The PageRank Eigenvalue Problem

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Outline

- The PageRank Problem
- PageRank as an eigenvalue problem
- Power method for PageRank vector
- Single vector iterations for eigenvalue problems

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The PageRank algorithm ranks web pages by solving the PageRank eigenvalue problem:

$$\mathbf{G}\mathbf{v} := \underbrace{(\alpha\mathbf{S} + (1-\alpha)\mathbf{E})}_{\text{Google matrix}}\mathbf{v} = \mathbf{v}.$$

The page with highest rank is displayed first. This is amazingly effective!

Non-negative matrices and vectors

Let $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x}$ is called

- non-negative $(\mathbf{x} \succeq \mathbf{0})$ if $x_j \geq 0$ for $j = 1 : n$.
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Let $\mathbf{e} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$ and $A \in \mathbb{R}^{m \times n}$ be non-negative. Then A is called

- column stochastic if $\mathbf{e}^\top A = \mathbf{e}^\top$, that is, $\mathbf{e}^\top A\mathbf{e}_j = 1$ for $j = 1 : n$.
- column sub-stochastic if $\mathbf{e}^\top A \preceq \mathbf{e}^\top$, that is, $\mathbf{e}^\top A\mathbf{e}_j \leq 1$ for $j = 1 : n$.

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Fact: If $A \succ \mathbf{0}$ then $\rho(A)$ is a simple eigenvalue of A. Thus

$$\text{algebraic multiplity of } \rho(A) = \text{ geometric multiplicity of } \rho(A) = 1.$$

Perron vector and Perron root

Thus $N(A - \rho(A)I) = \mathrm{span}(\mathbf{v})$ and either $\mathbf{v} \succ \mathbf{0}$ or $-\mathbf{v} \succ \mathbf{0}$.

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Proof: Let $A\mathbf{y} = \lambda \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{0}$. Let $\mathbf{x} \succ 0$ be the Perron vector of $A^\top$. Then $\mathbf{x}^\top \mathbf{y} > 0$. Hence

$$\rho(A)\mathbf{x}^\top = \mathbf{x}^\top A \Longrightarrow \rho(A)\mathbf{x}^\top \mathbf{y} = \mathbf{x}^\top A\mathbf{y} = \lambda \mathbf{x}^\top \mathbf{y} \Longrightarrow \rho(A) = \lambda. \blacksquare$$

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- Collatz-Weilandt formula

$$\rho(A) = \max\left\{ \min_{i, x_i \neq 0} \frac{(A\mathbf{x})_i}{x_i} : \mathbf{x} \succeq \mathbf{0} \text{ and } \mathbf{x} \neq \mathbf{0} \right\}$$

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Google's PageRank algorithm assigns ranks to all the web pages and is formulated as a matrix eigenvalue problem.

World Wide Web as a Graph

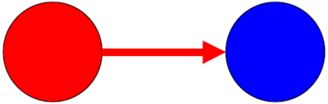
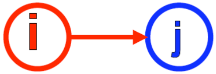


Figure : Link from one web page to another web page.

Web graph:

- Nodes = Web pages
- Edges = links

Ranking of web pages



The web is an example of a directed graph. Let all the web pages be ordered as $P_1, \ldots, P_n$. A link from P_i to P_j represents an arrow. Google assigns rank to a page based on its in-links (incoming links) and out-links (outgoing links).

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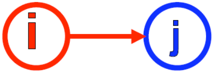

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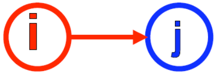

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However, a ranking system based only on the number of inlinks is easy to manipulate. Google overcomes this problem as follows.

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Let L_k be the set of in-links of P_k. Then the rank x_k of P_k is defined by

$$x_k = \sum_{j \in L_k} \frac{x_j}{n_j} \tag{1}$$

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$$\text{Eigenvalue problem} \quad \mathbf{Hv} = \mathbf{v} \quad ,$$

where $\mathbf{H}$ is the hyperlink matrix and $\mathbf{v}$ is the PageRank vector.

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Let n_j be the number of outlinks of the web page P_j. Then

$$\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \dots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix},$$

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Number of nonzero entries in $\mathbf{H}\mathbf{e}_j$ = number of outlinks of P_j.
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Thus $\mathbf{H}$ is non-negative: $\mathbf{H} \succeq 0$ and $\mathbf{He}_k = \mathbf{0}$ if P_k has no outlink.
On the other hand, $\mathbf{e}^\top \mathbf{He}_k = 1$ if the page P_k has outlinks.

Hence $\mathbf{H}$ is column sub-stochastic: $\mathbf{e}^\top \mathbf{He}_j \leq 1$ for $j = 1 : n$.

World Wide Web as a Hyperlink matrix

Let n_j be the number of outlinks of the web page P_j. Then

$$\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \cdots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix},$$

$h_{ij} = 1/n_j$ if there is a link from P_j to page P_i else $h_{ij} = 0$.

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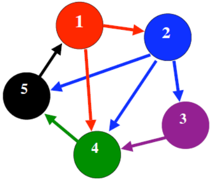
Dangling node: If a page P_k has no outlink ($\mathbf{He}_k = \mathbf{0}$) then it is called a dangling node. If there are no dangling nodes then $\mathbf{e}^\top \mathbf{H} = \mathbf{e}^\top$. Hence $\mathbf{H}$ is column stochastic.

Example

Column stochastic hyperlink matrix $\mathbf{H}$.



$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{e}^\top \mathbf{H} = \mathbf{e}^\top.$$

There is a unique PageRank vector $\mathbf{v} \succ 0$ such that $\mathbf{H}\mathbf{v} = \mathbf{v}$ and $\mathbf{e}^\top \mathbf{v} = 1$. The entries of $\mathbf{v}$ are the ranks of web pages.

Nonunique ranking

If $\mathbf{H}$ is column stochastic then $\mathbf{e}^\top \mathbf{H} = \mathbf{e}^\top$ which shows that $\mathbf{e}$ is a left eigenvector of $\mathbf{H}$ corresponding the eigenvalue 1.

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Theorem: Suppose that $\mathbf{H} \succ \mathbf{0}$ and $\mathbf{e}^\top \mathbf{H} = \mathbf{e}^\top$. Then

- 1 is the simple eigenvalue of $\mathbf{H}$ and there exists $\mathbf{v} \succ 0$ such that $\mathbf{H}\mathbf{v} = \mathbf{v}$. If $\mathbf{v} \in N(\mathbf{H} - I)$ is nonzero then either $\mathbf{v} \succ \mathbf{0}$ or $-\mathbf{v} \succ \mathbf{0}$.

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- 1 is the dominant eigenvalue, that is, if $\lambda_2, \ldots, \lambda_n$ are eigenvalues of $\mathbf{H}$ then $|\lambda_j| < 1$ for $j = 2 : n$.
- any other positive eigenvector of $\mathbf{H}$ is a scalar multiple of $\mathbf{v}$.

Adjustment of hyperlink matrix

The web is full of dangling nodes (e.g., pdf flies, image files, etc.). To fix this problem, Brin and Page replaced zero columns of $\mathbf{H}$ with $\frac{1}{n}\mathbf{e}$. Define

$$\mathbf{S} := \mathbf{H} + \frac{1}{n}\mathbf{e}\,\mathbf{a}^\top,$$

where $a_j = 1$ if P_j is a dangling node (i.e., $\mathbf{He}_j = \mathbf{0}$) and $a_j = 0$ else.

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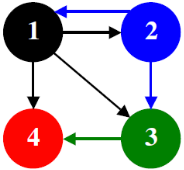
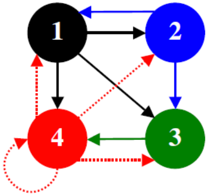
The binary vector $\mathbf{a}$ is called the dangling node vector. Note that $\mathbf{S}$ is column stochastic and the adjustment amounts to adding artificial links to the dangling nodes.





The Google matrix

The matrix $\mathbf{S}$, however, cannot guarantee existence of a unique PageRank vector. So, Brin and Page made the final adjustment to obtain the Google matrix

$$\mathbf{G} := \alpha\mathbf{S} + (1-\alpha)\mathbf{E}, \; 0 \leq \alpha \leq 1,$$

where $\mathbf{E} = \dfrac{1}{n}\mathbf{e}\mathbf{e}^{\top}$ is the teleportation matrix which enables jumping to any page.

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Fact: The Google matrix $\mathbf{G}$ has the following properties.

- $\mathbf{G}$ is positive $(\mathbf{G} \succ \mathbf{0})$ and column stochastic $(\mathbf{e}^\top\mathbf{G} = \mathbf{e}^\top)$.
- There is a unique vector $\mathbf{v}$ such that $\mathbf{G}\mathbf{v} = \mathbf{v}, \; \mathbf{v} \succ \mathbf{0}$ and $\mathbf{e}^\top\mathbf{v} = 1.$
- $\mathbf{G}$ is the rank-1 update of $\mathbf{H}$:

$$\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\frac{1}{n}\mathbf{e}\mathbf{e}^\top = \alpha\mathbf{H} + \frac{1}{n}\mathbf{e}(\alpha\mathbf{a} + (1-\alpha)\mathbf{e})^\top.$$

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Theorem: Let $\mathbf{G} := \alpha\mathbf{S} + (1-\alpha)\mathbf{E}$ be the Google matrix.

- If $\Lambda(\mathbf{S}) = \{1, \lambda_2, \ldots, \lambda_n\}$ then $\Lambda(\mathbf{G}) = \{1, \alpha\lambda_2, \ldots, \alpha\lambda_n\}$.

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$$\mathbf{v} = \alpha \mathbf{S}\mathbf{v} + (1-\alpha)\frac{1}{n}\mathbf{e} \Longrightarrow (I - \alpha\mathbf{S})\mathbf{v} = (1-\alpha)\frac{1}{n}\mathbf{e}.$$

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$$\mathbf{Gx} - \mathbf{Gv} = \alpha\mathbf{S}(\mathbf{x}-\mathbf{v}).$$

Consequently, we have

$$\|\mathbf{G}(\mathbf{x}-\mathbf{v})\|_1 = \alpha\|\mathbf{S}(\mathbf{x}-\mathbf{v})\|_1 \leq \alpha\|\mathbf{x}-\mathbf{v}\|_1,$$

where $\|\mathbf{x}\|_1 = |x_1| + \cdots + |x_n|$.

The Power method

Power method: Choose $\mathbf{v}_0 \succeq \mathbf{0}$ such that $\mathbf{e}^\top \mathbf{v}_0 = 1$. Define $\mathbf{u}_j = \alpha \mathbf{S} \mathbf{v}_{j-1} + (1-\alpha)\frac{1}{n}\mathbf{e}$ and $\mathbf{v}_j = \mathbf{u}_j / \mathbf{e}^\top \mathbf{u}_j$ for $j = 1, 2, \ldots$. Then

$$\|\mathbf{v}_j - \mathbf{v}\|_1 \leq \alpha^j \|\mathbf{v} - \mathbf{v}_0\|_1.$$

Algorithm: The Power Method
Input: Vector $\mathbf{v}_0 \succeq \mathbf{0}$ such that $\mathbf{e}^\top \mathbf{v}_0 = 1$ and number of iterations ℓ
Output: PageRank vector $\mathbf{v}_\ell$)

---

for $j = 1 : \ell$
  $\mathbf{u}_j := \alpha \mathbf{S} \mathbf{v}_{j-1} + (1-\alpha)\mathbf{e}/n$ (application of $\mathbf{G}^j$)
  $\mathbf{v}_j := \mathbf{u}_j / \mathbf{e}^\top \mathbf{u}_j$ (normalization)
end

Further reading

Want to know more about Google search engine?

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Read the following paper/book.

- K. Bryan and T. Leise, The $25,000,000,000 Eigenvector: The Linear Algebra behind Google, SIAM Review, 48(2006), pp.569-581.
- Amy N. Langville and Carl D. Meyer, *Google's PageRank and Beyond: The Science of Search Engine Rankings,* Princeton University Press, 2006

The power method

The power method (or power iteration) is one of the oldest and simplest iterative method for computing an eigenpair of A. It is designed for a matrices A with one dominant eigenvalue. Starting with an arbitrary nonzero vector $v_0 \in \mathbb{C}^n$, the power method computes the sequence

$$v_0,\ Av_0,\ \ldots,\ A^j v_0,\ \ldots$$

Setting $v_j := A^j v_0$, we have $v_j = Av_{j-1}$ for $j = 1, 2, \ldots$.

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Setting $v_j := A^j v_0$, we have $v_j = Av_{j-1}$ for $j = 1, 2, \ldots.$

Algorithm: The Power Method
Input: Nonzero vector v_0 and the number of iterations ℓ
Output: Dominant eigenpair (μ_ℓ, v_ℓ)

---

for $j = 1 : \ell$
  $w_j := Av_{j-1}$ (application of A^j)
  $v_j := w_j / \|w_j\|_2$ (normalization)
  $\mu_j := v_j^* A v_j$ (Rayleigh quotient)
end

The inverse iteration

The power method can only be used to compute a dominant eigenpair of A. It turns out that a simple *shift-and-invert* strategy can be utilized to compute any desired eigenvalue of A and an associated eigenvector. Assume for the moment that A is invertible. Then

$$Av = \lambda v \Longleftrightarrow A^{-1}v = \frac{1}{\lambda}v.$$

This shows that $1/\lambda$ is the dominant eigenvalue of $A^{-1} \iff \lambda$ is the smallest eigenvalue of A.

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If s is not an eigenmvalue of A then

$$Av = \lambda v \iff (A - sI)v = (\lambda - s)v \iff (A - sI)^{-1}v = \frac{1}{\lambda - s}v.$$

Thus with the help of a shift s that is closer to a simple eigenvalue λ than to the rest of the spectrum of A, the transformed eigenvalue $(\lambda - s)^{-1}$ can be made dominant for the transformed matrix $(A - sI)^{-1}$. Hence we can apply the power iteration to the shifted matrix $(A - sI)^{-1}$ and arrive at the shifted inverse iteration.

The inverse iteration

Algorithm: Shifted inverse iteration
Input: Nonzero vector v_0, a shift s and the number of iterations ℓ
Output: An eigenpair (μ_ℓ, v_ℓ) such that μ_ℓ closest to s

```
for j = 1:ℓ
    (A − sI)w_j = v_{j−1}      (application of (A − sI)^{−j})
    v_j := w_j/‖w_j‖_2         (normalization)
    μ_j := v_j^* A v_j         (Rayleigh quotient)
end
```

$$\begin{array}{ll}
\text{for } j = 1{:}\,\ell & \\
\quad (A - sI)w_j = v_{j-1} & \text{(application of } (A - sI)^{-j}\text{)} \\
\quad v_j := w_j/\|w_j\|_2 & \text{(normalization)} \\
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\text{end} &
\end{array}$$

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The computation of w_j requires solution of $(A - sI)w_j = v_j$. Since $(A - sI)$ is not changed during the iteration, we compute LU factorization $P(A - sI) = LU$ only once, where P is a permutation matrix, L is a unit lower triangular matrix and U is an upper triangular matrix.

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This will reduce the cost of computation to $2n^2$ flops per iteration. This is an enormous saving as solving the system without exploiting LU factorization would cost $2n^3/3$ flops per iteration. Hence the total cost of performing ℓ iterations of the inverse iteration is $(2n^3/3 + 4n^2\ell)$ flops.

Rayleigh quotient iteration (RQI)

The shifted inverse iteration converges rapidly if the shift parameter $s \in \mathbb{C}$ is close to an eigenvalue of A. The Rayleigh quotient $q(x) := x^*Ax/x^*x$ approximates an eigenvalue A. Hence it makses sense to use Rayleigh quotient as shift in the shifted inverse iteration. The resulting algorithm is called the *Rayleigh quotient iteration* (RQI).

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Algorithm: Rayleigh quotient iteration (RQI)
Input: Nonzero vector v_0 and the number of iterations ℓ
Output: Approximate eigenpair (μ_ℓ, v_ℓ)

$v_0 := v_0/\|v_0\|_2$ (normalize v_0)
$\mu_0 := v_0^* A v_0$ (Rayleigh quotient)
for $j = 1:\ell$
  $(A - \mu_{j-1} I)w_j = v_{j-1}$ (application of $(A - \mu_{j-1} I)^{-j}$)
  $v_j := w_j/\|w_j\|_2$ (normalization)
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end

Remark: RQI requires a new LU factorization in each iteration.