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MA-473: COMPUTATIONAL FINANCE

ENDSEM → PAPER-II

(4) PDE modelling Asian option with Arithmetic Avg. is :

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0}$$

The Payoff for Arithmetic Avg Strike call Asian option is given by.

$$\text{Payoff} = \left(S_T - \frac{1}{T}A_T\right)^+ \quad \text{where } A_t = \int_0^t S_\sigma \, d\sigma$$

$$= S_T\left(1 - \frac{1}{TS_T}\int_0^T S_\sigma \, d\sigma\right)$$

Now Define R_t as follows

$$\boxed{R_t = \frac{1}{S_t}\int_0^t S_\sigma \, d\sigma} \Rightarrow \boxed{R = \frac{A}{S}} \text{ (as given)}$$

Then $$\boxed{\text{Payoff} = S_T\left(1 - \frac{R_T}{T}\right)^+}$$

Now Using the transformation that is given

$$V(S, A, t) = S \cdot H(R, t) \quad \text{with } R = A/S$$

we have following

$$\boxed{\frac{\partial R}{\partial A} = \frac{1}{S}} \quad \text{and} \quad \boxed{\frac{\partial R}{\partial S} = -\frac{A}{S^2}}$$

Now

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left(S \cdot H(R,t)\right) = S\frac{\partial H}{\partial t}$$

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left(S \cdot H(R,t)\right) = H + S \cdot \frac{\partial H}{\partial S}$$

$$= H + S \cdot \frac{\partial H}{\partial R} \cdot \frac{\partial R}{\partial S}$$

$$= H + S \cdot \left(\frac{-A}{S^2}\right) \cdot \frac{\partial H}{\partial R} \qquad \left\{ \text{Using } \frac{\partial R}{\partial S} = \frac{-A}{S^2} \right\}$$

$$\boxed{\frac{\partial V}{\partial S} = H - R\frac{\partial H}{\partial R}} \qquad \left\{ \text{Using } R = A/S \right\}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(H - R\frac{\partial H}{\partial R}\right) = \frac{\partial}{\partial R}\left(H - R\frac{\partial H}{\partial R}\right) \cdot \frac{\partial R}{\partial S}$$

$$= -\frac{A}{S^2}\left\{\frac{\partial H}{\partial R} - \frac{\partial}{\partial R}\left(R \cdot \frac{\partial H}{\partial R}\right)\right\} \qquad \left(\text{Using } \frac{\partial R}{\partial S} = -A/S^2\right)$$

$$= -\frac{A}{S^2}\left\{\frac{\partial H}{\partial R} - \frac{\partial H}{\partial R} - R\frac{\partial^2 H}{\partial R^2}\right\}$$

$$= \frac{AR}{S^2}\frac{\partial^2 H}{\partial R^2} \Rightarrow \boxed{\frac{\partial^2 V}{\partial S^2} = \frac{R^2}{S}\frac{\partial^2 H}{\partial R^2}} \qquad \left(\text{Using } A = RS\right)$$

$$\frac{\partial V}{\partial A} = \frac{\partial}{\partial A}\left(S \cdot H\right) = S \cdot \frac{\partial H}{\partial A} = S \cdot \frac{\partial H}{\partial R} \cdot \frac{\partial R}{\partial A}$$

$$\rightarrow \boxed{\frac{\partial V}{\partial A} = \frac{\partial H}{\partial R}} \qquad \left(\text{Using } \frac{\partial R}{\partial A} = \frac{1}{S}\right)$$

Now substituting all the values in PDE, we get

$$S\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{R^2}{S}\frac{\partial^2 H}{\partial R^2} + S\left(r\left(H - R\frac{\partial H}{\partial R}\right) + \frac{\partial H}{\partial R}\right) - rSH = 0$$

$$\Rightarrow \frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + rH - rR\frac{\partial H}{\partial R} + \frac{\partial H}{\partial R} - rH = 0$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial t} + \frac{1}{2}\sigma R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0} \rightarrow (\#)$$

Which is the required transformed PDE.

Now for $R \to \infty$, we have from Payoff that

$$H(R_T, T) = \left(1 - \frac{1}{T}R_T\right)^+$$

$$\Rightarrow \boxed{H(R_T, T) = 0 \quad \text{for } R_T \to \infty}$$

Now for PDE (#), when $R \to 0$, we have.

$$\boxed{\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} = 0} \rightarrow (*)$$

<u>Claim</u>: If H is bounded, then $R^2\frac{\partial^2 H}{\partial R^2} \to 0$ as $R \to 0$

<u>Proof</u>: Assuming $R^2\frac{\partial^2 H}{\partial R^2}$ is non-zero, we have.

$$\frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right)$$

Integrating this twice, we get:

$$H = O(\log R) + C_1 R + C_2.$$

$\Rightarrow$ H is not bounded for $R \to 0$

Hence this contradicts our assumption that H is bounded.

$\Rightarrow$ H is bounded

Hence we have $\boxed{R^2 \frac{\partial^2 H}{\partial R^2} \to 0 \quad \text{as } R \to 0.}$

Using this in (*), we have the boundary condition at left boundary as follows:

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \quad \text{for } R \to 0}$$

Hence final transformed PDE is as follows.

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - \gamma R)\frac{\partial H}{\partial R} = 0$$

$$H(R_T, T) = \left(1 - \frac{R_T}{T}\right)^+$$

$$H = 0 \quad \text{for } R \to \infty$$

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \quad \text{for } R = 0$$

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5 Strong-order of convergence

A discretization $\overset{n}{\hat{X}}$ has strong order of convergence $\beta > 0$ if

$$\boxed{E\left[\|\hat{X}(nh) - X(T)\|\right] \le c h^{\beta}}$$

for some constant c and sufficiently small h.

A discretization scheme has weak order of convergence β if

$$\boxed{\left|E[f(\hat{X}(nh))] - E[f(X(T))]\right| \le c h^{\beta}}$$

for some constant c and all sufficiently small h. and $\forall f \in C_p^{2\beta+2}$. where c may depend on f.

$C_p^{2\beta+2}$ consists of functions from $\mathbb{R}^d$ to $\mathbb{R}$ whose derivatives of order $0, 1, \ldots, 2\beta+2$ are polynomially bounded. A function $g: \mathbb{R}^d \to \mathbb{R}$ is polynomially bounded if

$$\boxed{|g(x)| \le K(1 + \|x\|^q)}$$

for some constants K and q and $\forall x \in \mathbb{R}^d$.

6. Given SDE

$$\begin{cases} dX(t) = a(X(t))\,dt + b(X(t))\,dW(t) \\ X(0) = X_0 \end{cases}$$

We define the following operators:

$$\mathcal{L}^0 = a\frac{d}{dx} + \frac{1}{2}b^2\frac{d^2}{dx^2}$$

(For functions that depend explicitly on time 't', we define $\mathcal{L}^0$ as follows:
$$\mathcal{L}^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2}$$
)

$$\mathcal{L}^1 = b\frac{d}{dx}$$

Hence for any twice differentiable function 'f', we have.

$$\mathcal{L}^0 f(x) = a(x)f'(x) + \frac{1}{2}b^2(x)f''(x)$$

and
$$\mathcal{L}^1 f(x) = b(x)f'(x)$$

Now approximating the evolution of X over an interval $[t, t+h]$, we have

$$\boxed{X(t+h) = X(t) + \int_t^{t+h} a(X(u))\,du + \int_t^{t+h} b(X(u))\,dW(u)} \quad -(1)$$

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Now using Ito's formula on $a(X(u))$, we have

$$\boxed{a(X(u)) = a(X(t)) + \int_t^u \mathcal{L}^0 a(X(s))\,ds + \int_t^u \mathcal{L}^1 a(X(s))\,dW(s)} \quad -(2)$$

Now using Euler approximation to each of the two integrals in above expression, we have

$$\left.\begin{aligned} \mathcal{L}^0 a(X(s)) &\approx \mathcal{L}^0 a(X(t)) \\ \text{and} \quad \mathcal{L}^1 a(X(s)) &\approx \mathcal{L}^1 a(X(t)) \end{aligned}\right\} \text{for } s \in [t,u]$$

Now putting these approximations in (2), we get

$$\boxed{a(X(u)) \approx a(X(t)) + \mathcal{L}^0 a(X(t)) \int_t^u ds + \mathcal{L}^1 a(X(t)) \int_t^u dW(s)} \quad -(3)$$

Now putting this approximation in (1) we get approximation of first integral

$$\int_t^{t+h} a(X(u))\,du \approx a(X(t))\,h + \mathcal{L}^0 a(X(t)) \int_t^{t+h}\int_t^u ds\,du + \mathcal{L}^1 a(X(t)) \int_t^{t+h}\int_t^u dW(s)\,du$$

$$\boxed{\int_t^{t+h} a(X(u))\,du \cong a(X(t))\,h + \mathcal{L}^0 a(X(t))\, I_{(0,0)} + \mathcal{L}^1 a(X(t))\, I_{(1,0)}} \quad -(4)$$

where $I_{(0,0)} = \int_t^{t+h}\int_t^u ds\,du$ and $I_{(1,0)} = \int_t^{t+h}\int_t^u dW(s)\,du$

are double integrals

Now for approximating the second integral in (2),
we have for approximation of integrand $b(X(u))$
for $u \in [t, t+h]$ as follows:

$$b(X(u)) = b(X(t)) + \int_t^u \mathcal{L}^0 b(X(s))\,ds + \int_t^u \mathcal{L}^1 b(X(s))\,dW(s)$$

$$\boxed{b(X(u)) \approx b(X(t)) + \mathcal{L}^0 b(X(t)) \int_t^u ds + \mathcal{L}^1 b(X(t)) \int_t^u dW(s)} \quad -(5)$$

where Euler approximation is used for approximating the two integrands involved i.e.

$\mathcal{L}^0 b(X(s)) \approx \mathcal{L}^0 b(X(t))$ and $\mathcal{L}^1 b(X(s)) \approx \mathcal{L}^1 b(X(t))$
for $s \in [t, u]$.

Then the second integral in (1) is approximated as:

$$\int_t^{t+h} b(X(u))\,dW(u) \approx b(X(t))[W(t+h) - W(t)]$$
$$+ \mathcal{L}^0 b(X(t)) \int_t^{t+h}\int_t^u ds\,dW(u) + \mathcal{L}^1 b(X(t)) \int_t^{t+h}\int_t^u dW(s)\,dW(u)$$

$$\Rightarrow \boxed{\int_t^{t+h} b(X(u))\,dW(u) \equiv b(X(t))[W(t+h) - W(t)] + \mathcal{L}^0 b(X(t))\, I_{(0,1)} + \mathcal{L}^1 b(X(t))\, I_{(1,1)}} \quad -(6)$$

where $I_{(0,1)} = \int_t^{t+h}\int_t^u ds\, dW(u)$ and $I_{(1,1)} = \int_t^{t+h}\int_t^u dW(s)\, dW(u)$

Now Calculating the double integrals involved as follows:

$$* \quad I_{(0,0)} = \int_t^{t+h}\int_t^u ds\, du = \int_t^{t+h} (u-t)du = \frac{u^2}{2}\Big|_t^{t+h} - ht$$

$$\Rightarrow \quad \boxed{I_{(0,0)} = \frac{h^2}{2}}$$

$$* \quad I_{(1,1)} = \int_t^{t+h} [W(u) - W(t)]dW(u) = \int_t^{t+h} W(u)dW(u) - W(t)\int_t^{t+h} dW(u)$$

$$= \frac{1}{2}\left[(\Delta W)^2 - h\right] \quad \text{where } \Delta W = W(t+h) - W(t)$$

{ Here taking $f(x) = x^2$ and using Ito's formula, we have

$$W^2(t+h) - W^2(t) = 2\int_t^{t+h} W(u)dW(u) + \frac{1}{2}\cdot 2\int_t^{t+h} dt$$

$$\Rightarrow \int_t^{t+h} W(u)\, dW(u) = \frac{1}{2}\left[W^2(t+h) - W^2(t) - h\right]$$

Substituting this and rearranging we will get req. form. }

$$* \quad I_{(0,1)} = \int_t^{t+h}\int_t^u ds\, dW(u) = \int_t^{t+h} (u-t)\, dW(u)$$

$$= (u-t)W(u)\Big|_t^{t+h} - \int_t^{t+h} 1\cdot \int dW(u)$$

{ Using Integrating by parts or applying Ito's lemma to $tW(t)$ }

$$= hW(t+h) - \int_t^{t+h} W(u)\,du$$

$$= h\left[W(t+h) - W(t)\right] - \int_t^{t+h} \left[W(u) - W(t)\right] du$$

$$\Rightarrow \boxed{I_{(0,1)} = h\,\Delta W - I_{(1,0)}}$$

$$* \quad I_{(1,0)} = \int_t^{t+h} \left[W(u) - W(t)\right] du$$

Now given $W(t)$, $I_{(1,0)}$ and increment $\Delta W = W(t+h) - W(t)$ are jointly Normal.

Now conditional mean of both ΔW and $I_{(1,0)}$ is 0.

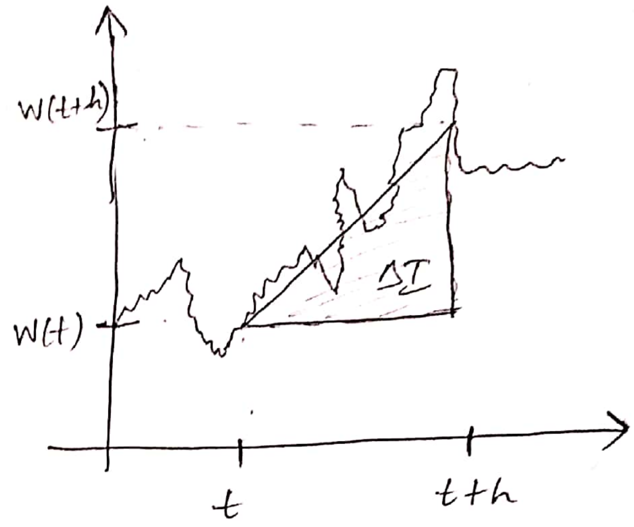
Conditional variance of $\Delta W = W(t+h) - W(t)$ is $t+h-t = h$.

conditional variance of $I_{(1,0)} = h^3/2$ $\left\{ \begin{array}{l} \text{bcoz} \\ W(t+h) - W(t) \sim N(0,h) \end{array} \right\}$

$$\left\{ \begin{array}{l} \text{bcoz} \quad Var\left[\int_0^T W(u)\,du\right] = 2\int_0^T \int_0^s Cov\left[W(u), W(s)\right] du\,ds \\ \qquad = 2\int_0^T \int_0^s u\,du\,ds = \int_0^T s^2\,ds = T^3/3 \end{array} \right\}$$

Now we have that

$$E\left[I_{(1,0)} \mid W(t), \Delta W\right] = \frac{h}{2}\Delta W$$



In the plot, shaded area is ΔI
Hence given $W(t)$ and $W(t+h)$, the conditional expectation of W at any intermediate time lies on straight line connecting these end-points.
$\Rightarrow$ conditional expectation of ΔI = Area of triangle $= \frac{1}{2} h \Delta W$

So, we have $E\left[I_{(1,0)} \Delta W\right] = h^2/2$

Hence we can simulate $\Delta W = W(t+h) - W(t)$ and $I_{(1,0)}$ as

$$\boxed{\begin{pmatrix} \Delta W \\ \Delta I \end{pmatrix} \sim N\left(0, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}\right)}$$

Now expanding $\mathcal{L}^0$ and $\mathcal{L}^1$ explicitly we have.

$\mathcal{L}^0 a = aa' + \frac{1}{2}b^2 a''$; $\mathcal{L}^1 a = ba'$

$\mathcal{L}^0 b = ab' + \frac{1}{2}b^2 b''$; $\mathcal{L}^1 b = bb'$

Then putting in (4), we have

$$\boxed{\int_t^{t+h} a(X(u))\,du \equiv ah + \left(aa' + \frac{1}{2}b^2 a''\right) I_{(0,0)} + ba'\left(I_{(1,0)}\right)}$$

Putting in (6), we have

$$\boxed{\int_t^{t+h} b(X(u))\,dW(u) \equiv b\Delta W + \left(ab' + \frac{1}{2}b^2 b''\right) I_{(0,1)} + bb' I_{(1,1)}}$$

Then putting in Eq^n (1), we have

$$\boxed{X(t+h) \approx X(t) + ah + b\Delta W + \left(aa' + \frac{1}{2}b^2 a''\right) I_{(0,0)} + \left(ab' + \frac{1}{2}b^2 b''\right) I_{(0,1)} + ba' I_{(1,0)} + bb' I_{(1,1)}}$$

with function a, b and their derivatives being evaluated at $X(t)$.

Finally using approximation of all double-integrals in above approximation, the 2nd order scheme is as follows:

$$\boxed{\hat{X}((i+1)h) = \hat{X}(ih) + ah + b\Delta W + \left(ab' + \frac{1}{2}b^2 b''\right)(\Delta W h - \Delta I) + ab'\Delta I + \frac{1}{2}bb'\left[\Delta W^2 - h\right] + \left(aa' + \frac{1}{2}b^2 a''\right)\frac{h^2}{2}}$$

with function a, b and their derivatives being evaluated at $\hat{X}(ih)$. This is called Milstein second order scheme.