MA668: Algorithmic and High Frequency Trading
Lecture 06

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Trading Costs: Modelling

1. Suppose that trader pays $\eta > 0$ per share, regardless of whether they are buying or selling the asset.
2. Further, assume that any remaining inventories after $t = 2$ are liquidated at $t = 3$ and that LT1 wants to sell $|i|$ units ($i > 0$) and LT2 wants to buy the same amount.
3. At $t = 2$, since MM's and LT1 enter the period with positive inventory, their optimal final holdings are:

$$q_2^j = \frac{E\left[(S_3 - \eta)|\epsilon_2\right] - (S_2 - \eta)}{\gamma\sigma^2},\ j \in \{MM, LT1\}$$

and the demand for shares by LT2 is:

$$q_2^{LT2} = \frac{E\left[(S_3 + \eta)|\epsilon_2\right] - (S_2 + \eta)}{\gamma\sigma^2}.$$

4. As everyone anticipates that their trading positions need to be liquidated anyway, the trading fees do not affect the price at $t = 2$, and we obtain $S_2 = E\left[S_3|\epsilon_2\right] = \mu + \epsilon_2$ (as before when $\eta = 0$ (no fees)).

Trading Costs: Modelling (Contd ...)

1. At $t = 1$, LT1 has a situation similar to that at $t = 2$, as any quantities not sold now, will have to be sold later. Accordingly,

$$q_1^{LT1} = \frac{E\left[(S_2 - \eta)\right] - (S_1 - \eta)}{\gamma\sigma^2}.$$

2. On the other hand, MM's anticipate that whatever they buy, they will have to sell later, which changes their asset demand function to:

$$q_1^{MM} = \frac{E\left[(S_2 - \eta)\right] - (S_1 + \eta)}{\gamma\sigma^2}.$$

3. The resulting market equilibrium condition is:

$$i = nq_1^{MM} + q_1^{LT1} = n\left(\frac{\mu - S_1 - 2\eta}{\gamma\sigma^2}\right) + \frac{\mu - S_1}{\gamma\sigma^2}.$$

4. This gives us:

$$i = (n+1)\left(\frac{\mu - S_1}{\gamma\sigma^2}\right) - \frac{2n\eta}{\gamma\sigma^2} \Rightarrow S_1 = \mu - \gamma\sigma^2\left(\frac{i}{n+1}\right) - 2\left(\frac{n}{n+1}\right)\eta.$$

Trading Costs: Modelling (Contd ...)

1. Recall that: For LT1, $i > 0$. Thus we conclude that the presence of trading fees introduces an extra liquidity discount to the initial price S_1.
2. What does this model tell us?: Almost all the trading fees are paid for by the liquidity trader initiating the transaction.
   - (A) She/he pays own trading fee of η, per share PLUS
   - (B) A substantial fraction $\dfrac{n}{n+1}$ of the two transaction fees paid by the MM's (2η), though indirectly, via a lower sale price, a lower S_1.
3. This also affects the immediacy that she/he obtains from the market, as her/his holdings at the end of $t = 1$ are no longer $q_1^{LT1,*} = \dfrac{i}{n+1}$, but $q_1^{LT1,*} = \dfrac{i}{n+1} + 2\left(\dfrac{n}{n+1}\right)\dfrac{\eta}{\gamma\sigma^2}$.

Trading Costs: Modelling (Contd ...)

1. Observation: Participation costs and fees have very different effects.
2. While participation costs enter directly through c, trading fees enter through expected future profits, which will be lower, as MM's must bear a fraction of the trading fees.
3. In particular, for each trade, the MM pays 2η, but recovers $2\left(\frac{n}{n+1}\right)\eta$, through the liquidity discount.
4. Summary: An increase in trading fees has a smaller effect on liquidity via competition, but a greater direct effect on immediacy and the liquidity discount.

Measuring Liquidity

1. We have already seen from the Grossman and Miller model that: Trading costs, whether setup costs or trading fees, are mostly paid by liquidity traders, either explicitly (as their own trading fees) or implicitly in the price (greater liquidity discount when selling and larger premium when buying).
2. We now consider how these divergences from “efficient” prices may be observed in electronic exchanges.
3. While the Grossman and Miller model is based on the premise of all trading taking place at one time and at a single price, but in electronic markets, not all decisions are taken at the same point of time.
4. Having said so, the equilibrium analysis can be easily reinterpreted in the context of an electronic market.
5. For example, suppose liquidity traders are very eager to trade and do so by sending MO’s into the exchange.
6. When the liquidity trader’s orders hit the market, they meet the LO’s that are posted by the patient MM’s and are resting in the LOB.

Measuring Liquidity (Contd ...)

The Grossman Miller model would correspond to the following sequence of events:

1. As LT1's MO's enter the market, they execute against LO's in the LOB which adjusts to the incoming MO.
2. As the execution price changes, so does LT1's strategy, and eventually, after selling $i\left(\frac{n}{n+1}\right)$ shares, the price has moved too far and LT1 stops trading.
3. Overall, LT1's MO executes at the average price of S_1, either because it was sent as a large order that walked the LOB or LOBs (in case of routing) OR because it was split up into several small orders that triggered a gradual move on the bid side in the LOB, away from the initial starting point.
4. Then the discount received by LT1 is the difference between:
   - A. Average price received, S_1 AND
   - B. Initial mid-price when the first MO hit the market (which is the usual proxy for the efficient price $E[S_2]$).

Measuring Liquidity (Contd ...)

1. Rewriting S_1 as a linear function of the quantity traded q^{LT1}:

$$S_1 = \mu + \lambda q^{LT1},$$

   so that in the Grossman Miller model of $S_1 = \mu - \gamma\sigma^2 \left(\frac{i}{n+1}\right)$, we would have:

$$\lambda = -\frac{\gamma\sigma^2}{n} \text{ and } q^{LT1} = i\left(\frac{n}{n+1}\right).$$

2. The parameter λ captures the market's price reaction to LT1's total order.
3. In particular: λ is used to describe the liquidity of the market for this asset.
4. A more liquid market will have a lower λ. (in absolute sense)

Measuring Liquidity (Contd ...)

1. A second popular way to measure liquidity is based on price changes and the measure is based on the autocovariance of the assets' return.

2. In order to see how this measure is constructed, let us introduce an additional date $t = 0$, prior to LT1's order submission ($t = 1$) and a random public news (announced between $t = 0$ and $t = 1$) event ϵ_1, that affects the assets' final liquidation price as follows:

$$S_3 = \mu + \epsilon_1 + \epsilon_2 + \epsilon_3.$$

3. Define the following constants:

$$\mu_0 = \overset{\mu}{E[S_3]},\ \mu_1 = \overset{\mu+\epsilon_1}{E[S_3|\epsilon_1]},\ \mu_2 = \overset{\mu+\epsilon_1+\epsilon_2}{E[S_3|\epsilon_1, \epsilon_2]} \text{ and } \mu_3 = S_3,$$

   and let $\epsilon_t, t = 1, 2, 3$ be normal with i.i.d. random variables having mean 0 and variance σ^2.

4. The discrete process μ_t is a martingale, and we refer to it as the efficient market price.