1)

a) $f(x) = \frac{1}{2} x^T A x + b^T x$

Let $f_1(x) = \frac{1}{2} x^T A x$

Let $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

$$x^T A x = [x_1 \; x_2 \; \dots \; x_n] \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [x^T a_1 \; x^T a_2 \; \dots \; x^T a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_i x_i x^T a_i$$

$$= \sum_i x_i \sum_j x_j a_{ji} = \sum_{i,j} x_i x_j a_{ij}$$

$$\therefore f_1(x) = \frac{1}{2} \sum_{i,j} x_i x_j a_{ij}$$

$$\frac{\partial f_1(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \frac{1}{2} \sum_{i,j} x_i x_j a_{ij}$$

$$= \frac{1}{2}\left[2x_k a_{kk} + \sum_{i: i \neq k} x_i a_{ik} + \sum_{j: j \neq k} x_j a_{kj} \right]$$

$$= \frac{1}{2}\left[\sum_i x_i a_{ik} + \sum_j x_j a_{kj} \right]$$

$$= \frac{1}{2}\left[a_k^T x + \bar{a}_k^T x \right]$$

$$\therefore \nabla f_1(x) = \begin{bmatrix} \frac{1}{2}(a_1^T + \bar{a}_1^T)x \\ \vdots \\ \frac{1}{2}(a_n^T + \bar{a}_n^T)x \end{bmatrix}$$

$$= \frac{1}{2}\left(\begin{bmatrix} a_1^T x \\ \vdots \\ a_n^T x \end{bmatrix} + \begin{bmatrix} \bar{a}_1^T x \\ \vdots \\ \bar{a}_n^T x \end{bmatrix} \right)$$

$$= \frac{1}{2}(Ax + A^T x)$$

$$\therefore \nabla f_1(x) = \frac{1}{2}(A + A^T)x$$

Since A is symmetric, $A = A^T$

$$\therefore \nabla f_1(x) = Ax$$

$$f_2(x) = b^T x$$
$$= b_1 x_1 + b_2 x_2 + \dots + b_n x_n$$

$$\nabla f_2(x) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b$$

$$\therefore \quad \nabla f(x) = \nabla f_1(x) + \nabla f_2(x) = Ax + b$$

$$\cancel{= \tfrac{1}{2}(A + A^T)x + b}$$

$\vec{b}x$

$$f(x) = g(h(x))$$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} g(h(x)) \\ \vdots \\ \frac{\partial}{\partial x_n} g(h(x)) \end{bmatrix}$$

$$= \begin{bmatrix} g'(h(x)) \cdot \frac{\partial}{\partial x_1} h(x) \\ \vdots \\ g'(h(x)) \cdot \frac{\partial}{\partial x_n} h(x) \end{bmatrix}$$

$$= g'(h(x)) \begin{bmatrix} \frac{\partial}{\partial x_1} h(x) \\ \vdots \\ \frac{\partial}{\partial x_n} h(x) \end{bmatrix} = g'(h(x)) \nabla h(x)$$

$$\nabla f(x) = g'(h(x)) \nabla h(x)$$

c) $f(x) = \frac{1}{2} x^T A x + b^T x$

$f_1(x) = \frac{1}{2} x^T A x = \frac{1}{2} \sum_{i,j} x_i x_j a_{ij}$ (Proven previously)

$$\frac{\partial f_1(x)}{\partial x_k \partial x_k} = \frac{1}{\partial x_k \partial x_k} \left(\frac{1}{2} \sum_{i,j} x_i x_j a_{ij} \right)$$

$$= \frac{1}{2} \cdot 2 \cdot a_{kk} = a_{kk}$$

When $k \neq l$,

$$\frac{\partial f_1(x)}{\partial x_k \partial x_l} = \frac{1}{\partial x_k \partial x_l} \left(\frac{1}{2} \sum_{i,j} x_i x_j a_{ij} \right)$$

$$= \frac{1}{2} \left(x_{kl} + x_{lk} \right) = x_{kl}$$

$$\nabla^2 f(x) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} = A$$

$f_2(x) = b^T x = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$

$$\frac{\nabla f_2(x)}{\partial x_i x_j} = 0$$

$\therefore \nabla^2 f(x) = A$

d) $f(x) = g(a^T x)$

$$\frac{\partial}{\partial x_j} f(x) = \frac{\partial}{\partial x_j} g(a^T x)$$

$$= g'(a^T x) \frac{\partial}{\partial x_j} \left(\sum_i a_i x_i \right)$$

$$= g'(a^T x) \cdot a_j$$

$$\frac{\partial}{\partial x_i \partial x_j} f(x) = \frac{\partial}{\partial x_i} \left(g'(a^T x)) a_j \right)$$

$$= g''(a^T x) \cdot \frac{\partial}{\partial x_i} (a^T x) \cdot a_j$$

$$= g''(a^T x) \cdot a_i a_j$$

$$\therefore \nabla^2 f(x) = \begin{bmatrix} g''(a^T x) a_1^2 & g''(a^T x) a_1 a_2 & \cdots & g''(a^T x) a_1 a_n \\ \vdots & \ddots & & \vdots \\ g''(a^T x) a_n a_1 & \cdots & & g''(a^T x) a_n a_n \end{bmatrix}$$

$$= g''(a^T x) \begin{bmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_n \\ \vdots & & & \\ a_n a_1 & \cdots & & a_n^2 \end{bmatrix}$$

$$= g''(a^T x) a a^T$$

2) a)

$$A = zz^T$$

$$x^T A x = x^T z z^T x$$

$$= (z^T x)^T \cdot (z^T x)$$

$$= \|z^T x\|^2 \geq 0$$

$$\therefore A \succeq 0$$

b)

$$A = zz^T$$

$$\text{Null}(A) = \{v : Av = 0\}$$

Note that $\text{Dim}(\text{Null}(A)) \leq n$

$$Av = 0 \Rightarrow zz^T v = 0$$

If $v \perp z$, $z^T v = 0 \Rightarrow Av = 0$

Since $z \in \mathbb{R}^n$, $\{v : v \perp z\}$ is of dim. $n-1$.

$$\therefore \text{Dim}(\text{Null}(A)) = n-1$$

By rank-nullity theorem,

$$\text{Dim}(\text{Rank}(A)) = 1$$

c) $A \succeq 0$. Is $BAB^T \succeq 0$

For any $x \in \mathbb{R}^m$, consider $x^T BAB^T x$

$$x^T (BAB^T) x = (x^T B) A (B^T x)$$

$$= (B^T x)^T A (B^T x)$$

$$= y^T A y \qquad y \in \mathbb{R}^n ; y = B^T x$$

$$\geq 0 \qquad (\because A \succeq 0)$$

$\therefore BAB^T \succeq 0$.

3) $A = T \Lambda T^{-1}$

To prove: $At^{(i)} = \lambda_i t^{(i)}$

$$A = T \Lambda T^{-1}$$

$$\Rightarrow AT = T\Lambda$$

$$\Rightarrow A \begin{array}{c} \begin{array}{ccccccc} & & & & i^{th} \text{ column} & & \end{array} \\ \begin{bmatrix} | & | & & | & & | \\ t_1^{(1)} & t_2^{(2)} & \cdots & t_i^{(i)} & \cdots & t_n^{(n)} \\ | & | & & | & & | \end{bmatrix} \end{array} = T\Lambda$$

Note that $At^{(i)}$ corresponds to the term corresponding to i^{th} column in T (in LHS) and Λ (in RHS)

We now look at i^{th} column of RHS

$$
\begin{array}{cc}
 & \begin{array}{c} i^{th} \text{ column} \\ \downarrow \end{array} \\
\begin{bmatrix} | & | & & | & & | \\ t^{(1)} & t^{(2)} & \cdots & t^{(i)} & \cdots & t^{(n)} \\ | & | & & | & & | \end{bmatrix} &
\begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_i & \\ 0 & & & & \ddots \ \lambda_n \end{bmatrix}
\end{array}
$$

$$\cancel{t^{(i)} \lambda_i t_1 + \lambda_i t_2^{(i)} + \cdots + \lambda_i}$$

$$= \begin{bmatrix} \lambda_i t_1^{(i)} \\ \lambda_i t_2^{(i)} \\ \vdots \\ \lambda_i t_3^{(i)} \end{bmatrix} = \lambda_i t^{(i)}$$

b) Since U is ~~orthonormal~~ orthogonal,

$$U^T U = I \Rightarrow U^T = U^{-1}$$

$$\therefore A = U \Lambda U^T = U \Lambda U^{-1}$$

Now, this is the same as last problem. (with $T = U$)

$\therefore$ $u^{(i)}$ is an eigenvector of A.

$$A u^{(i)} = \lambda_i u^{(i)}$$

c) $A \succeq 0$

Let $(\lambda_i, u^{(i)})$ be the eigenvalue, eigenvector pair

$$u^{(i)T} A u^{(i)} \geq 0$$

$$\Rightarrow u^{(i)T} \lambda_i u^{(i)} \geq 0 \qquad (\because Au^{(i)} = \lambda_i u^{(i)})$$

$$\Rightarrow \lambda_i \|u^{(i)}\|^2 \geq 0$$

$$\Rightarrow \lambda_i \geq 0 \qquad \forall i$$