MA668: Algorithmic and High Frequency Trading
Lecture 32

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Incorporating Order Flow: The Model Setup (Contd ...)

1. In this manner, the action of the agent's trades and other traders' actions are treated symmetrically.
2. We can define the net order flow as: $\mu_t := \mu_t^+ - \mu_t^-$ and a short computation shows that:

$$d\mu_t = -\kappa(\mu_t^+ - \mu_t^-)dt + \eta(dL_t^+ - dL_t^-) = -\kappa\mu_t dt + \eta(dL_t^+ - dL_t^-).$$

3. Hence, if the permanent impact functions $g(x) = bx$ are linear (with $b \geq 0$), we can use the net order flow as a state process rather than having to keep track of order flow in both directions separately.
4. Overall, we have:

$$dS_t^\nu = \sigma dW_t + b\,(\mu_t - \nu_t)\,dt.$$

5. The remainder of the agent's optimization problem is as before, that is:
   - A. The agent's inventory is: $dQ_t^\nu = -\nu_t dt$.
   - B. The agent's cash process is: $dX_t^\nu = (S_t^\nu - k\nu_t)\nu_t dt$.

Incorporating Order Flow: The Model Setup (Contd ...)

1. Agent's performance criteria:

$$H^{\nu}(t,x,S,\mu,q) = \mathbb{E}_{t,x,S,\mu,q}\left[X_T^{\nu} + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right) - \phi\int_t^T \left(Q_u^{\nu}\right)^2 du\right].$$

2. The value function (based on DPP) is:

$$H(t,x,S,\mu,q) = \sup_{\nu\in\mathcal{A}} H^{\nu}(t,x,S,\mu,q).$$

The Resulting DPE

1. The DPP for the value function suggests that the value function $H(t,x,S,\mu,q)$ satisfies the DPE (the value function now has an additional state variable, μ):

$$\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H + \mathcal{L}^{\mu}H - \phi q^2 \\ &+ \sup_{\nu}\left[\left(\nu\left(S - k\nu\right)\partial_x + b(\mu-\nu)\partial_S - \nu\partial_q\right)H\right]. \end{aligned}$$

The Resulting DPE (Contd ...)

1. The terminal condition is $H(T, x, S, \mu, q) = x + q(S - \alpha q)$, where the infinitesimal generator for the net order acts on the value function as follows:

$$\begin{aligned} \mathcal{L}^{\mu}(t, x, S, \mu, q) &= -\kappa\mu\partial_{\mu}H, \\ &+ \lambda\left[H(t, x, S, \mu + \eta, q) - H(t, x, S, \mu, q)\right], \\ &+ \lambda\left[H(t, x, S, \mu - \eta, q) - H(t, x, S, \mu, q)\right]. \end{aligned} \tag{1}$$

2. Ansatz: $H(t, x, S, \mu, q) = x + qS + h(t, \mu, q)$.
3. Non-linear PDE for h:

$$0 = \partial_t h + \mathcal{L}^{\mu} h + b\mu q - \phi q^2 + \sup_{\nu}\left[-k\nu^2 - (bq + \partial_q h)\nu\right],$$

   with terminal condition: $h(T, \mu, q) = -\alpha q^2$.
4. Recall that $x + qS$ represents the cash from the sale of shares so far plus the book value (at mid-price) of the shares the agent still holds and aims to liquidate.

The Resulting DPE (Contd ...)

1. The optimal control in feedback form is the same as seen earlier, but the function h satisfies a new equation.
2. More specifically, the first order conditions imply that:

$$\nu^* = -\frac{1}{2k}\left(bq + \partial_q h\right).$$

3. Upon substitution back into the previous equation we find that h satisfies the non-linear partial-integral differential equation (PIDE):

$$\left(\partial_t + \mathcal{L}^{\mu}\right) h + b\mu q - \phi q^2 + \frac{1}{4k}\left(bq + \partial_q h\right)^2 = 0. \qquad (2)$$

Solving the DPE

1. Due to the existence of linear and quadratic terms in q in (2) and its terminal conditions, we expect $h(t, \mu, q)$ to be a quadratic form in q.
2. Accordingly, we assume the ansatz:

$$h(t, \mu, q) = h_0(t, \mu) + qh_1(t, \mu) + q^2 h_2(t, \mu).$$

3. Substituting this into (2) and collecting like terms in q leads to the following coupled system of PIDEs:

$$\begin{aligned}
(\partial_t + \mathcal{L}^\mu)h_0 + \frac{1}{4k}h_1^2 &= 0, &(3)\\
(\partial_t + \mathcal{L}^\mu)h_1 + b\mu + \frac{1}{2k}h_1(b + 2h_2) &= 0, &(4)\\
(\partial_t + \mathcal{L}^\mu)h_2 - \phi + \frac{1}{4k}(b + 2h_2)^2 &= 0, &(5)
\end{aligned}$$

subject to the terminal conditions $h_0(T, \mu) = 0$, $h_1(T, \mu) = 0$ and $h_2(T, \mu) = -\alpha$.

Solving the DPE (Contd ...)

1. For (5): h_2 contains no source terms in μ and its terminal condition is independent of μ. Therefore the solution must be independent of μ, and accordingly, h_2 is a function only of time. Thus:

$$h_2(t,\mu) = \chi(t) - \frac{b}{2}, \text{ where } \chi(t) = \sqrt{k\phi}\,\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}.$$

   Recall that $\gamma = \sqrt{\dfrac{\phi}{k}}$ and $\zeta = \dfrac{\alpha - \frac{b}{2} + \sqrt{k\phi}}{\alpha - \frac{b}{2} - \sqrt{k\phi}}$.

2. For (4): To solve for h_1, we exploit the affine structure of the model for the net order flow and write:

$$h_1(t,\mu) = l_0(t) + \mu l_1(t),$$

   in which case:

$$\mathcal{L}^{\mu} h_1 = -\kappa\mu l_1 + \lambda(\eta l_1) + \lambda(-\eta l_1) = -\kappa\mu l_1,$$

   with terminal condition:

$$l_0(T) = l_1(T) = 0.$$

Solving the DPE (Contd ...)

1. Equation (4) reduces to:

$$\left[\partial_t l_0 + \frac{1}{k}\chi(t) l_0\right] + \left[\partial_t l_1 + \left(\frac{1}{k}\chi(t) - \kappa\right) l_1 + b\right]\mu = 0.$$

2. Since this must hold for every value of μ, each term in the braces must vanish individually and we obtain two simple ODEs for l_0 and l_1.
3. Since $l_0(T) = 0$ and its ODE is linear in l_0, the solution is $l_0(t) = 0$.
4. For l_1, due to the source term b, the solution is non-trivial and can be written as:

$$l_1(t) = b \int_t^T e^{-\kappa(s-t)} e^{\frac{1}{k}\int_t^s \chi(u)du} ds. \tag{6}$$

5. It can be shown that:

$$l_1(t) = b\bar{l}_1(T-t) \geq 0, \tag{7}$$

where $\bar{l}_1(\tau) = \dfrac{1}{\zeta e^{\gamma\tau} - e^{-\gamma\tau}}\left[e^{\gamma\tau}\dfrac{1-e^{-(k+\gamma)\tau}}{k+\gamma}\zeta - e^{-\gamma\tau}\dfrac{1-e^{-(k-\gamma)\tau}}{k-\gamma}\right]$
and $\tau = T - t$ represents the time remaining to the end of the trading horizon.

Solving the DPE (Contd ...)

1. For (3): The solution of h_0 can be obtained in a similar manner.
2. However: Note that the optimal speed of trading does not depend on h_0.
3. Reason:

$$\nu^* = -\frac{(bq + \partial_q h)}{2k} \text{ and } \partial_q h(t, \mu) = h_1(t, \mu) + 2q h_2(t, \mu).$$

4. Putting these results together we find that the optimal speed of trading is:

$$\nu_t^* = -\frac{1}{k}\chi(t) Q_t^{\nu^*} - \frac{b}{2k}\bar{l}_1(t)\mu_t. \tag{8}$$

Solving the DPE (Contd ...)

1. The optimal trading speed above differs from the earlier case solution by the second term on the right-hand side of (8), which represents the perturbations to the trading speed due to excess order flow.
2. Recall that: In the limit as $\alpha \to \infty$, we have $\chi \leq 0$.
3. Further, from the explicit solution above, we have $\bar{l}_1 \geq 0$.
4. Hence:
   - A. When the excess order flow is tilted to the buy side ($\mu_t > 0$), the agent slows down trading since she/he anticipates that excess buy order flow will push the prices upwards, and therefore will receive better prices when she/he eventually speeds up trading to sell assets later on.
   - B. Contrastingly, she/he increases her/his trading speed when order flow is tilted to the sell side ($\mu_t < 0$), since other traders are pushing the price downwards and she/he aims to get better prices now, rather than waiting for other traders to push it further down.

Solving the DPE (Contd ...)

1. Another interpretation is that she/he attempts to hide her/his orders by trading when order flow moves in her/his direction.
2. Finally, recall that $l_1(\tau) \to 0$ as $t \to T$: Hence, the order flow influences the agent's trading speed less and less as maturity approaches because there is little time left to take advantage of directional trends in the mid-price.
3. Somewhat surprisingly, the volatility of the order flow process η does not appear explicitly in the optimal strategy.
4. It does, however, affect the way the agent trades through its influence on the path which order flow takes.
5. When the order flow path is volatile, the optimal trading speed will be volatile as well.
6. It is also interesting to observe that if the jumps η in the order flow at the Poisson times were random and not constant, the resulting strategy would be identical.

Solving the DPE (Contd ...)

1. If we add a Brownian component to μ_t, the resulting optimal strategy in terms of μ_t would be identical, that is, (8) remains true.
2. Naturally, the actual path taken by the order flow, and therefore also that of trading, would be altered by these modifications to the model.
3. A final point we make about this optimal trading strategy is that ν_t is not necessarily strictly positive.
4. If the order flow μ_t is sufficiently positive, then the agent may be willing to purchase the asset to make gains from the increase in asset price, that is, her/his liquidation rate becomes negative.
5. This is because the way we have introduced order flow into the model generates predictability in the price process, which can be exploited, even if the agent is not executing a trade.
6. If the agent has liquidated the target $\mathfrak{R}$ at $t < T$, the optimal strategy is not to stop, but continue trading and exploit the effect of the order flow, and then her/his inventory can become negative at intermediate times.
7. If there is sufficient selling pressure, that is μ_t is sufficiently negative, then by shorting the asset, she/he may benefit from the downward price movement.

Solving the DPE (Contd ...)

1. One approach to avoid such scenarios is to simply restrict the trading strategy in a naive manner, by setting:

$$\nu^{\dagger} = \max\left(-\frac{1}{k}\chi(t)Q_t^{\nu^{\dagger}} - \frac{b}{2k}\bar{l}_1(t)\mu_t, 0\right)\mathbb{1}_{\left\{Q_t^{\nu^{\dagger}}>0\right\}}. \tag{9}$$

2. In other words, we can follow the unrestricted optimal solution whenever the trading rate is positive and the agent has positive inventory, otherwise we impose a trading stop.
3. This trading strategy, $\nu_t^{\dagger}$, is not the true optimal strategy.
4. To obtain the true optimal strategy we would need to go back to the DPE and impose the constraint $\nu \geq 0$ in the supremum and add an additional boundary condition along $q = 0$.
5. In this case, a numerical schemes can be used to solve the problem.
6. Nonetheless, the $\nu_t^{\dagger}$ strategy provides a reasonable approximation that is easy to implement.

Simulations of the Strategy With Order Flow

1. Simulations: To show the behaviour of the optimal strategy in this model.
2. Throughout, we use the following parameters:
   - A. $T = 1$ day.
   - B. $k = 10^{-3}$.
   - C. $b = 10^{-4}$.
   - D. $\phi = 0.01$.
   - E. $\lambda = 1000$.
   - F. $\kappa = 10$.
   - G. $\eta \sim \text{Exp}(5)$ [a].
   - H. $\sigma = 0.1$.
3. Henceforth "AC" will denoted the classical solution due to Almgren and Chriss [2000].

[a] $\eta \sim \text{Exp}(\eta_0)$ denotes the exponential distribution with mean size $\mathbb{E}[\eta] = \eta_0$

Figure 7.4

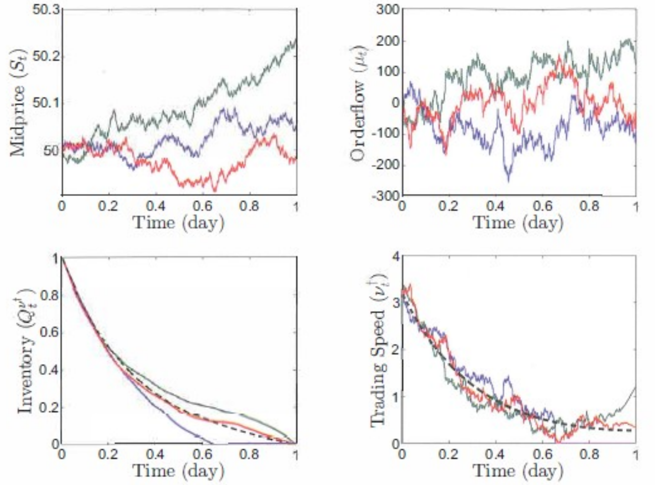

Figure 7.4 Optimal trading in the presence of order flow. The dashed lines show the classical AC solution.

Figure 7.4 (Contd ...)

1. Figure 7.4: Three scenarios when the agent uses the augmented strategy $\nu_t^\dagger$ in (9) for: `The mid-price, The order flow, The optimal inventory, The optimal speed of trading`.
2. As the figure shows, when the order flow is positive/negative the agent trades more slowly/quickly than the AC trading speed.
3. For example: The large order flow in the buy direction ($\mu_t > 0$), shown by the green path, causes the agent to trade more slowly in the initial stages of the trade.
4. As the end of the trading horizon approaches, the order flow influences her/his strategy less, but she/he must speed up her/his trading, since there is little time remaining in which to liquidate the remaining shares.
5. The red path has order flow that fluctuates mostly around zero, and as shown in the diagrams, she/he follows closely the AC strategy, but locally adjusts her/his trades relative to the path.
6. Finally, the blue path has a bias towards sell order flow, and the agent adds to this flow by trading more quickly throughout most of the trading horizon and eventually liquidates her/his shares early.

Figure 7.5

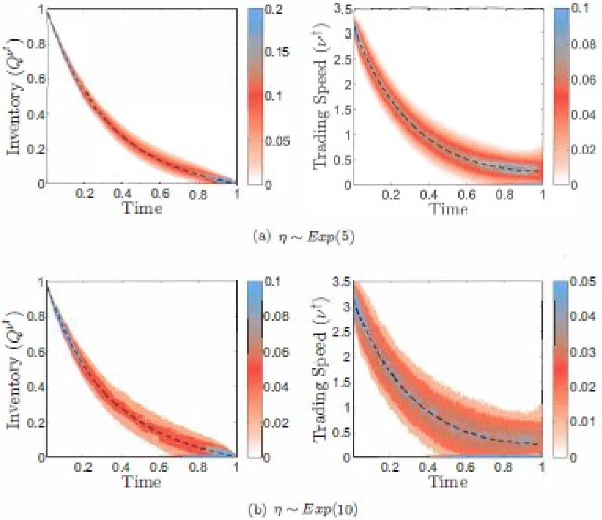

Figure 7.5 Heat-maps of the optimal trading in the presence of order flow for two volatility levels. The dashed lines show the classical AC solution.

Figure 7.5 (Contd ...)

1. Further insight: Figure 7.5 shows heat-maps from $5,000$ scenarios of the optimal inventory to hold and the optimal speed of trading.
2. Panels:
   - (A) Panel (a) shows the results when $\eta \sim \text{Exp}(5)$ (as in Figure 7.4).
   - (B) Panel (b) shows the results when $\eta \sim \text{Exp}(10)$.
3. As expected, the optimal trading strategy in scenario (b) is more volatile than in scenario (a), despite the optimal strategy (as seen in (9)) having no explicit dependence on this volatility.