MA668: Algorithmic and High Frequency Trading
Lecture 38

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

A Prelude

1. Now, we model how a market maker (MM) maximizes terminal wealth by trading in and out of positions using limit orders (LOs).
2. The MM provides liquidity to the limit order book (LOB) by posting buy and sell LOs and the control variable is the depth, which is measured from the mid-price at which these LOs are posted.
3. To formalize the problem, we list the relevant variables that we will use in the subsequent discussion:
    - $S = (S_t)_{\{0 \le t \le T\}}$, denotes the mid-price, with $S_t = S_0 + \sigma W_t, \sigma > 0$ and $W = (W_t)_{\{0 \le t \le T\}}$ is a standard Brownian motion.
    - $\delta^{\pm} = (\delta_t^{\pm})_{\{0 \le t \le T\}}$ denote the depth at which the agent posts LOs.
        - Sell LOs are posted at a price of $S_t + \delta_t^+$.
        - Buy LOs are posted at a price of $S_t - \delta_t^-$.
    - $M = (M_t)_{\{0 \le t \le T\}}$ denote the counting processes corresponding to the arrival of other participants' buy $(+)$ and sell $(-)$ market orders (MOs) which arrive at Poisson times with intensities $\lambda^{\pm}$.
    - $N = (N_t)_{\{0 \le t \le T\}}$ denote the controlled counting processes for the agent's filled sell $(+)$ and buy $(-)$ LOs.

A Prelude (Contd ....)

1. Definition of variables (contd ...):
   - Conditional on a market order (MO) arrival, the posted LO is filled with probability $e^{-\kappa_\pm \delta_t^\pm}$, with $\kappa_\pm \geq 0$.
   - $X^\delta = (X_t^\delta)_{\{0 \leq t \leq T\}}$ denotes the MM's cash process and satisfies the SDE:
   $$dX_t^\delta = (S_{t-} + \delta_t^+) dN_t^{\delta,+} - (S_{t-} - \delta_t^-) dN_t^{\delta,-}, \tag{1}$$
   which accounts for the cash inflow/increase when a sell LO is lifted by a buy MO and the cash outflow/decrease when a buy LO is hit by an incoming sell MO, respectively.
   - $Q^\delta = (Q_t^\delta)_{\{0 \leq t \leq T\}}$ denotes the agent's inventory process and:
   $$Q_t^\delta = N_t^{\delta,-} - N_t^{\delta,+}. \tag{2}$$
2. Whenever the process $N^{\delta,\pm}$ jumps, the process $M^{\delta,\pm}$ must also jump, but when $M^{\delta,\pm}$ jumps, $N^{\delta,\pm}$ will jump only if the MO is large enough to fill the agent's LO and $N^{\delta,\pm}$ is not a Poisson process.
3. Moreover, note that the fill rate of LOs can be written as $\Lambda_t^{\delta,\pm} = \lambda^\pm e^{-\kappa_\pm \delta_t^\pm}$, which is the rate of execution of an LO.

Market Making

1. To simplify notation, in the rest of this discussion we suppress the superscript δ in the counting process for filled LOs, cash, and inventory.
2. We assume that the MM seeks the strategy $(\delta_s^{\pm})_{\{0\leq s\leq T\}}$ that maximizes cash at the terminal date T.
3. We also assume that at time T the MM liquidates her/his terminal inventory Q_T using a MO at a price which is worse than the mid-price to account for liquidity taking fees as well as the MO walking the LOB.
4. Finally, the MM caps her/his inventory so that it is bounded above by $\bar{q} > 0$ and below by $\underline{q} < 0$, both finite, and also includes a running inventory penalty so that the performance criterion is:

$$H^{\delta}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T + Q_T^{\delta}(S_T^{\delta} - \alpha Q_T^{\delta}) - \phi\int_t^T (Q_u)^2 du\right],$$

where $\alpha \geq 0$ represents the fees for taking liquidity (that is, using an MO) as well as the impact of the MO walking the LOB. Further, $\phi \geq 0$ is the running inventory penalty parameter.

Market Making (Contd ...)

1. The MM's value function is:

$$H(t, x, S, q) = \sup_{\delta^\pm \in \mathcal{A}} H^\delta(t, x, S, q), \tag{3}$$

where $\mathcal{A}$ denotes the set of admissible strategies, that is, $\mathcal{F}$-predictable and bounded from below.

2. To solve the optimal control problem, a dynamic programming principle holds and the value function satisfies the DPE:

$$\begin{aligned} 0 &= \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2, \\ &+ \lambda^+ \sup_{\delta^+} \left[e^{-\kappa^+ \delta^+} \Big(H(t, x + (S + \delta^+), q - 1, S) - H(t, x, q, S) \Big) \right] \mathbb{1}_{\{q > \underline{q}\}}, \\ &+ \lambda^- \sup_{\delta^-} \left[e^{-\kappa^- \delta^-} \Big(H(t, x - (S - \delta^-), q - 1, S) - H(t, x, q, S) \Big) \right] \mathbb{1}_{\{q < \bar{q}\}}, \end{aligned} \tag{4}$$

where $\mathbb{1}$ is the indicator function, with the terminal condition:

$$H(T, x, S, q) = x + q(S - \alpha q). \tag{5}$$

Market Making (Contd ...)

1. Recall that the set of admissible strategies imposes bounds on q_t, so that when $q_t = \bar{q}$ ($\underline{q}$), the optimal strategy is to post one-sided LOs which are obtained by solving (4) with the term proportional to $\lambda^-(\lambda^+)$ absent as enforced by the indicator function $\mathbb{1}$ in the DPE.
2. Alternatively, one can view these boundary cases as imposing $\delta^- = +\infty$ and $\delta^+ = +\infty$ when $q = \underline{q}$ and $q = \bar{q}$, respectively.
3. Intuitively, the various terms in the DPE equation represent the arrival of MOs that may be filled by LOs together with the diffusion of the asset price through the term $\frac{1}{2}\sigma^2\partial_{SS}H$ and the effect of penalizing deviations of inventories from zero along the entire path of the strategy which is captured by the term ϕq^2.
4. In the second line of the DPE the sup over δ^+ contain the terms due to the arrival of a market buy order (which is filled by a limit sell order) and here we see the change in the value function H due to the arrival of the MO which fills the LO, so that cash increases by $(S + \delta^+)$ and the inventory decreases by one unit.
5. Similarly, in the last line in the DPE, the sup over δ^- contain the analogous terms for the market sell orders which are filled by limit buy orders.

Market Making (Contd ...)

1. To solve the DPE we use the terminal condition (5) to make an ansatz for H.
2. In particular, we write:

$$H(t, x, q, S) = x + qS + h(t, q), \tag{6}$$

   which has a simple interpretation.

3. The first term is the accumulated cash, the second term is the book value of the inventory marked-to-market (that is, the value of the shares at the current mid-price) and the last term is the added value from following an optimal market making strategy up to the terminal date.
4. We proceed by substituting the ansatz into (4) to obtain:

$$\begin{aligned} \phi q^2 &= \partial_t h(t, q) \\ &+ \lambda^+ \sup_{\delta^+} \left[e^{-\kappa^+ \delta^+} \left(\delta^+ + h(t, q-1) - h(t, q) \right) \right] \mathbb{1}_{\{q > \underline{q}\}} \\ &+ \lambda^- \sup_{\delta^-} \left[e^{-\kappa^- \delta^-} \left(\delta^- + h(t, q+1) - h(t, q) \right) \right] \mathbb{1}_{\{q < \bar{q}\}}, \end{aligned} \tag{7}$$

   with terminal condition $h(T, q) = -\alpha q^2$.

Market Making (Contd ...)

1. Then the optimal depths in feedback form are given by:

$$\delta^{+,*}(t,q) = \frac{1}{\kappa^+} - h(t,q-1) + h(t,q), \;\; q \neq \underline{q}, \qquad (8)$$

$$\delta^{-,*}(t,q) = \frac{1}{\kappa^-} - h(t,q+1) + h(t,q), \;\; q \neq \bar{q}, \qquad (9)$$

   and the boundary cases are $\delta^{+,*}(t,q) = +\infty$ and $\delta^{-,*}(t,q) = +\infty$ when $q = \bar{q}$ and $q = \underline{q}$, respectively.

2. To understand the intuition behind the feedback controls we first note that the optimal $\delta^{\pm}$ can be decomposed into two components.

Market Making (Contd ...)

1. Component 1: The first component, $\dfrac{1}{\kappa^{\pm}}$, is the optimal strategy for a MM who does not impose any restrictions on inventory ($\alpha = \phi = 0$ and $|\underline{q}| = \bar{q} = \infty$ (discussed later).
2. Component 2: The second component, the term $-h(t, q-1) + h(t, q)$, controls for inventories through time. As expected, if inventories are long, then the strategy consists in posting LOs that increase the probability of limit sell orders being hit. Moreover, the function $h(t, q)$ also induces mean reversion to an optimal inventory level, as a result of penalizing accumulated inventories throughout the entire trading horizon and the strategy approaching T as well as the other parameters of the model, including ϕ.
3. Substituting the optimal controls into the DPE, we obtain:

$$
\begin{aligned}
\phi q^2 \quad &= \quad \partial_t h(t,q) \\
&+ \quad \frac{e^{-1}\lambda^+}{\kappa^+} e^{-\kappa^+} \left(-h(t,q-1) + h(t,q)\right) \mathbb{1}_{\{q > \underline{q}\}} \\
&+ \quad \frac{e^{-1}\lambda^-}{\kappa^-} e^{-\kappa^-} \left(-h(t,q+1) + h(t,q)\right) \mathbb{1}_{\{q < \bar{q}\}} . \qquad (10)
\end{aligned}
$$

Solving the DPE

1. It is possible to find an analytical solution to the DPE if the fill probabilities of LOs are the same on both sides of the LOB.
2. In this case, if $\kappa = \kappa^+ = \kappa^-$, then we write:

$$h(t,q) = \frac{1}{\kappa} \log \omega(t,q),$$

and stack $\omega(t,q)$ into a vector:

$$\omega(t) = \left[\omega(t,\bar{q}), \omega(t,\bar{q}-1), \ldots, \omega(t,\underline{q})\right]^\top .$$

3. Now, let $\mathbf{A}$ denote the $(\bar{q} - \underline{q} + 1)$ square matrix whose rows are labeled from $\bar{q}$ to $\underline{q}$ and whose entries are given by:

$$\mathbf{A}_{i,q} = \begin{cases} -\phi\kappa q^2, \ i = q, \\ \lambda^+ e^{-1}, \ i = q-1, \\ \lambda^- e^{-1}, \ i = q+1, \\ 0, \ \text{otherwise}, \end{cases} \tag{11}$$

with terminal and boundary conditions $\omega(T,q) = e^{-\alpha\kappa q^2}$

Solving the DPE (Contd ...)

1. Now (10) becomes:
$$\partial_t \omega(t) + \mathbf{A}\omega(t) = \mathbf{0}.$$
2. The solution of this matrix ODE is straightforward and we finally have:
$$\omega(t) = e^{\mathbf{A}(T-t)}\mathbf{z}, \tag{12}$$
where $\mathbf{z}$ is a $(\bar{q} - \underline{q} + 1)$ dim vector, with each component being $z_j = e^{-\alpha\kappa j^2}$, $j = \bar{q}, \bar{q} - 1, \ldots, \underline{q}$.
3. The behavior of the strategy is elaborated upon in the following discussion.