MA423 Matrix Computations

Lectures 1&2: Floating-point system

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Lecture outline

- Floating-point system
- Machine precision and unit roundoff
- Rounding errors
- Floating-point arithmetic

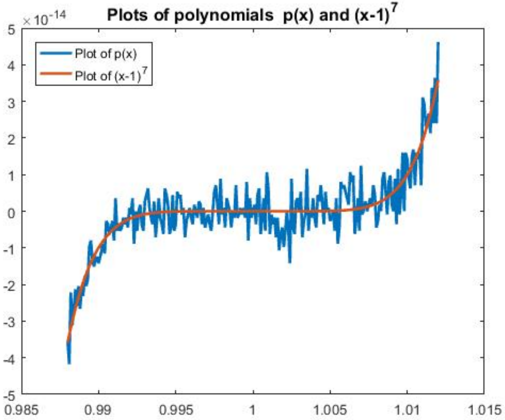
Anomalous arithmetic

Arithmetic operations on computers are inexact and can produce surprising results. For example, MATLAB produces

$$
\begin{aligned}
(\frac{4}{3} - 1) * 3 - 1 &= -2.2204 \times 10^{-16} \\
5 \times \frac{(1 + \exp(-50)) - 1}{(1 + \exp(-50)) - 1} &= \textbf{NaN} \\
\frac{\log(\exp(750))}{100} &= \textbf{Inf}
\end{aligned}
$$

Example 2

Let $p(x) = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1 = (x-1)^7$.



Does the plot look like a polynomial? What explains this behaviour?

Normalized floating-point representation

Let $x \in \mathbb{R}$ be nonzero and $\beta > 1$ be an integer. Then we have the normalized floating-point representation

$$x = \pm(\cdot d_1 d_2 \cdots d_t \cdots)_\beta \times \beta^e, \; 0 \le d_i < \beta, \;\; d_1 \ne 0, \;\; e \in \mathbb{Z}.$$

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Note that $0 < f \leq 1$ and $x = \pm f \times \beta^e$ is a unique representation. The number of digits allowed in f is called precision of the floating-point representation, which in the present case is infinite.

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The fraction $0 < f := (\cdot d_1 d_2 \cdots d_t)_\beta = \sum_{j=1}^{t} \frac{d_j}{\beta^j} \le 1 - \beta^{-t} < 1.$

Example: $(\cdot 101)_2 \times 2^2 = (1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}) \times 2^2 = 5/2.$

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Note that $\#F(\beta, t, e_{\min}, e_{\max}) = 2(\beta - 1)\beta^{t-1} \times (e_{\max} - e_{\min} + 1) + 1.$

Floating-Point System

Underflow threshold (smallest positive floating point number):

- $\texttt{realmin} := (\cdot 10\cdots 0)_\beta \times \beta^{e_{\min}} = \beta^{e_{\min}-1}$

Overflow threshold (largest positive floating point number):

- $\texttt{realmax} := (\cdot(\beta-1)\cdots(\beta-1))_\beta \times \beta^{e_{\max}} = (1-\beta^{-t})\beta^{e_{\max}}$

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Machine epsilon/precision: $\mathbf{eps} := \beta^{1-t}$.

- $\mathbf{eps} = (\cdot 10\cdots 01)_\beta \times \beta^1 - (\cdot 10\cdots 0)_\beta \times \beta^1 = \beta^{1-t}$

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Machine epsilon (eps) is defined as the distance (gap) between 1 and the next largest floating point number.

Machine epsilon/precision: **eps** $:= \beta^{1-t}$.

- **eps** $= (\cdot 10\cdots 01)_\beta \times \beta^1 - \underbrace{(\cdot 10\cdots 0)_\beta \times \beta^1}_{=1} = \beta^{1-t}$

For MATLAB we have the following

	Binary	Decimal
eps	2^{-52}	2.2204×10^{-16}
realmin	2^{-1022}	2.2251×10^{-308}
realmax	$(2-\text{eps}) \times 2^{1023}$	1.7977×10^{308}

IEEE single precision floating-point representation

IEEE specifies $F(2, t, e_{\min}, e_{\max})$ as follows.

sign (1 bit)	exponent (8 bits)	mantissa (23 bits)

Table : Single precision 32 bits

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Table : Single precision 32 bits

\# Exponents: $2^8 = 256$ Largest exponent: $2^8 - 1 = 255$
Exponent range: $0 \leq b \leq 255$ and $-127 \leq e \leq 128$

$$x = (-1)^s \times (1 \cdot d_1 \cdots d_{23})_2 \times 2^{b-127} \text{ where } e = b - 127.$$

Effective exponent range: $-126 \leq e \leq 127$ as $e = -127$ is reserved for ± 0 and $e = 128$ for $\pm$`inf` and `NaN`

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How is eps 2^-23 and not 2^-22

Why not 2^(-127)

$\texttt{eps} = 2^{-23} \approx 1.192 \times 10^{-7}$, $\texttt{realmin} = 2^{-126} \approx 1.175 \times 10^{-38}$, $\texttt{realmax} = (2 - \texttt{eps}) \times 2^{127} \approx 3.403 \times 10^{38}$

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$\texttt{eps} = 2^{-52} \approx 2.2204 \times 10^{-16}$, $\texttt{realmin} = 2^{-1022} \approx 2.2251 \times 10^{-308}$,
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Gap between floating-point numbers

Let $x \in F(\beta, t, e_{\min}, e_{\max})$ be given by $x = (\cdot d_1 d_2 \cdots d_t)_\beta \times \beta^e$.

Define $\mathrm{ulp}(x) := \beta^{e-t}$. Then $\mathrm{next}(x) := x + \mathrm{ulp}(x)$ is the next floating point number larger than x, that is, $\mathrm{next}(x)$ is the smallest floating-point number larger than x. Hence x and $\mathrm{next}(x)$ are consecutive floating-point numbers.

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This shows that $\mathrm{next}(x) - x = \mathrm{ulp}(x) = \beta^{e-t}$. So, if x is a large number then e is large which implies large gap between x and $\mathrm{next}(x)$. Hence the gap between consecutive floating-point number is nonuniform.

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Note that $\mathrm{eps} = \mathrm{ulp}(1) = \beta^{1-t}$ and $\mathrm{next}(1) = 1 + \mathrm{eps}$. Hence eps is the gap between 1 and the next floating-point number $\mathrm{next}(1)$ larger than 1.

Relative gap between floating-point numbers

Let $x \in F(\beta, t, e_{\min}, e_{\max})$ be given by $x = (\cdot d_1 d_2 \cdots d_t)_\beta \times \beta^e$. Then $\mathrm{next}(x) = x + \mathrm{ulp}(x) = x + \beta^{e-t}$ implies that the relative gap

$$\frac{\mathrm{next}(x) - x}{x} = \frac{\beta^{e-t}}{(.d_1 \cdots d_t)_\beta \times \beta^e} \leq \beta^{1-t} = \mathrm{eps}.$$

Hence the relative gap between two consecutive floating-point numbers is almost uniform and the relative gap is at most eps.

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Example: Consider $x \in F(10, 4, -30, 30)$ given by $x = 10^{-9} = .1000 \times 10^{-8}$. Then $\mathrm{ulp}(x) = 10^{-8-4} = 10^{-12}$ and $\mathrm{next}(x) - x = 10^{-12}$.

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$$\text{Define } \mathrm{fl}(x) := \begin{cases} x_L, & \text{round down} \\ x_R, & \text{round up} \\ x_L \text{ or } x_R \text{ whichever is closer to } x, & \text{round to nearest} \end{cases}$$

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Unit roundoff: $$\mathbf{u} := \begin{cases} \beta^{1-t} & \text{round down} \\ \frac{1}{2}\beta^{1-t}, & \text{round to nearest} \end{cases}$$

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Proof: Let $x = (\cdot d_1 d_2 \cdots d_t \cdots)_\beta \times \beta^e$. Set $\delta := \frac{\mathrm{fl}(x)-x}{x}$. Then $\mathrm{fl}(x) = x(1+\delta)$. Note that $|\mathrm{fl}(x) - x| \leq \beta^{e-t}$ for round down, and $|\mathrm{fl}(x) - x| \leq \beta^{e-t}/2$ for round to nearest. Also $x \geq \beta^{e-1}$.

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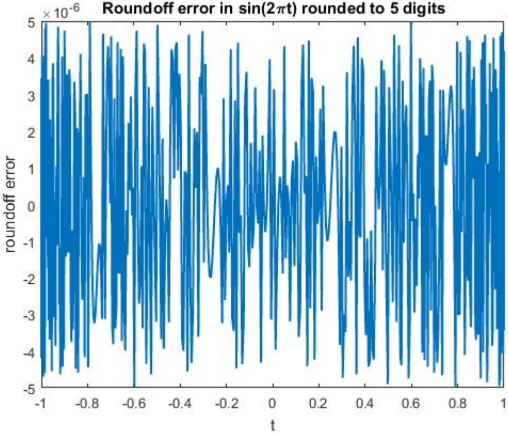
$$|\delta| \leq \frac{|\text{fl}(x) - x|}{|x|} \leq \begin{cases} \beta^{1-t}, & \text{round down} \\ \frac{1}{2}\beta^{1-t}, & \text{round to nearest.} \end{cases}$$

Rounding error

Let $f(t) := \sin(2\pi t)$ for $t \in [0, 1]$. Consider $F(10, 5, -10, 10)$. Then
`t = 0:.001:1; tt = sin(2*pi*t); rt = round(tt, 5);`
`roundoff = tt-rt; plot(tt, roundoff, 'LineWidth',1.5)` produces the following



Floating-point arithmetic

Arithmetic model: If $x, y \in F(\beta, t, e_{\min}, e_{\max})$ and $\oplus \in \{+, -, \times, /\}$ and $\mathbf{u}$ is the unit roundoff then

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Loss of significance (or cancellation): If x and y are not floating-point numbers then we have to start with $\widehat{x} := \mathrm{fl}(x) = x(1+\delta_1)$ and $\widehat{y} := \mathrm{fl}(y) = y(1+\delta_2)$. Then

$$\widehat{x} \pm \widehat{y} = x(1+\delta_1) \pm y(1+\delta_2) = (x \pm y)\left(1 + \frac{x\delta_1 \pm y\delta_2}{x \pm y}\right)$$

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Arithmetic model: If $x, y \in F(\beta, t, e_{\min}, e_{\max})$ and $\oplus \in \{+, -, \times, /\}$ and $\mathbf{u}$ is the unit roundoff then

$$\mathrm{fl}(x \oplus y) = (x \oplus y)(1+\delta), \text{ where } |\delta| \leq \mathbf{u}.$$

Loss of significance (or cancellation): If x and y are not floating-point numbers then we have to start with $\widehat{x} := \mathrm{fl}(x) = x(1+\delta_1)$ and $\widehat{y} := \mathrm{fl}(y) = y(1+\delta_2)$. Then

$$\widehat{x} \pm \widehat{y} = x(1+\delta_1) \pm y(1+\delta_2) = (x \pm y)\left(1 + \frac{x\delta_1 \pm y\delta_2}{x \pm y}\right)$$

Now, if x and y have the same sign then the relative error $\left|\frac{x\delta_1 + y\delta_2}{x+y}\right| \leq |\delta_1| + |\delta_2| \leq 2\mathbf{u}$ is small but $\left|\frac{x\delta_1 - y\delta_2}{x-y}\right|$ can be arbitrarily large when $x - y$ is very small.

Example

Problem: Solve $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$

Classical method: Naive use of the formula yields

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

does not avoid subtraction whereas the modified formula

$$x_1 = -\frac{b + \mathrm{sign}(b)\sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{c}{ax_1}$$

avoid subtractions and hence avoid loss of significance.

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For $10^{-3}x^2 + 10^7 x + 3 = 0$, the classical method in MATLAB yields $x_1 = -10^{10}$ and $x_2 = 0$ whereas the modified method yields the accurate answer $x_1 = -10^{10}$ and $x_2 = -3.0 \times 10^{-7}$.

Cancellation and remedy

Let $f(x) := \dfrac{1-\cos(x)}{\sin(x)}$ for $x \neq 0$. If a is close to 0 then evaluation of $f(x)$ at a causes cancellation as $\cos(a) \approx 1$. The remedy is to rewrite $f(x)$ as

$$f(x) = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)}$$

which avoids cancellation at $a \approx 0$.