Homework-3

MA423 : Matrix Computations 2023 R. Alam

Linear Systems

1. Suppose that $A \in \mathbb{C}^{n\times n}$ is given in the block form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with $A_{11} \in \mathbb{C}^{m\times m}$ being nonsingular. Then verify the block elimination formula

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

which eliminates the block A_{21}. The matrix $S(A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the **Schur complement** of A_{11} in A.

2. **Assignment:** Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ be such that $A + uv^\top$ is nonsingular. **10 marks**

   (a) **Sherman-Morrison formula:** Show that $1 + v^\top A^{-1}u \neq 0$ and that

$$(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}uv^\top A^{-1}}{1 + v^\top A^{-1}u}.$$

   More generally, let $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{n\times m}$ (with $m \leq n$) be such that $A + UV^\top$ is nonsingular. Let $\Sigma := (I + V^\top A^{-1}U)$. Then show that Σ is nonsingular and that

$$(A + UV^\top)^{-1} = A^{-1} - A^{-1}U\Sigma^{-1}V^\top A^{-1}.$$

   (b) Suppose that LU factorization $A = LU$ is given so that solving $Ax = b$ for any $b \in \mathbb{R}^n$ costs just $\mathcal{O}(n^2)$ flops. Use Sherman-Morrison formula and describe an efficient algorithm (outline only the steps) for solving $(A + uv^\top)x = b$ and determine the total flop count of the algorithm.

   Submission date: 5th September 2023 (Tuesday)
   Time: 8:00 PM

3. Let $A \in \mathbb{C}^{n\times n}$ be banded with bandwidth $2p+1$, that is, $a_{ij} = 0$ for $|i - j| > p$. Suppose that a factorization $PA = LU$ is computed by Gaussian elimination with partial pivoting (GEPP). What can you say about the band patterns of L and U? What can you say about the cost of computing L and U?

4. Let $A \in \mathbb{R}^{n\times n}$. Show that $\text{rank}(A) = r$ if and only if at the $(r+1)$-th step of Gaussian elimination with complete pivoting (GECP), the largest entry found in the submatrix $A(r+1:n, r+1:n)$ is zero (in exact arithmetic).

5. Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular. Describe Gaussian elimination based efficient algorithms for solving the following problems.

   (a) Compute $c^T A^{-1}b$, for $c, b \in \mathbb{R}^n$.

(b) Solve $A^m x = b$, where $b \in \mathbb{R}^n$ and m is a natural number.
[**Hint:** Use LU factorization of A]

You should describe your algorithm (outline only the steps) and determine the flop count.

6. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Partition A as 2-by-2 block matrix as given below and use the fact that

$$A = \begin{bmatrix} a_{11} & w^T \\ w & B \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ w/\sqrt{a_{11}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B - ww^\top/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & w^\top/\sqrt{a_{11}} \\ 0 & I \end{bmatrix}$$

to prove the existence of Cholesky factorization of A, where $w \in \mathbb{R}^{n-1}$ and $B \in \mathbb{R}^{(n-1)\times(n-1)}$. Also show that $|a_{ij}| < \sqrt{a_{ii}a_{jj}}$ for all $i \neq j$. What does this statement say about $\max_{ij} |a_{ij}|$?

7. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Partition A as 2-by-2 block matrix

$$A = \begin{bmatrix} A_{n-1} & b \\ b^\top & a_{nn} \end{bmatrix}, \text{ where } A_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)} \text{ and } b \in \mathbb{R}^{n-1},$$

and prove the existence and uniqueness of Cholesky factorization of A.

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