MA 423 Matrix Computations 2022

Mid Semester Examination

Q.1. ALG is backward stable if there exist $\Delta x, \Delta y \in \mathbb{R}^n$ such that

$$ALG(x, y) = fl(xy^T) = (x+\Delta x)(y+\Delta y)^T$$

and $\frac{\|\Delta y\|}{\|y\|} = O(u)$, $\frac{\|\Delta x\|}{\|x\|} = O(u)$.

$\boxed{\text{1 Mark.}}$

Now $fl(xy^T) = [fl(x_i y_j)]_{n\times n}$

$= [x_i y_j (1+\delta_{ij})]_{n\times n}$, where

$|\delta_{ij}| \leq u$ for $i,j = 1:n$.

$$\Rightarrow \quad fl(xy^T) = xy^T + [x_i y_j \delta_{ij}]_{n\times n}$$

$$\Rightarrow \quad |fl(x,y^T) - xy^T| \leq u\,|xy^T|$$

$\boxed{\text{2 Marks}}$

Since δ_{ij} are rounding errors, the matrix $[x_i y_j (1+\delta_{ij})]_{n\times n}$ cannot be expected to be a rank-1 matrix. Hence $fl(xy^T)$ cannot be written as $(x+\Delta x)(y+\Delta y)^T$ for some $\Delta x, \Delta y \in \mathbb{R}^n$.

$\boxed{\text{2 Marks}}$

Q.2 (a) Since P and L are nonsingular,

$\text{rank}(A) = \text{rank}(PA) = \text{rank}(LU) = \text{rank}(U)$

However, $\text{rank}(U) \neq$ number of nonzero diagonal entries of U.

Counter example: $U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Then $\text{rank}(A) = 1$ but # nonzero diagonal entries of $U = 0$.

$\boxed{\text{2 Marks}}$

(b) x is a solution of the LSP $Ax \approx b$

$\Longleftrightarrow A^* A x = A^* b \Longleftrightarrow A^*(Ax - b) = 0$

$\Longleftrightarrow A^* r = 0$ $\boxed{\text{1 Mark}}$

Hence x is a solution of LSP $Ax \approx b$

$\Longleftrightarrow Ax - b = r$ and $A^* r = 0$

$$\Longleftrightarrow \begin{bmatrix} I_m & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} -r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\boxed{\text{2 Marks}}$

Q.3 Let $x \in \mathbb{R}^n$ be nonzero. Then

$$x^T(A + vv^T)x = x^TAx + (v^Tx)^2$$
$$\geq x^TAx > 0.$$

$\Rightarrow$ $A + vv^T$ is SPD $\because$ $(A + vv^T)$ is symmetric.

$\boxed{\text{2 Marks.}}$

Since G is nonsingular, we have

$$\text{rank}\left(\begin{bmatrix} G^T \\ v^T \end{bmatrix}\right) = \text{rank}(G^T) = n.$$

Since Q is orthogonal, we have

$$\text{rank}(R) = \text{rank}\left(\begin{bmatrix} R \\ 0 \end{bmatrix}\right) = \text{rank}\left(Q\begin{bmatrix} R \\ 0 \end{bmatrix}\right)$$
$$= \text{rank}\left(\begin{bmatrix} G^T \\ v^T \end{bmatrix}\right) = n$$

$\Rightarrow$ R is nonsingular $\boxed{\text{2 Marks}}$

Let $r_{11} \ldots r_{nn}$ be the diagonal entries of R. Define $\hat{Q} = \left[\begin{array}{c|c} \text{diag}(r_{11} \ldots r_{nn}) & 0 \\ \hline 0 & I_n \end{array}\right]$

and $\hat{R} = \text{diag}(r_{11}, \ldots, r_{nn})R$. Then $\hat{R}$ has positive diagonal entries. and

$$\begin{bmatrix} G^T \\ v^T \end{bmatrix} = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = Q\hat{Q}\left(\hat{Q}\begin{bmatrix} R \\ 0 \end{bmatrix}\right)$$
$$= (Q\hat{Q})\begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$$

This shows that R can be chosen to have positive diagonal entries

$\boxed{2 \text{ Marks}}$

Suppose that R has positive diagonal entries. Then

$$\begin{bmatrix} G^T \\ v^T \end{bmatrix}^T \begin{bmatrix} G^T \\ v^T \end{bmatrix} = R^T R$$

$$\Rightarrow \begin{bmatrix} G & v \end{bmatrix} \begin{bmatrix} G^T \\ v^T \end{bmatrix} = R^T R$$

$$\Rightarrow GG^T + vv^T = R^T R$$

$$\Rightarrow A + vv^T = R^T R.$$

Since R^T is ~~upper~~ lower triangular with positive diagonal entries, $R^T R$ is the Cholesky factorization of $A + vv^T$.

$\boxed{2 \text{ Marks}}$

Q.4

$$(A+\Delta A)\hat{x} = b+\Delta b$$

$$\Rightarrow (I+A^{-1}\Delta A)\hat{x} = A^{-1}b + A^{-1}\Delta b$$
$$= x + A^{-1}\Delta b.$$

$$\Rightarrow \hat{x}-x = A^{-1}\Delta b - A^{-1}\Delta A x$$

$\boxed{\text{2 Marks}}$

$$\therefore \|\hat{x}-x\| \leq \|A^{-1}\| \|\Delta b\| + \|A^{-1}\|.\|\Delta A\| \|x\|$$

$$\Rightarrow \frac{\|\hat{x}-x\|}{\|x\|} \leq \frac{\|A^{-1}\| \|\Delta b\|}{\|x\|} + \|A^{-1}\| \|\Delta A\|$$

$$\leq \text{Cond}(A)\left[\frac{\|\Delta b\|}{\|A\| \|x\|} + \frac{\|\Delta A\|}{\|A\|}\right].$$

$\boxed{\text{2 Marks}}$

Q.5 $A = LU$

$\Rightarrow \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = L \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$

Equating rows, we have

$$A_i = \sum_{j=1}^{n} L(i,j) U_j$$

Since L is lower triangular and $L(j,j)=1$ for $j = 1, 2, \dots n$, we have

$$A_i = \sum_{j=1}^{i} L(i,j) U_j = \sum_{j=1}^{i-1} L(i,j) U_j + U_i$$

$$\Rightarrow U_i = A_i - \sum_{j=1}^{i-1} L(i,j) U_j$$

$\boxed{2 \text{ Marks}}$

Note that $\|A\|_\infty = \max_{1 \le i \le n} \|A_i\|_1$ and

$$\|U\|_\infty = \max_{1 \le i \le n} \|U_i\|_1 .$$

Since $|L(i,j)| \le 1$, we have

$U_1 = A_1 \Rightarrow \|U_1\|_1 \le \|A_1\|_1 \le \|A\|_\infty$.

$U_2 = A_2 - L(2,1) U_1$

$\Rightarrow \|U_2\|_1 \le \|A_2\|_1 + \|U_1\|_1 \le 2\|A\|_\infty$.

$\boxed{2 \text{ Marks}}$

Next, $U_3 = A_3 - L(3,1)U_1 - L(3,2)U_2$

$\Rightarrow \|U_3\|_1 \le \|A_3\|_1 + \|U_1\|_1 + \|U_2\|_1$

$\le \|A\|_\infty + \|A\|_\infty + 2\|A\|_\infty$

$\le 2^2 \|A\|_\infty$.

$\boxed{\text{2 Marks}}$

Continuing this process, we have

$\|U_n\| \le 2^{n-1} \|A\|_\infty$

$\Rightarrow \|U\|_\infty = \max_{1\le j\le n} \|U_j\| \le 2^{n-1} \|A\|_\infty$.

$\boxed{\text{1 Mark}}$

Since $|L(i,j)| \le 1$, we have

$\|L\|_\infty \le n$. Hence

$$PG(A) = \frac{\|L\|_\infty \|U\|_\infty}{\|A\|_\infty} \le n\, 2^{n-1}$$

$\boxed{\text{2 Marks}}$

Q. 6 Since $\hat{R}$ is $m \times (n+1)$ upper triangular and $m > n$, we have

$$\hat{R} = \begin{bmatrix} R & c \\ 0 & d \\ 0 & 0 \end{bmatrix}$$ where $R \in \mathbb{R}^{n \times n}$ is upper triangular and $d \in \mathbb{R}$.

$\boxed{\text{1 Mark}}$

Since $\operatorname{rank}(A) = n$, the first n columns of $[A, b]$ are linearly independent.

Since Q unitary and $[A, b] = Q\hat{R}$, the first n columns of $\hat{R}$ are linearly independent.

$\Rightarrow$ Columns of R are linearly independent

$\Rightarrow$ R is nonsingular.

$\boxed{\text{3 Marks}}$

Now $\|Ax - b\|_2 = \left\| [A, b] \begin{bmatrix} x \\ -1 \end{bmatrix} \right\|_2$

$= \left\| Q\hat{R} \begin{bmatrix} x \\ -1 \end{bmatrix} \right\|_2 = \left\| \hat{R} \begin{bmatrix} x \\ -1 \end{bmatrix} \right\|_2$

$= \sqrt{\|Rx - c\|_2^2 + |d|^2}$

$\boxed{\text{3 Marks}}$

$\Rightarrow$ ~~MER~~ ||

$\Rightarrow \cdot \min \|Ax - b\|_2 = |d| \iff$

$Rx = c$

$\boxed{1 \text{ Mark}}$

Algorithm.

1. Compute $[A, b] = Q \begin{bmatrix} R & c \\ 0 & d \\ 0 & 0 \end{bmatrix}$

2. Solve $Rx = c$

3. Compute the resedual $|d|$.

$\boxed{2 \text{ Marks}}$

Q.7 Suppose that $y^* x \in \mathbb{R}$.

Then $x^* y = (y^* x)^* \in \mathbb{R}$

and

$$\langle x+y, x-y \rangle = \|x\|_2^2 - y^* x + x^* y - \|y\|_2^2$$

$$= 0.$$

$\Rightarrow (x+y) \perp (x-y)$ $\boxed{1 \text{ Mark}}$

Define $u := x-y$. Then

$$H = I - \frac{2uu^*}{\|u\|_2^2}$$ is a reflector

and $Hx = H\left(\frac{x+y}{2} + \frac{(x-y)}{2}\right)$

$$= \frac{x+y}{2} - \left(\frac{x-y}{2}\right) = y$$ $\boxed{2 \text{ Marks}}$

Conversely, suppose that H is a reflector such that $Hx = y$. Since $H^* = H$,

$$y^* x = (Hx)^* x = x^* H^* x = x^* H x \in \mathbb{R}.$$

$\boxed{2 \text{ Marks}}$

Note that $(\cos\theta/2, \sin\theta/2)$ is a point on the line $y = (\tan\theta/2)x$.

Also note that $\left(\sin\theta/_2, -\cos\theta/_2\right)$ is orthogonal to $\left(\cos\theta/_2, \sin\theta/_2\right)$.

Hence $u := \left(\sin\theta/_2, -\cos\theta/_2\right)$ is a normal to the line $y = \left(\tan\theta/_2\right)x$. $\boxed{\text{2 Marks}}$

Therefor $H = I - 2uu^T$ is the unique reflector that reflects a vector through the line $y = \left(\tan\theta/_2\right)x$. $\boxed{\text{1 Mark}}$

Now $H = I - 2uu^T$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2\sin^2\theta/_2 & 2\sin\theta/_2\cos\theta/_2 \\ 2\sin\theta/_2\cos\theta/_2 & 2\cos^2\theta/_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2\sin^2\theta/_2 & \sin\theta \\ \sin\theta & 1-2\cos^2\theta/_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$\boxed{\text{2 Marks}}$