# Singular Value Decomposition (SVD)

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# Outline

- Singular Value Decomposition (SVD)

# Spectral Theorem

Spectral theorem: Let $A \in \mathbb{C}^{n\times n}$. Then $A$ is Hermitian if and only if

$$A = V\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,V^*,$$

where $V \in \mathbb{C}^{n\times n}$ is unitary and $\lambda_j \in \mathbb{R}$ for $j = 1 : n$. ■

If $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ then $Av_j = \lambda_j v_j$ for $j = 1 : n$. Hence $v_j$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_j$ for $j = 1 : n$.

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**Example:**

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^* . ■$$

# Singular Value Decomposition (SVD)

In a sense, SVD generalizes spectral theorem for Hermitian matrices to the case of arbitrary $m \times n$ matrices.

Theorem: Let $A \in \mathbb{C}^{m \times n}$. Then there exist unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ such that $A = U\Sigma V^*$,

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$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathbb{R}^{m\times n},$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ , $\Sigma_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ and $r = \mathrm{rank}(A)$.

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Trimmed SVD: Let $U$ and $V$ be given by $U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$. Then

$$A = U_r\Sigma_r V_r^* = \sigma_1 u_1 v_1^* + \cdots + \sigma_r u_r v_r^*,$$

where $U_r := \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}$ and $V_r := \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}$.

# Eexample

The MATLAB commands `[U, S, V] = svd(A)` and `[U, S, V] = svd(A,0)` compute full and trimmed SVD of an $m \times n$ matrix $A$, respectively.

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The SVD of the 2-by-2 Hermitian matrix is given by

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*,$$

whereas the spectral decomposition is given by

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*. \blacksquare$$

# Properties of SVD

Assume for the moment that the SVD $A = U\Sigma V^*$ exists. Then $AV = U\Sigma$ yields

$$\begin{array}{lcl} Av_j = \sigma_j u_j, \ j = 1:r & \Rightarrow & R(A) = \mathrm{span}(u_1, \ldots, u_r) \\ Av_j = 0, \ j = r+1:n & \Rightarrow & N(A) = \mathrm{span}(v_{r+1}, \ldots, v_n) \end{array}$$

and $A^*U = V\Sigma^*$ yields

$$\begin{array}{ll} A^*u_j = \sigma_j v_j, \ j = 1:r, & R(A^*) = \mathrm{span}(v_1, \ldots, v_r) \\ A^*u_j = 0, \ j = r+1:m & N(A^*) = \mathrm{span}(u_{r+1}, \ldots, u_m) \end{array}$$

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We have

$$A^*A = V \begin{bmatrix} \Sigma_r^2 & 0 \\ 0 & 0 \end{bmatrix} V^* \implies \begin{array}{l} A^*Av_j = \sigma_j^2 v_j, \; j = 1:r \\ A^*Av_j = 0, \; j = r+1:n \end{array}$$

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$$AA^* = U \begin{bmatrix} \Sigma_r^2 & 0 \\ 0 & 0 \end{bmatrix} U^* \quad \Longrightarrow \quad \begin{array}{l} AA^*u_j = \sigma_j^2 u_j, \; j = 1:r \\ AA^*u_j = 0, \; j = r+1:m \end{array}$$

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This shows that

- the right singular vectors $v_1, \ldots, v_n$ are orthonormal eigenvectors of the positive semi-definite matrix $A^*A$
- the left singular vectors $u_1, \ldots, u_m$ are orthonormal eigenvectors of the positive semi-definite matrix $AA^*$
- and the nonzero singular values $\sigma_1, \ldots, \sigma_r$ are the square roots of the nonzero eigenvalues of $A^*A$ (or equivalently of $AA^*$).

# Existence of SVD

Consider the special case when $A \in \mathbb{C}^{n\times n}$ is nonsingular. Then $A^*A$ is positive definite and by spectral theorem $A^*A$ has positive eigenvalues:

$$A^*A = V\mathrm{diag}(\lambda_1, \cdots, \lambda_n)V^*,$$

where $\lambda_1 \geq \ldots \geq \lambda_n > 0$.

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Define $\Sigma := \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ and $U := AV\Sigma^{-1}$. Then $A = U\Sigma V^*$.

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# Existence of SVD

as rank(A) = n = r, so Ax != 0, implies,
(Ax)*.(Ax) > 0, implies x^(*)Ax > 0.

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(Vj*)(Vj) = 1, as Vj's are orthonormal

where $\lambda_1 \geq \ldots \geq \lambda_n > 0$. Indeed, $A^*Av_j = \lambda_j v_j \Longrightarrow \lambda_j = v_j^* A^* A v_j > 0$ for $j = 1 : n$.

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METHOD 1. SVD of an $n \times n$ nonsingular matrix $A$

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1. Compute spectral decomposition $A^*A = V\mathrm{diag}(\lambda_1, \cdots, \lambda_n)V^*$.
2. Define $\Sigma := \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$.
3. Compute $U := AV\Sigma^{-1}$. Then $A = U\Sigma V^*$ is an SVD of $A$.

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# Existence of SVD

The SVD of an $m$-by-$n$ matrix $A$ exists and can be computed from spectral decompositions of $A^*A$ and $AA^*$ in four steps:

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METHOD 2. SVD of an $m \times n$ matrix $A$

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1. Compute $A^*A = V\mathrm{diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0)V^*$. Set $\Sigma_r := \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ and let $V_r$ denote the first $r$ columns of $V$.
2. Compute $U_r := AV_r\Sigma_r^{-1}$, that is, $U_r := [Av_1/\sigma_1, \ldots, Av_r/\sigma_r]$, where $v_j := Ve_j$, $j = 1 : r$.

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3. Compute $AA^* = Z\mathrm{diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0)Z^*$. Let $U_{m-r}$ denote the last $m-r$ columns of $Z$. Then $R(U_{m-r}) = N(A^*)$.

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2. Compute $U_r := AV_r\Sigma_r^{-1}$, that is, $U_r := [Av_1/\sigma_1, \ldots, Av_r/\sigma_r]$, where $v_j := Ve_j$, $j = 1 : r$. Then $U_r^*U_r = \Sigma_r^{-1}V_r^*A^*AV_r\Sigma_r^{-1} = \Sigma_r^{-1}\Sigma_r^2\Sigma_r^{-1} = I_r$. Hence $U_r$ is an isometry.
3. Compute $AA^* = Z\mathrm{diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0)Z^*$. Let $U_{m-r}$ denote the last $m - r$ columns of $Z$. Then $R(U_{m-r}) = N(A^*)$. ? As R(U(m-r)) = L.C. of ui's, for i = r+1.....m, which equals N(A*) which is span(u(r+1), u(r+2), u(r+3).......... u(m))
4. Set $U := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix}$ and $\Sigma := \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$. Then $U$ is unitary and $A = U\Sigma V^*$ is an SVD of $A$.

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# Example

Let $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $A^*A = [2]$ and

$$AA^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^*.$$

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Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} [1]^*$ is an SVD of $A$.

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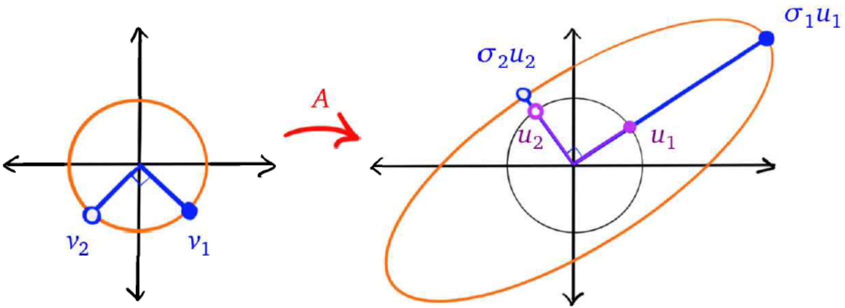
Similarly,

$$B := \begin{bmatrix} 1 & 2 \end{bmatrix} = [1] \begin{bmatrix} \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}^*$$

is an SVD of $B$.

# SVD in action




The image of a unit circle in $\mathbb{R}^2$ under the action of a $2 \times 2$ nonsingular matrix $A$. It follows that the image of a unit circle is an ellipse with semi-major axis $\sigma_1 u_1$ and semi-minor axis $\sigma_2 u_2$.

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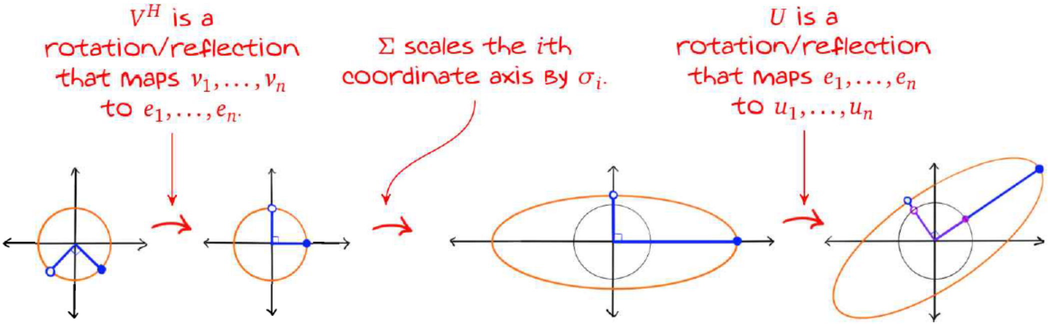


Let $\mathbb{T}$ denote the unit circle in $\mathbb{R}^2$. Then $V^*(\mathbb{T})$ is again a unit circle. Now $\Sigma$ maps the unit circle $V^*(\mathbb{T})$ to the ellipse $\mathbb{E} := \Sigma V^*(\mathbb{T})$. Finally, $U$ maps the ellipse $\mathbb{E}$ to the ellipse $U(\mathbb{E})$.

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