MA423 Matrix Computations

Lectures 6 & 7: System of Linear Equations-I

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Outline

- Solution of triangular system
- Gaussian elimination
- LU decomposition

Linear system

Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and $b \in \mathbb{R}^n$.

Problem: Solve $Ax = b$ for $x \in \mathbb{R}^n$.

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The MATLAB command

```
>> x= A\b
```

solves the system $Ax = b$ using Gaussian elimination.

Lower triangular linear system

Consider the lower triangular linear system of equations

$$\begin{bmatrix} \ell_{11} & & & \\ \ell_{21} & \ell_{22} & & \\ \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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By forward substitution, we have

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_i &= \left(b_i - \textstyle\sum_{j=1}^{i-1} \ell_{ij} x_j \right) / \ell_{jj}, \quad i = 2:n. \end{aligned}$$

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Indeed, $\sum_{i=1}^{n} 2i = \int_0^n 2x dx +$ lower order terms $\simeq n^2$.

Column-oriented forward substitution

Writing $Lx = b$ as

$$\mathtt{L}(:,1)\mathtt{x}(1) + \cdots + \mathtt{L}(:,\mathtt{n})\mathtt{x}(\mathtt{n}) = \mathtt{b}$$

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```
x = zeros(n,1);
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    x(j) =b(j)/L(j,j);
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- Solving a lower triangular system costs n^2 flops.

Upper triangular linear system

Consider the upper triangular system

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If $u_{11}, \ldots, u_{nn}$ are nonzero, then by back substitution, we have a unique solution

$$\begin{aligned} x_n &= b_n/u_{nn} \\ x_i &= \left(b_i - \textstyle\sum_{j=i+1}^{n} u_{ij}x_j\right)/u_{jj}, \quad i = n-1, \ldots, 1. \end{aligned}$$

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Cost: An upper triangular system is solved by back substitution and costs n^2 flops.

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$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned} \iff \underbrace{\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]}_{\text{augmented matrix}}$$

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Use first equation to eliminating x from 2nd and 3rd equation

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned} \iff \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right].$$

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Now interchange 2nd and 3rd equations

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned} \iff \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right] \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Gaussian elimination (GE)

Gaussian elimination can be rewritten as a method that factorizes a matrix. We consider three variants of GE. These variants yield three matrix factorizations, namely,

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Here P and Q are permutation matrices. An $n \times n$ permutation matrix is obtained by permuting rows of the identity matrix I_n.

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Here P and Q are permutation matrices. An $n \times n$ permutation matrix is obtained by permuting rows of the identity matrix I_n.

The matrix L is unit lower triangular and U is upper triangular. A lower triangular matrix L is called unit lower triangular if the diagonal entries of L are 1, that is, $\ell_{jj} = 1$ for $j = 1 : n$.

LU Decomposition

Definition: An LU decomposition of a matrix $A \in \mathbb{R}^{n\times n}$ is a factorization of the form $A = LU$, where L is unit lower triangular and U is upper triangular. Thus

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ \ell_{n1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ & \ddots & \vdots \\ & & u_{nn} \end{bmatrix} = LU.$$

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Then

$$MA = U \Longrightarrow L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_1^{-1} A = U \Longrightarrow A = LU,$$

where L is unit lower-triangular and U is upper-triangular.

LU factorization

Suppose A is 4×4 matrix. Then schematically

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \end{bmatrix}}_{L_1^{-1}A}$$

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Example

Let $A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Consider $L_1 := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$. Then

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$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \text{ and } L_1^{-1}A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -21 \end{bmatrix}.$$

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Now consider $L_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Then $L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$,

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$$U := L_2^{-1}L_1^{-1}A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix} \text{ and } L := L_1L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}.$$

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Thus we obtain $A = LU$.

Elimination matrix

Define $\ell_k := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \ell_{k+1,k} \\ \vdots \\ \ell_{nk} \end{bmatrix}$ and $L_k := I + \ell_k e_k^\top$ for $k = 1 : (n-1)$.

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$$\begin{aligned} L_k &= I + \begin{bmatrix} 0 & \cdots & 0 & \ell_k & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & \ell_{nk} & & & 1 \end{bmatrix} \end{aligned}$$

is unit lower triangular.

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$$\underbrace{(I + \ell_k e_k^\top)}_{L_k}(I - \ell_k e_k^\top) = I + \ell_k e_k^\top - \ell_k e_k^\top - \ell_k e_k^\top \ell_k e_k^\top = I.$$

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This shows that $L_k^{-1} = I - \ell_k e_k^\top$. Next observe that

$$L_k L_{k+1} = (I + \ell_k e_k^\top)(I + \ell_{k+1} e_{k+1}^\top) = I + \ell_k e_k^\top + \ell_{k+1} e_{k+1}^\top.$$

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$$\underbrace{(I + \ell_k e_k^\top)}_{L_k}(I - \ell_k e_k^\top) = I + \ell_k e_k^\top - \ell_k e_k^\top - \ell_k e_k^\top \ell_k e_k^\top = I.$$

This shows that $L_k^{-1} = I - \ell_k e_k^\top$. Next observe that

$$L_k L_{k+1} = (I + \ell_k e_k^\top)(I + \ell_{k+1} e_{k+1}^\top) = I + \ell_k e_k^\top + \ell_{k+1} e_{k+1}^\top.$$

Consequently

$$L = L_1 L_2 \cdots L_{n-1} = I + \ell_1 e_1^\top + \ell_2 e_2^\top + \cdots + \ell_{n-1} e_{n-1}^\top$$

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Consequently

$$\begin{aligned} L &= L_1 L_2 \cdots L_{n-1} = I + \ell_1 e_1^\top + \ell_2 e_2^\top + \cdots + \ell_{n-1} e_{n-1}^\top \\ &= I + \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_{n-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix}. \end{aligned}$$

Creating zeros via elimination matrix

Applying L_k^{-1} to the k-th column of an $n \times n$ matrix A, we have

$$L_k^{-1} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{kk} \\ a_{k+1,k} \\ \vdots \\ a_{nk} \end{bmatrix} = (I - \ell_k e_k^\top) \begin{bmatrix} a_{1k} \\ \vdots \\ a_{kk} \\ a_{k+1,k} \\ \vdots \\ a_{nk} \end{bmatrix} = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{kk} \\ a_{k+1,k} \\ \vdots \\ a_{nk} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \ell_{k+1,k} \\ \vdots \\ \ell_{nk} \end{bmatrix} a_{kk}$$

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$$= \begin{bmatrix} a_{k1} \\ \vdots \\ a_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ when } \ell_{ik} = a_{ik}/a_{kk},\ i = k+1 : n.$$

This shows that if $a_{kk} \neq 0$ then L_k can be used to create zeros in the k-th column of A below a_{kk}.

Multiplying a matrix by an elimination matrix

Let $A \in \mathbb{R}^{n\times n}$ and $L_k := I + \ell_k e_k^\top \in \mathbb{R}^{n\times n}$. Then

$$L_k^{-1}A = (I - \ell_k e_k^\top)A = A - \ell_k e_k^\top A = A - \ell_k \begin{bmatrix} a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

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The outer product shows that the first k rows of A remain unchanged when L_k^{-1} is multiplied to the left of A. Let $B := L_k^{-1}A$. In MATLAB, B can be written compactly as a rank-1 update (outer product from)

$$\mathtt{B} = \mathtt{A}(\mathtt{k}+1:\mathtt{n},:) - \ell(\mathtt{k}+1:\mathtt{n}) * \mathtt{A}(\mathtt{k},:)$$

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Cost: $2(n-1)^2$ flops.

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Thus overwriting A, in MATLAB notation, we have

```
A(2:n,1) = A(2:n,1)/A(1,1);   % multipliers
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Cost: $2(n-1)^2 + 2(n-2)^2 + \cdots + 2 \simeq 2n^3/3$ flops.

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This gives

$$A = L_1L_2 \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Gaussian Elimination with No Pivoting (GENP)

```
function [L, U] = GENP(A);
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    j = k+1:n;
    A(j,j) = A(j,j)-A(j,k)*A(k,j);
end
% strict lower triangle of A, plus I
L = eye(n,n)+ tril(A,-1);
U = triu(A); % upper triangle of A
```

Solution of $Ax = b$ by LU factorization

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Question: What will be the complexity if the system $Ax = b$ is solved as $\mathtt{x} = \mathtt{A}^{-1} * \mathtt{b}$?

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- Compute LU factorization $A = LU$. Cost: $\frac{2n^3}{3}$ flops.
- Solve $Ly = b$ for y. Cost: n^2 flops. bcoz of forward substitution
- Solve $Ux = y$ for x. Cost: n^2 flops. bcoz of back substitution

Thus the cost for solving system $Ax = b$ is $2n^3/3$ flops.

Question: What will be the complexity if the system $Ax = b$ is solved as $\mathtt{x} = \mathtt{A}^{-1} * \mathtt{b}$? Answer: $2n^4/3$ flops.

Solution of $Ax = b$ by LU factorization

An $n \times n$ linear system $Ax = b$ can be solved in three steps:

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Theorem: Let A be nonsingular. Then A admits a unique LU factorization $\Leftrightarrow$ all leading principal submatrices of A are nonsingular, that is, $A(1:j, 1:j)$ is nonsingular for $j = 1:n$.

Existence of LU factorization

Proof: Suppose that $A = LU$ exists and unique. Then writing

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Finally, $0 \neq \det(A) = \det(\hat{U})d \Rightarrow d \neq 0$. This completes the proof. ■