Homework - 9

MA423 : Matrix Computations 2023 R. Alam

Eigenvalue problems

1. Let $A \in \mathbb{R}^{n\times n}$ and $s \in \mathbb{C}$. Perform two steps of explicit QR algorithm on A with shifts s and $\bar{s}$:

$$\begin{aligned} A - sI &= Q_1R_1,\ A_1 := R_1Q_1 + sI \\ A_1 - \bar{s}I &= Q_2R_2,\ A_2 := R_2Q_2 + \bar{s}I. \end{aligned}$$

Show that $A_1 = Q_1^*AQ_1$ is complex and $A_2 = Q_2^*A_1Q_2$ is real. Outline an algorithm to compute A_2 directly from A without computing A_1. Describe an implicit implementation of your algorithm.

2. Consider the matrix $H := \begin{bmatrix} H_1 & C \\ 0 & H_2 \end{bmatrix}$, where H_1 and H_2 are irreducible upper hessenberg matrices. Is H an irreducible upper hessenberg matrix? Suppose that H is nonsingular. Show that performing a QR step on H is equivalent to performing a QR step on H_1 and H_2 separately and modifying C suitably.

3. Let $T \in \mathbb{C}^{n\times n}$ be upper triangular with distinct diagonals. Outline an algorithm for computing n linearly independent eigenvectors of T and determine the total flop count.

4. Let $u \in \mathbb{R}^n$ be a unit vector. Consider the Householder reflector $H = I - 2uu^\top$. What are the eigenvalues of H?

5. Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let v_1 be an eigenvector of A corresponding to λ_1. Let H be the Householder reflector such that $Hv_1 = \|v_1\|_2 e_1$.

   (a) Show that $H^*AH = \begin{bmatrix} \lambda_1 & h^\top \\ 0 & S \end{bmatrix}$, where $h \in \mathbb{C}^{n-1}$ and $S \in \mathbb{C}^{(n-1)\times(n-1)}$ with eigenvalues $\lambda_2, \ldots, \lambda_n$.

   (b) Let x_2 be an eigenvector of S corresponding to the eigenvalue λ_2. Set $v_2 := H \begin{bmatrix} \frac{h^\top u_2}{\lambda_2 - \lambda_1} \\ x_2 \end{bmatrix}$ when $\lambda_1 \neq \lambda_2$. Show that v_2 is an eigenvector of A corresponding to the eigenvalue λ_2. This process is called deflation and can be repeated to find additional eigenvalues and eigenvectors of A by solving smaller and smaller size eigenvalue problems.

6. Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let y_1 and x_1 be left and right eigenvectors of A corresponding to λ_1 such that $y_1^*x_1 = 1$. Show that the matrix $S := A - \lambda_1 x_1 y_1^*$ has eigenvalues $0, \lambda_2, \ldots, \lambda_n$.

7. Let $A \in \mathbb{C}^{n\times n}$. Consider the disks

$$D(a_{ii}, r_i) := \{z \in \mathbb{C} : |a_{ii} - z| \leq r_i\},\ \text{where } r_i := \sum_{j\neq i}^{n} |a_{ij}|,\ i = 1 : n.$$

Show that $\Lambda(A) \subset \cup_{i=1}^n D(a_{ii}, r_i)$.

8. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n$. Consider the shift $\mu := (\lambda_2 + \lambda_n)/2$. Determine the rate of convergence of the power method applied to $A - \mu I$. One can show that this shift gives the fastest converegnce.

9. Let $A \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$. Show, for every $\lambda \in \mathbb{R}$ and nonzero $v \in \mathbb{R}^n$, that $\min\limits_{1\leq j\leq n} |\lambda - \lambda_j| \leq \dfrac{\|\lambda v - Av\|_2}{\|v\|_2}$.

10. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Consider the Rayleigh quotient function

$$q_A : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}, x \longmapsto x^\top Ax/x^\top x.$$

Show that the gradient of $q_A(x)$ is given by $\nabla q_A(x) = \dfrac{2(Ax - q_A(x)x)}{x^\top x}$. Deduce that $q_A(v)$ is a stationary point of $q_A(x)$ if and only if $(q_A(v), v)$ is an eigenpair of A.

********** End ***********