Applications of SVD/PCA

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Outline

- SVD/PCA based algorithm for face recognition
- SVD and Polar decomposition

Face recognition

Figure: Library of twelve Presidents of the United States.

We store these images as vectors or matrices and apply PCA (Principal Component Analysis) to process the images.

Face recognition

Assume that each image has the same pixel resolution with m rows and n columns. Let $P_1, \ldots, P_N$ be $m \times n$ matrices representing N images. Define

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Usually 50 or 100 eigenfaces are required to create strong approximations to a set of $10,000$ faces.

Face recognition

(a) label 1

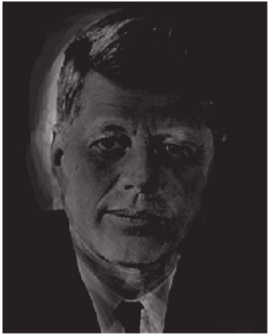

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Figure: (a) The average image of twelve U.S. Presidents and (b) the average image subtracted from the image of President Kennedy .

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For the presidential library images, the eigenfaces are obtained from eigenvectors of the 12×12 matrix $A^\top A$.

Figure: A half dozen eigenfaces of the library of twelve U.S. Presidents.

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This is part of the power of PCA in facial recognition. One can use a large library of images, reduce it to a much smaller set of eigenfaces, and then use it to recognize a face or create an approximation, even with some disguising.

Face recognition

(a) label 1

(b) label 2

Figure: (a) An altered image of President Kennedy and (b) the image recreated using the six eigenfaces.

Polar decomposition of a matrix

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Remark: Note that polar decomposition is unique when A is nonsingular.

Theorem: Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Then there exists an isometry $W \in \mathbb{C}^{m\times n}$ and a positive semidefinite matrix $R \in \mathbb{C}^{n\times n}$ such that $A = WR$.

Polar decomposition

Proof: Consider the trimmed SVD $A = U_n \Sigma_n V_n^*$, where $U_n \in \mathbb{C}^{m \times n}$ is an isometry, $V_n \in \mathbb{C}^{n \times n}$ is unitary, and $\Sigma_n = \mathrm{diag}(\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^{n \times n}$.

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Example: Consider $A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$. We now compute polar decomposition $A = WR$ from SVD of A.

Example

The trimmed SVD of A is given by

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>> [U, S, V] = svd(A, 0)
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$$U = \begin{bmatrix} -0.1409 & 0.8247 & 0.5477 \\ -0.3439 & 0.4263 & -0.7276 \\ -0.5470 & 0.0278 & -0.1880 \\ -0.7501 & -0.3706 & 0.3679 \end{bmatrix}, \quad S = \begin{bmatrix} 25.4624 & 0 & 0 \\ 0 & 1.2907 & 0 \\ 0 & 0 & 0.0000 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.5045 & -0.7608 & -0.4082 \\ -0.5745 & -0.0571 & 0.8165 \\ -0.6445 & 0.6465 & -0.4082 \end{bmatrix}$$

Example

The desired polar factors are given by

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>> W = U*V', R = V*S*V'
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$$W = \begin{bmatrix} -0.7799 & 0.4810 & 0.4004 \\ 0.1463 & -0.4208 & 0.7943 \\ 0.3316 & 0.1592 & 0.4473 \\ 0.5102 & 0.7525 & 0.0936 \end{bmatrix}, \quad R = \begin{bmatrix} 7.2286 & 7.4367 & 7.6448 \\ 7.4367 & 8.4085 & 9.3804 \\ 7.6448 & 9.3804 & 11.1160 \end{bmatrix}.$$

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