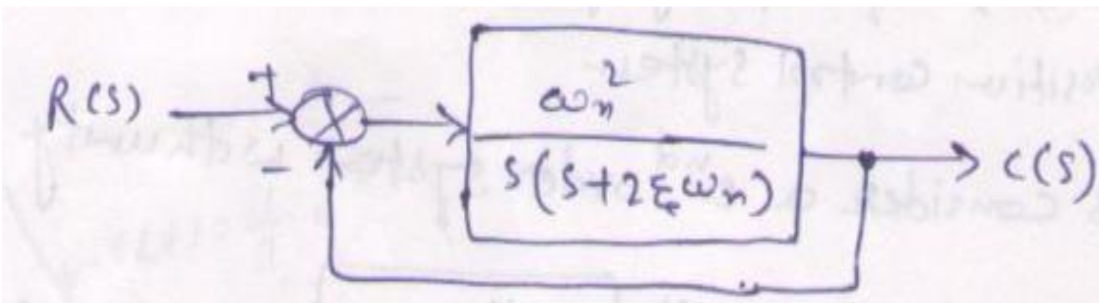
Transient State Analysis

Time Response of the Second Order (2^{nd} Order) System:

- A control system in which the highest power of 's' in the denominator of its transfer function is equal to two is called the 2^{nd} order control system.
- Consider the following block diagram of the closed loop control system. Here, an open loop transfer function (OLTF) $\omega_n^2/s(s+2\xi\omega_n)$ is connected with a unity negative feedback.



Here, $G(s) = \text{OLTF} = \omega_n^2/s(s+2\xi\omega_n)$ and $H(s) = 1$, therefore, the closed loop transfer function (CLTF) is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\xi\omega_n)}} \quad (1)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \quad (1)$$

2^{nd} order low pass filter

Standard form of 2^{nd} order CLTF

Where, C(s) is the Laplace transform of the output signal, c(t)

R(s) is the Laplace transform of the input signal, r(t)

ω_n represents the natural frequency

and ξ represents the damping ratio

$$T(s=0) = \text{Low freq}^{y}\ \text{gain} = \text{DC gain} = \frac{\omega_n^2}{0+0+\omega_n^2} = 1.0$$

$$T(s=\infty) = \text{High freq}^{y}\ \text{gain} = 0.0$$

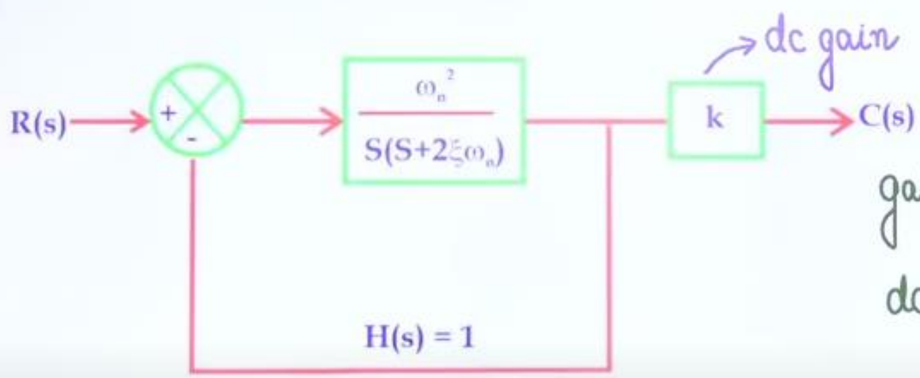
$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The characteristic equation is-

$$1 + G(s)H(s) = 0$$

$$\therefore s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

i.e., Roots of Ch. Eq.=Eigen values=poles



gain at 0 freq. = dc gain

$$\text{dc gain} = T(s)\Big|_{s=0} = K$$

$$T(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$T(s) = f[\omega n, \xi]$ Normal Thought ✓

Dc gain = k

$T(s) = f[\omega n, \xi, k]$

$$T(s) = \frac{k\omega n^2}{s^2 + 2\xi\omega n s + \omega n^2}$$

$$T(s=0) = k$$

$$T(s) = \frac{100}{s^2+10s+100}$$

$$2\xi\omega_n = 10$$

$$\omega_n^2 = 100$$

$$2\xi \times \cancel{10} = \cancel{10}$$

$$\xi = 0.5$$

$$T(s) = \frac{200}{s^2+10s+100}$$

$\omega_n = 10\ \text{r/sec}$

$\xi = 0.5$

DC gain $= 2.0$

$$T(s) = \frac{K\omega_n^2 e^{-t_d s}}{s^2+2\xi\omega_n s+\omega_n^2}$$

Assume $t_d = 0$ sec, $K = 1.0$

$$T(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Here, td represents the transportation delay

Note: The roots of characteristic equation represent the closed loop poles.

By solving the characteristics equation, the roots of characteristics equation are obtained as follows:

$$s = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1})}{2}$$

$$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \quad = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2} \quad = \sigma \pm j\omega_d$$

Where, $\omega_d = \omega_n\sqrt{1 - \xi^2}$ represents the damped frequency.

From the roots of the characteristic equations, the following conclusions can be drawn:

- The two roots (or poles) are complex with only the imaginary part when $\xi = 0$
- The two roots (or poles) are real and equal when $\xi = 1$
- The two roots (or poles) are real but not equal when $\xi > 1$
- The two roots (or poles) are complex conjugate when $0 < \xi < 1$

Damping Ratio (ξ):

- The damping ratio is a measure describing how rapidly the oscillations decay from one bounce to the next. The damping ratio is a system parameter which is denoted by ξ (zeta).
- Mathematically, it is defined as the ratio of actual damping to the critical damping ($\xi = 1$).
- It is a dimensionless quantity.

$$\text{Damping ratio} = \frac{\text{Actual damping}}{\text{Critical damping}} = \frac{\xi\omega_n}{1 * \omega_n} = \xi$$

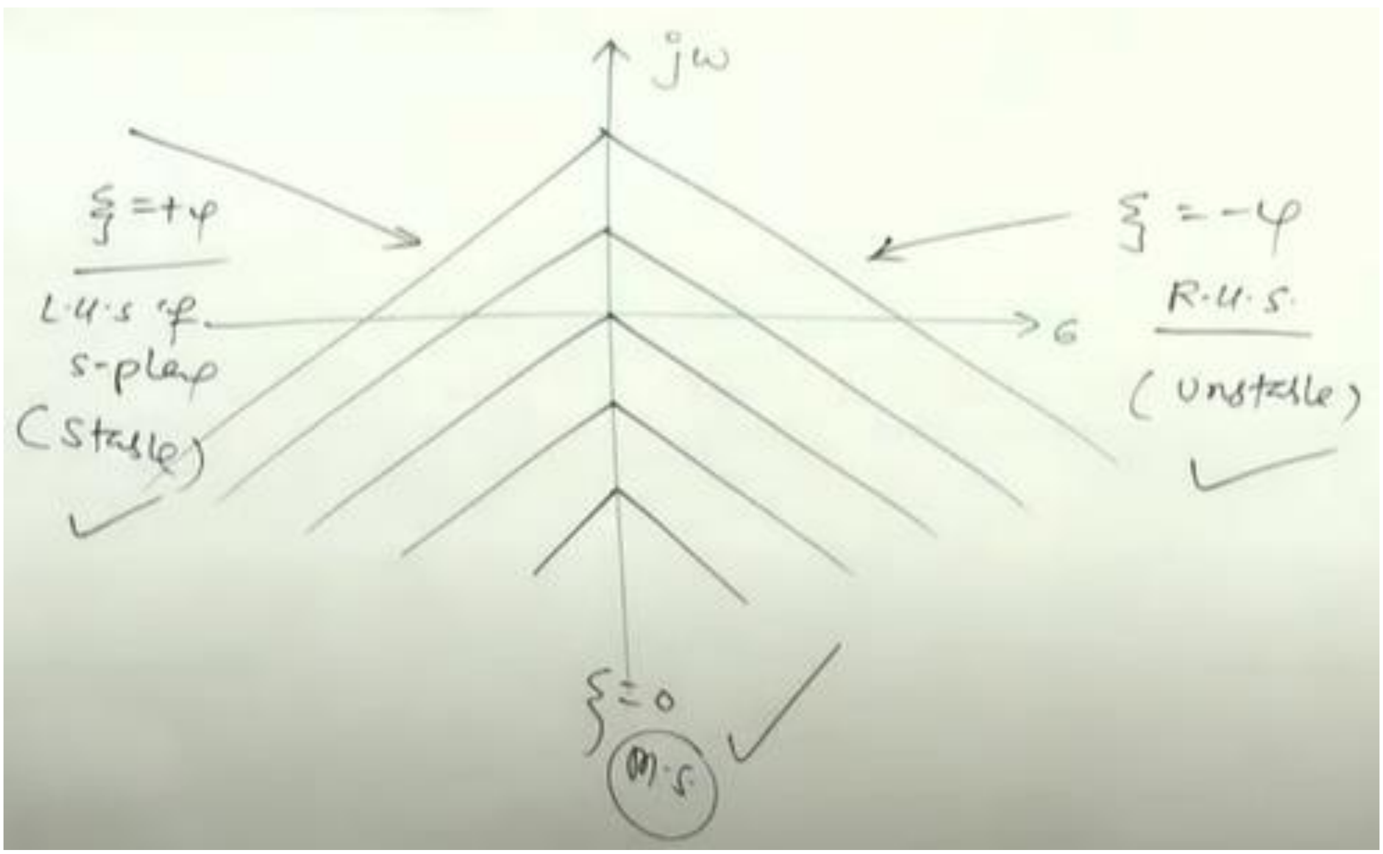
Note: For $\xi = 1$, the actual damping $= \omega_n$.

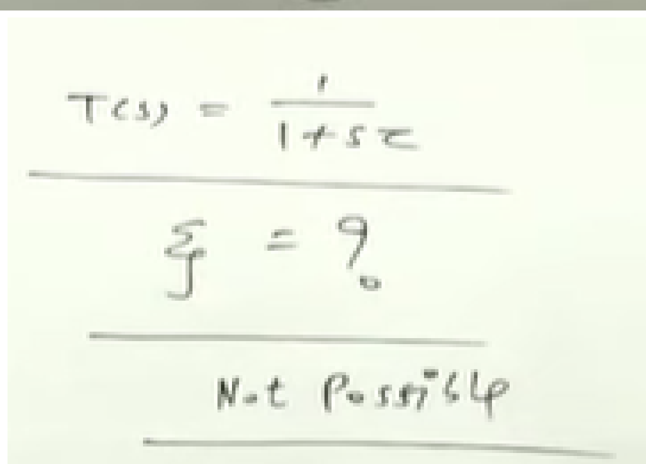
This actual damping when $\xi = 1$ is known as ***critical damping***.

Damping Factor ($\xi\omega_n$): In an underdamped 2nd order system, damping is provided by the real part of poles i.e., $-\xi\omega_n$. So, this factor is called damping factor or damping coefficient or actual damping.

Key points:

- Damping ratio describe the performance of 2nd order system.
- Damping ratio describe the stability of 2nd order system.
- For ξ = 0, all roots lie on the imaginary axis in the form of complex conjugate. i.e., $\xi = 0$ represents marginal stable system.
- For ξ = +ve, all roots lie in the left hand side (LHS) of s-plane. i.e., ξ = +ve represents stable system.
- For ξ = -ve, all roots lie in the right hand side (RHS) of s-plane. i.e., ξ = -ve represents unstable system.





Natural frequency/Undamped frequency (ω_n):

- The natural frequency of a 2nd order system is the frequency of oscillation without damping (when $\xi = 0$).
- In another words, It is the frequency of oscillation in output response when $\xi = 0$.

↳ no loss of energy
↳ oscillations never die out
↳ amplitude of oscillations = const around SS value

For example, the frequency of oscillation of a series RLC circuit with the resistance shorted would be the natural frequency.

Damped frequency (ω_d): It is the frequency of oscillation in output response when $0 < \xi < 1$.

i.e., $\omega_d = \omega_n\sqrt{1 - \xi^2}$ & therefore, $\omega_d < \omega_n$.

Some Practical Examples of 2nd Order Systems:

- All indicating instruments
- RLC network
- Position control system, etc.

Poles of second order system

$$T(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Characteristic equation

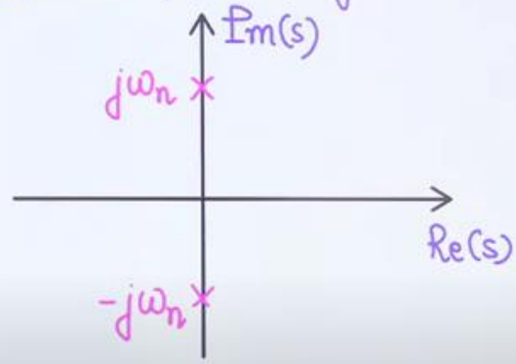
$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - \omega_n^2}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case-1: undamped system

$\xi = 0$; poles $= \pm j\omega_n$



purely imaginary

Case-2: underdamped system

$\xi < 1$

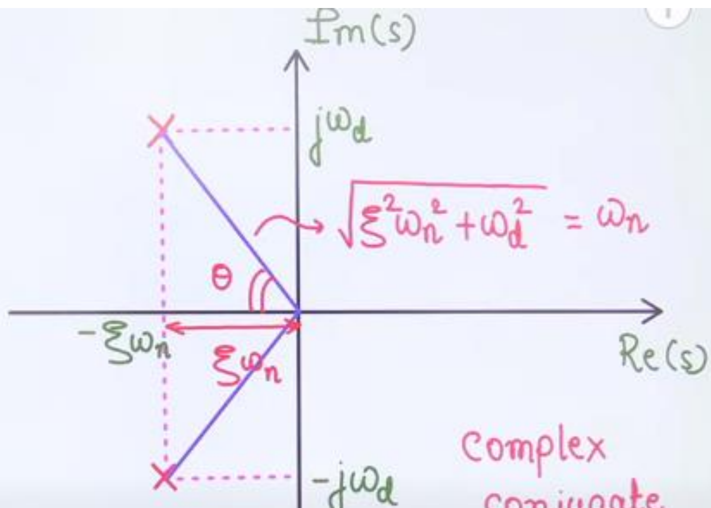
$$\text{poles} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm j\omega_d$$

ω_d : damping freq.

= freq. of damped oscillations

$= \omega_n\sqrt{1-\xi^2}$



complex conjugate poles

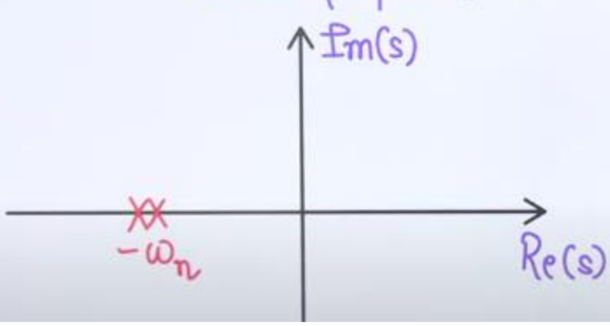
$$\cos\theta = \frac{\xi\omega_n}{\omega_n} = \xi$$

$$\theta = \cos^{-1}\xi$$

Case-3: Critically damped system ($\xi = 1$)

poles $= -\xi\omega_n = -\omega_n$

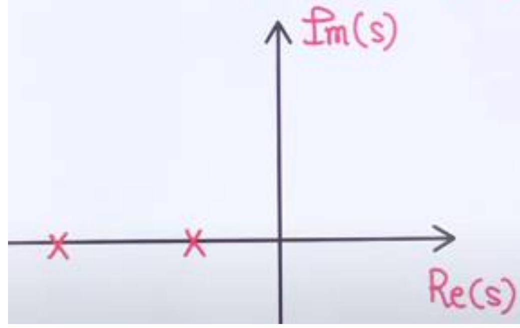
↳ real & equal poles



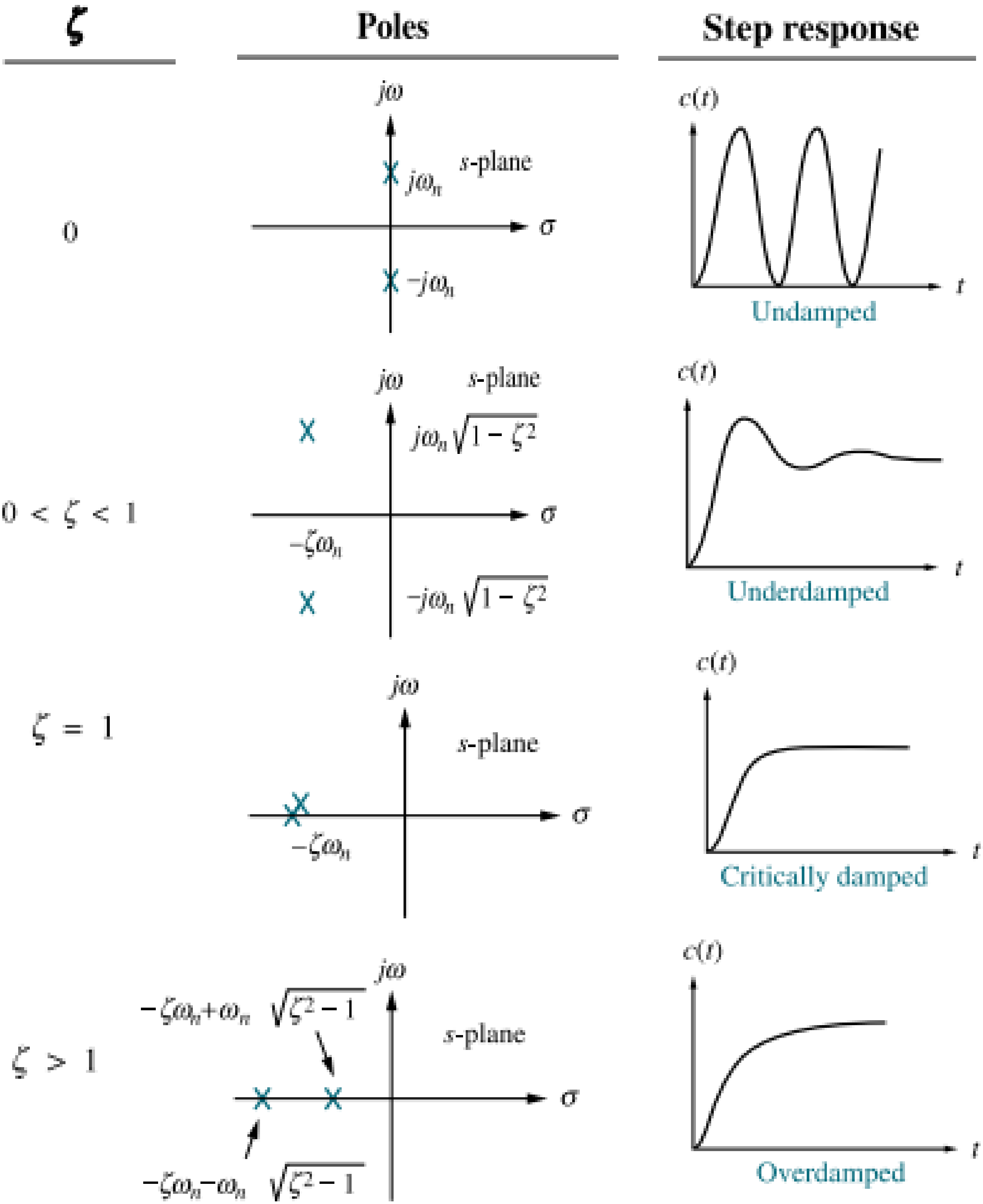
Case-4: overdamped system ($\xi > 1$)

poles $= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

real & distinct poles



➢ **These four cases of 2nd order response are a function of ξ; they are summarized in figure below:**



Effect of Damping Ratio (ξ) On 2nd Order System:

1. <u>Effect of ξ on pole location:</u>

(i) <u>case 1 : when $0 < \xi_e < 1$</u> (underdamped)

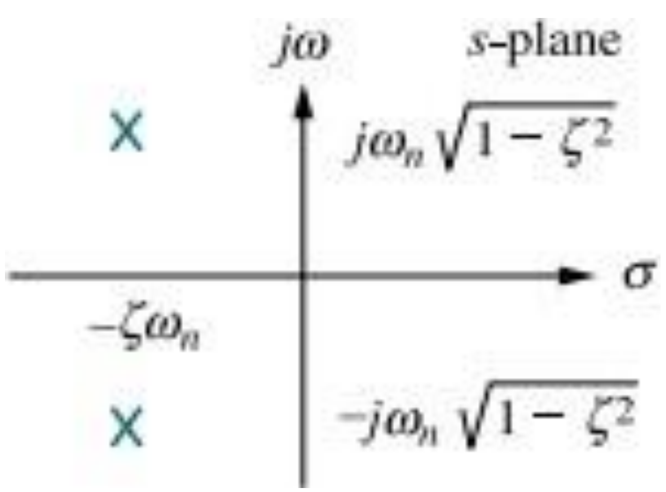
(a) the poles are complex conjugates of each other and are given by —

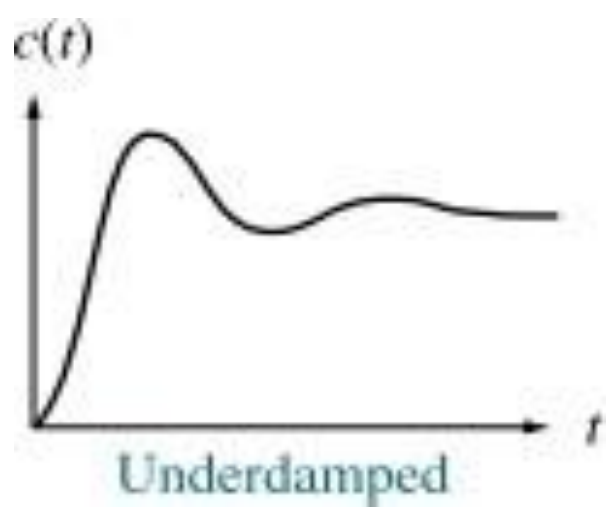
$$s_1, s_2 = -\xi_e \omega_n \pm j\omega_n \sqrt{1-\xi_e^2}$$

(b) the poles are located on the second and third co-ordinates due to the existence of both real and imaginary parts.

(c) The response is underdamped and having damped oscillation with overshoot and undershoot.

(d) This is stable system.





(ii) Case 2: when $\xi = 1$ (critically damped)

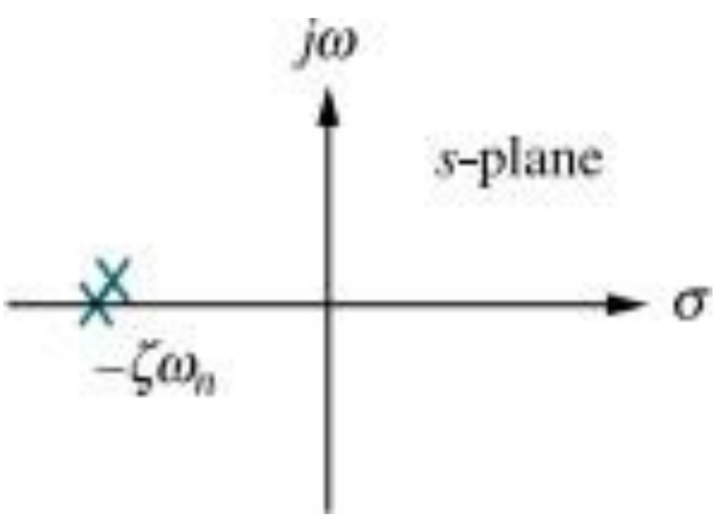
(a) the poles are real and equal. the poles are

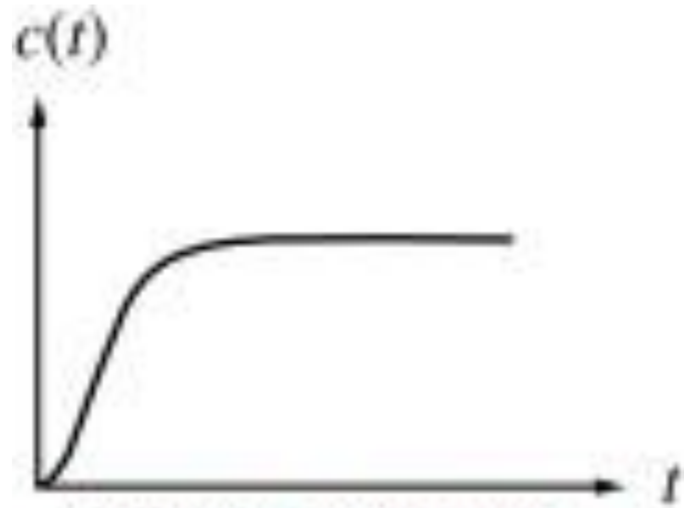
$$s_1 = s_2 = -\xi\omega_n$$

(b) they lie on the negative real axis (σ-axis).

(c) The response is critically damped and in this this case the system will reach steady-state value in the minimum time without overshoot.

(d) This is stable system.





Critically damped

(iii) Case 3: when $\xi > 1$ (over damped)

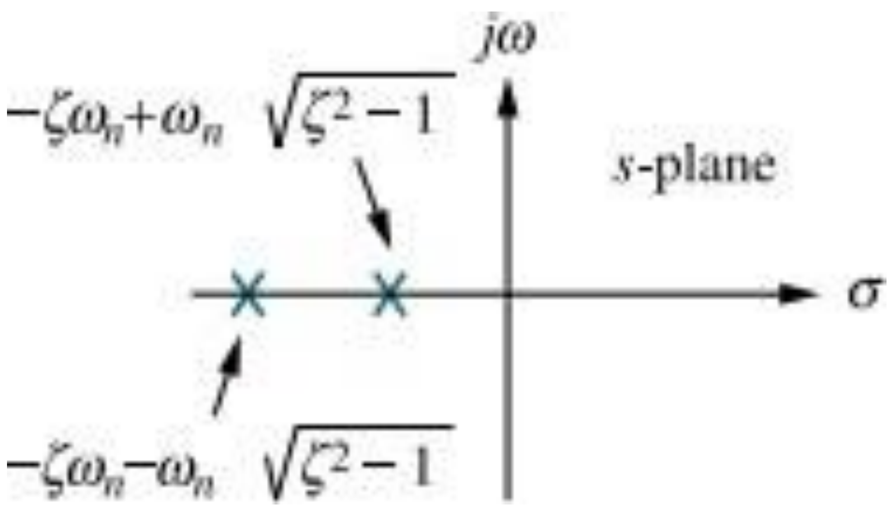
(a) the poles are real and unequal. these are given by—

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

(b) Since there are no imaginary terms ($\xi > 1$), the poles are lie on the -ve real axis and at unequal places.

(c) The response is overdamped.

(d) This is stable system.



(iv) Case 4: when $\xi = 0$ (undamped)

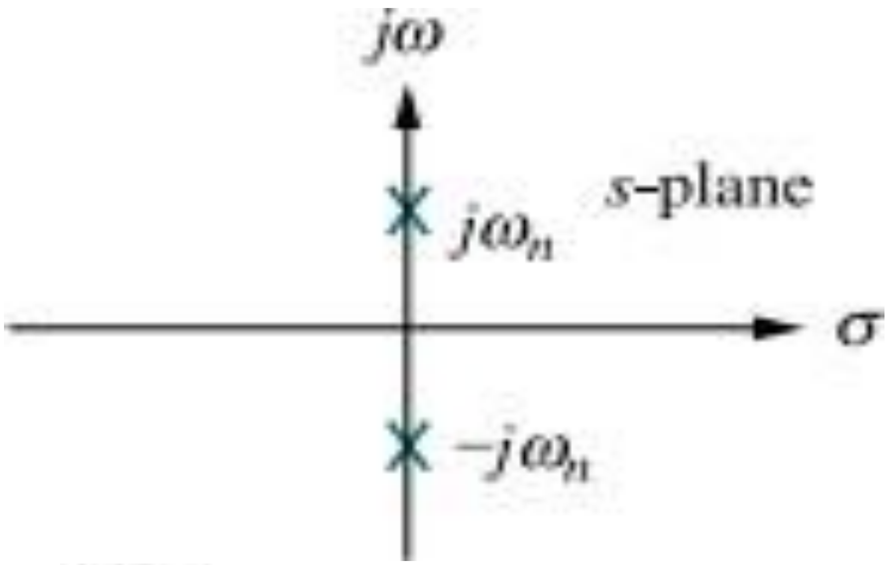
(a) the poles are complex with only the imaginary part and lie on $j\omega$ - axis. The poles are conjugates of each other

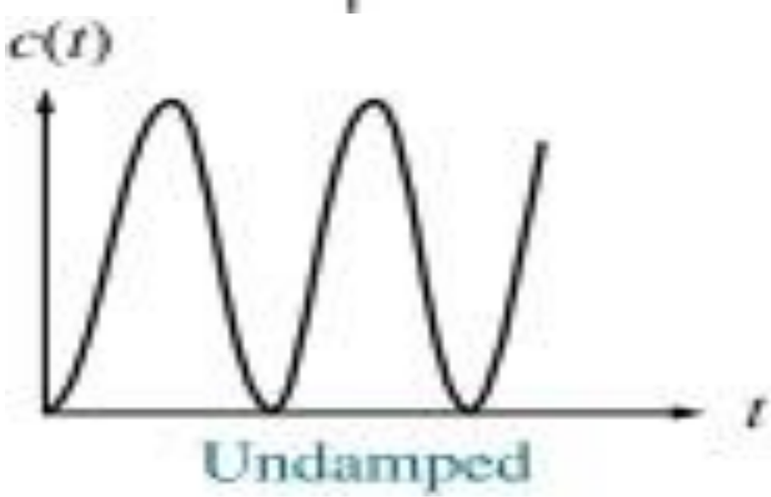
(b) The poles are given by

$$s_1, s_2 = \pm j\omega_n$$

(c)The response is undamped.

(d)This is marginal stable system.



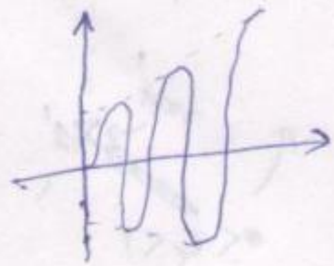


case 5: when $-1 < \xi_e < 0$ (-ve damping)

(a) Since ξ_e is negative, the poles are given by

$$s_1, s_2 = -\xi_e \omega_n \pm j\omega_n \sqrt{1-\xi_e^2}$$

(b) The real part of the poles are +ve, the poles lie on the right half plane (RHP) of s-plane.



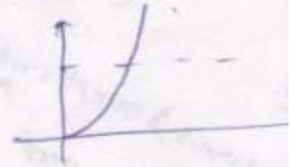
(c) This is an unstable condition

(vi) case 6: when $\xi_e = -1$ (-ve damping)

(a) the poles are given by $s_1, s_2 = -\xi_e \omega_n$

(b) the poles are located in the RHP of s-plane

(c) The system is unstable



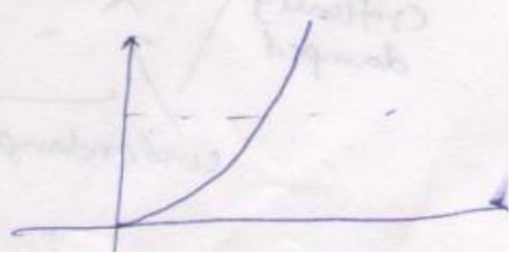
(vii) case 7: when $\xi_e < -1$ (-ve damping)

(a) The poles are given by

$$s_1, s_2 = -\xi_e \omega_n \pm \omega_n \sqrt{\xi_e^2 - 1}$$

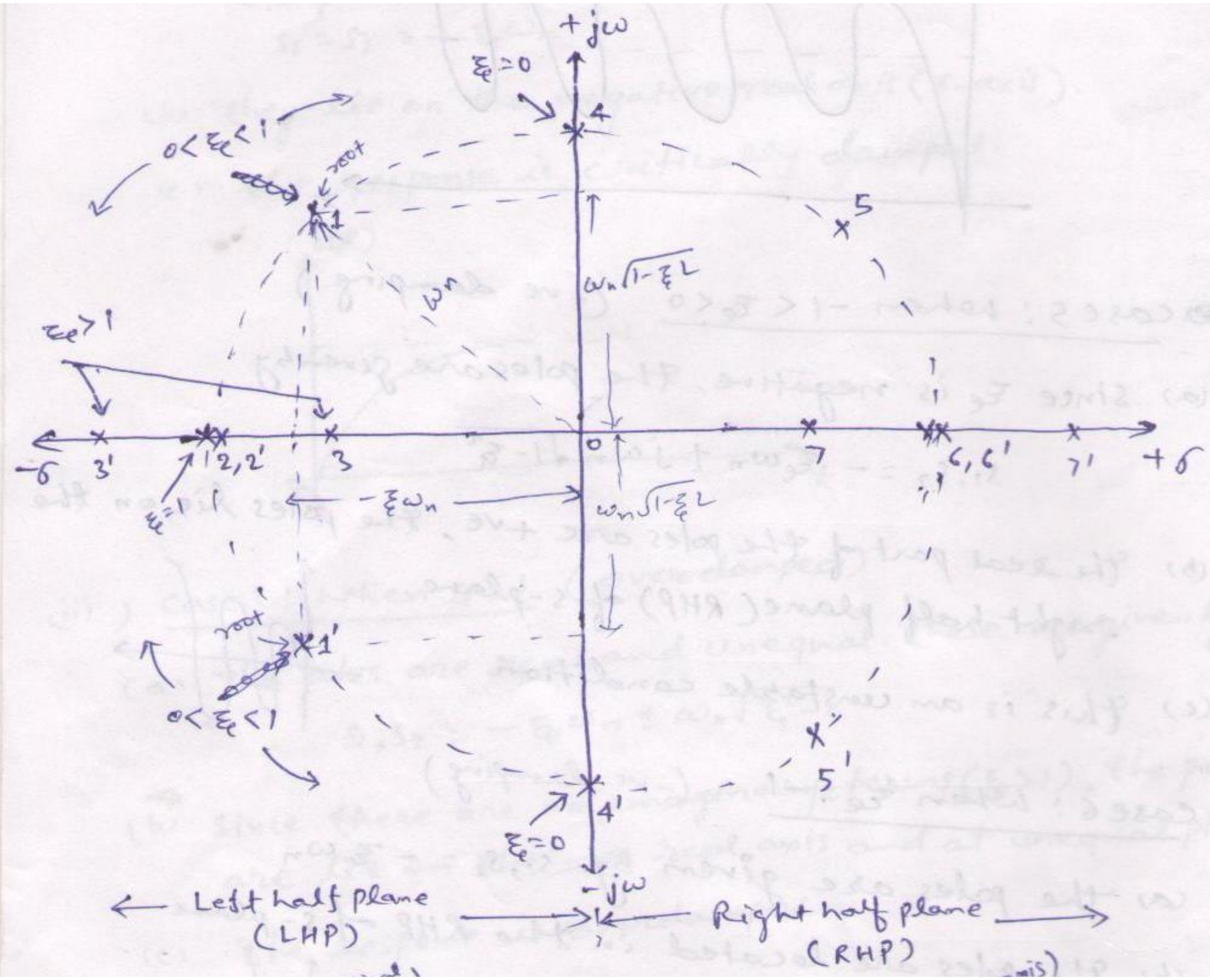
(b) the poles are located in the RHP of s-plane

(c) The system is unstable.



- If ξ = -ve , then amplitude exponentially increases with time (unbounded) and hence output does not settle to a steady-state value
- System will be always unstable.

Now For all cases, the location of poles for a 2^{nd} order system is —



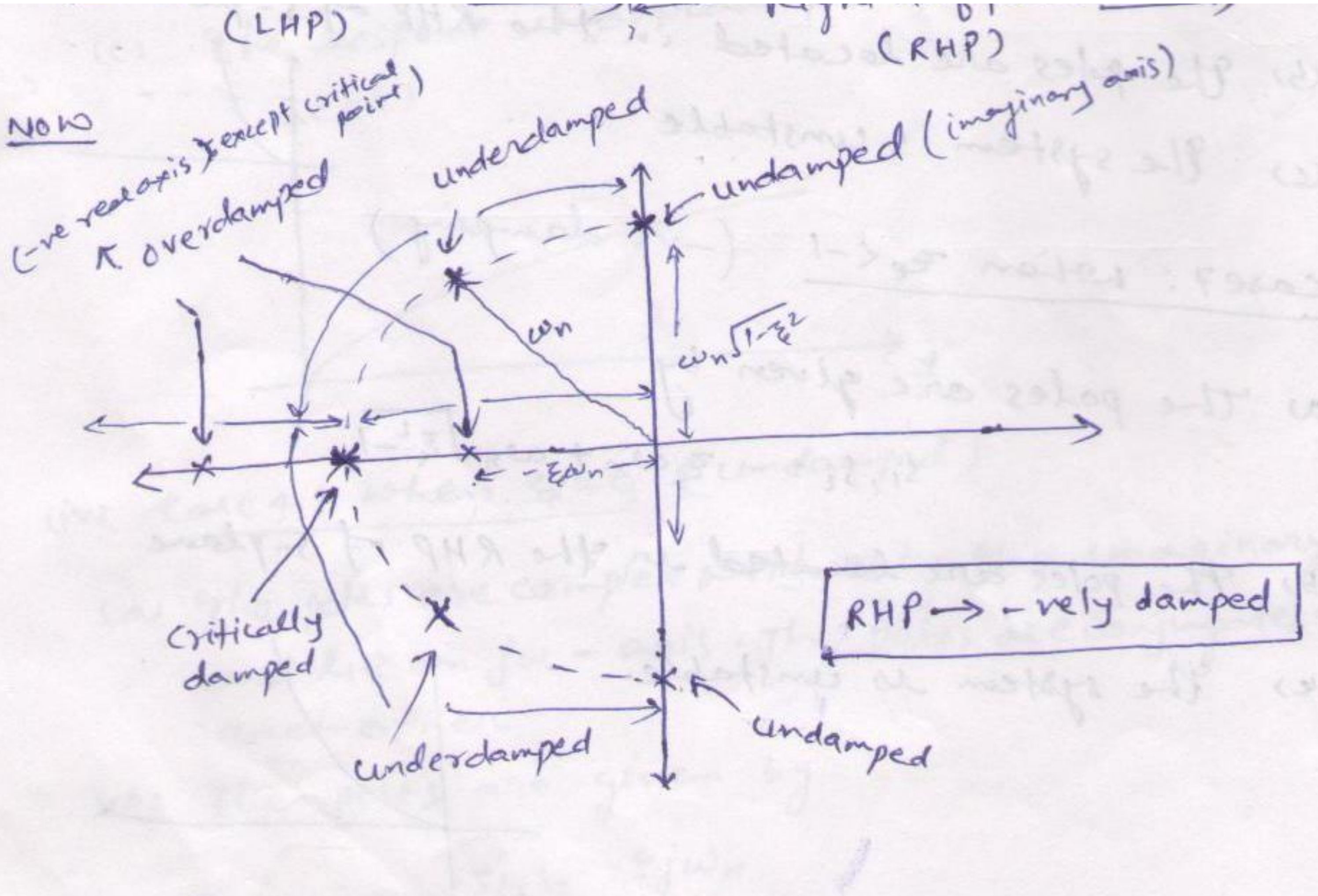


fig: ξ and nature of response

- $0<\xi<1$: underdamped
  ↳ oscillations of reducing amplitude
- $\xi=1$: critically damped
  ↳ no oscillations
- $\xi>1$: over damped
  ↳ no oscillations but slower than critically damped
- $\xi=0$: undamped
  ↳ oscillations of const amplitude