MA423 Matrix Computations

Lecture 11: Perturbation Theory for Linear Systems

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Outline

- Condition numbers
- Perturbation and sensitivity analysis of linear systems

Condition number

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Notice that columns A are orthogonal whereas columns of B are nearly linearly dependent. Indeed, $\cos\theta = \langle Be_1, \, Be_2 \rangle / \|Be_1\|_2 \|Be_2\|_2 = 10^{10}/\sqrt{1 + 10^{20}} \simeq 1$.

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Remark: There is a ΔA such that $\|\Delta A\| \, \|A^{-1}\| = 1$ and $A + \Delta A$ is singular. In other words, the distance to nearest singular matrix $\propto \dfrac{1}{\mathrm{cond}(A)}$.

Sensitivity analysis of linear systems

Consider the linear system

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}}_{\text{Hilbert matrix H}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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To see what this means, set $x := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top \in \mathbb{R}^n$ and define $b := Hx$. Then x is the solution of $Hx = b$.

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Now we use MATLAB to solve the linear system and compare the computed solution with the known solution x.

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The Hilbert matrix is SPD but the computed solutions differ drastically from true solutions. Is it the fault of the algorithm?

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$\dfrac{\|x - \hat{x}\|}{\|x\|} \leq \mathrm{cond}(A)\dfrac{\|r\|}{\|b\|}$.

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Observe that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$ and $A^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1997 \\ -1999 \end{bmatrix}$.

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$$\mathrm{cond}(H_n) = \frac{\sigma_{\max}(H_n)}{\sigma_{\min}(H_n)} \approx \frac{\pi}{\sigma_{\min}(H_n) + \mathsf{eps}} \approx \frac{\pi}{\mathsf{eps}}.$$

Now $\pi/\mathsf{eps} = 1.4148e+16$ and $\mathrm{cond}_2(H_{13}) = 4.7864e+17$.

Growth of condition number of Hilbert matrix