MA668: Algorithmic and High Frequency Trading
Lecture 04

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Grossman-Miller Market Making Model

1. When a MM provides liquidity by accepting one side of the trade (say buying from someone who wants to sell), the MM will hold the asset for an uncertain period of time, the time it takes for another person to come to the market with a matching demand (to buy the asset of MM acquired in the previous trade).
2. Intervening time: The MM is exposed to the risk that the price movement is adverse (price decline leading to loss, for the above example).
3. Recall: MM does not have the need or desire to hold onto the assets. That is: they buy/sell based on anticipation of subsequent sale/purchase.
4. Model of Grossman and Miller: Described how MM's obtain a liquidity premium that exactly compensates MM's for the price risk, resulting from holding the inventory.

Grossman-Miller Market Making Model (A Simplified Version)

1.
   A. For a finite number "n": Identical MM's for some given asset.
   B. Three dates: $t \in \{1, 2, 3\}$.
   C. To simplify: No uncertainty about arrival of matching orders.
2. If at date $t = 1$, a liquidity trader (LT1) comes to the market to sell i units of asset, there will be another liquidity trader (LT2) who will arrive at the market to purchase i units of asset.
   A. LT1 is said to trade $+i$ units.
   B. LT2 is said to trade $-i$ units.
   C. Note: LT2 does not arrive until time $t = 2$.
3.
   A. Let all agents start with an initial cash W_0.
   B. MM's hold no assets, LT1 holds "$+i$" units and LT2 holds "$-i$" units.
4. Assumption: No trading cost or direct cost for holding inventory.

Grossman-Miller Market Making Model (A Simplified Version) (Contd ..)

Focus on price change:

A. At $t = 3$, cash value of asset is:

$$S_3 = \mu + \epsilon_2 + \epsilon_3,$$

with μ constant and $\epsilon_2, \epsilon_3 \sim \mathcal{N}(0, \sigma^2)$.

B. Further ϵ_2 is announced between time $t = 1$ and $t = 2$, while ϵ_3 is announced between $t = 2$ and $t = 3$.

C. We ignore the fact that in some cases the above model could lead to negative asset prices, and instead focus on the illustrative part.

Grossman-Miller Market Making Model (A Simplified Version) (Contd ..)

1. Let us assume that all traders (MM's as well as liquidity traders) are risk averse.
2. Let us suppose that they have the following expected utility, for future random cash value of the asset, *i.e.,* $E[U(X_3)]$, where $U(X) = -e^{-\gamma X}$, with $\gamma > 0$ being the risk aversion parameter.
3. Problem: Solved backwards for a description of trading behaviour and prices.
4.
   A. At $t = 3$, the cash value of asset is $S_3 = \mu + \epsilon_2 + \epsilon_3$.
   B. At $t = 2$, the "n" MM's and LT1 come into the period with asset holdings of q_1^{MM} and q_1^{LT1}, respectively.
   C. LT2 comes in with $-i$ and they all exit with asset holding q_2^j, where $j \in \{MM, LT1, LT2\}$.

The Problem Statement

At $t = 2$, agent j chooses q_2^j, so as to maximize the expected utility, knowing the realization of ϵ_2:

$$\max_{q_2^j} E\left[U\left(X_3^j\right)\middle|\epsilon_2\right],$$

subject to the constraints:

- (A) $X_3^j = X_2^j + q_2^j S_3$: Cash value of the agent's assets at $t = 3$ is equal to agent's cash holdings at $t = 2$ plus cash value of agent's asset inventory.
- (B) $X_2^j + q_2^j S_2 = X_1^j + q_1^j S_2$: Cash value of agent's asset, when exiting $t = 2$ is equal to the cash value of agent's asset, when entering $t = 2$.

The Problem Statement (Contd ...)

Given the normality assumption and properties of expected utility [a], it can be shown that:

$$E\left[U\left(X_3^j\right)|\epsilon_2\right] = -\exp\left\{-\gamma\left(X_2^j + q_2^j E\left[S_3|\epsilon_2\right]\right) + \frac{1}{2}\gamma^2\left(q_2^j\right)^2\sigma^2\right\}.$$

[a] $E\left[e^{-\gamma\epsilon}\right] = e^{-\gamma\mu+\frac{\gamma^2}{2}\sigma^2}$, $\epsilon \sim \mathcal{N}(\mu, \sigma^2)$

The Problem Solution

The problem solution is characterized by:

$$q_2^{j,*} = \frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2},$$

for all agents, namely, "*n*" MM's, LT1 and LT2.

The Problem Solution (Contd ...)

1. At date $t = 2$, the demand and supply for the asset have to be equal to each other. Therefore we can solve for the equilibrium price S_2:

$$nq_1^{MM} + q_1^{LT1} + q_1^{LT2} = nq_2^{MM} + q_2^{LT1} + q_2^{LT2}. \tag{1}$$

2. We have used the convention that q_1^{LT2}, which are the assets that LT2 came in with at period 2, is equal to their desired trade given by "$-i$".

3. As seen, all q_2 are equal and therefore:

$$nq_2^{MM} + q_2^{LT1} + q_2^{LT2} = (n+2)\left[\frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2}\right]. \tag{2}$$