MA668: Algorithmic and High Frequency Trading
Lecture 34

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

A Prelude

1. So far: We focused on execution strategies which relied on market orders (MOs) only.
2. One of the advantages of sending MOs is that execution is guaranteed.
3. However: The execution price, is generally worse than the mid-price due to both the existence of non-zero spread and the fact that orders may walk the book.
4. **In practice**: The agent also employs limit orders (LOs) because instead of picking up liquidity-taking fees and incurring market impact costs, the prices at which LOs are filled are better than the mid-price, though there is no guarantee that a matching order will arrive.
5. To address these issues: We look at optimal execution problems when the agent employs LOs and possibly also MOs.

A Prelude (Contd ...)

1. Assumption: When the agent posts LOs to liquidate a position, she/he posts a limit sell order for a fixed volume (e.g., some percentage of the average size of an MO, or a fixed amount of, say, 10 shares) at a price of $S_t + \delta_t$, where S_t is the mid-price.
2. Hence, δ is a premium that the agent demands for providing liquidity to the market.
3. The larger the δ is, the larger is the premium.
4. But: The probability that an order arrives and walks the limit order book (LOB), up to the posted depth, decreases with δ.
5. The strategy used by the agent relies on speed to post-and-cancel LOs.
6. At every instant in time: The agent reassesses market conditions, cancels any LO resting in the book, posts a new LO at the optimal level, and so on.
7. To do this, requires software, hardware, and connection to the exchange so that the strategy does not have stale quotes in the LOB and can quickly process information.

A Prelude (Contd ...)

1. The probability of being filled when posting at a given depth δ, conditional on the arrival of an MO, is called the **fill probability**, which we denote by the function $P(\delta)$.
2. Naturally, P must be decreasing, it changes throughout the day, and it is sensitive to the current status of the LOB.
3. Left panel of Figure 8.1: Shows a block-shaped LOB together with:
   - (A) A post at $\delta = 10$ (the dashed line).
   - (B) The depth to which an MO of volume 700 lifts sell orders (dark green region).
   - (C) The depth to which an MO of volume 1,500 lifts sell LOs (dark plus light green region).
4. The deeper the LO is posted (that is, further away from the mid-price), the less likely it is that MOs large enough walk the LOB up to that price level.
5. Hence, the probability of being filled decreases as δ increases.

Figure 8.1

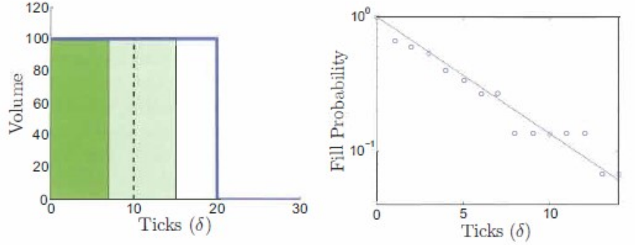

Figure 8.1 (Left) A flat (or block shaped) LOB. (Right) Empirical fill probabilities for NFLX on June 21, 2011 for the time interval 12:55pm to 1:00pm using 500 millisecond resting times. The straight line shows the fit to an exponential function.

Figure: Figure 8.1

A Prelude (Contd ...)

1. If we assume that the volume of individual MOs, denoted by V, is exponentially distributed with mean volume of η, and that the LOB is block shaped with height A, that is, the posted volume at a price of $S + \delta$ is equal to a constant A out to a maximum price level of $S + \bar{\delta}$, then the probability of fill is exponential.
2. That is, conditional on the arrival of an MO of volume V, the probability that the sell LO is lifted is given by:

$$\mathbb{P}(\text{order posted at level } \delta \text{ is lifted}) = \mathbb{P}(V > A\delta) = \exp\left(-\frac{A\delta}{\eta}\right). \quad (1)$$

3. One could in principle also use power law fill probabilities.
4. However: To keep the analysis consistent and self-contained we use the exponential fill probability throughout.

Liquidation With Only Limit Orders

1. The agent posts only LOs and the setup of the problem is similar to the earlier case.
2. But: Now we must track not only the agent's inventory, but also the arrival of other traders' MOs, which is what will (possibly) lift the agent's posted sell LOs.
3. We summarize the model ingredients and the notation

Notation

1. $\mathfrak{R}$: The amount of shares that the agent wishes to liquidate.
2. T: The terminal time at which the liquidation programme ends.
3. $S = (S_t)_{\{0 \leq t \leq T\}}$ is the asset's mid-price with $S_t = S_0 + \sigma W_t$, $\sigma > 0$, and $W = (W_t)_{\{0 \leq t \leq T\}}$ is a standard Brownian motion.
4. $\delta = (\delta_t)_{\{0 \leq t \leq T\}}$ denotes the depth at which the agent posts limit sell orders, that is, the agent posts LOs at a price of $S_t + \delta_t$ at time t.

Notation (Contd ...)

1. $M = (M_t)_{\{0 \leq t \leq T\}}$ denotes a Poisson process (with intensity λ) corresponding to the number of market buy orders (from other traders) that have arrived.
2. $N^\delta = (N_t^\delta)_{\{0 \leq t \leq T\}}$ denotes the (controlled) counting process corresponding to the number of market buy orders which lift the agent's offer, that is, MOs which walk the sell side of the book to a price greater than or equal to $S_t + \delta_t$.
3. $P(\delta) = e^{-\kappa\delta}$ with $\kappa > 0$ is the probability that the agent's LO will be lifted when a buy MO arrives.
4. $X^\delta = (X_t^\delta)_{\{0 \leq t \leq T\}}$ is the agent's cash process and satisfies the SDE:

$$dX_t^\delta = (S_t + \delta_t) dN_t^\delta. \tag{2}$$

5. $Q_t^\delta = \mathfrak{R} - N_t^\delta$ is the agent's inventory which remains to be liquidated.

Liquidation With Only Limit Orders

1. Note that whenever the process N jumps, the process M must also jump, but when M jumps, N will jump only if the MO is large enough to walk the book and lift the agent's posted LO.
2. Moreover, conditional on an MO arriving (that is, M jumps), N jumps with probability $P(\delta_t) = e^{-\kappa \delta_t}$.
3. However N is not a Poisson process since its activity reacts to the depth at which the agent posts.
4. Moreover, in contrast to the setup earlier, when the agent's orders are executed, she/he receives better than mid-prices.
5. Finally, the filtration $\mathcal{F}$ on which the problem is setup is the natural one generated by S, N and M.
6. Moreover, the agent's depth postings (or strategy) δ will be $\mathcal{F}$-predictable and in particular will be left-continuous with right limits.

The Agent's Optimization Problem

1. The agent wishes to maximize the profit from liquidating $\mathfrak{N}$ shares.
2. Also requires that most, if not all, of the shares are sold by the terminal time T.
3. If the agent has inventory remaining at the end of the trading horizon, she/he liquidates it using an MO for which she/he obtains worse prices than the mid-price.
4. As argued previously, a linear impact function on MOs is a reasonable first order approximation of market impact.
5. Hence the agent's optimization problem is to find:

$$H(x, S) = \sup_{\delta \in \mathcal{A}} \mathbb{E}\left[X_\tau^\delta + Q_\tau^\delta\left(S_\tau - \alpha Q_\tau^\delta\right) \middle| X_{0^-}^\delta = x, S_0 = S, Q_{0^-}^\delta = \mathfrak{N}\right], \tag{3}$$

where $\alpha \geq 0$ is the liquidation penalty (linear impact function).

The Agent's Optimization Problem (Contd ...)

1. Moreover, the admissible set $\mathcal{A}$ consists of strategies δ which are bounded from below, and the stopping time:

$$\tau = T \wedge \min\{t : Q_t^\delta = O\}$$

is the minimum of T or the first time that the inventory hits zero, because then no more trading is necessary.

2. The corresponding value function is:

$$H(t, x, S, q) = \sup_{\delta \in \mathcal{A}} \mathbb{E}_{t,x,S,q} \left[X_\tau^\delta + Q_\tau^\delta \left(S_\tau - \alpha Q_\tau^\delta \right) \right],$$

where the notation $\mathbb{E}[\cdot]$ represents expectation conditional on $X_{0^-}^\delta = x$, $S_0 = S$ and $Q_{0^-}^\delta = \mathfrak{R}$.

3. In this setup, the agent does not have any urgency, that is, does not penalize inventories different from zero.

The Resulting DPE

1. The dynamic programming principle (DPP) suggests that the value function solves the following dynamic programming equation (DPE):

$$
\begin{aligned}
0 &= \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H \\
&+ \sup_{\delta} \Big[\lambda e^{-\kappa\delta} \left[H(t, x + (S + \delta), S, q - 1) - H(t, x, S, q) \right] \Big] ,
\end{aligned}
$$

along with $H(t, x, S, 0) = x$ and $H(T, x, S, q) = x + q(S - \alpha q)$.

2. We have an optimal trading problem where the state variables jump and the resulting DPE results in a non-linear partial integral differential equation (PIDE) rather than a non-linear PDE.

The Resulting DPE (Contd ...)

1. Interpretation of the various terms of the PIDE.
    1. The operator ∂_{SS} corresponds to the generator of the Brownian motion which drives the mid-price.
    2. The supremum takes into account the agent's ability to control the depth of her sell LOs.
    3. The term $\lambda e^{-\kappa\delta}$ represents the rate of arrival of other market participants' buy MOs which lift the agent's posted sell LO at price $S + \delta$.
    4. The difference (jump) term $H(t, x + (S + \delta), S, q - 1) - H(t, x, S, q)$ represents the change in the agent's value function when an MO fills the agent's LO: The agent's cash increases by $S + \delta$ and her/his inventory decreases by 1.
2. The terminal condition at $t = T$ represents the cash the agent has acquired up to that point in time "plus" the value of liquidating the remaining shares at the worse than mid-price of $(S - \alpha q)$ per share.
3. The boundary condition along $q = 0$ represents the cash the agent has at that stopping time and since $q = 0$ there is no liquidation value, and the agent simply walks away with x in cash.

The Resulting DPE (Contd ...)

1. The terminal and boundary conditions suggest that the ansatz for the value function is:

$$H(t, x, s, q) = x + qS + h(t, q), \tag{4}$$

for a yet to be determined function $h(t, q)$.

2. This ansatz has three terms.

    A. The first term is the accumulated cash.
    B. The second term denotes the book value of the remaining inventory which is marked-to-market using the mid-price.
    C. Finally the function $h(t, q)$ represents the added value to the agent's cash from optimally liquidating the remaining shares.

3. Now $h(t, q)$ satisfies the coupled system of non-linear ODEs:

$$0 = \partial_t h + \sup_{\delta} \left[\lambda e^{-\kappa\delta} \left[\delta + h(t, q-1) - h(t, q) \right] \right], \tag{5}$$

with $h(t, 0) = 0$ and $h(T, q) = -\alpha q^2$.

The Resulting DPE (Contd ...)

1. The optimal depth can be found in feedback form by focusing on the first order conditions for the supremum.
2. This provides us with the following:

$$\begin{aligned} 0 &= \partial_\delta \left[\lambda e^{-\kappa\delta}\left[\delta + h(t,q-1) - h(t,q)\right]\right], \\ &= \lambda\left(-\kappa e^{-\kappa\delta}\left[\delta + h(t,q-1) - h(t,q)\right] + e^{-\kappa\delta}\right), \\ &= \lambda e^{-\kappa\delta}\left(-\kappa\left[\delta + h(t,q-1) - h(t,q)\right] + 1\right). \end{aligned}$$

3. Hence the optimal strategy δ^* in feedback control form is given by:

$$\delta^* = \frac{1}{\kappa} + \left[h(t,q) - h(t,q-1)\right]. \tag{6}$$