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# Repetition CALCULUS II

## A) Integration

### I. Riemann integral

$f$ bounded function on $[a,b]$

$P$ partition of $[a,b]$: $a=x_0<x_1<x_2<\dots<x_n=b$

$$L(f,P)=\sum_{k=1}^{n}\Big[\inf_{x\in[x_{k-1},x_k]} f(x)\Big](x_k-x_{k-1})$$ lower Riemann sum

$$U(f,P)=\sum_{k=1}^{n}\Big[\sup_{x\in[x_{k-1},x_k]} f(x)\Big](x_k-x_{k-1})$$ upper Riemann sum

If $\sup_P L(f,P)=\inf_P U(f,P)$, then $f$ is integrable on $[a,b]$

and common value is denoted by

$$I=\int_a^b f(x)\,dx$$ ← Riemann integral

Properties

i) $f$ continuous on $[a,b]$ $\Rightarrow$ $f$ integrable on $[a,b]$

ii) $f,g$ integrable on $[a,b]$ $\Rightarrow$

a) $\int_a^b [Af(x)+Bg(x)]\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx \quad \forall A,B\in\mathbb{R}$

b) $f(x)\le g(x)\ \forall x\in[a,b] \Rightarrow \int_a^b f(x)\,dx\le\int_a^b g(x)\,dx$

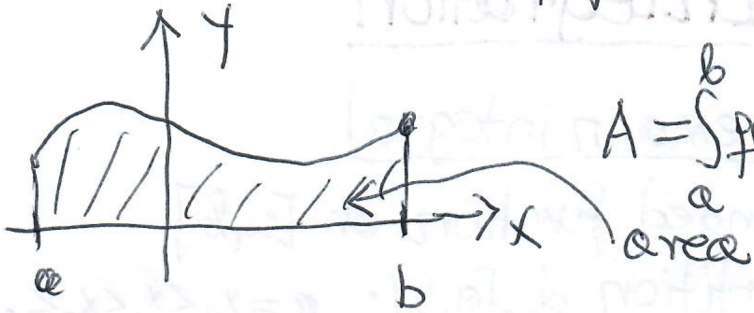
iii) $f$ odd on $\mathbb{R}$, i.e. $f(-x)=-f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx=0$

iv) $f$ even on $\mathbb{R}$, i.e. $f(-x)=f(x) \Rightarrow$

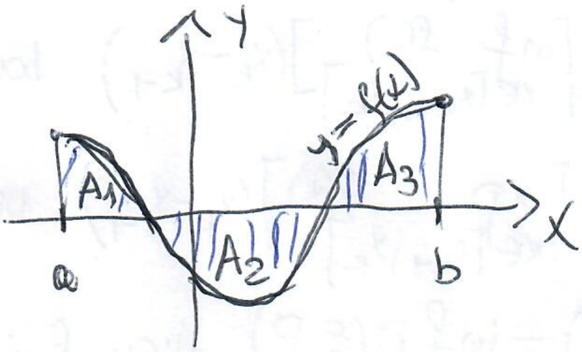
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

## II Geometric interpretation

$y = f(x) \geq 0$ for all $x \in [a,b]$




$$A = \int_a^b f(x)\,dx$$




$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2 + A_3$$

## III Fundamental Theorem of Calculus

$f$ continuous on $[a,b] \Rightarrow F(x) = \int_a^x f(t)\,dt$ differentiable on $[a,b]$ and $F'(x) = f(x)$ $\forall x \in [a,b]$

Moreover,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

# IV Methods of integration

## 1. Substitution

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\int f'(g(x)) \cdot g'(x)\,dx = \left\{ \begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right\} = \int f'(u)\,du = f(u) + C$$
$$= f(g(x)) + C$$

Remark: $\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C$

## 2. Integration by parts

$$(uv)' = u'v + uv'$$

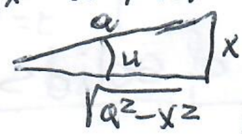
$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx.$$

## 3. Partial fractions

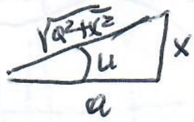
$$\int \frac{P(x)}{Q(x)}dx, \quad \text{degree } P < \text{degree } Q$$

## 4. Inverse substitutions

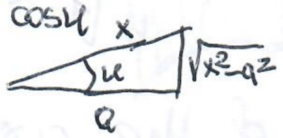
a) $\sqrt{a^2-x^2} \longrightarrow \left\{ x = a\sin u \right\}$ $\quad \sqrt{a^2-x^2} = a\cos u$




b) $\sqrt{a^2+x^2}$ or $\frac{1}{a^2+x^2} \longrightarrow \left\{ x = a\tan u \right\}$ $\quad \sqrt{a^2+x^2} = \frac{a}{\cos u}$




c) $\sqrt{x^2-a^2} \longrightarrow \left\{ x = \frac{a}{\cos u} \right\}$ $\quad \sqrt{x^2-a^2} = a|\tan u|$



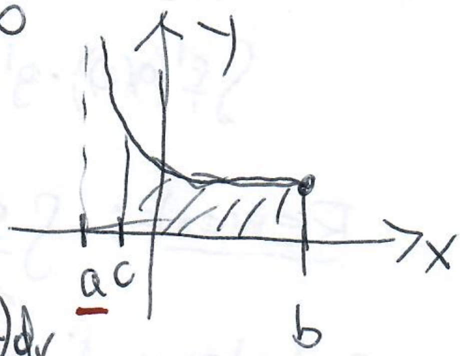

d) $$\int R(\sin u, \cos u)\,du = \left\{ \begin{array}{l} x = \tan\frac{u}{2} \text{ or } u = 2\arctan x \\ \sin u = \frac{2x}{1+x^2} \\ \cos u = \frac{1-x^2}{1+x^2} \\ du = \frac{2}{1+x^2}dx \end{array} \right\}$$

# V. Improper integrals

a) $\int_a^{\infty} f(x)\,dx = \lim\limits_{b \to \infty} \int_a^b f(x)\,dx$

Similarly with $-\infty$

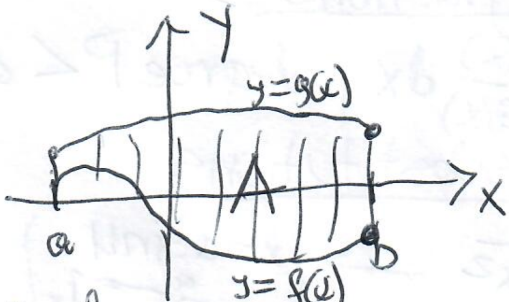
b) $\lim\limits_{x \to a^+} f(x) = \pm\infty$




$\int_a^b f(x)\,dx = \lim\limits_{c \to a^+} \int_c^b f(x)\,dx$

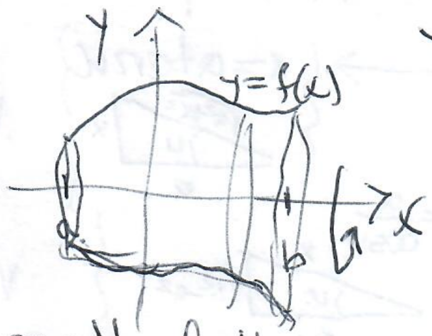
# VI. Applications of integration

## A. Areas between two curves




$A = \int_a^b [g(x) - f(x)]\,dx$

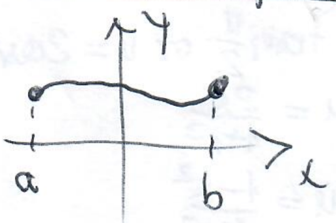
## B. Volume of solids of revolution about x-axis




$y = f(x),\ x \in [a, b]$

$V_x = \pi \int_a^b f(x)^2\,dx$

## C. Length of the curve




$y = f(x),\ x \in [a, b]$

$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$

# B Series

$a_k \in \mathbb{R}$, $k \in \mathbb{N}$

Series $\sum_{k=1}^{\infty} a_k$ converges to $S$ if

$$\lim_{n\to\infty} S_n = S, \text{ where } S_n = \underbrace{\sum_{k=1}^{n} a_k}_{\text{partial sums}}$$

Thm 1 A geometric series

$$a + ar + ar^2 + \dots, \quad a \neq 0$$

converges $\Longleftrightarrow$ $|r|<1$. In convergence case the sum is

$$a + ar + ar^2 + \dots = \frac{a}{1-r}.$$

Thm 2. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Thm 3. Series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \Longleftrightarrow p>1.$$

## Tests for convergence of positive series

Thm 4 (comparison test) $0 \le a_k \le b_k$ $\forall k \in \mathbb{N}$

i) $\sum_{k=1}^{\infty} b_k < \infty \Longrightarrow \sum_{k=1}^{\infty} a_k < \infty$.

ii) $\sum_{k=1}^{\infty} a_k = \infty \Longrightarrow \sum_{k=1}^{\infty} b_k = \infty$.

Thm 5 (integral test) $f$ decreasing, continuous on $[1,\infty)$ and $f(k) = a_k > 0$ $\forall k \in \mathbb{N}$

$$\Downarrow$$

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

both converge or both diverge.

**Example** $p \in \mathbb{R}$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots \text{ converges} \iff p > 1$$

In particular, harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \text{ diverges to } \infty.$$

**Thm 6** (ratio test) If $a_k > 0 \ \forall k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = l, \text{ then}$$

i) $l < 1$ implies that $\sum_{k=1}^{\infty} a_k$ converges.

ii) $l > 1$ implies that $\sum_{k=1}^{\infty} a_k$ diverges.

iii) $l = 1$ the series may converge or diverge so that another test must be tried.

**Thm 7** (root test) If $a_k > 0 \ \forall k \in \mathbb{N}$ and

$$R = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} (a_k)^{1/k}, \text{ then}$$

i) $R < 1$ implies that $\sum_{k=1}^{\infty} a_k$ converges.

ii) $R > 1$ implies that $\sum_{k=1}^{\infty} a_k$ diverges.

iii) $R = 1$ no conclusion.

## Alternating series

**Thm 8** If i) $a_1 \geq a_2 \geq a_3 \geq \dots$, i.e. $\{a_k\}$ is nonincreasing

ii) $\lim_{k\to\infty} a_k = 0$, then

alternating series

$$a_1 - a_2 + a_3 - a_4 + \dots \text{ converges.}$$

**Example** Alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ converges.}$$

## Power series in $x$:

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \dots, \quad c_k \in \mathbb{R}$$

**Thm 1** Power series in $x$ is absolutely convergent for $|x| < R$ and divergent for $|x| > R$, where

$$R = \frac{1}{\varrho}, \quad \varrho = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right|.$$


## Power series in $x - a$

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Interval of convergence:

$$\{x \in \mathbb{R} : |x - a| < R = \frac{1}{\varrho}\}$$

Fact (ratio test):

$$\lim_{k\to\infty} \left| \frac{c_{k+1}(x-a)^{k+1}}{c_k (x-a)^k} \right| = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| |x - a| < 1$$

## Taylor's formula

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(f,a,x)}$$

Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

# Ⓒ Functions of several variables

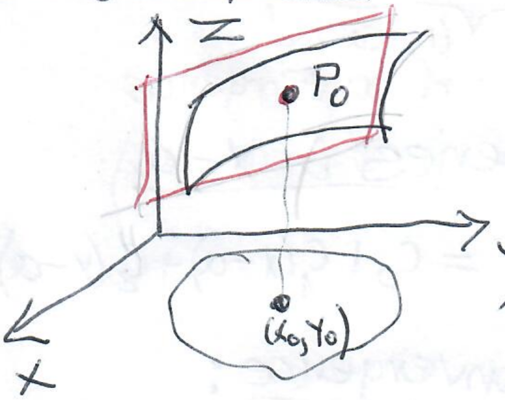
## Ⓘ Partial derivatives $z=f(x,y)$

$f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ etc.

Schwarz Theorem

$f_x, f_y, f_{xy}, f_{yx}$ continuous in $D \subset \mathbb{R}^2$ $\Rightarrow$ $\boxed{f_{xy} = f_{yx}}$ in $D$

## Ⓘⓘ Tangent plane




$P_0(x_0, y_0, z_0)$ point on the surface $z=f(x,y)$

$\vec{n} = [f_x(x_0,y_0), f_y(x_0,y_0), -1]$ normal vector to the surface $z=f(x,y)$ at $P_0$

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0}$$

equation of the tangent plane at $P_0$ to the surface $z=f(x,y)$.

## III Chain rule

1° $z = z(t) = f(x(t), y(t))$

$$\Downarrow$$

$$\frac{dz}{dt} = f_x \cdot x' + f_y \cdot y' = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

2° $z(s,t) = f(x(s,t), y(s,t))$

$$\Downarrow$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

## IV Taylor approximation

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k + \\ + \frac{f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2}{2} + \dots$$

## V Extreme value problems of $z = f(x,y)$

1° $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Longrightarrow (x_0, y_0)$ critical point and candidate for extremum

2° $D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

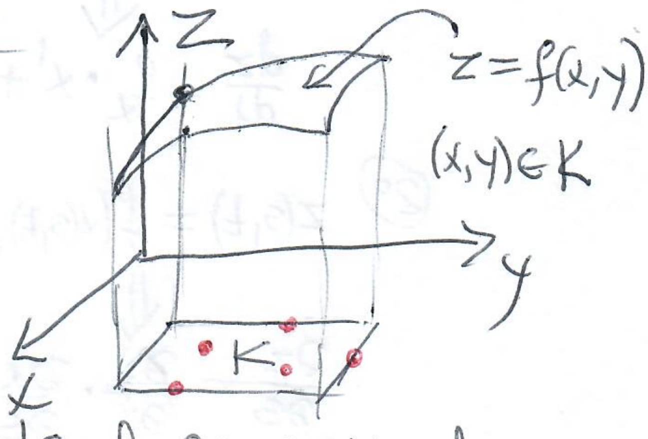
$f_{xx} > 0$ and $D > 0$ local minimum at $(x_0, y_0)$

$f_{xx} < 0$ and $D > 0$ local maximum at $(x_0, y_0)$

$D < 0$ no extremum i.e. $(x_0, y_0)$ is a saddle point

$D = 0$ no conclusion

# VI Absolute extrema of a continuous function $z=f(x,y)$ on a closed and bounded set $K$




1°. Find critical points of $f(x,y)$ that lie in the interior of $K$.

2°. Find all boundary points at which the absolute extrema can occur.

3°. Evaluate $f(x,y)$ at the points obtained in 1° and 2°.

The largest of these values is the absolute maximum and the smallest the absolute minimum

this follows from the Weierstrass theorem.

# D Ordinary differential equations

## I Linear differential equations of the first order

$$\boxed{(1)\quad y' + p(x)y = r(x), \quad x \in I}$$

a) Method of variation of constant

1°. Homogeneous equation

$$(1_H)\quad y' + p(x)y = 0 \Rightarrow y = Ce^{-P(x)},$$ where $P(x) = \int p(x)dx$

2° Variation of constant

$$y = C(x)e^{-P(x)}$$ and put to (1) to get function $C(x)$.

b) Method of integrating factor

Multiply (1) by $e^{P(x)}$

$$\underbrace{y'e^{P(x)} + p(x)ye^{P(x)}} = r(x)e^{P(x)}$$

$$(ye^{P(x)})' = r(x)e^{P(x)} \Rightarrow$$ we can get $y(x)$.

## II Linear differential equations of the second order

$$\boxed{(2)\quad y'' + p(x)y' + q(x)y = r(x), \quad x \in I}$$

THM1 All solutions of (2) are

$$y(x) = y_H(x) + y_p(x),$$

where $y_H$ are all solutions of homogeneous equation

$$(2_H)\quad y'' + p(x)y' + q(x)y = 0$$

and $y_p$ is a particular solution of (2).

## III Homogeneous case

$$(2_H) \quad y'' + p(x)y' + q(x)y = 0, \ x \in I$$

1° solutions $y_1, y_2$ of $(2_H)$ are linearly independent if

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0 \text{ at some } x \in I$$

$$\Downarrow$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall c_1, c_2 \in \mathbb{R}$$

are all solutions of $(2_H)$.

2° If $y_1$ is solution of $(2_H)$, then we can get second solution $y_2$ by variation of constant

$$y_2(x) = C(x) y_1(x)$$

such that $y_1, y_2$ are linearly independent.

## IV Homogeneous case with constant coefficients

$$(3_H) \quad \boxed{y'' + py' + qy = 0}, \text{ where } p, q \in \mathbb{R} \quad \Big\{ y = e^{\lambda x}$$

$$\boxed{\lambda^2 + p\lambda + q = 0} \quad \text{characteristic equation}$$

1° $\lambda_1 \neq \lambda_2$ real

$$y_H = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

2° $\lambda_1 = \lambda_2 = \lambda$ real with multiplicity 2

$$y_H = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

3° $\lambda_{1/2} = \alpha \pm \beta i, \ \beta \neq 0$

$$y_H = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \sin(\beta x)\right]$$

## V Nonhomogeneous case (particular solution)

$$(2) \quad y'' + p(x)y' + q(x)y = r(x), \quad x \in I$$

| Equation (2) | Initial GUESS for $y_p$ |
|---|---|
| $r(x) = ke^{\alpha x}$ | $y_p(x) = Ae^{\alpha x}$ |
| $r(x) = a_0 + a_1 x + \dots + a_n x^n$ | $y_p(x) = A_0 + A_1 x + \dots + A_n x^n$ |
| $r(x) = a_1 \cos bx + a_2 \sin bx$ | $y_p(x) = A_1 \cos bx + A_2 \sin bx$ |

## VI Euler differential equation

$$(3) \quad x^2 y'' + axy' + by = r(x), \quad x > 0$$

1° Homogeneous case

$$(3_H) \quad x^2 y'' + axy' + by = 0 \qquad \} \; y = x^\lambda$$

$$\boxed{\lambda(\lambda - 1) + a\lambda + b = 0}$$ characteristic equation

1° $\lambda_1 \neq \lambda_2$ real

$$y_H = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$$

2° $\lambda_1 = \lambda_2 = \lambda$ real

$$y_H = C_1 x^\lambda + C_2 x^\lambda \ln x$$

3° $\lambda_{1,2} = \alpha \pm \beta i, \; \beta \neq 0$

$$y_H = x^\alpha \left[ C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x) \right]$$

Particular solution $y_p$ by guess or variation of constants.

# VII Linear differential equations of higher order

(4) $y''' + p_2 y'' + p_1 y' + p_0 y = r(x)$, $x \in I$

$p_0, p_1, p_2 \in \mathbb{R}$

1° Homogeneous case

$(4_H)$ $y''' + p_2 y'' + p_1 y' + p_0 y = 0$

$$\boxed{\lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 = 0}$$ characteristic equation

Higher order d.e. we do similarly.

Particular solution $y_p$ of (4) we can get by guess.