Lech M.

Lecture 9

Functions of Two or More Variables
(Domain, Limit and Continuity)

$f: \mathbb{R}^2 \supset D \longrightarrow \mathbb{R}$ function of two-variables

$f: \mathbb{R}^n \supset D \longrightarrow \mathbb{R}$ function of n-variables

Domain D = set of points at which the function is defined.

Ex. 1 Let

$$f(x,y) = \sqrt{4-x^2-y^2} + \ln(x^2+y^2-1).$$

Find $f(1,1)$, $f(0,2)$ and the domain (natural domain) of f.

$$f(1,1) = \sqrt{4-1-1} + \ln(1+1-1) = \sqrt{2}$$

$$f(0,2) = \sqrt{4-0-4} + \ln(0+4-1) = \ln 3$$

D: $4-x^2-y^2 \geq 0$ and $x^2+y^2-1 > 0$

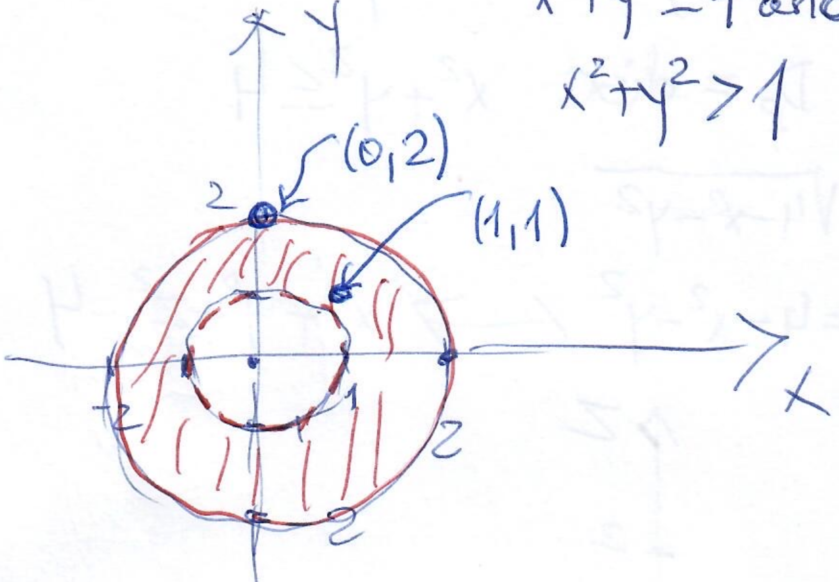
$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$x^2+y^2 \leq 4$ $\qquad$ $x^2+y^2 > 1$

$x^2+y^2=4$ $\qquad$ $x^2+y^2=1$

$x^2+y^2\leq 4$ and

$x^2+y^2>1$



$D=\{(x,y): x^2+y^2\leq 4 \text{ and } x^2+y^2>1\}$

<u>EX. 2</u> $\quad f(x,y)=\arcsin\frac{x}{2}+\sqrt{xy}$.

Find $f(2,2)$, $f(1,4)$ and domain (natural domain)

$f(2,2)=\arcsin 1+\sqrt{4}=\frac{\pi}{2}+2$

$f(1,4)=\arcsin\frac{1}{2}+\sqrt{4}=\frac{\pi}{6}+2$

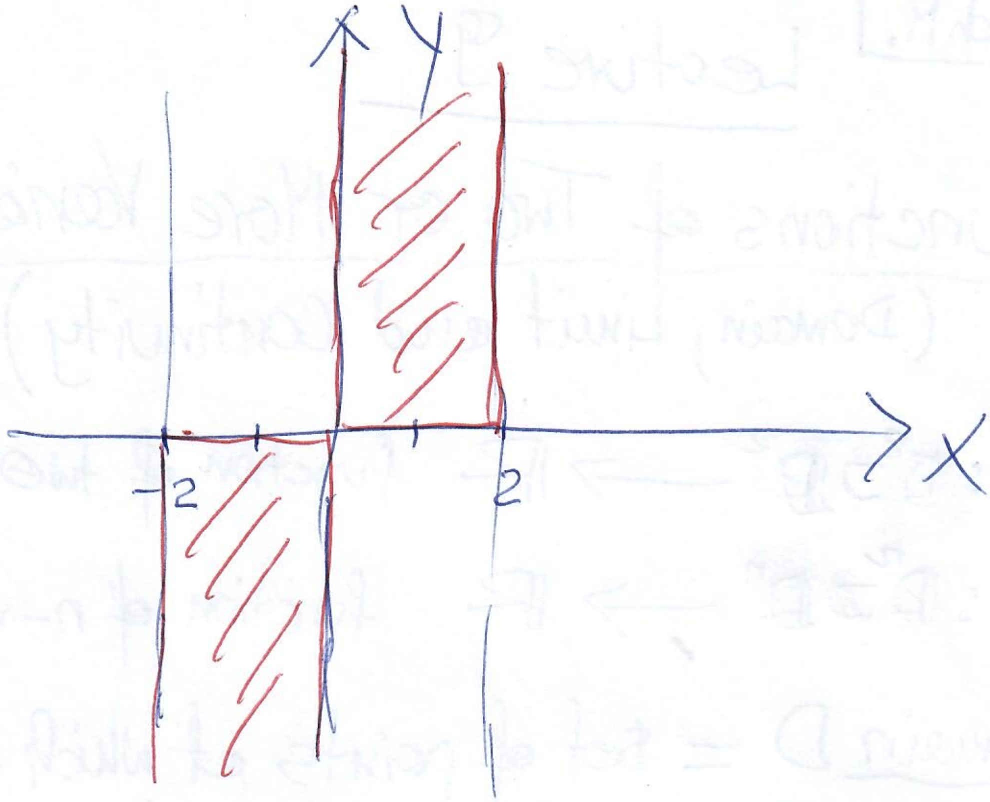
<u>Domain</u>:

$-1\leq\frac{x}{2}\leq 1$ $\quad$ because

$\Updownarrow$ $\qquad$ $\arcsin\frac{x}{2}=\alpha \Leftrightarrow \sin\alpha=\frac{x}{2}$

$-2\leq x\leq 2$ $\qquad$ and $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$

Also $xy\geq 0 \Leftrightarrow x\geq 0$ and $y\geq 0$ or $x\leq 0$ and $y\leq 0$

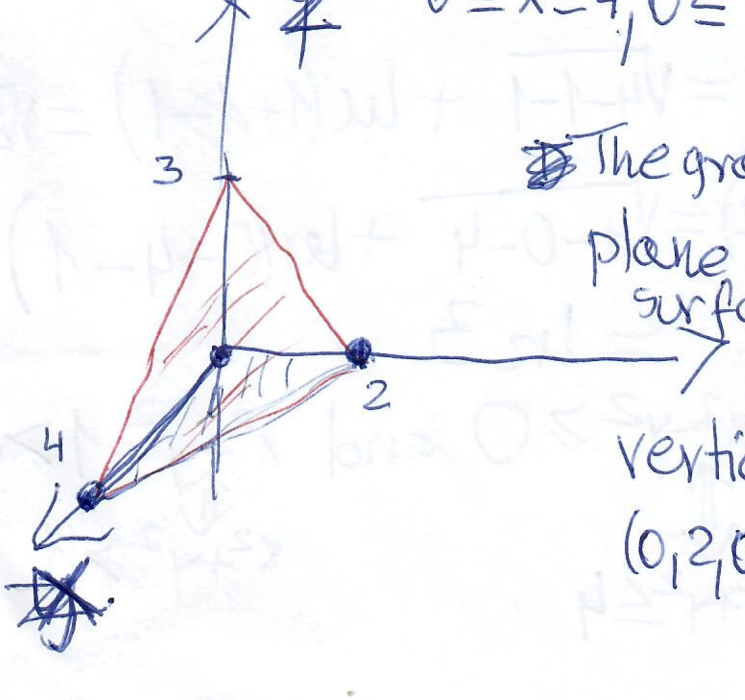


<u>Graph</u> of the function $z=f(x,y)$ is in $\mathbb{R}^3$.

<u>Ex. 3</u> Consider

$$f(x,y)=3\left(1-\frac{x}{4}-\frac{y}{2}\right),$$

$$0\le x\le 4,\ 0\le y\le 2-\frac{x}{2}.$$



The graph is the plane triangular surface with vertices at $(4,0,0)$, $(0,2,0)$ and $(0,0,3)$.

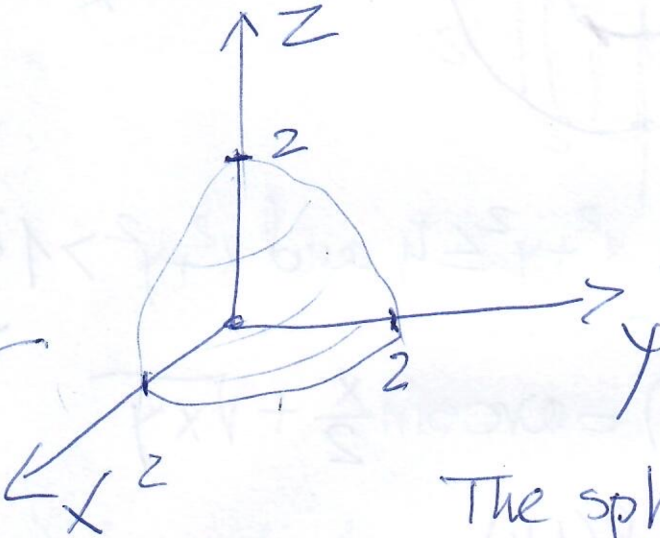
Ex. 4

$$f(x,y)=\sqrt{4-x^2-y^2}$$

$D_f = \text{disc } x^2+y^2\le 4$

$z=\sqrt{4-x^2-y^2}$

$z^2=4-x^2-y^2 \Longleftrightarrow x^2+y^2+z^2=4$



The sphere of radius 2 centred at origin

In the xy-plane the curves $f(x,y)=C$ for various constant C are called <u>level curves of f</u>

Level curves

$C=\sqrt{4-x^2-y^2} \Longrightarrow x^2+y^2=4-C^2$

circles

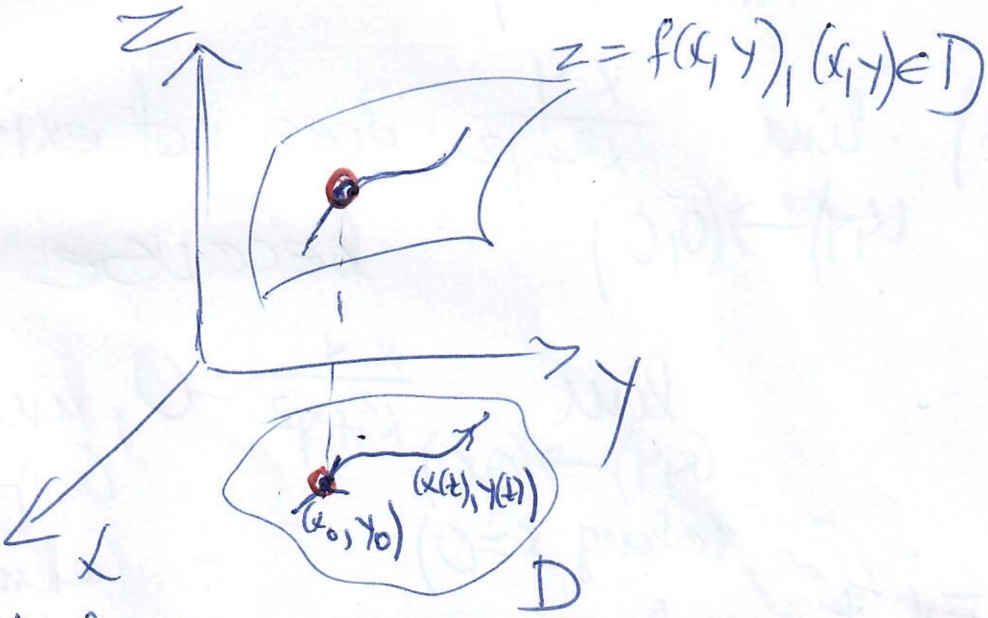
Limit along curves

Let $f(x,y)$ be a function of two variables and C a smooth ~~or~~ parametric curve in the xy-plane

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad t \in I$$

and $(x_0, y_0) = (x(t_0), y(t_0))$ point on the curve C.
Then

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (\text{along } C)}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t)).$$



EX. 5 Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

Find the limit of $f(x,y)$ as $(x,y)\to(0,0)$ along

(a) the x-axis

(b) the y-axis

(c) the parabola $y = x^2$

(d) the line $y = x$

Remark

If we can find two different smooth curves containing (x_0, y_0) along which $f(x,y)$ has different limit as $(x,y) \to (x_0, y_0)$ then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Ex. 6

a) $$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2} = \frac{1\cdot 2}{1^2+2^2} = \frac{2}{5}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$ does not exist because

$$\lim_{\substack{(x,y)\to(0,0) \\ (\text{along } x=0)}} \frac{xy}{x^2+y^2} = 0, \quad \lim_{\substack{(x,y)\to(0,0) \\ (\text{along } y=x)}} \frac{xy}{x^2+y^2} = \frac{1}{2}$$

Ex. 7 Find

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = 0$$

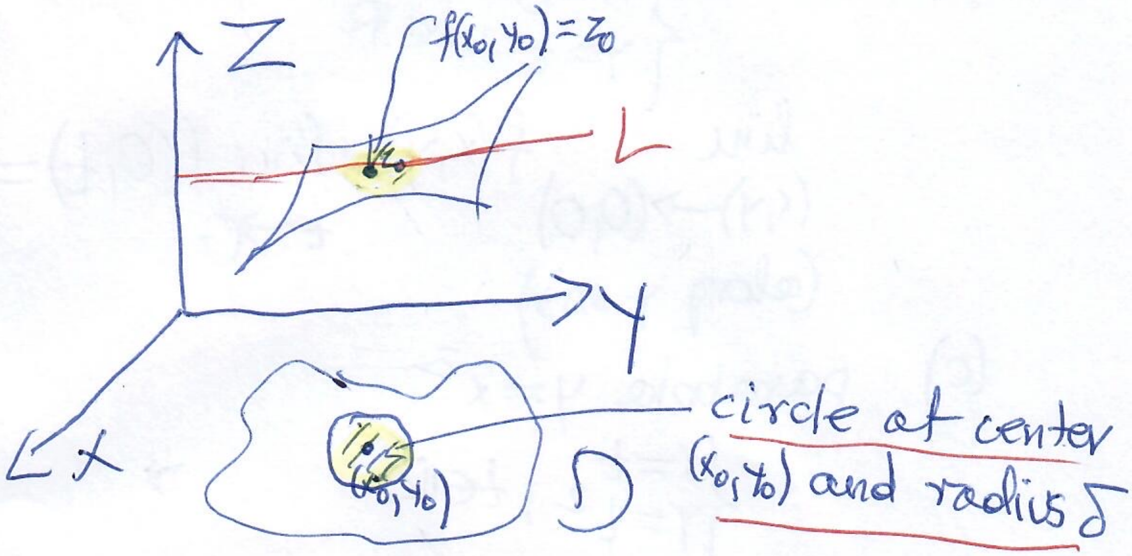
Limit of functions of two variables

Def. 1

We write $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if

$\forall \varepsilon>0\ \exists \delta>0\ (x,y)\in D_f$

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \Longrightarrow |f(x,y)-L|<\varepsilon$$



THM 1

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L \Longrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)=L.$$

[f has a limit L at (x_0,y_0)]

(along any smooth curve in D_f)

$$(P\Rightarrow Q) \Longleftrightarrow (\sim Q \Longrightarrow \sim P)$$

Solutions

$$\boxed{f(x,y)=\frac{xy}{x^2+y^2}}$$

(a) The x-axis has parametric equation

$$\begin{cases} x=t \\ y=0 \end{cases}, \quad t\in\mathbb{R}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(along x-axis)}}} f(x,y)=\lim_{t\to 0} f(t,0)=0$$

(b) The y-axis

$$\begin{cases} x=0 \\ y=t \end{cases}, \quad t\in\mathbb{R}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(along y-axis)}}} f(x,y)=\lim_{t\to 0} f(0,t)=0$$

(c) parabola $y=x^2$

$$\begin{cases} x=t \\ y=t^2 \end{cases}, \quad t\in\mathbb{R}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(along } y=x^2)}} f(x,y)=\lim_{t\to 0} f(t,t^2)=\lim_{t\to 0}\frac{t^3}{t^2+t^4}=$$

$$=\lim_{t\to 0}\frac{t}{1+t^2}=\frac{0}{1}=0$$

(d) $y=x$

$$\begin{cases} x=t \\ y=t \end{cases}, \quad t\in\mathbb{R}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(along } y=x)}} f(x,y)=\lim_{t\to 0} f(t,t)=\lim_{t\to 0}\frac{t^2}{t^2+t^2}=\frac{1}{2}$$

Continuity

A function $f(x,y)$ is called continuous at (x_0, y_0) if:

1. $f(x_0, y_0)$ is defined
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists (is a number)
3. $\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)}$

A function $z = f(x,y)$ is continuous on region R if is continuous at each point of the region R.

Continuity Theorems

1. $g(x)$, $h(y)$ continuous of one variable $\Longrightarrow f(x,y) = g(x)h(y)$ is continuous
2. $g(x)$ continuous, $h(x,y)$ continuous $\Longrightarrow f(x,y) = g(h(x,y))$ is continuous
3. The sum, difference and product of continuous functions of two variable is also continuous.

EX. 8 Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

but

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} \text{ does not exist.}$$

Solution

a) $0 < \sqrt{x^2+y^2} < \delta \implies \left| \frac{x^2 y}{x^2+y^2} - 0 \right| < \varepsilon$

$$\left| \frac{x^2 y}{x^2+y^2} - 0 \right| = \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1 \text{ since } x^2 \leq x^2+y^2} |y| \leq |y| \leq \sqrt{x^2+y^2} < \delta = \varepsilon$$

Why?

$$\sqrt{x^2+y^2} \geq \sqrt{0+y^2} = |y|$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

b) Take limits on the lines $\qquad f(x,y) = \frac{x^2 y}{x^4+y^2}$

$$\begin{cases} x(t) = \alpha t \\ y(t) = \beta t \end{cases} \qquad \lim_{t\to 0} \frac{\alpha^2 t^2 \cdot \beta t}{\alpha^4 t^4 + \beta^2 t^2} = \lim_{t\to 0} \frac{\alpha^2 \beta t}{\alpha^4 t^2 + \beta^2} = 0$$

but on parabola $y = x^2$

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \qquad \lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4 + t^4} = \frac{1}{2}$$

4. The quotient of continuous functions is continuous except the points where the denominator is zero.

Ex. 9

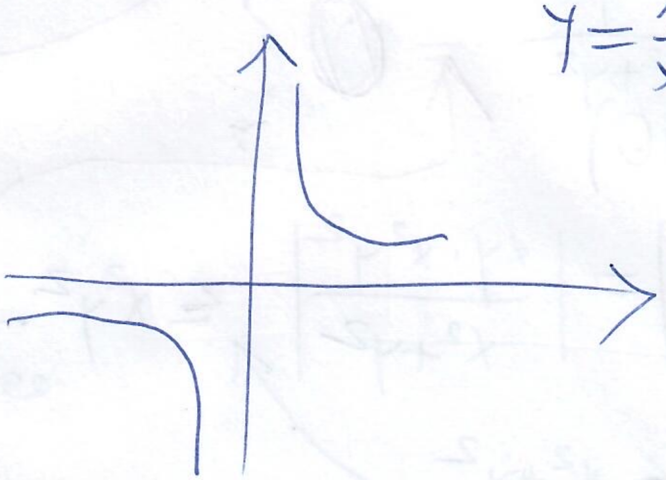
a) $f(x,y) = \underbrace{2x^2y}_{\text{cont.}} \underbrace{\sin(x^2+y)}_{\text{cont}} + \underbrace{\cos(xy^2)}_{\text{cont}}$

continuous function on $\mathbb{R}$

b) $g(x,y) = \dfrac{x^2y^2}{1-xy}$ is continuous except at

$1-xy=0$

$y=\dfrac{1}{x}$



Ex. 10 How can the function

$$f(x,y)=\frac{x^2+y^2-x^3y^3}{x^2+y^2},\ (x,y)\neq(0,0)$$

be defined at the origin so that it becomes continuous at all points of the xy-plane?

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2-x^3y^3}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\left[1-\frac{x^3y^3}{x^2+y^2}\right]=1$$

$$\lim_{(x,y)\to(0,0)}\frac{x^3y^3}{x^2+y^2}=0\ ?$$

$$\left|\frac{x^3y^3}{x^2+y^2}\right|=\left|\frac{xy\cdot x^2y^2}{x^2+y^2}\right|\leq x^2y^2\longrightarrow 0$$

as $x^2+y^2\to 0$

Why?

$$|xy|\leq x^2+y^2$$

$$0\leq(|x|-|y|)^2=|x|^2-2|x||y|+|y|^2=x^2-2|xy|+y^2$$

$$|xy|\leq 2|xy|\leq x^2+y^2$$

$$f(0,0)=1$$