MA423 Matrix Computations

Lectures 4&5: Vectors and Matrices

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Outline

Topics:

- Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$
- Matrix-vector multiplication
- Matrix-matrix multiplication
- Block matrices
- Outer product of vectors

Vectors in $\mathbb{R}^n$

We define $\mathbb{R}^n$ to be the set of all ordered n-tuples of real numbers. Thus an n-tuple in $\mathbb{R}^n$ (also called an n-vector) is of the form

$$\text{row vector:} \quad \vec{\mathbf{v}} = [v_1, \dots, v_n] \quad \text{or} \quad \text{column vector:} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

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We always write a vector in $\mathbb{R}^n$ as a column vector. Thus

$$\mathbb{R}^n := \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} : v_1, \ldots, v_n \in \mathbb{R} \right\}.$$

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Transpose: $[v_1, \ldots, v_n]^\top = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}^\top = [v_1, \ldots, v_n]$.

Vectors in $\mathbb{C}^n$

We define $\mathbb{C}^n$ to be the set of all ordered n-tuples of complex numbers. Thus an n-tuple in $\mathbb{C}^n$ (also called an n-vector) is of the form

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Conjugate transpose: Here $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$.

$$[v_1, \ldots, v_n]^* = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix} \text{ and } \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}^* = [\bar{v}_1, \ldots, \bar{v}_n].$$

Algebraic properties of vectors in $\mathbb{F}^n$

Define addition and scalar multiplication on $\mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) as follows:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \text{ and } \alpha \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{bmatrix} \text{ for } \alpha \in \mathbb{F}.$$

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1. Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
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4. Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. Distributivity : $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
6. Distributivity : $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
7. Associativity: $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
8. Identity: $1\mathbf{u} = \mathbf{u}$.

Examples of vectors

Standard vectors: The vectors $\mathbf{e}_1 := \begin{bmatrix} 1 & 0 & \cdots 0 \end{bmatrix}^\top$, $\mathbf{e}_2 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^\top, \ldots, \mathbf{e}_n := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^\top$ are called standard vectors or canonical vectors in $\mathbb{F}^n$.

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For instance, the word count vector $[25, 2, 0]^\top$ means that the first dictionary word appears 25 times, the second one twice, and the third one not at all.

Inner product

Angle, Length, and Distance can all be described by using the notion of inner product (dot product) of two n-vectors.

Definition: If $\mathbf{u} := [u_1, \ldots, u_n]^\top$ and $\mathbf{v} := [v_1, \ldots, v_n]^\top$ are n-vectors then the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle \quad := \quad u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad \text{when } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

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Example: If $\mathbf{u} := [1, 2, -3]^\top$ and $\mathbf{v} := [-3, 5, 2]^\top$ then

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1.$$

Matrices

Definition: A matrix is an array of numbers. An $m \times n$ matrix A has m rows and n columns and is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

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The j-th column of A : $\mathbf{a}_j := \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$ for $j = 1 : n$.

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$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right] = \begin{bmatrix} - \hat{\mathbf{a}}_1 - \\ \vdots \\ - \hat{\mathbf{a}}_m - \end{bmatrix}.$$

Special matrices

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Example: $I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{O} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 4 & 0 \\ -2 & 6 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -1 \\ -2 & 6 & 7 \end{bmatrix} \\ 2A &= \begin{bmatrix} 2 & 8 & 0 \\ -4 & 12 & 10 \end{bmatrix} \text{ and } (-1)A = \begin{bmatrix} -1 & -4 & 0 \\ 2 & -6 & -5 \end{bmatrix}. \end{aligned}$$

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7. $\alpha\,(\beta\, A) = (\alpha\beta)A$.
8. $1A = A$.

Transpose and Conjugate transpose

Transpose: The transpose of an $m \times n$ matrix $A = [a_{ij}]_{m\times n}$ is the $n \times m$ matrix denoted by $A^\top$ and is given by $A^\top = [a_{ji}]_{n\times m}$.

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Example: $\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$ and $\begin{bmatrix} 1+i & 2 \\ 3 & 4+5i \end{bmatrix}^\top = \begin{bmatrix} 1+i & 3 \\ 2 & 4+5i \end{bmatrix}$

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Conjugate transpose: The conjugate transpose of an $m \times n$ complex matrix $A = [a_{ij}]_{m\times n}$ is the $n \times m$ matrix denoted by A^* and is given by

$$A^* = [\bar{a}_{ji}]_{n\times m} = ([\bar{a}_{ij}]_{m\times n})^\top = (\bar{A})^\top,$$

where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Transpose and Conjugate transpose

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Example: $\begin{bmatrix} i & 4 & 1+i \\ 3 & 4+5i & 0 \end{bmatrix}^* = \begin{bmatrix} -i & 3 \\ 4 & 4-5i \\ 1-i & 0 \end{bmatrix}$

Transpose and conjugate transpose

Exercise: Let $A, B \in \mathbb{F}^{m\times n}$ and $\alpha \in \mathbb{F}$. Then show that

$$(a)\,(A+B)^\top = A^\top + B^\top \quad (b)\,(\alpha A)^\top = \alpha A^\top \text{ and } (\alpha A)^* = \bar{\alpha} A^* \quad (c)\,(A^\top)^\top = A.$$

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Remark: Let $A := [a_{ij}]_{n\times n}$. If $A^\top = -A$ then $a_{jj} = 0$ for $j = 1 : n$. On the other hand, if $A^* = -A$ then $\mathrm{Re}(a_{jj}) = 0$ for $j = 1 : n$.

Matrix-vector multiplication

Let $A := \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{F}^{m\times n}$ and $\mathbf{x} := [x_1, \ldots, x_n]^\top \in \mathbb{F}^n$. We define the matrix-vector multiplication $A\mathbf{x}$ as the linear combination of columns of A.

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Definition: Matrix-vector multiplication

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

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Matrix-vector multiplication

A row vector $\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}$ is a $1 \times n$ matrix. Therefore

$$\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_{i1}x_1 + \cdots + a_{in}x_n.$$

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Example: Matrix-vector multiplication in two ways

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} \\ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x} \\ \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \mathbf{x} \end{bmatrix} \end{aligned}$$

Row and column oriented matrix-vector multiplication

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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Writing $A := \begin{bmatrix} \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_n \end{bmatrix}$ and $A = \begin{bmatrix} - \hat{\mathbf{a}}_1 - \\ \vdots \\ - \hat{\mathbf{a}}_m - \end{bmatrix}$, we have

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_1\mathbf{x} \\ \vdots \\ \hat{\mathbf{a}}_m\mathbf{x} \end{bmatrix}.$$

Matrix-matrix multiplication

Let $A \in \mathbb{F}^{m \times n}$ and $B := \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix} \in \mathbb{F}^{n \times p}$.

Definition: Define the matrix-matrix multiplication AB by

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$$\mathbf{c}_j = C\mathbf{e}_j = (AB)\mathbf{e}_j = A(B\mathbf{e}_j) = A\mathbf{b}_j \Longrightarrow C = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

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Fact: Let $A \in \mathbb{F}^{m\times n}$. Let $\mathbf{e}_i \in \mathbb{F}^m$ and $\mathbf{e}_j \in \mathbb{F}^n$ be standard unit vectors. Then

- $A\mathbf{e}_j$ is the j-th column of A.
- $\mathbf{e}_i^\top A$ is the i-th row of A.

Matrix-matrix multiplication

Let $A = \begin{bmatrix} -\hat{\mathbf{a}}_1- \\ \vdots \\ -\hat{\mathbf{a}}_m- \end{bmatrix} \in \mathbb{F}^{m\times n}$, $B := \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix} \in \mathbb{F}^{n\times p}$. Then

$$AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_1\mathbf{b}_1 & \cdots & \hat{\mathbf{a}}_1\mathbf{b}_p \\ \vdots & \cdots & \vdots \\ \hat{\mathbf{a}}_m\mathbf{b}_1 & \cdots & \hat{\mathbf{a}}_m\mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_1 B \\ \vdots \\ \hat{\mathbf{a}}_m B \end{bmatrix}.$$

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Thus if $A := [a_{ij}]_{m\times n}$, $B := [b_{ij}]_{n\times p}$ and $C := AB = [c_{ij}]_{m\times p}$ then

$$c_{ij} = \hat{\mathbf{a}}_i\mathbf{b}_j = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

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Remark: If A and B are $n \times n$ matrices then in general $AB \neq BA$.

Example

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Block matrices

Definition: An $m \times n$ block matrix (or a partitioned matrix) is a matrix of the form

$$A := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \cdots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

where each A_{ij} is a $p_i \times q_j$ matrix for $i = 1 : m$ and $j = 1 : n$.

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Then $\begin{bmatrix} A_{i1} & \cdots & A_{in} \end{bmatrix}$ is the i-th block row of A and $\begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix}$ is the j-th block column of A.

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Example: $\left[\begin{array}{cc|ccc|c} 1 & 2 & 2 & 0 & 1 & 4 \\ 3 & 4 & 1 & 2 & 3 & 5 \\ \hline 5 & 7 & 2 & 7 & 8 & 8 \\ 3 & 4 & 1 & 9 & 2 & 2 \end{array}\right]$ has 2 block rows and 3 block columns.

Block matrix operations

Block matrix addition: Let $A := [A_{ij}]_{m \times n}$ and $B := [B_{ij}]_{m \times n}$ be block matrices such that size of A_{ij} = size of B_{ij} for $i = 1 : m$ and $j = 1 : n$. Then $A + B := [A_{ij} + B_{ij}]_{m \times n}$.

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Block matrix multiplication: Let $A := [A_{ij}]_{m\times n}$ and $B := [B_{ij}]_{n\times p}$ be block matrices. If the matrix multiplication $C_{ij} := \sum_{k=1}^{n} A_{ik}B_{kj}$ is well defined for $i = 1 : m$ and $j = 1 : p$ then AB is an $m \times p$ block matrix given by $AB = [C_{ij}]_{m\times p}$.

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Block matrix addition: Let $A := [A_{ij}]_{m\times n}$ and $B := [B_{ij}]_{m\times n}$ be block matrices such that size of A_{ij} = size of B_{ij} for $i = 1 : m$ and $j = 1 : n$. Then $A + B := [A_{ij} + B_{ij}]_{m\times n}$.

Block matrix multiplication: Let $A := [A_{ij}]_{m\times n}$ and $B := [B_{ij}]_{n\times p}$ be block matrices. If the matrix multiplication $C_{ij} := \sum_{k=1}^{n} A_{ik}B_{kj}$ is well defined for $i = 1 : m$ and $j = 1 : p$ then AB is an $m \times p$ block matrix given by $AB = [C_{ij}]_{m\times p}$.

Conformal partition: If an operation on block matrices A and B are well defined then A and B are said to be partitioned conformably.

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Example: $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} =$

$$\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{bmatrix}.$$

Block matrix multiplication

Example:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|cc} 8 & 6 & 4 & 5 \\ \hline 10 & 9 & 6 & 7 \\ 18 & 15 & 10 & 12 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|cc} 8 & 6 & 4 & 5 \\ 10 & 9 & 6 & 7 \\ \hline 18 & 15 & 10 & 12 \end{array}\right]$$

Outer product

Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we have seen that $\mathbf{y}^\top \mathbf{x}$ is the inner product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n = \mathbf{y}^\top \mathbf{x}.$$

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Outer product: The matrix product $\mathbf{x}\mathbf{y}^\top$ is an $n \times n$ matrix and is given by

$$\mathbf{x}\mathbf{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{bmatrix}.$$

The product $\mathbf{x}\mathbf{y}^\top$ is called the outer product of $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

Outer product

Example: If $\mathbf{x} := \begin{bmatrix} 4 & 1 & 3 \end{bmatrix}^\top$ and $\mathbf{y} := \begin{bmatrix} 3 & 5 & 2 \end{bmatrix}^\top$ then

$$\mathbf{x}\mathbf{y}^\top = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 20 & 8 \\ 3 & 5 & 2 \\ 9 & 15 & 6 \end{bmatrix}.$$

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Let $X := \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ and $Y := \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \end{bmatrix} \in \mathbb{R}^{p \times n}$. Then $XY^\top \in \mathbb{R}^{m \times p}$ can be written as sum of outer products of vectors

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1. Associative Law: $(AB)C = A(BC)$
2. Left Distributive Law: $A(B+C) = AB + AC$
3. Right Distributive Law: $(A+B)C = AC + BC$
4. Scalar multiplication: $\alpha(AB) = (\alpha A)B = A(\alpha B)$
5. Multiplicative identity: If A is an $m \times n$ matrix then $I_m A = A = AI_n$.

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Definition: An $n \times n$ matrix A is said to be invertible if there exists a matrix B such that $AB = I_n = BA$. The matrix B is called an inverse of A.

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$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

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- The $n \times n$ zero matrix $\mathbf{O}$ is not invertible.
- If an $n \times n$ matrix A has a zero row or a zero column, then A is NOT invertible.

Floating-Point Operation (FLOP) count

Vector-vector operations: Let $\alpha \in \mathbb{R}$. Let $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ and $\mathbf{y} := \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^\top \in \mathbb{R}^n$.

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- $\mathbf{z} \leftarrow A^\top\mathbf{x} = \begin{bmatrix} \mathbf{a}_1^\top\mathbf{x} & \cdots & \mathbf{a}_n^\top\mathbf{x} \end{bmatrix}^\top$ and $\mathbf{d} \leftarrow \alpha \cdot A^\top\mathbf{x} + \beta \cdot \mathbf{y}$ require $2n^2$ flops

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- $D \leftarrow AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_n \end{bmatrix}$ and $D \leftarrow \alpha \cdot AB + \beta \cdot C$ require $2n^3$ flops

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- $\mathbf{z} \leftarrow A^\top \mathbf{x} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{x} & \cdots & \mathbf{a}_n^\top \mathbf{x} \end{bmatrix}^\top$ and $\mathbf{d} \leftarrow \alpha \cdot A^\top \mathbf{x} + \beta \cdot \mathbf{y}$ require $2n^2$ flops

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- $D \leftarrow A^\top B$ or $D \leftarrow AB^\top$ and $D \leftarrow \alpha \cdot A^\top B + \beta \cdot C$ require $2n^3$ flops