MA668: Algorithmic and High Frequency Trading
Lecture 33

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Optimal Liquidation in Lit and Dark Markets

1. So far: The agent has been trading on transparent (lit) markets, where all agents can observe quantities being offered for sale or purchase at different prices, that is, the LOB is visible to all interested parties.
2. We now consider the possibility that the agent can also trade in what are known as **dark pools**.
3. Dark pools are trading venues which, in contrast to traditional (or lit) exchanges, do not display bid and ask quotes to their clients.
4. Trading may occur continuously, as soon as orders are matched, or consolidated and cleared periodically (sometimes referred to as throttling).
5. We focus on a particular kind of dark pool known as a **crossing network** defined by the Securities Exchange Commission (SEC) as: `"... systems that allow participants to enter unpriced orders to buy and sell securities, these orders are crossed at a specified time at a price derived from another market ...`

Optimal Liquidation in Lit and Dark Markets (Contd ...)

1. Typically, the price at which transactions are crossed is the mid-price in a corresponding lit trading venue.
2. When a trader places an order in a dark pool, she/he may have to wait for some time until a matching order arrives so that her/his order is executed.
3. Thus: On the one hand the trader who sends orders to the dark pool is exposed to execution risk, but on the other hand does not receive the additional temporary price impact of walking the LOB.
4. Here we analyze the case when the agent trades continuously in the lit market and simultaneously posts orders in the dark pool with the aim to liquidate $\mathfrak{N}$ shares.

Model Setup

1. On the lit market, we assume, as before, that the agent is exposed to a temporary market impact from her/his market orders.
2. So when trading $\nu_t dt$ in the lit market, she/he receives $\widehat{S}_t = S_t - k\nu_t$ per share, with $k > 0$, where the mid-price S_t is a Brownian motion.
3. In addition to trading in the lit market, the agent posts $y_t \leq q_t$ units of inventory in the dark pool.
4. Here $q_t \leq \mathfrak{R}$ are the remaining shares to be liquidated, and she/he may continuously adjust this posted order.
5. Matching orders in the dark have no price impact because they are pegged to the lit market's mid-price.
6. So the agent receives S_t per share for each unit executed in the dark pool, which is not necessarily the whole amount y_t.

Model Setup (Contd ...)

1. Furthermore, other market participants send matching orders to the dark pool which are assumed to arrive at Poisson times and the volumes associated with the orders are independent.
2. More specifically, let N_t denote a Poisson process with intensity λ and let $\{\xi_j : j = 1, 2, \dots\}$ be a collection of independent and identically distributed random variables corresponding to the volume of the various matching orders which are sent by other market participants into the dark pool.
3. The total volume of buy orders (which may match the agent's posted sell order) placed in the dark pool up to time t is the compound Poisson process:
$$V_t = \sum_{n=1}^{N_t} \xi_n.$$
4. When a matching order arrives, it may be larger or smaller than the agent's posted sell order, and hence the agent's inventory (accounting for both the continuous trading in the lit market and her/his post in the dark pool) satisfies the SDE:
$$dQ_t^{\nu,y} = -\nu_t dt - \min\left(y_t, \xi_{1+N_{t-}}\right) dN_t.$$

Model Setup (Contd ...)

1. Recall that the agent's aim is to liquidate $\mathfrak{R}$ shares on or before the terminal date T.
2. In the preceding equation above:
   - (A) The first term on the right-hand side represents the shares that the agent liquidates using MOs in the lit market.
   - (B) The second term on the right-hand side represents the orders the agent sends to the dark pool.
3. We assume that the agent is at the front of the sell queue in the dark pool, so that she/he is the first to execute against any new orders coming into that market.
4. The model can be modified to account for the agent not being at the front.
5. This can be done by introducing another random variable representing the volume of orders in front of the agent.
6. This complicates but does not alter the approach in a fundamental way.

Model Setup (Contd ...)

1. Hence, the agent's cash process $X_t^{\nu,y}$ satisfies the SDE:

$$dX_t^{\nu,y} = (S_t - k\nu_t)\nu_t dt + S_t \min\left(y_t, \xi_{1+N_{t-}}\right) dN_t.$$

2. Her/his performance criteria is, as usual, given by:

$$H^{\nu,y}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_\tau + Q_\tau^{\nu,y}(S_\tau - \alpha Q_\tau^{\nu,y}) - \phi \int_t^\tau (Q_u^{\nu,y})^2 du\right].$$

3. Here $\mathbb{E}_{t,x,S,q}$ is conditioned on $X_{t-} = x$, $S_t = S$ and $Q_{t-} = q$.
4. The stopping time $\tau = T \wedge \inf\{t : Q_t = 0\}$, represents the time until the agent's inventory is completely liquidated or the terminal time has arrived.
5. The value function is:

$$H(t,x,S,q) = \sup_{\nu,y\in\mathcal{A}} H^{\nu,y}(t,x,S,q),$$

where the set of admissible strategies consists of $\mathcal{F}$-predictable processes bounded from above, and her/his posted volume in the dark pool is at most her/his remaining inventory, that is, $y_t \leq Q_t^{\nu,y}$.

The Resulting DPE

The resulting DPE is given by:

$$
\begin{aligned}
0 &= \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2, \\
&+ \sup_{\nu} \left[(S - k\nu)\nu \partial_x H - \nu \partial_q H \right], \\
&+ \sup_{y \le q} \left[\lambda \mathbb{E} \left[H\left(t, x + S \min(y, \xi), S, q - \min(y, \xi)\right) - H(t, x, S, q) \right] \right],
\end{aligned}
$$

subject to the terminal condition:

$$H(T, x, S, q) = x + q(S - \alpha q).$$

The Resulting DPE (Contd ...)

1. The expectation represents an expectation over the random variable ξ and the various terms in the DPE carry the following interpretations:
   - A. The term ∂_{SS} represents the diffusion of the midprice.
   - B. The term $-\phi q^2$ represents the running penalty which penalises inventories different from zero.
   - C. The term $\sup_\nu[\cdot]$ represents optimizing over continuous trading in the lit market.
   - D. The term $\sup\limits_{y \leq q}$ represents optimizing over the volume posted in the dark pool. The expectation is there to account for the fact that buy volume coming into the dark pool from other traders is random.
2. The terminal condition once again suggests the ansatz: $H(t, x, S, q) = x + qS + h(t, q)$.
   - A. $x + qS$: Represents the cash from sales so far, in both lit and dark markets, plus the book value (at mid-price) of the shares the agent still holds and aims to liquidate.
   - B. Hence, h represents the value of optimally trading beyond the book value of cash and assets.

The Resulting DPE (Contd ...)

1. The DPE then reduces to a simpler equation for h:

$$\begin{aligned} 0 &= \partial_t h - \phi q^2 + \sup_{\nu} \left[-k\nu^2 - \nu \partial_q h \right] \\ &+ \lambda \sup_{y \le q} \mathbb{E} \left[h(t, q - \min(y, \xi)) - h(t, q) \right], \end{aligned} \tag{1}$$

subject to the terminal condition $h(T, q) = -\alpha q^2$.

2. Next: The first order condition for ν implies that the optimal speed to trade in feedback control form is:

$$\nu^* = -\frac{1}{2k} \partial_q h. \tag{2}$$

3. Therefore:

$$\sup_{\nu} \left[-k\nu^2 - \nu \partial_q h \right] = \frac{1}{4k} (\partial_q h)^2.$$

4. To determine the optimal over y, that is, the optimal volume to post in the dark pool), we need to either resort to numerics or place more structure on the random variable ξ.

Explicit Solution When Dark Pool Executes in Full

1. To obtain an explicit solution to the problem we assume that the agent's desired execution (the liquidation order) is small relative to the volume coming into the dark pool, $\xi_i \geq \mathfrak{R},\ i = 1, 2, \ldots$.
2. This assumption ensures that when a matching buy order arrives in the dark pool, the agent's order is executed in full, as the incoming buy order is larger than the amount posted in the agent's sell order.
3. Note that the agent's inventory at any point in time is at most $\mathfrak{R}$ (the initial amount that must be liquidated).
4. As the agent's posts are always filled entirely, in (1), we have, $\min(\xi, y) = y$.

Explicit Solution When Dark Pool Executes in Full (Contd ...)

1. We hypothesize that the ansatz is a polynomial in q.
2. Before proposing the ansatz, note that the DPE contains an explicit q^2 penalty, the optimum over ν is quadratic in $\partial_q h$, and the terminal condition is $-\alpha q^2$.
3. Thus this suggests the following ansatz for $h(t, q)$:

$$h(t, q) = h_0(t) + h_1(t)q + h_2(t)q^2,$$

with terminal conditions $h_0(T) = h_1(T) = 0$ and $h_2(T) = -\alpha$.

4. The supremum over y becomes:

$$\begin{aligned}
& \sup_{y \le q} \mathbb{E}\left[h(t, q - \min(y, \xi)) - h(t, q)\right], \\
= \quad & \sup_{y \le q} \left[h(t, q - y) - h(t, q)\right], \\
= \quad & \sup_{y \le q} \left[-yh_1 + (y^2 - 2qy)h_2\right], \\
= \quad & -\frac{1}{4h_2}\left(h_1 - 2qh_2\right)^2.
\end{aligned}$$

Explicit Solution When Dark Pool Executes in Full (Contd ...)

1. The optimal dark pool volume in feedback form is:

$$y^* = q + \frac{1}{2}\frac{h_1}{h_2}.$$

2. From the terminal condition, $h_2(t) < 0$.
3. It remains to be seen that $h_1(t) \geq 0$ so that indeed $y^* \leq q$ and the admissibility criteria are satisfied.
4. Notice that both y^* and ν^* are independent of h_0, so while h_0 is important in determining the value function, it is irrelevant for obtaining the optimal strategy.
5. Inserting the above feedback controls into the DPE (1), leads to a coupled system of ODEs for h_0, h_1 and h_2 (on the lines of an earlier discussion).
6. These ODEs upon application of the boundary conditions leads to $h_0(t) = 0$ and $h_1(t) = 0$.

Explicit Solution When Dark Pool Executes in Full (Contd ...)

1. The equation for h_2 given by:

$$\partial_t h_2 - \phi - \lambda h_2 + \frac{1}{k} h_2^2 = 0,$$

is of Riccati type and can be solved explicitly.

2. Let $\zeta^{\pm}$ denote the roots of the polynomial $\phi + \lambda p - \frac{1}{k} p^2 = 0$.

3. Then we get:

$$\partial_t h_2 = -\frac{1}{k}(h_2 - \zeta^+)(h_2 - \zeta^-).$$

4. Using partial fractions and integrating from t to T gives:

$$\begin{aligned} \partial_t h_2 \left(\frac{1}{h_2 - \zeta^+} - \frac{1}{h_2 - \zeta^-} \right) &= -\frac{1}{k} \left(\zeta^+ - \zeta^- \right) \\ \Rightarrow \log \left(\frac{h_2 - \zeta^-}{h_2 - \zeta^+} \right) - \log \left(\frac{\alpha + \zeta^-}{\alpha + \zeta^+} \right) &= -\frac{1}{k} \left(\zeta^+ - \zeta^- \right) (T - t), \end{aligned}$$

where we have used the terminal condition $h_2(T) = -\alpha$.

Explicit Solution When Dark Pool Executes in Full (Contd ...)

1. Finally we arrive at:

$$h_2(t) = \frac{\zeta^- - \zeta^+ \beta e^{-\gamma(T-t)}}{1 - \beta e^{-\gamma(T-t)}},$$

where $\zeta^\pm = \frac{k\lambda}{2} \pm \sqrt{\frac{k^2\lambda^2}{4} + k\phi}$, $\beta = \frac{\alpha + \zeta^-}{\alpha + \zeta^+}$ and $\gamma = \frac{1}{k}\left(\zeta^+ - \zeta^-\right)$.

2. Therefore, the optimal trading strategy is:

$$\nu_t^* = -\frac{1}{k} h_2(t) Q_t^{\nu^*, y^*} \text{ and } y_t^* = Q_t^{\nu^*, y^*}. \tag{3}$$

3. The optimal inventory to hold up to the arrival of matching order in the dark pool:

$$Q_t^{\nu^*, y^*} = e^{(\zeta^-/k)t} \left(\frac{1 - \beta e^{-\gamma(T-t)}}{1 - \beta e^{-\gamma T}} \right) \mathfrak{R}. \tag{4}$$

Explicit Solution When Dark Pool Executes in Full (Contd ...)

1. In the limit in which the terminal penalty α is very large, that is, $\alpha \to \infty$ (so that the agent guarantees full execution by the end of the trading horizon), $\beta \to 1$ and hence:

$$Q_t^{\nu^*,y^*} \to e^{(\zeta^-/k+\gamma/2)t} \left(\frac{\sinh\left(\frac{\gamma}{2}\,(T-t)\right)}{\sinh\left(\frac{\gamma}{2}\,T\right)} \right) \mathfrak{R},$$

   as $\alpha \to \infty$.

2. Furthermore, in the limit $\lambda \to 0$, $\zeta^- \to -\sqrt{k\phi}$ and $\gamma \to 2\sqrt{\phi/k}$ and hence:

$$Q_t^{\nu^*,y^*} \to \frac{\sinh\left(\sqrt{\frac{\phi}{k}}\,(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{k}}\,T\right)} \mathfrak{R},$$

   as $(\alpha, \lambda) \to (\infty, 0)$.

3. This recovers the results from the AC case without the dark pool.

Liquidation Strategy With Dark Pool

1. It is clear that the optimal amount to send to the dark pool is always what remains to be liquidated.
2. This is reasonable because in our model there is no market impact in the dark pool so the agent obtains the midprice for orders that are crossed in the dark pool.
3. However: The more interesting part of the liquidation strategy is how much the agent should send to the lit markets now that she/he has access to a dark pool.
4. To answer this question it is useful to compare the lit market liquidation rate ν^* in (3) with the optimal liquidation strategy when there is no dark pool, that is, $\lambda = 0$.
5. Recall that when $\lambda = 0$, the optimal speed of trading in the lit market is that given by the AC solution.
6. It is not immediately clear whether the modified rate at which the agent is trading in the lit market is larger or smaller than the liquidation rate when the agent does not have access to a dark pool.
7. Also: Not clear whether the trading rate is decreasing as in the AC case.

Figure 7.6

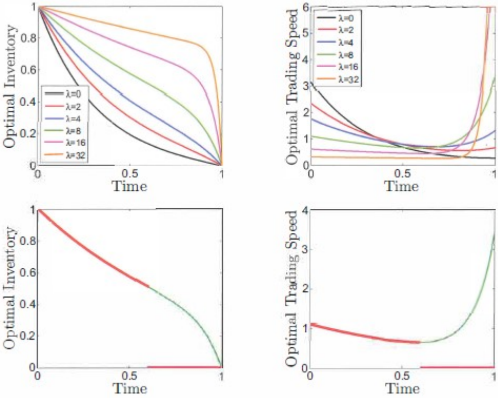

Figure 7.6 The top panels show optimal inventory path and speed of trading prior to a matching order in the dark pool. The bottom panels show the optimal inventory and trading speed where we assume that the dark pool matching order arrives at $t = 0.6$, right after which inventory drops to zero.

Figure: Figure 7.6

Figure 7.6 (Contd ...)

1. Figure 7.6: We plot the optimal liquidation rate in the lit market given in (3) for different levels of the rate of arrival of matching orders in the dark pool.
2. The figure shows that the trading rate may be larger or smaller than the AC case, and it may be increasing or decreasing.
3. Also, the optimal inventory to hold (up to the time at which a matching order arrives) may be either convex or concave or neither.
4. In particular, the top two panels of Figure 7.6 show the optimal inventory path and optimal speed of trading, where we also assume that the liquidation penalty at time T is $\alpha \to \infty$, and the paths shown are prior to the order posted in the dark pool being executed.
5. Other model parameters are $k = 0.001$ and $\phi = 0.01$.
6. The bottom two panels of the figure show the case where the order in the dark pool was executed at time $t = 0.6$ and the agent's inventory drops to zero.

Figure 7.6 (Contd ...)

1. Since the execution of the orders in the dark pool occur according to a Poisson process with intensity λ, the time at which this occurs is exponentially distributed with mean $1/\lambda$.
2. Thus, when $\lambda = 0$, there are no executions in the dark pool and the liquidation strategy corresponds to the AC solution.
3. When $\lambda > 0$, the agent starts trading slower than the AC speed in the lit market, to allow for the potential of dark pool execution, but then as time runs out and no execution occurs, her/his rate of trading increases to compensate for the initially slow trading.
4. Interestingly, the optimal trading curve ceases to be convex and its convexity changes signs.
5. In the limiting case, when $\lambda \to \infty$, the agent does not trade at all in the lit market, since execution in the dark pool is guaranteed.
6. In this case, the optimal inventory path flows along $Q_t^* = \mathfrak{R}\mathbb{1}_{\{t<T\}}$, but is then infinitely fast at T to rid herself/himself of the assets.