QR factorization by Givens rotations

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Outline

- Rotation in $\mathbb{R}^2$
- Givens rotation in $\mathbb{R}^n$
- QR factorization by Givens rotations

Rotation in $\mathbb{R}^2$

Definition: Let $\theta \in [0, 2\pi]$. A rotation in $\mathbb{R}^2$ is a matrix $G(\theta) \in \mathbb{R}^{2\times 2}$ that rotates each vector in $\mathbb{R}^2$ by an angle θ in the anti-clock-wise direction.

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Consequently, $G(\theta)e_2 = G(\pi/2 + \theta)e_1 = \begin{bmatrix} \cos(\pi/2 + \theta) \\ \sin(\pi/2 + \theta) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

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Hence we have $G(\theta) = \begin{bmatrix} G(\theta)e_1 & G(\theta)e_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Plane rotation in $\mathbb{R}^2$

We have $G(\theta)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}.$

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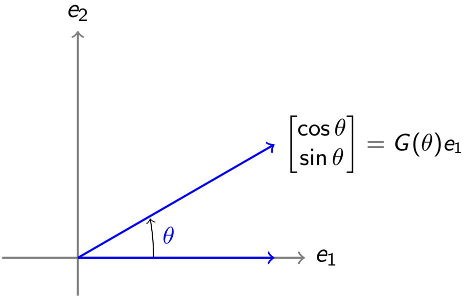

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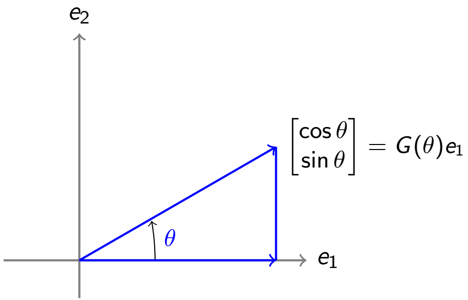

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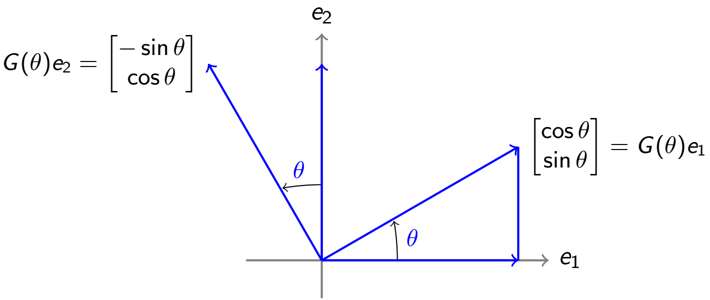

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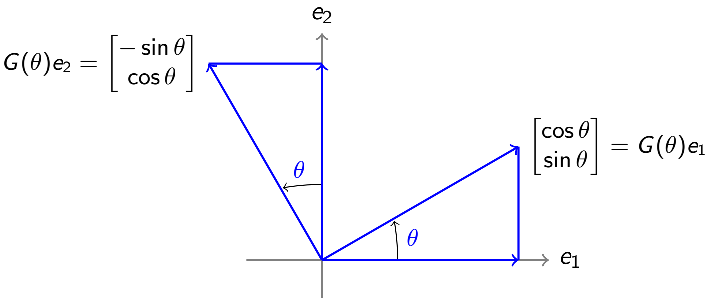

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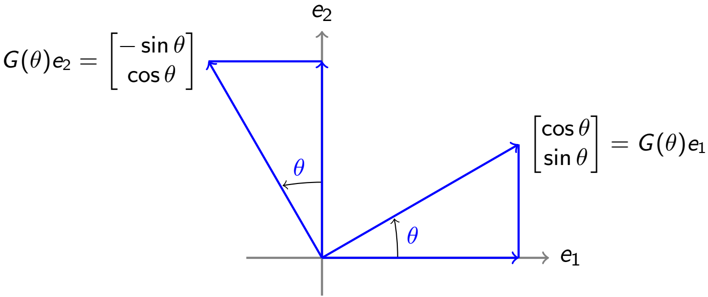

The rotation $G(\theta)$. Note that $G(\theta)e_2 = G(\pi/2+\theta)e_1$.

Properties of rotation

- Consider $G(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then $G(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $G(\theta)^\top = G(-\theta)$. Thus $G(\theta)^\top$ is a clock-wise rotation by an angle θ.

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- Proof: Note that $G(\phi)G(\theta)v$ is the anti-clock-wise rotation of v by an angle θ followed by an angle ϕ. Thus $G(\phi)G(\theta)v = G(\phi + \theta)v$ for all $v \in \mathbb{R}^2$. Hence $G(\phi)G(\theta) = G(\theta + \phi)$. ■

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Example: $G(\pi/6) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $G(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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Similarly, we can construct a rotation G such that $Gv = \|v\|_2 e_2$ or $G^\top v = -\|v\|_2 e_2$.

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Example: Consider $v := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}^\top \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$, where $c = 1/\sqrt{2} = s$. Note that $\theta = \pi/4$.

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Let $\theta \in [0, 2\pi]$. Then a Givens rotation in the x_1-x_2 plane in $\mathbb{R}^3$ is a matrix $G_{12}(\theta) \in \mathbb{R}^{3\times 3}$ that rotates each vector in the x_1-x_2 plane in $\mathbb{R}^3$ by an angle θ in the anti-clock-wise direction and $G_{12}(\theta)e_3 = e_3$.

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Consequently, the rotation $G_{12}(\theta)$ is given by

$$G_{12}(\theta) = \left[\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} c & -s & 0 \\ s & c & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

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The Givens rotation $G_{13}(\theta)$ is a rotation in x_1-x_3 plane. Hence we have

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- It follows that $G_{13}(\theta)G_{13}(\phi) = G_{13}(\theta+\phi)$ and $G_{13}(0) = I$ which shows that $G_{13}(\theta)$ is unitary and $G_{13}(\theta)^{-1} = G_{13}(-\theta) = G_{13}(\theta)^\top$.
- Ditto for the rotations $G_{12}(\theta)$ and $G_{23}(\theta)$.

Givens rotations in $\mathbb{R}^3$

The Givens rotation $G_{13}(\theta)$ is a rotation in x_1-x_3 plane. Hence we have

$$G_{13}(\theta)e_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \text{ and } G_{13}(\theta)e_3 = \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix}.$$

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- Ditto for the rotations $G_{12}(\theta)$ and $G_{23}(\theta)$.
- There are three Givens rotations in $\mathbb{R}^3$, namely, $G_{12}(\theta)$, $G_{23}(\theta)$ and $G_{13}(\theta)$.

Givens rotations in $\mathbb{R}^n$

Definition: A Givens rotation in $\mathbb{R}^n$ is a rotation in the coordinate plane, that is, in the plane spanned by two coordinate axes.

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Note that if $x := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ then the map $x \longmapsto G_{ij}(\theta)x$ alters only the components x_i and x_j by their linear combinations

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$$\begin{bmatrix} x_i \\ x_j \end{bmatrix} \longrightarrow \begin{bmatrix} x_i \cos\theta - x_j \sin\theta \\ x_i \sin\theta + x_j \cos\theta \end{bmatrix}$$

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Hence $G_{ij}(\theta)e_i = e_i \cos\theta + e_j \sin\theta$ and $G_{ij}(\theta)e_j = -e_i \sin\theta + e_j \cos\theta$.

Givens rotations in $\mathbb{R}^n$

For simplicity, we denote a Givens rotation in the x_i-x_j plane in $\mathbb{R}^n$ by G_{ij}. For $i < j$, we have

$$G_{ij} = \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
 & & & c & & -s & & & \\
\hline
 & & & & 1 & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & & s & & c & & & \\
\hline
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
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Note that G_{ij} differs from the identity matrix I in four enties (i, i), (i, j), (j, i) and (j, j). These entries are $c, -s, s$ and c, respectively.

Properties of Givens rotations in $\mathbb{R}^n$

- Let $G_{ij}(\theta)$ and $G_{ij}(\phi)$ be Givens rotations in $\mathbb{R}^n$. Then $G_{ij}(\theta)G_{ij}(\phi) = G_{ij}(\phi)G_{ij}(\theta) = G_{ij}(\theta + \phi)$ and $G_{ij}(0) = I$.

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$$G_{ij}(\theta)\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ cx_i - sx_j \\ \vdots \\ sx_i + cx_j \\ \vdots \\ x_n \end{bmatrix}.$$

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4 multiplication ops, c with xi, xj & s with xi, xj
2 subtraction ops
2 ops to calculate c and s.

- The transformation $A \longmapsto G_{ij}(\theta)A$ alters only the i-th and j-th rows of $A \in \mathbb{R}^{n\times p}$ and requires $8p$ flops as

$$e_i^\top(G_{ij}(\theta)A) = c(e_i^\top A) - s(e_j^\top A) \text{ and } e_j^\top(G_{ij}(\theta)A) = s(e_i^\top A) + c(e_j^\top A).$$

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Then the j-th component of $y := G_{ij}^\top x$ is zero. In fact, we have

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QR factorization by Givens rotations

We now use rotations for introducing zeros below the diagonal entries of a matrix. For $i < j$, we denote a rotation in the x_i-x_j plane G_{ji}.

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Let $A \in \mathbb{R}^{n\times n}$. First, we choose rotations $G_{21}, \ldots, G_{n1}$ that introduce zeros at (2,1), (3, 1), . . . , (n, 1) entries of A. Schematically

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Next, choose rotations $G_{32}, \ldots, G_{n2}$ that introduce zeros at (3, 2), (4, 2), ..., (n, 2) entries of $G_{n1}^\top \cdots G_{21}^\top A$ and so on. Schematically

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Theorem: Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$. Then $A = QR$, where $Q \in \mathbb{R}^{m\times m}$ is unitary and $R \in \mathbb{R}^{m\times n}$ is upper triangular.

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Proof: Choose a rotation G_{21} in the x_1-x_2 plane such that

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Next, choose a rotation G_{31} that creates a zero at $(3, 1)$ entry of $G_{21}^\top A$. Since G_{31} is a rotation in x_1-x_3 plane, it leaves the zero at $(2, 1)$ entry of $G_{21}^\top A$ unchanged.

QR factorization by Givens rotations

Theorem: Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$. Then $A = QR$, where $Q \in \mathbb{R}^{m\times m}$ is unitary and $R \in \mathbb{R}^{m\times n}$ is upper triangular.

Proof: Choose a rotation G_{21} in the x_1-x_2 plane such that

$$G_{21}^\top \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ 0 \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}. \text{ Then } G_{21}^\top A = \begin{bmatrix} \sigma_1 & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

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Continuing in this manner, choose rotations $G_{41}, \ldots, G_{m1}$ such that $G_{m1}^\top \cdots G_{21}^\top A$ has zeros in the first column at (2, 1), ..., (m, 1) entries.

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Similarly, choose rotations $G_{32}, \ldots, G_{m2}$ that create zeros at (3, 2), ..., (m, 2) entries of $G_{m1}^{\top} \cdots G_{21}^{\top} A$.

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Similarly, choose rotations $G_{32}, \ldots, G_{m2}$ that create zeros at (3, 2), ..., (m, 2) entries of $G_{m1}^\top \cdots G_{21}^\top A$. Then $G_{m2}^\top \cdots G_{32}(G_{m1}^\top \cdots G_{21}^\top A)$ has zeros in the first and second columns below the diagonals.

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$$\sum_{j=1}^{n} 8(m-j)(n-j+1) = 8\int_0^n (m-x)(n-x)dx = 4(mn^2 - \frac{n^3}{3}).$$

Stable generation of rotations

A naive method to generate a rotation such that $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}^{\top} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$ is to define $c := \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}}$ and $s := \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}}$.

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```
function [c,s] = Rotation(x1,x2);
% Input:  x1,x2 scalars
% Output:  c,s such that c^2 + s^2 = 1
% and -s*x1 + c*x2 = 0.
    if x2 == 0 c = 1; s = 0;
    else
    if abs(x2)>=abs(x1) k = x1/x2; % computes cot(θ)
    s = 1/sqrt(1+kˆ2); c = s*k;
    else
    t = x2/x1; % computes tan(θ)
    c = 1/sqrt(1+ tˆ2); s = c*t;
    end
end
```

Example

Let $A := \begin{bmatrix} 4 & 5 & 8 \\ 6 & 7 & 9 \\ 3 & 6 & 4 \end{bmatrix}$. We use rotation to compute QR factorization of A.

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The rotation G_{21} with $c := 0.5547$ and $s := 0.8321$ gives

$$G_{21}^{\top} A = \begin{bmatrix} 7.2111 & 8.5979 & 11.9261 \\ 0 & -0.2774 & -1.6641 \\ 3.0000 & 6.0000 & 4.0000 \end{bmatrix}.$$

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$$G_{31}^\top G_{21}^\top A = \begin{bmatrix} 7.8102 & 10.2430 & 12.5476 \\ 0 & -0.2774 & -1.6641 \\ 0 & 2.2372 & -0.8878 \end{bmatrix}.$$

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Finally, the rotation G_{32} with $c := -0.1230$ and $s := 0.9924$ gives

$$G_{32}^{\top} G_{31}^{\top} G_{21}^{\top} A = \begin{bmatrix} 7.8102 & 10.2430 & 12.5476 \\ 0 & 2.2543 & -0.6763 \\ 0 & 0 & 1.7607 \end{bmatrix}.$$