MA668: Algorithmic and High Frequency Trading
Lecture 25

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Control for Counting Processes

1. In the previous discussion, diffusion processes were the driving sources of uncertainty in the control problem.
2. However, in many circumstances, and in particular for problems related to algorithmic and high-frequency trading, counting processes will be used to drive uncertainty.
3. There are many features that can be incorporated into the analysis, but the general approach remains the same, and as such only the case of a single counting process with controlled intensity will be investigated.
4. This amounts to treating doubly stochastic Poisson processes, or Cox processes, which are counting processes with intensity that itself is a stochastic process and in this case at least partially controlled.

Control for Counting Processes (Contd ...)

1. Consider the situation in which the agent can control the frequency of the jumps in a counting process N and does so to maximize some target.

2. In this case, the control problem is of the general form:

$$H(n) = \sup_{u \in \mathcal{A}_{0,T}} \mathbb{E}\left[G\left(N_T^u\right) + \int_0^T F\left(s, N_s^u, u_s\right) ds \right], \tag{1}$$

   where $u = (u_t)_{\{0 \le t \le T\}}$ is the control process, $(N_t^u)_{\{0 \le t \le T\}}$ is a controlled doubly stochastic Poisson process (starting at $N_{0-} = n$) with intensity $\lambda_t^u = \lambda\left(t, N_{t-}^u, u_t\right)$ so that $(\widehat{N}_t^u)_{\{0 \le t \le T\}}$ where $\widehat{N}_t^u = N_t - \int_0^t \lambda_s^u ds$, is a martingale, $\mathcal{A}$ is a set of $\mathcal{F}$-predictable processes such that $\widehat{N}$ is a true martingale, $G : \mathbb{R} \to \mathbb{R}$ is a terminal reward and $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a running reward/penalty.

3. As before, the functions G and F are assumed to be uniformly bounded.

Control for Counting Processes (Contd ...)

1. For an arbitrary admissible control u, the performance criteria $H^u(n)$ is given by:

$$H^u(n) = \mathbb{E}\left[G\left(N_T^u\right) + \int_0^T F\left(s, N_s^u\right) ds \right]. \tag{2}$$

2. The agent seeks to maximize this performance criteria:

$$H(n) = \sup_{u \in \mathcal{A}_{0,T}} H^u(n). \tag{3}$$

The Dynamic Programming Principle

As before, the original problem is embedded into a larger class of problems indexed by time $t \in [0, T]$ by first defining:

$$H(t,n) := \sup_{u \in \mathcal{A}_{t,T}} H^u(t,n), \tag{4}$$

$$H^u(t,n) := \mathbb{E}_{t,n}\left[G\left(N_T^u\right) + \int_t^T F\left(s, N_s^u, u_s\right) ds \right], \tag{5}$$

where the notation $\mathbb{E}_{t,n}[\cdot]$ represents the expectation conditional on $N_{t-} = n$.

Dynamic Programming Principle for Counting Processes

The value function satisfies the DPP:

$$H(t,n) = \sup_{u \in \mathcal{A}} \mathbb{E}_{t,n}\left[H\left(\tau, N_\tau^u\right) + \int_t^\tau F\left(s, N_s^u, u_s\right) ds \right], \tag{6}$$

for all $(t,n) \in [0,T] \times \mathbb{Z}_+$ and all stopping times $\tau \leq T$.

DPE/HJB

1. Stopping time: $\tau = T \wedge \inf\{s > t : (s-t, |N_s^u - n|) \notin [0,h) \times [0,\epsilon)\}$.
2. Value function:

$$\begin{aligned} H(\tau, N_\tau^u) &= H(t,n) + \int_t^\tau (\partial_s + \mathcal{L}_s^u) H(s, N_s^u)\, ds \\ &+ \int_t^\tau [H(s, N_{s-}^u + 1) - H(s, N_{s-}^u)]\, d\widehat{N}_s^u, \end{aligned}$$

where $\mathcal{L}_t^u$ represents the infinitesimal generator of N_t^u and acts on functions $h : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ as follows:

$$\mathcal{L}_t^u h(t,n) = \lambda(t,u,n)\,[h(t,n+1) - h(t,n)]\,.$$

Hamilton-Jacobi-Bellman Equation for Counting Process

$$\partial_t H(t,n) + \sup_{u \in \mathcal{A}} \left(\mathcal{L}_t^u H(t,n) + F(t,n,u)\right) = 0, \;\; H(T,n) = G(n). \tag{7}$$

Using the Poisson Process to Drive a Secondary Controlled Process

1. Let $(X_t^u)_{\{0\le t\le T\}}$ denote a controlled process satisfying the SDE:

$$dX_t^u = \mu\,(t, X_t^u, N_t^u, u_t)dt + \sigma(t, X_{t-}^u, N_{t-}^u, u_t)\,dN_t^u. \tag{8}$$

2. The DPP for this problem is:

$$H(t,x,n) = \sup_{u\in\mathcal{A}} \mathbb{E}_{t,x,n}\left[H(\tau, X_\tau^u, N_\tau^u) + \int_t^\tau F(s, X_s^u, N_s^u, u_s)ds\right]. \tag{9}$$

Hamilton-Jacobi-Bellman Equation

$$\partial_t H(t,x,n) + \sup_{u\in\mathcal{A}}\left(\mathcal{L}_t^u H(t,x,n) + F(t,x,n,u)\right) = 0,\;\; H(T,x,n) = G(x,n), \tag{10}$$

where,

$$\begin{aligned}\mathcal{L}_t^u H(t,x,n) &= \mu(t,x,n,u)\partial_x H(t,x,n),\\ &+ \lambda(t,x,n,u)\left[H(t, x+\sigma(t,x,n,u), n+1) - H(t,x,n)\right].\end{aligned}$$

Example: Maximizing Expected Wealth Using Round-Trip Trades

1. We look at an example of an agent who uses a Market Order (MO) to purchase one share at the best offer and then seeks to unwind her/his position by posting a Limit Order (LO) at the mid price plus the depth u which she/he controls.
2. She/he repeats this operation over and over again until a future date T.
3. Her/his cost from acquiring the share is $S_t + \Delta/2$, where Δ is the spread between the best bid and best ask and is assumed constant, since S_t is the midprice and the best ask is resting in the LOB at $S_t + \Delta/2$.
4. The revenue from selling (if the LO is lifted by an MO) is $S_t + u_t$.
5. Therefore, the wealth that is accrued to the agent from this round-trip trade is: $u_t - \Delta/2$.
6. Now: $dX_t^u = \left(u_t - \dfrac{\Delta}{2}\right) dN_t^u$ which implies that $\mu = 0$ and $\sigma_t^u = \left(u_t - \dfrac{\Delta}{2}\right)$.
7. Here N_t counts the number of round-trip trades that the agent has completed up until time t.

Example: Maximizing Expected Wealth Using Round-Trip Trades (Contd ...)

1. First, we need to assume an arrival rate for the buy MOs that are sent by other market participants.
2. Here we assume, for simplicity, that this rate is a constant $\Lambda > 0$.
3. Second, we need the probability of the LO being filled, conditional on the MO arriving.
4. A popular choice in the literature is to assume that when posted "$u \geq 0$" away from the mid-price, the probability of being filled, given that an MO arrives, is $P(u) = e^{-\kappa u_t}$ and another is $P(u) = (1 + \kappa u_t)^{-\gamma}$ where κ and γ are positive constants.
5. The corresponding fill probabilities are $\lambda_t^u = e^{-\kappa u_t}\Lambda$ and $\lambda_t^u = (1 + \kappa u_t)^{-\gamma}\Lambda$, respectively.
6. Further we assume that $F = 0$ and $G(x, n) = x$.
7. If we take the fill rate as $\lambda_t^u = e^{-\kappa u_t}\Lambda$, then the DPE/HJB becomes:

$$\partial_t H + \sup_{u \geq 0} \left(H\left(t, x + \left(u - \frac{\Delta}{2}\right), n + 1\right) - H(t, x, n)\right)\lambda_t^u = 0,$$

subject to $H(T, x, n) = x$.

Example: Maximizing Expected Wealth Using Round-Trip Trades (Contd ...)

1. Since there is no explicit dependence on n itself, we can assume that $H(t, x, n) = h(t, x)$, so the value function depends solely on wealth and time.
2. Furthermore, due to the linear nature of the problem, we can write $h(t, x) = x + g(t)$, for some deterministic function $g(t)$, with terminal condition $g(T) = 0$.
3. Hence, from the above we obtain:
$$u^* = \frac{\Delta}{2} + \frac{1}{\kappa}.$$
4. Note that: $\Delta/2$ is the half-spread cost incurred when using an MO for acquisition and $1/\kappa$ is how deep an agent can post in the LOB.
5. Further, we get:
$$\partial_t g + \frac{\Lambda}{\kappa} e^{-\kappa\left(\frac{\Delta}{2} + \frac{1}{\kappa}\right)} = 0.$$
6. Therefore:
$$H(t, x, n) = x + \frac{\Lambda}{\kappa} e^{-\kappa\left(\frac{\Delta}{2} + \frac{1}{\kappa}\right)} (T - t)\,.$$