Implicit QR Algorithm for Eigenvalue Problems

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Outline

- Implicit QR algorithm and bulge chasing
- Implicit single and double shift QR algorithm

Explicit QR step

Let A be proper Hessenberg. Then a shifted explicit QR step with shit μ is given by

- $A - \mu I = Q_1 Q_2 \cdots Q_{n-1} R$ % Givens QR factorization of $A - \mu I$.

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Therefore, by Implicit-Q Theorem, $G = QD$ and $\widehat{A}_1 = G^* A G = D^* Q^* A Q D = D^* A_1 D$.

Single shift Implicit QR algorithm

Algorithm. (Single shift implicit QR algorithm)
Input: An $n \times n$ Hessenberg matrix A **Output:** Upper triangular matrix $T = Q^*AQ$

Repeat until convergence

(i) Choose a shift μ and construct a Givens rotation Q_1 such that

$$Q_1^*(A - \mu I)e_1 = Q_1^* \begin{bmatrix} a_{11} - \mu \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_1^*AQ_1$. This destroys Hessenberg form.

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(ii) Construct Givens rotations $Q_2, \ldots, Q_{n-1}$ such that

$$A \longleftarrow Q_{n-1}^* \cdots Q_2^* Q_1^* A Q_1 Q_2 \cdots Q_{n-1}$$

is upper Hessenberg.

Double shift explicit QR Algorithm

Let A be Hessenberg and let μ_1 and μ_2 be scalars. Consider two steps of QR iteration on A.

- $A - \mu_1 I = Q_1 R_1, \;\; A_1 = R_1 Q_1 + \mu_1 I$ % first QR step

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$$A^2 - (\mu_1 + \mu_2)A + \mu_1 \mu_2 I \;\; = \;\; (A - \mu_2 I)(A - \mu_1 I) = (A - \mu_2 I) Q_1 R_1$$

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$$\begin{aligned} A^2 - (\mu_1 + \mu_2)A + \mu_1 \mu_2 I &= (A - \mu_2 I)(A - \mu_1 I) = (A - \mu_2 I) Q_1 R_1 \\ &= Q_1 Q_1^* (A - \mu_2 I) Q_1 R_1 = Q_1 (A_1 - \mu_2 I) R_1 \end{aligned}$$

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A^2 - (\mu_1 + \mu_2)A + \mu_1 \mu_2 I &= (A - \mu_2 I)(A - \mu_1 I) = (A - \mu_2 I) Q_1 R_1 \\
&= Q_1 Q_1^* (A - \mu_2 I) Q_1 R_1 = Q_1 (A_1 - \mu_2 I) R_1 \\
&= Q_1 Q_2 R_2 R_1 = QR.
\end{aligned}
$$

Double shift explicit QR Algorithm

Let A be Hessenberg and let μ_1 and μ_2 be scalars. Consider two steps of QR iteration on A.

- $A - \mu_1 I = Q_1 R_1, \;\; A_1 = R_1 Q_1 + \mu_1 I$ % first QR step
- $A_1 - \mu_2 I = Q_2 R_2, \;\; A_2 = R_2 Q_2 + \mu_2 I$ % 2nd QR step

Define $Q := Q_1 Q_2$ and $R := R_2 R_1$. Then $A_2 = Q_2^* A_1 Q_2 = Q_2^* Q_1^* A Q_1 Q_2 = Q^* A Q$ and

$$\begin{aligned} A^2 - (\mu_1 + \mu_2)A + \mu_1\mu_2 I &= (A - \mu_2 I)(A - \mu_1 I) = (A - \mu_2 I) Q_1 R_1 \\ &= Q_1 Q_1^* (A - \mu_2 I) Q_1 R_1 = Q_1 (A_1 - \mu_2 I) R_1 \\ &= Q_1 Q_2 R_2 R_1 = QR. \end{aligned}$$

This yields double shift QR algorithm.

Double shift explicit QR Algorithm:

- Choose shifts μ_1 and μ_2
- $A^2 - (\mu_1 + \mu_2)A + \mu_1\mu_2 I = QR = Q_1 Q_2 \cdots Q_{n-1} R$ % QR factorization

Double shift explicit QR Algorithm

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Double shift explicit QR Algorithm:

- Choose shifts μ_1 and μ_2
- $A^2 - (\mu_1 + \mu_2) A + \mu_1 \mu_2 I = QR = Q_1 Q_2 \cdots Q_{n-1} R$ % QR factorization
- $A \longleftarrow Q^* A Q = Q_{n-1}^* \cdots Q_2^* (Q_1^* A Q_1) Q_2 \cdots Q_{n-1}$ % similarity transformation

Double shift Implicit QR algorithm

Algorithm. (Double shift implicit QR algorithm)
Input: An $n \times n$ Hessenberg matrix A **Output:** Upper triangular matrix $T = Q^* AQ$

(i) Choose shifts μ_1 and μ_2 and construct a Householder reflector Q_1 such that

$$Q_1^*(A^2 - (\mu_1 + \mu_2)A + \mu_1\mu_2 I)e_1 = Q_1^* \begin{bmatrix} a_{21}[\frac{a_{11}^2 - (\mu_1+\mu_2)a_{11} + \mu_1\mu_2}{a_{21}} + a_{12}] \\ a_{21}[a_{11} + a_{22} - (\mu_1 + \mu_2)] \\ a_{21}a_{32} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_1^* AQ_1$. This destroys Hessenberg form.

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and compute $Q_1^*AQ_1$. This destroys Hessenberg form.
(ii) Construct Householder reflectors $Q_2, \ldots, Q_{n-1}$ such that

$$A \longleftarrow Q_{n-1}^* \cdots Q_2^* Q_1^* A Q_1 Q_2 \cdots Q_{n-1}$$

is upper Hessenberg.