MA668: Algorithmic and High Frequency Trading
Lecture 35

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

The Resulting DPE (Contd ...)

1. This form for the optimal depth has an interesting interpretation.
2. Consider the first term $\frac{1}{\kappa}$.
3. It stems from optimizing the instantaneous expected profits from selling one share.
4. The profit is given by the revenue of selling one share at $(S + \delta)$, minus the cost S, which results in δ.
5. Hence, the expected profit is $\delta P(\delta)$ and when the fill probability is $P(\delta) = e^{-\kappa\delta}$, the maximum is attained (as seen earlier) at $\delta^\dagger = \frac{1}{\kappa}$.
6. The difference term $h(t, q) - h(t, q - 1)$ can be viewed as the agent's correction to this static optimization taking into account her/his future optimal behaviour.
7. In particular, it represents a reservation price, which is defined as the price p such that $H(t, x + p, S, q - 1) = H(t, x, S, q)$, that is, it is the additional wealth the agent demands for selling the asset such that her/his value function remains unchanged.

The Resulting DPE (Contd ...)

1. We expect that $\delta^*(t, q)$ is decreasing in q, since the more inventory the agent has, the more urgent she/he should be in getting rid of her/his holdings and hence the closer to the mid-price she/he should post.
2. It may be that for large enough q the optimal depth becomes negative and the solution to the control problem is no longer financially meaningful.
3. We should instead solve the constrained problem, with $\delta^* \geq 0$ being enforced in the set of admissible strategies.
4. One naive approach, which avoids solving the constrained problem, is to view negative depths as an indicator that the agent should execute a market order instead of positing a limit order.
5. The sound approach to addressing the optimal posting of LOs versus MOs is deliberated upon later in the discussion on this topic.

Solving the DPE

1. It can be shown that $h(t,q) = \frac{1}{\kappa} \log \omega(t,q)$ where:

$$\omega(t,q) = \sum_{n=0}^{q} \frac{\widetilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-n)^2} (T-t)^n,$$

with $\widetilde{\lambda} = \lambda e^{-1}$.

2. This solution provides the function $h(t,q)$, which upon substitution gives the optimal depth:

$$\delta^*(t,q) = \frac{1}{\kappa} \left[1 + \log \frac{\sum_{n=0}^{q} \frac{\widetilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-n)^2} (T-t)^n}{\sum_{n=0}^{q-1} \frac{\widetilde{\lambda}^n}{n!} e^{-\kappa\alpha(q-1-n)^2} (T-t)^n} \right], \tag{1}$$

for $q > 0$.

Solving the DPE (Contd ...)

1. The optimal depth at which to post is:
   - A. A decreasing function of time for any model parameter.
   - B. A decreasing function of the agent's inventory q.
   - C. Increases with the rate of arrival of MOs
2. The increasing behaviour in activity rate is intuitive, since as market order arrival rates increase, the agent is willing to post deeper in the book so that her/his effective rate of filled LOs remains essentially constant, while reaping more profits if a matching arrives.
3. Figure 8.2: The optimal depths are shown as a function of time for several inventory levels as well as penalty parameter α.
4. MOs arrive at the rate of 50/min and the agent is attempting to liquidate $\mathfrak{R} = 5$ shares, and is hence 10% of the average market volume.
5. These plots show several interesting features of the optimal depths, as described.

Figure 8.2

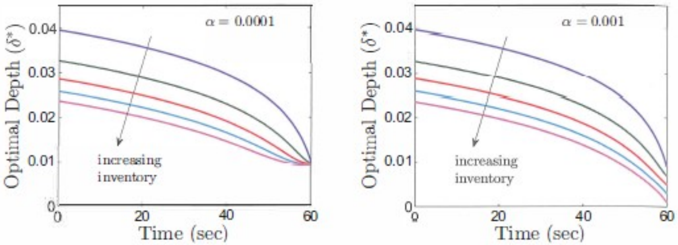

Figure 8.2 The optimal depths δ^* at which the agent posts LOs as a function of time and current inventory. The parameters are $\lambda = 50/\min$, $\kappa = 100$, and $\mathfrak{N} = 5$ with the penalty α shown in each panel. The lowest depth corresponds to $q = 5$ and the highest depth to $q = 0$.

Figure: Figure 8.2

Figure 8.2 (Contd ...)

1. The depths are decreasing in inventory. This is natural, as if the agent's inventory is large, she/he is willing to accept a lower premium δ, for providing liquidity, to increase the probability that her/his order is filled.
2. At the same time, this ensures that she/he may complete the liquidation of the $\mathfrak{R}$ shares by end of the time horizon and avoid crossing the spread (that is, using MOs) and paying a terminal penalty.
3. However, if inventories are low, the agent is willing to hold on to it in exchange for large δ, because with low inventory the terminal penalty she/he picks up when crossing the spread will be moderate.
4. For fixed inventory level, the depths all decrease in time. Once again, this is due to the agent becoming more averse to holding inventories as the terminal time approaches, due to the penalty they will receive from crossing the spread.

Figure 8.2 (Contd ...)

1. As the penalty parameter α increases, all depths decrease because increasing the penalty induces the trader to liquidate her/his position faster, but at lower prices.
2. We point out that if α or q is large, then the optimal depths can become negative.
3. In practice one cannot post LOs which improve the best quote on the other side of the LOB, so one may want to interpret this as the agent being very keen to get her/his LO filled, but here we do not allow the agent to submit MOs.
4. The depths keep increasing as one moves further from the end of the trading horizon.
5. The reason is that the agent is only being penalized by her/his terminal inventory, so far from terminal time, that there is no incentive to liquidate her/his position.
6. If the agent instead penalizes inventories through time, the strategies will become asymptotically constant far from maturity.

Figure 8.2 (Contd ...)

1. Far from the terminal time, that is, when $\tau = T - t \gg 1$, the ratio appearing in the logarithm becomes:

$$\frac{\omega(t,q)}{\omega(t,q-1)} = e^{-\kappa\alpha} + \frac{\widetilde{\lambda}}{q}\tau + O(\tau^{-1}).$$

2. Therefore, far from the terminal time, the agent posts at depths that grow logarithmically as follows:

$$\delta^*(t,q) = \frac{1}{\kappa}\left[1 + \log\left(e^{-\kappa\alpha} + \frac{\lambda e^{-1}}{q}\tau\right) + o(\tau^{-1})\right].$$

3. In this expression, the dependence of the optimal depth on the parameters becomes clear:
`It is increasing in activity rate and decreasing in inventory, time, fill probability and terminal penalty.`

Liquidation With Limit and Market Orders

1. In the preceding discussion: The agent considers posting only LOs and as shown, posts more aggressively (that is, depth δ decreases so LOs are posted nearer the mid-price) as maturity approaches when her/his inventory is held fixed.
2. Now: We consider the situation in which the agent is allowed to post MOs in addition to LOs.
3. In this case, when she/he is far behind schedule, that is, when maturity is approaching but she/he still has many shares to liquidate, then she/he could be willing to execute an MO in order to place her/his strategy back on target.
4. In this case, the agent searches for both an optimal control and a sequence of optimal stopping times at which to execute MOs.
5. To formalize the problem, we now need to keep track of the agent's posted MOs, in addition to other traders' MOs, and her/his executed LOs.

The Agent's Optimization Problem

1. Below we list the additional stochastic processes and changes to the cash process to account for executing MOs.
2. All other stochastic processes, including the mid-price S, other trader's MO's M, and the agent's filled LOs N, posted at depth δ, remain unaltered in their definition.
   - A. $M^a = (M_t^a)_{\{0 \le t \le T\}}$ denotes the counting process for the agent's MOs.
   - B. The corresponding increasing sequence of stopping times at which the agent executes MOs is denoted by $\tau = \{\tau_k : k = 1, \ldots, K\}$, with $K \le \mathfrak{N}$, so that $M_t^a = \sum_{k=1}^{K} \mathbb{1}_{\{\tau_k \le t\}}$. Note that the agent may place fewer, but never more, than $\mathfrak{N}$ number of MOs.
   - C. ξ denotes the half-spread, that is, half-way distance between the best ask and best bid.
   - D. $X = (X_t)_{\{0 \le t \le T\}}$ denotes the agent's cash process and satisfies the SDE:
$$dX_t^{\tau,\delta} = (S_t + \delta_{t-})dN_t^{\delta} + (S_t - \xi_t)dM_t^{a,\tau}.$$

The Agent's Optimization Problem (Contd ...)

1. Agent's cash process:
   - A. The first term on the right-hand side of the cash process denotes the cash received from having an LO lifted.
   - B. The second term is the cash received from selling a share using an MO.
   - C. Note that when the agent executes a sell MO she/he crosses the spread, which is why the proceeds from selling one unit of the asset is the mid-price minus the half-spread, that is, the best bid.
   - D. Furthermore, we assume that the size of the MOs is small enough not to walk the LOB.
2. We assume that the agent is averse to holding inventory throughout the strategy.
3. To achieve this, we apply an urgency penalty to her/his performance criteria.

The Agent's Optimization Problem (Contd ...)

1. The agents' performance criteria is:

$$H^{(\tau,\delta)}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T^{\tau,\delta} + Q_T^{\tau,\delta} S_T - l\left(Q_T^{\tau,\delta}\right) - \phi \int_t^T \left(Q_u^{\tau,\delta}\right)^2 du\right], \tag{2}$$

where as usual $\mathbb{E}_{t,x,S,q}[\cdot]$ denotes expectation conditional on $X_{t^-}^{\tau,\delta} = x$, $S_{t^-} = S$ and $Q_{t^-}^{\tau,\delta} = q$, and the terminal liquidation penalty $l(q) = q(\xi + \alpha q)$.

2. The terminal liquidating cost per share of the shares remaining at the end is written as $(S_T - \xi - \alpha Q_T)$ because the agent must cross the spread and then walk the LOB to liquidate the remaining shares.

3. Recall that we assumed that the MOs sent during the liquidation strategy before the terminal date did not walk the LOB.

The Agent's Optimization Problem (Contd ...)

1. Since the agent may execute MOs, her/his inventory is reduced each time an LO is filled or an MO is executed, so that:

$$Q_t^{\tau,\delta} = \mathfrak{R} - N_t - M_t^a.$$

2. The set of admissible strategies $\mathcal{A}$ now includes seeking over all $\mathcal{F}$-stopping times, in addition to the set of $\mathcal{F}$-predictable, bounded from below, depths δ.

3. Accordingly, the value function is:

$$H(t, x, S, q) = \sup_{(\tau,\delta)\in\mathcal{A}} H^{(\tau,\delta)}(t, x, S, q).$$

The Resulting DPE

1. Now, the DPP implies that the value function should satisfy the quasi-variational inequality (QVI), rather than the usual non-linear PDE:

$$
\begin{aligned}
0 \;=\; & \max\Big[\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\
& + \sup_{\delta} \lambda e^{-\kappa\delta} \left[H(t, x + (S+\delta), S, q-1) - H(t, x, S, q)\right], \\
& \left[H(t, x + (S-\xi), S, q-1) - H(t, x, S, q)\right]\Big],
\end{aligned}
$$

   with boundary and terminal conditions $H(t, x, S, 0) = x$ and $H(T, x, s, q) = x + qS - l(q)$, respectively.

2. Note that the first part of the maximization above is identical to the previous discussion where we have limit orders only. The various terms in the QVI may be interpreted as described in the following presentation.

3. The overall max operator represents the agent's choice to either post an LO (the continuation region) resulting in the first term in the max operator, or to execute an MO (the stopping region) resulting in a value function change of $[H(t, x + (S-\xi), S, q-1) - H(t, x, S, q)]$: The agent's cash increases by $(S-\xi)$ and inventory decreases by 1 upon executing an MO.

The Resulting DPE (Contd ...)

1. Within the continuation region where the agent posts LOs (the first term in the max):
   - A. The operator ∂_{SS} corresponds to the generator of the Brownian motion which drives mid-price.
   - B. The term $-\phi q^2$ corresponds to the contribution of the running inventory penalty.
   - C. The supremum over δ takes into account the agent's ability to control the posted depth.
   - D. The $\lambda e^{-\kappa\delta}$ coefficient represents the arrival rate of MOs which fill the agents posted LO at the price $S + \delta$.
   - E. The difference term $[H(t, x + (S + \delta), S, q - 1) - H(t, x, S, q)]$ represents the change in the value function when an MO fills the agent's LO: The agent's cash increases by $S + \delta$ and her/his inventory decreases by 1.

The Resulting DPE (Contd ...)

1. As before, the terminal and boundary conditions suggest the ansatz for the value function: $H(t, x, S, q) = x + qS + h(t, q)$.
2. Making this substitution, we find that h(t, q) satisfies the much simplified QVI:

$$\begin{aligned} 0 \quad = \quad & \max \left[\partial_t h - \phi q^2 + \sup_{\delta} \lambda e^{-\kappa\delta} \left[\delta + h(t, q-1) - h(t, q) \right], \right. \\ & \left. -\xi + h(t, q-1) - h(t, q) \right], \end{aligned} \tag{3}$$

with terminal condition,

$$h(T, q) = -l(q), \;\; q = 1, \ldots, \mathfrak{R}, \tag{4}$$

and boundary condition,

$$h(t, 0) = 0. \tag{5}$$

3. Focusing on the supremum term, the optimal posting in feedback control form is:

$$\delta^* = \frac{1}{\kappa} + \left[h(t, q) - h(t, q-1) \right]. \tag{6}$$

The Resulting DPE (Contd ...)

1. In this feedback control form, the optimal posting is identical to the one without MOs, but the precise function $h(t, q)$ which enters into its computation is different.
2. The first term $\frac{1}{\kappa}$ has the same interpretation as before: It is the optimal depth to post to maximize the expected instantaneous profit from a round-trip liquidated at mid-price, that is, the δ that maximizes $\delta P(\delta)$ (and recall that $P(\delta)$ is the probability of the LO being filled conditional on an MO arriving).
3. The difference term is the correction to this static optimization to account for the agent's ability to optimally trade.

The Resulting DPE (Contd ...)

1. The timing of MO executions also have a simple feedback form.
2. From (3), we see that an MO will be executed at time τ_q whenever

$$h(\tau_q, q-1) - h(\tau_q, q) = \xi. \tag{7}$$

3. This can be interpreted as executing an MO whenever doing so increases the value function by the half-spread.
4. Combining this observation with the feedback form for the optimal depth above, we can place a simple lower bound on δ^* of

$$\delta^* \geq \frac{1}{\kappa} - \xi.$$

5. Thus, it is clear that if we require $\delta > 0$, so that the strategy never posts sell LOs below the mid price, we must require that $\xi < \dfrac{1}{\kappa}$.