MA668: Algorithmic and High Frequency Trading
Lecture 24

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DPE/HJB Equation (Contd ...)

1. Since the above inequality holds for arbitrary $\mathbf{v} \in \mathcal{A}$, it follows that:

$$\partial_t H(t, \mathbf{x}) + \sup_{\mathbf{u} \in \mathcal{A}} \left(\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) + F(t, \mathbf{x}, \mathbf{u}) \right) \leq 0. \tag{1}$$

2. Next, we show that the inequality is indeed an equality.
3. To show this, suppose that $\mathbf{u}^*$ is an optimal control, then we have:

$$H(t, \mathbf{x}) = \mathbb{E}_{t,\mathbf{x}} \left[H(\tau, \mathbf{X}_\tau^{\mathbf{u}^*}) + \int_t^\tau F\left(s, \mathbf{X}_s^{\mathbf{u}^*}, \mathbf{u}^*\right) ds \right].$$

4. As above, by applying Ito's lemma to write $H(\tau, \mathbf{X}_\tau^{\mathbf{u}^*})$ in terms of $H(t, \mathbf{x})$ plus the integral of its increments, taking expectations, and then taking the limit as $h \downarrow 0$, we find that:

$$\partial_t H(t, \mathbf{x}) + \mathcal{L}_t^{\mathbf{u}^*} H(t, \mathbf{x}) + F(t, \mathbf{x}, \mathbf{u}^*) = 0.$$

DPE/HJB Equation (Contd ...)

1. Combined with (1), we finally arrive at the DPE (also known in this context as the Hamilton-Jacobi-Bellman equation):

$$\partial_t H(t, \mathbf{x}) + \sup_{\mathbf{u} \in \mathcal{A}} \left(\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) + F(t, \mathbf{x}, \mathbf{u}) \right) = 0, \; H(T, \mathbf{x}) = G(\mathbf{x}). \tag{2}$$

2. The terminal condition above follows from the definition of the value function from which we see that the running reward/penalty drops out and $G(\mathbf{X}_T^{\mathbf{u}})$ is $\mathcal{F}_T$-measurable.
3. Notice that the optimization of the control in (2) is only over its value at time t, rather than over the whole path of the control.
4. Hence, it appears that the optimal control can be obtained point-wise. Treating the value function as known, the optimal control can often be found in feedback control form in terms of the value function itself.
5. Substituting the feedback control back into (2) results in non-linear PDEs.

Example: The Merton Problem

1. Consider now the Merton optimization problem described earlier.
2. The optimization problem is as given earlier and has the associated time dependent performance criteria:

$$H^{\pi}(t,x,S) = \mathbb{E}_{t,x,S}\left[U\left(X_T^{\pi}\right)\right], \tag{3}$$

where X^{π} (investor wealth) and S (risky asset price) satisfy the SDEs already seen, with π representing the dollar value of wealth invested in the risky asset S.

3. The infinitesimal generator of the pair of processes $(X_t^{\pi}, S_t)_{\{0\le t\le T\}}$ is then:

$$\mathcal{L}_t^{\pi} = (rx + (\mu - r)\pi)\,\partial_x + \frac{1}{2}\sigma^2\pi^2\partial_{xx} + (\mu - r)\,S\partial_S + \frac{1}{2}\sigma^2 S^2\partial_{SS} + \sigma\pi\partial_{xS}.$$

Example: The Merton Problem (Contd ...)

1. According to (2), the value function $H(t,x,S) = \sup_{\pi \in \mathcal{A}_{[t,T]}} H^{\pi}(t,x,S)$ should satisfy the equation:

$$
\begin{aligned}
0 &= \left(\partial_t + rx\partial_x + \frac{1}{2}\sigma^2 S^2 \partial_{SS}\right) H \\
&+ \sup_{\pi} \left\{ \pi \left((\mu - r)\partial_x + \sigma \partial_{xS} \right) H + \frac{1}{2}\sigma^2 \pi^2 \partial_{xx} H \right\},
\end{aligned}
$$

subject to the terminal condition $H(T,x,S) = U(x)$.

2. Note that the argument of the sup is quadratic in π and as long as $\partial_{xx} H(t,x,S) < 0$, the sup attains a maximum.

3. By completing the squares we have:

$$
\pi \left((\mu - r)\partial_x + \sigma \partial_{xS} \right) H + \frac{1}{2}\sigma^2 \pi^2 \partial_{xx} H = \frac{1}{2}\sigma^2 \partial_{xx} H \left((\pi - \pi^*)^2 - \pi^{*2} \right),
$$

where:

$$
\pi^* = -\frac{(\mu - r)\partial_x H + \sigma \partial_{xS} H}{\sigma^2 \partial_{xx} H},
$$

is the optimal control in feedback form *i.e.,* it is the optimal control given the known value function $H(t,x,S)$.

Example: The Merton Problem (Contd ...)

1. Substituting this optimum back into the DPE yields the non-linear PDE for the value function:

$$0 = \left(\partial_t + rx\partial_x + \frac{1}{2}\sigma^2 S^2 \partial_{SS}\right) H - \frac{((\mu - r)\partial_x H + \sigma \partial_{xS} H)^2}{2\sigma^2 \partial_{xx} H}.$$

2. This simplifies somewhat by observing that the terminal condition $H(t, x, S) = U(x)$ is independent of S.

3. Hence, it suggests the ansatz $H(t, x, S) = h(t, x)$ in which case we obtain a simpler, but still non-linear, equation for $h(t, x)$:

$$0 = (\partial_t + rx\partial_x)\, h(t, x) - \frac{\lambda}{2\sigma} \frac{(\partial_x h(t, x))^2}{\partial_{xx} h(t, x)},$$

with terminal condition $h(T, x) = U(x)$ and where $\lambda := \dfrac{(\mu - r)^2}{\sigma}$.

Example: The Merton Problem (Contd ...)

1. Moreover, the optimal control simplifies to:

$$\pi^* = -\frac{\lambda}{\sigma}\left(\frac{\partial_x h}{\partial_{xx} h}\right).$$

2. The explicit solution of the non-linear PDE depends on the precise form of the utility function $U(x)$.
3. We consider one classic example, namely that of exponential utility:

$$U(x) = -e^{-\gamma x}, \gamma > 0,$$

which is defined for all $x \in \mathbb{R}$.

4. For the exponential utility, we have the ansatz:

$$h(t, x) = -\alpha(t)e^{-\gamma x \beta(t)},$$

where $\alpha(t)$ and $\beta(t)$ are yet to be determined.

In physics and mathematics, an ansatz is an educated guess or an additional assumption made to help solve a problem, and which may later be verified to be part of the solution by its results

Example: The Merton Problem (Contd ...)

1. From the terminal condition $h(T, x) = -e^{-\gamma x}$, we have that $\alpha(T) = \beta(T) = 1$ and upon substitution into the non-linear PDE, we find that:
$$\left(\partial_t \alpha - \frac{\lambda}{2\sigma}\alpha\right) = 0 \text{ and } (\partial_t \beta + r\beta) = 0.$$
2. These together with the terminal conditions, are easily solved to find:
$$\alpha(t) = e^{-\frac{\lambda}{2\sigma}(T-t)} \text{ and } \beta(t) = e^{r(T-t)}.$$
3. Upon back substitution, we find that the optimal amount to invest in the risky asset is a deterministic function of time (as follows):
$$\pi^*(t) = \frac{\lambda}{\gamma\sigma} e^{-r(T-t)}.$$