MA668: Algorithmic and High Frequency Trading
Lecture 20

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Table 4.17

Distance to Midprice	At Post			At Exit		
	Posts	% Exec	Exec	Posts	% Exec	Exec
< 2	905	78.6	3.3	988	88.4	4.1
2	3,053	64.7	9.2	3,508	76.4	12.5
3	5,193	55.4	13.5	6,236	67.5	19.7
4	5,617	44.4	11.7	6,448	51.5	15.6
5	6,374	34.9	10.4	7,557	45.5	16.1
6	7,626	27.6	9.8	7,586	29.9	10.6
7	7,996	20.2	7.6	7,624	20.4	7.3
8	7,826	15.9	5.8	8,062	14.2	5.4
9	7,675	12.3	4.4	7,946	7.5	2.8
10	7,967	8.6	3.2	7,487	6.1	2.1
> 10	195,415	2.3	21	192,205	0.4	3.8

Table 4.17 Messages by distance to midprice at post and at exit (AAPL 2013-07-30).

Figure: Table 4.17

Figure 4.13

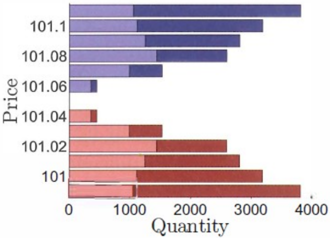

Figure 4.13 Illustration of orders posted and executed as described in Table 4.17.

Figure: Figure 4.13

Table 4.17 and Figure 4.13 (Contd ...)

1. For intraday trading, it is very important to understand the posting and cancellation dynamics, especially around the bid and ask.
2. Table 4.17: Looks at the orders posted by their distance to the mid-price ("Distance to Mid", k) for AAPL on July 30th, 2013.
3. Figure 4.13: Illustrates the contents of the table.
4. Table 4.17: The second column ("Posts") counts the number of messages posted k ticks (cents) from the mid price.
5. Figure 4.13: Displays this visually using a hypothetical mid-price of \$101.05, and split the quantities evenly between the bid and the ask.
6. Thus, for example, the total quantity posted two ticks from the mid-price (3,053 units) is displayed as 1,527 units posted at \$101.07 and 1,527 units posted at \$101.03 (the total length of the bars).

Table 4.17 and Figure 4.13 (Contd ...)

1. Table 4.17: The third column ("% Exe") looks at the percentage of those posted messages that were executed, that is, the posted order was crossed with an incoming MO.
2. Figure 4.13: This is illustrated by using a lighter color for the orders that were executed (and a darker one for those cancelled).
3. Thus, for example, of the 1,527 units posted at $101.07, 64.7% (988 units) were executed.
4. Table 4.17: Finally, the fourth column ("Exe") describes the percentage of the total number of executed orders that were executed at that level.
5. So, these 988 orders executed at $101.07 plus the 988 executed at $101.03, represent 9.2% of the total number of orders executed that day.

Table 4.17 and Figure 4.13 (Contd ...)

1. If we consider that the (one-minute time-average) quoted spread for that day is 10.3 cents on average (Q1: 8.5, median: 10.2, Q3: 11.7), most of the time the distance to the mid-price (half of the quoted spread) is between 4 and 6 cents.
2. Table 4.17: Using the information we compute that 26% of orders executed were initially posted at between 1 and 3 cents from the mid-price, but only 32% are posted between 4 and 6 and the remaining 42% of orders executed had been originally posted relatively far from the mid-price (7+ cents away).
3. If, on the other hand, we look at the distance from the mid-price at the time the trade was executed, not posted, in Table 4.17, we find that the distance to the mid-price is (naturally) shorter, and we can compute that 36% of orders were executed at prices between 1 and 3 cents from the mid-price, and 42% at prices between 4 and 6, so that only 22% of executions were relatively far from the mid-price (7+ cents away).
4. This illustrates the point made earlier (when comparing the volatility of the effective and the quoted spreads) that executions tend to occur more often when the spread is narrower, and hence the effective spread will naturally be less volatile than the quoted spread.

Figure 4.14

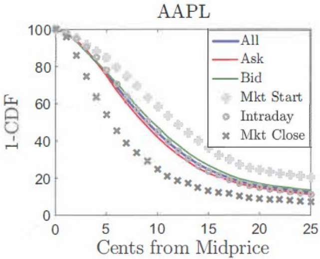

Figure 4.14 Survivor function for executions as a function of distance from midprice.

Figure: Figure 4.14

Figure 4.14

1. Figure 4.14: Displays the survivor function, $S(x)$ (one minus the CDF: $S(x) = Pr(X > x) = I - F(x)$) of total executions, as the distance from the price at which the original LO was posted increases.
2. This represents an approximation to the "fill probability": The probability that a posted order is executed. The thick blue line describes the distribution in Table 4.17.
3. We have also included the same distribution separating executions on the bid and ask side, and it is interesting that the distribution for bid (ask) side executions lies systematically below (above) the one for all executions.
4. This indicates that market buy orders tended to occur much closer to the mid-price than market sell orders on this particular day, which had an overall positive order flow for AAPL shares and a slight price increase from market open to market close.

Figure 4.14 (Contd ...)

1. Figure 4.14: We have also included total executions separated by the time of day: The first half hour after the market opens (Mkt Start), the last half hour before the market closes (Mkt Close), and the time in between (Intraday).
2. We observe that Mkt Close tends to be below that of Intraday, implying that during the last half hour of trading, executions tend to be close to the mid-price, which is consistent with the pattern of the quoted spread in Figure 4.6.
3. But the difference does not seem to be very large and may be statistically insignificant.
4. What happens at the market open does look very different, as the distribution is above and quite far away from that for Intraday.
5. It appears that the wider spreads we observed in Figure 4.6 and the uncertainty from Figure 4.7 combine to generate executions for orders posted quite far from the midprice.

Figure 4.15

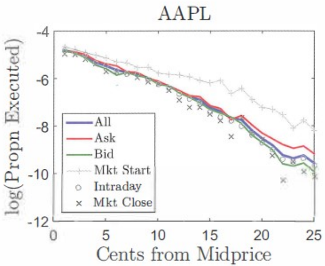

Figure 4.15 Log of the proportion of posted orders that are executed as a function of distance from midprice.

Figure: Figure 4.15

Figure 4.15 (Contd ...)

1. Figure 4.15: Looks at the same data from a different angle.
2. In it we consider (in logs) the proportion of orders posted a certain distance from the mid price, that were eventually executed.
3. Interpreting this proportion as a probability, the figure displays the natural decreasing relationship between the distance from the mid-price and the probability of the order being executed.
4. We have drawn these curves for:
   - A. All executions, aggressive buys and sells and executions by time of day.
   - B. Around the market open, the market close and the rest of the day.
5. All of them are very similar with only one exception, namely, that for the first half hour of the trading day (Mkt Start).

Figure 4.15 (Contd ...)

1. What we observe (looking at the underlying data) is that, at Mkt Start, an unusually high proportion of trades which were posted six cents from the mid-price were later executed.
2. This generates the shift in the CDF we observe in Figure 4.15.
3. Looking at the quoted spreads during that time, we find that the mean was 15.2 cents on average (Q1: 12.5, median: 14.2, Q3: 19.0).
4. This suggests that as early morning uncertainty over the "true market price" was reduced, the quoted spread was slow to react and a relatively large number of executions occurred.
5. Further, this happened when the quoted spread had fallen to around 12 cents.

Hidden Orders

When discussing market quality earlier (Section 4.3), and spreads in particular, we saw that one of the reasons why the quoted spread is generally greater than the effective spread is the presence of posted orders that are not visible to market participants, but that will match with incoming MOs ahead of existing visible ones (at a price at or better than the current bid/ask). These are **hidden orders**

Table 4.18

Asset	Mean	StdDev	P01	Q1	Median	Q3	P99
ISNS	4	59	0	0	0	0	100
FARO	31	154	0	0	0	0	600
MENT	117	568	0	0	0	0	2,150
AAPL	3,849	5,905	0	1,052	2,220	4,504	26,547
ISNS	1.2	10.7	0.0	0.0	0.0	0.0	99.6
FARO	9.9	27.1	0.0	0.0	0.0	0.0	100.0
MENT	9.4	24.1	0.0	0.0	0.0	0.0	100.0
AAPL	44.6	16.9	0.0	33.5	44.9	56.0	83.7

Table 4.18 Execution against hidden orders (volume (Q) and percentage).

Figure: Table 4.18

Table 4.18 (Contd ...)

1. Table 4.18 is split into two panels.
2. The top panel of the table describes the quantity executed against hidden orders in NASDAQ per minute, for each minute of 2013.
3. As we can see, for the less traded assets, ISNS, FARO and MENT, there is little trading taking place against hidden orders (less than 25 percent of the time), though when it happens it can be quite significant. But for AAPL, the case is quite different.
4. We find trading against hidden orders more than 75 percent of the time, and for a substantial amount of shares (more than 1,000 units per minute).
5. Note that these large quantities are not indicative of large trades, but rather of quite frequent ones: the distribution of the average size of an MO executed against a hidden order (per minute) has a mean of 127, with Q1 equal to 94 and Q3 to 148 shares per trade.

Table 4.18 (Contd ...)

1. The bottom panel of Table 4.18 considers the same variable, the quantity of shares executed against hidden orders, but rather than in absolute numbers, as a proportion of the total number of shares executed (in that minute).
2. For ISNS, FARO and MENT, executions are relatively infrequent, and when they occur against hidden orders they tend to be isolated trades.
3. In those cases, the hidden order is a large proportion, if not one hundred percent, of all shares traded during that minute.
4. For AAPL, execution against hidden orders is a common phenomenon and half the time they represent between 33 and 56 percent of all trades.
5. An agent posting visible offers for AAPL at the bid and ask (during 2013) found her offers trumped by more aggressive hidden ones relatively often.

Prelude

1. Stochastic control problems arise in many facets of financial modelling.
2. The classical example is the optimal investment problem introduced and solved in continuous-time by Merton (1971).
3. Of course there is a multitude of other applications, such as optimal dividend setting, optimal entry and exit problems, utility indifference valuation and so on.
4. In general: The all-encompassing goal of stochastic control problems is to maximize (or minimize) some expected profit (cost) function by tuning a strategy which itself affects the dynamics of the underlying stochastic system and to find the strategy which attains the maximum (minimum).

Prelude (Contd ...)

1. For example, in the simplest form of the Merton problem, the agent is trying to maximize the expected utility of future wealth by trading a risky asset and a risk-free bank account.
2. The agent's actions affect her/his wealth, but at the same time the uncertain dynamics in the traded asset modulates the agent's wealth in a stochastic manner.
3. The resulting optimal strategies are tied to the dynamics of the asset and perhaps also to the agent's wealth.
4. It is a surprising fact that, in many cases, the optimal strategies turn out to be Markov in the underlying state variables, even if the agent is considering non-Markovian controls (which may depend on the entire history of the system).

Prelude (Contd ...)

1. Tools which keeps coming to the forefront when solving stochastic control problems:
   - (A) The dynamic programming principle (DPP).
   - (B) The related non-linear partial differential equation (PDE) known as the Hamilton-Jacobi-Bellman (HJB) equation, which is also called the dynamic programming equation (DPE).
2. The DPP allows a stochastic control problem to be solved from the terminal date backwards.
3. The HJB equation/DPE can be viewed as its infinitesimal version.
4. In our discussion: The subtle mathematical issues are not addressed and focus is instead placed on the mechanics which allow for immediate application to algorithmic trading problems.

Examples of Control Problems in Finance

1. We next provide a few examples of financially motivated stochastic control problems.
   - A. The first example is a classical one in finance and pertains to optimal investment over long time horizons.
   - B. The second is one of the first algorithmic trading control problems and pertains to the optimal liquidation of assets.
   - C. The third refers to optimal placement of orders in a Limit Order Book (LOB)
2. All of these are essentially toy models and the last two will encompass the focus of the discussion in subsequent chapters.

The Merton Problem

1. As a first example let us consider the classical portfolio optimization problem of Merton (1971), in which the agent seeks to maximize expected (discounted) wealth by trading in a risky asset and the risk-free bank account.
2. Specifically: At time t, she/he places π_t dollars of her/his total wealth X_t in the risky asset S_t and seeks to obtain the so-called value function:

$$H(S,x) = \sup_{\pi \in \mathcal{A}_{0,T}} \mathbb{E}_{S,x}\left[U\left(X_T^{\pi}\right)\right], \tag{1}$$

which depends on the current wealth x and asset price S and the resulting optimal trading strategy π, where,

$$\begin{aligned} dS_t &= (\mu - r)\, S_t dt + \sigma S_t dW_t,\ S_0 = S,\ \text{(Risky Asset)} && (2)\\ dX_t^{\pi} &= \left(\pi_t\left(\mu - r\right) + rX_t^{\pi}\right) dt + \pi_t \sigma dW_t,\ X_0^{\pi} = x.\ \text{(Wealth)} && (3) \end{aligned}$$

3. In the above, μ represents the (expected) continuously compounded rate of growth of the traded asset, while r is the continuously compounded rate of return of the risk-free bank account.

The Merton Problem (Contd ...)

1. $W = (W_t)_{\{0 \le t \le T\}}$ is a Brownian motion.
2. $S = (S_t)_{\{0 \le t \le T\}}$ is the discounted price process of a traded asset.
3. $\pi = (\pi_t)_{\{0 \le t \le T\}}$ is a self-financing trading strategy corresponding to having π_t invested in the risky asset at time t (with the remaining funds in the risk-free bank account).
4. $X^\pi = (X_t^\pi)_{\{0 \le t \le T\}}$ is the agent's discounted wealth process given that she/he follows the self-financing strategy π.
5. $U(x)$ is the agent's utility function.
6. $\mathcal{A}_{t,T}$ is a set of strategies, called the "admissible set", corresponding to all $\mathcal{F}$-predictable self-financing strategies that have $\int_t^T \pi_s^2 ds < \infty$. This constraint allows strong solutions of (3) to exist.