Sofya Aksenyuk, 150284.

IT – Lab 2

Introduction to measures of information

I. (Logarithms) Prove the following properties of logarithms:

a) $\log_b a \cdot \log_a b = 1$ **b)** $(\log_b a)^{-1} = \log_a b$ **c)** $\frac{\log_p a}{\log_p b} = \log_b a$

a) $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \Rightarrow \log_b a \cdot \log_a b = \frac{1}{\log_a b} \cdot \log_a b = 1.$

b) $(\log_b a)^{-1} = \left(\frac{\log_a a}{\log_a b}\right)^{-1} = \left(\frac{1}{\log_a b}\right)^{-1} = \log_a b.$

c) Let $\log_b a = x \Rightarrow a = b^x$:

$\log_p a = \log_p b \cdot x \Rightarrow x = \frac{\log_p a}{\log_p b} \Rightarrow \frac{\log_p a}{\log_p b} = \log_b a.$

b) Compute a product of the following numbers: 1024, 1/512, 16, 8, 2, $\frac{1}{2048}$, but do not use for this multiplication. Historically, the main use of logarithms (values of which were precomputed in logarithm tables) was avoiding costly and error-prone multiplications before effective calculating machines.

$\log_2 1024 + \log_2 \frac{1}{512} + \log_2 16 + \log_2 8 + \log_2 2 + \log_2 \frac{1}{2048} = \log_2 x.$ $x - ?$

$\log_2\left(1024 \cdot \frac{1}{512} \cdot 16 \cdot 8 \cdot 2 \cdot \frac{1}{2048}\right) = \log_2 x \Rightarrow x = \frac{1}{4} = 0.25.$

II. (Hartley information) Let's say that we have a message source with m symbols which generates sequences of length n. Fill the equations.

$L(m, n) = n \cdot \log_p m.$

$L(m, k \cdot n) = k \cdot L(m,n) = k \cdot n \cdot \log_p m.$

$L(k \cdot m, n) = L(k,n) + L(m,n) = n \cdot \log_p (k \cdot m).$

III. (Shannon information)

a) Fill the missing parts:

$H([p_1, p_2, \ldots, p_n]) = -\sum_i p_i \cdot \log_p p_i = \sum_i p_i \cdot \log_p \frac{1}{p_i}.$

$H([1/n, 1/n, \ldots, 1/n]) = -\sum_n \frac{1}{n} \cdot \log_p \frac{1}{n} = \sum_n \frac{1}{n} \log_p n = \log_p n.$

$H([0, 1]) = 0 \cdot \log_p 0 + 1 \cdot \log_p 1 = \log_p 1 = 0.$

$H([1/2, 1/2]) = \log_p 2.$

$H([1/4, 1/4, 1/4, 1/4]) = \log_p 4.$

$H([1/2, 1/4, 1/4]) = \frac{1}{2} \cdot \log_p 2 + \frac{1}{2} \log_p 4 + \frac{1}{2} \log_p 4 = \frac{1}{2}(\log_p 2 + \log_p 4) = \frac{1}{2} \log_p 8.$

b) What is the entropy (in bits) of a message composed of two independent parts: a random number from $[0, 7]$ and a result of a coin flip.

$S = [\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}]$

$H(S) + H([\frac{1}{2}, \frac{1}{2}]) = \log_2 8 + \log_2 2 = 4.$

c) Use vector subdivision property of entropy to show the result of subdiving the entropy of a 6-sided fair dice with values $\{1, 2, 3, 4, 5, 6\}\}$ into subsets $\{1, 2\}$ and $\{3, 4, 5, 6\}$.

$S = [\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$

$H(S) = \log 6 = x.$

$H([\frac{1}{3}, \frac{2}{3}]) + \frac{1}{3} H(3[\frac{1}{6}, \frac{1}{6}]) + \frac{2}{3} H(\frac{3}{2}[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]) = \frac{1}{3}(\log 3 + 2 \log \frac{3}{2} +$

$+ H([\frac{1}{2}, \frac{1}{2}])) + 2H([\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]) = \frac{1}{3}(\log 3 + 2 \log \frac{3}{2} + \log 2 + 2 \log 4) = \frac{1}{3} \log (3 \cdot \frac{9}{4} \cdot 2 \cdot 4 \cdot 4)$

$= \frac{1}{3} \log (3^3 \cdot 2^3) = \log (3 \cdot 2) = \log 6 = x.$

d) We have 10 bits of information. How many trits (a unit of information for the logarithm of base 3) is it?

$2^{10} = 3^x$

$x = \log_3 2^{10} = 10 \log_3 2 = 10 \cdot 0.63 = 6.3.$

Answer: 7 trits.

IV. Write **'T'** when a sentence is true, and **'F'** when it is false.

Shannon information can be greater than Hartley information	T
Hartley information can be greater than Shannon information	T
1 dit > 1 nat > 1 bit	T
A text file taking 1024 bits in memory has 1024 bits of Shannon information	F
The task of compression is to decrease the entropy of messages	F
Sending a message always increases the amount of information the recipient has (we ignore the possibility of a message being malware)	F