = \frac{1}{e} < 1$$

series converges.

Alternating series

Def.

A series of the form

$$(*) \quad \sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

or

$$\sum_{k=1}^{\infty} (-1)^k a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

where the a_k's are all positive is called an alternating series.

For example

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots$$

THM 9 (Alternating series test)

An alternating series $(*)$ converges if the following two conditions are satisfied:

(a) $a_1 \geq a_2 \geq a_3 \geq \cdots$, i.e. $\{a_k\}$ is decreasing

(b) $\lim_{k \to \infty} a_k = 0$.

EX. 9

a) Alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

converges

since

a) $\frac{1}{k} > \frac{1}{k+1}$, $\{a_k\}$ is decreasing.

b) $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k} = 0$

b) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}=\frac{4}{2}-\frac{5}{6}+\frac{6}{12}-\frac{7}{20}+\dots$

b) $\displaystyle\lim_{k\to\infty}\frac{k+3}{k(k+1)}=\lim_{k\to\infty}\frac{\frac{1}{k}+\frac{3}{k^2}}{1+\frac{1}{k}}=\frac{0}{1}=0$

a) $\displaystyle\frac{a_{k+1}}{a_k}=\frac{\frac{k+4}{(k+1)(k+2)}}{\frac{k+3}{k(k+1)}}=\frac{k+4}{\cancel{(k+1)}(k+2)}\cdot\frac{k\cancel{(k+1)}}{k+3}$

$\displaystyle=\frac{k^2+4k}{k^2+5k+6}<1$

$\{a_k\}$ decreasing

series converges

Def. A series

$$\sum_{k=1}^{\infty}a_k=a_1+a_2+a_3+\dots$$

is said to converge absolutely if the series of the absolute values

$$\sum_{k=1}^{\infty}|a_k|=|a_1|+|a_2|+\dots \text{ converges}$$

series absolutely convergent $\Longrightarrow$ is convergent

$$\left|\sum_{k=1}^{n}a_k\right|\overset{\text{triangle inequality}}{\leq}\sum_{k=1}^{n}|a_k|\leq\sum_{k=1}^{\infty}|a_k|$$

Ex. 10

a) Series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ converges

but it is not absolutely convergent because

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k} = \infty$$

diverges to ∞

b) Series $\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{1}{2^k} = \frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\dots$

is absolutely convergent since

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\cdot\frac{1}{2^k}\right| = \sum_{k=1}^{\infty}\frac{1}{2^k} = \frac{1/2}{1-1/2} = 1$$

Remark

A series which is convergent but not absolutely convergent is called <u>conditionally convergent</u>.

Alternating harmonic series is conditionally convergent.