MA668: Algorithmic and High Frequency Trading
Lecture 28

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Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. Focusing on the above non-linear PDE for h, we see that writing a separation of variables in the form $h(t,q) = q^2 h_2(t)$ (Note: The subscript 2 represents that this function is the coefficient of q^2) allows us to factor out q and obtain a simple non-linear ODE for $h_2(t)$:

$$\partial_t h_2 + \frac{1}{k} h_2^2 = 0. \tag{1}$$

2. Solving by integrating between t and T, we obtain:

$$h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{k}(T - t) \right)^{-1}.$$

3. As discussed above, the optimal strategy must ensure that the terminal inventory is zero and this is equivalent to requiring that $h_2(t) \to -\infty$ as $t \to T$.
4. In this way, the value function heavily penalizes the non-zero final inventory.

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. An alternative way to obtain this condition is to calculate the inventory path along the optimal strategy and impose the condition that the terminal inventory be zero.
2. To see this, we use the ansatz to obtain:

$$\nu_t^* = -\frac{1}{k} h_2(t) Q_t^{\nu^*}. \tag{2}$$

3. Then, we integrate $dQ_t^{\nu^*} = -\nu_t^* dt$ over $[0, t]$ to obtain the inventory profile along with the optimal strategy:

$$\int_0^t \frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} = \int_0^t \frac{h_2(s)}{k} ds.$$

4. This implies that:

$$Q_t^{\nu^*} = \frac{(T-t) - k/h_2(T)}{T - k/h_2(T)} \mathfrak{R}.$$

Liquidation Without Penalties: Only Temporary Impact (Contd ...)

1. To satisfy the terminal inventory condition $Q_T^{\nu^*} = 0$ and also ensure that the correction $h(t, q)$ to the book value of the outstanding shares that need to be liquidated is negative, we must have:

$$h_2(t) \to -\infty, \text{ as } t \to T. \tag{3}$$

2. Returning to solving the optimal problem, we have that:

$$h_2(t) = -k\,(T - t)^{-1}.$$

3. Then the optimal inventory to hold is:

$$Q_t^{\nu^*} = \left(1 - \frac{t}{T}\right)\mathfrak{R}. \tag{4}$$

4. Therefore the optimal speed of trading is:

$$\nu_t^* = \frac{\mathfrak{R}}{T}. \tag{5}$$

5. Finally: The shares must be liquidated at a constant rate and this strategy is the same as that of the time weighted average price (TWAP).

Optimal Acquisition With Terminal Penalty and Temporary Impact

1. The problem now is to acquire (not liquidate) $\mathfrak{N}$ by time T, starting with $Q_0^\nu = 0$.
2. As in the preceding discussion, the agent's MOs walk the LOB so that her/his execution price is described by:

$$\widehat{S}_t^\nu = S_t^\nu \pm \left(\frac{\Delta}{2} + f(\nu_t) \right), \ \widehat{S}_0^\nu = \widehat{S},$$

   with $f(\nu) = k\nu$, $k > 0$.
3. Although the agent's objective is to complete the acquisition programme by time T, she/he allows for strategies that fall short of this target, namely, $Q_T < \mathfrak{N}$ and in this case she/he must execute a buy MO for the remaining amount and pick up an additional penalty.
4. This terminal inventory penalty is parameterized by $\alpha > 0$, which includes the cost of walking the book at T and any other additional penalties that the agent must incur for the execution of the trade at the terminal date.

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Thus, the agent's expected costs from strategy ν_t is:

$$EC^{\nu} = \mathbb{E}\left[\int\limits_t^T \widehat{S}_u^{\nu}\nu_u du + (\mathfrak{R} - Q_T^{\nu})\, S_T + \alpha\,(\mathfrak{R} - Q_T^{\nu})^2\right]. \tag{6}$$

   - A. $\int\limits_t^T \widehat{S}_u^{\nu}\nu_u du$: Terminal Cash.
   - B. $(\mathfrak{R} - Q_T^{\nu})\, S_T$: Terminal Execution at Mid-Price.
   - C. $\alpha\,(\mathfrak{R} - Q_T^{\nu})^2$: Terminal Penalty.

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Compared to the expected costs in the previous discussion, we now have two additional terms.
2. In the liquidation problem of the previous discussion, the agent seeks a strategy that ensures all shares are liquidated by T and the expected costs arise exclusively from continuous trading.
3. Now, the agent can reach T, short of her/his target, but this generates the additional terms that incorporate the sale "plus" the penalty to purchase the remaining shares at the terminal date.

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. To simplify notation, we introduce a new stochastic process $Y = (Y_t)_{\{0 \le t \le T\}}$ to denote the shares remaining to be purchased between t and the end of the trading horizon T:

$$Y_t^{\nu} = \mathfrak{R} - Q_t^{\nu} \Rightarrow dY_t^{\nu} = -\nu_t dt.$$

2. Accordingly, we write the value function as:

$$H(t, S, y) = \inf_{\nu \in \mathcal{A}} \mathbb{E}_{t,S,y} \left[\int_t^T \widehat{S}_u^{\nu} \nu_u du + Y_T^{\nu} S_T + \alpha \left(Y_T^{\nu} \right)^2 \right],$$

where it is clear that the strategy seeks to minimize the cash paid to acquire the shares.

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Applying the DPP, we expect that the value function should satisfy the DPE:
$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \inf_{\nu} \left[(S + k\nu)\nu - \nu \partial_y H \right], \tag{7}$$
with terminal condition $H(T, S, y) = yS + \alpha y^2$.

2. Solving for the first order conditions, the optimal speed of trading in feedback form is given by:
$$\nu^* = \frac{1}{2k} \left(\partial_y H - S \right). \tag{8}$$

3. Upon substitution into the DPE above, we obtain:
$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \frac{1}{4k} \left(\partial_y H - S \right)^2 = 0.$$

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. To solve this DPE, we can write the value function in terms of the book value of the assets remaining to be acquired and the excess value function from optimally acquiring these shares.
2. From looking at the terminal condition and the way y enters into the DPE, we hypothesize that the excess value function can be written in terms of a quadratic function in y.
3. The corresponding ansatz is:

   $$H(t, S, y) = yS + h_0(t) + h_1(t)y + h_2(t)y^2, \tag{9}$$

   where $h_0(t)$, $h_1(t)$, $h_2(t)$ are (yet to be determined) deterministic functions of time.
4. Note that the subscripts on the functions indicate the power of y which multiplies them in the full ansatz.
5. Recall that the value function at the terminal date T is: $H(T, S, y) = yS + \alpha y^2$.
6. Then:

   $$h_2(T) = \alpha \text{ and } h_1(T) = h_0(T) = 0.$$

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Moreover, upon substituting the ansatz into the above non-linear PDE we find that:

$$0 = \left[\partial_t h_2 - \frac{1}{k} h_2^2\right] y^2 + \left[\partial_t h_1 - \frac{1}{2k} h_2 h_1\right] y + \left[\partial_t h_0 - \frac{1}{4k} h_1^2\right].$$

2. Since this equation must be valid for each y, therefore each term in the brackets must individually vanish.
3. This provides us with three equations for the three functions h_0, h_1 and h_2.
4. Due to the terminal condition $h_1(T) = 0$, we see that the solution we get for h_1 (by setting the second term in brackets to zero) is $h_1(t) = 0$.
5. Similarly, due to the terminal condition $h_0(T) = 0$, we see that the solution we get for h_0 (by setting the third term in brackets to zero, and knowing that $h_1(t) = 0$) is $h_0(t) = 0$.
6. Indeed we could have begun with the ansatz:

$$H(t, S, y) = yS + h_2(t)y^2,$$

and have ended up with the same equation for h_2.

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. The final equation (obtained by setting the first term in brackets to zero) allows us to obtain $h_2(t)$ and in this case, since $h_2(T) = \alpha$, we obtain the following non-trivial solution:

$$h_2(t) = \left(\frac{1}{k} (T - t) + \frac{1}{\alpha} \right)^{-1}.$$

2. Putting this together with the ansatz for the value function, we find that the optimal trading speed is:

$$\nu_t^* = \left((T - t) + \frac{k}{\alpha} \right)^{-1} Y_t^{\nu^*}. \tag{10}$$

Optimal Acquisition With Terminal Penalty and Temporary Impact (Contd ...)

1. Here we see that as the terminal penalty parameter $\alpha \to \infty$, the acquisition rate converges to that of TWAP.
2. Similarly, the smaller the value of α, all else being equal, the slower the acquisition rate will be.
3. Furthermore, in the limiting case $\alpha \to 0$, the optimal strategy is not to purchase any shares until the terminal date is reached, at which point all the $\mathfrak{N}$ shares are purchased.
4. In this limiting case, there are no costs of walking the book at date T, and so it is optimal to purchase all the inventory at the end.
5. In general, however, we expect that $\alpha \gg k$.
6. As before, we can solve for the optimal inventory path explicitly by integrating $dY_t^{\nu^*} = -\nu_t^* dt$ over $[0, t]$, that is, by solving:

$$dY_t^{\nu^*} = -\left((T - t) + \frac{k}{\alpha}\right)^{-1} Y_t^{\nu^*} dt,$$

for $Y_t^{\nu^*}$.