Lech M.

Lecture 3

Techniques of integration

$$\int_a^b f(x)\,dx = \text{primitive function } \underbrace{F(x)}\Big|_a^b = F(b)-F(a)$$

f given $\Downarrow$ find F such that $\boxed{F'(x) = f(x)}$

(A) Method of substitution

(= the integral version of the Chain Rule)

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$. Thus

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C$$

Formally: Let $u = g(x)$. Then $\frac{du}{dx} = g'(x)$ and so $du = g'(x)\,dx$

$$\int f'(g(x))\,g'(x)\,dx = \left\{\begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}\right\}$$

$$= \int f'(u)\,du = f(u) + C = \underline{f(g(x)) + C}$$

Ex. 1

a) $a \neq 0$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}\,dx$$

$$= \left\{\begin{array}{l} u = \frac{x}{a} \\ du = \frac{1}{a}\,dx \end{array}\right\} = \frac{1}{a^2}\int \frac{1}{1+u^2}\,a\,du = \frac{a}{a^2}\underbrace{\int \frac{1}{1+u^2}\,du}$$

$$= \frac{1}{a}\arctan u + C = \frac{1}{a}\arctan\frac{x}{a} + C$$

b) $$\int \frac{x}{x^2+1}\,dx = \left\{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}\right\} = \int \frac{1}{u}\,\frac{du}{2} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$$

c) $$\int e^x\sqrt{1+e^x}\,dx = \left\{\begin{array}{l} u = 1+e^x \\ du = e^x\,dx \end{array}\right\} = \int \sqrt{u}\,du$$

$$= \frac{u^{3/2}}{3/2} + C = \underline{\frac{2}{3}(1+e^x)^{3/2} + C}$$

d) $\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx = \left\{ \begin{array}{l} u = x+2 \\ du = dx \end{array} \right\}$

$= \int \frac{1}{u^2+1}\,du = \arctan u + C = \arctan(x+2) + C$

<u>Thm 1</u> (The method of substitution in definite integral)

1° g differentiable on $[a,b]$, $g(a)=A$, $g(b)=B$

2° f continuous in the range of g

$\Downarrow$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_A^B f(u)\,du$$

<u>Proof</u> If F is a primitive function of f, i.e. $F'(u)=f(u)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big|_a^b$$

$$= F(g(b)) - F(g(a)) = F(B) - F(A)$$

$$= F(u)\Big|_A^B = \int_A^B f(u)\,du$$

Ex. 2

a) $$\int_0^1 \sqrt{3x+1}\,dx = \left\{\begin{array}{l} u=3x+1 \\ du=3dx \\ x=0 \Rightarrow u=1 \\ x=1 \Rightarrow u=4 \end{array}\right\} = \int_1^4 \sqrt{u}\,\frac{1}{3}du$$

$$= \frac{1}{3}\cdot\left(\frac{u^{3/2}}{3/2}\Big|_1^4\right) = \frac{1}{3}\cdot\frac{2}{3}\left(4^{3/2}-1^{3/2}\right)$$

$$= \frac{2}{9}(8-1) = \frac{14}{9}$$

b) $$\int_0^2 \frac{2x}{2x^2+1}\,dx = \left\{\begin{array}{l} u=2x^2+1 \\ du=4x\,dx \\ x=0 \Rightarrow u=1 \\ x=2 \Rightarrow u=9 \end{array}\right\} = \int_1^9 \frac{1}{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\int_1^9 \frac{1}{u}du = \frac{1}{2}\left(\ln|u|\Big|_1^9\right) = \frac{1}{2}(\ln 9 - \ln 1)$$

$$= \frac{1}{2}\ln 9 = \ln 9^{1/2} = \underline{\ln 3}$$

EX. 2

c) $$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx = \left\{\begin{array}{l} u=\sqrt{x+1} \\ du=\frac{1}{2\sqrt{x+1}}\,dx \\ x=0 \Rightarrow u=1 \\ x=8 \Rightarrow u=3 \end{array}\right\}$$

$$= \int_1^3 \cos u \cdot 2\,du = 2\int_1^3 \cos u\,du$$

$$= 2\left(\sin u\Big|_1^3\right) = 2(\sin 3 - \sin 1)$$

EX. 3

a) First method
$$\int \sin 2x\,dx = \left\{\begin{array}{l} u=2x \\ du=2dx \end{array}\right\} = \int \sin u\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \sin u\,du = \frac{1}{2}(-\cos u)+C =$$

$$= -\frac{\cos 2x}{2}+C$$

b) Second method
$$\int \sin 2x\,dx = 2\int \sin x\cos x\,dx = \left\{\begin{array}{l} u=\sin x \\ du=\cos x\,dx \end{array}\right\}$$

$$= 2\int u\,du = 2\left(\frac{u^2}{2}+D\right) = \sin^2 x + D$$

Remark:

$$-\frac{\cos 2x}{2}+C = -\frac{\cos^2 x-\sin^2 x}{2}+C =$$

$$= -\frac{1-\sin^2 x-\sin^2 x}{2}+C = \sin^2 x + C-\frac{1}{2}\ D$$

EX. 4

$$\int \frac{g'(x)}{g(x)}\,dx = \left\{\begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}\right\} = \int \frac{1}{u}\,du$$

$$\left(\ln|g(x)|\right)' = \frac{g'(x)}{g(x)} \qquad = \ln|u| + C$$

$$= \ln|g(x)| + C$$

In particular,

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$$

$$\sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t|$$

B) Integration by parts

$$(uv)' = u' \cdot v + u \cdot v'$$

Integrating

$$\underline{u(x)v(x)} = \int (uv)'(x)\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

or

$$(*) \quad \boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx}$$

integration by parts

$$\int u\,dv = uv - \int v\,du$$

Ex. 1

a) $\int x \underline{e^x\,dx} = \left\{ \begin{array}{ll} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{array} \right.$

$$\left\{ \begin{array}{ll} u(x) = x & v'(x) = e^x \\ u'(x) = 1 & v(x) = e^x \end{array} \right\} = xe^x - \int e^x\,dx$$

$$= \underline{xe^x - e^x + C}$$

Integration by parts

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx}$$

Ex. 1

b) $\int x\cos x\,dx = \left\{\begin{matrix} u(x) = x & v'(x) = \cos x \\ u' = 1 & v = \sin x \end{matrix}\right\}$

$= x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

c) $\int x^2\cos x\,dx = \left|\begin{matrix} u = x^2 & v' = \cos x \\ u' = 2x & v = \sin x \end{matrix}\right\}$

$= x^2\sin x - 2\int x\sin x\,dx = \left\{\begin{matrix} u_1 = x & v_1' = \sin x \\ u_1' = 1 & v_1 = -\cos x \end{matrix}\right\}$

$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right)$

$= x^2\sin x + 2x\cos x - 2\int \cos x\,dx$

$= x^2\sin x + 2x\cos x - 2\sin x + C$

d) $\int x \arctan x \, dx = \left\{ \begin{matrix} u = x & v' = \arctan x \\ u' = 1 & v = \end{matrix} \right\}$

not cleaver ?

$= \left\{ \begin{matrix} u = \arctan x & v' = x \\ u' = \frac{1}{1+x^2} & v = \frac{x^2}{2} \end{matrix} \right\}$

$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{x^2}{2} \arctan x - \frac{1}{2} (x - \arctan x) + C$

$\int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= x - \arctan x$

Integration by parts for definite integral

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)dx$$

$$u(b)v(b) - u(a)v(b)$$

Why?

$$u(x)v(x)\Big|_a^b = \int_a^b (uv)'(x)dx = \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx$$

Ex. 2

$$\int_e^{e^2} \ln x\,dx = \left\{\begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix}\right\} = x\ln x\Big|_e^{e^2} - \int_e^{e^2} dx$$

$$= e^2\ln e^2 - e\ln e - (e^2 - e)$$

$$= 2e^2 - \cancel{e} - e^2 + \cancel{e}$$

$$= e^2$$

$$\int \ln x\,dx = \left\{\begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix}\right\}$$

Ex. 3

$$\int_e^{e^2} \frac{\ln(\ln x)}{x}\,dx = \left\{\begin{matrix} u=\ln(\ln x) & v'=\frac{1}{x} \\ u'=\frac{1}{\ln x}\cdot\frac{1}{x} & v=\ln x \end{matrix}\right\}$$

$$= \ln x\,\ln(\ln x)\Big|_e^{e^2} - \int_e^{e^2} \frac{1}{\cancel{\ln x}}\cdot\frac{1}{x}\cdot\cancel{\ln x}\,dx$$

$$= \ln e^2\,\ln(\ln e^2) - \ln e\,\ln(\ln e)$$

$$- \left(\ln x\Big|_e^{e^2}\right) = 2\ln 2 - 0 - (\ln e^2 - \ln e)$$

$$= 2\ln 2 - 2 + 1 = 2\ln 2 - 1$$

$$= \ln 4 - 1$$