Homework - 7

MA423 : Matrix Computations 2023 R. Alam

Rotations, Gram-Schmidt orthogonalization

1. Let $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$ and $r := \sqrt{x_1^2 + x_2^2}$. Determine a rotation $G \in \mathbb{R}^{2\times 2}$ and $\begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top \in \mathbb{R}^2$ such that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y_1 = y_2$.

2. Let $A := \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$. Then A satisfies the Hadamard's determinant inequality
$$|\det(A)| \leq \prod_{j=1}^{n} \|\mathbf{a}_j\|_2.$$
Your task is to prove the Hadamard's inequality using QR factorization of A. Let $A = QR$ be a QR factorization of A and let $R = \begin{bmatrix} \mathbf{r}_1 & \cdots & \mathbf{r}_n \end{bmatrix}$ be the column partition of R.
   (a) Show that $\|\mathbf{a}_j\|_2 = \|\mathbf{r}_j\|_2$ for $j = 1 : n$.
   (b) Show that $|\det(Q)| = 1$.
   (c) Show that $|\det(A)| = \prod_{j=1}^n |r_{jj}| \leq \prod_{j=1}^n \|\mathbf{a}_j\|_2$, where $r_{11}, \ldots, r_{nn}$ are diagonal entries of R.
   (d) Suppose the columns of A are nonzero. Show that $|\det(A)| = \prod_{j=1}^n \|\mathbf{a}_j\|_2 \iff$ the columns of A are orthogonal.

3. Let $A \in \mathbb{R}^{m\times n}$. Suppose that $\text{rank}(A) = n$. Consider the compact QR factorization $A = QR$, where $Q \in \mathbb{R}^{m\times n}$ is an isometry and $R \in \mathbb{R}^{n\times n}$ is upper triangular with positive diagonal entries. Partitioning A, Q and R as
$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}, \tag{$**$}$$
where $A_1 \in \mathbb{R}^{m\times k}$, we obtain a block algorithm for the QR factorization $(**)$:
   1. Compute the unique compact QR factorization $A_1 = Q_1 R_{11}$
   2. Compute $R_{12} \leftarrow Q_1^T A_2$
   3. Compute $A_2 \leftarrow A_2 - Q_1 R_{12}$
   4. Recursively continue with A_2

   Show that for $k = 1$ the resulting algorithm is the same as the Modified Gram-Schmidt method (MGS). Show that for $k = n - 1$ the resulting algorithm is the same as the Classical Gram-Schmidt method (CGS).

********** End ***********