ME 620: Fundamentals of Artificial Intelligence

Lecture 17: First Order Logic – Part III

Shyamanta M Hazarika
Biomimetic Robotics and Artificial Intelligence Lab
Mechanical Engineering and M F School of Data Sc. & AI
IIT Guwahati

First-order Logic

- In the PREVIOUS Lecture on First-order Logic, we discussed
  - Syntax of First-order Logic
  - Translating to LOGIC statements
  - Negation of Quantified Sentences
  - Distributivity of the Quantifiers over $\land$ and $\lor$

- The treatment of semantics, however, was quite informal; in THIS Lecture, we provide
  - Precise definition of meaning called Declarative Semantics
  - Understand *conceptualization.*
  - Blocks World Example; A Simple Genealogy KB.

Conceptualization

We have argued that intelligent behaviour depends on knowledge an entity has about its environment.

- Declarative Knowledge

  Type of knowledge that is, by its very nature, expressed in declarative sentences or indicative propositions

  - Much of the knowledge of the environment is descriptive and can be expressed in declarative form.

- Formalization of knowledge in a declarative form begins with a *conceptualization.*
  - Includes the objects presumed or hypothesized to exist in the world and their interrelationships.
  - Objects can be anything about which we want to say something!

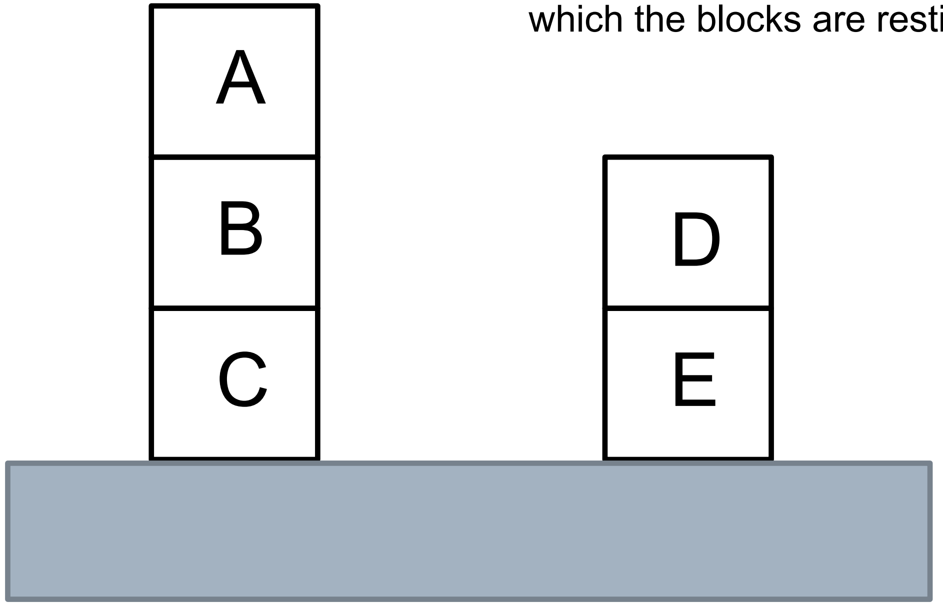
Conceptualization

Blocks World

Not all knowledge representation tasks require that we consider all the objects in the world.

Universe of Discourse – set of objects about which knowledge is expressed.

Apart from the BLOCKS; many conceptualize the TABLE on which the blocks are resting as an object as well.



BLOCKS WORLD scene

In this example, there are finitely many elements in our universe of discourse. This need not always be the case.

Universe of discourse = {A, B, C, D, E}

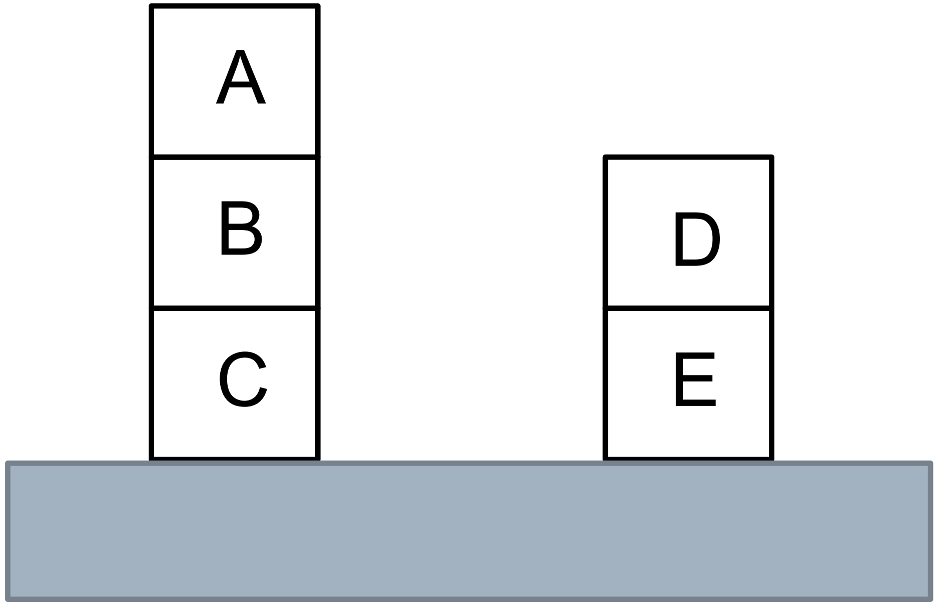
It is common in MATHEMATICS for example to consider the set of ALL INTEGERS as universe with infinitely many elements.

Conceptualization

Blocks World

Function – one kind of interrelationship among objects in a universe of discourse.

Many functions could be defined; The set of functions emphasized in an conceptualization is called the **functional basis set**.



BLOCKS WORLD scene

In this example, it would make sense to conceptualize the partial function *hat* that maps a block into block on top of it, if any exists

Tuples corresponding to the *hat* function

hat: { ⟨B, A ⟩,⟨C, B⟩ ,⟨E, D⟩ }

Conceptualization

Blocks World

Relation – second kind of interrelationship among objects in a universe of discourse.

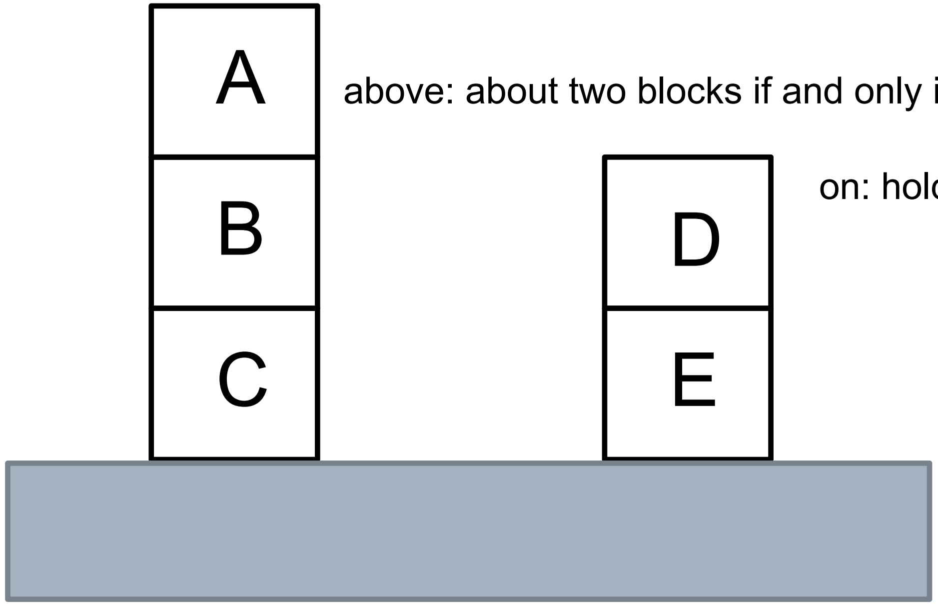
Many relations could be defined; The set of relations emphasized in an conceptualization is called the **relational basis set**.

clear: to mean no block is on top of the other block

In a spatial configuration of the Blocks World, there are a number of meaningful relations.

above: about two blocks if and only if one is above the other.

on: holds if and only if one is immediately above the other.



BLOCKS WORLD scene

For the scene, elements corresponding to the different relations are

on: {⟨A, B⟩, ⟨B, C⟩, ⟨D, E⟩ }

above: {⟨A, B ⟩, ⟨B, C⟩ , ⟨A, C ⟩, ⟨D, E⟩ }

clear: {⟨A, D⟩}

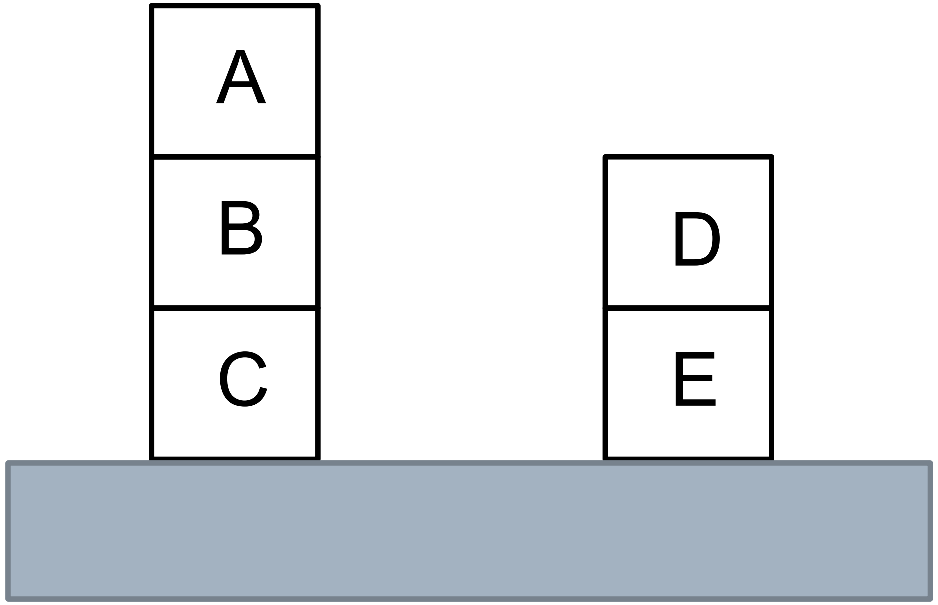
table: {⟨C, E⟩}

Conceptualization

Blocks World

The generality of relations can be determined by comparing their elements.

Many relations could be defined; The set of relations emphasized in an conceptualization is called the **relational basis set**.



BLOCKS WORLD scene

The `on’ relation is less general than the `above’ relation; when viewed as a set of tuples it is subset of the `above’ relation.

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For the scene, elements corresponding to the different relations are

on: {⟨A, B⟩, ⟨B, C⟩, ⟨D, E⟩ }

above: {⟨A, B ⟩, ⟨B, C⟩ , ⟨A, C ⟩, ⟨D, E⟩ }

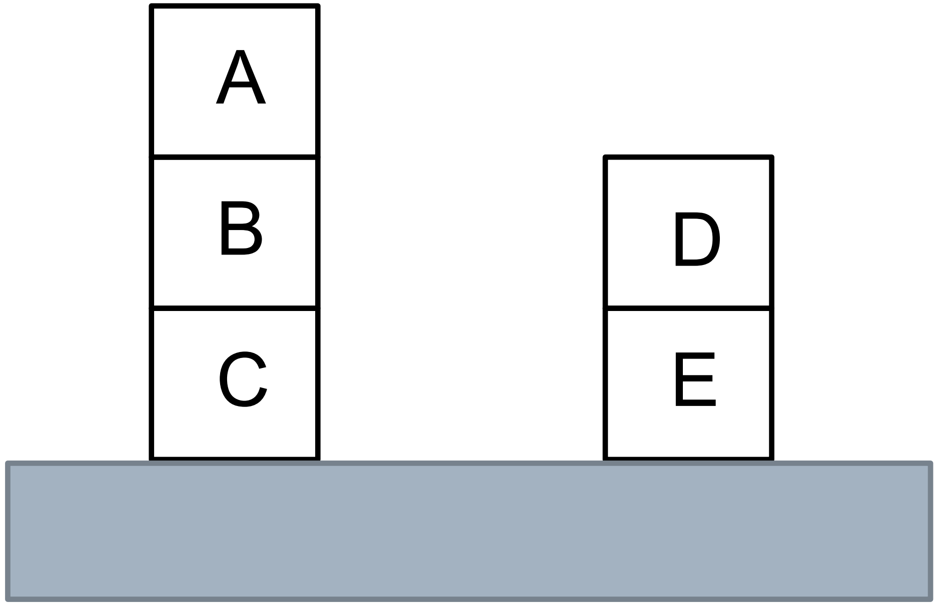
clear: {A, D}

table: {C, E}

Conceptualization

Blocks World

Formally, a **conceptualization** is a triple consisting of a universe of discourse, a functional basis set for that universe of discourse, and a relational basis set.



BLOCKS WORLD scene

The following is one conceptualization of the BLOCKS world here,

$\langle$ {A, B, C, D, E},
{hat},
{on, above, clear, table} $\rangle$

Although we have written names of objects, functions and relations here, the conceptualization consists of the objects, functions and relations themselves.

Conceptualization

What makes one conceptualization better than another?

No comprehensive answer!

Blocks World

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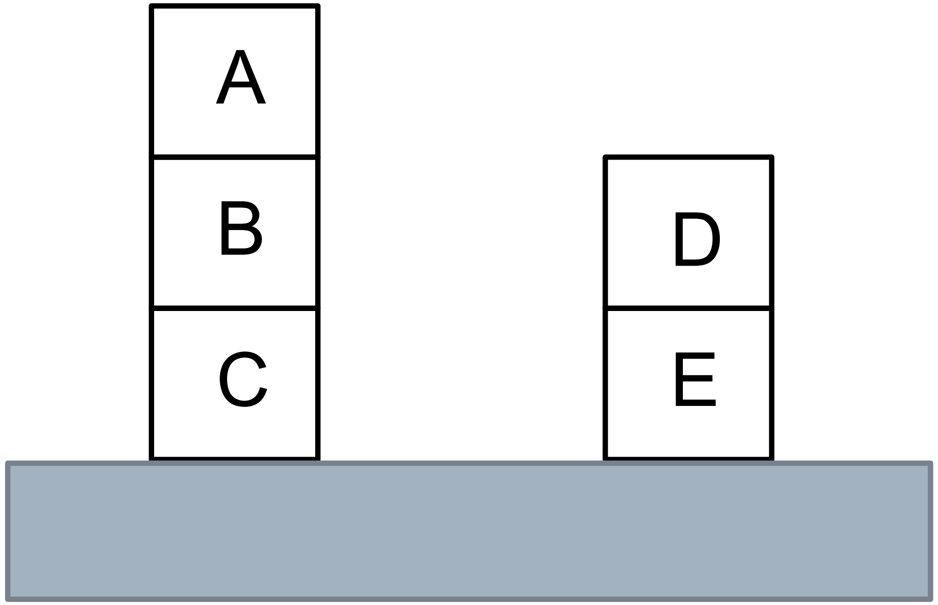


BLOCKS WORLD scene

The following is one conceptualization of the BLOCKS world here,

$$\langle \{A, B, C, D, E\},\ \{hat\},\ \{on, above, clear, table\}\rangle$$

Noteworthy issues include – GRANULARITY or grain size. Choosing too small a grain size can make knowledge representation tedious. E.g. Think of objects in U here as atoms!

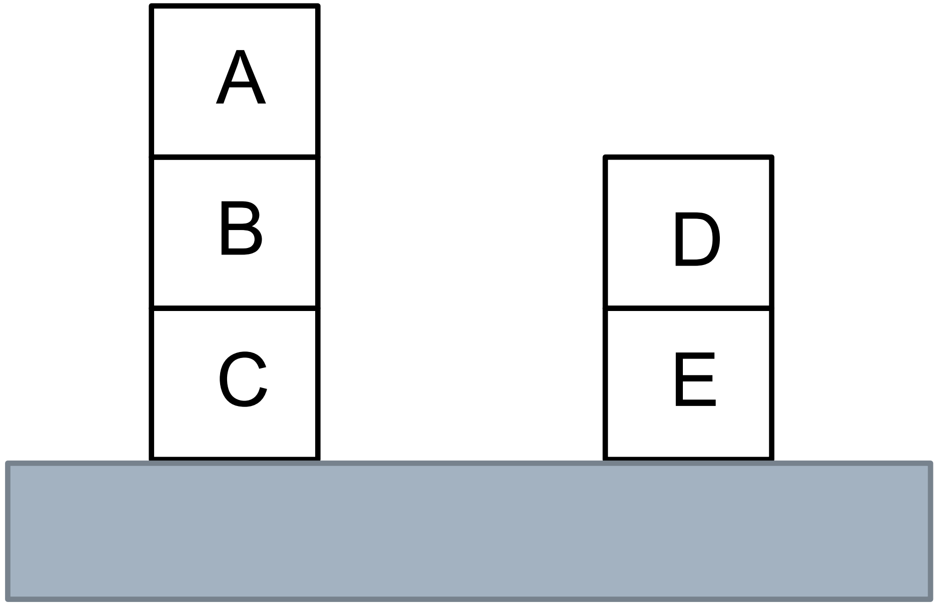
Conceptualization

Blocks World

What makes one conceptualization better than another?

No comprehensive answer!

Formally, a **conceptualization** is a triple consisting of a universe of discourse, a functional basis set for that universe of discourse, and a relational basis set.



BLOCKS WORLD scene

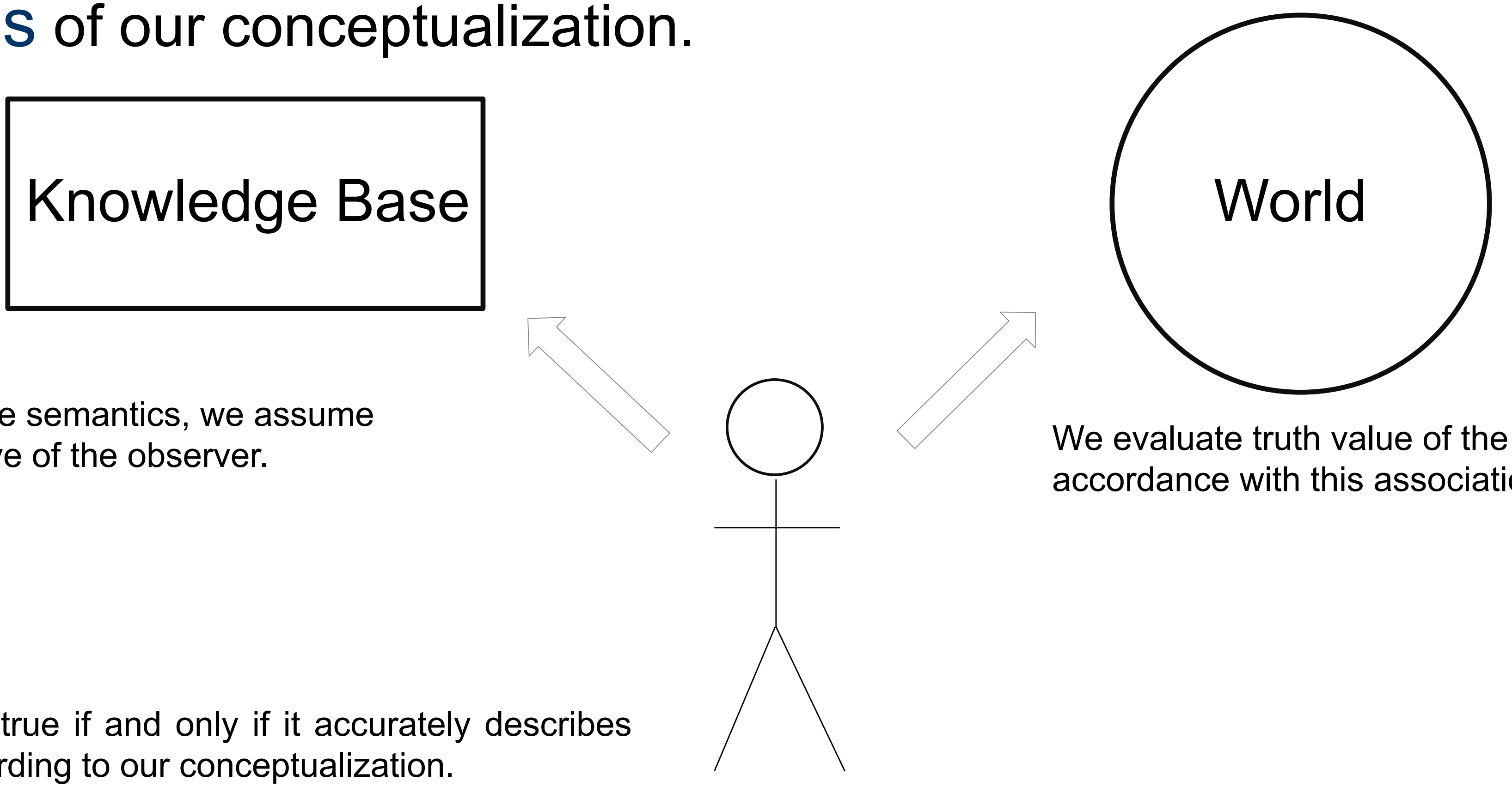
The following is one conceptualization of the BLOCKS world here,

⟨{A, B, C, D, E},
{hat},
{on, above, clear, table}⟩

Noteworthy issues include – GRANULARITY or grain size. Choosing too large a grain size can make knowledge representation impossible. E.g. Think of chemist interested in the objects in U here!

Declarative Semantics

We have a set of sentences and a conceptualization of the world; we associate symbols used in the sentences with objects, functions and relations of our conceptualization.



For declarative semantics, we assume the perspective of the observer.

We evaluate truth value of the sentences in accordance with this association

A sentence is true if and only if it accurately describes the world according to our conceptualization.

Interpretation

Definition: An **interpretation** I is a mapping between elements of the language and elements of a conceptualization. The mapping is represented by the function $I(\sigma)$, where σ is an element of the language. Abbreviate $I(\sigma)$ to σ^I ; the universe of discourse is represented as $|I|$.

For I to be an interpretation, it must satisfy the following properties.

used to name a specific element of a universe of discourse.

1. If σ is an object constant, then $\sigma^I \in |I|$.

used to designate a function on members of the universe of discourse.

2. If π is an n-ary function constant, then $\pi^I : |I|^n \rightarrow |I|$.

used to name a relation on the universe of discourse.

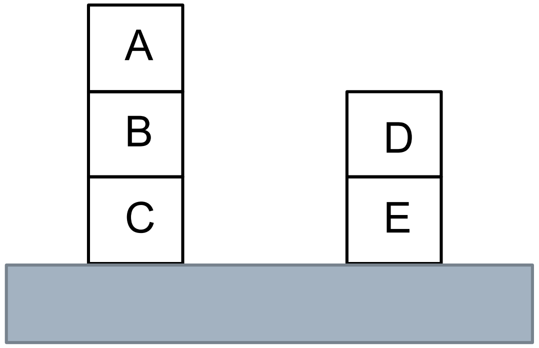
3. If ρ is an n-ary relation constant, then $\rho^I \subseteq |I|^n$.

Interpretation

Blocks World

Predicate-calculus language has the five object constants: A, B, C, D, AND E.

The following mapping correspond to our usual interpretation for these symbols.



BLOCKS WORLD scene

A^I = A

B^I = B

Function constant hat; Relational constant on, above, clear and table

C^I = C

This is the intended interpretation; the one suggested by the names of the constant.

D^I = D

These constants can equally well be interpreted in other ways!

E^I = E

hat^I = {⟨B, A ⟩,⟨C, B⟩ ,⟨E, D⟩ }

$table^I$ ={C,E}.

on^I = {⟨A, B⟩, ⟨B, C⟩, ⟨D, E⟩ }

$clear^I$ ={D,A}.

$above^I$ = {⟨A, B ⟩, ⟨B, C⟩ , ⟨A, C ⟩, ⟨D, E⟩ }

Interpretation

Definition: A **variable assignment** U is a function from the variables of a language to objects in the universe of discourse.

Example: In the Blocks World Example

$$x^U = A$$
$$y^U = A$$
$$z^U = B$$

Definition: Given an interpretation I and a variable assignment U, the **term assignment** T_{IU} corresponding to I and U is a mapping from terms to objects.

Example: For above U, term hat(C) designates block B. I maps C to block C and tuple ⟨C, B⟩ is a member of the function designated by hat.

Satisfiability

- The notions of interpretation and variable assignment are important because they allow us to define a relative notion of truth called **satisfaction**.
  - The fact that a sentence ϕ is satisfied by an interpretation I and a variable assignment U is written as $\vDash_I \phi(U)$.

    $A^I = A$; $B^I = B$; $\langle A, B\rangle \in On^{I}$; we can write $\vDash_I On(A,B)[U]$.

  - We say that the sentence ϕ is true relative to the interpretation I and the assignment U.

The definitions for *satisfaction* differs from one type of sentence to another. We have highlighted the main idea; working through each of the type of sentence is left for the read as self-study.

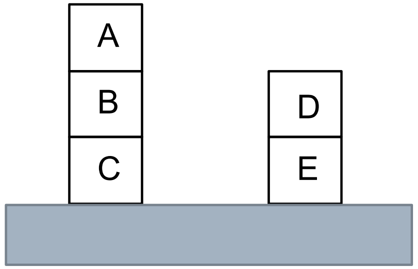
Satisfiability

Satisfiability is also dependent on interpretation. Under some interpretation a sentence could be true; under other interpretations, it can be false.

- The satisfiability of logical sentences depends on the logical operators involved.
  - Universally quantified sentence is satisfied if and only if the enclosed statement is satisfied for all assignments of the quantified variable.
  - Existentially quantified sentence is satisfied if and only if the enclosed statement is satisfied for some assignments of the quantified variable.

Model

Definition: If an interpretation I satisfies a sentence ϕ for all variable assignments, then I is said to be a **model** of ϕ, written $\vDash_I \phi$.



BLOCKS WORLD scene

Consider: on(x,y) → above(x,y)

Interpretation I from our Blocks World example is a model of the sentence.

Consider the variable assignment U that maps x to block A and y to block B; under this assignment, on(x,y) and above(x,y) are both satisfied, They satisfy the implication. As an alternative consider variable assignment U that maps x and y to block A. Under this above(x,y) is not satisfied; but neither is on(x,y). The implication is satisfied.

Model

We can easily extend the definitions to set of sentences.

Definition: A set Γ of sentences is satisfied by an interpretation I and a variable assignments, written as $\vDash_I \Gamma(U)$, if and only if every member of Γ is satisfied by I and U.

Definition: An interpretation I is a model of a set $\Gamma$of sentences, written as $\vDash_I \Gamma$, if and only if it is a model of every member of Γ.

Knowledge Representation

- Conceptualization is followed by selecting a vocabulary of object constants, function constants and relation constants.
  - Associate these constants with the objects, functions and relations in our conceptualization.
- Write sentences to constitute the machine's declarative knowledge.
  - It is generally true that as one writes more sentences the number of possible models decreases.

Is it possible to define symbols so thoroughly that no interpretation is possible except the one intended?

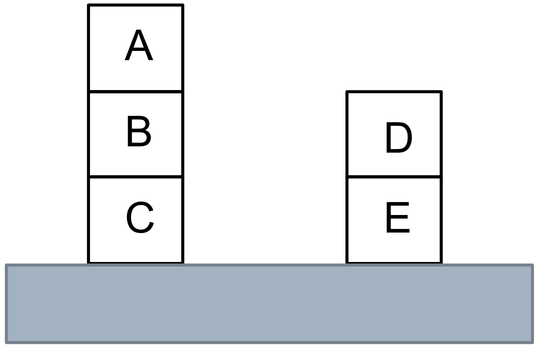
  - However there is no way in general of ensuring a unique interpretation, no matter how many sentences we write!

Knowledge Representation

Blocks World Example

Essential Information

on(A,B)	above(A,B)	clear(A)
on(B,C)	above(B,C)	clear(D)
on(D,E)	above(A,C)	table(C)
	above(D,E)	table(E)



BLOCKS WORLD scene

Note that all of these sentences are true under the intended interpretation.

Knowledge Representation

Blocks World Example

Essential Information

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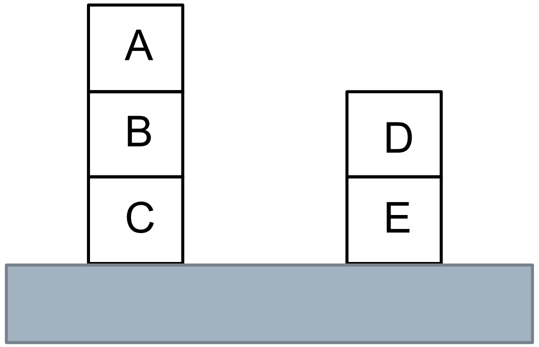


BLOCKS WORLD scene

Note that all of these sentences are true under the intended interpretation.

Encode some more general facts.

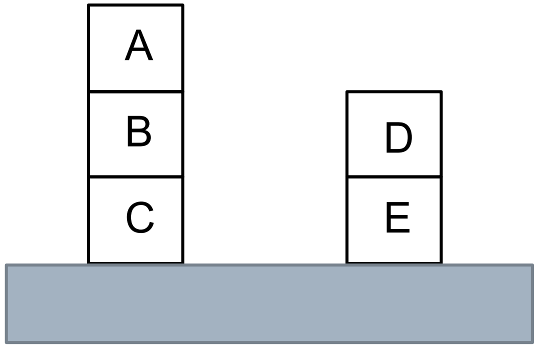
General Sentences

$$\forall x\ \forall y(on(x,y) \rightarrow above(x,y))$$

If one block is on another block; then that block is above the other.

Knowledge Representation

Blocks World Example



BLOCKS WORLD scene

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Note that all of these sentences are true under the intended interpretation.

Encode some more general facts.

If one block is on another block; then that block is above the other.

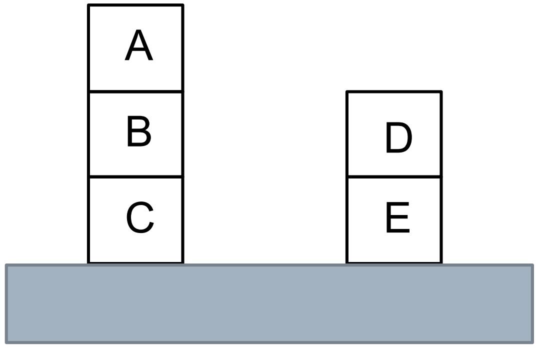
General Sentences

$\forall x\ \forall y (on(x,y) \rightarrow above(x,y))$

$\forall x \forall y \forall z\ (\ above(x,y) \land above(y,z) \rightarrow above(x,z))$

Knowledge Representation

Blocks World Example



BLOCKS WORLD scene

Essential Information

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	above(D,E)	table(E)

Note that all of these sentences are true under the intended interpretation.

Encode some more general facts.

General Sentences

If one block is on another block; then that block is above the other.

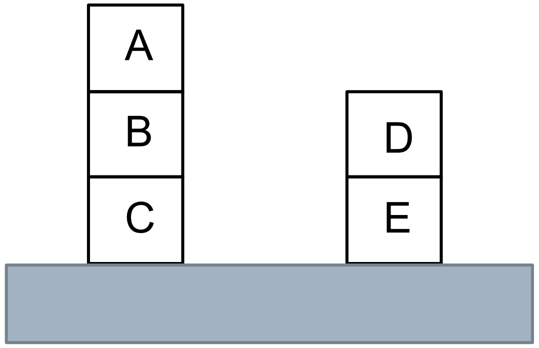
$$\forall x\ \forall y(on(x,y) \rightarrow above(x,y))$$

$$\forall x \forall y \forall z\ (\ above(x,y) \wedge above(y,z) \rightarrow above(x,z))$$

The relation `Above(x,y) is transitive. If one block is above a second, and second is above a third, then the first is also above the third.

Knowledge Representation

Blocks World Example



BLOCKS WORLD scene

Essential Information

on(A,B)	above(A,B)	clear(A)
on(B,C)	above(B,C)	clear(D)
on(D,E)	above(A,C)	table(C)
	above(D,E)	table(E)

Note that all of these sentences are true under the intended interpretation.

Encode some more general facts.

An advantage of writing such general statements is economy!

General Sentences

Record information on ON and encode relation between ON and above; No need to have and ABOVE information explicitly.

$\forall x\ \forall y(on(x,y) \rightarrow above(x,y))$

$\forall x \forall y \forall z\ (\ above(x,y) \wedge above(y,z) \rightarrow\ above(x,z))$

These general statements ALSO apply to Blocks World scenes other than the one pictured here. It is possible to have NONE of the specific sentences TRUE but the general statements are still correct.

Knowledge Representation

A Simple Genealogy KB

- **Axioms** are facts and rules that attempt to capture all of the (important) facts and concepts about a domain; axioms can be used to prove **theorems**
  - Mathematicians don't want any unnecessary (dependent) axioms –ones that can be derived from other axioms
  - Dependent axioms can make reasoning faster, however
  - Choosing a good set of axioms for a domain is a kind of design problem
- A **definition** of a predicate is of the form "$P(x) \leftrightarrow \ldots$." and can be decomposed into two parts
  - Necessary description: $P(x) \rightarrow$
  - Sufficient description: $P(x) \leftarrow$

Knowledge Representation

A Simple Genealogy KB

Define father(x, y) from parent(x, y) and male(x)

- parent(x, y) is a necessary (**but not sufficient**) description of father(x, y)

  $$father(x, y) \rightarrow parent(x, y)$$

- parent(x, y); male(x) AND age(x, 35) is a **sufficient** (**but not necessary**) description of father(x, y)

  $$father(x, y) \leftarrow (parent(x, y) \wedge male(x) \wedge age(x, 35))$$

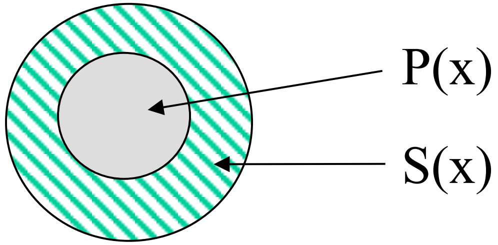
- parent(x, y) AND male(x) is a **necessary and sufficient** description of father(x, y)

  $$father(x, y) \leftrightarrow (parent(x, y) \wedge male(x))$$

Knowledge Representation

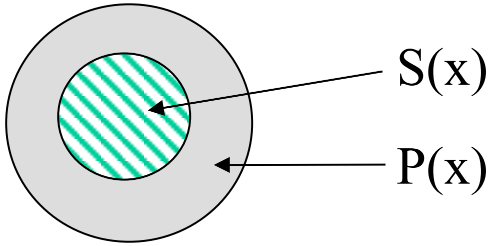
A Simple Genealogy KB

S(x) is a necessary condition of P(x)



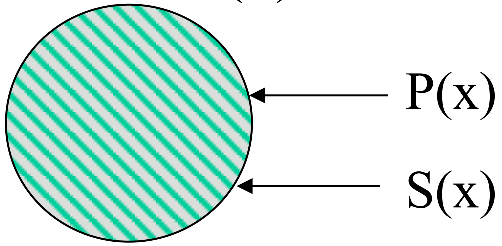
$$\forall x\ (P(x) \rightarrow S(x))$$

S(x) is a sufficient condition of P(x)



$$\forall x\ (P(x) \leftarrow S(x))$$

S(x) is a necessary and sufficient condition of P(x)



$$\forall x\ (P(x) \leftrightarrow S(x))$$

Knowledge Representation

A Simple Genealogy KB

Predicates:

- parent(x, y)
- child(x, y)
- father(x, y)
- daughter(x, y)
- spouse(x, y)
- husband(x, y); wife(x,y)
- ancestor(x, y); descendant(x, y)
- male(x); female(y)
- relative(x, y)

Knowledge Representation

A Simple Genealogy KB

- $\forall x \, \forall y \; parent(x,y) \leftrightarrow child(y,x)$
- $\forall x \forall y \; father(x,y) \leftrightarrow (parent(x, y) \wedge male(x))$

  similarly for mother(x, y)
- $\forall x \forall y \; daughter(x,y) \leftrightarrow (child(x, y) \wedge female(x))$

  similarly for son(x,y)
- $\forall x \forall y \; husband(x,y) \leftrightarrow (spouse(x, y) \wedge male(x))$

  similarly for wife(x,y)

Knowledge Representation

A Simple Genealogy KB

- $\forall x \forall y$ parent(x, y) $\rightarrow$ ancestor(x, y)
- $\forall x \forall y \exists z$ parent(x, z) $\wedge$ ancestor(z, y) $\rightarrow$ ancestor(x, y)
- $\forall x \forall y$ descendant(x, y) $\leftrightarrow$ ancestor(y, x)
- $\forall x \forall y \exists z$ ((ancestor(z, x) $\wedge$ ancestor(z, y)) $\rightarrow$ relative(x, y))
  related by common ancestry
- $\forall x \forall y$ spouse(x, y) $\rightarrow$ relative(x, y)
  related by marriage

Knowledge Representation

- While representing declarative knowledge we write sentences we believe to be true; ones that are satisfied by our intended interpretation.

  Intended interpretation is the model of the sentences we write.

- In describing a domain, we seldom start with a complete conceptualization.
  - Rarely list the tuples for every function and relation.
  - Start with an idea of a conceptualization and attempt to make it precise by adding more sentences.
  - Many of these sentences are reduntant; they are entitled by the preceding sentences. This is part of the notion of logical entitlement.