MA 668 | End-Sem

Algorithmic and High Frequency Trading | 09:00 AM–12:00 PM

IIT Guwahati | 29th April 2024

Instructions

1. Write your name and roll number on the answerscript.
2. Your writing should be legible and neat.
3. This End-Sem has 8 questions, for a total of 40 marks.

Questions

[5 marks] 1. State TRUE or FALSE

(A) The probability of a sell Limit Order, placed at depth δ, being lifted on arrival of a Market Order of volume V, increases with respect to δ.

(B) The fill probability, *i.e.,* the probability of being filled when posting at a given depth δ, conditional on the arrival of a Market Order, is an decreasing function with respect to δ.

(C) A trader who sends orders to the Dark Pool is exposed to execution risk.

(D) A trader who sends orders to the Dark Pool experiences the additional temporary price impact resulting from walking the Limit Order Book.

(E) Pairs trading strategy produces returns in excess of the risk-free rate, while having zero risk.

Answer:

(A) FALSE ...(1 mark)

(B) TRUE ...(1 mark)

(C) TRUE ...(1 mark)

(D) FALSE ...(1 mark)

(E) FALSE ...(1 mark)

[2 marks] 2. Fill in the blanks:

(A) The trading venue that does not display the bid and ask quotes to their clients is commonly known as ________.

(B) A trading strategy that involves the simultaneous buying of one stock and selling of another stock with the goal of generating profit from the relative price movements of the two stocks is commonly known as ________.

Answer:

(A) Dark Pools ...(1 mark)

(B) Pairs Trading ...(1 mark)

[2 marks] 3. (A) Write down the equation for mid-price dynamics, in terms of permanent price impact $g : \mathbb{R}_+ \to \mathbb{R}_+$ and the standard Brownian motion.

(B) If $f : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the temporary price impact, then, what is the relation between execution price, mid-price, bid-ask spread and temporary price impact.

Answer:

(A) The mid-price dynamics is given by:

$$dS_t^\nu = \pm g(\nu_t)dt + \sigma dW_t, \;\; S_0^\nu = S. \ldots (1 \text{ mark})$$

(B) The execution price dynamics is given by:

$$\widehat{S}_t^\nu = S_t^\nu \pm \left(\frac{\Delta}{2} + f(\nu_t)\right), \;\; \widehat{S}_0^\nu = \widehat{S}. \ldots (1 \text{ mark})$$

[4marks] 4. Assume that the controlled processes $\mathbf{X}^{\mathbf{u}} = (\mathbf{X}_t^{\mathbf{u}})_{0\le t\le T}$ satisfies the SDEs:

$$d\mathbf{X}_t^{\mathbf{u}} = \mu_t^{\mathbf{u}} dt + \sigma_t^{\mathbf{u}} d\mathbf{W}_t + \gamma_t^{\mathbf{u}} d\mathbf{N}_t^{\mathbf{u}},$$

The value function is given by:

$$H(t, \mathbf{x}) = \sup_{\mathbf{u}\in\mathcal{A}_{[t,T]}} H^{\mathbf{u}}(t, \mathbf{x}),$$

where the agent's performance criteria is given by:

$$H^{\mathbf{u}}(t, \mathbf{x}) = \mathbb{E}_{t,\mathbf{x}}\left[G\left(\mathbf{X}_T^{\mathbf{u}}\right) + \int_t^T F\left(s, \mathbf{X}_s^{\mathbf{u}}, \mathbf{u}_s\right) ds\right].$$

Then write the DPP and the DPE for this problem.

Answer:

The DPP is given by:

$$H(t, \mathbf{x}) = \sup_{\mathbf{u}\in\mathcal{A}_{[t,T]}} \mathbb{E}_{t,\mathbf{x}}\left[H(\tau, \mathbf{X}_\tau^{\mathbf{u}}) + \int_t^\tau F\left(s, \mathbf{X}_s^{\mathbf{u}}, \mathbf{u}_s\right) ds\right],$$

for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^m$ and all stopping times $\tau \le T$. $\ldots$(2 marks)

The DPE is given by:

$$\partial_t H(t, x) + \sup_{\mathbf{u}\in\mathcal{A}_t} \left(\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) + F(t, \mathbf{x}, \mathbf{u})\right) = 0, \;\; H(T, \mathbf{x}) = G(\mathbf{x}),$$

where:

$$\begin{aligned}\mathcal{L}_t^{\mathbf{u}} H(t, \mathbf{x}) &= \mu(t, \mathbf{x}, \mathbf{u}) \cdot \mathcal{D}_x H(t, \mathbf{x}) + \frac{1}{2}\sigma(t, \mathbf{x}, \mathbf{u})\sigma(t, \mathbf{x}, \mathbf{u})' \mathcal{D}_x^2 H(t, \mathbf{x}) \\ &+ \sum_{j=1}^{p} \lambda_j(t, \mathbf{x}, \mathbf{u}) \left[H(t, \mathbf{x} + \gamma_{\cdot j}(t, \mathbf{x}, \mathbf{u})) - H(t, \mathbf{x})\right].\end{aligned}$$

$\ldots$(2 marks)

[5marks] 5. (A) Obtain the value function for the Market Maker seeking the strategy $(\delta_s^{\pm})_{0\le s\le T}$, which maximizes the cash at the terminal date, under the condition that the Market Maker liquidates the terminal inventory Q_T using a Market Order.

(B) Hence write the DPE for the problem with the boundary conditions.

Answer:

(A) The MM's cash process $X = (X_t^\delta)_{0\leq t\leq T}$ satisfies the SDE:

$$dX_t^\delta = (S_{t^-} + \delta_t^+)dN_t^{\delta,+} - (S_{t^-} - \delta_t^-)dN_t^{\delta,-},$$

which accounts for the cash inflow/increase when a sell LO is lifted by a buy MO and the cash outflow/decrease when a buy LO is hit by an incoming sell MO, respectively. The fill rate of LOs can be written as $\Lambda_t^{\delta,\pm} = \lambda^\pm e^{-\kappa_\pm \delta_t^\pm}$, which is the rate of execution of an LO. ...(1 mark)

Then the performance criterion is:

$$H^\delta(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T + Q_T^\delta(S_T^\delta - \alpha Q_T^\delta) - \phi\int_t^T (Q_u)^2 du\right],$$

where $\alpha \geq 0$ represents the fees for taking liquidity (that is, using an MO) as well as the impact of the MO walking the LOB. Further, $\phi \geq 0$ is the running inventory penalty parameter. Consequently the MM's value function is:

$$H(t,x,S,q) = \sup_{\delta^\pm\in\mathcal{A}} H^\delta(t,x,S,q),$$

where $\mathcal{A}$ denotes the set of admissible strategies, that is, $\mathcal{F}$-predictable and bounded from below. ...(2 marks)

(B) Thus the DPE is given by:

$$\begin{aligned}
0 &= \partial_t H + \frac{1}{2}\sigma^2\partial_{SS}H - \phi q^2, \\
&+ \lambda^+ \sup_{\delta^+}\left[e^{-\kappa^+\delta^+}\left(H(t,x+(S+\delta^+),q-1,S) - H(t,x,q,S)\right)\right]\mathbb{1}_{\{q>\underline{q}\}}, \\
&+ \lambda^- \sup_{\delta^-}\left[e^{-\kappa^-\delta^-}\left(H(t,x-(S-\delta^-),q+1,S) - H(t,x,q,S)\right)\right]\mathbb{1}_{\{q<\bar{q}\}},
\end{aligned}$$

where $\mathbb{1}$ is the indicator function, with the terminal condition of $H(T,x,S,q) = x + q(S-\alpha q)$. ...(2 marks)

[4 marks] 6. Consider the problem of agent's execution strategy targeting a percentage of the speed at which the other market participants are trading, focusing on the liquidation strategy with Market Orders only. The agent's inventory is given by the process $dQ_t^\nu = -\nu_t dt$, with $Q_0^\nu = \mathfrak{R}$. Since the agent's objective is to seek an optimal liquidation speed ν_t which targets the Percentage of Volume (POV) $\rho\mu_t$ at time t, with $0<\rho<1$, the performance criteria is:

$$H^\nu(t,x,S,\mu,q) = E_{t,x,S,\mu,q}\left[X_T^\nu + Q_T^\nu(S_T^\nu - \alpha Q_T^\nu) - \phi\int_t^T (\nu_u - \rho\mu_u)^2 du\right].$$

(A) What do X_T^ν, α and ϕ represent.

(B) Write down the DPE for the problem.

(C) Hence, write down the optimal ν^* in terms of H and other parameters.

Answer:

(A) X_T^ν is terminal cash, $\alpha \geq 0$ is a liquidation penalty and $\phi \geq 0$ is the target penalty parameter. ...(1 mark)

(B) The DPE for the problem is given by:

$$\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS} + \mathcal{L}^\mu\right) H \\ &+ \sup_\nu \left[(S - k\nu)\nu\partial_x H - \nu\partial_q H - b\nu\partial_S H - \phi(\nu - \rho\mu)^2\right], \end{aligned}$$

subject to the terminal condition $H(T, x, S, \mu, q) = x + q(S - \alpha q)$. ...(2 marks)

(C) The supremum is attained at the optimal value of:

$$\nu^* = \frac{S\partial_x H - \partial_q H - b\partial_S H + 2\phi\rho\mu}{2(k + \phi)} \text{...(1 mark)}$$

[9marks] 7. (A) Construct the following DPE for optimal liquidation in lit and dark markets:

$$\begin{aligned} 0 &= \partial_t H + \frac{1}{2}\sigma^2\partial_{SS} H - \phi q^2, \\ &+ \sup_\nu \left[(S - k\nu)\nu\partial_x H - \nu\partial_q H\right], \\ &+ \sup_{y \leq q} \left[\lambda\mathbb{E}\left[H\left(t, x + S\min(y, \xi), S, q - \min(y, \xi)\right) - H(t, x, S, q)\right]\right], \end{aligned}$$

subject to the terminal condition:

$$H(T, x, S, q) = x + q(S - \alpha q).$$

(B) Hence provide the interpretation of the terms in the DPE.

Answer:

(A) When trading $\nu_t dt$ in the lit market, the agent receives $\widehat{S}_t = S_t - k\nu_t$ per share, with $k > 0$, where the mid-price S_t is a Brownian motion. In addition to trading in the lit market, the agent posts $y_t \leq q_t$ units of inventory in the dark pool. Here $q_t \leq \mathfrak{R}$ are the remaining shares to be liquidated, and the agent may continuously adjust this posted order. Matching orders in the dark have no price impact because they are pegged to the lit market's mid-price and hence the agent receives S_t per share for each unit executed in the dark pool, which is not necessarily the whole amount y_t. ...(1 mark)
The other market participants send matching orders to the dark pool which are assumed to arrive at Poisson times and the volumes associated with the orders are independent. More specifically, let N_t denote a Poisson process with intensity λ and let $\{\xi_j : j = 1, 2, \dots\}$ be a collection of independent and identically distributed random variables corresponding to the volume of the various matching orders which are sent by other market participants into the dark pool. The total volume of buy orders (which may match the agent's posted sell order) placed in the dark pool up to time t is the compound Poisson process:

$$V_t = \sum_{n=1}^{N_t} \xi_n \text{...(1 mark)}$$

When a matching order arrives, it may be larger or smaller than the agent's posted sell order, and hence the agent's inventory (accounting for both the continuous trading in the lit market and the agent's post in the dark pool) satisfies the SDE:

$$dQ_t^{\nu,y} = -\nu_t dt - \min\left(y_t, \xi_{1+N_{t^-}}\right) dN_t \ldots \text{(1 mark)}$$

The agent's aim is to liquidate $\mathfrak{N}$ shares on or before the terminal date T. In the preceding equation above:

(a) The first term on the right-hand side represents the shares that the agent liquidates using MOs in the lit market.

(b) The second term on the right-hand side represents the orders the agent sends to the dark pool.

We assume that the agent is at the front of the sell queue in the dark pool, so that she/he is the first to execute against any new orders coming into that market. ...(1 mark)
Then, the agent's cash process $X_t^{\nu,y}$ satisfies the SDE:

$$dX_t^{\nu,y} = (S_t - k\nu_t)\nu_t dt + S_t \min\left(y_t, \xi_{1+N_{t^-}}\right) dN_t.$$

The agent's performance criteria is given by:

$$H^{\nu,y}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_\tau + Q_\tau^{\nu,y}(S_\tau - \alpha Q_\tau^{\nu,y}) - \phi \int_t^\tau (Q_u^{\nu,y})^2 du\right] \ldots \text{(1 mark)}$$

Here $\mathbb{E}_{t,x,S,q}$ is conditioned on $X_{t^-} = x$, $S_t = S$ and $Q_{t^-} = q$. The stopping time $\tau = T \wedge \inf\{t : Q_t = 0\}$, represents the time until the agent's inventory is completely liquidated or the terminal time has arrived. The value function is:

$$H(t,x,S,q) = \sup_{\nu,y\in\mathcal{A}} H^{\nu,y}(t,x,S,q),$$

where the set of admissible strategies consists of $\mathcal{F}$-predictable processes bounded from above, and her/his posted volume in the dark pool is at most her/his remaining inventory, that is, $y_t \le Q_t^{\nu,y}$. ...(1 mark)
The resulting DPE is given by:

$$\begin{aligned} 0 &= \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2, \\ &+ \sup_\nu \left[(S - k\nu)\nu \partial_x H - \nu \partial_q H\right], \\ &+ \sup_{y\le q} \left[\lambda \mathbb{E}\left[H\left(t, x + S\min(y,\xi), S, q - \min(y,\xi)\right) - H(t,x,S,q)\right]\right], \end{aligned}$$

subject to the terminal condition:

$$H(T,x,S,q) = x + q(S - \alpha q) \ldots \text{(1 mark)}$$

(B) The various terms in the DPE carry the following interpretations:

(a) The term ∂_{SS} represents the diffusion of the midprice.

(b) The term $-\phi q^2$ represents the running penalty which penalizes inventories different from zero.

(c) The term $\sup_\nu[\cdot]$ represents optimizing over continuous trading in the lit market.

(d) The term $\sup_{y\le q}$ represents optimizing over the volume posted in the dark pool: The expectation is there to account for the fact that buy volume coming into the dark pool from other traders is random.

...(2 marks)

[9 marks] 8. (A) Construct the following DPE for the problem of liquidation with Limit Orders and Market Orders:

$$
\begin{aligned}
0 \;=\; \max \Big[& \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\
& + \sup_{\delta} \lambda e^{-\kappa\delta} \left[H(t, x + (S+\delta), S, q-1) - H(t,x,S,q) \right], \\
& \left[H(t, x + (S-\xi), S, q-1) - H(t,x,S,q) \right] \Big],
\end{aligned}
$$

with boundary and terminal conditions $H(t, x, S, 0) = x$ and $H(T, x, s, q) = x + qS - l(q)$, respectively.

(B) Hence provide the interpretation of the terms in the DPE.

Answer:

(A) The agent's cash process $X = (X_t)_{0 \le t \le T}$ satisfies the SDE:

$$
dX_t^{\tau,\delta} = (S_t + \delta_{t^-}) dN_t^{\delta} + (S_t - \xi_t) dM_t^{a,\tau} \ldots \text{(2 marks)}
$$

The agents' performance criteria is:

$$
H^{(\tau,\delta)}(t, x, S, q) = \mathbb{E}_{t,x,S,q} \left[X_T^{\tau,\delta} + Q_T^{\tau,\delta} S_T - l\left(Q_T^{\tau,\delta}\right) - \phi \int_t^T \left(Q_u^{\tau,\delta}\right)^2 du \right],
$$

where $\mathbb{E}_{t,x,S,q}[\cdot]$ denotes expectation conditional on $X_{t^-}^{\tau,\delta} = x$, $S_{t^-} = S$ and $Q_{t^-}^{\tau,\delta} = q$, and the terminal liquidation penalty $l(q) = q(\xi + \alpha q)$. ...(2 marks)

Since the agent may execute MOs, her/his inventory is reduced each time an LO is filled or an MO is executed, so that:

$$
Q_t^{\tau,\delta} = \mathfrak{R} - N_t - M_t^a \ldots \text{(1 mark)}
$$

The set of admissible strategies $\mathcal{A}$ now includes seeking over all $\mathcal{F}$-stopping times, in addition to the set of $\mathcal{F}$-predictable, bounded from below, depths δ. Accordingly, the value function is:

$$
H(t, x, S, q) = \sup_{(\tau,\delta) \in \mathcal{A}} H^{(\tau,\delta)}(t, x, S, q) \ldots \text{(1 mark)}
$$

The DPP implies that the value function should satisfy the quasi-variational inequality (QVI), rather than the usual non-linear PDE:

$$
\begin{aligned}
0 \;=\; \max \Big[& \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\
& + \sup_{\delta} \lambda e^{-\kappa\delta} \left[H(t, x + (S+\delta), S, q-1) - H(t,x,S,q) \right], \\
& \left[H(t, x + (S-\xi), S, q-1) - H(t,x,S,q) \right] \Big],
\end{aligned}
$$

with boundary and terminal conditions $H(t, x, S, 0) = x$ and $H(T, x, s, q) = x + qS - l(q)$, respectively. ...(1 mark)

(B) The overall max operator represents the agent's choice to either post an LO (the continuation region) resulting in the first term in the max operator, or to execute an MO (the stopping region) resulting in a value function change of
$[H(t, x + (S-\xi), S, q-1) - H(t,x,S,q)]$: The agent's cash increases by $(S - \xi)$ and inventory decreases by 1 upon executing an MO. Within the continuation region where the agent posts LOs (the first term in the max):

(a) The operator ∂_{SS} corresponds to the generator of the Brownian motion which drives mid-price.

(b) The term $-\phi q^2$ corresponds to the contribution of the running inventory penalty.

(c) The supremum over δ takes into account the agent's ability to control the posted depth.

(d) The $\lambda e^{-\kappa\delta}$ coefficient represents the arrival rate of MOs which fill the agents posted LO at the price $S + \delta$.

(e) The difference term $[H(t, x + (S + \delta), S, q - 1) - H(t, x, S, q)]$ represents the change in the value function when an MO fills the agent's LO: The agent's cash increases by $S + \delta$ and her/his inventory decreases by 1.

...(2 marks)