Homework-4

MA423 : Matrix Computations 2023 R. Alam

Stability of algorithms and pivot growth

1. For each of the following problems state with justification whether the given algorithm is backward stable or not.

   (a) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = 1 + d, \text{ALG}(d) = \text{round}(1 + d)$.

   (b) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = 2d, \text{ALG}(d) = \text{round}(d + d)$.

   (c) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = d^2, \text{ALG}(d) = \text{round}(d * d)$.

   (d) Data: $d_1, \ldots, d_m \in F(\beta, t, L, U)$, Solution: $f(d) = \sum_{j=1}^{m} d_j$ and $\text{ALG}(d_1, \ldots, d_m)$ is given by

   $s = d_1$
   for $j = 2\colon m$
   $\qquad s = \text{round}(s + d_j)$
   end

2. Let x and y be nonzero vectors in $\mathbb{R}^n$. Consider the rank-1 matrix $A := xy^\top$ also referred to as the outer product of x and y. Show that the computation of A in finite precision arithmetic is NOT backward stable.

3. Let $A = LU$ be the LU factorization of a matrix $A \in \mathbb{C}^{n\times n}$ with $L(i,j)| \leq 1$. Let A_i and U_i denote the i-th row of A and U, respectively. Show that $U_i = A_i - \sum_{j=1}^{i-1} L(i,j)U_j$ and use it to show that $\|U\|_\infty \leq 2^{n-1}\|A\|_\infty$. Define $PG(A) := \|L\|_\infty \|U\|_\infty / \|A\|_\infty$ and show that $PG(A) \leq n2^{n-1}$.

4. Suppose that $A \in \mathbb{R}^{n\times n}$ is SPD and that $A = GG^T$. Show that $\|G\|_2\|G^T\|_2 = \|A\|_2$ and that $\|G\|_\infty \|G^T\|_\infty \leq n^{3/2}\|A\|_\infty$. [Hint: Use relation between $\|x\|_\infty$ and $\|x\|_2$ for $x \in \mathbb{R}^n$.]

   Conclude that the spectral norm pivot growth $PG_2(A) := \frac{\|G\|_2\|G^T\|_2}{\|A\|_2} = 1$ and the ∞-norm pivot growth $PG_\infty(A) := \frac{\|G\|_\infty\|G^T\|_\infty}{\|A\|_\infty} \leq n^{3/2}$.

5. Let $\mathbf{u}$ be the unit roundoff of a floating-point system. Assume that the entries of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ are floating-point number. Show that barring overflow/underflow

   $$|\text{round}(AB) - AB| \leq n\mathbf{u}|A| \cdot |B| + \mathcal{O}(\mathbf{u}^2).$$

   Here $|X|$ denotes the matrix obtained by taking absolute values of the entries of X. The product $|A| \cdot |B|$ is the usual matrix multiplication.

   Further, deduce that $\text{round}(AB) = \hat{A}B = A\hat{B}$ for some matrices $\hat{A}$ and $\hat{B}$ such that $|A - \hat{A}| \leq n\mathbf{u}|A| + \mathcal{O}(\mathbf{u}^2)$ and $|B - \hat{B}| \leq n\mathbf{u}|B| + \mathcal{O}(\mathbf{u}^2)$, where the inequality $|A| \leq |B|$ holds componentwise, that is, $|A(i,j)| \leq |B(i,j)|$ for all i, j.

   [**Hint:** Use the fact that if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are floating-point numbers then

   $$\text{round}(\sum_{j=1}^{n} x_j y_j) = \sum_{j=1}^{n} x_j y_j (1 + \delta_j)$$

   where $|\delta_j| \leq n\mathbf{u} + \mathcal{O}(\mathbf{u}^2)$ and $n\mathbf{u} < 1$.]

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