Homework-5

MA-423 : Matrix Computations 2023 R. Alam

Norms of vectors and matrices

1. Let $x \in \mathbb{C}^n$. Show that
$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty \text{ and } \frac{\|x\|_1}{\sqrt{n}} \leq \|x\|_2 \leq \|x\|_1.$$

2. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Show that $|\,\|x\| - \|y\|\,| \leq \|x - y\|$ for all $x, y \in \mathbb{C}^n$.

3. Plot the closed unit ball $\mathcal{S}_p := \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$ for $p = 1, 2, \infty$.

4. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and $W \in \mathbb{C}^{n\times n}$ be nonsingular. Show that $\|x\|_W := \|Wx\|$ is a norm on $\mathbb{C}^n$.

5. Let $A \in \mathbb{C}^{m\times n}$. Show that $\|A\| := \max\limits_{\|x\|=1} \|Ax\| = \max\limits_{\|x\|\leq 1} \|Ax\| = \max\limits_{x\neq 0} \dfrac{\|Ax\|}{\|x\|}$.

6. Show that $\|\text{diag}(\lambda_1, \cdots, \lambda_n)\|_p = \max_{1\leq i\leq n} |\lambda_i|$ for $p = 1, 2, \infty$.

7. Let $A \in \mathbb{C}^{m\times n}$. Show that A^*A is positive semi-definite and hence there is a unitary matrix Q such that $A^*A = Q\text{diag}(\lambda_1, \cdots, \lambda_n)Q^*$ with $\lambda_j \geq 0, j = 1 : n$. Set $\lambda_{\max}(A^*A) := \max_j |\lambda_j|$. Show that $\|A^*A\|_2 = \lambda_{\max}(A^*A)$. Also show that $\|Ax\|_2^2 = x^*A^*Ax$ and deduce that $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)} = \sqrt{\|A^*A\|_2}$.

8. Let $A \in \mathbb{C}^{m\times n}$. Let $\sigma_1(A), \ldots, \sigma_n(A)$ denote the square roots of the n eigenvalues of A^*A. Show that $\|A\|_F = \|[\sigma_1(A), \ldots, \sigma_n(A)]^\top\|_2$.

9. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Show that $\|AB\|_F \leq \|A\|_2\|B\|_F$ and $\|AB\|_F \leq \|A\|_F\|B\|_2$.

10. Let $A \in \mathbb{C}^{m\times n}$. Define $\|A\|_{\max} := \max_{i,j} |e_i^\top Ae_j|$. Show that $\|\cdot\|_{\max}$ is a matrix norm but it is not a submultiplicative norm. Show that $\|A\|_{\max} \leq \|A\|_p \leq n\|A\|_{\max}$ when $m = n$, where $p = 1, 2, \infty$.

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