MA668: Algorithmic and High Frequency Trading
Lecture 37

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Solving the DPE When Targeting Rate of Trading (Contd ...)

1. Ansatz: $H(t,x,S,\mu,q) = x + qS + h(t,\mu,q)$.
2. Now: $(\partial_t + \mathcal{L}^{\mu})\, h + \dfrac{1}{4(k+\varphi)}\left(\partial_q h + bq - 2\varphi\rho\mu\right)^2 - \varphi\rho^2\mu^2$, subject to the terminal condition $h(T,\mu,q) = -\alpha q^2$.
3. Ansatz: $h(t,\mu,q) = h_0(t,\mu) + q h_1(t,\mu) + q^2 h_2(t,\mu)$.
4. The optimal trading speed in feedback form reduces to:

$$\nu^* = \frac{1}{k+\varphi}\left[\left[\varphi\rho\mu - \frac{1}{2}h_1(t,\mu)\right] - \left[\frac{1}{2}b + h_2(t,\mu)\right] q\right].$$

Solving the DPE When Targeting Rate of Trading (Contd ...)

1. After a fairly elaborate exercise one can arrive at:
   - (A) $$h_2(t,\mu) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}b.$$
   - (B) $$h_1(t,\mu) = 2\varphi\rho\,\frac{\int_t^T \mathbb{E}\left[\mu_u \mid \mu_t = \mu\right] du}{(T-t)+\zeta}, \text{ where, } \zeta = \frac{k+\varphi}{\alpha - \frac{1}{2}b}[a].$$

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[a] The integral appearing above is precisely the **expected total volume** over the remainder of the trading horizon. This is because it is the integral of the expected trading rate μ_t between the current time and the end of the strategy's trading horizon. Moreover, the integral term combined with the factor $((T-t)+\zeta)^{-1}$ is approximately the average expected trading rate for the remaining time horizon. This combination would be exactly the average expected trading rate if $k = \varphi = 0$ and/or $\alpha \to \infty$

Solving the DPE When Targeting Rate of Trading (Contd ...)

1. The agent's optimal trading speed can be represented as:

$$\nu_t^* = \frac{\varphi\rho}{k+\varphi}\left[\mu_t - \frac{\int_t^T \mathbb{E}\left[\mu_u | \mathcal{F}_t^\mu\right] du}{(T-t)+\zeta}\right] + \frac{Q_t^{\nu^*}}{(T-t)+\zeta}, \tag{1}$$

where the conditional expectation is now with respect to the filtration $\mathcal{F}_t^\mu$ generated by μ.

2. By inserting the general result for h_1 into the optimal inventory to hold, we also have the compact representation:

$$\begin{aligned} Q_t^{\nu^*} &= \left(1 - \frac{t}{T+\zeta}\right)\mathfrak{R} \\ &- \frac{\varphi\rho}{k+\varphi}\int_0^t \frac{(T-t)+\zeta}{(T-s)+\zeta}\left[\mu_s - \frac{\int_s^T \mathbb{E}\left[\mu_u | \mathcal{F}_s^\mu\right] du}{(T-s)+\zeta}\right] ds. \end{aligned} \tag{2}$$

Solving the DPE When Targeting Rate of Trading (Contd ...)

1. The optimal trading speed is given by:

$$\nu_t^* = \frac{\varphi}{k+\varphi}\rho\mu_t \tag{3}$$

$$+ \frac{1}{(T-t)+\zeta}\left(Q_t^{\nu^*} - \frac{\varphi\rho}{k+\varphi}\int_t^T \mathbb{E}\left[\mu_u | \mathcal{F}_t^{\mu}\right] du\right). \tag{4}$$

2. The first component of the strategy (3) accounts for the trading rate that must be achieved in order to meet the POV target, taking into account the trade-off between the POV target penalty φ and the costs stemming from temporary price impact k.

Solving the DPE When Targeting Rate of Trading (Contd ...)

1. Although the POV target is $\rho\mu_t$, the strategy targets a lower amount since $\frac{\varphi\rho}{k+\varphi} \leq \rho$, where equality is achieved if the costs of missing the target are $\varphi \to \infty$ and k remains finite or there is no temporary impact as $k \downarrow 0$.
2. The second component is a TWAP-like strategy (the first term in the braces in (4)) with a downward adjustment (the second term in the braces) because throughout the trading horizon there is the component targeting the POV.
3. This is why we see that the TWAP-like strategy is applied to the remaining inventory $Q_t^{\nu^*}$ (minus the number of shares that are expected to be liquidated as part of the POV target, which will be done by the first term of the strategy (3) over the remaining time of the strategy.
4. We continue our discussion of the optimal strategy by looking at some limiting cases.

Limiting Case 1

1. The limiting case in which the agent wishes to always track a fraction ρ of the rate μ_t is obtained by letting the target penalty parameter $\varphi \to \infty$.
2. In this case, the liquidation speed and inventory path become $\nu_t^* \to \rho\mu_t$ and $Q_t^{\nu^*} \to \mathfrak{R} - \rho \int_0^t \mu_s ds$, respectively, since $\zeta \to \infty$ as $\varphi \to \infty$.
3. Regardless of what the inventory target is, the strategy liquidates at a rate of $\rho\mu_t$.
4. Clearly, as shown by the inventory path, the strategy could liquidate an amount of shares which exceeds or falls short of the initial target $\mathfrak{R}$.
5. When the strategy reaches T, any outstanding shares, short or long, are liquidated with an MO at the mid-price and receive a finite penalty of αq_T^2 which, in this limit, the agent prefers "to" picking up the more onerous running penalty which would be infinite if she/he did not liquidate at the rate $\rho\mu_t$.

Limiting Case 2

1. The limiting case in which the agent wishes to fully liquidate her/his inventory leads to a finite trading strategy, with finite inventory paths.
2. In particular, since $\zeta \to 0$ as $\alpha \to \infty$, so we have,

$$\nu_t^* \to \frac{\varphi\rho}{k+\varphi}\left[\mu_t - \frac{\int_t^T \mathbb{E}\left[\mu_u | \mathcal{F}_t^\mu\right] du}{T-t}\right] + \frac{Q_t^{\nu^*}}{T-t},$$

and

$$Q_t^{\nu^*} \to \left(1 - \frac{t}{T}\right)\mathfrak{R} - \frac{\varphi\rho}{k+\varphi}\int_0^t \frac{T-t}{T-s}\left[\mu_s - \frac{\int_s^T \mathbb{E}\left[\mu_u | \mathcal{F}_s^\mu\right] du}{T-s}\right] ds.$$

Limiting Case 3

1. Suppose that, in addition to requiring full liquidation, the agent is also very averse to trading at a rate different from $\rho\mu_t$.
2. As φ increases, she/he will target more and more closely the required trading rate.
3. However, due to the constraint that she/he must fully liquidate, she/he will not be able to match the required trading rate at all times.
4. Therefore, in the limit in which $\varphi \to \infty$, after we have already taken $\alpha \to \infty$, the value function will become arbitrarily large and negative, and will not be finite.
5. The limiting optimal strategy, however, does remain finite, as does her/his optimal inventory path, and the net value of liquidating her/his shares remains finite and well behaved.

Limiting Case 3 (Contd ...)

1. The optimal speed of trading and inventory position in this second double limiting case ($\varphi \to \infty$ and $\alpha \to \infty$) are:

$$\nu_t^* \to \rho \left[\mu_t - \frac{\int_t^T \mathbb{E}\left[\mu_u | \mathcal{F}_t^\mu\right] du}{T - t} \right] + \frac{Q_t^{\nu^*}}{T - t},$$

and

$$Q_t^{\nu^*} \to \left(1 - \frac{t}{T}\right) \mathfrak{R} - \rho \int_0^t \frac{T - t}{T - s} \left[\mu_s - \frac{\int_s^T \mathbb{E}\left[\mu_u | \mathcal{F}_s^\mu\right] du}{T - s} \right] ds.$$

2. Interestingly, when the other agents trade at a constant rate, that is, μ_t is a constant the POV correction terms in the above cancel and the agent's strategy becomes TWAP.

Stochastic Mean-Reverting Trading Rate

1. We begin with a mean-reverting model for the volume, for which we assume that the sell volume rate μ_t is a mean reverting process which satisfies the SDE:

$$d\mu_t = -\kappa\mu_t dt + \eta_{1+N_{t-}} dN_t, \tag{5}$$

where $\kappa \geq 0$ is the mean reversion rate, N_t is a homogeneous Poisson process with intensity λ, and $\{\eta_1, \eta_2, \dots\}$ are non-negative i.i.d. random variables with distribution function F, with finite first moment, independent of N_t.

2. The solution to (5) is:

$$\begin{aligned} \mu_t &= e^{-\kappa t}\mu_0 + \int_0^t e^{-\kappa(t-u)}\eta_{1+N_{t-}} dN_u, \\ &= e^{-\kappa t}\mu_0 + \sum_{m=1}^{N_t} e^{-\kappa(t-\tau_m)}\eta_m, \end{aligned} \tag{6}$$

where τ_m denotes the time of the m-th arrival of the Poisson process.

Simulations

1. We focus on the double limiting case of first ensuring that all inventory is liquidated ($\alpha \to \infty$) and second that the agent wishes to trade very close to POV ($\varphi \to \infty$).
2. For the simulations, we use the following modelling parameters: $S_0 = 20$, $\sigma = 0.5$, $T = 1$, $\mu_0 = \dfrac{\psi}{\kappa}$, $\eta \sim \text{Exp}(10)$, $\lambda = 50$, $\kappa = 20$, $k = 0.1$, $b = 0.1$ and $\rho = 0.05$.
3. We assume that the agent is attempting to liquidate ρ percentage of the volume she/he expects to arrive in the market over her/his trading horizon.

Figure 9.4

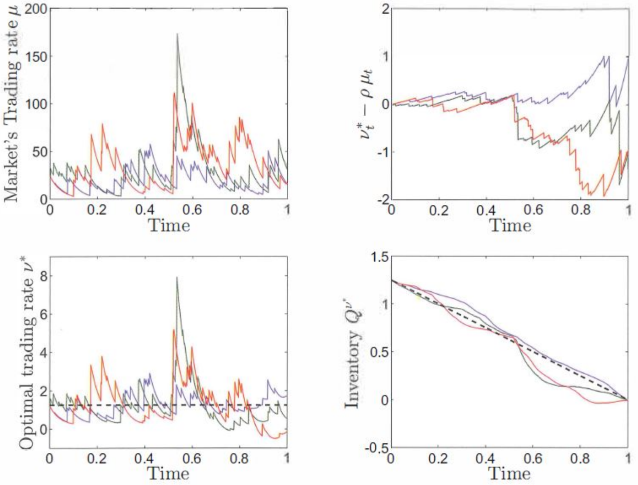

Figure 9.4 Three sample paths of the market's trading rate, the optimal trading rate, the difference between the optimal trading rate and the targeted rate, and the agent's inventory.

Figure 9.4 (Contd ...)

1. Figure 9.4: We show three sample paths of the trading rate of other market participants μ_t, the optimal trading rate ν_t^*, the difference between the optimal rate and the target rate $\nu_t^* - \rho\mu_t$, and the agent's inventory $Q_t^{\nu^*}$.
2. In the bottom left panel, the dotted line is the expected trading rate equal to ψ/κ and in the bottom right panel, the dotted line is TWAP.
3. Note that ν_t^* and μ_t are strongly correlated.

Figure 9.5

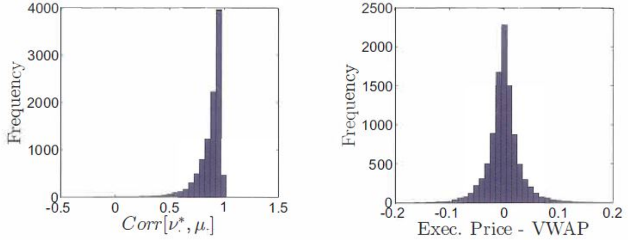

Figure 9.5 Left: histogram of the correlation between the agent's trading rate ν_t^* and μ_t. Right: histogram of the difference between the execution price per share and the VWAP.

Figure: Figure 9.5

Figure 9.5 (Contd ...)

1. Figure 9.5: In the left panel, we show the histogram of the correlation between ν_t^* and μ_t viewed as a time series, along $10,000$ sample paths.
2. The mean correlation is quite high at 0.88, illustrating the fact that the trading rate tracks the rate of order flow.
3. There are, however, deviations from the targeted rate of $\rho\mu_t$.
4. These differences appear most notably towards the end of the trading horizon (as seen in the top right panel in Figure 9.4) where the agent's main concern is to drive her/his inventory to zero and she/he is less concerned about targeting other participants' trading rate.
5. Figure 9.5: Right panel shows the difference between the executed price per share and the VWAP.
6. If the agent is closely targeting a fraction ρ of the trading volume then this results in strategies which do indeed target VWAP on average.
7. The deviation around VWAP is symmetric (skewness of 0.06) with mean -1.4×10^{-4} and standard error $\pm 3 \times 10^{-4}$.

Figure 9.6

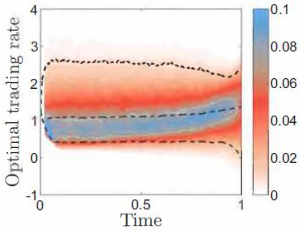

Figure 9.6 Heat-maps of the optimal trading rate and inventory. The dotted lines show the 5%, 50% and 95% quantiles.

Figure: Figure 9.6

Figure 9.6 (Contd ...)

1. Figure 9.6: We illustrate a heat-map of the agent's trading rate and her/his inventory.
2. The dotted lines here indicate the 5%, 50% and 95% quantiles.
3. Interestingly, the median inventory path is TWAP, while the agent's inventory may deviate both above and below this trajectory in her/his attempt to match the POV target.
4. The median path of her/his optimal trading rate is essentially constant through time, although the mode of this trajectory tends to increase towards maturity.
5. This suggests there is a slight bias towards first trading a little more slowly compared with TWAP and then trading faster to catch up.

Percentage of Cumulative Volume

1. Now: We assume that the agent's execution strategy targets a percentage of cumulative volume (POCV) and the liquidation strategy relies on MOs only.
2. Here the accumulated volume V of sell orders, excluding the agent's own trades, is given by:

$$V_t = \int_0^t \mu_u du,$$

where as above μ_t denotes other market participants' rate of trading.
3. The agent's performance criteria is now modified to:

$$\begin{aligned} H^{\nu}(t,x,S,\mu,V,q) &= \mathbb{E}_{t,x,S,\mu,V,q}\Big[X_T^{\nu} + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right) \\ &- \varphi \int_t^T \left((\mathfrak{R} - Q_u^{\nu}) - \rho V_u\right)^2 du\Big], \end{aligned} \tag{7}$$

where $\mathfrak{R}$ is the number of shares that the agent wishes to liquidate by the terminal date T.

Percentage of Cumulative Volume (Contd ...)

1. The running target penalty $\varphi \int_t^T \left((\mathfrak{R} - Q_u^\nu) - \rho V_u\right)^2 du$ is not a financial cost that the agent incurs.
2. Rather, its purpose is to allow the agent to seek for optimal liquidation rates where the total amount that has been liquidated up to time t is not too far away from a percentage of what the entire market has sold.
3. For example: When the penalty parameter $\varphi \to \infty$, the optimal strategy is forced to liquidate shares so that at any point in time the number of shares that have already been liquidated, $\mathfrak{R} - q_t$, equals ρV_t.
4. In this manner, the agent devises a strategy where the cumulative sum of her/his own sell MOs is a fraction $0 < \rho < 1$ of the market.

Percentage of Cumulative Volume (Contd ...)

1. The agent's value function is:

$$H(t, x, S, \mu, V, q) = \sup_{\nu \in \mathcal{A}} H^{\nu}(t, x, S, \mu, V, q). \tag{8}$$

2. This leads to the supremum being attained at:

$$\nu^* = \frac{S\partial_x H - \partial_q H - b\partial_S H}{2k}.$$

3. Ansatz 1:

$$H(t, x, S, \mu, V, q) = x + qS + h(t, \mu, V, q).$$

4. Ansatz 2:

$$h(t, \mu, V, q) = h_0(t, \mu, V) + h_1(t, \mu, V)q + h_2(t, \mu, V)q^2.$$

Percentage of Cumulative Volume (Contd ...)

1. The optimal speed of trading in feedback form can be obtained as:

$$\nu_t^* = -\frac{1}{2k}\left[h_1(t,\mu_t,V_t) + 2\left(h_2(t) + \frac{1}{2}b\right)Q_t^{\nu^*}\right]. \tag{9}$$

2. For h_1 and h_2:

   A. We have: $h_2(t) = \sqrt{k\varphi}\,\frac{1+\zeta e^{2\xi(T-t)}}{1-\zeta e^{2\xi(T-t)}} - \frac{1}{2}b$, where $\xi = \sqrt{\frac{\varphi}{k}}$ and $\zeta = \frac{\alpha - \frac{1}{b} + \sqrt{k\varphi}}{\alpha - \frac{1}{b} - \sqrt{k\varphi}}$.

   B. We have: $h_1(t,\mu,V) = 2\varphi \int_t^T g(t,u)\left(\mathfrak{R} - \rho\mathbb{E}_{t,\mu,V}\left[V_u\right]\right) du$, where $g(t,u) = \left(\frac{\zeta e^{\xi(T-u)} - e^{-\xi(T-u)}}{\zeta e^{\xi(T-t)} - e^{-\xi(T-t)}}\right)$.