MA668: Algorithmic and High Frequency Trading
Lecture 30

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Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty

1. So far: We solved the general case when the agent's trades have temporary impact on the execution price and permanent impact on the mid price.
2. We assumed that these two impacts were linear in the speed of trading, $f(\nu) = k\nu$ and $g(\nu) = b\nu$ for constants $k \geq 0$ and $b \geq 0$.
3. One typically observes that $b \ll k$ and we also assume that the liquidation penalty parameter $\alpha \gg k$.
4. **Next:** We discuss the relationship between the liquidation penalty parameter α and the permanent price impact parameter b.
5. The discussion for acquisition problems is very similar.
6. Observation: In the optimal speed of trading, the permanent impact and the liquidation penalty always appear in the form $\alpha - \dfrac{b}{2}$ (see definition of ζ).

Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty (Contd ...)

1. This implies that in the current model, where the permanent impact is linear in the speed of trading and the liquidation of terminal inventory is quadratic, αQ_T^2, one could define a single parameter $c := \alpha - \frac{1}{2}b$ (so that $c = \chi(T)$) to describe how both the permanent impact and the liquidation penalty affect the optimal speed of trading.
2. Obviously, we cannot do this for other variables in the model, such as for the cash obtained from liquidating shares.
3. The impact of the permanent price impact parameter on this variable is quite distinct from that of the liquidation penalty.

Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty (Contd ...)

1. To see this, we consider how the proceeds from selling the $\mathfrak{N}$ shares are affected by the permanent impact that the agent's trades have on the mid-price.
2. First, we calculate the agent's terminal cash when she/he follows an arbitrary strategy ν_t.
3. Recall that the agent's cash position satisfies the SDE:

$$dX_t^\nu = (S_t^\nu - k\nu_t)\,\nu_t dt,$$

where,

$$dS_t^\nu = -b\nu_t dt + \sigma dW_t.$$

4. For simplicity, we assume that $X_0 = 0$, $k = 0$ and $S_0 = 0$.

Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty (Contd ...)

1. Then, the revenue from liquidating her shares, including the liquidation of the terminal inventory, is:

$$
\begin{aligned}
R^{\nu} &= \int_0^T S_t^{\nu} \nu_t dt + Q_T^{\nu} \left(S_T^{\nu} - \alpha Q_T^{\nu} \right), \\
&= \int_0^T \left[-b \int_0^t \nu_u du + \sigma W_t \right] \nu_t dt + Q_T^{\nu} \left(S_T^{\nu} - \alpha Q_T^{\nu} \right), \\
&= \int_0^T \left[-b \left(\mathfrak{R} - Q_t^{\nu} \right) + \sigma W_t \right] \left(-dQ_t^{\nu} \right) + Q_T^{\nu} \left(S_T^{\nu} - \alpha Q_T^{\nu} \right), \\
&= -b \int_0^T \left(\mathfrak{R} - Q_t^{\nu} \right) d \left(\mathfrak{R} - Q_t^{\nu} \right) - \sigma \int_0^T W_t dQ_t^{\nu} + Q_T^{\nu} \left(S_T^{\nu} - \alpha Q_T^{\nu} \right).
\end{aligned}
$$

Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty (Contd ...)

1. Thus:

$$R^{\nu} = -\frac{b}{2}\left(\mathfrak{R} - Q_T^{\nu}\right)^2 + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right) - \sigma \int_0^T W_t dQ_t^{\nu}.$$

2. Having expressed R^{ν} in this way, we see that both α and b appear together with $(Q_T^{\nu})^2$ and both act to penalize inventories different from zero.

3. Nevertheless, if we isolate the terms in R^{ν} that are affected by α and b, we obtain:

$$R^{\nu} = -\frac{b}{2}\left(\mathfrak{R}^2 - 2\mathfrak{R} Q_T^{\nu}\right) - \left(\frac{b}{2} + \alpha\right)\left(Q_T^{\nu}\right)^2 + Q_T^{\nu} S_T^{\nu} - \sigma \int_0^T W_t dQ_t^{\nu}.$$

4. It is now clear that not only do α and b affect the revenue process in a very different way than they do the speed of trading, but also that the effect of the parameter of the permanent price impact cannot be absorbed into the liquidation penalty.

Equivalence Between Permanent Price Impact and Terminal Liquidation Penalty (Contd ...)

1. Indeed, b shows up explicitly in the value function separately from α.
2. First note that α and b do appear in $\chi(t)$ together in the form $c = \alpha - \frac{b}{2}$ (through ζ)
3. But, b appears separately through the relationship of $h_2(t) = \chi(t) - \frac{b}{2}$.
4. Since $\chi(t)$ is what determines the optimal trading strategy, we see that b can be absorbed into α for the purpose of the trading strategy.
5. But this effect does not extend to the revenue process.
6. We can see this most clearly when the agent follows the optimal strategy in the limiting case where $\alpha \to \infty$.
7. In this limiting case, the agent will complete the trade by the terminal date, hence $Q_T^{\nu^*} = 0$, and any terminal penalty would be applied to a terminal quantity equal to zero.
8. Nevertheless, the impact of the agent's trades on the mid-price will be strictly positive: A loss of $\frac{b}{2}\mathfrak{R}^2$.

Non-Linear Temporary Price Impact

1. Agent's performance criteria:

$$H^{\nu}(t,x,S,q)=\mathbb{E}_{t,x,S,q}\left[X_T^{\nu}+Q_T^{\nu}\left(S_T^{\nu}-\alpha Q_T^{\nu}\right)-\phi\int_t^T\left(Q_u^{\nu}\right)^2du\right].$$

2. Dynamics of S^{ν} , X^{ν} and Q^{ν}:

$$\begin{aligned} dS_t^{\nu} &= -b\nu_t dt+\sigma dW_t, \\ dX_t^{\nu} &= \left(S_t^{\nu}-f(\nu_t)\right)\nu_t dt, \\ dQ_t^{\nu} &= -\nu_t dt. \end{aligned}$$

3. Accordingly, the value function (based on DPP) is:

$$H(t,x,S,q)=\sup_{\nu\in\mathcal{A}}H^{\nu}(t,x,S,q).$$

Non-Linear Temporary Price Impact (Contd ...)

1. The DPP implies that the value function should satisfy the HJB equation:

$$\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 \\ &+ \sup_{\nu}\left[\left(\nu\left(S - f(\nu)\right)\partial_x - b\nu\partial_S - \nu\partial_q\right)H\right], \end{aligned}$$

   with the terminal condition being $H(T, x, S, q) = x + q(S - \alpha q)$.

2. Ansatz: $H(t, x, S, q) = x + qS + h(t, q)$.

3. Non-linear PDE for h:

$$0 = \partial_t h - \phi q^2 + \sup_{\nu}\left[-\nu f(\nu) - (bq + \partial_q h)\nu\right],$$

   with terminal condition: $h(T, q) = -\alpha q^2$.

4. Denoting $F(\nu) = \nu f(\nu)$, and assuming that $\nu f(\nu)$ is convex leads to the following non-linear PDE (which needs to be solved numerically):

$$\partial_t h - \phi q^2 + F^*(-(bq + \partial_q h)) = 0, \ \ h(t, q) = -\alpha q^2,$$

   where F^* is the Legendre transform of the function F.

Recap

1. Previously: We studied the problem of optimal execution for an agent who aims to liquidate/acquire a considerable proportion of the average daily volume (ADV) of shares.
2. There we saw how the agent trades off the impact on prices that her/his trades would have if she/he traded quickly "WITH" the uncertainty in prices she/he would receive/pay if she/he traded slowly.
3. We observed that the agent's optimal strategy is to trade quickly initially (ensuring that she/he receives a price close to the arrival price, but with a non-trivial impact).
4. Then slow down as time goes by (to reduce her/his overall impact, but increase price uncertainty).
5. Surprisingly, the optimal strategies we obtain are deterministic and in particular are independent of the mid-price process.
6. This is irrespective of the level of urgency required to complete her/his trade.

A Prelude

1. NEXT: We incorporate a number of other important aspects of the problem that the agent may wish to include in her/his optimization decision, and explore how her/his trading behaviour adjusts to account for them.
2. Specifically, we look at three distinct aspects of the optimal execution problem:

An Upper Price Limit

1. Section 7.2: We study the problem of an agent wishing to acquire a large position, who has an upper price limit on what she/he is willing to pay.
2. We find that the optimal strategy in this case is no longer independent of the mid-price, beyond the obvious change that the agent stops trading when the upper limit price is breached.

Informative Order Flow

1. Section 7.3: We study the problem of an agent wishing to liquidate a large position, taking into account that the order flow from other traders in the market also impacts the mid-price.
2. We show that the agent alters her/his strategy so that when the net effect of other market participants is to trade in her/his direction, she/he increases her/his trading speed.
3. Conversely, if the net effect of other agents is to trade in the opposite direction, she decreases her/his trading speed.

Dark Pools

1. Section 7.4: The agent has access to a (standard) lit market and also to a dark pool.
2. Trading in the dark pool exposes her/his to execution risk, but removes some of the price impact.
3. We find that the optimal strategy is still deterministic.
4. Initially the agent trades in the lit market at speeds below a specified level [Almgren and Chriss (2000)] and posts the whole of the remaining order in the dark pool, in the hope of it being filled there.