Lech M.

Lecture 12

Introduction to ordinary differential equations and linear differential equations (variation of constant and integrating factor).

Equation where as the unknown function is derivative (or derivatives) is called differential equation (d. e.)

Ex. 1

a) $y' - y = x$

or more precisely

$$\boxed{y'(x) - y(x) = x \quad \forall x \in I}$$

We want to find $y = y(x)$.

b) $y'' + (y')^2 \sin(xy) + 2\sin y = e^{xy}$

is d. e.

Solution of differential equation is any function which satisfies given differential equation.

Ex. 2 a) $y(x) = e^{-x^2}$ is a solution of d.e

$$y' + 2xy = 0.$$

Why?

$$y' = e^{-x^2}(-x^2)' = -2x e^{-x^2}$$

$$y' + 2xy = -2xe^{-x^2} + 2x \cdot e^{-x^2} = 0 \quad \forall x \in \mathbb{R}$$

OK

b) $y(x) = Ce^x - x - 1$ for all $C \in \mathbb{R}$ are solutions of d. e.

$$y' - y = x.$$

Why?

$$y' = Ce^x - 1$$

$$y' - y = \cancel{Ce^x} - \cancel{1} - (\cancel{Ce^x} - x - \cancel{1})$$

$$= x \quad \forall x \in \mathbb{R}$$ OK

Initial value problem

$$\begin{cases} y'-y=x \\ y(0)=0 \end{cases} \longrightarrow y=Ce^x-x-1$$

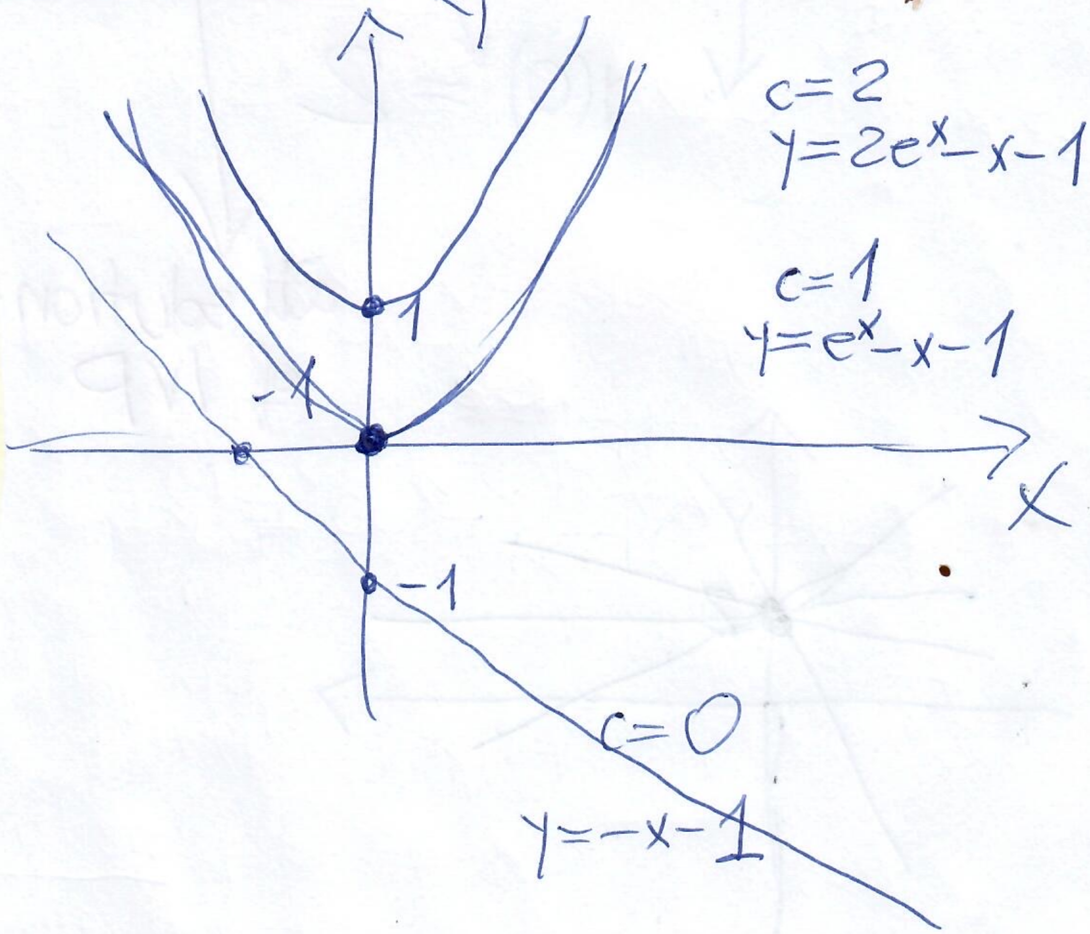
$$y(0)=C-1=0$$

$$\Downarrow$$

$$C=1$$

$$\Downarrow$$

$$\boxed{y=e^x-x-1}$$ unique



Ex. 3 Solution of initial value problem might not be unique.

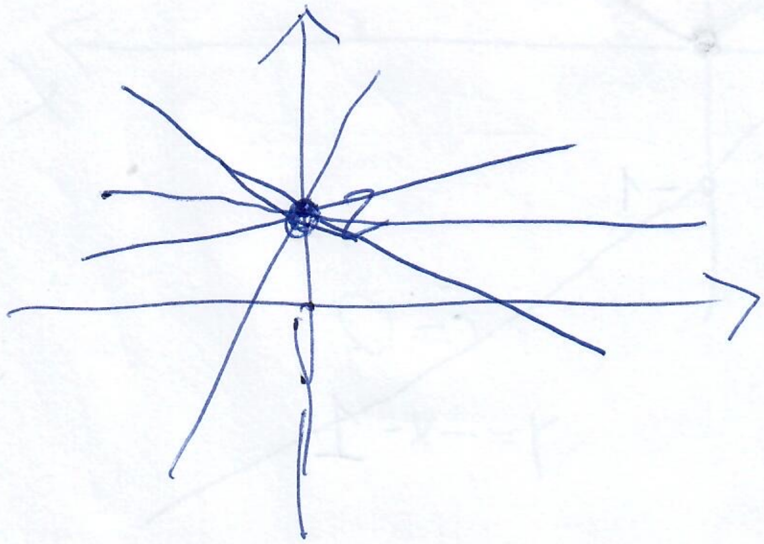
$$\begin{cases} xy' - y + 2 = 0 \\ y(0) = 2 \end{cases}$$

Solution of d. e. later on!

They are $y = Cx + 2 \ \forall C \in \mathbb{R}$

$y(0) = 2$

all solutions of IVP



Ex.4 Newton's Second Law of Motion

$$m\frac{d^2x}{dt^2}=F(t) \qquad mx''(t)=F(t)$$

Force

$x(t)$ position of object of constant mass m

Ex. 5

$m(t)$ bacterias at time t

$$\frac{dm}{dt}=k\,m(t) \implies m(t)=Ce^{kt}$$

Order of differential equation = highest order of the derivative in the d. e.

Differential equation of the first order has form

$$F(x,y,y')=0 \text{ or } \underbrace{F(x,y(x),y'(x))=0}_{\text{very complicated}}$$

we take simple d.e. of the first order

$$\boxed{y'=f(x,y)}$$

$y' = f(x, y)$ or precisely

$$\boxed{y'(x) = f[x, y(x)],\ x \in I}$$

<u>Ex. 6</u> Find all solutions of d. e.

a) $y' = -2y$

$\left\{ y'(x) = -2y(x) \right.$

$y = 0$ or $y \neq 0$

$$\frac{y'}{y} = -2$$

$$(\ln|y|)' = -2$$

$$(\ln|y|)' = \frac{1}{|y|} |y|' \cdot y'$$

$$= \frac{\operatorname{sgn} y}{|y|} y' = \frac{\operatorname{sgn} y}{y \operatorname{sgn} y} y' = \frac{y'}{y}$$

$$(\ln|y|)' = \frac{1}{|y|} \cdot |y|' \cdot y'$$

$$\Downarrow$$

$$\ln|y| = -2x + A$$

$$|y| = e^{-2x+A} = e^A e^{-2x}$$

$$|y| = \underbrace{e^A}\, e^{-2x}$$

$$y = Ce^{-2x} \quad \forall C \in \mathbb{R} \setminus \{0\}$$

All solutions

$$y = Ce^{-2x} \quad \forall C \in \mathbb{R}$$

b) $y' = -xy$ $\quad \Big\{\, y'(x) = -x\,y(x) \quad \forall x \in \mathbb{R}$

$y = 0$ or $y \neq 0$

$$\frac{y'}{y} = -x$$

$$(\ln|y|)' = -x$$

$$\ln|y| = -\frac{x^2}{2} + A$$

$$|y| = e^{-\frac{x^2}{2}+A} = \underbrace{e^A}\, e^{-x^2/2}$$

$$y = Ce^{-x^2/2} \qquad C \neq 0$$

$$y = Ce^{-x^2/2} \quad \forall C \in \mathbb{R}$$

Cauchy Problem

Solve the initial value problem

$$(CP)\quad \begin{cases} y'(x) = f(x, y(x)), & x \in [a,b] \\ y'(x_0) = y_0, & x_0 \in (a,b) \end{cases}$$

f, x_0, y_0 are given

we are looking for $y = y(x)$

Thm 1 (Peano 1896)

f is continuous in $[a,b] \times [c,d]$

$\Longrightarrow \exists\, y$ with continuous derivative which satisfy IVP = Cauchy Problem.

Thm 2 (Picard – Lindelöf)
1890 1894

f and $\boxed{\frac{\partial f}{\partial y}}$ are continuous in $[a,b] \times [c,d]$

$\Longrightarrow \exists$ unique solution of (CP)

Proof Picard iterations

B) Linear differential equation of order n

degree

(1) $g_n(x)\,y^{(n)} + g_{n-1}(x)\,y^{(n-1)} + \dots + g_2(x)\,y'' + g_1(x)\,y' +$
$g_0(x)\,y = h(x)$

If $h(x)=0$, then equation (1) is called homogeneous.

Degree 1 $\quad g_1(x)y' + g_0(x)y = h(x)$

Degree 2 $\quad g_2(x)y'' + g_1(x)y' + g_0(x)y = h(x)$

EX. 7 a)

$$y'' + 3x\,y' - (x-2)\,y = e^x$$

linear d. e. of order 2

b) $y \cdot y'' + y' = 2x$ nonlinear

$y'' = 2x\cos(3y)$ — " —

$(y')^2 = y \Leftrightarrow y' \cdot y' = y$ nonlinear

Thm 3 (Linear superposition principle)
If y_1, y_2 are solutions of the homogeneous linear differential equation of degree n, then

$$y = c_1 y_1 + c_2 y_2$$

are also solutions for any $c_1, c_2 \in \mathbb{R}$.

Proof for $n=2$

By assumption
$$\begin{cases} g_2(x) y_1'' + g_1(x) y_1' + g_0(x) y_1 = 0 \\ g_2(x) y_2'' + g_1(x) y_2' + g_0(x) y_2 = 0 \end{cases}$$

$$\Downarrow$$

$$g_2(x) y'' + g_1(x) y' + g_0(x) y = g_2(x)[c_1 y_1 + c_2 y_2]'' + g_1(x)[c_1 y_1 + c_2 y_2]'$$
$$+ g_0(x)[c_1 y_1 + c_2 y_2] =$$
$$= c_1 [g_2(x) y_1'' + g_1(x) y_1' + g_0(x) y_1]$$
$$+ c_2 [g_2(x) y_2'' + g_1(x) y_2' + g_0(x) y_2$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

Ex. 8 Equation (nonlinear)

$$(*) \qquad y'' - 3y^2 = 0$$

has solutions $y_1 = \frac{2}{x^2}$ and $y_2 = \frac{2}{(x+1)^2}$

but $y = y_1 + y_2$ is not a solution of $(*)$.

Proof

$$y_1 = \frac{2}{x^2}, \quad y_1' = \frac{-4}{x^3}, \quad y_1'' = \frac{12}{x^4}$$

$$y_1'' - 3y_1^2 = \frac{12}{x^4} - 3\left(\frac{2}{x^2}\right)^2 = \frac{12}{x^4} - \frac{12}{x^4} = 0$$

OK

$$y_2 = \frac{2}{(x+1)^2}, \quad y_2' = \frac{-4}{(x+1)^3}, \quad y_2'' = \frac{12}{(x+1)^4}$$

$$y_2'' - 3y_2^2 = \frac{12}{(x+1)^4} - 3\left[\frac{2}{(x+1)^2}\right]^2 = \frac{12}{(x+1)^4} - \frac{12}{(x+1)^4} = 0$$

OK

$$y = \frac{2}{x^2} + \frac{2}{(x+1)^2}$$

$$\begin{cases} y'' = \dfrac{12}{x^4} + \dfrac{12}{(x+1)^4} \\ y^2 = \left[\dfrac{2}{x^2} + \dfrac{2}{(x+1)^2}\right]^2 = \dfrac{4}{x^4} + \dfrac{8}{x^2(x+1)^2} + \dfrac{4}{(x+1)^4} \end{cases}$$

$$y'' - 3y^2 = \cancel{\frac{12}{x^4}} + \cancel{\frac{12}{(x+1)^4}} - \cancel{\frac{12}{x^4}} - \frac{24}{x^2(1+x)^2} - \cancel{\frac{12}{(x+1)^4}}$$

$$= -\frac{24}{x^2(1+x)^2} \quad \text{NOT } 0$$

C) Linear differential equations of the first order

(1) $y' + p(x)y = r(x)$

If $r(x) = 0$, then (1) is said to be **homogeneous**

otherwise is said to be **nonhomogeneous**

I Method of variation of constant

First – homogeneous equation

$$\boxed{y' + p(x)y = 0}$$

$y = 0$ or $y \neq 0$

$$y' = -p(x)y$$

$$\underbrace{\frac{y'}{y}} = -p(x)$$

$$(\ln|y|)' = -p(x) \Longrightarrow \ln|y| = -\underbrace{\int p(x)dx}_{P(x)} + C_1$$

$$|y| = e^{-P(x)+C_1}$$

$$|y| = \underbrace{e^{C_1}}\, e^{-P(x)}$$

$$\boxed{y = C e^{-P(x)} \quad \forall C \in \mathbb{R}}$$

<u>Second</u> – variation of constant and put to nonhomogeneous equation

$$y = \underline{C(x)} e^{-P(x)}$$

$$y' = C'(x) e^{-P(x)} + C(x) e^{-P(x)} \cdot (-p(x))$$

$$= C'(x) e^{-P(x)} - p(x) C(x) e^{-P(x)}$$

$$y' + p(x) y = C'(x) e^{-P(x)} \cancel{- p(x) C(x) e^{-P(x)}} + \cancel{p(x) C(x) e^{-P(x)}}$$

$$= C'(x) e^{-P(x)}$$

$$\Downarrow$$

$$C'(x) e^{-P(x)} = r(x)$$ We need to calculate $C(x)$.

$$C'(x) = r(x) e^{P(x)}$$

$$C(x) = \int r(x) e^{P(x)} dx + A$$

Solution of (1) is

$$y = \left[\int r(x) e^{P(x)} dx + A \right] e^{-P(x)}$$

II Method of integrating factor

(1) $y' + p(x)y = r(x)$

Multiply (1) by $\boxed{e^{P(x)}}$ where $P(x)$ is a primitive function of $p(x)$

$$\underbrace{y' \cdot e^{P(x)} + p(x) y e^{P(x)}} = r(x) e^{P(x)}$$

Nice trick !

$$\frac{d}{dx}\left(y e^{P(x)}\right) = \underbrace{y' \cdot e^{P(x)} + y e^{P(x)} \cdot p(x)}$$

$$\frac{d}{dx}\left(y e^{P(x)}\right) = r(x) e^{P(x)}$$

$$y e^{P(x)} = \int r(x) e^{P(x)} dx + A$$

$$y = e^{-P(x)} \left[\int r(x) e^{P(x)} dx + A\right]$$