Data Structures and Algorithms

(CSE102)

Slides courtesy: Prof. Surender Baswana, CSE, IIT Kanpur

Lecture 16

Graphs

- Notations and terminologies
- Data structures for graphs
- A few algorithmic problems in graphs

Why **Graphs** ??

Finding shortest route between cities

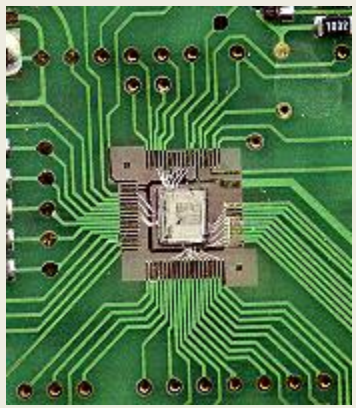
Given a network of **roads** connecting various cities,

compute the <u>shortest route</u> between any two **cities**.

Just imagine how you would solve/approach this problem.

Embedding

an integrated circuit on mother board

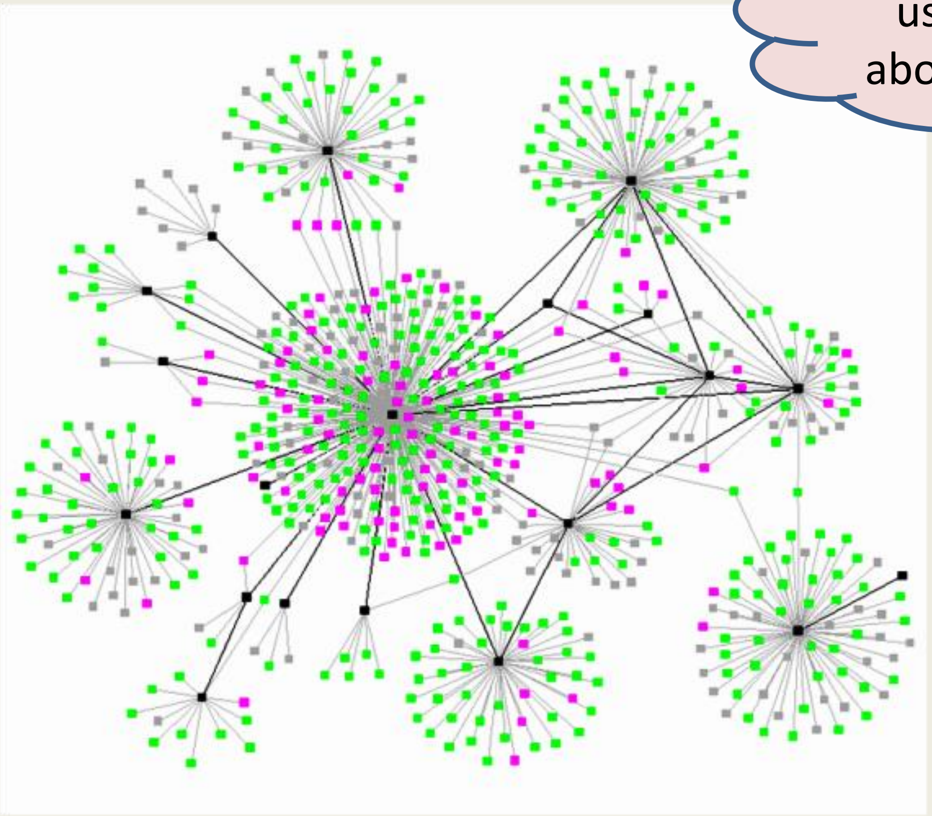


How to embed **ports** of various ICs on a plane and make **connections** among them so that

- No two connections **intersect** each other
- The **total length** of all the connections is **minimal**

A social network or world wide web (WWW)

Can we make some useful observations about such networks ?



diameter

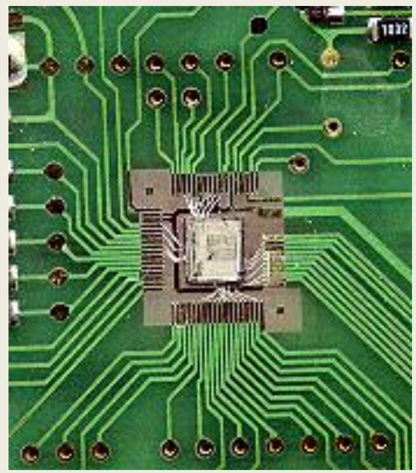
degree distribution

Do you know about the "6 degree of separation principle" of the world ?

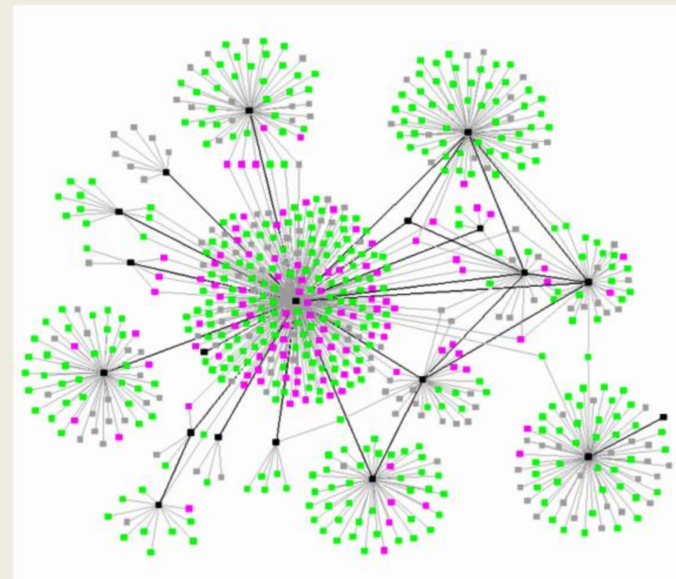
Visit the site https://en.wikipedia.org/wiki/Six_degrees_of_separation

How will you **model** these problems ?

I

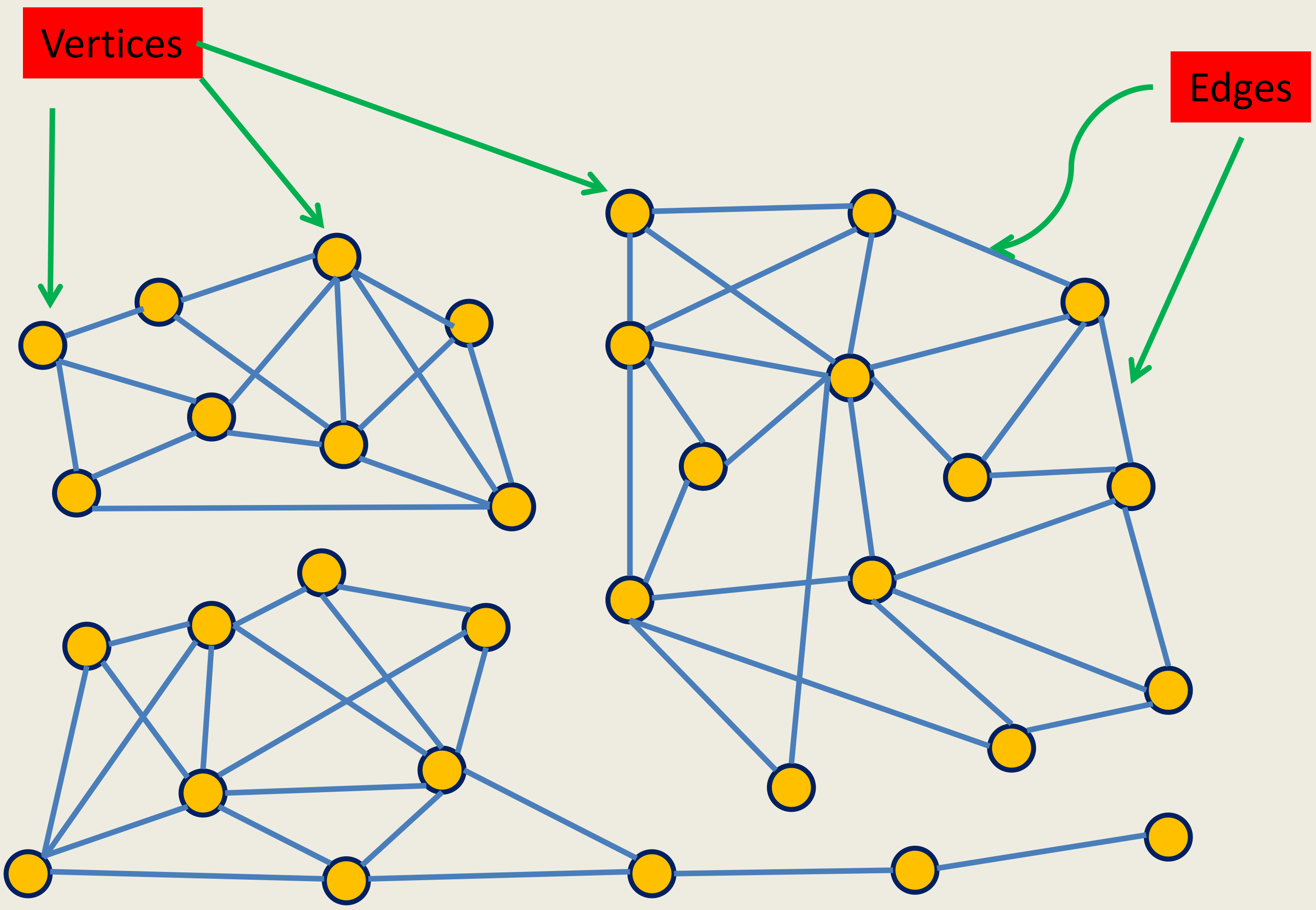


II



III

Graph



Graph

Definitions, notations, and terminologies

Graph

A graph G is defined by two sets

- V : set of vertices
- E : set of edges

Is there any relation between E and V ?

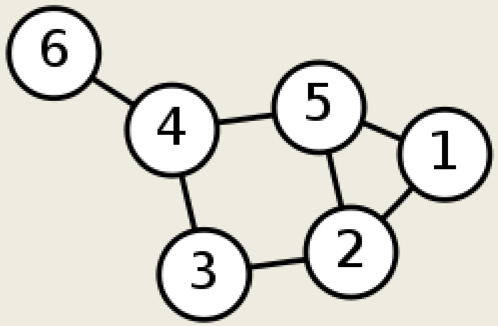
$E \subseteq (V \times V)$

Notation:

- A graph G consisting of vertices V and edges E is denoted by (V,E)

Types of graphs

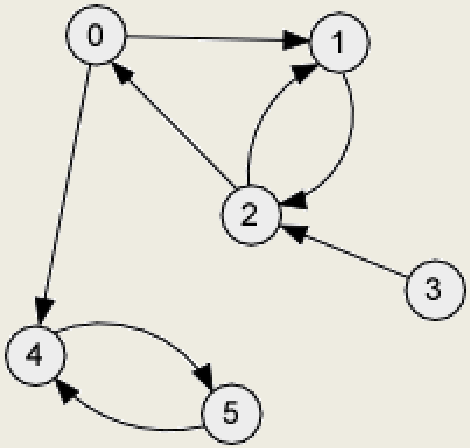
Undirected Graph



V= {1,2,3,4,5,6}

E= {(1,2), (1,5),
(2,5), (2,3),
(3,4),
(4,5), (4,6)}

Directed Graph



V= {0,1,2,3,4,5}

E= { (0,1), (0,4),
(1,2),
(2,0), (2,1),
(3,2),
(4,5),
(5,4) }

Notations

Notations:

- $n = |V|$
- $m = |E|$

Note: For directed graphs, $m \leq \boxed{n(n-1)}$

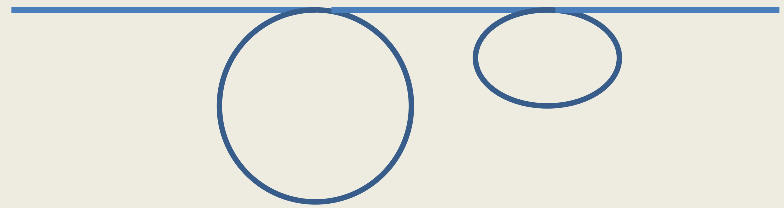
For undirected graphs, $m \leq \boxed{n(n-1)/2}$

Walks, paths, and cycles

Walk:

A sequence <v_0, v_1, …, v_k> of vertices is said to be a **walk** from x to y

- $x = v_0$
- $y = v_k$
- For each $i<k$, $(v_i, v_{i+1}) \in E$

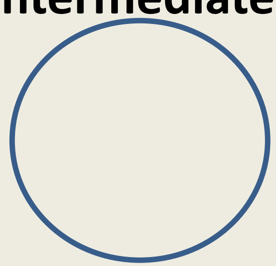


Path:

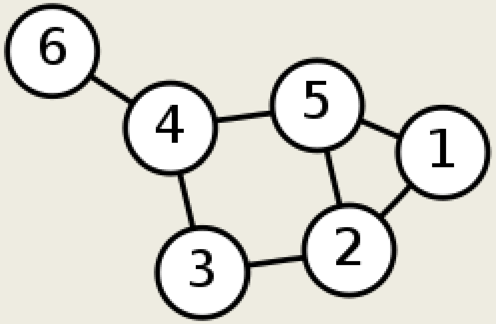
A walk <v_0, v_1, …, v_k> on which <u>no vertex appears twice</u>.

Cycle:

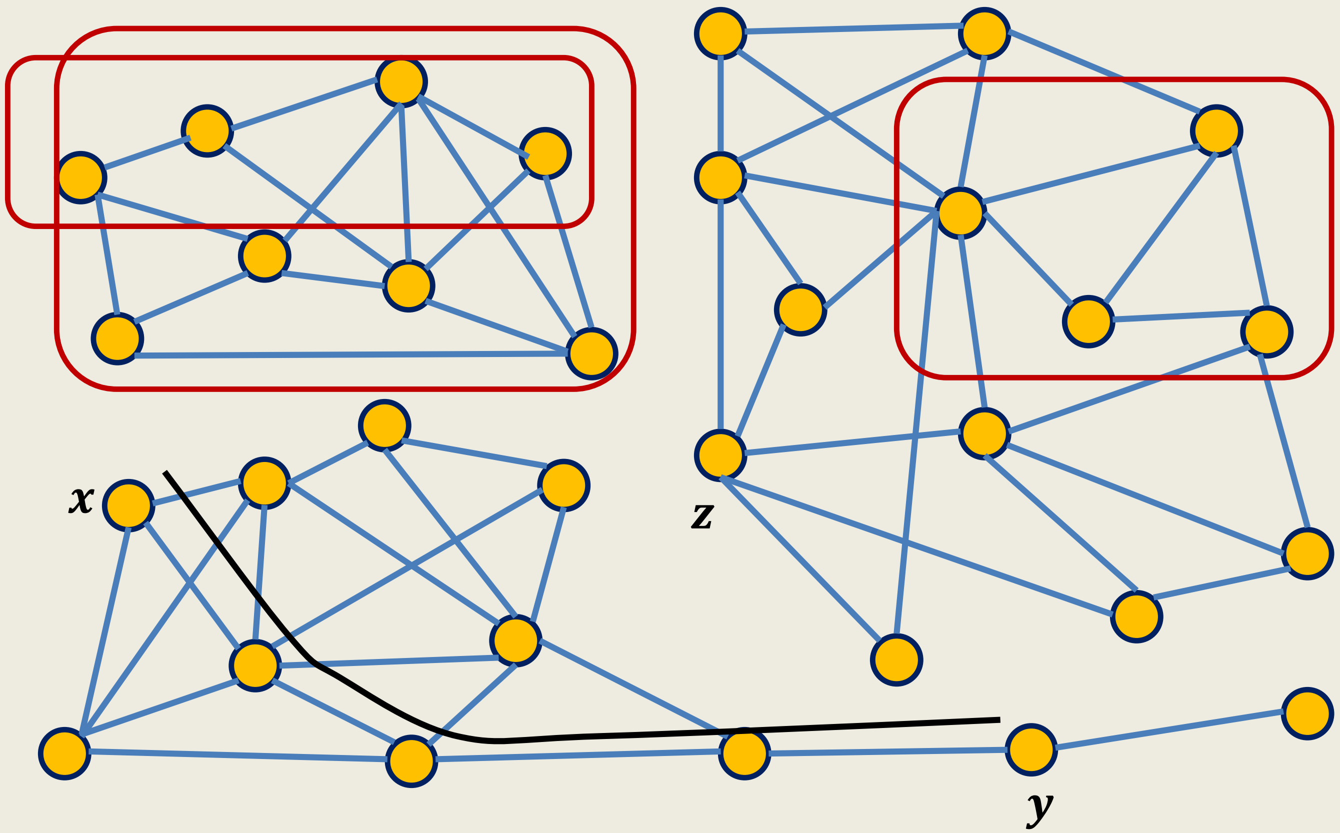
A walk <v_0, v_1, …, v_k> where no **intermediate** vertex gets repeated and $v_0 = v_k$



Examples



- **<1,5,4>** is a **walk** from **1** to **4.**
- <**1,3,2,5**> is **not** a **walk.**
- <**1,2,5,2,3,4,5,4,6**> is a **walk** from **1** to **6.**
- <**1,2,5,4,6**> is a **path** from **1** to **6.**
- <**2,3,4,5,2**> is a **cycle.**



two vertices are said to be ***connected*** if there is a **path** between them

Connected component:

- **A subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph**

Data Structures for Graphs

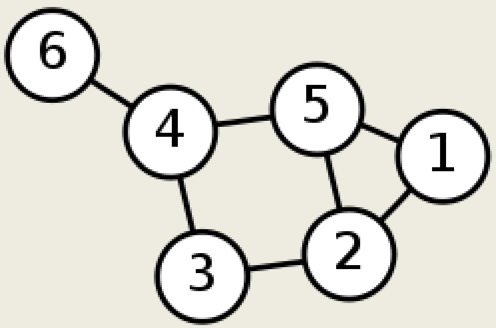
Vertices are always numbered

$\mathbf{1, \ldots, n}$

Or $\mathbf{0, \ldots, n-1}$

Link based data structure for graph

Undirected Graph



V= {1,2,3,4,5,6}

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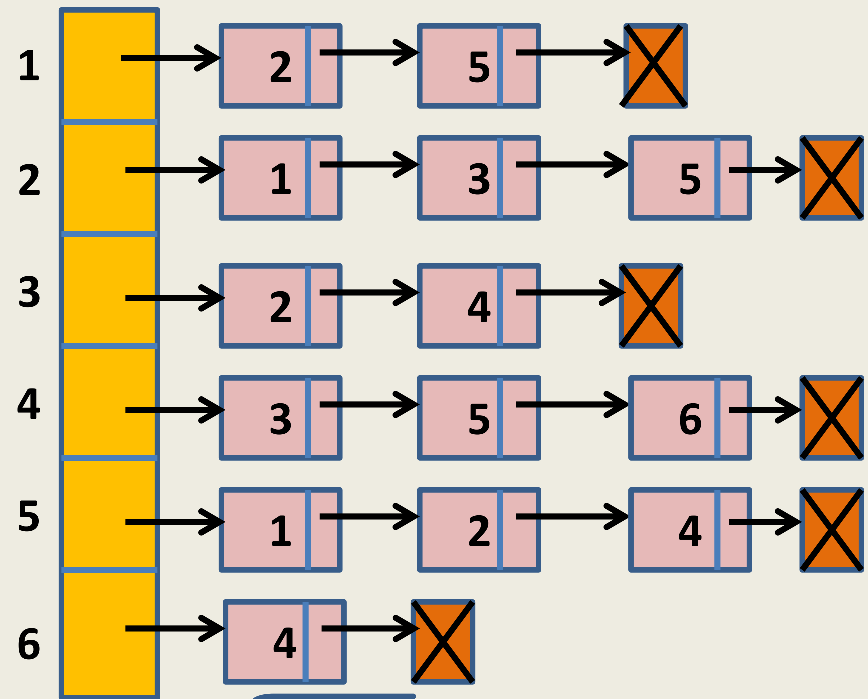
Adjacency Lists



Size = O($n + m$)

Link based data structure for graph

Advantage of Adjacency Lists :

- Space efficient
- Computing all the neighbors of a vertex in optimal time.

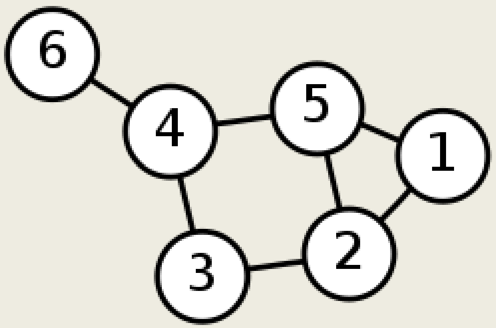
Disadvantage of Adjacency Lists :

- How to determine if there is an edge from $\boldsymbol{x}$ to $\boldsymbol{y}$?

(O(n) time in the worst case).

Array based data structure for graph

Undirected Graph



V= {1,2,3,4,5,6}

E= {(1,2), (1,5),
(2,5), (2,3),
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Adjacency Matrix

	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	1	0	0
4	0	0	1	0	1	1
5	1	1	0	1	0	0
6	0	0	0	1	0	0

Size = O(n^2)

Array based data structure for graph

Advantage of Adjacency Matrix :

- Determining whether there is an edge from $\boldsymbol{x}$ to $\boldsymbol{y}$ for any two vertices $\boldsymbol{x}$ and $\boldsymbol{y}$.

in $\mathbf{O}(1)$ time

Disadvantage of Adjacency Matrix :

- Computing all neighbors of a given vertex $\boldsymbol{x}$
- It takes $\mathbf{O}(n^2)$ space.

in $\mathbf{O}(n)$ time

Which data structure is commonly used for storing graphs ?

Adjacency lists

Reasons:

- Graphs in real life are sparse ($\boldsymbol{m} \ll \boldsymbol{n}^2$).
- Most algorithms require processing neighbors of each vertex.

➔Adjacency matrix will enforce **O**($\boldsymbol{n}^2$) bound on time complexity for such algorithm.

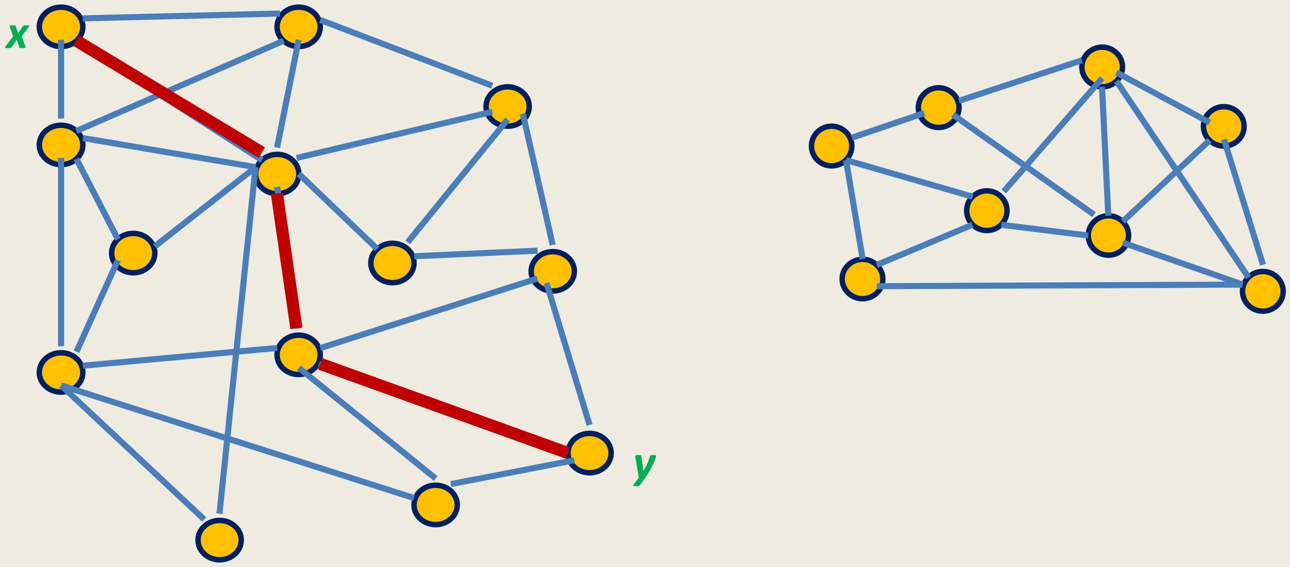
But, there are a few **exceptions**

Graph traversal

Graph traversal

Definition:

A vertex ***y*** is said to be reachable from ***x*** if there is a **path** from ***x*** to ***y***.



Graph traversal from vertex *x*: Starting from a given vertex ***x***, the aim is to visit all vertices which are reachable from ***x***.

Non-triviality of graph traversal

- **Avoiding loop**:

  How to avoid visiting a vertex multiple times ?

  (keeping track of vertices already visited)

- **Finite number of steps** :

  The traversal **must stop** in finite number of steps.

- **Completeness** :

  We must visit **all** vertices reachable from the start vertex ***x***.

Spend some time on these points and see how can you design such a traversal algorithm

A sample of Graph algorithmic Problems

- Are two vertices x and y connected ?

- Find all connected components in a graph.

- Is there is a cycle in a graph ?

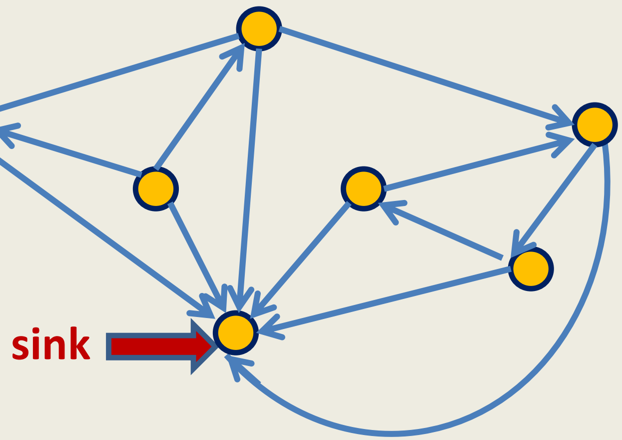
- Compute a path of shortest length between two vertices ?

- Is there is a cycle <u>passing through all vertices</u> ?

An interesting problem
(Finding a sink)

Definition: A vertex ***x*** in a given directed graph is said to be a universal **sink** if

- There is no edge **emanating from** (leaving) ***x***
- Every other vertex has an edge **into** ***x***.



How many **sinks** can there be in ***G*** ?

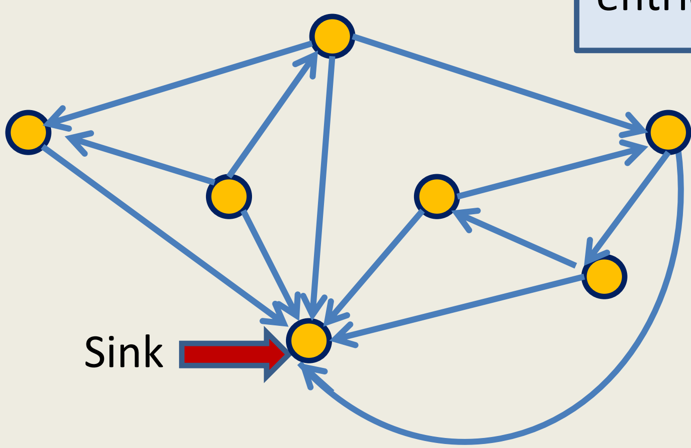
At most **1**.

An **interesting** problem

(Finding a sink)

Problem: Given a directed graph $G=(V,E)$ in an **adjacency matrix** representation, design an $\mathbf{O}(n)$ time algorithm to determine if there is any **sink** in G.

We are allowed to look into only $\mathbf{O}(n)$ entries of the **Adjacency matrix M**.

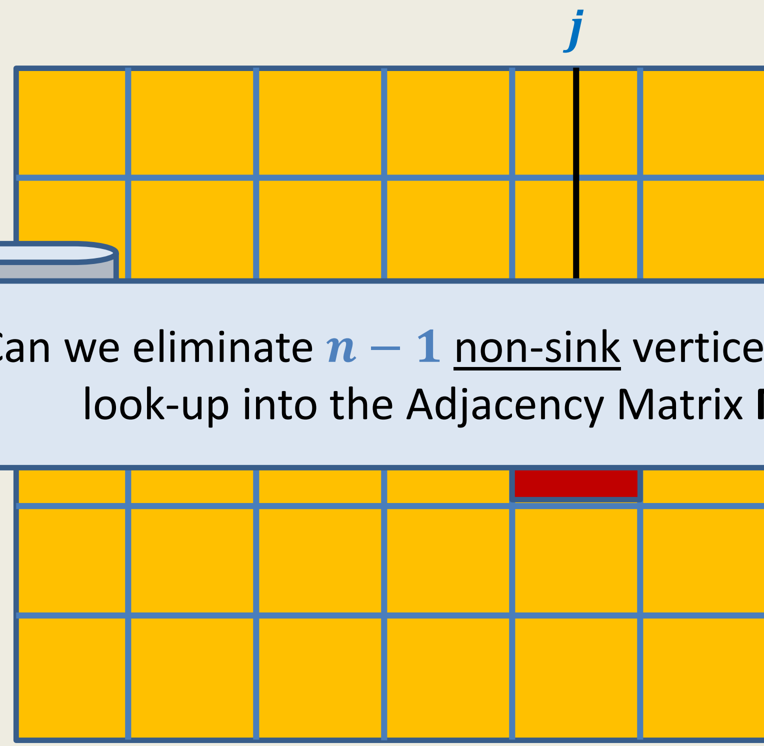


Question: Can we verify efficiently whether any given vertex i is a sink ?

Answer: Yes, in $\mathbf{O}(n)$ time only ☺

Look at ith **row** and ith **column** of **M**.

Key idea



Can we eliminate $n-1$ non-sink vertices in **O**(n) look-up into the Adjacency Matrix **M** ?

If **M**[i, j] = **0**, then j can not be sink

If **M**[i, j] = **1** , then i can not be sink

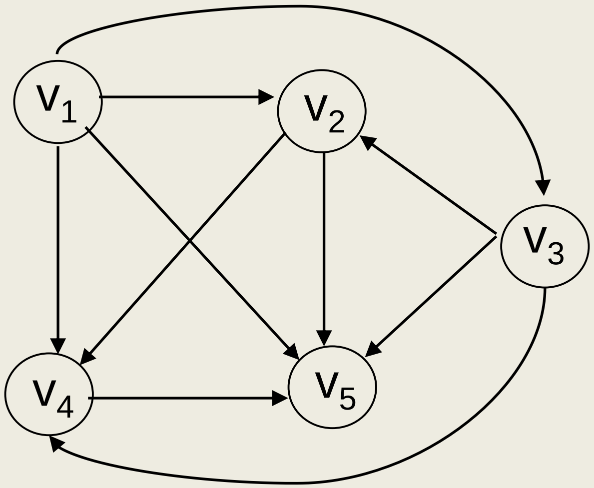
Algorithm to find a sink in a graph

```
Universal_Sink (A)
Let A be |V| x |V|
i = 1; j = 1;
while i <= |V| and j <= |V|
    do if A[i,j] == 1
            then i = i + 1
            else j = j + 1
if (i > |V|)
    then print "there is no universal sink"
else if is_sink (A, i) == False
    then print "there is no universal sink"
else
    print i "is the universal sink"
```

Algorithm to find a sink in a graph (Analysis)

- Loop terminates when i > |V| or j > |V|
- Upon termination, the only vertex that could possibly be sink is i.
  - if (i > |V|), there is no sink
  - if (i <= |V|), then j > |V|
    - Vertices k where 1 <= k < i can not be sinks
    - Vertices k where i < k <= |V| can not be sinks

Algorithm to find a sink in a graph



$$
\begin{array}{c c}
 & \begin{matrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} & \mathbf{v_5} \end{matrix} \\
\begin{matrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \mathbf{v_4} \\ \mathbf{v_5} \end{matrix} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$