MA668: Algorithmic and High Frequency Trading
Lecture 29

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Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Recalling that $Y_t^\nu = \mathfrak{R} - Q_t^\nu$, it is straightforward to obtain the optimal inventory path as:

$$Q_t^{\nu^*} = \frac{t}{T + \frac{k}{\alpha}} \mathfrak{R}. \tag{1}$$

2. From this equation we can see that for any finite $\alpha > 0$ and finite $k > 0$, it is always optimal to leave some shares to be executed at the terminal date and the fraction of shares left to execute at the end decreases with the relative price impact at the terminal date, namely, $\dfrac{k}{\alpha}$.

3. To obtain the optimal speed of acquisition, we substitute for $Q_t^{\nu^*}$, the expression for ν_t^*, so that:

$$\nu_t^* = \frac{\mathfrak{R}}{T + \frac{k}{\alpha}}. \tag{2}$$

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Comparing this with the result from the previous problem, we see that the agent acquires at a constant, but slower rate than that of an agent who heavily penalizes ($\alpha \to \infty$) paths which do not complete the execution fully.
2. Moreover, the agent trades at a constant speed and this speed is the same as that of an agent who must execute everything by the end of the period, but who has a terminal date T' that is further into the future, and is given by, $T' = T + \dfrac{k}{\alpha}$.

Liquidation With Permanent Price Impact

1. We now switch from acquisition back to liquidation.
2. The agent continues to use only MOs to liquidate a total of $\mathfrak{N}$ shares, but now her/his trades have both a temporary and a permanent price impact.
3. The mid-price dynamics is given by:

$$dS_t^\nu = \pm g(\nu_t)dt + \sigma dW_t, \; S_0^\nu = S,$$

   with drift $g(\nu_t) > 0$, which enters the equation with negative sign because the agent's sell trades exert a permanent downward pressure.
4. The execution price is given by:

$$\widehat{S}_t^\nu = S_t^\nu \pm \left(\frac{\Delta}{2} + f(\nu_t)\right), \; \widehat{S}_0^\nu = \widehat{S},$$

   with $f(\nu_t) > 0$, which enters the equation with a negative sign because the sell trades have an adverse temporary impact.

Liquidation With Permanent Price Impact (Contd ...)

1. Here we assume that if the agent's strategy reaches the terminal date T with inventory left, then she/he must execute an MO to reach $\mathfrak{R}$ for a total revenue of $Q_T^\nu\,(S_T^\nu - \alpha Q_T^\nu)$, where $\alpha \geq 0$ is the terminal liquidation penalty parameter.
2. The agent's objective is to minimize the execution cost:

$$EC^\nu = \mathfrak{R} S_0 - \mathbb{E}\left[X_T^\nu + Q_T^\nu\,(S_T^\nu - \alpha Q_T^\nu)\right].$$

3. The process corresponding to the investor's wealth X_t^ν is given by:

$$dX_t^\nu = \widehat{S}_t^\nu\,\nu_t dt,\ X_0^\nu = x.$$

4. Here we have switched from writing out the cash process explicitly in terms of the integrated execution costs, to including the cash process directly.
5. This way the cash process becomes a state variable.

Liquidation With Permanent Price Impact (Contd ...)

1. Naturally, we could, in principle keep using the integrated costs representation.
2. However, it is sometimes easier to motivate the choice of ansatz for the forthcoming problems when the value functions are written in terms of X as a state variable.
3. We also introduce another element into the model, by way of a running inventory penalty of the form $\phi \int_t^T (Q_u^\nu)^2 \, du$, with $\phi \geq 0$.
4. This running inventory penalty is not (and should not be considered) a financial cost to the agent's strategy.
5. The parameter ϕ allows us to incorporate the agent's urgency for executing the trade.
6. The higher the value of ϕ, the quicker the optimal strategy liquidates the shares, as it increases the penalty for the late liquidation of shares and incentivises strategies that front load the liquidation of inventory.

Liquidation With Permanent Price Impact (Contd ...)

1. The agent's performance criterion is:

$$H^{\nu}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T^{\nu} + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right) - \phi\int_t^T \left(Q_u^{\nu}\right)^2 du\right]. \quad (3)$$

2. Accordingly, the value function is:

$$H(t,x,S,q) = \sup_{\nu\in\mathcal{A}} H^{\nu}(t,x,S,q).$$

3. The DPP implies that the value function should satisfy the HJB equation:

$$\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 \\ &+ \sup_{\nu}\left[\left(\nu\left(S - f(\nu)\right)\partial_x - g(\nu)\partial_S - \nu\partial_q\right)H\right], \end{aligned} \quad (4)$$

with the terminal condition being $H(T,x,S,q) = x + Sq - \alpha q^2$.

Liquidation With Permanent Price Impact (Contd ...)

1. We use the simplifying assumption that permanent and temporary price impact functions are linear in the speed of trading, that is, $f(\nu) = k\nu$ and $g(\nu) = b\nu$, for finite constants $k \geq 0$ and $b \geq 0$.
2. The first order condition allows us to obtain the optimal speed of trading in feedback control form as:

$$\nu^* = \frac{1}{2k} \frac{\left(S\partial_x - b\partial_S - \partial_q\right) H}{\partial_x H}. \tag{5}$$

   Substituting the optimal feedback control into the DPE leads to:

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS}\right) H - \phi q^2 + \frac{1}{4k} \frac{\left[\left(S\partial_x - b\partial_S - \partial_q\right) H\right]^2}{\partial_x H}.$$

Liquidation With Permanent Price Impact (Contd ...)

1. By inspecting the terminal condition $H(T, x, S, q) = x + Sq - \alpha q^2$, we consider the ansatz:

$$H(t, x, S, q) = x + Sq + h(t, S, q), \tag{6}$$

   where $h(t, S, q)$ with the terminal condition $h(T, S, q) = -\alpha q^2$, is yet to be determined.

2. The first term of the ansatz is the accumulated cash of the strategy.
3. The second is the marked-to-market book value (at mid-price) of the remaining inventory.
4. Finally, the third term is the extra value stemming from optimally liquidating the rest of the shares.

Liquidation With Permanent Price Impact (Contd ...)

1. Using this ansatz in the equation above and simplifying, we find the following non-linear PDE for h:

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS}\right) h - \phi q^2 + \frac{1}{4k}\left[b(q + \partial_S h) + \partial_q h\right]^2 .$$

2. Since the above PDE contains no explicit dependence on S and the terminal condition is independent of S, it follows that $\partial_S h(t, S, q) = 0$, and we can write $h(t, S, q) = h(t, q)$ (with a slight flexibility in notation).
3. The equation then simplifies even further to:

$$0 = \partial_t h(t, q) - \phi q^2 + \frac{1}{4k}\left[bq + \partial_q h(t, q)\right]^2 .$$

4. Furthermore, the optimal control in feedback form from (5) takes on the much more compact form:

$$\nu^* = -\frac{1}{2k}\left(\partial_q h(t, q) + bq\right) . \tag{7}$$

Liquidation With Permanent Price Impact (Contd ...)

1. In this form, it appears that the solution admits a separation of variables $h(t,q) = h_2(t)q^2$ where $h_2(t)$ satisfies the non-linear ODE (recall that the subscript 2 represents that this function is the coefficient of q^2):

$$0 = \partial_t h_2 - \phi + \frac{1}{k}\left[h_2 + \frac{b}{2}\right]^2, \tag{8}$$

   subject to the terminal condition $h_2(T) = -\alpha$.

2. This ODE is of Riccati type and can be integrated exactly.
3. First, let $h_2(t) = -\frac{1}{2}b + \chi(t)$.
4. Then re-arranging the ODE, we obtain:

$$\frac{\partial_t \chi}{k\phi - \chi^2} = \frac{1}{k},$$

   subject to $\chi(T) = \frac{b}{2} - \alpha$.

Liquidation With Permanent Price Impact (Contd ...)

1. Next, integrating both sides of the above, over $[t, T]$ yields:

$$\log \frac{\sqrt{k\phi} + \chi(T)}{\sqrt{k\phi} - \chi(T)} - \log \frac{\sqrt{k\phi} + \chi(t)}{\sqrt{k\phi} - \chi(t)} = 2\gamma(T - t).$$

2. Therefore:

$$\chi(t) = \sqrt{k\phi} \left(\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}} \right),$$

where $\gamma = \sqrt{\frac{\phi}{k}}$ and $\zeta = \frac{\alpha - \frac{b}{2} + \sqrt{k\phi}}{\alpha - \frac{b}{2} - \sqrt{k\phi}}$.

3. At this point the solution of the DPE is fully determined and the optimal speed of trading can now be explicitly shown in terms of the state variables rather than in feedback form.

4. Specifically, from (7), the optimal speed to trade at is given by:

$$\nu_t^* = \gamma \left(\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} \right) Q_t^{\nu^*}. \tag{9}$$

Liquidation With Permanent Price Impact (Contd ...)

1. Interestingly, the optimal speed to trade is still proportional to the investor's current inventory level, as we found in the previous simpler models, but now the proportionality factor depends non-linearly on time.
2. From this expression, it is also possible to obtain the agent's inventory $Q_t^{\nu^*}$ that results from following this strategy.
3. Recall that the agent's inventory satisfies $dQ_t^{\nu} = -\nu_t dt$.
4. Hence:

$$dQ_t^{\nu^*} = \frac{\chi(t)}{k} Q_t^{\nu^*} dt,$$

so that:

$$Q_t^{\nu^*} = \mathfrak{R} \exp\left\{ \int_0^t \frac{\chi(s)}{k} ds \right\}.$$

Liquidation With Permanent Price Impact (Contd ...)

1. To obtain the inventory along the optimal strategy we first solve the integral:

$$\int_0^t \frac{\chi(s)}{k} ds = \frac{1}{k} \int_0^t \sqrt{k\phi} \left(\frac{1 + \zeta e^{2\gamma(T-s)}}{1 - \zeta e^{2\gamma(T-s)}} \right) ds,$$

$$= \log \left(e^{-\gamma(T-s)} - \zeta e^{\gamma(T-s)} \right) \Big|_0^t, \tag{10}$$

$$= \log \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}}. \tag{11}$$

2. Hence:

$$Q_t^{\nu^*} = \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \mathfrak{R}. \tag{12}$$

3. Substituting this expression into (9) allows us to write the optimal speed to trade as a simple deterministic function of time:

$$\nu_t^* = \gamma \frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \mathfrak{R}.$$

Liquidation With Permanent Price Impact (Contd ...)

1. In the limit in which the quadratic liquidation penalty goes to infinity, that is, as $\alpha \to \infty$, we get $\zeta \to 1$.
2. Then, the optimal inventory to hold and the optimal speed to trade simplify to:
$$Q_t^{\nu^*} \to \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)}\mathfrak{R}, \text{ as } \alpha \to +\infty,$$
and
$$\nu_t^* \to \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma T)}\mathfrak{R}, \text{ as } \alpha \to +\infty,$$
respectively.
3. Both of these expressions are independent of b. For other values of α the relationship between α and the permanent price impact parameter b is more complex and we look at it after considering some numerical examples.

Figure 6.2

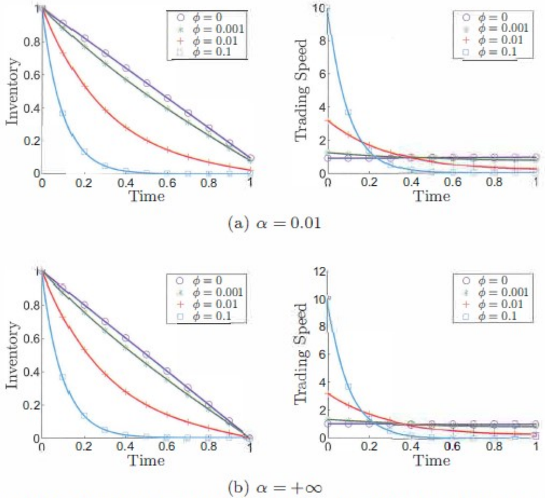

Figure 6.2 The investor's inventory along the optimal path for various levels of the running penalty ϕ. The remaining model parameters are $k = 10^{-3}$, $b = 10^{-3}$.

Figure 6.2 (Contd ...)

1. Figure 6.2: Plots of the inventory level under the optimal strategy for two levels of the liquidation penalty "α" and several levels of the running penalty "ϕ".
2. Note that with no running penalty, that is, $\phi = 0$, the strategies are straight lines and in particular, with $\alpha \to \infty$ the strategy is equivalent to a TWAP strategy.
3. As the running penalty ϕ increases, the trading curves become more convex and the optimal strategy aims to sell more assets sooner.
4. This is an intuitive result, since ϕ represents the agent's urgency to liquidate the position, and therefore as it increases she/he initially liquidates more quickly.
5. Naturally, as the liquidation penalty increases, the terminal inventory is pushed to zero.
6. One can check that in the limit in which the running penalty vanishes, $\phi \to 0$, the analog of the result from the previous section is recovered: $Q_t^{\nu^*} \to \left(1 - \frac{t}{T + \frac{k}{\alpha}}\right) \mathfrak{R}$, as $\phi \to 0$.