MA423 Matrix Computations

Lecture 8: System of Linear Equations-II

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Outline

- Gaussian elimination with pivoting
- Permutated LU decomposition

Pivoting

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However, interchanging the rows we obtain an upper triangular matrix

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Theorem (GEPP): Let A be an $n \times n$ matrix. Then there is a permutation matrix P such that

$$PA = LU$$

where L is unit lower triangular and U is upper triangular.

Gaussian elimination with partial pivoting

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$$L_1^{-1} P_1 A = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{array}\right], \text{ where } a_{ij}^{(1)} = a_{ij} - \ell_{i1} a_{1j}. \textbf{ Cost: } 2(n-1)^2 \text{ flops.}$$

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If $a_{22}^{(1)} = 0$ then elimination breaks down.

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$$P_2L_1^{-1}P_1A = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \end{array}\right].$$

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Here P_2 is the permutation matrix that interchanges second row with n-th row of $L_1^{-1}A$.

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Repeating the process we have $L_{n-1}^{-1} P_{n-1} L_{n-2}^{-1} \cdots P_2 L_1^{-1} P_1 A = U$.

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$$PA = \underbrace{\hat{L}_1 \hat{L}_2 \cdots \hat{L}_{n-2} L_{n-1}}_{L} U = LU,$$

where L is unit lower triangular, $\hat{L}_j$'s are obtained from L_j's by permutating their multipliers and $P := P_{n-1} P_{n-2} \cdots P_2 P_1$.

Gaussian Elimination with Partial Pivoting (GEPP):

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function [L, U, p] = GEPP(A);
% [L U, p] = GEPP(A) produces a unit
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[n, n] = size(A);
p = (1:n)';
for k = 1:n-1
    % find largest element in A(k:n,k)
    [r, m] = max( abs( A(k:n,k) ) );
    m = m+k-1;
    if (m ~=k) % swap rows
        A([k m],:)  = A([m k],:);
        p([k m]) = p([m k]);
    end
```

GEPP (cont.)

```
    if (A(k,k) ~= 0)
        % compute multipliers for k-th step
        A(k+1:n,k) = A(k+1:n,k)/A(k,k);
        % update A(k+1:n,k+1:n)
        j = k+1:n;
        A(j,j) = A(j,j)-A(j,k)*A(k,j);
    end
end
% strict lower triangle of A, plus I
L = eye(n,n)+ tril(A,-1);
U = triu(A); % upper triangle of A
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The search for the largest entry in each column guarantees that the denominator `A(k,k)` in the entries `L(k+1:n,k) = A(k+1:n,k)/A(k,k)` is at least as large as the numerators.

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This ensures that $|L(i,j)| \leq 1$ for all i, j. This is crucial for stability.

Example

Consider

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} = \begin{bmatrix} 4 & 9 & -3 \\ 2 & 4 & -2 \\ -2 & -3 & 7 \end{bmatrix}.$$

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Then $L_1 = I + \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} e_1^\top, \quad L_1^{-1}A = \begin{bmatrix} 4 & 9 & -3 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & \frac{11}{2} \end{bmatrix}.$

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Then $L_2 = I + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{bmatrix} e_2^\top, \quad L_2^{-1}P_2L_1^{-1}P_1A = \begin{bmatrix} 4 & 9 & -3 \\ 0 & \frac{3}{2} & \frac{11}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$

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Thus we have $L_2^{-1}P_2L_1^{-1}P_1A = U \Longrightarrow A = MU$, where $M := (L_2^{-1}P_2L_1^{-1}P_1)^{-1} = P_1L_1P_2L_2$. Now

$$\begin{aligned}
P_1L_1 &= P_1\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \\
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where $L := P_2L_1P_2L_2$ is unit lower triangular.

Example (cont.)

Thus we have $L_2^{-1}P_2L_1^{-1}P_1A = U \Longrightarrow A = MU$, where $M := (L_2^{-1}P_2L_1^{-1}P_1)^{-1} = P_1L_1P_2L_2$. Now

$$P_1L_1 = P_1\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$P_2L_2 = P_2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

shows that M is not lower triangular. Next, observe that

$$U = L_2^{-1}P_2L_1^{-1}P_1A = L_2^{-1}P_2L_1^{-1}P_2P_2P_1A \Longrightarrow P_2P_1A = P_2L_1P_2L_2U = LU$$

where $L := P_2L_1P_2L_2$ is unit lower triangular. Indeed

$$P_2L_1P_2L_2 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix}.$$

Permutated LU decomposition (PA = LU)

By GEPP we have $L_{n-1}^{-1}P_{n-1}\cdots L_2^{-1}P_2L_1^{-1}P_1A = U$, where P_k is the permutation matrix and $L_k := I + \ell_k e_k^\top$ is the elimination matrix at the k-th step.

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Theorem: Set $L(\ell_k) := L_k$. Then $P_m L(\ell_k) = L(P_m \ell_k)P_m$ for $m > k$. Hence

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Set $\widehat{L}_k := L(P_{n-1}\cdots P_{k+1}\ell_k)$. Then $\widehat{L}_k$ is unit lower triangular and

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Thus, setting $P := P_{n-1}P_{n-2}\cdots P_1$ and $L := \widehat{L}_1\cdots\widehat{L}_{n-2}L_{n-1}$, we have $PA = LU$.

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Proof: Note that the first $m-1$ rows of P_m (P_m is used at the m-th step of elimination) are the same as the first $m-1$ rows of I_n. Hence $e_k^\top P_m = e_k^\top$ for $k = 1 : m-1$.

Permutated LU decomposition ($PA = LU$)

Since $e_k^\top P_m = e_k^\top$ for $m > k$, we have
$P_m L(\ell_k) = P_m(I + \ell_k e_k^\top) = P_m + P_m \ell_k e_k^\top = P_m + P_m \ell_k e_k^\top P_m = L(P_m \ell_k) P_m.$

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Continuing this process, we have

$$L_{n-1}^{-1} P_{n-1} \cdots L_2^{-1} P_2 L_1^{-1} P_1 A = L_{n-1}^{-1} \widehat{L}_{n-2}^{-1} \cdots \widehat{L}_2^{-1} \widehat{L}_1^{-1} P_{n-1} \cdots P_1 A.$$

Hence the results follow. ∎

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GEPP is the standard method used in practice for solving a linear system. GEPP is a default method in MATLAB for solution of a linear system. The command `x = A\b` solves $Ax = b$ using GEPP. The command `[L, U, P] = lu(A)` computes $PA = LU$.

Gaussian elimination with complete pivoting

Gaussian elimination with complete pivoting (GECP) is a variant of GE. At the k-step, GECP searches not just column `A(k:n,k)` but the entire submatrix `A(k:n,k:n)` for the largest entry and then swaps rows and columns to put that entry into `A(k,k)`.

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GECP guarantees $|L(i,j)| <= 1$ and $|U(i,j)| <= |U(i,i)|$.

Cost: $2n^3/3 + n^3/3 = n^3$ flops. Additional $n^3/3$ flops is due to finding maximum element at each step.

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- We still do not fully understand why GEPP and GECP work so well in the presence of roundoff.

***