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MA-473 Midsem-Exam PAPER-1

1) Given transformation

$$x = \ln S.$$

$$\tau = \frac{\sigma^2}{2}(T-t)$$

$$V(S,t) = e^{\alpha x + \beta \tau} u(x,\tau) \quad \text{where} \quad \alpha = -\left(\frac{2r}{\sigma^2}-1\right)/2$$

$$\beta = -\left(\frac{2r}{\sigma^2}+1\right)^2/4$$

We have

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left[e^{\alpha x+\beta\tau}u(x,\tau)\right] = \frac{\partial}{\partial \tau}\left[e^{\alpha x+\beta\tau}u(x,\tau)\right]\cdot\frac{d\tau}{dt}$$

$$\boxed{\frac{\partial v}{\partial t} = \left(\beta e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}\right)\left(\frac{-\sigma^2}{2}\right)}$$

$$\frac{\partial v}{\partial S} = \frac{\partial}{\partial S}\left[e^{\alpha x+\beta\tau}u(x,\tau)\right] = \frac{\partial}{\partial x}\left[e^{\alpha x+\beta\tau}u(x,\tau)\right]\cdot\frac{\partial x}{\partial S}.$$

$$\boxed{\frac{\partial v}{\partial S} = \left(\alpha e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right)\frac{1}{S}}$$

$$\frac{\partial^2 v}{\partial S^2} = \frac{\partial}{\partial S}\left(\left[\frac{\alpha}{\cancel{\beta}} e^{\alpha x+\beta\tau} u + \cancel{\beta} e^{\alpha x+\beta\tau} \frac{\partial u}{\partial x}\right]\frac{1}{S}\right)$$

$$= -\frac{1}{S^2}\left[\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right] + \frac{1}{S}\frac{\partial}{\partial S}\left[\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right]$$

$$= -\frac{1}{S^2}\left[\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right] + \frac{1}{S}\left\{\alpha\frac{\partial}{\partial x}\left(e^{\alpha x+\beta\tau} u\right)\cdot\frac{\partial x}{\partial S} + \frac{\partial}{\partial x}\left(e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right)\cdot\frac{\partial x}{\partial S}\right\}$$

$$= -\frac{e^{\alpha x+\beta\tau}}{S^2}\left[\alpha u + \frac{\partial u}{\partial x}\right] + \frac{1}{S}\left\{\frac{\alpha}{S}\cdot\left[\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right] + \frac{1}{S}\left[\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}\right]\right\}$$

$$= -\frac{e^{\alpha x+\beta\tau}}{S^2}\left[\alpha u + \frac{\partial u}{\partial x}\right] + \frac{\alpha e^{\alpha x+\beta\tau}}{S^2}\left[\alpha u + \frac{\partial u}{\partial x}\right] + \frac{e^{\alpha x+\beta\tau}}{S^2}\left[\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right]$$

$$\boxed{\frac{\partial^2 v}{\partial S^2} = \frac{e^{\alpha x+\beta\tau}}{S^2}\left\{-\alpha u - \frac{\partial u}{\partial x} + \alpha^2 u + \alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right\}}$$

Now putting the values in B.S. PDE i.e.,

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rV = 0$$

we get;

$$\frac{-\sigma^2}{2} e^{\alpha x+\beta\tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right) + \frac{1}{2}\sigma^2 S^2 \cdot \frac{1}{S^2}\left\{-\alpha u - \frac{\partial u}{\partial x} + \alpha^2 u + \alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right\} e^{\alpha x+\beta\tau}$$

$$+ r S \cdot \frac{e^{\alpha x+\beta\tau}}{S}\left(\alpha u + \frac{\partial u}{\partial x}\right) - r e^{\alpha x+\beta\tau} u = 0$$

$$\Rightarrow -\frac{\sigma^2}{2}\left(\beta u + \frac{\partial u}{\partial \tau}\right) + \frac{\sigma^2}{2}\left(-\alpha u - \frac{\partial u}{\partial x} + \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) + r\left(\alpha u + \frac{\partial u}{\partial x}\right) - r u = 0$$

$$\Rightarrow -\frac{\sigma^2}{2}\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) + \left(-\frac{\sigma^2\beta}{2}u - \frac{\sigma^2\alpha}{2}u - \frac{\sigma^2}{2}\frac{\partial u}{\partial x} + \frac{\sigma^2\alpha^2}{2}u + \sigma^2\alpha\frac{\partial u}{\partial x} + r\alpha u + r\frac{\partial u}{\partial x} - r u\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty, \ \tau \geq 0.}$$

Now we will show:

$$-\frac{\sigma^2}{2}\beta u - \frac{\sigma^2\alpha}{2}u - \frac{\sigma^2}{2}\frac{\partial u}{\partial x} + \frac{\sigma^2\alpha^2}{2}u + \sigma^2\alpha\frac{\partial u}{\partial x} + r\alpha u + r\frac{\partial u}{\partial x} - r u = 0$$

$$\Rightarrow \left(-\frac{\sigma^2}{2}\beta - \frac{\sigma^2\alpha}{2} + \frac{\sigma^2\alpha^2}{2} + r\alpha - r\right)u + \left(\frac{-\sigma^2}{2} + \sigma^2\alpha + r\right)\frac{\partial u}{\partial x}$$

$$= \left(\frac{-\sigma^2}{2}\beta + \frac{\sigma^2}{2}\alpha(\alpha-1) + r(\alpha-1)\right)u + \left(\frac{-\sigma^2}{2} + \frac{\sigma^2}{2}\left(\cancel{\frac{-2r}{\sigma}} + 1\right) + r\right)\frac{\partial u}{\partial x}$$

0

$$\Rightarrow \quad -\frac{\sigma^2}{2}\beta + \frac{\sigma^2}{2}\alpha(\alpha-1) + r(\alpha-1)$$

on putting values of $\alpha = -\frac{1}{2}\left(\frac{2r}{\sigma^2}-1\right)$

and $\beta = -\frac{1}{4}\left(\frac{2r}{\sigma^2}+1\right)^2$

we will get

$$\boxed{-\frac{\sigma^2}{2}\beta + \frac{\sigma^2}{2}\alpha(\alpha-1) + r(\alpha-1) = 0}$$

Hence Proved

Now ~~Bou~~ Final condition

$$V(S,T) = V_T(S)$$

$$u(x,0)\cdot e^{\alpha x} = V(S,T) = V_T(S)$$

$$\Rightarrow \quad \boxed{u(x,0) = e^{-\alpha x} V_T(S)}$$

Hence B.S. PDE is converted to

1-D Heat-conduction PDE.

(2) Given BVP:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \ \tau > 0 \\ u(x,0) = u_0(x) \end{cases} \rightarrow *$$

consider solution of form:

$$u(x,\tau) = \tau^{-1/2} U(\eta) \text{ where } \eta = \frac{x-\xi}{\sqrt{\tau}} \text{ and } \xi \rightarrow \text{paramete}$$

Then

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau}\left(\tau^{-1/2} U(\eta)\right) = \frac{\partial}{\partial \tau}(\tau^{-1/2}) U(\eta) + \tau^{-1/2} \frac{\partial}{\partial \tau} U(\eta)$$

$$= -\frac{1}{2}\tau^{-3/2} U(\eta) + \tau^{-1/2} \frac{\partial u}{\partial \eta} \cdot \frac{d\eta}{d\tau}$$

$$= -\frac{1}{2}\tau^{-3/2} U(\eta) + \tau^{-1/2} \frac{du}{d\eta}(x-\xi)\frac{\tau^{-3/2}}{-2}$$

$$= -\frac{1}{2}\tau^{-3/2}\left(U + \eta \frac{du}{d\eta}\right)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial \tau} = -\frac{1}{2}\tau^{-3/2} \frac{d}{d\eta}\left(\eta U(\eta)\right)}$$

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$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\tau^{-1/2} U(\eta)\right] = \tau^{-1/2}\frac{dU}{d\eta}\cdot\frac{d\eta}{dx}$$

$$\boxed{\frac{\partial u}{\partial x} = \tau^{-1}\frac{dU}{d\eta}}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\tau^{-1}\frac{dU}{d\eta}\right] = \tau^{-1}\frac{d^2 U}{d\eta^2}\cdot\frac{d\eta}{dx}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = \tau^{-3/2}\frac{d^2 U}{d\eta^2}}$$

Putting the values in $*$, we get

$$-\frac{\tau^{-3/2}}{2}\frac{d}{d\eta}\left[\eta U(\eta)\right] = \tau^{-3/2}\frac{d^2 U}{d\eta^2}$$

$$\Rightarrow \boxed{\frac{d^2 U}{d\eta^2} + \frac{1}{2}\frac{d}{d\eta}\left[\eta U\right] = 0}$$

Integrating this ~~so~~ ODE we get

$$\boxed{\frac{dU}{d\eta} + \frac{\eta}{2}U = C_1} \rightarrow \text{const of Integration}$$

Choose $C_1 = 0$

Then we have linear Homogenous Eqn

$$\frac{dU}{d\eta} + \frac{\eta}{2} U = 0$$

$$\frac{dU}{U} = -\frac{\eta}{2} d\eta \Rightarrow \ln\left(\frac{U}{C}\right) = -\eta^2/4$$

$$\Rightarrow \boxed{U = c e^{-\eta^2/4}} \rightarrow c \text{ is const.}$$

Hence for Heat Eqn we have soln of form.

$$\boxed{u(x, \tau) = c\, \tau^{-1/2} e^{-(x-\xi)^2/4\tau}}$$

Now we need

$$\int_{-\infty}^{\infty} c\, \tau^{-1/2} e^{-(x-\xi)^2/4\tau} d\xi = 1$$

Putting $\eta = \dfrac{x-\xi}{\sqrt{2}\sqrt{\tau}}$ $\Rightarrow \boxed{\xi = x - (2\tau)^{1/2} \eta}$

$$d\xi = (2\tau)^{1/2} d\eta \Rightarrow \tau^{-1/2} d\xi = \sqrt{2}\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} c\sqrt{2}\, e^{-\eta^2/2}\, d\eta = 1$$

$$\Rightarrow c\sqrt{2}\cdot\sqrt{2\pi}\ \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\eta^2/2}\, d\eta}_{=1 \text{ by standard } N(0,1) \text{ CDF}} = 1$$

$$\Rightarrow \boxed{c = \frac{1}{2\sqrt{\pi}}}$$

then we get special solution as.

$$\boxed{u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\, e^{-(x-\xi)^2/4\tau}} \rightarrow \text{fundamental sol}^n \text{ OR Green function for heat Eq}^n$$

let $g(\xi; x, \tau)$ be class of green func. with ξ parameter

Then

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} \quad \text{for any } \xi$$

thus for $u_0(\xi)$, we have

$$\int_{-\infty}^{\infty} u_0(\xi)\cdot \frac{\partial g}{\partial \tau}\, d\xi = \int_{-\infty}^{\infty} u_0(\xi)\, \frac{\partial^2 g}{\partial x^2}\, d\xi$$

$$\Rightarrow \frac{\partial}{\partial \bar{\tau}}\left[\int_{-\infty}^{\infty} u_0(\xi) g \, d\xi\right] = \frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{\infty} u_0(\xi) g \, d\xi\right]$$

$$\Rightarrow \boxed{u(x,\bar{\tau}) = \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} \, d\xi}$$

is also solⁿ of $*$ with I.C. $u_0(x)$

Now

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = \begin{cases} 0 & , x-\xi \neq 0 \\ \infty & , x-\xi = 0 \end{cases}$$

and also

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1 \quad \text{is true } \forall \bar{\tau}$$

we have.

$$\boxed{\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)/4\bar{\tau}} = \delta(x-\xi)}$$

Dirac-Delta function

$$\Downarrow$$

$$\boxed{\lim_{\bar{\tau}\to 0} \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = u_0(x)}$$

I.C.

Now for European call option

using $V(S,t) = e^{-r(T-t)} u(x, \bar{\tau})$.

we have for B·S· PDE

$$V(S,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi) \cdot \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} \, d\xi .$$

$$\boxed{V(S,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} V_T(e^{\xi}) \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(\xi-x)^2/4\bar{\tau}} \, d\xi}$$

$$V(S,t) = e^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \times \int_{0}^{\infty} V_T(S') e^{-\left\{\ln S' - \left[\ln S + \left(r - D_0 - \frac{\sigma^2}{2}\right)(T-t)\right]\right\}^2 / 2\sigma^2(T-t)} \cdot \frac{dS'}{S'}$$

where
$$\boxed{G(S',T;S,t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\left\{\ln S' - \left[\ln S + \left(r - D_0 - \frac{\sigma^2}{2}\right)(T-t)\right]\right\}^2 / 2\sigma^2(T-t)}}$$

is Green's function for B·S·PDE.

for call option $\boxed{V_T(S') = \max\{S' - K, 0\}}$