Lab Session 10

MA 453 : Matrix Computations Lab 2023 R. Alam

Singular values and eigenvalues

1. This problem illustrates SVD based image compression algorithm. The MATLAB commdand `A = im2double(imread('photo.jpg'))` reads an image (color as well as black and white) and convets it into a matrix `A`. The command `AG = rgb2gray(A)` converts the image to a grayscale image `AG`. On the other hand, the command `image(X)` displays the image (color or gray) stored in a matrix `X`.

   The MATLAB command `load clown` loads gray scale image of a clown in an array `X`. You may use `X` in place of `AG`.

   A compressed image of the grayscale image `AG` is computed as follows. Compute the SVD $AG = U\Sigma V^\top$ and the best k rank approximation $A_k := U \begin{bmatrix} \text{diag}(\sigma_1, \ldots, \sigma_k) & 0 \\ 0 & 0 \end{bmatrix} V^T$ for a chosen value of $k \leq \text{rank}(A)$. Then A_k represents a compressed image. The compressed image can be displayed by the command `image(`A_k`)`. Use the following commands:

   ```
   [U, S, V] = svd(AG); ; Ak = U(:, 1:k)*S(1:k, 1:k)*V(:,1:k)'; image(Ak)
   ```

   Write a MATLAB function or an `m-file` that takes an image 'photo.jpg' and rank k as input and returns a compressed image as output.

   The storage required for A_k is $k(m+n)$ whereas the storage required for the full image is mn. Therefore, $\frac{(m+n)k}{mn}$ gives the compression ratio for the compressed image. Also the error in the representation is $\frac{\sigma_{k+1}}{\sigma_1}$.

   Choose a photo (jpg, png or any format) and run the above commands for various choices of k and make a table that records the relative errors and compression ratios for each choice.

   How does the choice of approximating rank k affect the visual qualities of the images? There are no precise answers here. Your results will depend upon the images you choose and the judgments you make.

   Color image: For an m-by-n color image the command
   `X = im2double(imread('photo.jpg'));`
   produces a three-dimensional $m \times n \times 3$ array X with m-by-n integer subarrays for the red, green, and blue intensities. It would be possible to compute three separate m-by-n singular value decompositions of the three colors, i.e., SVD of `X(:, :, r)` for $r = 1, 2, 3$. Then rank k approximation of the three matrices can be used to compress the image.

   An alternative that requires less work involves altering the dimensions of X with
   `X = reshape(X,m,3*n);` and then computing one m-by-3n SVD.

   Choose a color image and perform SVD based image compression.

2. A random matrix with normalized columns and specified singular values is generated by MATLAB command `gallery('randcolu', x),` where tt x is a n-vector containing

singular values in descending order and satisfies $\|\mathrm{x}\|_2^2 = n$. For more information type `help private/randcolu`. If σ_j's are specified singular values and $\hat{\sigma}_j$'s are computed singular values using `svd` command then compute the relative errors $|\sigma_j - \hat{\sigma}_j|/\sigma_j$ for $j = 1 : n$. Does `svd` compute singular values with high relative accuracy?

3. Write a MATLAB function `mypolar` for computing polar decomposition of an $n \times n$ matrix

```
function [U, R] = myploar(A)
% myploar computes polar decomposition A = UR using svd of A,
% where U is unitary and R is positive semidefinite.
```

   Compute polar decomposition of the matrix `magic(10)` and the Frank matrix given by `A = gallery('frank', 12)`.

4. The MATLAB command `eig` computes eigenvalues and eigenvectors of a square matrix and the command `schur` computes Schur decomposition of a square matrix. Type help eig and schur for more information. What is the largest eigenvalue of `magic(n)` for $n = 4, 5, 6$. Can you explain the results? Compute Schur decomposition of `magic(5)`.

5. The MATLAB command `svd` computes SVD of a matrix. What is the largest singular value of `magic(n)` for $n = 4, 5, 6$. Can explain the results?

6. The MATLAB commands `P = gallery('pascal',12)` and `F = gallery('frank',12)` generate Pascal and Frank matrices of size 12. Both the matrices have the property that if λ is an eigenvalue then so is $1/\lambda$. If `e = eig(P)` then type `[ e'; 1./e']`. Can you see $(\lambda, 1/\lambda)$ pairing? How well do the computed eigenvalues preserve this property? Repeat the same for Frank matrix F.

7. Consider the matrix given by the MATLAB command `A = gallery(5)`. Compute $B := A^5$. What are the the eigenvalues of B? Now compute the eigenvalues of A using MATLAB command `eig`. What are the eigenvalues of A? Now plot the eigenvalues with the following commands:

```
A = gallery(5)
e = eig(A)
plot(real(e),imag(e),'r*',0,0,'ko')
axis(.1*[-1 1 -1 1])
axis square
```

   What do you observe? Now repeat the experiment with a matrix where each element is perturbed by a single roundoff error. The elements of `gallery(5)` vary over four orders of magnitude, so the correct scaling of the perturbation is obtained with

```
e = eig(A + eps*randn(5,5).*A)
```

   Put this statement, along with the plot and axis commands, on a single line and use the up arrow to repeat the computation several times. You will see that the pentagon flips orientation and that its radius varies between 0.03 and 0.07, but that the computed eigenvalues of the perturbed problems behave pretty much like the computed eigenvalues of the original matrix.

   This experiment provides evidence for the fact that the computed eigenvalues are the exact eigenvalues of a matrix $A + E$ where the elements of E are on the order of roundoff error compared to the elements of A. This is the best we can expect to achieve with floating-point computation.

8. Numerically, an eigenvalue is real if the imaginary part is negligible. Set `tol =` 10^{-6} and consider MATLAB code

```
A = randn(8); sum( abs( imag ( eig(A) ) ) < tol).
```

What output does this code return? Justify your answer.

Now generate 1000 random (normally distributed) matrices of size 10. Run the code fragment on these matrices and prepare a histogram. Type `help hist` for information about histogram. What information do you get from the histogram? The histogram gives a distribution of the number of real eigenvalues.

*** End ***