Pg (1)

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MA-473 Midsem-Exam Paper-(2)

(5)

TO Show: $S_f(t) < \lim_{t \to T} S_f(t) = \min\left(K, \frac{r}{\delta}K\right)$

Proof: for $t = T$, value of V_P^{Am} equals payoff.

i.e. $V_P^{Am}(S,T) = K - S$. for $S < K$

Putting this in B.S. Eq^n, we get

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 (0) + (r-\delta) S(-1) - rV = 0$$

$$\left\{ \frac{\partial V}{\partial S} = \frac{\partial}{\partial S}(K-S) = -1 \ , \ \frac{\partial^2 V}{\partial S^2} = 0 \right\}$$

$$\Rightarrow \frac{\partial V}{\partial t} + 0 - (r-\delta) S - r(K-S) = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} V(S,T) = rK - \delta S} \quad -(*)$$

Now we have

$$\boxed{\frac{\partial V(S,T)}{\partial t} \leq 0} \rightarrow$$ bcoz if this does not happen then for t ~~cat~~ close to T a contradiction occurs for result $\boxed{V \geq \text{Payoff}}$ for American options

(#)

Using * and #, for $t = T$ and $S < K$, we have

$$rK - \delta S \leq 0 \Rightarrow \boxed{S \geq \frac{r}{\delta}K}$$

Now this is meaningfull only when $\delta > r$.

Case-I : $\boxed{\delta > r}$

Then

$$S_f(T) = \lim_{\substack{t \to T \\ t < T}} S_f(t) \text{ satisfies } \boxed{S_f(T) = \frac{r}{\delta}K}$$

To prove this we will show that other two scenarios viz

(i) $S_f(T) < \frac{r}{\delta}K$

(ii) $S_f(T) > \frac{r}{\delta}K$

are not possible

(i) $\exists\ S$ s.t. $S_f(T) < S < \frac{r}{\delta}K$.

Then $\frac{\partial V(S,T)}{\partial t} = rK - \delta S > 0$

which contradicts that $\frac{\partial V(S,T)}{\partial t} \leq 0$.

Hence this scenario is NOT possible.

(ii) Here $\exists S$ s.t $\frac{\gamma}{\delta}K < S < S_f(T)$

Then $\gamma K < \delta S$.

We have for small time dt : $\boxed{K(e^{\gamma dt}-1) < S(e^{\delta dt}-1)}$

This means dividend earns more than interest on K and early exercise is NOT optimal which contradicts the meaning of $S < S_f(T)$

Hence this scenario is also NOT possible.

Thus we have $\boxed{S_f(T) = \frac{\gamma}{\delta}K \quad \text{for } \delta > \gamma}$

Case-II : $\delta \leq \gamma$

We have by def.^{n} of S_f that

$S_f(T) > K$ cannot occur.

Hence we assume $\boxed{S_f(T) < K}$

then for $S_f(T) < S < K$ and $t \approx T$, we have.

$$\frac{dV}{dt} = \gamma K - \delta S \qquad \left\{ \begin{array}{l} \gamma \geq \delta. \\ \Rightarrow \gamma K \geq \delta K > \delta S. \\ \Rightarrow \gamma K > \delta S. \\ \Rightarrow \gamma K - \delta S > 0 \end{array} \right.$$

We have LHS $= \frac{dV}{dt} \leq 0$

and RHS $= rK - \delta S > 0$

which leads to contradiction

Hence we have

$$\boxed{S_f(T) = K \text{ for } \delta \leq r}$$

Combining result of case-I and case-II, we have

$$\boxed{S_f(t) < \lim_{\substack{t \to T \\ t < T}} S_f(t) = \min\left(K, \frac{r}{\delta} K\right)}$$

$\Downarrow$

For $\boxed{r = 0}$ American Put is identical to European Put

i.e. Early Exercise will never be optimal.

(4) Cryer Problem $\begin{cases} \text{find vectors } x \text{ and } y \text{ s.t. for } \hat{b} = b - Ag. \\ Ax - y = \hat{b} \;,\; y \geq 0 \;,\; x^T y = 0 \end{cases}$

is Equivalent to

Min Problem $\begin{cases} \min\limits_{x \geq 0} G(x) \;, \text{ where } G(x) = \frac{1}{2}(x^T A x) - \hat{b}^T x \\ \text{is strictly convex.} \end{cases}$

Proof: Consider $G(x) = \frac{1}{2}(x^T A x) - \hat{b}^T x$.

Then. $G_x = Ax - \hat{b}$

$G_{xx} = A$ (Hessian Matrix)

Now we have the structure of A as follows.

$$A = \begin{pmatrix} 1+2\lambda\theta & -\lambda\theta & \cdots & 0 \\ -\lambda\theta & 1+2\lambda\theta & -\lambda\theta & \cdots 0 \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots \end{pmatrix}$$

i.e. A is tri-diagonal matrix and

θ decides which of FTCS, BTCS or CN we are using.

$\Rightarrow$ Eigen values of A are given by

$$\mu_K = (1+2\lambda\theta) + 2(-\lambda\theta)\sqrt{\frac{-\lambda\theta}{-\lambda\theta}}\cos\left(\frac{K\pi}{N+1}\right)$$

for $K = 1, 2, --, N$

$$= 1+2\lambda\theta - 2\lambda\theta\cos\left(\frac{K\pi}{N+1}\right)$$

$$= 1+2\lambda\theta\left\{1-\cos\left(\frac{K\pi}{N+1}\right)\right\}$$

$$= 1+2\lambda\theta \cdot 2\sin^2\left(\frac{K\pi}{2(N+1)}\right)$$

$$\boxed{\mu_K = 1+4\lambda\theta\sin^2\left(\frac{K\pi}{2(N+1)}\right), \quad K = 1, 2, \cdots N}$$

$\Rightarrow$ We clearly have μ_K is positive for all K.

i.e. A has positive eigen values.

Also Hessian matrix $G_{xx} = A$ is symmetric

Thus G_{xx} is Positive Definite $\Rightarrow$ G is strictly convex.

Since G is strictly convex, it has unique minimum on each convex set in $\mathbb{R}^n$, Ex: $x \geq 0$ (or $-x \leq 0$)

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KKT - Theorem

KKT - theorem minimizes G under conditions

$H_i(x) \leq 0$ for $i = 1, --, m$

According to KKT - Theorem, for a vector x_0 to be a minimum is equivalent to existence of Lagrange Multipliers. $y \geq 0$, s.t.

$$\boxed{\operatorname{grad} G(x_0) + \left(\frac{\partial H(x_0)}{\partial x}\right)^{tr} y = 0, \quad y^{tr} H(x_0) = 0}$$

Now we want to minimize $G(x)$ over set $x \geq 0$ (or $-x \leq 0$). Hence accordingly we can define $H(x) = -x$.

Then $\operatorname{grad} G(x) = Ax - \hat{b}$

$$\left(\frac{\partial H(x)}{\partial x}\right)^{tr} = (-I)^{tr}$$

Hence KKT conditions become

$$\boxed{Ax - \hat{b} + (-I)^{tr} y = 0, \quad y^{tr} x = 0}$$

Finally we have

$$\boxed{Ax - \hat{b} = y, \quad y^{tr} x = 0, \quad x \geq 0, \quad y \geq 0}$$

which is Cryer's Problem.

(3) PDE : $u_t = u_{xx}$

In FTCS scheme, we approximate u_t using Forward time approximation and u_{xx} using Central space approximation

$$u_t = \frac{u_{i,n+1} - u_{i,n}}{\Delta \tau} + O(\Delta \tau)$$

$$u_{xx} = \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{\Delta x^2} + O(\Delta x^2)$$

Ignoring the error terms, we can write

$u_t = u_{xx}$ as

$$\frac{w_{i,n+1} - w_{in}}{\Delta \tau} = \frac{w_{i+1,n} - 2w_{i,n} + w_{i-1,n}}{\Delta x^2}$$

for approximation w

$$\Rightarrow \boxed{w_{i,n+1} = w_{in} + \frac{\Delta \tau}{\Delta x^2}\left(w_{i+1,n} - 2w_{in} + w_{i-1,n}\right)}$$

taking $\lambda = \frac{\Delta \tau}{\Delta x^2}$, we get

$$\boxed{w_{i,n+1} = \lambda w_{i-1,n} + (1-2\lambda) w_{in} + \lambda w_{i+1,n}}$$

For $v = 0$, we have I.C. (say $y(x)$)

$$\boxed{w_{i0} = y(x_i, 0)} \qquad 0 \le i \le m$$

For Boundary conditions, we take $\boxed{w_{0n} = w_{mn} = 0}$

Now we can combine Eq^n s at all time steps into a matrix Eq^n as.

$$\boxed{w^{(n+1)} = A w^{(n)}} \qquad \text{for } n = 0, 1, 2, \ldots$$

where $w^{(n)} = (w_{1n}, w_{2n}, \ldots, w_{m-1,n})^{tr}$

and $A = \begin{pmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & \lambda & 1-2\lambda \end{pmatrix}$

we can rewrite A as

$$A = I - \lambda \underbrace{\begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix}}$$

Consider this matrix as T.

Then the eigen-values of G are:

$$l^{G}_{1K} = 2 - 2\sqrt{\frac{-1}{-1}}\cos\left(\frac{K\pi}{N+1}\right), \quad K = 1, 2, \ldots, N$$

$$\Rightarrow l^{G}_{1K} = 2 - 2\cos\left(\frac{K\pi}{N+1}\right)$$

$$\boxed{l^{G}_{1K} = 4\sin^2\left(\frac{K\pi}{2(N+1)}\right)}$$

Hence we have $A = I - \lambda G$.

$\Rightarrow$ eigen values of A are

$$\boxed{l^{A}_{1K} = 1 - \lambda \cdot 4\sin^2\left(\frac{K\pi}{2(N+1)}\right)}$$

Now for the stability of FTCS scheme, we ~~have~~ need

$$|l^{A}_{1K}| < 1$$

$$\Rightarrow \left|1 - 4\lambda\sin^2\left(\frac{K\pi}{2(N+1)}\right)\right| < 1, \quad K = 1, 2, \ldots, m-1$$

$$\Rightarrow -1 < 1 - 4\lambda\sin^2\left(\frac{K\pi}{2(N+1)}\right) < 1, \quad K = 1, 2, \ldots, m-1$$

$$\Rightarrow -1 < 1 - 4\lambda\sin^2\left(\frac{K\pi}{2(N+1)}\right) \text{ and } \lambda > 0.$$

We have $-1 < 1 - 4\lambda \sin^2\left(\frac{k\pi}{2(N+1)}\right)$

$$\Rightarrow \boxed{\frac{1}{2} > \lambda \sin^2\left(\frac{k\pi}{2(N+1)}\right)} \quad k = 1, 2, \ldots, m-1$$

Now the largest Sin-term is $\sin\left(\frac{(m-1)\pi}{2m}\right)$

for increasing m tends monotonically to 1

we have.

$$\boxed{0 < \lambda \le \frac{1}{2} \text{ is required for } w^{(n+1)} = Aw^{(n)} \text{ to be Stable}}$$

$$\Downarrow$$

$$0 < \frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2} \Rightarrow \boxed{0 < \Delta\tau \le \frac{\Delta x^2}{2}}$$