Homework-8

MA-571 : Numerical Linear Algebra 2023 R. Alam

Singular Value Decomposition

1. Let $A \in \mathbb{C}^{n\times n}$ and $\sigma_1, \ldots, \sigma_n$ be the singular values of A. Show that $|\det(A)| = \prod_{j=1}^{n} \sigma_j$.

2. Show that if $A \in \mathbb{C}^{n\times n}$ is positive semi-definite then its singular values are the same as its eigenvalues. What is the relationship between the eigenvectors and the singular vectors of A?

3. Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Show that the singular values of A are the absolute values of the eigenvalues of A. Let $A = UDU^*$ be a spectral decomposition of A, where U is unitary and $D := \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Show that $A = U\,|D|\,(U\,\mathrm{sign}(D))^*$ is a singular value decomposition of A, where $|D|$ and $\mathrm{sign}(D)$ are diagonal matrices with diagonal entries $|\lambda_j|$ and $\mathrm{sign}(\lambda_j)$, respectively.

4. Find a 2-by-2 matrix A such that $\sigma_{\max}(A) > \max(|\lambda_1|, |\lambda_2|)$, where λ_1 and λ_2 are eigenvalues of A and $\sigma_{\max}(A)$ is the largest singular value of A.

5. Let $\in \mathbb{C}^{m\times n}$ be such that $\mathrm{rank}(A) = n$. Let $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ denote the smallest and the largest singular values of A, respectively. Then show that the following hold:

$$\|A^*A\|_2 = \sigma_{\max}(A)^2, \quad \|(A^*A)^{-1}\|_2 = \sigma_{\min}(A)^{-2}, \quad \|(A^*A)^{-1}A^*\|_2 = \sigma_{\min}(A)^{-1},$$
$$\|A(A^*A)^{-1}\|_2 = \sigma_{\min}(A)^{-1}, \quad \|A(A^*A)^{-1}A^*\|_2 = 1.$$

6. Let $\sigma_1 \geq \ldots \geq \sigma_r > 0$ be nonzero singular values of an m-by-n matrix A. Show that $\|A\|_2 = \sigma_1$ and $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$. Further, show that $\|A\|_F \leq \sqrt{\mathrm{rank}(A)}\,\|A\|_2$.

7. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Let $\sigma_1, \ldots, \sigma_n$ be singular values of A. Show that the singular values of $\begin{bmatrix} I_n \\ A \end{bmatrix}$ are equal to $\sqrt{1+\sigma_j^2}$ for $j = 1 : n$.

8. Let $A \in \mathbb{C}^{n\times n}$ and $\sigma > 0$. Show that σ is a singular value of $A \iff \begin{bmatrix} A & -\sigma I_n \\ -\sigma I_n & A^* \end{bmatrix}$ is singular.

9. Let $A = BC$ where $B \in \mathbb{C}^{m\times n}$ has $\mathrm{rank}(B) = n$ and $C \in \mathbb{C}^{n\times n}$ is nonsingular. Show that $A^+ = C^{-1}B^+$. Deduce that if $\mathrm{rank}(A) = n$ and $A = QR$ is a compact QR factorization then $A^+ = R^{-1}Q^*$, where $Q \in \mathbb{C}^{m\times n}$ is an isometry and $R \in \mathbb{C}^{n\times n}$ is upper triangular.

10. Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Let $x \in \mathbb{C}^n$ be nonzero and set $r := b - Ax$. Define $E := rx^+$. Show that $(A+E)x = b$ and $\|E\|_2 = \dfrac{\|r\|_2}{\|x\|_2}$.

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