i.e. $x(n) \circledast_N y(n)$

$$= \sum_{m=0}^{N-1} x((m))_N \cdot y((n-m))_N$$

$$\longleftrightarrow X(k) \cdot Y(k)$$

Proof: $DFT[x(n) \circledast_N y(n)]$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x((m))_N \, y((n-m))_N \, \omega_N^{kn}$$

$$= \sum_{m=0}^{N-1} x((m))_N \underbrace{\sum_{n=0}^{N-1} y((n-m))_N \cdot \omega_N^{kn}}_{\text{circular-shift property}}$$

$$= \sum_{m=0}^{N-1} x((m))_N \cdot \omega_N^{km} \, Y(k)$$

$$= X(k) \cdot Y(k)$$

(vi) $x(n) \cdot y(n) \longleftrightarrow \frac{1}{N}[X(k) \circledast_N Y(k)]$

(vii) $x^*(n) \longleftrightarrow X^*((-k))_N$

(viii) $x^*(-n) \longleftrightarrow X^*(k)$

(ix) $Re[x(n)] \longleftrightarrow \underbrace{X_e(k)}_{\text{even}} = \frac{1}{2}[X(k) + X^*(-k)]$

(x) $j\,Img[x(n)] \longleftrightarrow \underbrace{X_o(k)}_{\text{odd}} = \frac{1}{2}[X(k) - X^*(-k)]$

(xi) $\frac{1}{2}[x(n) + x^*(-n)] = x_e(n) \longleftrightarrow Re[X(k)]$

(xii) $\frac{1}{2}[x(n) - x^*(-n)] = x_o(n) \longleftrightarrow j\,Img[X(k)]$.

Proof (vi):

$$x(n)\cdot y(n) \longleftrightarrow \frac{1}{N}\left[X(k) \circledast_N Y(k)\right]$$

$$DFT\left[x(n)\cdot y(n)\right]$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n) \cdot \underbrace{y(n)} \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot \frac{1}{N} \sum_{m=0}^{N-1} Y(m) \cdot W_N^{-mn} \cdot W_N^{kn}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} Y(m) \cdot \sum_{n=0}^{N-1} x(n) \cdot W_N^{(k-m)n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} Y(m) \cdot X((k-m))_N$$

$$= \frac{1}{N}\left[X(k) \circledast_N Y(k)\right]$$

Proof (ix):-

since $Re(x(n)) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$

$$X^*((-k))_N \qquad X^*(k)$$

Remarks:

- If $x[n]$ is real, i.e. $x[n]=x^*[n]$.

then,

i) $X(k)=X^*((-k))_N$

<u>Note</u>: If $x[n]$ is real, then there is no compulsion for $X[k]$ to be real.

ii) $Re[X(k)]=Re[X((-k))_N]$

iii) $Img[X(k)]=-Img[X((-k))_N]$

iv) $|X(k)|=|X((-k))_N|$

v) $\angle X(k)=-\angle X((-k))_N$

For proofs, use

$$\boxed{X(k)=X_r(k)+jX_{img}(k)}$$

$$\Rightarrow X(-k)=X_r(-k)+jX_{img}(-k)$$

$$\therefore \boxed{X^*(-k)=X_r(-k)-jX_{img}(-k)}$$

<u>Q</u> Let $X(k)$ denotes the n-point DFT of a sequence $x(n)$. $X(k)$ is itself an N-pt sequence. If $DFT[X(k)]=x_1(n)$, determine $x_1(n)$ in terms of $x(n)$.

<u>Soln.</u>

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k).W_N^{-kn}$$

$$\therefore N.x(n)=\sum_{k=0}^{N-1}X(k).W_N^{-kn}$$

or

$$N.(x(-n))_N=\sum_{k=0}^{N-1}X(k).W_N^{kn}$$

$$=DFT[X(k)].$$

$$=x_1(n)$$

$$\therefore \boxed{x_1(n)=N.x((-n))_N}$$

Q Consider a finite length sequence

$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$

a] compute its 5-point DFT.

b] compute 5-point IDFT of $Y(k) = X^2(k)$ to obtain the sequence $y(n)$.

Sol^n. (a) $x(n) = \{2, 1, 0, 1\}$ → since $\delta[n-2]$ is not there.
$\to N=4$

$x(n) = \{2, 1, 0, 1, 0\}$ → to make it to have length 5.
$\to N=5$

$$X(k) = \sum_{n=0}^{4} x[n] W_5^{kn}$$

$0 \le k \le 4$

$$= \sum_{n=0}^{4} \left(2\delta[n] + \delta[n-1] + \delta[n-3]\right) \cdot W_5^{kn}$$

$$\therefore \boxed{X(k) = 2 + W_5^{k} + W_5^{3k}}$$

then,

$X(0) = 4$

$X(1) = 2 + W_5^{1} + W_5^{3}$

$X(2) = 2 + W_5^{2} + W_5^{6}$

$X(3) = 2 + W_5^{3} + W_5^{9}$

$X(4) = 2 + W_5^{3} + W_5^{2}$

$e^{-j\frac{2\pi}{5} \times 5} = 1$

[since $W_5^{5} = 1$
so, $W_5^{6} = W_5^{1}$]

(b) $y(k) = X^2(k)$

$= \left(2 + W_5^{k} + W_5^{3k}\right)^2$

$e^{-j\frac{2\pi}{5} \times 5k} = 1$

$= 4 + W_5^{2k} + \underline{W_5^{6k}} + 4W_5^{k} + 4W_5^{3k} + 2W_5^{4k}$

$= \underbrace{4 + W_5^{2k} + W_5^{k} + 4W_5^{k}}_{5W_5^{k}} + 4W_5^{3k} + 2W_5^{4k}$

$Y(k) = 4 + 5W_5^{k} + W_5^{2k} + 4W_5^{3k} + 2W_5^{4k}$

$\therefore\; y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4).$

$$y(n) = \frac{1}{5}\sum_{k=0}^{4} Y(k)\, \omega_5^{-kn}$$

Q. If $x[n]$ is real, then prove that

$$X(k) = X^*(N-k).$$

Sol^n. $X^*(N-k) = \left[\sum_{n=0}^{N-1} x[n]\cdot \omega_N^{(N-k)n}\right]^*$

$\rightarrow \omega_N^{Nn} = 1$
$\rightarrow \omega_N^{-kn}$

$$= \sum_{n=0}^{N-1} \underbrace{x^*[n]}_{\text{Being real} \equiv x[n]}\cdot \omega_N^{kn}$$

$$= X(k)$$

Q. (Based on above proof); if some of $X[n]$ are provided find others

$x[n]$ is real, $N=7$

$X(0)$
$X(1)$
$X(2)^*$
$X(3)^*$
$X(4)$
~~$X(5)$~~
$X(5)$
$X(6)$

$*$

$X[n]$ is real, $N=6$

$X(0)$
$X(1)$
$X(2)^*$
$X(3)$ → $\boxed{\text{Real}}$
$X(4)$
$X(5)$

$*$

- Relation b/w z-transform and DFT:

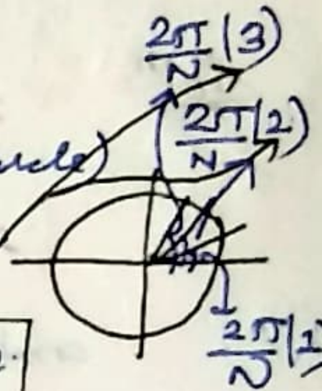
$$X(z) = \sum_{n=0}^{N-1} x[n]\cdot z^{-n}$$

and,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x[n]\cdot W_N^{kn} = \sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi}{N}kn}}$$

since $z = re^{+j\omega}$

for $|z|=1 \Rightarrow r=1$ (unit circle)



$$\therefore X(z) = \sum_{n=0}^{N-1} x[n]\cdot e^{-j\omega n}$$

$$\boxed{X(z) = \sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi k}{N} n}}$$

Note:- Sampling the z-transform on a unit circle separated by $\angle$ of $\frac{2\pi}{N}$ for N samples would gave the DFT.

Q. By using the synthesis and analysis equation, show that $\boxed{\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2}$ given that $x(n)$ is a N-point sequence while $X(k)$ is its N-point DFT.

Parseval's theorem for DFT

Sol^n.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \omega_N^{-kn}$$

$$\cancel{x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1}}$$

and,

$$X(k) = \sum_{k=0}^{N-1} x[n] \cdot \omega_N^{+kn}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \omega_N^{-kn}.$$

Then,

$$\frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot X^*(k)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \sum_{n=0}^{N-1} x^*(n) \omega_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) \cdot \underbrace{\sum_{k=0}^{N-1} X(k) \cdot \omega_N^{-kn}}_{x(n)}$$

$$= \sum_{n=0}^{N-1} |x(n)|^2$$

[on interchanging the Σ)