MA668: Algorithmic and High Frequency Trading
Lecture 27

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Figure 6.1 (Contd ...)

1. The second way in which the agent's execution can affect the mid-price is through $g(\nu_t)$.
2. We refer to this as the permanent impact of the trading rate.
3. If $g(\nu_t) > 0$, then a trade of size $\nu dt > 0$ moves the mid-price of the asset upwards.
4. An interpretation of this modelling assumption is that the agent is trading on information that reflects permanent changes in the value of the firm, and market participants adjust their quotes in response to the agent's trades.
5. The model can also be modified to incorporate the situation where the agent's trades exert pressure on the mid-price and then the pressure subsides after the agent has completed her/his execution target.
6. But, as we are focusing on an agent's execution of a single block of shares, this is not relevant as she/he will never receive any of the benefits of the "price correction" once she/he stops trading and we exclude it from the analysis.

The Model (Contd ...)

1. To conclude the description of the model, we turn to the agent's cash process X_t^ν.
2. This process satisfies the SDE:

$$dX_t^\nu = \widehat{S}_t^\nu \nu_t dt, \ X_0^\nu = x. \tag{1}$$

3. The expected revenue from the sale is:

$$R^\nu = \mathbb{E}\left[X_T^\nu - x\right] = \mathbb{E}\left[\int_0^T \widehat{S}_t \nu_t dt\right], \tag{2}$$

which is easy to see if we look at the sales proceeds over discrete time-steps.

The Model (Contd ...)

1. Suppose that the agent must liquidate $Q_0 = \mathfrak{R}$ amount of shares over the time period $[0, T]$.
2. Now, we split this trading horizon into equally spaced time intervals $t_0 = 0 < t_1 < t_2 < \cdots < t_N = T$, where $t_n - t_{n-1} = \Delta t$, for $n = 1, 2, \ldots, N$.
3. Next, assuming that over the time interval $[0, t_1]$, the agent sells $Q_0 - Q_{t_1}$ shares at the price $\widehat{S}_0$, over the interval $[t_1, t_2]$ the agent sells $Q_{t_1} - Q_{t_2}$ shares at the price $\widehat{S}_{t_1}$, and so on.
4. Then the total expected revenue from selling shares is:

$$R^{\nu}_{\Delta t} = \mathbb{E}\left[(Q_0 - Q_{t_1})\widehat{S}_0 + (Q_{t_1} - Q_{t_2})\widehat{S}_{t_1} + \cdots + (Q_{t_{N-1}} - Q_T)\widehat{S}_{t_{N-1}}\right].$$

5. Recalling the formulation of speed of trading, we observe that as $\Delta t \to 0$, we obtain (2).

The Model (Contd ...)

1. The reminder of the discussion in Chapter 6 looks at different optimal strategies to trade a block of shares using only MOs.
2. In each section, the setup of the control problem makes different assumptions about:
   - A. How the agent penalizes and/or controls inventory.
   - B. How her/his rate of trading affects her/his execution price as well as the mid-price of the asset.
3. We also alternate between share liquidation and share acquisition problems.

The Model (Contd ...)

1. Section 6.3: The agent must liquidate a block of shares and the agent's trades affect her/his execution price but do not affect the mid-price of the asset ($g(\nu) = 0$). The setup of the problem assumes that the execution strategy is designed so that all shares are liquidated by the terminal date.
2. Section 6.4: The agent solves for the optimal acquisition rate where any remaining unacquired inventory may be purchased at the terminal date but subject to a penalty (and ($g(\nu) = 0$)).
3. Section 6.5: The agent has to liquidate a block of shares and the agent's actions have both a permanent effect ($g(\nu) \geq 0$) on the execution price and a temporary effect ($f(\nu) \geq 0$) on the mid-price of the asset. In addition, we incorporate a parameter for the agent's urgency to execute the trade, through a penalization exposure to inventory, throughout the entire life of the strategy.

Liquidation Without Penalties: Only Temporary Impact

1. We start by discussing how an agent uses only MOs to optimally liquidate $\mathfrak{N}$ shares between $t = 0$ and $t = T$.
2. We assume that the agent's own trades do not affect the mid price of the asset. Accordingly, the stock's mid price is given by:

   $$dS_t^\nu = \pm g(\nu_t)dt + \sigma dW_t, \; S_0^\nu = S,$$

   with $g(\nu_t) = 0$.
3. On the other hand, the agent's trades have temporary impact on her/his own execution price because these MOs walk the LOB.
4. We assume that the temporary impact is linear in the speed of trading, so that $f(\nu_t) = k\nu_t$ with $k > 0$ in:

   $$\widehat{S}_t^\nu = S_t^\nu \pm \left(\frac{\Delta}{2} + f(\nu_t)\right), \; \widehat{S}_0^\nu = \widehat{S},$$
5. Recall that the speed of trading "ν_t" is what the agent controls.

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. For simplicity, we assume that the bid-ask spread $\Delta = 0$ or equivalently that S_t represents the best bid price [a].
2. Finally, we also assume that the agent is insistent that all $\mathfrak{N}$ shares are liquidated by time T.
3. The agent's objective is to choose the rate at which she/he liquidates $\mathfrak{N}$ shares so that she/he obtains the maximum amount of revenue from the sale.
4. Consequently, her/his strategy must be such that all shares are liquidated by time T, that is, the agent cannot reach expiry with any inventory left.
5. In other words, the agent wishes to find, among all admissible liquidation strategies ν, the one that minimizes the execution cost:

$$EC^{\nu} = \mathfrak{N} S_0 - \mathbb{E}\left[\int_0^T \widehat{S}_t^{\nu} \nu_t dt\right],$$

which is equivalent to maximizing the expected revenues from the target sale of the $\mathfrak{N}$ shares.

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[a] It is a simple matter to include a non-zero spread

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. Thus the agent's value function is:

$$H(t, S, q) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,S,q} \left[\int_t^T (S_u - k\nu_u)\, \nu_u du \right],$$

   where $\mathbb{E}_{t,S,q}$ denotes expectation conditional on $S_t = S$ and $Q_t = q$. Also $\mathcal{A}$ is the set of admissible strategies: $\mathcal{F}$-predictable non-negative bounded strategies. This constraint excludes repurchasing of shares and keeps the liquidation rate finite.

2. To solve this optimal control problem, we use the dynamic programming principle (DPP) which suggests that the value function satisfies the dynamic programming equation (DPE):

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \sup_{\nu} \left[(S - k\nu)\nu - \nu \partial_q H \right] = 0. \tag{3}$$

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. The agent requires that the optimal strategy liquidates all the inventory by time T.
2. Thus the value function reflects this by "penalizing" any terminal inventory which is not zero. So we require:
   - (A) $H(T, S, q) \to -\infty$ as $t \to T$, for $q > 0$.
   - (B) $H(T, S, 0) \to 0$ as $t \to T$.
3. The first order condition applied to DPE (3) shows that it attains a supremum at:

$$\nu^* = \frac{1}{2k}\left(S - \partial_q H\right), \tag{4}$$

   which is the optimal trading speed in feedback control form.

4. Upon substitution into the DPE, we obtain the non-linear partial differential equation:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \frac{1}{4k}\left(S - \partial_q H\right)^2 = 0, \tag{5}$$

   for the value function.

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. To propose an ansatz for the above equation it is helpful to look at the boundary conditions, so as to get an idea of which variables are relevant in the value function.
2. We know that if the strategy reaches the terminal date with a non-zero inventory, the value function must become arbitrarily large and negative: Because the optimal strategy must ensure that all shares are liquidated.
3. We propose that the value function be written in terms of the book value of the current inventory (marked-to-market, using the mid-price as reference) plus the excess value due to optimally liquidating the remaining shares, that is,

$$H(t, S, q) = qS + h(t, q), \tag{6}$$

where $h(t, q)$ is still to be determined, though we know that it must blow up as t approaches T.

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. The way the problem is set up, the best that the agent can do, is to achieve the mid-price.
2. Hence, the correction $h(t, q)$ to the book value must be negative and the agent's objective is to minimize this downward adjustment.
3. Substituting this ansatz into the DPE (5), we arrive at the following equation for $h(t, q)$:
$$\partial_t h + \frac{1}{4k} \left(\partial_q h\right)^2 = 0.$$
4. Interestingly, the volatility of the asset's mid-price drops out of the problem.
5. The reason for this is that the Brownian component is a martingale and hence on an average, it contributes zero to the value of liquidating shares.