MA668: Algorithmic and High Frequency Trading
Lecture 31

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Optimal Acquisition With a Price Limiter

1. We solve the problem for an agent whose target is to acquire $\mathfrak{R}$ shares over a trading horizon of T.
2. There is a cap on the price at which she/he acquires shares, set equal to $\bar{S}$.
3. If the mid-price reaches this limit price before T, all remaining shares are immediately purchased and the acquisition programme stops.
4. We assume the earlier mid-price dynamics with $g(\nu_t) = b\nu_t$, $b \geq 0$.
5. We assume the earlier execution price with linear price impact $f(\nu_t) = k\nu_t$, $k > 0$.
6. The agent will stop trading if any one of the following events occur:
    - (A) The agent's inventory reaches the target level $\mathfrak{R}$.
    - (B) The terminal time T is reached.
    - (C) The mid-price S_t reaches the upper limit price $\bar{S}$.

Optimal Acquisition With a Price Limiter (Contd ...)

1. Accordingly, these define the following stopping time, τ:

$$\tau = T \wedge \inf\{t : S_t = \bar{S}\} \wedge \{t : Q_t = \mathfrak{R}\}.$$

When either of events (B) or (C) occur, the agent acquires the remaining $(\mathfrak{R} - Q_\tau^\nu)$ units of the security and pays $S_\tau + \alpha(\mathfrak{R} - Q_\tau^\nu)$ per unit, where $\alpha > 0$.

2. To simplify notation, we let $Y_t^\nu = \mathfrak{R} - Q_t^\nu$ denote the remaining shares to be acquired, satisfying:

$$dY_t^\nu = -\nu_t dt,$$

where ν_t is the (positive) rate of trading.

3. The agent's performance criteria is given by:

$$H^\nu(t,S,y) = \mathbb{E}_{t,S,y}\left[\int_t^\tau (S_u + k\nu_u)\nu_u du + y_\tau(S_\tau + \alpha y_\tau) + \phi\int_t^\tau y_u^2 du\right], \tag{1}$$

where $\phi \int_t^\tau y_u^2 du$, with $\phi \geq 0$ being a running inventory penalty of the remaining shares to be acquired.

Optimal Acquisition With a Price Limiter (Contd ...)

1. The value function is:

$$H(t,S,y) = \inf_{\nu\in\mathcal{A}} H^{\nu}(t,S,y),\ 0 \le t \le T, S \le \bar{S}, 0 \le y \le Q,$$

   with $\mathcal{A}$ being the admissible set of trading strategies in which ν is non-negative and uniformly bounded from above.

2. The resulting DPE is given by:

$$\begin{aligned} 0 &= \partial_t H + \frac{1}{2}\sigma^2\partial_{SS}H + \phi y^2 \\ &+ \inf_{\nu}\left[-\nu\partial_y H + b\nu\partial_S H + (S+k\nu)\nu\right], \end{aligned} \tag{2}$$

   where, $H(T,S,y) = (S+\alpha y)y$, $H(t,\bar{S},y) = (\bar{S}+\alpha y)y$ and $H(t,S,0) = 0$.

3. The terminal and boundary conditions reflect the fact that the agent acquires the remaining shares at the stopping time.

Optimal Acquisition With a Price Limiter (Contd ...)

1. Note that when her/his inventory equals the target $\mathfrak{R}$ at $t < T$, she/he stops acquiring and there is no penalty.
2. Hence the value function equals zero along $y = 0$.
3. From the first order conditions, we obtain the optimal acquisition strategy in feedback form as:

$$\nu^*(t, S, y) = -\frac{1}{2k}\left(b\partial_S H - \partial_y H + S\right).$$

4. This results in:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \frac{1}{4k}\left(b\partial_S H - \partial_y H + S\right)^2 + \phi y^2 = 0, \qquad (3)$$

subject to the terminal and boundary conditions as above.

Dimensionality Reduction Without Permanent Price Impact

1. In general, the DPE (3) will have to be solved numerically.
2. However, in practice it is normally the case that the effect of permanent impact is much smaller than the temporary impact from walking the LOB.
3. Therefore in order to reduce the dimension of the problem, we set $b = 0$.
4. Accordingly, in this case, due to the form of the DPE (3) and its terminal and boundary conditions in (2), it is possible to solve for the dependence in q exactly by using the ansatz: $H(t, S, y) = yS + y^2 h(t, S)$.
5. Then the function h satisfies the Fisher-type PDE:

$$\partial_t h + \frac{1}{2}\sigma^2 \partial_{SS} h - \frac{1}{k} h^2 + \phi = 0, \ (t, S) \in [0, T) \times (-\infty, \bar{S}), \tag{4}$$

subject to conditions $h(T, s) = \alpha, \ S \leq \bar{S}$ and $h(t, \bar{S}) = \alpha, \ t \leq T$.

6. The optimal acquisition strategy ν^* reduces to:

$$\nu^*(t, S, y) = \frac{1}{k} y h(t, S). \tag{5}$$

7. As inventory Q increases (so that Y decreases), all else being equal, the optimal rate of acquisition slows down.

Incorporating Order Flow

1. Previously: We assumed that in the absence of the agent's trades, the mid-price process is a martingale.
2. Further: Assume that when the agent begins to liquidate (acquire) shares, her/his actions induce a downward (upward) drift in the mid-price process.
3. The act of her/his selling (buying) shares induces the market as a whole to adjust prices downwards (upwards).
4. However, at the same time we are ignoring the trades of other market participants, implicitly assuming that on an average their actions even out to yield a net of zero drift.
5. This may be acceptable at an aggregate level, but over short time horizons, there may be order flow imbalance, which very often results in prices trending upwards or downwards over short intervals in time.
6. Next: We show how to incorporate the order flow from the remainder of the market into the mid-price dynamics and how the agent modifies her/his strategy to adapt to it locally.

Incorporating Order Flow: The Model Setup

1. In addition to the usual state variables and stochastic processes already introduced, we now also model the dynamics of the buy and sell rate of order flow $\mu_t^{\pm}$ and assume that they satisfy the SDE:

$$d\mu_t^{\pm} = -\kappa \mu_t^{\pm} dt + \eta L_t^{\pm}, \tag{6}$$

where $L_t^{\pm}$ are independent Poisson processes (assumed independent of all other processes as well) with equal intensity λ.

2. This assumption implies that the buy and sell order flows arrive independently at Poisson times with rate λ, and induce an increase in the order flow rate by η and jumps in order flow rate decay at the speed κ.

3. Next, we incorporate the order flow in the mid-price process S_t, which now satisfies the SDE:

$$dS_t^{\nu} = \sigma dW_t + \left(g(\mu_t^{+}) - g(\mu_t^{-} + \nu_t)\right) dt,$$

where W_t is a Brownian motion independent of the Poisson processes and g is an impact function which dictates how the mid-price drift is affected by the buy/sell order flow.