Householder QR factorization

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Outline

- Reflector
- Householder reflector
- Householder QR factorization

Orthogonal projection

Let $P \in \mathbb{C}^{n\times n}$. Then P is called a projection (or idempotent) if $P^2 = P$. If P is a projection then $\mathbb{C}^n = R(P) \oplus N(P)$. Indeed, $x = Px + (I - P)x$.

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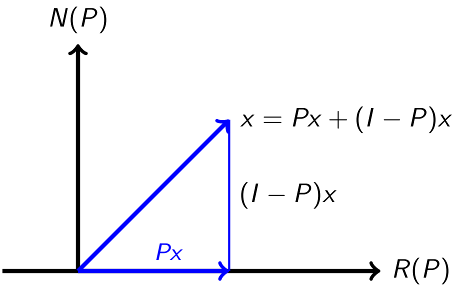

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Centering projection: Set $\mathbf{e} := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^{\top} \in \mathbb{C}^n$. Then $P := I - \dfrac{1}{n}\mathbf{e}\mathbf{e}^{\top}$ is an orthogonal projection. The projection P is also called a centering matrix as it projects a vector to a zero mean vector.

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Note that if $\mathbf{y} = P\mathbf{x}$ then $\mu_y := \dfrac{y_1 + \cdots + y_n}{n} = \dfrac{\mathbf{e}^\top \mathbf{y}}{n} = 0$. Hence $\mathbf{y}$ has zero mean.

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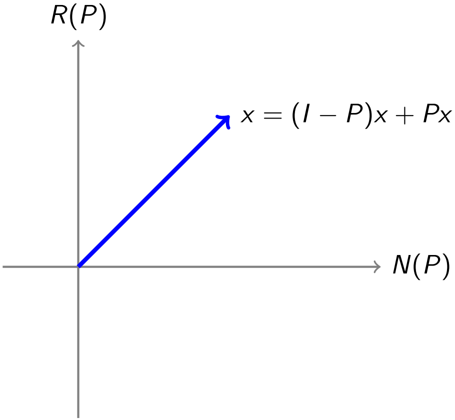

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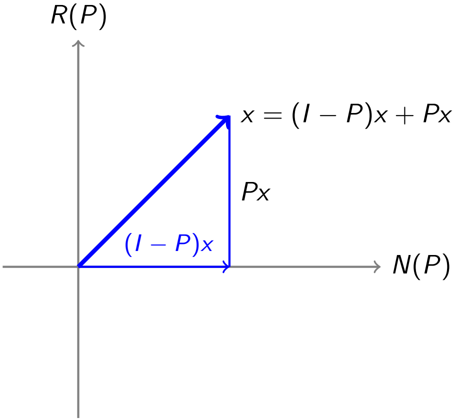

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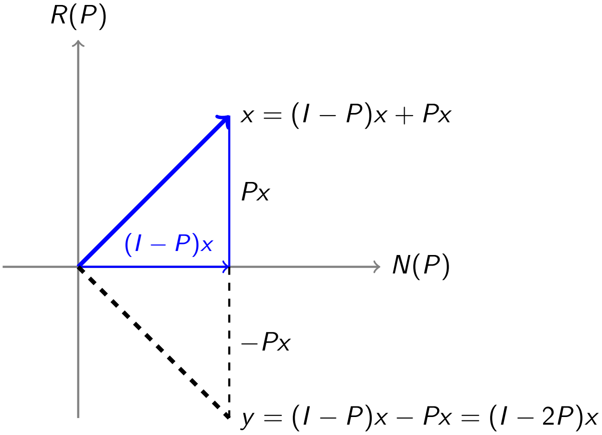

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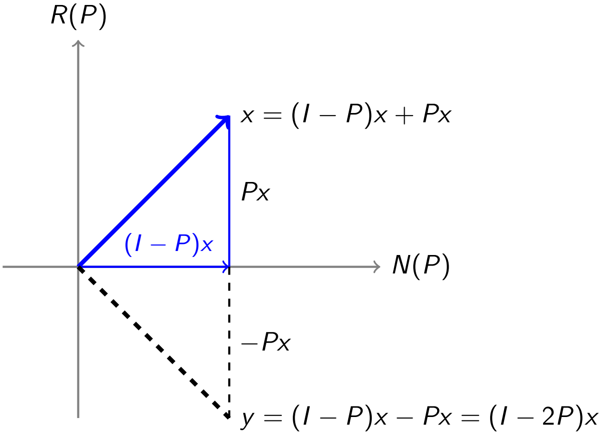

Then $H := I - 2P$ is a reflector. Hx is a reflection of x through $N(P)$.

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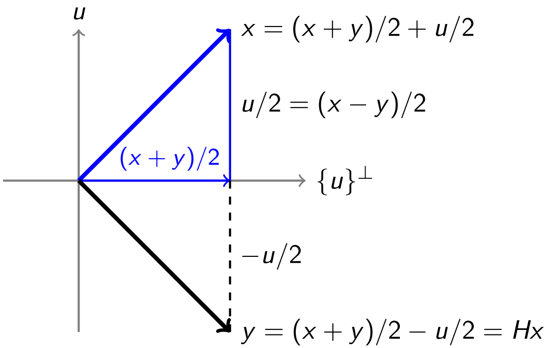

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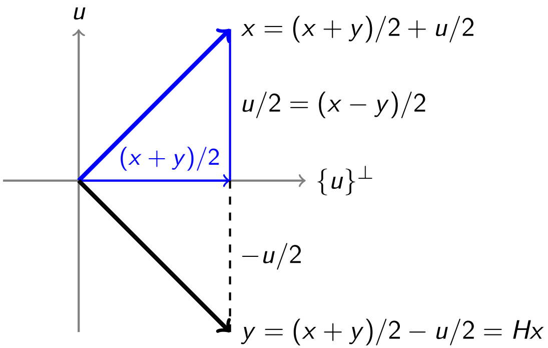

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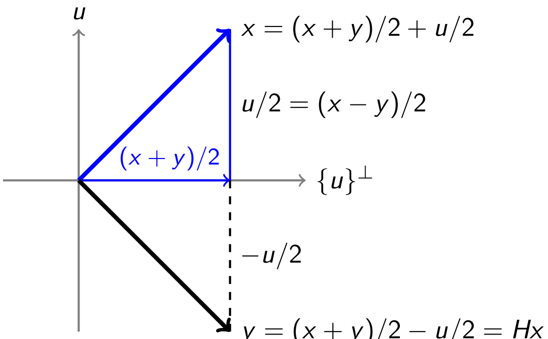

Note that $u \perp (x + y)$. Hence $Hu = -u$ and $H(x + y) = x + y$. Now $x = \frac{1}{2}(u + x + y) \Rightarrow Hx = \frac{1}{2}(Hu + H(x + y)) = \frac{1}{2}(-u + x + y) = y$. ■

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(2n-1) + 1 flops

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Set $u := x - y = \begin{bmatrix} 0 & \cdots & 0 & x_i + \sigma_i & x_{i+1} & \cdots & x_n \end{bmatrix}^\top$. Then $H := I - \alpha u u^\top$ is a unique reflector such that $Hx = y$, where $\alpha = 2/\|u\|_2^2 = 1/\sigma_i(x_i + \sigma_i) = 1/\sigma_i u_i$.

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Let $x := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ and $y := \begin{bmatrix} x_1 & \cdots x_{i-1} & -\sigma_i & 0 & \cdots & 0 \end{bmatrix}^\top$, where $\sigma_i := \mathrm{sign}(x_i)\sqrt{x_i^2 + \cdots + x_n^2}$. Then $\|x\|_2 = \|y\|_2$ and $x \neq y$.

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This shows that for $w \in \mathbb{R}^n$, the first $i-1$ components of Hw remain unchanged. Similarly, for $A \in \mathbb{R}^{n \times n}$, the first $i-1$ rows of HA and the first $i-1$ columns of AH remain unchanged.

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Hence we have $H_n \cdots H_2 H_1 A = R \Rightarrow A = H_1 H_2 \cdots H_n R = QR$.

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Input: An $m \times n$ matrix A.
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Since $H_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & \widehat{H}_k \end{bmatrix}$, $A \leftarrow H_k A$ performs transformation on $A(k:m, k:n)$, which dominates the computation and requires $4(m-k)(n-k)$ flops.

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$$\sum_{k=1}^{n} 4(m-k)(n-k) = 4\int_0^n (m-x)(n-x)dx = 2mn^2 - 2n^3/3 \text{ flops.}$$

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    % Update the k-th column.
    A(k:m,k) = 0; A(k,k) = -sigma;
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The MATLAB function `hqr` can be modified to store the Householder vectors in the lower triangular part of A and $-\sigma_1, \ldots, -\sigma_n$ can be stored in an additional array `sigma`.

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R = diag(sigma) + triu(A, 1).
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Solution of LSP by Householder QR factorization

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The total cost for solving the LSP $Ax = b$ by Householder QR factorization is $2mn^2 - \dfrac{2n^3}{3}$ flops.