MA668: Algorithmic and High Frequency Trading
Lecture 21

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

The Merton Problem (Contd ...)

1. Note the change*: *In the last slide of Lecture 20:* $X^\pi = (X_t^\pi)_{\{0\le t\le T\}}$ *is the agent's wealth process given that she/he follows the self-financing strategy* π.
2. In this classical example, the agent's trading decisions affect only her/his wealth process, but not the dynamics of the asset which she/he is trading.
3. On long time scales and if the agent's strategy does not change "too quickly", this is a reasonable assumption.
4. However, if an agent is attempting to acquire (or sell) a large number of shares in a short period of time, her/his actions most certainly affect the dynamics of the price itself, in addition to her/his wealth process.
5. This issue is ignored in the Merton problem, but is at the heart of research into algorithmic trading and specifically optimal execution problems which we introduce next.

The Optimal Liquidation Problem

1. As mentioned above, imagine that an agent has a large number of shares $\mathfrak{N}$ of an asset whose price is S_t.
2. Furthermore, suppose her/his fundamental analysis on the asset shows that it is no longer a valuable investment to hold.
3. Therefore she/he wishes to liquidate these shares by the end of the day, say at time T.
4. The fact that the market does not have infinite liquidity (to absorb a large sell order) at the best available price implies that the agent will obtain poor prices if she/he attempts to liquidate all units immediately.
5. Instead, she/he should spread this out over time and solve a stochastic control problem to address the issue.
6. She/he may also have a certain sense of urgency to get rid of these shares, represented by penalizing the holding of inventories different from zero throughout the strategy.

The Optimal Liquidation Problem (Contd ...)

If ν_t denotes the rate at which the agent sells her/his shares at time t, then the agent seeks the value function:

$$H(x, S, q) = \sup_{\nu \in \mathcal{A}_{0,T}} \mathbb{E}\left[X_T^{\nu} + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right) - \phi \int_0^T \left(Q_s^{\nu}\right)^2 ds \right], \qquad (1)$$

and the resulting optimal liquidation trading strategy ν^*, where [a],

$$\begin{aligned} dQ_t^{\nu} &= -\nu_t dt,\ Q_0^{\nu} = q,\ \text{(Agent's Inventory)} && (2)\\ dS_t^{\nu} &= -g(\nu_t)dt + \sigma dW_t,\ S_0^{\nu} = S,\ \text{(Fundamental Asset Price)} && (3)\\ \widehat{S}_t^{\nu} &= S_t^{\nu} - h(\nu_t),\ \widehat{S}_0^{\nu} = S,\ \text{(Execution Price)} && (4)\\ dX_t^{\nu} &= \nu_t \widehat{S}_t^{\nu} dt,\ X_0^{\nu} = x.\ \text{(Agent's Cash)} && (5) \end{aligned}$$

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[a] α term: Terminal liquidity penalty parameter which reflects the penalty per share and ϕ-term: Running inventory penalty, which reflects the agent's urgency of executing the trade

The Optimal Liquidation Problem (Contd ...)

1. $\nu = (\nu_t)_{\{0 \le t \le T\}}$ is the (positive) rate at which the agent trades (liquidation rate) and is what the agent can control.
2. $Q^\nu = (Q_t^\nu)_{\{0 \le t \le T\}}$ is the agent's inventory.
3. $W = (W_t)_{\{0 \le t \le T\}}$ is a Brownian motion.
4. $S^\nu = (S_t^\nu)_{\{0 \le t \le T\}}$ is the fundamental price process.
5. $g : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the permanent (negative) impact that the agent's trading action has on the fundamental price.
6. $\widehat{S}^\nu = (S_t^\nu)_{\{0 \le t \le T\}}$ corresponds to the execution price process at which the agent can sell the asset.
7. $h : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the temporary (negative) impact that the agent's trading action has on the price they can execute the trade at.
8. $X^\nu = (X_t^\nu)_{\{0 \le t \le T\}}$ is the agent's cash process.
9. $\mathcal{A}_{t,T}$ is the admissible set of strategies: $\mathcal{F}$-predictable non-negative bounded strategies. This constraint excludes repurchasing of shares and keeps the liquidation rate finite.

Optimal Limit Order Placement

1. In the optimal liquidation problem above, the agent is assumed to post market orders spread through time to liquidate her/his shares.
2. Such a strategy is intuitively sub-optimal since she/he will consistently be crossing the spread and potentially walking the book in order to sell her/his shares.
3. Since she/he may also place limit orders, she/he can at least save the cost of crossing the spread, and perhaps even achieve better performance by posting deeper in the LOB: At a depth of δ_t relative to the mid-price S_t.
4. The risk in doing so is that she/he may not execute her/his shares.
5. Conditional on a market sell order arriving, the probability that it lifts the agent's posted offer at a price of $S_t + \delta_t$ can be modelled as a function of δ_t which we call the "fill probability" and denote by $P(\delta_t)$.
6. The agent therefore can pose a control problem to decide how deep she/he must post in the LOB to optimize the value of liquidating her/his shares, subject to crossing the spread at the end of the trading horizon.

Optimal Limit Order Placement (Contd ...)

In this case, the agent's value function is given by:

$$H(x,S,q) = \sup_{\delta \in \mathcal{A}_{0,T}} \mathbb{E}\left[X_T^\delta + Q_T^\delta \left(S_T^\delta - \alpha Q_T^\delta \right) - \phi \int_0^T \left(Q_s^\delta \right)^2 ds \right],$$

where,

$$\begin{aligned} M_t & \quad \text{(Market Sell Orders)} && (6) \\ S_t &= S_0 + \sigma W_t, \ \text{(Asset Mid-Price)} && (7) \\ dX_t^\delta &= (S_t + \delta_t)(-dQ_t^\delta), \ X_0^\delta = x, \ \text{(Agent's Cash)} && (8) \\ dQ_t^\delta &= -\mathbb{1}_{\left\{ U_{M_t^{-1}+1} > P(\delta_t) \right\}} dM_t, \ Q_0^\delta = q. \ \text{(Agent's Inventory)} && (9) \end{aligned}$$

Control for Diffusion Processes

We now consider control problems of the form:

$$H(\mathbf{x}) = \sup_{\mathbf{u}\in\mathcal{A}_{0,T}} \mathbb{E}\left[G\left(\mathbf{X}^{\mathbf{u}}_T\right) + \int_0^T F\left(s, \mathbf{X}^{\mathbf{u}}_s, \mathbf{u}_s\right) ds \right], \tag{10}$$

where:

1. $\mathbf{u} = (\mathbf{u}_t)_{\{0\le t\le T\}}$ is the vector (dim p) valued control process.
2. $\mathbf{X}^{\mathbf{u}} = (\mathbf{X}_t)_{\{0\le t\le T\}}$ is the vector (dim n) valued controlled process assumed (in this disucussion) to be an Ito diffusion satisfying:

   $$d\mathbf{X}^{\mathbf{u}}_t = \mu(t, \mathbf{X}^{\mathbf{u}}_t, \mathbf{u}_t)dt + \sigma(t, \mathbf{X}^{\mathbf{u}}_t, \mathbf{u}_t)d\mathbf{W}_t,\ \ \mathbf{X}^{\mathbf{u}}_0 = \mathbf{x}, \tag{11}$$

   where $(\mathbf{W}_t)_{\{0\le t\le T\}}$ is a vector of independent Brownian motions.
3. $\mathcal{A}_{0,T}$ is a set (called the "admissible set") of $\mathcal{F}$-predictable processes such that (11) admits a strong solution (and may contain other constraints such as the process being bounded).
4. $G : \mathbb{R}^n \to \mathbb{R}$ is a terminal reward.
5. $F : \mathbb{R}_+ \times \mathbb{R}^{n+p} \to \mathbb{R}$ is a running penalty/reward.