MA668: Algorithmic and High Frequency Trading
Lecture 39

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Figure 10.1

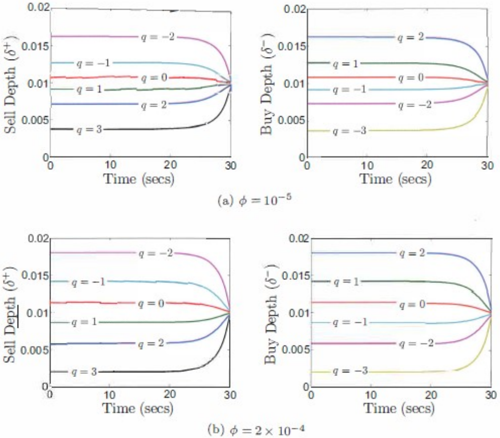

Figure 10.1 The optimal depths as a function of time for various inventory levels and $T=30$. The remaining model parameters are: $\lambda^{\pm}=1$, $\kappa^{\pm}=100$, $\overline{q}=-\underline{q}=3$, $\alpha=0.0001$, $\sigma=0.01$, $S_0=100$.

Figure: Figure 10.1

Figure 10.1 (Contd ...)

1. Figure 10.1: Shows the behaviour of the optimal depths as a function of time for different inventory levels.
2. In the examples the arrival rate of MOs is $\lambda^{\pm} = 1$ (there are on an average 1 buy and 1 sell MO per second), $\bar{q} = -\underline{q} = 3$, with $\phi = 10^{-5}$ in panel (a) and $\phi = 2 \times 10^{-4}$ in panel (b).
3. In the left of panel of (a) we show the optimal sell postings δ^{+}, that is upon the arrival of a market buy order the MM is willing to sell one unit of the asset at the price $S_t + \delta^{+}$, and in the right of panel of (a) we show the optimal buy postings δ^{-}.
4. For example: When the strategy is far away from expiry and inventories are close to the allowed minimum, the optimal sell posting is furthest away from the mid-price because only at a very "high" price is the MM willing to decrease her/his inventories further, and at the same time the optimal buy posting is very close to the mid-price because the strategy would like to complete round-trip trades (that is, a buy followed by a sell or a sell followed by a buy) and push the inventories to zero.

Figure 10.1 (Contd ...)

1. We also observe that as the strategy approaches T and $q_t < 0$ ($q_t > 0$), the optimal sell (buy) depth δ^+ (δ^-) decreases (increases).
2. To understand the intuition behind the optimal strategy note that if the terminal inventory $q_T < 0$ is liquidated at the price $S_T - \alpha Q_T$, then when α is sufficiently low, as well as being fractions of a second away from expiry, it is optimal to post nearer the mid-price to increase the chances of being filled (that is, selling one more unit of the asset) because the price is not expected to move too much before expiry and the entire position will be unwound at the mid-price, making a profit on the last unit of the asset that was sold.
3. It is also interesting to see that the optimal strategy induces mean reversion in inventories.
4. For example: If $q_t = 2$ then the sell depth is lower than the buy depth ($\delta^+ < \delta^-$), so that it is more likely for the strategy to sell, than to buy, one unit of the asset.
5. This asymmetry in the optimal depths is what induces mean reversion to zero in the inventory.

Figure 10.1 (Contd ...)

1. Moreover, in panel (b) the strategy's running inventory penalty is much higher and it is clear that higher ϕ is, the quicker the inventories will revert to zero.
2. We also see that the strategy $\delta^{*,\pm}(t,q)$ induces mean reversion in inventories, by observing that the expected drift in inventories is given by the difference in the arrival rates of filled orders.
3. Thus, given the pair of optimal strategies $\delta^{*,+}(t,q)$ and $\delta^{*,-}(t,q)$, the expected drift in inventories is given by:

$$\begin{aligned} \mu(t,q) \quad &\triangleq \quad \lim_{s\to t} \frac{1}{s-t}\mathbb{E}\left[Q_s - Q_t | Q_{t^-} = q\right], \\ &= \quad \lambda^- e^{-\kappa^- \delta^{-,*}(t,q)} - \lambda^+ e^{-\kappa^+ \delta^{+,*}(t,q)}. \end{aligned} \tag{1}$$

4. Note that the drift $\mu(t,q)$ depends on time.
5. For instance, it is clear that for the same level of inventory the speed will be different depending on how near or far the strategy is from the terminal date, because at time T the strategy tries to unwind all outstanding inventory.

Figure 10.2

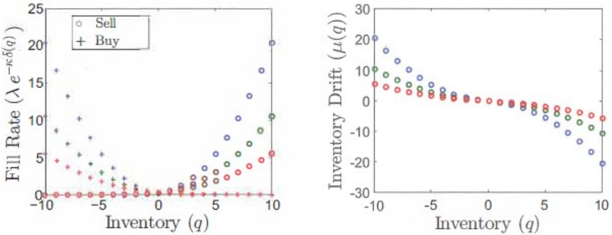

Figure 10.2 Long-term inventory level. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\bar{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$, and $\phi = \{2 \times 10^{-3}, 10^{-3}, 5 \times 10^{-4}\}$.

Figure: Figure 10.2

Figure 10.2 (Contd ...)

1. Figure 10.2: Shows the optimal level of inventory to which the strategy reverts, where we assume that we are far away from $T - t \to \infty$ in the expression for $\mu(t, q)$.
2. The model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\bar{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$ and we vary the running penalty $\phi = \{2 \times 10^{-3}, 10^{3}, 5 \times 10^{-4}\}$.
3. Note that for the set of parameters we are using here it suffices to be a few seconds away from the terminal date so that the optimal postings are not affected by the proximity to T.
4. In the left panel of the figure we plot the fill rate probabilities of both sides of the LOB which are given by $\lambda^{\pm} e^{-\kappa^{\pm}\delta^{\pm}(t,q),*}$
5. Blue circles and blue crosses are the fill rate probabilities for the sell side and buy side of the LOB respectively when $\phi = 2 \times 10^{-3}$.

Figure 10.3

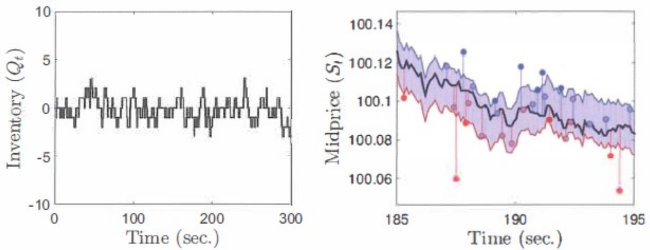

Figure 10.3 Inventory and midprice path. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\phi = 2 \times 10^{-4}$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Figure: Figure 10.3

Figure 10.3 (Contd ...)

1. Figure 10.3: Shows the inventory and price path for one simulation of the strategy.
2. The model parameters are $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\bar{q} = -\underline{q} = 10$ and $\phi = 2 \times 10^{-4}$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.
3. In the left panel we see how the inventory is mean reverting to zero and for this particular path we see that although the maximum and minimum amount of inventory that the strategy is allowed to hold is 10, it never goes beyond five units of the asset short or long.
4. The right panel of Figure 10.3 shows a window of the mid-price path along with MM's buy and sell LOs.
5. Solid circles in the figure show the incoming MO's which are filled by the MM's resting LOs (a red circle is a sell MO filled by the MM's buy LO and a blue circle is a buy MO filled by the MM's sell LO) and grey circles represent MOs that were filled by other market participants.
6. The distance between the mid-price and the MOs that arrive shows how far the MOs are walking into the LOB.

Figure 10.3 (Contd ...)

1. At the beginning of the window, the agent's inventory is zero and we observe that the strategy acquires two units (one at 185.3s and another at 187.5s) before the first sell order (at 187.8s) is filled and then closed out an instant later (at 187.9s).
2. After the first filled buy order the strategy remains asymmetric and the agent posts closer to the mid-price on the sell side of the book, compared to the buy side of the book, to rid herself/himself of her/his inventory. At 189s, 190.2s, 190.8s, 191.1s and 191.9s, a sequence of sell orders is filled and the agent holds a short position of 2 assets after the last sale at 191.9s.
3. Her/his strategy is therefore to post closer to the mid-price on the buy side of the book to increase her/his chance of unwinding her/his position.
4. These shifts in her/his posts, which induce the unwinding of any inventory she/he acquires (long or short), continues until the end of the trading horizon.

Figure 10.4

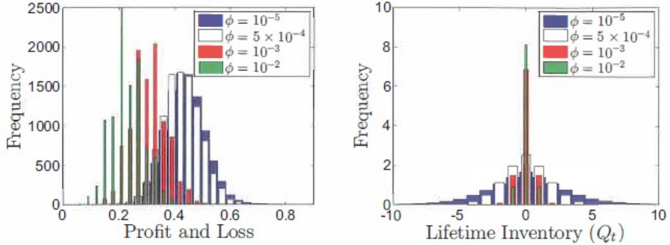

Figure 10.4 P&L and Life Inventory of the optimal strategy for 10,000 simulations. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, and $S_0 = 100$.

Figure: Figure 10.4

Figure 10.4 (Contd ...)

1. Now we turn to discussing the financial performance of the strategy.
2. The left panel of Figure 10.4 shows the profit and loss (P&L) of the optimal strategy and the right panel shows the lifetime inventory for different running penalty parameters $\phi = \{10^{-5}, 5 \times 10^{-4}, 10^{-3}, 10^{-2}\}$.
3. We observe that when ϕ increases, the histogram of P&L shifts to the left because the strategy does not allow inventory positions to stray away from zero and hence expected profits decrease.
4. The lifetime inventory histogram shows how much time the strategy holds an inventory of n.
5. For example: When $\phi = 10^{-2}$ we know that the strategy heavily penalizes deviations of running inventory from zero, so the strategy spends most of the time at inventory levels of -1, 0, 1.
6. As the running inventory penalty becomes smaller, the strategy spends more time at levels away from zero.

Market Making With No Inventory Restrictions

1. If we assume that the MM does not penalize running inventories and does not pick up a terminal inventory penalty, that is $\phi = \alpha = 0$, and there are no constraints on the amount of inventory the strategy may accumulate, that is, $|\underline{q}|, \bar{q} \to \infty$, then the MM's strategy simplifies to:

$$\delta^{+,*}(t,q) = \frac{1}{\kappa^+} \text{ and } \delta^{-,*}(t,q) = \frac{1}{\kappa^-}. \tag{2}$$

2. This optimal strategy tells the MM to post in the LOB so that the probability of the LOs being filled is maximized.
3. To see this we observe that if there are no penalties for liquidating terminal inventory, by assuming $\alpha = 0$, the terminal inventory is unwound at the mid-price and there is no running penalty for inventories straying away from zero. Then we make the ansatz:

$$H(t,x,q,S) = x + qS + h(t). \tag{3}$$

Market Making With No Inventory Restrictions (Contd ....)

1. This is similar to the one proposed earlier, but here $h(t)$ does not depend on q because the MM does not pose any restrictions on the inventory throughout the life of the strategy and can liquidate terminal inventory at the mid-price.

2. Thus, substituting the ansatz into the DPE:

$$0 = \partial_t h + \lambda^+ \sup_{\delta^+} \left[e^{-\kappa^+ \delta^+} \delta^+ \right] + \lambda^- \sup_{\delta^-} \left[e^{-\kappa^- \delta^-} \delta^- \right], \tag{4}$$

   with terminal condition $h(T) = 0$, delivers the result (2).

3. Furthermore, we can show that:

$$h(t) = e^{-1} \left(\frac{\lambda^+}{\kappa^+} + \frac{\lambda^-}{\kappa^-} \right) (T - t).$$

Market Making With No Inventory Restrictions (Contd ....)

1. This result is simple to interpret.
2. A MM who does not penalize inventories and who unwinds terminal inventory at the mid-price, will make markets by maximizing the probability of her/his LOs being filled at every instant in time regardless of the inventory position or how close the terminal date is.
3. Therefore, the MM's problem reduces to choosing $\delta^{\pm}$ to maximize the expected depth conditional on an MO hitting or lifting the appropriate side of the LOB, that is, to maximize $\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}}$.
4. The first order condition of this optimization problem is:

$$e^{-\kappa^{\pm}\delta^{\pm}} - \kappa^{\pm}\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}} = 0^{a}. \tag{5}$$

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[a] Refer to (2)