Transient State Analysis

2. Effect of ξ on nature of response:

(i) Time Response of the Second Order (2nd Order) System for Step Response Input:

Consider the unit step signal as an input to the first order system.

So, r(t)=u(t) and therefore, R(s)=1/s

From equation (1), we can write C(s) as,

$$C(s) = \left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right) R(s) \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$C(s) = \left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

Do partial fraction of C(s).

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\Rightarrow \frac{\omega_n^2}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}} = \frac{A(s^2+2\xi\omega_n s+\omega_n^2)+(Bs+C)s}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}}$$

$$\Rightarrow \omega_n^2 = (A+B)s^2 + (2A\xi\omega_n + C)s + A\omega_n^2$$

$$\therefore (A+B) = 0 \; ; \quad 2A\xi\omega_n + C = 0 \; ; \quad A\omega_n^2 = \omega_n^2$$

$$\Rightarrow B = -1 \qquad \Rightarrow C = -2\xi\omega_n \qquad \Rightarrow A = 1.$$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$ (Making denominator Perfect square)

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \qquad \left[\because \omega_d = \omega_n\sqrt{1-\xi^2}\right]$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

Taking Laplace inverse transform :-

$$\therefore c(t) = 1 - e^{-\xi\omega_n t} \cdot \cos\omega_d t - \frac{\xi\omega_n}{\omega_d} \cdot e^{-\xi\omega_n t} \cdot \sin\omega_d t$$

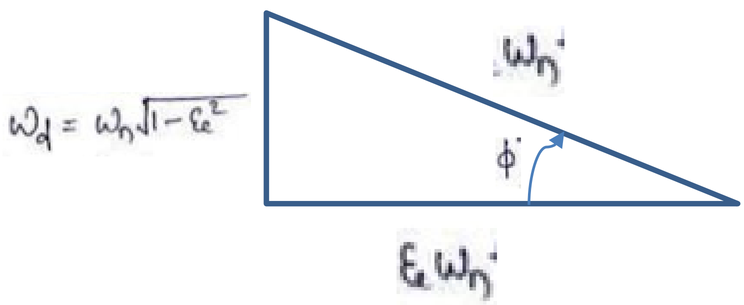
$$\Rightarrow c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\left(\sqrt{1-\xi^2}\right)\cos\omega_d t + \xi \sin\omega_d t\right].$$

Put $\cos\phi = \xi \Rightarrow \sin\phi = \sqrt{1-\xi^2}$

$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\phi \cdot \cos\omega_d t + \cos\phi \cdot \sin\omega_d t\right)$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi).}$$

where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}\xi$



$\sqrt{1-\xi^2} = \sin\phi$

$\therefore \cos\phi = \xi, \quad \tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$

Now, Put the values of ξ and ω_d in the expression of c(t), we get

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[(\omega_n\sqrt{1-\zeta^2 t}) + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \quad \ldots\ldots(3)$$

Steady state

Transient

Time constant of exponential decay is

$$e^{-t/T} = e^{-\varepsilon\omega_n t} \longrightarrow \boxed{T = \frac{1}{\xi\omega_n}}^{**}$$

The error is given as $e(t) = r(t) - c(t)$

and $$r(t) = 1$$

$$\therefore \quad e(t) = 1 - \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

or $$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \sin\left[\omega_n \sqrt{1-\zeta^2}\, t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \qquad \ldots\ldots(4)$$

the steady state error is —

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$\boxed{e_{ss} = 0}$$

therefore, at steady state there is no error between input and output.

Note:- *As the time response of* 2^{nd} *order system is influenced by* ξ*, therefore, there are four possible cases for positively damped systems* ($\xi > 0$). *(system will be stable or marginal stable system)*

$0 < \xi < 1$: under damped

$\xi = 1$: critically damped

$\xi > 1$: over damped

$\xi = 0$: undamped

Also, there are three possible cases for negatively damped systems ($\xi < 0$). *(system will be unstable)*

(a) Case I: Underdamped Case ($0 < \xi < 1$)

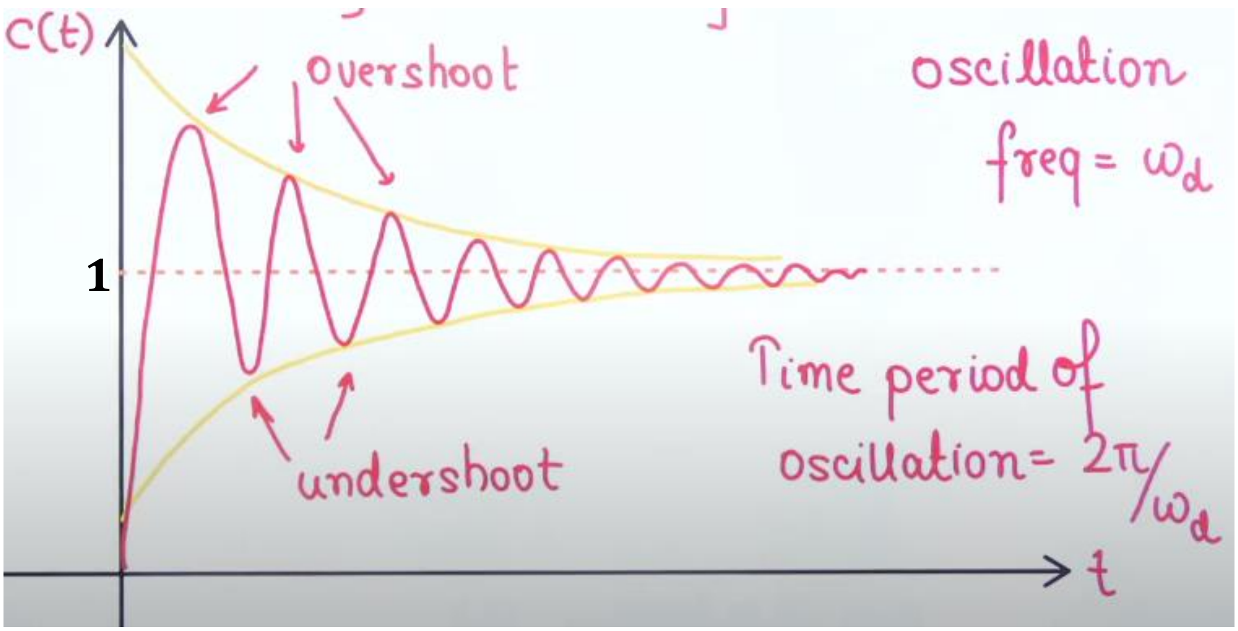
In this case, the response is given below:

$$\therefore\ c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\phi \cdot \cos\omega_d t + \cos\phi \cdot \sin\omega_d t\right)$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi).}$$

where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}\xi$

$$\cos\phi = \xi \Rightarrow \sin\phi = \sqrt{1-\xi^2}$$



→ The response is oscillatory with oscillating frequency 'ω_d' but decreasing amplitude due to exponential term '$e^{-\xi\omega_n t}$'.

→ This type of response is called underdamped response.

→ Steady state value $= 1$.

→ error is given by, $e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$

→ steady state error is $e_{ss} = 0$.

→ Time constant is $\boxed{T = 1/\xi\omega_n}$.

Note :→ The damped frequency always less than the undamped frequency because of factor ξ_e.

i.e., $\boxed{\omega_d < \omega_n}$ $\left[\because \omega_d = \omega_n\sqrt{1-\xi_e^2}\right]$

Note :→ As ξ increased, the response becomes progressively less oscillatory till it becomes critically damped (just non-oscillatory) for $\xi_e = 1$ and becomes overdamped for $\xi_e > 1$.

(b) Case II: Undamped Case (ξ = 0)

The time response for $\xi_e = 0$ will be :-

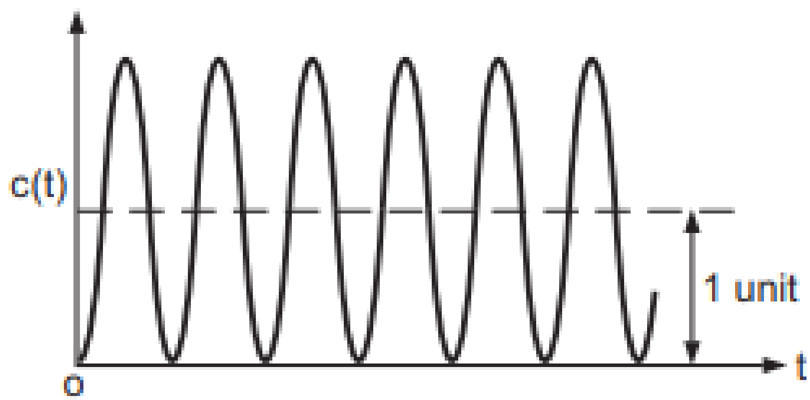
$$C(t) = 1 - \frac{e^{-\xi_e\omega_n t}}{\sqrt{1-\xi_e^2}} \sin\left[(\omega_n\sqrt{1-\xi_e^2})t + \cos^{-1}\xi_e\right]$$

$$c(t) = 1 - \frac{e^{-o\omega_n t}}{\sqrt{1-o^2}} \sin\left[\omega_n\sqrt{1-o^2}\,t + \tan^{-1}\left(\frac{\sqrt{1-o^2}}{o}\right)\right]$$

$$c(t) = 1 - \sin(\omega_n t + \tan^{-1}\infty)$$

$$c(t) = 1 - \sin\left(\omega_n t + \frac{\pi}{2}\right)$$

$$c(t) = (1 - \cos\omega_n t)$$



∴ So the response at $\xi_e = 0$, is called as undamped oscillation or sustained oscillation, with frequency 'ω_n'.

Or, for ($\xi = 0$, undamped case)

$$\frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2} = \frac{A(s^2+\omega_n^2)+Bs^2+Cs}{s(s^2+\omega_n^2)}$$

$$\omega_n^2 = (A+B)s^2 + Cs + \omega_n^2 A$$

$$A+B=0 \Rightarrow A=-B$$

$$C=0, \quad A=1$$

$$\therefore B=-1$$

$$\therefore C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$$c(t) = 1 - \cos\omega_n t$$

→ steady state value always lie between 0 to 2.

→ Error is given by $e(t) = r(t) - c(t) = \cos\omega_n t$

→ steady state error is between -1 to 1.

Note:- Since there is no time damping and therefore

- ➢ Oscillations never die out with time.
- ➢ Amplitude of oscillation = constant around steady-state.
- ➢ This response is known as undamped response.
- ➢ There is no loss of energy.

(c) Case III: Critically damped Case ($\xi = 1$)

The time response at $\xi = 1$ will be :-

$$c(t) = \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n \sqrt{1-\xi^2}) t + \phi \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left[\sin (\omega_n \sqrt{1-\xi^2}) t \cdot \cos\phi + (\cos \omega_n \sqrt{1-\xi^2} t) \sin\phi \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left[(\sin \omega_n \sqrt{1-\xi^2} t) \cdot \xi + (\cos \omega_n \sqrt{1-\xi^2} t) \sqrt{1-\xi^2} \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left[(\omega_n \sqrt{1-\xi^2}) t + 1 \cdot \sqrt{1-\xi^2} \right] \right\}$$

$$\lim_{\zeta \to 1} \sin (\omega_n \sqrt{1-\zeta^2}\, t) \to \omega_n \sqrt{1-\zeta^2}\, t$$

$$\lim_{\zeta \to 1} \cos (\omega_n \sqrt{1-\zeta^2}\, t) \to 1$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi \omega_n t}}{\cancel{\sqrt{1-\xi^2}}} \left[\cancel{\sqrt{1-\xi^2}} (\omega_n t + 1) \right] \right\}$$

$$\Rightarrow \boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) = 1 - e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}}$$

→ So, output response consists of two exponentials but with same time constant $T = 1/\omega_n$.

→ steady state value is $c(t)|_{t \to \infty} = 1$.

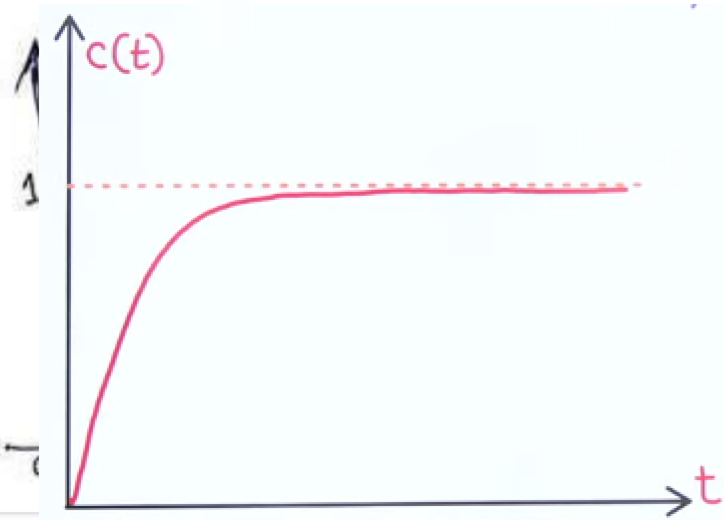
→ error is given by

$$e(t) = r(t) - c(t)$$

$$= e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}.$$

→ steady state error,

$$e_{ss} = e(t)|_{t \to \infty} = 0.$$



→ For $\xi = 1$, oscillations in output response are just dissappeared. This type of response is called as Critically damped response.

→ Characteristic eq^n : $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$$\Rightarrow s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \text{ or } -\xi\omega_n \pm j\omega_d.$$

→ For $\xi = 1$; The roots are $-\omega_n, -\omega_n$.

→ System is Absolute stable.

Or, for ($\xi = 1$, critical damped case)

$$c(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

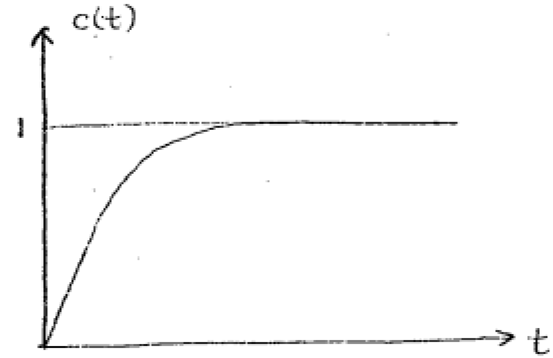
$$\omega_n^2 = A(s^2+2\omega_n s+\omega_n^2) + Bs(s+\omega_n) + Cs$$

compare both side

A = 1; B = -1 and C = $-\omega_n$

$$c(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t \cdot e^{-\omega_n t}$$



(d) Case IV: Overdamped Case ($\xi > 1$)

The time response is given by

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_n^2(\xi^2-1)}$$

$$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2} \quad \left[\text{Put } \omega_d^2 = \omega_n^2(\xi^2-1)\right]$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n+\omega_d)(s+\xi\omega_n-\omega_d)} = \frac{A}{s} + \frac{B}{s+\xi\omega_n+\omega_d} + \frac{C}{s+\xi\omega_n-\omega_d}.$$

$$\therefore A = \left.\frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2}\right|_{s=0} = \frac{\omega_n^2}{(\xi\omega_n)^2 - \omega_n^2(\xi^2-1)} = 1.$$

$$B = \left.\frac{\omega_n^2}{s(s+\xi\omega_n-\omega_d)}\right|_{s=-\xi\omega_n-\omega_d} = \frac{\omega_n^2}{(\xi\omega_n+\omega_d)(2\omega_d)} = \frac{\omega_n^2}{2\xi\omega_n\omega_d + 2\omega_d^2}$$

$$\Rightarrow B = \frac{\omega_n^2}{2\xi\omega_n \cdot \omega_n\sqrt{\xi^2-1} + 2\omega_n^2(\xi^2-1)} = \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)}.$$

Similarly, $C = \dfrac{-1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)}$.

Now,
$$C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)(s+\xi\omega_n+\omega_d)} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)(s+\xi\omega_n-\omega_d)}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\left[s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right]} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)\left[s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right]}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\left[s+\omega_n\left(\xi+\sqrt{\xi^2-1}\right)\right]} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)\left[s+\omega_n\left(\xi-\sqrt{\xi^2-1}\right)\right]}$$

Taking Inverse laplace transform :-

$$\boxed{c(t) = 1 + \frac{e^{-\left(\xi+\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)} - \frac{e^{-\left(\xi-\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)}}.$$

→ Now above response has two time constants :-

$$T_1 = \frac{1}{\left(\xi+\sqrt{\xi^2-1}\right)\omega_n} \quad \& \quad T_2 = \frac{1}{\left(\xi-\sqrt{\xi^2-1}\right)\omega_n}.$$

Since, here $\xi > 1$, then $T_1 \ll T_2$.

As a result the first exponential term decaying much faster than the other exponenrtial term. So, for time response neglect the term having the pole at $-(\xi + \sqrt{\xi^2 - 1})\omega_n$.

$$\boxed{c(t) = 1 - \frac{1}{2\sqrt{\xi_e^2 - 1}\,(\xi_e - \sqrt{\xi_e^2 - 1})} \cdot e^{-(\xi_e - \sqrt{\xi_e^2-1})\omega_n t}}$$

the time constant is —

$$\boxed{T = \frac{1}{(\xi_e - \sqrt{\xi_e^2 - 1})\omega_n}}$$

Observations :-

1. steady state value

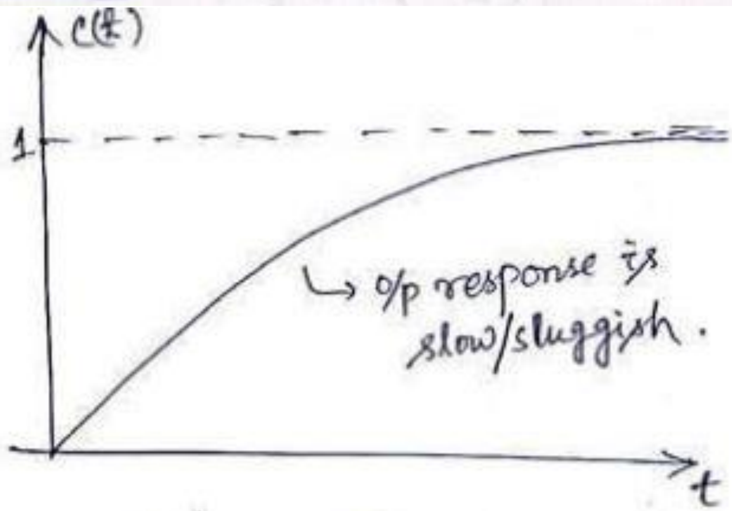
$$c(t)\big|_{t\to\infty} = 1.$$

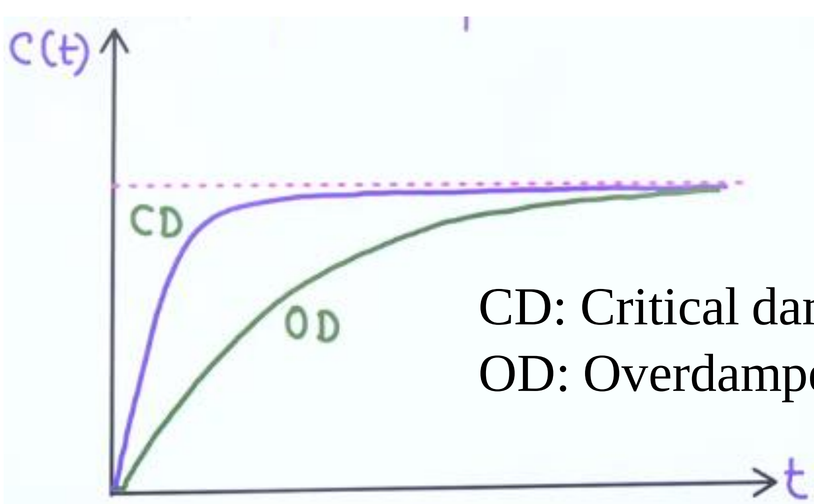
2. error is given by :-

$$e(t) = r(t) - c(t)$$

$$\Rightarrow e(t) = \frac{e^{-(\xi_e - \sqrt{\xi_e^2 - 1})\omega_n t}}{2\sqrt{\xi_e^2 - 1}\,(\xi_e - \sqrt{\xi_e^2 - 1})}$$

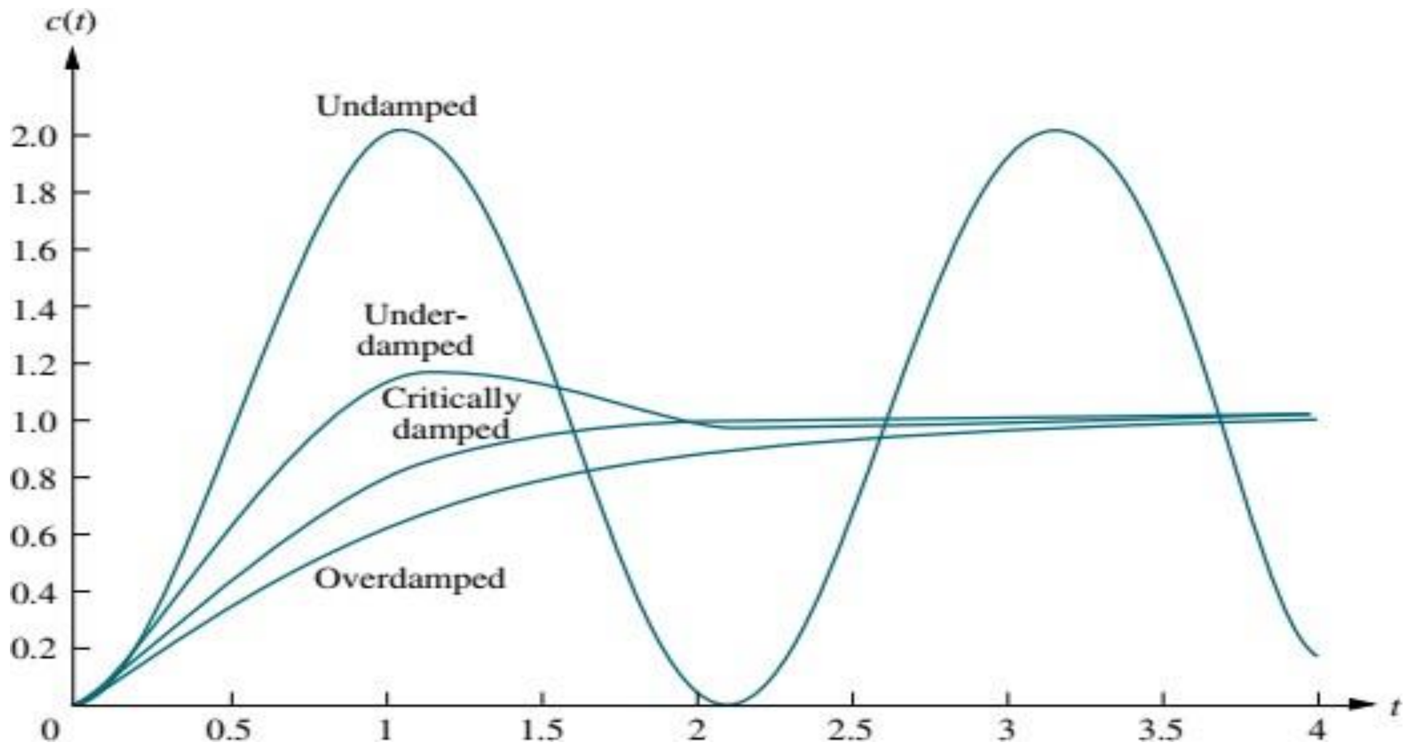
3. steady state error is $e_{ss} = e(t)\big|_{t\to\infty} = 0$.





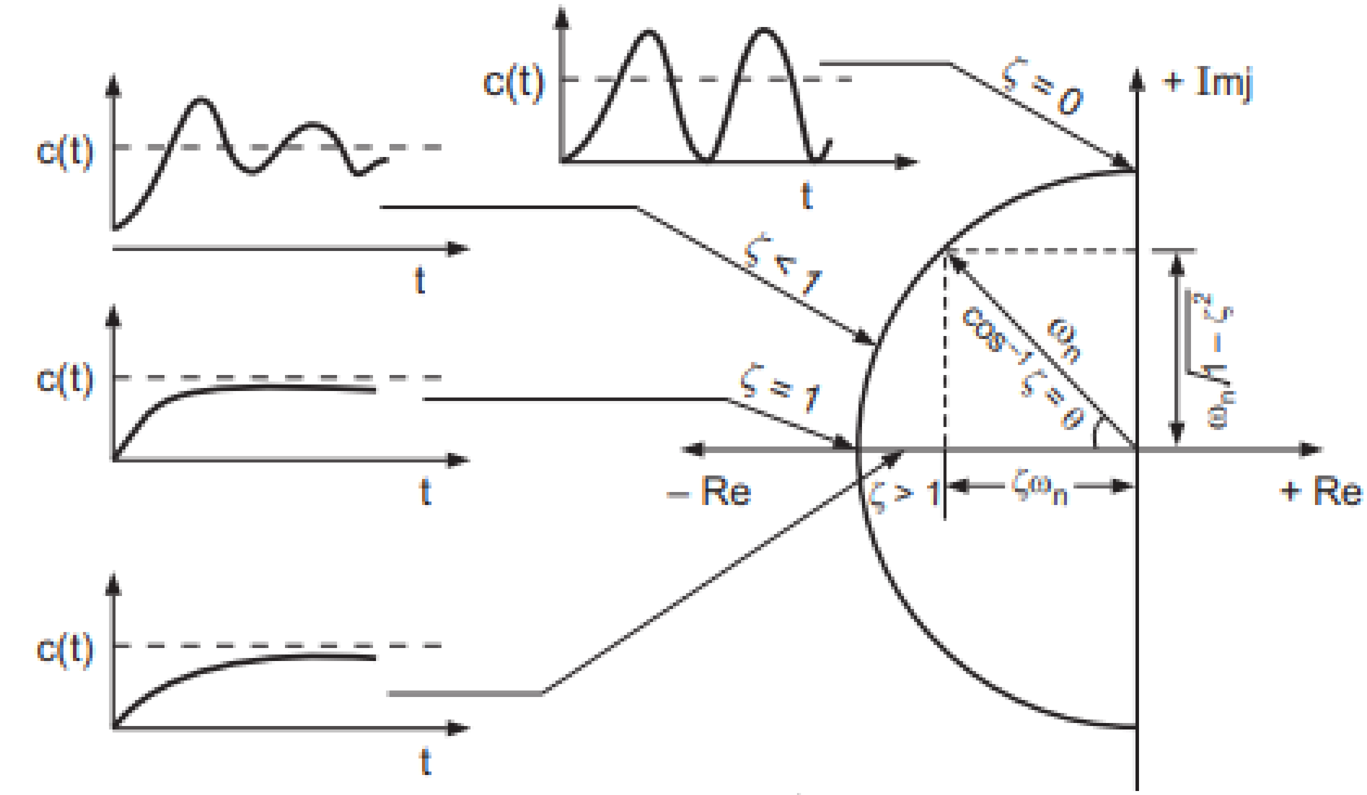
CD: Critical damped
OD: Overdamped

- Finally, the step response for the four cases of damping discussed in the above section are superimposed in figure below.
- Notice that the critically damped caes is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.



- $0<\xi<1$: under damped
  ↳ oscillations of reducing amplitude
- $\xi=1$: critically damped
  ↳ no oscillations
- $\xi>1$: over damped
  ↳ no oscillations but slower than critically damped
- $\xi=0$: undamped
  ↳ oscillations of const amplitude

The location of roots of the characteristic equation for various values of ζ (keeping ωn fixed) and the corresponding time response for a second order control system is shown here.



ORDER OF TIME CONSTANTS

Largest sluggish Resp

Medium Resp

Smallest τ

$$\tau_{\text{UNDAMPED}} > \tau_{\text{OVERDAMPED}} > \tau_{\text{UNDERDAMPED}} > \tau_{\text{CRITICAL DAMPER}}$$

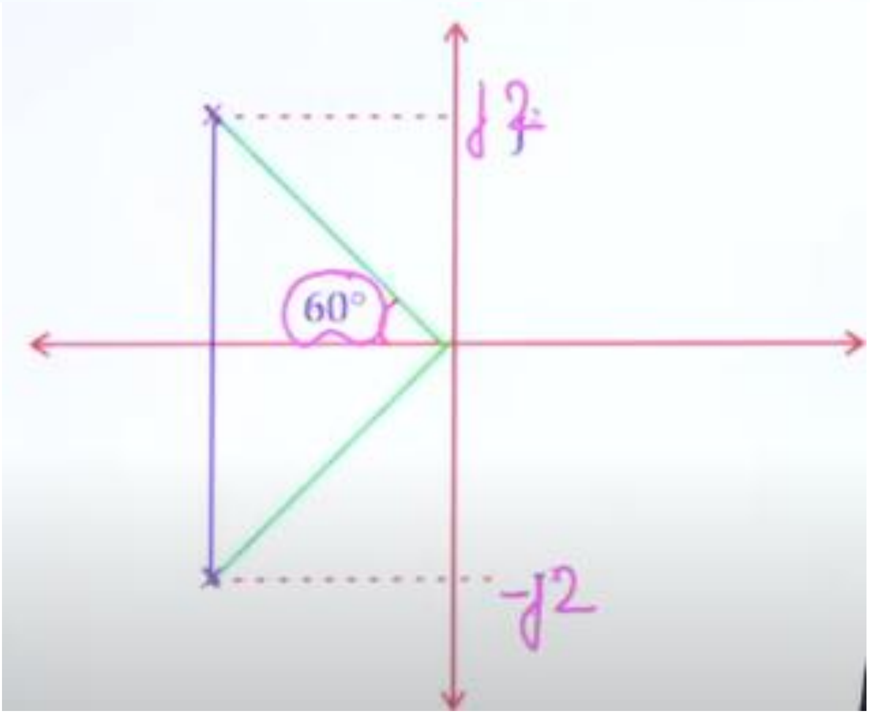
MARGINALLY STABLE

STABLE

$$\infty > \frac{1}{\xi\omega_n - \omega_n\sqrt{\xi^2-1}} > \frac{1}{\xi\omega_n} > \frac{1}{\omega_n}$$

Ques 1:

Calculate time constant of the system whose pole zero diagram is given.



Solution: This is underdamped system because from the given diagram, it can be seen that both the poles are lying left half of s-plane and both are conjugate complex (2nd and 3rd quadrant)

$$\cos 60 = \xi \Rightarrow \xi = 0.5$$

$$2 = \omega_n \sqrt{1-\xi^2}$$

$$\omega_n = \frac{2}{\sqrt{0.75}} = \frac{4}{\sqrt{3}}$$

under damped system

$$\tau = \frac{1}{|Re(pole)|} = \frac{1}{\xi \omega_n} = \frac{\sqrt{3}}{2} = 0.866$$

Ques 2: The transfer function of a system is given as $\frac{100}{s^2+20s+100}$, This system is

(a) An over damped system
(b) An under damped system
(c) A critically damped system
(d) An unstable system

Solution:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = 100$$

$$\omega_n = 10$$

$$2\xi\omega_n = 20$$

$\xi = 1$: Critically damped

Ques 3:

A unity negative feedback system has an open loop transfer function $G(s) = \frac{K}{s(s+10)}$, The gain K for the system to have a damping ratio of 0.25 is ________.

Solution:

$$T(s) = \frac{K}{s^2 + 10s + K}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$K = \omega_n^2$$

$$2\xi\omega_n = 10$$

$$2 \times 0.25 \times \omega_n = 10$$

$$\omega_n = 20$$

$$K = \omega_n^2 = 400$$

Ques 4:

The open loop transfer function of a unity feedback control system is given by $G(s) = \frac{K}{s(s+1)}$. If the system becomes critically damped, then the system gain 'K' tends to become ....

Solution: $\xi = 1$

$$T(s) = \frac{K}{s^2 + s + K}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$K = \omega_n^2$$

$$2\xi\omega_n = 1$$

$$2\omega_n = 1$$

$$\omega_n = 0.5$$

$$K = \omega_n^2 = 0.25$$

Ques 5:

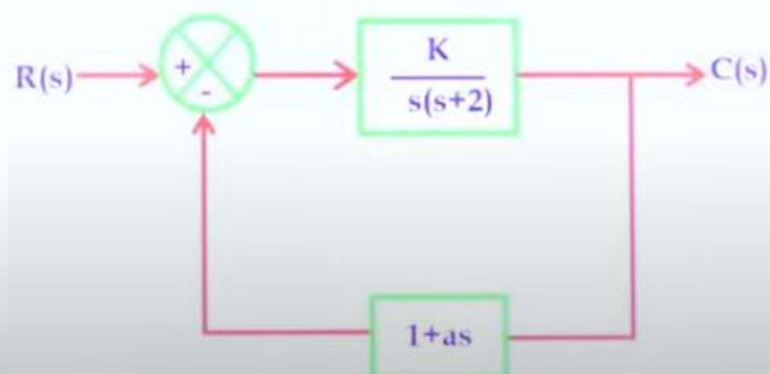
For the system shown in figure with a damping ratio ξ of 0.7 and an un-damped natural frequency ω_n of 4 rad / sec, the values of K and a are

(A) K = 4, a = 0.35

(B) K = 8, a = 0.455

(C) K = 16, a = 0.225

(D) K = 64, a = 0.9



Solution: (C)

$$T(s) = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)} \cdot (1+as)}$$

$$= \frac{K}{s^2 + 2s + K + Kas}$$

$$T(s) = \frac{K}{s^2 + (2+Ka)s + K}$$

$$2\xi\omega_n = 2 + Ka$$

$$4\ \omega_n^2 = K$$

$$\Rightarrow K = 4^2 = 16$$

$$2 \times 0.7 \times 4 = 2 + 16a$$

$$\Rightarrow 1.4 \times 4 - 2 = 16a$$

$$\Rightarrow a = \frac{5.6 - 2}{16}$$

$$= \frac{3.6}{16}$$

$$\boxed{a = 0.225}$$

& K = 16

Ques:- For the unity feedback system having $G(S) = \frac{K}{S(ST+2)}$,

Find the following:—

(i) the factor by which the gain K should be multiplied to increase the damping ratio from 0.15 to 0.6.

(ii) The factor by which the time constant T should be multiplied to reduce the damping ratio from 0.8 to 0.4.

Solⁿ :→ Here, $G(S) = \frac{K}{S(ST+2)}$ and $H(S) = 1$

$$\therefore \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\cdot H(S)} = \frac{K}{S^2T + 2S + K} = \frac{K/T}{S^2 + \frac{2}{T}\cdot S + K/T}$$

NOW

$$\omega_n^2 = \frac{K}{T} \quad \therefore \omega_n = \sqrt{K/T}$$

and $$2\xi\omega_n = 2/T \quad \therefore \xi = \frac{\cancel{2}}{T\cdot\cancel{2}\omega_n} = \frac{1}{\sqrt{KT}}$$

i) Let $\xi_1 = \frac{1}{\sqrt{K_1 T}}$

& $\xi_2 = \frac{1}{\sqrt{K_2 T}}$ for $\xi = 0.15$ and $\xi = 0.6$ respectively

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{K_2 T}{K_1 T}} = \frac{0.15}{0.6} = \frac{1}{4}$$

$$\therefore \frac{K_2}{K_1} = \frac{1}{16} \quad \therefore \boxed{K_2 = \frac{K_1}{16}}$$

the gain must be multiplied by factor 1/16 to increase the damping ratio 0.15 to 0.6.

(ii) Let $\xi_1 = \frac{1}{\sqrt{T_1 K}}$

and $\xi_2 = \frac{1}{\sqrt{T_2 K}}$ for $\xi = 0.8$ and $\xi = 0.4$ respectively

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{T_2 K}{T_1 K}} = \frac{0.8}{0.4} = 2$$

$$\Rightarrow \frac{T_2}{T_1} = 4 \Rightarrow \boxed{T_2 = 4T_1}$$

The time constant T must be multiplied by factor 4 to reduce the damping ratio from 0.8 to 0.4.

Example *The forward path transfer function of a unity feedback control system is given by*

$$G(s) = \frac{2}{s(s+3)}$$

Obtain an expression for unit step response of the system.

Solution. The overall transfer function for the system is

$$\frac{C(s)}{R(s)} = \frac{\frac{2}{s(s+3)}}{1 + \frac{2}{s(s+3)} \cdot 1} \quad \text{or} \quad \frac{C(s)}{R(s)} = \frac{2}{(s^2 + 3s + 2)}$$

It is noted that the denominator of the above expression can be factored as $[(s + 1)(s + 2)]$

$$\therefore \qquad C(s) = R(s) \frac{2}{[(s+1)(s+2)]}$$

As the input is a unit step

$$R(s) = 1/s$$

$$\therefore \qquad C(s) = \frac{1}{s} \cdot \frac{2}{[(s+1)(s+2)]}$$

The R.H.S. of the above expression can be expanded into partial fraction as follows :

$$\frac{1}{s} \cdot \frac{2}{(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{(s+1)} + \frac{K_3}{(s+2)}$$

The coefficients K_1, K_2 and K_3 can be determined as

$$K_1 = 1, K_2 = -2 \text{ and } K_3 = 1$$

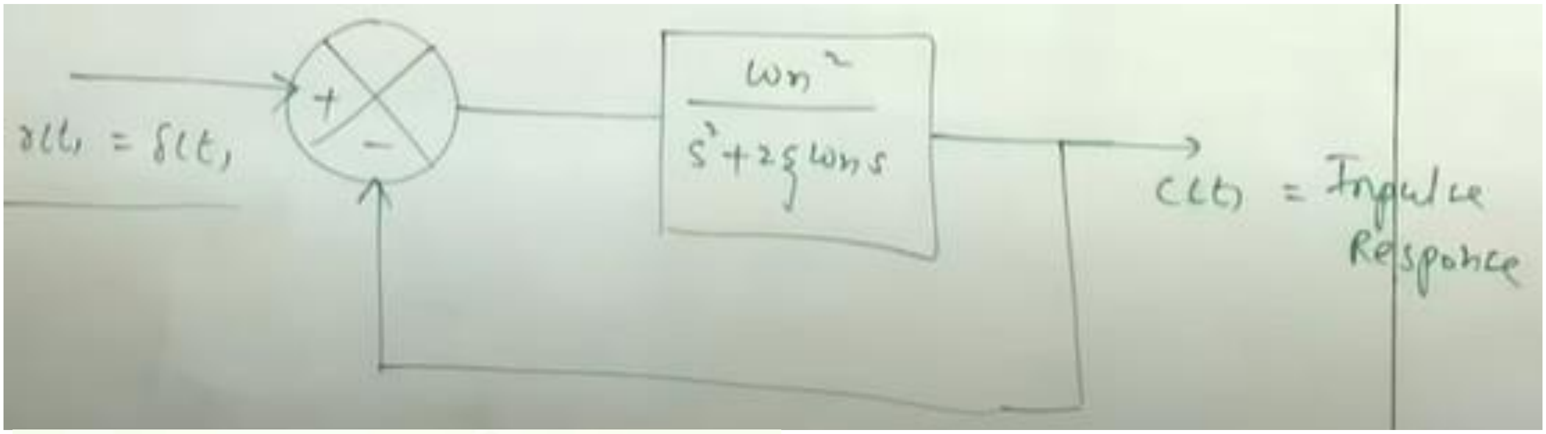
$$\therefore \qquad C(s) = \frac{1}{s} - \frac{2}{(s+1)} + \frac{1}{(s+2)}$$

Taking inverse Laplace transform on both sides $c(t) = (1 - 2e^{-t} + e^{-2t})$. **Ans.**

Assignment:

1. **Time Response of the Second Order (2^{nd} Order) System for unit Impulse Input.**
2. **Time Response of the Second Order (2^{nd} Order) System for unit Ramp Input.**

***Time Response of the Second Order (2nd Order) System for Impulse Response Input:**



$$\frac{C(s)}{R(s)} = CLTF = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$r(t) = \delta(t)$$

$$R(s) = 1.0$$

$$C(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Case 1. $\xi = 0$ (Undamped oscillation)

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$s^2 + \omega_n^2 = 0$$

$$s = \pm j\omega_n$$

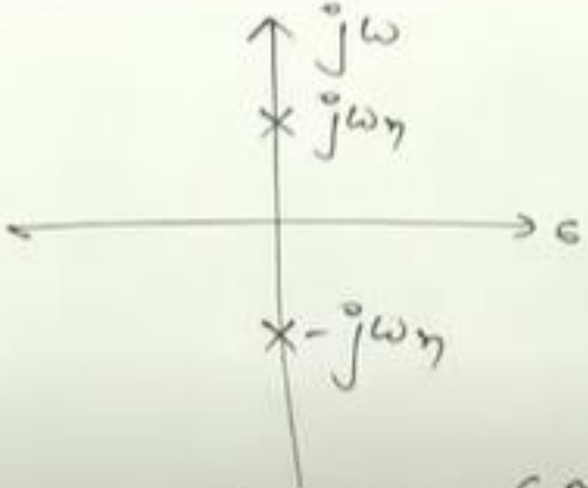


fig $\xi = 0$ (m.s.)

freq^n of oscillation $= \omega_n$ ✓

$$\tau = \frac{1}{0} = \infty$$ ✓

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$\mathcal{L}^{-1}$

$$c(t) = \omega_n \sin \omega_n t$$

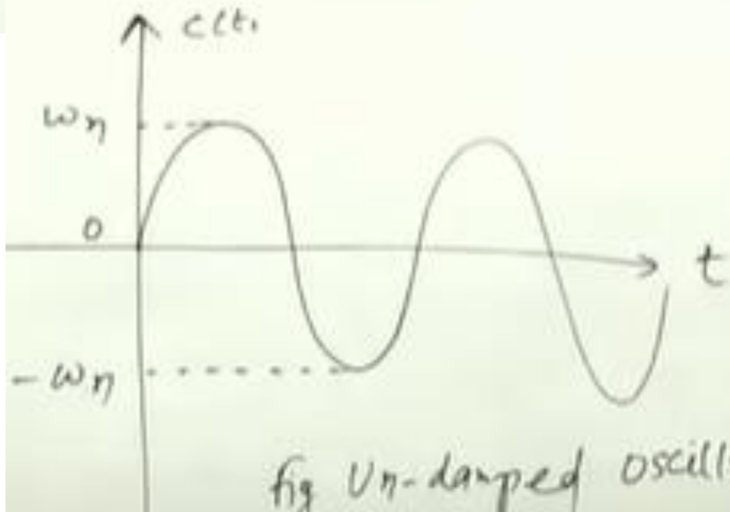


fig Un-damped oscillation

[Constant oscillation]

Case 02 $0 < \xi < 1$ (Under-damped oscillation)

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s = \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2 - 1}}{2}$$

$$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$\xi^2 - 1 = $ Neg

$\sqrt{\xi^2 - 1} = $ Imag

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$

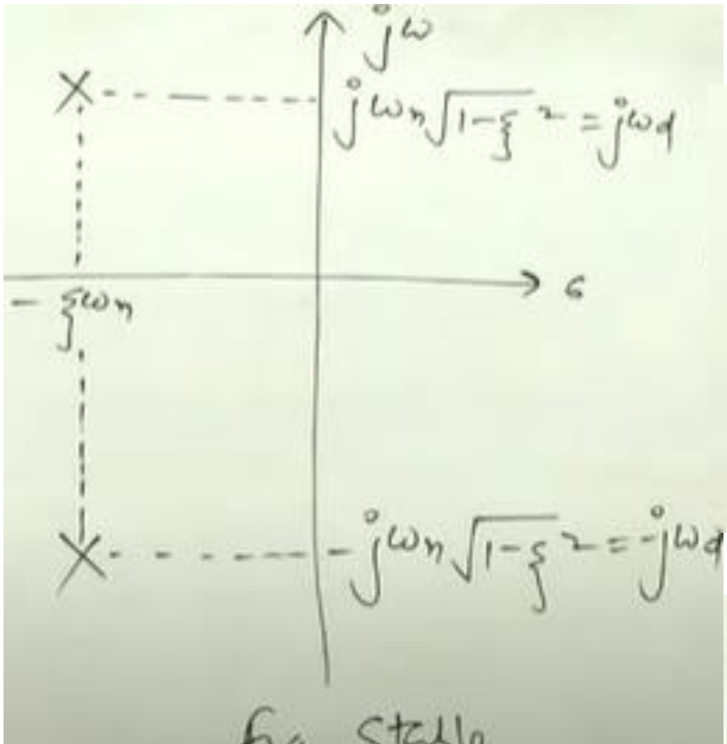


fig Stable

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \text{Damped freq}^n$$

$$\xi = 0$$

$$\omega_d = \omega_n$$

freq^n of oscillation $= \omega_d$ (r/sec)

$$C(s) = \frac{\omega_n^2}{(s+\alpha-j\omega_d)(s+\alpha+j\omega_d)}$$

$$C(s) = \frac{\omega_n^2 \times \omega_d \times 1/\omega_d}{(s+\alpha)^2 + \omega_d^2}$$

$$(a-jb)(a+jb) = a^2 - (jb)^2 = a^2 + b^2$$

$$C(s) = \frac{\omega_n^2 \times \omega_d \times 1/\omega_d}{(s+\alpha)^2 + \omega_d^2}$$

$$\frac{\omega_n^2}{\omega_d} = \frac{\omega_n^{\cancel{2}}}{\omega_n\sqrt{1-\xi^2}} = \frac{\omega_n}{\sqrt{1-\xi^2}}$$

$$C(s) = \frac{\omega_n}{\sqrt{1-\xi^2}} \left[\frac{\omega_d}{(s+\alpha)^2 + \omega_d^2} \right]$$

$$\mathcal{L}^{-1}$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\alpha t} \sin\omega_d t = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\omega_d t$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\omega_d t$$

$$\tau = \frac{1}{\xi\omega_n} \text{ sec}$$

$$\omega_0 = \omega_d$$

$$c(0) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-0} \sin 0 = 0.0$$

$$c(\infty) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\infty} \sin\infty = 0.0$$

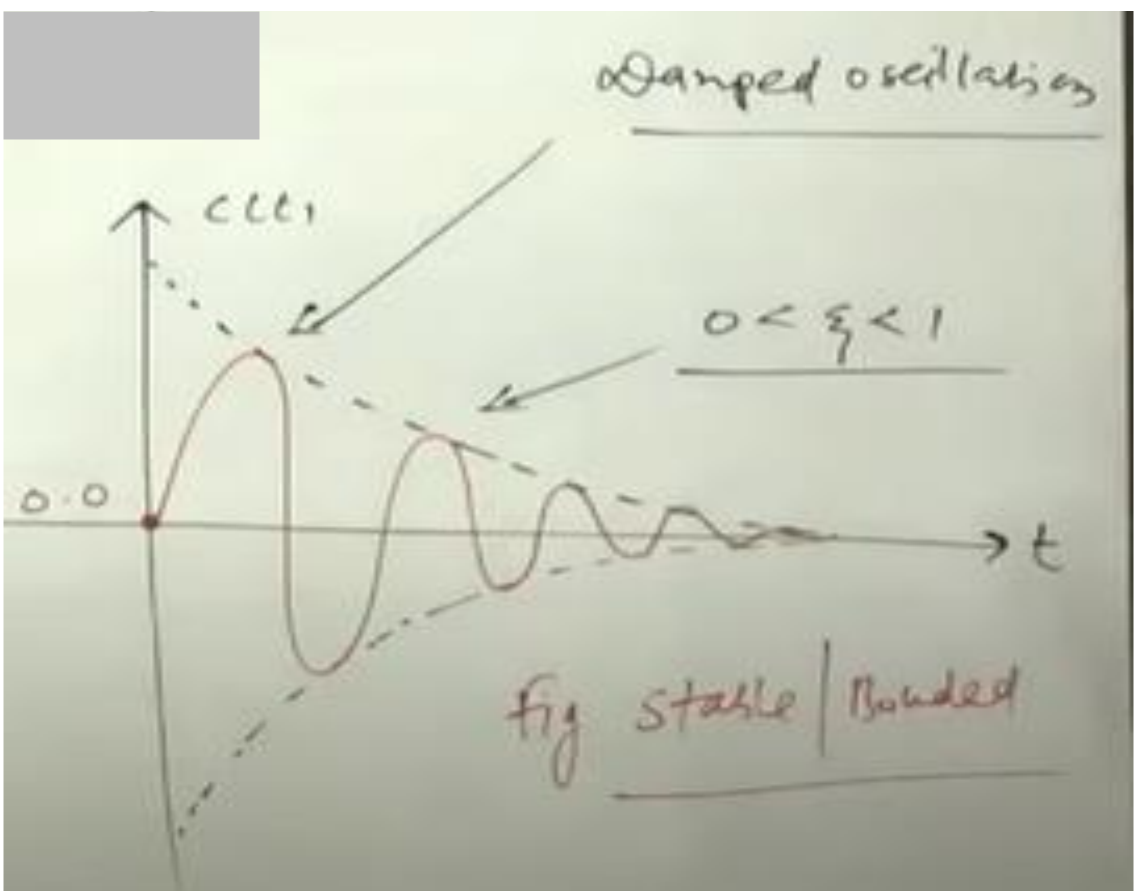


fig Stable | Bounded