NAME: MOHAMMAD HUMAM KHAN

ROLL NO: 180123057

MA-473 : COMPUTATIONAL FINANCE

END SEM → PAPER-I

(1) Given PDE

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2_{(S)} S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \; t \le T \\ V(S,T) = V_T(S), & 0 \le S \end{cases} \to (1)$$

Using following transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m} \\ \tau = T - t \\ V(S,t) = (S+P_m)\, \bar{V}(\xi,\tau) \end{cases}$$

then we have $\boxed{S = \dfrac{P_m \xi}{1-\xi}}$ -(2) and $\boxed{S+P_m = \dfrac{P_m}{1-\xi}}$ -(3)

Now $$\frac{d\xi}{dS} = 1 \cdot \frac{1}{S+P_m} + S \cdot \left(\frac{-1}{(S+P_m)^2} \right) \cdot 1$$

$$\Rightarrow \frac{d\xi}{dS} = \frac{P_m}{(S+P_m)^2}$$

$$\Rightarrow \boxed{\frac{d\xi}{dS} = \frac{(1-\xi)^2}{P_m}} \to (4) \quad \{\text{Using } (3)\}$$

Now we have

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[(S+P_m)\,\bar{V}(\xi,\tau)\right] = (S+P_m)\,\frac{\partial \bar{V}(\xi,\tau)}{\partial t}$$

$$= (S+P_m)\,\frac{\partial \bar{V}}{\partial \tau}\cdot\frac{d\tau}{dt} \Rightarrow \boxed{\frac{\partial V}{\partial t} = -(S+P_m)\frac{\partial \bar{V}}{\partial \tau}}$$

Using 3, we have

$$\boxed{\frac{\partial V}{\partial t} = -\frac{P_m}{1-\xi}\,\frac{\partial \bar{V}}{\partial \tau}}$$

Now $\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[(S+P_m)\bar{V}(\xi,\tau)\right] = 1\cdot\bar{V}(\xi,\tau) + (S+P_m)\frac{\partial \bar{V}(\xi,\tau)}{\partial S}$

$$= \bar{V} + (S+P_m)\,\frac{\partial \bar{V}}{\partial \xi}\cdot\frac{d\xi}{dS}$$

$$= \bar{V} + \frac{P_m}{1-\xi}\cdot\frac{\partial \bar{V}}{\partial \xi}\cdot\frac{(1-\xi)^2}{P_m} \quad \left\{\begin{array}{l}\text{Using } 3\\ \text{and } 4\end{array}\right\}$$

$$\Rightarrow \boxed{\frac{\partial V}{\partial S} = \bar{V} + (1-\xi)\frac{\partial \bar{V}}{\partial \xi}}$$

Now $\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{\partial V}{\partial S}\right] = \frac{\partial}{\partial S}\left[\bar{V} + (1-\xi)\frac{\partial \bar{V}}{\partial \xi}\right]$

$$= \frac{\partial \bar{V}}{\partial \xi}\cdot\frac{d\xi}{dS} + \frac{\partial}{\partial S}\left[(1-\xi)\cdot\frac{\partial \bar{V}}{\partial \xi}\right]$$

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$$\Rightarrow \frac{\partial^2 V}{\partial S^2} = \frac{\partial \bar{V}}{\partial \xi_1} \cdot \frac{d\xi_1}{dS} + \frac{\partial}{\partial \xi_1}\left[(1-\xi_1)\frac{\partial \bar{V}}{\partial \xi_1}\right] \cdot \frac{d\xi}{dS}$$

$$= \frac{\partial \bar{V}}{\partial \xi_1} \cdot \frac{d\xi_1}{dS} - \frac{\partial \bar{V}}{\partial \xi_1} \cdot \frac{d\xi_1}{dS} + (1-\xi_1) \cdot \frac{\partial^2 \bar{V}}{\partial \xi_1^2} \cdot \frac{d\xi_1}{dS}$$

$$\Rightarrow \boxed{\frac{\partial^2 V}{\partial S^2} = \frac{(1-\xi_1)^3}{P_m} \frac{\partial^2 \bar{V}}{\partial \xi_1^2}}$$

Now let $\bar{\sigma}(\xi_1) = \sigma(S(\xi_1)) = \sigma\left(\frac{P_m \xi_1}{1-\xi_1}\right)$

Now putting all values in PDE (1), we get

$$\boxed{\begin{aligned} \frac{\partial \bar{V}}{\partial \tau} &= \frac{\bar{\sigma}^2(\xi_1) \cdot \xi_1^2 (1-\xi_1)^2}{2} \cdot \frac{\partial^2 \bar{V}}{\partial \xi_1^2} + (r-\delta)\, \xi_1 (1-\xi_1) \frac{\partial \bar{V}}{\partial \xi_1} \\ &\quad - \left[r(1-\xi_1) + \delta \xi_1\right] \bar{V} \\ & 0 \le \xi_1 < 1, \quad \tau \ge 0 \end{aligned}}$$

Now assuming $\bar{V}$ is smooth function of ξ_1.
then Eq^n also holds at $\xi_1 = 1$.

Now $V(S,T) = (S+P_m)\, \bar{V}(\xi_1, 0) = \bar{V}(\xi_1, 0) \frac{P_m}{1-\xi_1}$

Hence the condition $V(S,T) = V_T(S)$ can be re-written as:

$$\boxed{\bar{V}(\xi_1,0) = V_T\left(\frac{P_m \xi_1}{1-\xi_1}\right)\cdot\frac{1-\xi_1}{P_m}}$$

This is ~~called~~ Initial condition for PDE 5.

Boundary Conditions

At $\xi_1 = 0$ and $\xi_1 = 1$, PDE 5 degenerates to O.D.E at boundaries

$\boxed{\text{At } \xi_1 = 0}$ $$\frac{\partial \bar{V}(0,\tau)}{\partial \tau} = -r\bar{V}(0,\tau).$$

$$\Rightarrow \frac{d\bar{V}(0,\tau)}{\bar{V}(0,\tau)} = -r\,d\tau$$

$$\Rightarrow \boxed{\bar{V}(0,\tau) = \bar{V}(0,0)\, e^{-r\tau}}$$

Similarly $\boxed{\text{at } \xi_1 = 1}$

$$\frac{\partial \bar{V}(1,\tau)}{\partial \tau} = -\delta\,\bar{V}(1,\tau)$$

$$\Rightarrow \frac{d\bar{V}(1,\tau)}{\bar{V}(1,\tau)} = -\delta\, d\tau$$

$$\Rightarrow \boxed{\bar{V}(1,\tau) = \bar{V}(1,0)\, e^{-\delta\tau}}$$

Hence for PDE 5, the two solution of ODE provide boundary values and no. B.C. are needed for the PDE to have unique solution.

Hence the final PDE becomes:

$$\begin{cases} \dfrac{\partial \bar{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\,\xi^2(1-\xi)^2\dfrac{\partial^2 \bar{V}}{\partial \xi^2} + (r-\delta)\xi(1-\xi)\dfrac{\partial \bar{V}}{\partial \xi} - [r(1-\xi)+\delta\xi]\bar{V} \\ \qquad 0 \le \xi \le 1, \; 0 \le \tau \\ \bar{V}(\xi,0) = \dfrac{1-\xi}{P_m} V_T\left(\dfrac{P_m \xi}{1-\xi}\right), \quad 0 \le \xi \le 1 \quad \rightarrow \text{I.C} \\ \left.\begin{aligned} \bar{V}(0,\tau) &= \bar{V}(0,0)\, e^{-r\tau} \\ \bar{V}(1,\tau) &= \bar{V}(1,0)\, e^{-\delta\tau} \end{aligned}\right\} \rightarrow \text{B.C.} \end{cases}$$

Hence B.S. PDE is transformed from infinite Domain $S \ge 0$ to finite domain $0 \le \xi \le 1$.

(2)

American option pricing problem can be formulated as variational problem with class of comparison functions defined as :

$$K = \{ v \in \mathcal{C}^0 : \frac{\partial v}{\partial x} \text{ is piecewise } \mathcal{C}^0,$$
$$v(x,\tau) \geq g(x,\tau) \ \forall\ x,\tau$$
$$v(x,0) = g(x,0)$$
$$v(x_{max},\tau) = g(x_{max},\tau)$$
$$v(x_{min},\tau) = g(x_{min},\tau) \}$$

where

$$g(x,\tau) = \begin{cases} \exp\{\frac{\tau}{4}((q_\delta-1)^2+4q)\} \max\{e^{\frac{x}{2}(q_\delta-1)} - e^{\frac{x}{2}(q_\delta+1)}, 0\} & \text{for put} \\ \exp\{\frac{\tau}{4}((q_\delta-1)^2+4q)\} \max\{e^{\frac{x}{2}(q_\delta+1)} - e^{\frac{x}{2}(q_\delta-1)}, 0\} & \text{for call} \end{cases}$$

with $q = \frac{2r}{\sigma^2}$ and $q_\delta = \frac{2(r-\delta)}{\sigma^2}$

Now, let y be the exact solution of American option Pricing problem formulated as Linear Complimentarity Problem. As solution of Partial Differential Inequality, y is $\mathcal{C}^2$-smooth on continuation region and $y \in K$.

Also for $v \in K$

$$v \geq g \quad , \quad \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$$

$$\Rightarrow \boxed{\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-g)\, dx \geq 0} \quad -(1)$$

Also from ~~can~~ complimentarity condition, we have:

$$\boxed{\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g)\cdot dx = 0} \quad -(2)$$

Now Subtracting (2) from (1), we get

$$\boxed{\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-y)\, dx \geq 0}$$

$$\Rightarrow \int_{x_{min}}^{x_{max}} \frac{\partial y}{\partial \tau}(v-y)\, dx - \int_{x_{min}}^{x_{max}} (v-y)\cdot\frac{\partial^2 y}{\partial x^2}\, dx \geq 0$$

$$\Rightarrow \boxed{\int_{x_{min}}^{x_{max}} \frac{\partial y}{\partial \tau}(v-y)\, dx - \left[(v-y)\frac{\partial y}{\partial x}\Bigg|_{x_{min}}^{x_{max}} - \int_{x_{min}}^{x_{max}} \left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial x}\, dx \right] \geq 0} \quad \downarrow (3)$$

{ Using Integration by Parts in 2^{nd} Integral
1^{st} func. $= v-y$, 2^{nd} func. $= \frac{\partial^2 y}{\partial x^2}$. }

Now at boundary points x_{min} and x_{max}, we have:

$$\boxed{\begin{aligned} &v(x_{min},\tau) = g(x_{min},\tau) \\ \text{and } &v(x_{max},\tau) = g(x_{max},\tau) \end{aligned}}$$

and

$$\boxed{\begin{aligned} &y(x_{min},\tau) = g(x_{min},\tau) \\ \text{and } &y(x_{max},\tau) = g(x_{max},\tau) \end{aligned}}$$

Hence, we have

$$v(x_{min}, \tau) = y(x_{min}, \tau) \quad \text{and} \quad v(x_{max}, \tau) = y(x_{max}, \tau)$$

Hence, the term $\boxed{\left.\frac{\partial y}{\partial x}(v-y)\right|_{x_{min}}^{x_{max}} = 0}$

Putting this in Eq^n 3, we get

$$\boxed{I(y;v) = \int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} \cdot (v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right) dx \geq 0 \quad \forall v \in K} \quad \to 4$$

Now, this inequality holds $\forall\, v \in K$. Hence choosing $v = y$ we have $I(y;y) = 0$ and the integral $I(y;v)$ takes its minimum value i.e.

$$\boxed{\min_{v \in K} I(y;v) = I(y;y) = 0}$$

2 (a) We now consider $\hat{y} \in K$ that is not C^2-smooth on continuation region. We want to find $\hat{y} \in K$

$$\left.\begin{array}{l} \text{s.t } I(\hat{y}; v) \geq 0 \quad \forall\, v \in K \\ \text{and } \inf_{v \in K} I(\hat{y}; v) = 0 \end{array}\right\} \text{This is called Weak Formulation}$$

Now, consider the following approximations for $\hat{y}$ and v:

$$\left.\begin{array}{l} \hat{y} = \sum_i w_i(\tau)\, \varphi_i(x) \\ v = \sum_i v_i(\tau)\, \varphi_i(x) \end{array}\right\} \text{where } \varphi_i \text{ are the basis functions.}$$

This formulation separates the independent variables τ and x. Hence the same x-grid is applied for all τ, which results in rectangular grid in (x, τ)-plane. The time dependance is incorporated in coeff. functions w_i and v_i. Since φ_i represent x_i-grid, we consider semi-discretization of eq^n 4 as follows:

$$\int \left\{ \left(\sum_i \frac{dw_i}{d\tau} \varphi_i\right)\left(\sum_j (v_j - w_j)\varphi_j\right) + \left(\sum_i w_i \varphi_i'\right)\left(\sum_j (v_j - w_j)\varphi_j'\right) \right\} dx \geq 0$$

$$\Rightarrow \boxed{\sum_i \sum_j \frac{dw_i}{d\tau}(v_j - w_j) \int \varphi_i \varphi_j \, dx + \sum_i \sum_j w_i (v_j - w_j) \int \varphi_i' \varphi_j' \, dx \geq 0}$$

Now writing the inequality into vector-matrix form, we have

$$\left(\frac{dw}{dt}\right)^T B(v-w) + w^T A(v-w) \geq 0$$

$$\Rightarrow \boxed{(v-w)^T \left(B\frac{dw}{dt} + Aw\right) \geq 0} \longrightarrow (\#)$$

where matrices A and B are constructed using assembling algorithm s.t.

$$a_{ij} = \int_{\mathcal{D}} \phi_i \phi_j \, dx \qquad \text{and} \qquad b_{ij} = \int_{\mathcal{D}} \phi_i' \phi_j' \, dx$$

where $\mathcal{D}$ is domain.

For equidistant steps, we have for ϕ_i as hat functions

$$A = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

$$B = \frac{h}{6}\begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & \ddots & \\ & & \ddots & \ddots & \\ & & & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}$$

Now time is discretized.

Define $w^{(n)} = w(\tau_n)$ and $v^{(n)} = v(\tau_n)$

Now considering:

$$\boxed{\frac{w_{i,n+1} - w_{in}}{\Delta\tau} = \theta\left(\frac{w_{i+1,n+1} - 2w_{i,n+1} + w_{i-1,n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{w_{i+1,n} - 2w_{in} + w_{i-1,n}}{\Delta x^2}\right)}$$

Now using this approximation in (#), we get

$$\left(v^{(n+1)} - w^{(n+1)}\right)^T \left(B\frac{1}{\Delta\tau}\left(w^{(n+1)} - w^{(n)}\right) + \theta A w^{(n+1)} + (1-\theta) A w^{(n)}\right) \geq 0$$

$\forall\, n.$

Now for $\theta = 1/2$, we have Crank-Nicolson method.

$$\Rightarrow \boxed{\left(v^{(n+1)} - w^{(n+1)}\right)^T \left((B + \Delta\tau\,\theta A) w^{(n+1)} + (\Delta\tau(1-\theta)A - B)\, w^{(n)}\right) \geq 0}$$

Now let $C = B + \Delta\tau\,\theta A$

$r = (B - \Delta\tau(1-\theta)A)\, w^{(n)}$

then we have:

$$\boxed{\left(v^{(n+1)} - w^{(n+1)}\right)^T \left(C w^{(n+1)} - r\right) \geq 0}$$

which is fully discretized version of $\mathcal{I}(\hat{y}; v) \geq 0$

Side conditions

We have $\hat{y}(x, \tau) \geq g(x, \tau)$

$$\Rightarrow \quad \sum w_i(\tau)\, \phi_i(x) \geq g(x, \tau)$$

For hat functions ϕ_i ($\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for $j \neq i$) and $x = x_j$, we have $w_j(\tau) \geq g(x_j, \tau)$.

Then with $\tau = \tau_n$, we have

$$\boxed{w^{(n)} \geq g^{(n)}}$$

and similarly $\boxed{v^{(n)} \geq g^{(n)}}$

(3) For showing equivalence of FEM and FDM, consider FEM and FDM version American option pricing problem as follows:

FEM

Construct w s.t.

$\forall\, v \geq g$.

$(v-w)^T (Cw - r) \geq 0, \quad w \geq g$

FDM

Find w s.t.

$Cw - r \geq 0, \quad w \geq g$

$(Cw - r)^T (w - g) = 0$

The proof of equivalence of FDM and FEM is as follows:

(FDM) $\Rightarrow$ (FEM)

Let w solve FDM, so $w \geq g$ and

$$(v-w)^T (Cw-r) = (v-g)^T (Cw-r) - (w-g)^T (Cw-r)$$

Now, $Cw - r \geq 0$ and $(w-g)^T (Cw-r) = 0$

$$\Rightarrow (v-w)^T (Cw-r) = (v-g)^T (Cw-r) \geq 0 \quad \forall\, v \geq g.$$

$$\Rightarrow \boxed{(v-w)^T (Cw-r) \geq 0 \quad \forall\, v \geq g}$$ Hence w solves FEM.

$(FEM) \Rightarrow (FDM)$

Let w solve FEM, so $w \geq g$ s.t.

$$v^T(Cw-r) \geq w^T(Cw-r) \quad \forall\, v \in K$$

Now suppose k^{th} component of $Cw-r$ is negative and make v_k arbitrarily large. then LHS becomes arbitrarily small, which is contradiction.

$$\Downarrow$$

$$Cw-r \geq 0.$$

Hence $w \geq g \Rightarrow \boxed{(w-g)^T(Cw-r) \geq 0} \rightarrow (*)$

Now in (FEM), we have

$$(v-w)^T(Cw-r) \geq 0$$

Put $v=g$, then $\boxed{(w-g)^T(Cw-r) \leq 0} \rightarrow (\#)$

Using $(*)$ and $(\#)$, we conclude that

$$\boxed{(w-g)'(Cw-r) = 0}$$

Hence w solves FDM.

Combining both parts, we have FEM and FDM formulation are equivalent i.e. solution of one problem satisfies another as well.