Homework - 6

MA423 : Matrix Computations 2023 R. Alam

Complex rotation, complex Householder reflector, and QR factorization

1. Let $A \in \mathbb{R}^{n\times n}$ be SPD with Cholesky decomposition $A = GG^\top$. Let $v \in \mathbb{R}^n$. Explain why $A+vv^\top$ has a unique Cholesky decomposition. Let $\begin{bmatrix} G^\top \\ v^\top \end{bmatrix} = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$ be a QR factorization, where $R \in \mathbb{R}^{n\times n}$ is upper triangular. Explain why R is nonsingular. Show that R can be chosen to have positive diagonal entries. Assuming that R has positive diagonal entries, deduce the Cholseky decomposition of $A + vv^\top$.

2. If $Q \in \mathbb{R}^{2\times 2}$ is orthogonal then show that there exists $\theta \in [0, 2\pi)$ such that either
$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$
Obviously, Q is a rotation in the first case. Show that, in the second case, Q is a Householder reflector in $\mathbb{R}^2$ which reflects a vector through the line $y = \tan(\theta/2)x$. Determine a unit Householder vector u and show that $Q = I - 2uu^\top$. Conclude that a 2×2 orthogonal matrix is either a rotation or a reflection.

3. **Complex rotator:** This problem is about how to construct a complex rotator. Given $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathbb{C}^2$, determine c, s and r such that $Q = \begin{bmatrix} c & -\bar{s} \\ s & \bar{c} \end{bmatrix}$ is unitary and $Q^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$. Such a matrix Q is called a complex rotator. Show that Q is not uniquely determined. Your task is to determine c, s and r so that your rotator has the following properties:

   > The definition for real and complex data should be consistent, so that real data passed to the complex algorithm (for rotator) should result in the same answer (modulo roundoff) as from the real algorithm.
   >
   > [**Hint:** See stable generation of rotation in the slides on Givens rotation.]

   Define a complex plane rotator in $\mathbb{C}^n$ that rotates a vector in the x_ix_j plane.

4. **Complex reflector:** Given $u \in \mathbb{C}^n$ with $\|u\|_2 = 1$, define $Q := I - 2uu^*$. Show that $Qu = -u$ and $Qv = v$ if $\langle u, v\rangle = 0$. Further show that $Q = Q^* = Q^{-1}$. The matrix Q is called a complex reflector.

   Let $x, y \in \mathbb{C}^n$ be such that $\|x\|_2 = \|y\|_2$. Suppose that x and y are linearly independent and $\langle x, y\rangle$ is real, that is, $\langle x, y\rangle \in \mathbb{R}$. Show that there exists a complex reflector Q such that $Qx = y$. If $x \in \mathbb{C}^n$ is nonzero then show that there is a complex reflector H such that
$$Hx = \begin{bmatrix} -e^{i\theta}\|x\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$
   where θ is such that $x_1 = |x_1|e^{i\theta}$ and $\theta = 0$ if $x_1 = 0$. Write an algorithm that generates the reflector H.

5. Let $A \in \mathbb{R}^{n\times n}$. Consider the QR algorithm

```
For k =1:n-1

    Compute reflector Q = I- aplha * u * u' such that
    (QA)(k,k) = - sigma and (QA)(k+1:n, k) = 0
    Compute w = A' * u and A = A - alpha * u * w'

end
```

   (a) Show that at the k-th step, the rank one update `A` $\leftarrow$ `A - alpha * u * w'` does not affect the first $k-1$ rows and columns of A.

   (b) Modify above algorithm so that the strict upper triangular part of A contains the strict upper triangular part of R (where $A = QR$), the diagonal entries of R are stored in an extra array and the lower triangular part of A contains the householder vectors (that is, vectors u's that determine the reflectors).

6. This problem is about how to accumulate reflectors to produce Q in the QR factorization of $A \in \mathbb{R}^{n\times n}$. Let $Q_1, \ldots, Q_{n-1}$ be reflectors such that $Q_{n-1}Q_{n-2}\cdots Q_2Q_1A = R$, where R is upper triangular and $Q_k = I - \alpha_k u_k u_k^*$ for $k = 1 : n-1$. Set $Q = Q_1Q_2\cdots Q_{n-1}$. Then $A = QR$. Suppose that the matrix Q is explicitly required. Describe an efficient algorithm to compute Q and give the total flop count.

********** End ***********