ME 620: Fundamentals of Artificial Intelligence

Lecture 16: First Order Logic – Part II

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Quantifiers

- The biggest change from propositional logic to first-order logic is the use of quantifiers.
- A **quantifier** is a statement that expresses that some property is true for some or all choices that could be made.
  - Turn predicates into propositions by assigning values to all variables:
    - Predicate P(x): x is even.
    - Proposition P(6): 6 is even.
  - The other way to turn a predicate into a proposition:
    - Add a quantifier like "All" or "Some" that indicates the number of values for which the predicate is true.

A formula that contains variables is not simply true or false unless each of these variables is **bound** by a quantifier

The Universal Quantifier

Definition: The symbol $\forall$ is the **universal quantifier**.

- The universal quantification of P(x) is the statement P(x) for all values x in the universe, which is written in logical notation as:

$$\forall x\, P(x) \text{ or sometimes } \forall x \in D,\, P(x).$$

- A statement of the form $\forall x\, P(x)$ asserts that for every choice of x in our domain, P(x) is true.
  - Example: All professors are people.

$$\forall x\, (\text{Professor}(x) \rightarrow \text{People}(x))$$

The Universal Quantifier

Definition: The **counterexample** for $\forall x\ P(x)$ is any $t \in U$, where U is the domain of discourse, such that P(t) is false.

- Example

  $\forall x,y,z$ sum(x,y,z): `z' is the sum of `x' and `y'.

  For U = non-negative integers.

  Proposition sum(1,7,8) is true.

  sum(5,1,8) is false.

The Existential Quantifier

- **Definition**: The symbol $\exists$ is the **existential quantifier.**
  - The existential quantification of P(x) is the statement

    P(x) for some values x in the universe, which is written in logical notation as

    $$\exists x\, P(x).$$

  - A statement of the form $\exists x\, P(x)$ asserts that for some choice of x in our domain, P(x) is true.
    - Even and prime in the number series.

      $\exists x.\ (Even(x) \wedge Prime(x))$

Bound and Free Variable

- **Definition:** A variable can occur as a term in a sentence without an enclosing quantifier.
  - When used in this way, a variable is said to be free.
  - All variables in a predicate must be **bound** to turn a predicate into a proposition. We bind a variable by assigning it a value or quantifying it. Variables which are not bound are **free.**

- **Definition:** If a sentence has no free variables, it is called a **closed sentence**. If it has neither free nor bound variables, it is called a **ground sentence**.

Translating Into First-Order Logic

1. All students are smart.

A universal quantification is a type of quantifier, a logical constant which is interpreted as "given any" or "for all

Incorrect Translation

$\forall x\ (Student(x) \land Smart(x))$

This should work for any choice of x, including things that aren't students.

- Although the original statement is true, this logical statement is false. It's therefore not a correct translation.

Correct Translation

$\forall x\ (Student(x) \rightarrow Smart(x))$

Translating Into First-Order Logic

2. There is a student who is smart.

Incorrect Translation

$$\exists x\ (\text{Student}(x) \rightarrow \text{Smart}(x))$$

- Under an interpretation that the original statement is false; this logical statement is true. It's therefore not a correct translation.

Correct Translation

$$\exists x\ (\text{Student}(x) \land \text{Smart}(x))$$

Translating Into First-Order Logic

- All P's are Q's

  translates as

  $\forall x\ (P(x) \rightarrow Q(x))$

  - $\forall$ quantifier usually is paired with $\rightarrow$

  - In the case of $\forall$, the $\rightarrow$ connective prevents the statement from being false when speaking about some object you don't care about.

Translating Into First-Order Logic

- Some P's are Q's

  translates as

  $\exists x\ (P(x) \land Q(x))$

  - $\exists$ quantifier usually is paired with $\land$

  - In the case of $\exists$, the $\land$ connective prevents the statement from being true when speaking about some object you don't care about.

De Morgan's Laws for Quantifiers

- $\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$
  - If $\neg \forall x\ P(x)$, then $P(x)$ is not true for every x,
  - For some value a, $P(a)$ is not true. This means that $\neg P(a)$ is true.
  - Since $\neg P(a)$ is true, it is certainly the case that there is some value of x that makes $\neg P(x)$ true.
  - $\exists x\ \neg P(x)$ is true.
- $\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$

Nesting Quantifiers

- For predicate $P(x,y)$:
  - Switching the order of universal quantifiers *does not* change the meaning

    $$\forall x \forall y P(x,y) \leftrightarrow \forall y \forall x P(x,y).$$

  - Similarly, one can switch the order of existential quantifiers

    $$\exists x \exists y P(x,y) \leftrightarrow \exists y \exists x P(x,y).$$

- Can not interchange the position of $\forall$ and $\exists$ like this!

Combining Quantifiers

3. Everyone loves someone else

Correct Translation

$\forall x\ \exists y\ Loves(x,y)$

Person(x) : `x' is a Person.

Loves(x,y): `x' loves `y'.

Different from him

$$\forall x\ (Person\ (x) \rightarrow \exists y\ (Person(y) \wedge x \neq y \wedge Loves(x,y)))$$

For EVERY person — There is SOMEONE — They LOVE

Combining Quantifiers

4. Someone everyone else loves.

Correct Translation

$\exists x \, \forall y \, Loves(y,x)$

Person(x) : `x' is a Person.

Loves(x,y): `x' loves `y'.

Different from him

$\exists x \, (Person(x) \wedge \forall y \, (Person(y) \wedge x \neq y \rightarrow Loves(y,x)))$

SOMEONE EVERYONE LOVES

Quantifier Ordering

- **Order** of the quantifiers is **important** when **mixing existential and universal quantifiers**!

  - $\forall x\ \exists y\ P(x,y)$

    For any choice x, there's some y where P(x, y) is true

  - $\exists x\ \forall y\ P(x,y)$

    There is some x where for any choice of y, we get that P(x, y) is true

    The inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Negations of Quantified Statements

Negation of a Universal Statement

1. All dogs bark.

   Incorrect Negation

   No dogs bark.

   If at least one dog does not bark, then the original statement is false.

   Correct Negation

   Some dogs do not bark.

The negation of a universal statement ($\forall x\ \varphi$) is logically equivalent to an existential statement ($\exists x\ \neg\varphi$).

Negations of Quantified Statements

Negation of a Existential Statement

2. Some snowflakes are the same.

Incorrect Negation

Some snowflakes are the different.

Correct Negation

No snowflakes are the same.

All snowflakes are different.

The negation of an existential statement ($\exists x\ \varphi$) is logically equivalent to a universal statement ($\forall x\ \neg\varphi$)

Negations of Quantified Statements

Negation – Pushing the NOT across

3. Everyone loves someone.

$\forall x\ \exists y\ Loves(x,y)$

Correct Negation

$\neg\forall x\ \exists y\ Loves(x,y)$ $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

$\exists x\ \neg\exists y\ Loves(x,y)$

$\exists x\ \forall y\ \neg Loves(x,y)$ $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$

There is someone who doesn't love anyone.

Negations of Quantified Statements

Negation of a Universal Conditional Statement

Negation of a conditional (if-then) statement is logically equivalent to an AND statement.

$$\neg (P \rightarrow Q) \equiv P \land \neg Q$$

$\neg (\neg P \lor Q)$
$\neg \neg P \land \neg Q$
$P \land \neg Q$

Negation of a universal statement is logically equivalent to an existential statement.

$$\neg \forall x\ \varphi \equiv \exists x\ \neg\varphi.$$

Substituting the conditional statement into the universal statement

$$\neg \forall x\ (P(x) \rightarrow Q(x)) \equiv \exists x\ (P(x) \land \neg Q(x))$$

Negations of Quantified Statements

Negation of a Universal Conditional Statement

4. If x is a rational number, then $\sqrt{x}$ is a rational number.

$\forall$x Rational ($\sqrt{x}$ is Rational)

Correct Negation

$\exists$x Rational ($\sqrt{x}$ is $\neg$ Rational)

There exist a rational number x, such that $\sqrt{x}$ is not a rational number.

Distributivity of ∀ over ∧

$\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x))$

∀ distributes over ∧

No matter what the domain is, these two propositions always have the same truth value.

This shouldn't be surprising, since for a finite domain, say {1,2,3},

$$\forall x\, P(x) \equiv (P(1) \wedge P(2) \wedge P(3))$$

Further ∧ is commutative and associative, so:

$$\forall x \in \{1, 2, 3\}(P(x) \wedge Q(x))$$

$$\equiv (P(1) \wedge Q(1)) \wedge (P(2) \wedge Q(2)) \wedge (P(3) \wedge Q(3))$$ For this example domain

$$\equiv (P(1) \wedge P(2) \wedge P(3)) \wedge (Q(1) \wedge Q(2) \wedge Q(3))$$ Commutativity/Associativity

$$\equiv \forall x \in \{1, 2, 3\}P(x) \wedge \forall x \in \{1, 2, 3\}Q(x)$$ For this specific example

Though this is only an example domain, the intuition extends to other domains as well, including infinite domains.

Distributing ∃ over ∧

$\exists x\, (P(x) \wedge Q(x)) \not\equiv \exists x\, P(x) \wedge \exists x\, Q(x)$.

The existential quantifier ∃ does not distribute over ∧

Find a counterexample - a universe and predicates P and Q - such that one of the propositions is true and the other is false:

Let U = N.

Set P(x): "x is prime" and Q(x): "x is composite" (i.e. not prime).

$\exists x\, (P(x) \wedge Q(x))$ is False,

$\exists x\, P(x) \wedge \exists x\, Q(x)$ is True.

Distributivity of ∃ over ∨

$$\exists x(P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$$

∃ distributes over ∨

This rule holds for arbitrary P and Q

Recall: $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Replace P by ¬ S and Q by ¬ R.

$$\forall x\,(\neg S(x) \wedge \neg R(x)) \equiv \forall x\, \neg S(x) \wedge \forall x\, \neg R(x)$$

Negate both sides.

$$\neg \forall x\,(\neg S(x) \wedge \neg R(x)) \equiv \neg(\forall x\, \neg S(x) \wedge \forall x\, \neg R(x))$$

$$\exists x\, \neg(\neg S(x) \wedge \neg R(x)) \equiv \neg \forall x\, \neg S(x) \vee \neg \forall x\, \neg R(x))$$

$$\exists x\,(\neg\neg S(x) \vee \neg\neg R(x)) \equiv \exists x\, \neg\neg S(x) \vee \exists x\, \neg\neg R(x))$$

$$\exists x(S(x) \vee R(x)) \equiv \exists x\, S(x) \vee \exists x\, R(x).$$