2. Effect of ξ on nature of response:

(i) Time Response of the Second Order (2nd Order) System for Unit Step Input:

Consider the unit step signal as an input to the first order system.

So, r(t)=u(t) and therefore, R(s)=1/s

From 2nd order standard block diagram, we can write C(s) as,

$$C(s) = \left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right) R(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$C(s) = \left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

Do partial fraction of C(s).

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\Rightarrow \frac{\omega_n^2}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}} = \frac{A(s^2+2\xi\omega_n s+\omega_n^2)+(Bs+C)s}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}}$$

$$\Rightarrow \omega_n^2 = (A+B)s^2 + (2A\xi\omega_n + C)s + A\omega_n^2$$

$$\therefore (A+B)=0 \quad ; \quad 2A\xi\omega_n + C = 0 \quad ; \quad A\omega_n^2 = \omega_n^2$$

$$\Rightarrow B=-1 \qquad \Rightarrow C = -2\xi\omega_n \qquad \Rightarrow A = 1.$$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$ (Making denominator Perfect square)

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \qquad \left[\because \omega_d = \omega_n\sqrt{1-\xi^2}\right]$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

Taking Laplace inverse transform :-

$$\therefore c(t) = 1 - e^{-\varepsilon_e \omega_n t} \cdot \cos\omega_d t - \frac{\varepsilon_e \omega_n}{\omega_d} \cdot e^{-\varepsilon_e \omega_n t} \cdot \sin\omega_d t$$

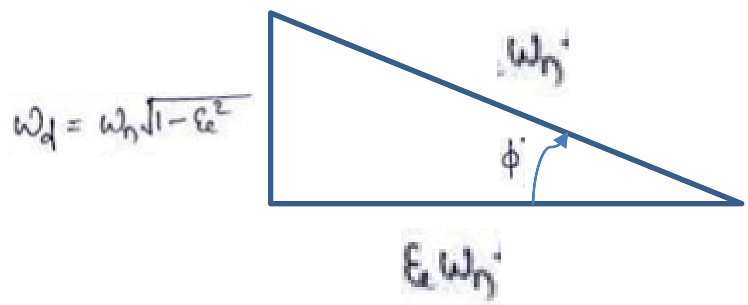
$$\Rightarrow c(t) = 1 - \frac{e^{-\varepsilon_e \omega_n t}}{\sqrt{1-\varepsilon_e^2}} \left[\left(\sqrt{1-\varepsilon_e^2}\right) \cos\omega_d t + \varepsilon_e \sin\omega_d t \right].$$

Put $\cos\phi = \varepsilon_e \Rightarrow \sin\phi = \sqrt{1-\varepsilon_e^2}$

$$\therefore c(t) = 1 - \frac{e^{-\varepsilon_e \omega_n t}}{\sqrt{1-\varepsilon_e^2}} \left(\sin\phi \cdot \cos\omega_d t + \cos\phi \cdot \sin\omega_d t \right)$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\varepsilon_e \omega_n t}}{\sqrt{1-\varepsilon_e^2}} \sin(\omega_d t + \phi).}$$

where, $\omega_d = \omega_n \sqrt{1-\varepsilon_e^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\varepsilon_e^2}}{\varepsilon_e}\right) = \cos^{-1}\varepsilon_e$



$\sqrt{1-\xi_e^2} = \sin\phi$

$\therefore \cos\phi = \xi_e, \tan\phi = \frac{\sqrt{1-\xi_e^2}}{\xi_e}$

Now, Put the values of ξ and ω_d in the expression of c(t), we get

$$\boxed{c(t) = \underbrace{1}_{\text{Steady state}} - \underbrace{\frac{e^{-\xi_e \omega_n t}}{\sqrt{1-\xi_e^2}} \sin\left[\left(\omega_n\sqrt{1-\xi_e^2}\right)t + \tan^{-1}\frac{\sqrt{1-\xi_e^2}}{\xi_e}\right]}_{\text{Transient}}} \quad \text{......(3)}$$

$e^{-t/T} = e^{-\varepsilon\omega_n t}$ $\longrightarrow$ Time constant T is -

$$\boxed{T = \frac{1}{\xi_e \omega_n}} \; **$$

the error signal for the system

$$e(t) = r(t) - c(t)$$

$$= \cancel{r} - \cancel{r} + \frac{e^{-\xi_e \omega_n t}}{\sqrt{1-\xi_e^2}} \sin\left[(\omega_n\sqrt{1-\xi_e^2})t + \tan^{-1}\frac{\sqrt{1-\xi_e^2}}{\xi_e}\right]$$

$$\Rightarrow \boxed{e(t) = \frac{e^{-\xi_e \omega_n t}}{\sqrt{1-\xi_e^2}} \cdot \sin\left[(\omega_n\sqrt{1-\xi_e^2})t + \tan^{-1}\frac{\sqrt{1-\xi_e^2}}{\xi_e}\right]} \quad \text{…..(4)}$$

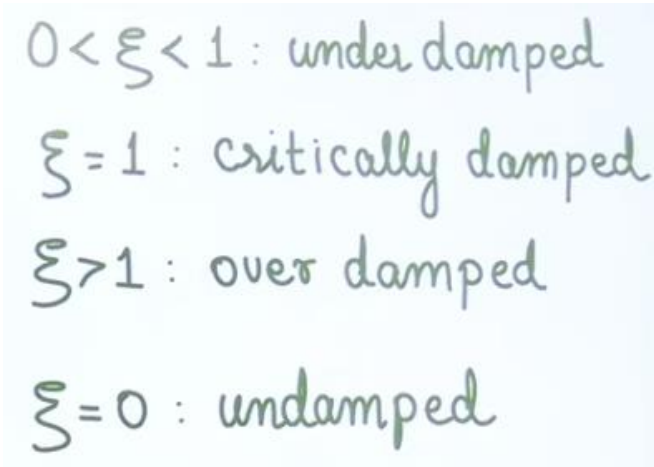
the steady state error is —

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$\boxed{e_{ss} = 0}$$

<u>therefore</u>, at steady state there is no error between input and output.

Note:- *As the time response of 2nd order system is influenced by ξ, therefore, there are four possible cases for positively damped systems ($\xi > 0$). (system will be stable or marginal stable system)*



Also, there are three possible cases for negatively damped systems ($\xi < 0$). (system will be unstable)

(a) Case I: Underdamped Case ($0 < \xi < 1$)

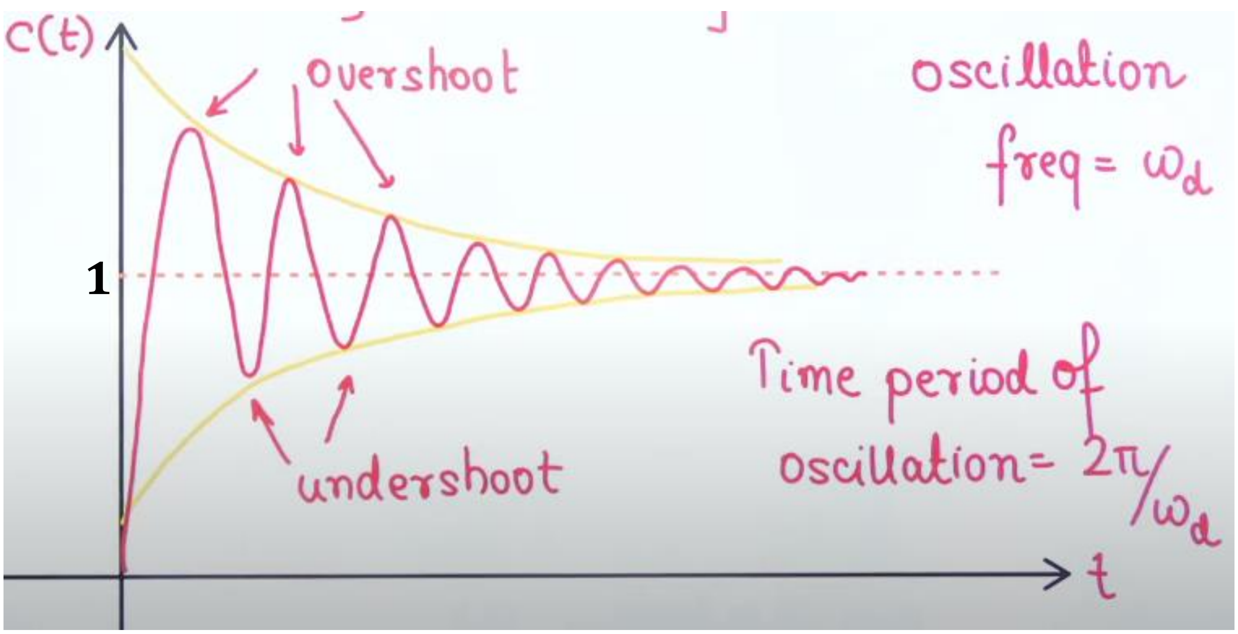
In this case, the response is given below:

$$\therefore\ c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\phi \cdot \cos\omega_d t + \cos\phi \cdot \sin\omega_d t\right)$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi).}$$

where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}\xi$

$$\cos\phi = \xi \Rightarrow \sin\phi = \sqrt{1-\xi^2}$$



→ The response is oscillatory with oscillating frequency 'ω_d' but decreasing amplitude due to exponential term '$e^{-\xi\omega_n t}$'.

→ This type of response is called underdamped response.

→ Steady state value $= 1$.

→ error is given by, $e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$

→ Steady state error is $e_{ss} = 0$.

→ Time constant is $\boxed{T = 1/\xi\omega_n}$.

Note :→ the damped frequency always less than the undamped frequency because of factor ξ_e.

i.e; $\boxed{\omega_d < \omega_n}$ $\left[\because \omega_d = \omega_n\sqrt{1-\xi_e^2}\right]$

Note :→ As ξ_e increased, the response becomes progressively less oscillatory till it becomes critically damped (just non-oscillatory) for $\xi_e = 1$ and becomes overdamped for $\xi_e > 1$.

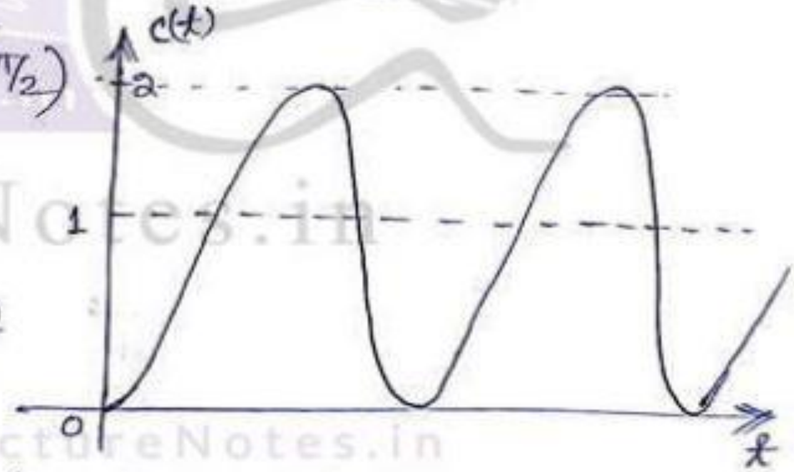
(b) Case II: Undamped Case ($\xi = 0$)

The time response for $\xi_e = 0$ will be :-

$$c(t) = 1 - \frac{e^{-\xi_e \omega_n t}}{\sqrt{1-\xi_e^2}} \sin\left[\left(\omega_n\sqrt{1-\xi_e^2}\right)t + \cos^{-1}\xi_e\right].$$

$$\Rightarrow c(t) = 1 - \frac{1}{1}\sin(\omega_n t + \pi/2)$$

$$\Rightarrow \boxed{c(t) = (1 - \cos\omega_n t)}$$



∴ So the response at $\xi_e = 0$, is called as undamped oscillation or sustained oscillation, with frequency 'ω_n'.

→ steady state value always lie between 0 to 2.

→ Error is given by $e(t) = r(t) - c(t) = \cos\omega_n t$

→ steady state error is between -1 to 1.

Note:- Since there is no time damping and therefore

- Oscillations never die out with time.
- Amplitude of oscillation = constant around steady-state.
- This response is known as undamped response.
- There is no loss of energy.

(c) Case III: Critically damped Case ($\xi = 1$)

The time response at $\xi = 1$ will be :-

$$c(t) = \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[(\omega_n\sqrt{1-\xi^2})t + \phi] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sin(\omega_n\sqrt{1-\xi^2})t \cdot \cos\phi + (\cos \omega_n\sqrt{1-\xi^2}\, t)\sin\phi \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[(\sin \omega_n\sqrt{1-\xi^2}\, t) \cdot \xi + (\cos \omega_n\sqrt{1-\xi^2}\, t)\sqrt{1-\xi^2} \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[(\omega_n\sqrt{1-\xi^2})t + 1 \cdot \sqrt{1-\xi^2} \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\cancel{\sqrt{1-\xi^2}}} \left[\cancel{\sqrt{1-\xi^2}}\,(\omega_n t + 1) \right] \right\}$$

$$\Rightarrow \boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) = 1 - e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}}$$

→ So, output response consists of two exponentials but with same time constant $T = 1/\omega_n$.

→ steady state value is $c(t)|_{t\to\infty} = 1$.

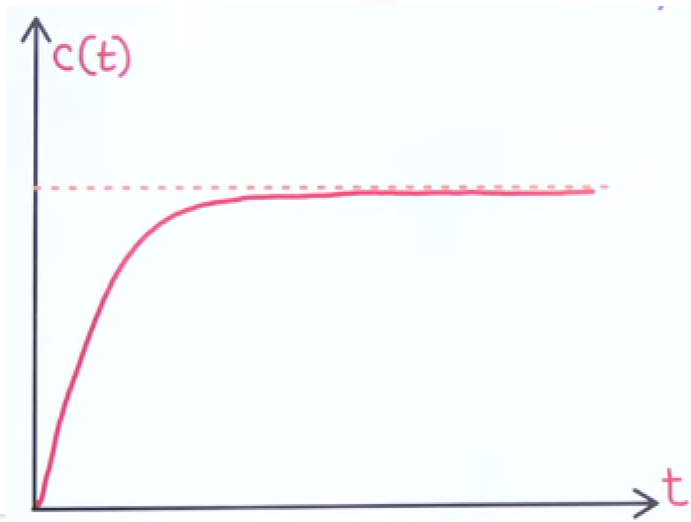
→ error is given by

$e(t) = r(t) - c(t)$

$= e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}$.

→ steady state error,

$e_{ss} = e(t)|_{t\to\infty} = 0$.



→ For $\xi = 1$, oscillations in output response are just dissappeared. This type of response is called as Critically damped response.

→ Characteristic eqⁿ : $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$$\Rightarrow s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \text{ or } -\xi\omega_n \pm j\omega_d .$$

→ For $\xi = 1$; The roots are $-\omega_n, -\omega_n$.

→ System is Absolute stable.

(d) Case IV: Overdamped Case ($\xi > 1$)

The time response is given by

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_n^2(\xi^2-1)}$$

$$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2} \quad \left[\text{Put } \omega_d^2 = \omega_n^2(\xi^2-1)\right]$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n+\omega_d)(s+\xi\omega_n-\omega_d)} = \frac{A}{s} + \frac{B}{s+\xi\omega_n+\omega_d} + \frac{C}{s+\xi\omega_n-\omega_d}.$$

$$\therefore A = \left.\frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2}\right|_{s=0} = \frac{\omega_n^2}{(\xi\omega_n)^2 - \omega_n^2(\xi^2-1)} = 1.$$

$$B = \left.\frac{\omega_n^2}{s(s+\xi\omega_n+\omega_d)}\right|_{s=-\xi\omega_n-\omega_d} = \frac{\omega_n^2}{(\xi\omega_n+\omega_d)(2\omega_d)} = \frac{\omega_n^2}{2\xi\omega_n\omega_d + 2\omega_d^2}$$

$$\Rightarrow B = \frac{\omega_n^2}{2\xi\omega_n \cdot \omega_n\sqrt{\xi^2-1} + 2\omega_n^2(\xi^2-1)} = \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)}.$$

Similarly, $C = \dfrac{-1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)}$.

Now,

$$C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)(s+\xi\omega_n+\omega_d)} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)(s+\xi\omega_n-\omega_d)}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\left[s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right]} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)\left[s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right]}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\left[s+\omega_n\left(\xi+\sqrt{\xi^2-1}\right)\right]} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)\left[s+\omega_n\left(\xi-\sqrt{\xi^2-1}\right)\right]}$$

Taking Inverse Laplace transform :-

$$\boxed{c(t) = 1 + \frac{e^{-\left(\xi+\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)} - \frac{e^{-\left(\xi-\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)}}.$$

→ Now above response has two time constants :-

$$T_1 = \frac{1}{\left(\xi+\sqrt{\xi^2-1}\right)\omega_n} \quad \& \quad T_2 = \frac{1}{\left(\xi-\sqrt{\xi^2-1}\right)\omega_n}.$$

Since, here $\xi > 1$, then $T_1 \ll T_2$.

As a result the first exponential term decaying much faster than the other exponenrtial term. So, for time response neglect the term having the pole at $-(\xi + \sqrt{\xi^2 - 1})\omega_n$.

$$\boxed{c(t) = 1 - \frac{1}{2\sqrt{\xi_e^2-1}\,(\xi_e - \sqrt{\xi_e^2-1})} \cdot e^{-(\xi_e - \sqrt{\xi_e^2-1})\omega_n t}}$$

the time constant is –

$$\boxed{T = \frac{1}{(\xi_e - \sqrt{\xi_e^2-1})\omega_n}}$$

<u>Observations :-</u>

1. steady state value

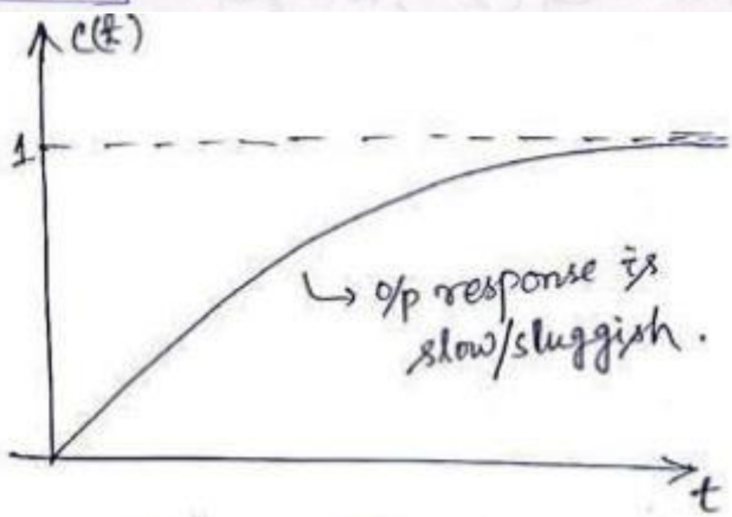
$$c(t)\big|_{t\to\infty} = 1 .$$

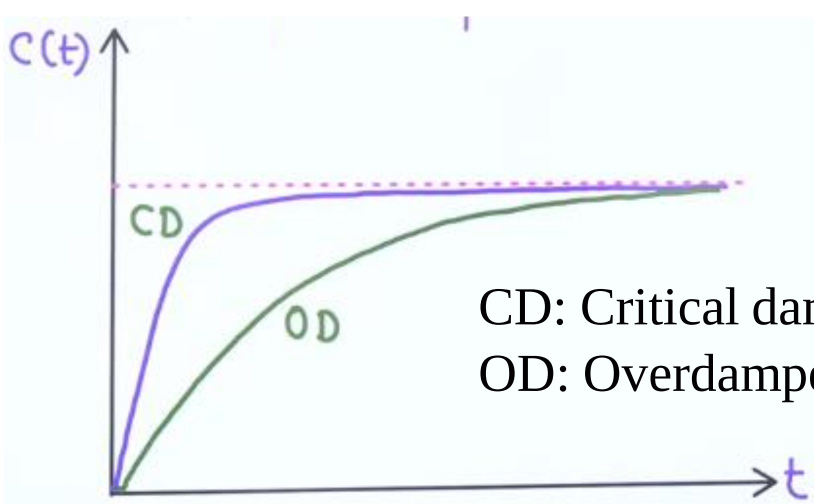
2. error is given by :-

$$e(t) = r(t) - c(t)$$

$$\Rightarrow e(t) = \frac{e^{-(\xi_e - \sqrt{\xi_e^2-1})\omega_n t}}{2\sqrt{\xi_e^2-1}\,(\xi_e - \sqrt{\xi_e^2-1})}$$

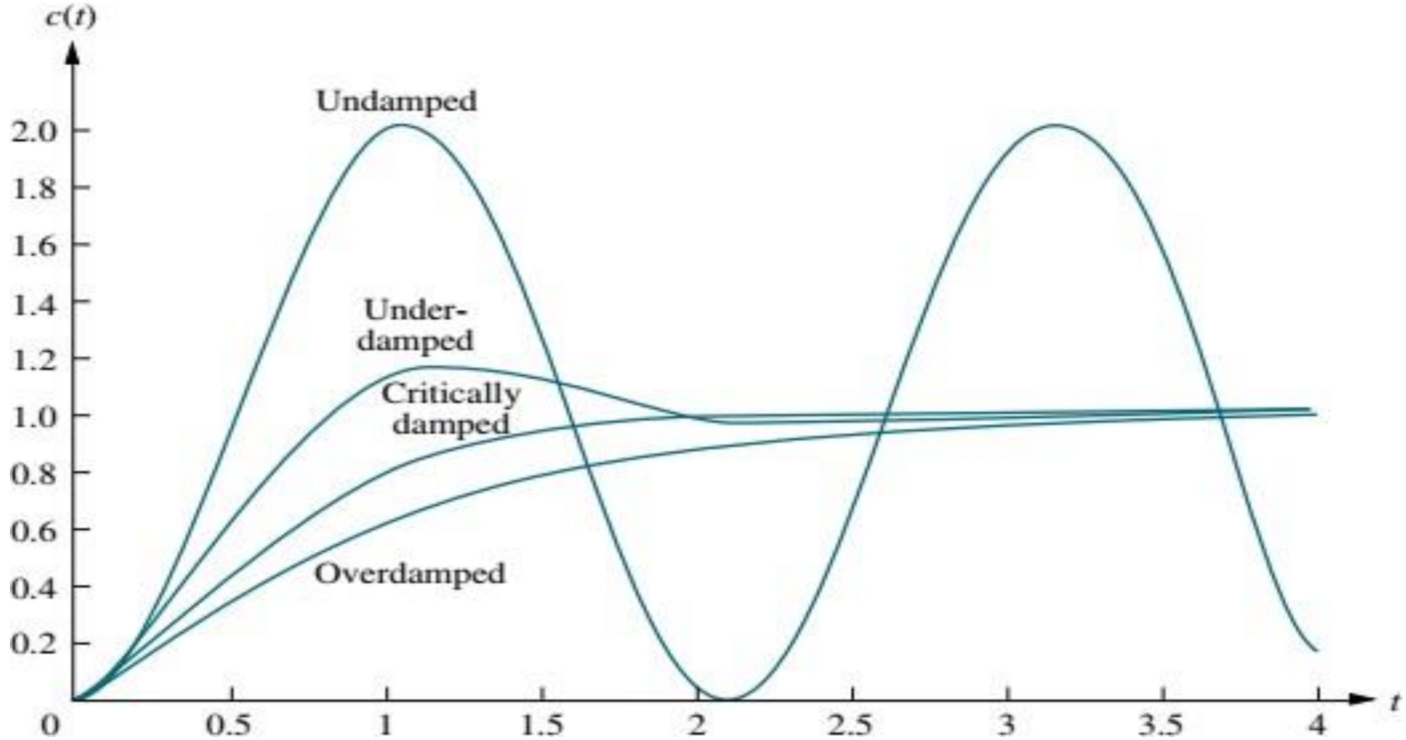
3. steady state error is $e_{ss} = e(t)\big|_{t\to\infty} = 0$.





CD: Critical damped
OD: Overdamped

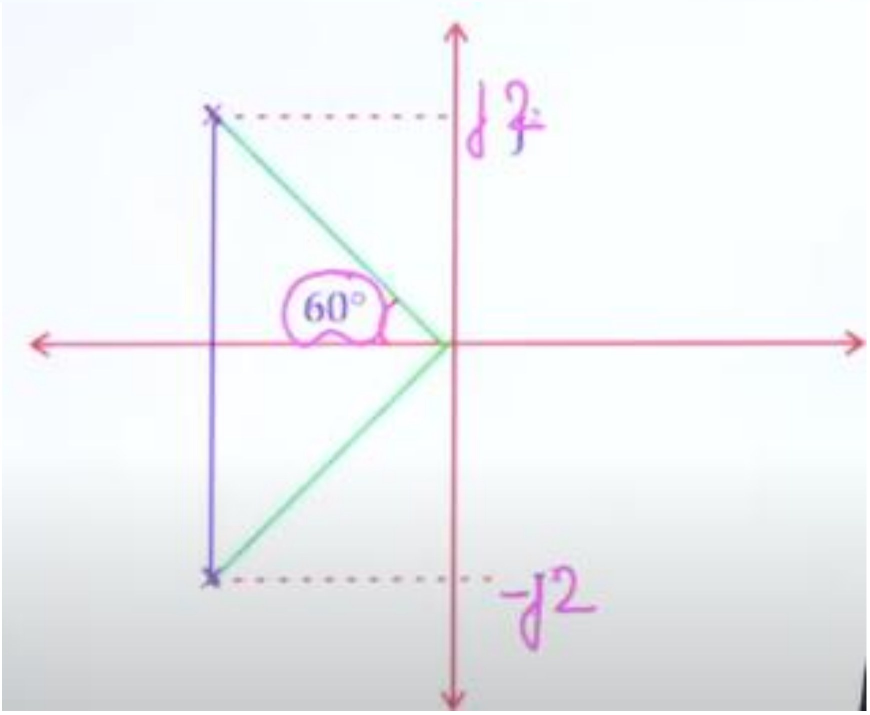
- Finally, the step response for the four cases of damping discussed in the above section are superimposed in figure below.
- Notice that the critically damped caes is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.



- $0 < \xi < 1$: underdamped
  ↳ oscillations of reducing amplitude
- $\xi = 1$: critically damped
  ↳ no oscillations
- $\xi > 1$: over damped
  ↳ no oscillations but slower than critically damped
- $\xi = 0$: undamped
  ↳ oscillations of const amplitude

Ques 1:

Calculate time constant of the system whose pole zero diagram is given.



Solution: This is underdamped system because from the given diagram, it can be seen that both the poles are lying left half of s-plane and both are conjugate complex (2nd and 3rd quadrant)

$$\cos 60 = \xi \Rightarrow \xi = 0.5$$

$$2 = \omega_n \sqrt{1-\xi^2}$$

$$\omega_n = \frac{2}{\sqrt{0.75}} = \frac{4}{\sqrt{3}}$$

under damped system

$$\tau = \frac{1}{|Re(pole)|} = \frac{1}{\xi \omega_n} = \frac{\sqrt{3}}{2} = 0.866$$

Ques 2: The transfer function of a system is given as $\frac{100}{s^2+20s+100}$, This system is

(a) An over damped system

(b) An under damped system

(c) A critically damped system

(d) An unstable system

Solution:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = 100$$

$$\omega_n = 10$$

$$2\xi\omega_n = 20$$

$\xi = 1$: Critically damped

Ques 3:

A unity negative feedback system has an open loop transfer function $G(s) = \frac{K}{s(s+10)}$, The gain K for the system to have a damping ratio of 0.25 is ________.

Solution:

$$T(s) = \frac{K}{s^2 + 10s + K}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$K = \omega_n^2$$

$$2\xi\omega_n = 10$$

$$2 \times 0.25 \times \omega_n = 10$$

$$\omega_n = 20$$

$$K = \omega_n^2 = 400$$

Ques 4:

The open loop transfer function of a unity feedback control system is given by $G(s) = \frac{K}{s(s+1)}$. If the system becomes critically damped, then the system gain 'K' tends to become ....

Solution: $\xi = 1$

$$T(s) = \frac{K}{s^2 + s + K}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$K = \omega_n^2$$

$$2\xi\omega_n = 1$$

$$2\omega_n = 1$$

$$\omega_n = 0.5$$

$$K = \omega_n^2 = 0.25$$

Ques 5:

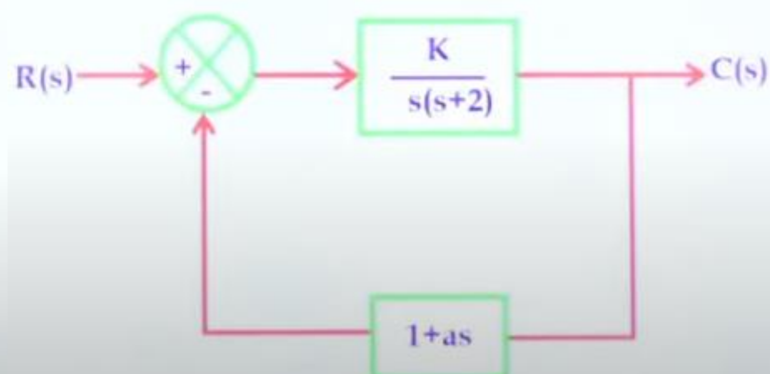
For the system shown in figure with a damping ratio ξ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec, the values of K and a are

(A) K = 4, a = 0.35

(B) K = 8, a = 0.455

(C) K = 16, a = 0.225

(D) K = 64, a = 0.9



Solution: (C)

$$T(s) = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)}\cdot(1+as)}$$

$$= \frac{K}{s^2 + 2s + K + Kas}$$

$$T(s) = \frac{K}{s^2 + (2+Ka)s + K}$$

$$2\xi\omega_n = 2 + Ka$$

$$\omega_n^2 = K$$

$$\Rightarrow K = 4^2 = 16$$

$$2 \times 0.7 \times 4 = 2 + 16a$$

$$\Rightarrow 1.4 \times 4 - 2 = 16a$$

$$\Rightarrow a = \frac{5.6 - 2}{16} = \frac{3.6}{16}$$

$$\boxed{a = 0.225}$$

$$\& \ K = 16$$

Ques:- For the unity feedback system having $G(S) = \frac{K}{S(ST+2)}$,

Find the following:—

(i) the factor by which the gain K should be multiplied to increase the damping ratio from 0.15 to 0.6.

(ii) The factor by which the time constant T should be multiplied to reduce the damping ratio from 0.8 to 0.4.

Solⁿ :→ Here, $G(S) = \frac{K}{S(ST+2)}$ and $H(S) = 1$

$$\therefore \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\cdot H(S)} = \frac{K}{S^2T + 2S + K} = \frac{K/T}{S^2 + \frac{2}{T}\cdot S + K/T}$$

NOW

$$\omega_n^2 = \frac{K}{T} \quad \therefore \omega_n = \sqrt{K/T}$$

and $$2\xi\omega_n = 2/T \quad \therefore \xi = \frac{\cancel{2}}{T\cdot\cancel{2}\omega_n} = \frac{1}{\sqrt{KT}}$$

i) Let $\xi_1 = \frac{1}{\sqrt{K_1 T}}$

& $\xi_2 = \frac{1}{\sqrt{K_2 T}}$ for $\xi = 0.15$ and $\xi = 0.6$ respectively

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{K_2 T}{K_1 T}} = \frac{0.15}{0.6} = \frac{1}{4}$$

$$\therefore \frac{K_2}{K_1} = \frac{1}{16} \quad \therefore \boxed{K_2 = \frac{K_1}{16}}$$

the gain must be multiplied by factor $1/16$ to increase the damping ratio 0.15 to 0.6.

(ii) Let $\xi_1 = \frac{1}{\sqrt{T_1 K}}$

and $\xi_2 = \frac{1}{\sqrt{T_2 K}}$ for $\xi_1 = 0.8$ and $\xi_2 = 0.4$ respectively

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{T_2 K}{T_1 K}} = \frac{0.8}{0.4} = 2$$

$$\Rightarrow \frac{T_2}{T_1} = 4 \Rightarrow \boxed{T_2 = 4T_1}$$

The time constant T must be multiplied by factor 4 to reduce the damping ratio from 0.8 to 0.4.

Assignment:

1. **Time Response of the Second Order (2^{nd} Order) System for unit Impulse Input.**
2. **Time Response of the Second Order (2^{nd} Order) System for unit Ramp Input.**