MA 668 | Quiz-2

Algorithmic and High Frequency Trading | 09:10 AM–09:55 AM

IIT Guwahati | 2^{nd} April 2024

Instructions

1. Write your name and roll number on the answerscript.
2. Your writing should be legible and neat.
3. This Quiz has 2 questions, for a total of 15 marks.

Questions

[8 marks] 1. Let us define:

$$H(t,\mathbf{x}) := \sup_{\mathbf{u}\in\mathcal{A}_{t,T}} H^{\mathbf{u}}(t,\mathbf{x}),$$

and

$$H^{\mathbf{u}}(t,\mathbf{x}) := \mathbb{E}_{t,\mathbf{x}}\left[G\left(\mathbf{X}_T^{\mathbf{u}}\right) + \int_t^T F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right],$$

where the notation $\mathbb{E}_{t,\mathbf{x}}[\cdot]$ represents the expectation conditional on $\mathbf{X}_t^{\mathbf{u}} = \mathbf{x}$.

(A) Prove that the value function satisfies the DPP:

$$H(t,\mathbf{x}) = \sup_{\mathbf{u}\in\mathcal{A}} \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right],$$

for all $(t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n$ and all stopping times $\tau \le T$.

(B) Hence establish the DPE:

$$\partial_t H(t,\mathbf{x}) + \sup_{\mathbf{u}\in\mathcal{A}}\left(\mathcal{L}_t^{\mathbf{u}} H(t,\mathbf{x}) + F(t,\mathbf{x},\mathbf{u})\right) = 0,\ \ H(T,\mathbf{x}) = G(\mathbf{x}).$$

Answer:

(A) Applying iterated expectations, on the given equation, we get:

$$\begin{aligned}
H^{\mathbf{u}}(t,\mathbf{x}) &= \mathbb{E}_{t,\mathbf{x}}\left[G\left(\mathbf{X}_T^{\mathbf{u}}\right) + \int_\tau^T F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right],\\
&= \mathbb{E}_{t,\mathbf{x}}\left[\mathbb{E}_{\tau,\mathbf{x}_\tau^{\mathbf{u}}}\left[G\left(\mathbf{X}_T^{\mathbf{u}}\right) + \int_\tau^T F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right] + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right],\\
&= \mathbb{E}_{t,\mathbf{x}}\left[H^{\mathbf{u}}(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right] \ldots\text{(1 mark)}
\end{aligned}$$

Now, $H(t,\mathbf{x}) \ge H^{\mathbf{u}}(t,\mathbf{x})$ for an arbitrary admissible control $\mathbf{u}$ (with equality holding if $\mathbf{u}$ is the optimal control $\mathbf{u}^*$ assuming that $\mathbf{u}^* \in \mathcal{A}_{t,T}$, *i.e.*, the supremum is attained by an admissible strategy. Hence:

$$
\begin{aligned}
H^{\mathbf{u}}(t,\mathbf{x}) &\leq \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right], \\
&\leq \sup_{\mathbf{u}\in\mathcal{A}}\mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right] \ldots\text{(1 mark)}
\end{aligned}
$$

Taking supremum over admissible strategies on the left-hand side, so that the left-hand side also reduces to the value function, we have that:

$$
H(t,\mathbf{x}) \leq \sup_{\mathbf{u}\in\mathcal{A}}\mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right].
$$

Next, we take an arbitrary admissible control $\mathbf{u}\in\mathcal{A}$ and consider what is known as an ϵ-optimal control denoted by $\mathbf{v}^\epsilon\in\mathcal{A}$ and defined as a control which is better than $H(t,\mathbf{x})-\epsilon$, but of course not as good as $H(t,\mathbf{x})$ i.e., a control such that

$$
H(t,\mathbf{x}) \geq H^{\mathbf{v}^\epsilon}(t,\mathbf{x}) \geq H(t,\mathbf{x}) - \epsilon.
$$

Consider next the modification of the ϵ-optimal control.

$$
\widetilde{\mathbf{v}}^\epsilon = \mathbf{u}_t\mathbb{1}_{t\leq\tau} + \mathbf{v}^\epsilon 1_{t>\tau}\ldots\text{(1 mark)}
$$

Then we have:

$$
\begin{aligned}
H(t,\mathbf{x}) &\geq H^{\widetilde{\mathbf{v}}^\epsilon}(t,\mathbf{x}) \\
&= \mathbb{E}_{t,\mathbf{x}}\left[H^{\widetilde{\mathbf{v}}^\epsilon}(\tau,\mathbf{X}_\tau^{\widetilde{\mathbf{v}}^\epsilon}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\widetilde{\mathbf{v}}^\epsilon},\widetilde{\mathbf{v}}_s^\epsilon\right)ds\right], \\
&= \mathbb{E}_{t,\mathbf{x}}\left[H^{\widetilde{\mathbf{v}}^\epsilon}(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right], \\
&\geq \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right] - \epsilon.
\end{aligned}
$$

Taking limit as $\epsilon\downarrow 0$, we have,

$$
H(t,\mathbf{x}) \geq \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right].
$$

Moreover, since the above holds true for every $\mathbf{u}\in\mathcal{A}$, we have that:

$$
H(t,\mathbf{x}) \geq \sup_{\mathbf{u}\in\mathcal{A}}\mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int\limits_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right] \ldots\text{(1 mark)}
$$

The upper bound and lower bound together give the required relation.

(B) We have:

$$
\begin{aligned}
H(t,\mathbf{x}) &\geq \sup_{\mathbf{u}\in\mathcal{A}} \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}},\mathbf{u}_s\right)ds\right], \\
&\geq \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{v}}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{v}},\mathbf{v}_s\right)ds\right], \\
&= \mathbb{E}_{t,\mathbf{x}}\left[H(t,\mathbf{x}) + \int_t^\tau \left(\partial_s + \mathcal{L}_s^{\mathbf{v}}\right)H(s,\mathbf{X}_s^{\mathbf{v}})ds\right. \\
&\quad + \left.\int_t^\tau \mathcal{D}_x H(s,\mathbf{X}_s^{\mathbf{v}})'\sigma_s^{\mathbf{v}} d\mathbf{W}_s + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{v}},\mathbf{v}_s\right)ds\right] \ldots\text{(2 marks)}
\end{aligned}
$$

This leads to:

$$
H(t,\mathbf{x}) \geq \mathbb{E}_{t,\mathbf{x}}\left[H(t,\mathbf{x}) + \int_t^\tau \left\{\left(\partial_s + \mathcal{L}_s^{\mathbf{v}}\right)H(s,\mathbf{X}_s^{\mathbf{v}}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{v}},\mathbf{v}\right)\right\}ds\right].
$$

Moving the $H(t,\mathbf{x})$ on the left-hand side over to the right-hand side, dividing by h and taking the limit as $h\downarrow 0$ yields:

$$
\begin{aligned}
0 &\geq \lim_{h\downarrow 0} \mathbb{E}_{t,\mathbf{x}}\left[\frac{1}{h}\int_t^\tau \left\{\left(\partial_s + \mathcal{L}_s^{\mathbf{v}}\right)H(s,\mathbf{X}_s^{\mathbf{v}}) + F\left(s,\mathbf{X}_s^{\mathbf{v}},\mathbf{v}\right)\right\}ds\right], \\
&= \left(\partial_t + \mathcal{L}_t^{v}\right)H(t,\mathbf{x}) + F(t,\mathbf{x},\mathbf{v}).
\end{aligned}
$$

Since the above inequality holds for arbitrary $\mathbf{v}\in\mathcal{A}$, it follows that:

$$
\partial_t H(t,\mathbf{x}) + \sup_{\mathbf{u}\in\mathcal{A}}\left(\mathcal{L}_t^{\mathbf{u}}H(t,\mathbf{x}) + F(t,\mathbf{x},\mathbf{u})\right) \leq 0 \ldots\text{(1 mark)}
$$

Next, we show that the inequality is indeed an equality. To show this, suppose that $\mathbf{u}^*$ is an optimal control, then we have:

$$
H(t,\mathbf{x}) = \mathbb{E}_{t,\mathbf{x}}\left[H(\tau,\mathbf{X}_\tau^{\mathbf{u}^*}) + \int_t^\tau F\left(s,\mathbf{X}_s^{\mathbf{u}^*},\mathbf{u}^*\right)ds\right].
$$

As above, by applying Ito's lemma to write $H(\tau,\mathbf{X}_\tau^{\mathbf{u}^*})$ in terms of $H(t,\mathbf{x})$ plus the integral of its increments, taking expectations, and then taking the limit as $h\downarrow 0$, we find that:

$$
\partial_t H(t,\mathbf{x}) + \mathcal{L}_t^{\mathbf{u}^*}H(t,\mathbf{x}) + F(t,\mathbf{x},\mathbf{u}^*) = 0.
$$

We finally arrive at the DPE (also known in this context as the Hamilton-Jacobi-Bellman equation):

$$
\partial_t H(t,\mathbf{x}) + \sup_{\mathbf{u}\in\mathcal{A}}\left(\mathcal{L}_t^{\mathbf{u}}H(t,\mathbf{x}) + F(t,\mathbf{x},\mathbf{u})\right) = 0,\ \ H(T,\mathbf{x}) = G(\mathbf{x}) \ldots\text{(1 mark)}
$$

The terminal condition follows from the value function.

[7 marks] 2. (A) Construct the DPE for an agent who uses only MO's to optimally liquidate $\mathfrak{N}$ shares between $t=0$ and $t=T$, with only temporary impact, that is, the permanent impact is

$g(\nu_t) = 0$ and the bid-ask spread is $\Delta = 0$.

(B) Further, assuming $H(t, S, q) = qS + q^2 h_2(t)$. derive the optimal trading speed.

Answer:

(A) The stock's mid price is given by:

$$dS_t^\nu = \pm g(\nu_t)dt + \sigma dW_t, \ S_0^\nu = S,$$

with $g(\nu_t) = 0$. The execution price is given by:

$$\widehat{S}_t^\nu = S_t^\nu \pm \left(\frac{\Delta}{2} + f(\nu_t)\right), \ \widehat{S}_0^\nu = \widehat{S},$$

with $f(\nu_t) = k\nu_t$ and $k > 0$. ...(1 mark)

It is given that $\Delta = 0$, and we assume that the agent is insistent that all $\mathfrak{R}$ shares are liquidated by time T. The agent's value function is:

$$H(t, S, q) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,S,q} \left[\int_t^T (S_u - k\nu_u)\, \nu_u du \right],$$

where $\mathbb{E}_{t,S,q}$ denotes expectation conditional on $S_t = S$ and $Q_t = q$. ...(1 mark)

To solve this optimal control problem, we use the dynamic programming principle (DPP) which suggests that the value function satisfies the dynamic programming equation (DPE):

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \sup_\nu \left[(S - k\nu)\nu - \nu \partial_q H\right] = 0 \ldots \text{(1 mark)}$$

We require that: $H(T, S, q) \to -\infty$ as $t \to T$, for $q > 0$ and $H(T, S, 0) \to 0$ as $t \to T$.

(B) The first order condition applied to DPE results in the supremum being attained at:

$$\nu^* = \frac{1}{2k}\left(S - \partial_q H\right),$$

which is the optimal trading speed in feedback control form. ...(1 mark)

Upon substitution into the DPE, we obtain the non-linear partial differential equation:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \frac{1}{4k}\left(S - \partial_q H\right)^2 = 0,$$

for the value function. We assume that:

$$H(t, S, q) = qS + h(t, q),$$

where $h(t, q)$ is still to be determined. Substituting, we arrive at the following equation for $h(t, q)$:

$$\partial_t h + \frac{1}{4k}\left(\partial_q h\right)^2 = 0.$$

Now choosing $h(t, q) = q^2 h_2(t)$, we arrive at the following non-linear ODE for $h_2(t)$:

$$\partial_t h_2 + \frac{1}{k} h_2^2 = 0,$$

whose solution is given by:

$$h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{k}(T - t)\right)^{-1} \ldots \text{(1 mark)}$$

Therefore:

$$\nu_t^* = -\frac{1}{k} h_2(t) Q_t^{\nu^*}.$$

We integrate $dQ_t^{\nu^*} = -\nu_t^* dt$ over $[0,t]$ to obtain the inventory profile along with the optimal strategy:

$$\int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} = \int_0^t \frac{h_2(s)}{k} ds.$$

This implies that:

$$Q_t^{\nu^*} = \frac{(T-t) - k/h_2(T)}{T - k/h_2(T)} \mathfrak{R}.$$

To satisfy the terminal inventory condition $Q_T^{\nu^*} = 0$ and also ensure that the correction $h(t,q)$ to the book value of the outstanding shares that need to be liquidated is negative, we must have:

$$h_2(t) \to -\infty, \text{ as } t \to T.$$

Returning to solving the optimal problem, we have that:

$$h_2(t) = -k\,(T-t)^{-1}.$$

Then the optimal inventory to hold is:

$$Q_t^{\nu^*} = \left(1 - \frac{t}{T}\right) \mathfrak{R} \ldots \text{(1 mark)}$$

Therefore the optimal speed of trading is:

$$\nu_t^* = \frac{\mathfrak{R}}{T} \ldots \text{(1 mark)}$$