# Homework-2

**MA423 :** Matrix Computations 2023 R. Alam

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### Block matrices and outer product

1. Let $X = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \end{bmatrix}$ and $Y := \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$.

   (a) Compute the outer product expansion of $XY^\top$.

   (b) Compute the outer product expansion of $YX^\top$. How is the outer product expansion of $YX^\top$ related to the outer product expansion of $XY^\top$?

2. Let $U := \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \in \mathbb{R}^{m\times m}$ and $V := \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$. Let

$$S := \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_p & \\ \hline & 0 & & 0 \end{array}\right] \in \mathbb{R}^{m\times n}$$

   be a diagonal matrix, where $\sigma_1, \ldots, \sigma_p$ are nonzero real numbers. Show that $A = USV^\top$ can be expressed as an outer product expansion of the form

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\top + \cdots + \sigma_p \mathbf{u}_p \mathbf{v}_p^\top.$$

3. Let $A := \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ be a block upper triangular matrix where each block is an $n \times n$ matrix. If $A_{11}$ and $A_{22}$ are nonsingular, then show that $A$ must also be nonsingular and that $A^{-1}$must be of the form $A^{-1} = \begin{bmatrix} A_{11}^{-1} & C \\ 0 & A_{22}^{-1} \end{bmatrix}$. Determine $C$.

4. Let $A$ and $B$ be $n \times n$ matrices and define $2n \times 2n$ matrices $S$ and $M$ by

$$S = \begin{bmatrix} I_n & A \\ 0 & I_n \end{bmatrix} \text{ and } M := \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}.$$

   Determine the block form of $S^{-1}$ and use it to compute the block form of the product $S^{-1}MS$.

5. Let $A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be such that $A_{11}$ is an $m \times m$ nonsingular matrix and $A_{22}$ is an $n \times n$ matrix. Show that $A$ can be factored as a product of block matrices

$$A = \begin{bmatrix} I_m & 0 \\ B & I_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

   where $B := A_{21}A_{11}^{-1}$ and $S := A_{22} - A_{21}A_{11}^{-1}A_{12}$. The matrix $S$ is called the Schur complement of $A_{11}$ in $A$. Show that $A$ is nonsingular $\iff$ $S$ is nonsingular. Also show that $\det(A) = \det(A_{11})\det(S)$.

6. Let $B$ be an $n \times n$ matrix such that $B^2 = 0$. Consider the block matrix $A = \begin{bmatrix} 0 & I_n \\ I_n & B \end{bmatrix}$. Determine the block form of $A^{-1} + A^2 + A^3$.

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