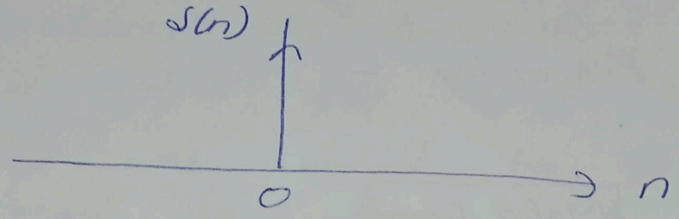
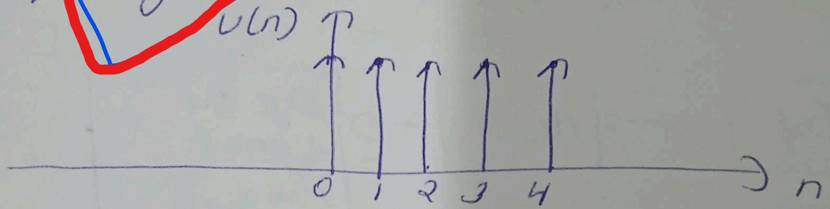
Digital Signal Processing

Unit sample signal – $\delta(t)$ or $\delta(n)$



$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Unit Step Signal – $u(n)$



$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\boxed{u(n) = \sum_{k=0}^{\infty} \delta(n-k)}$$

and, delta(n)=U(n)-U(n-1)

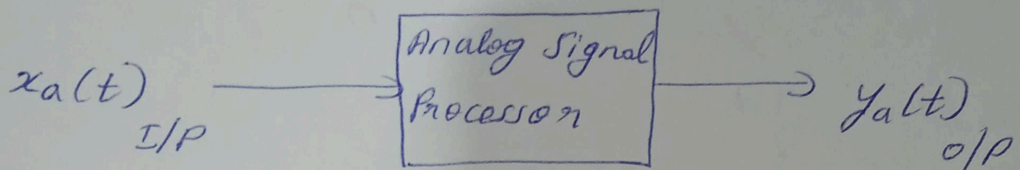
$$\sum_{k=0}^{\infty} \delta(n-k) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

Putting $n = 0$, $u(0) = 1$

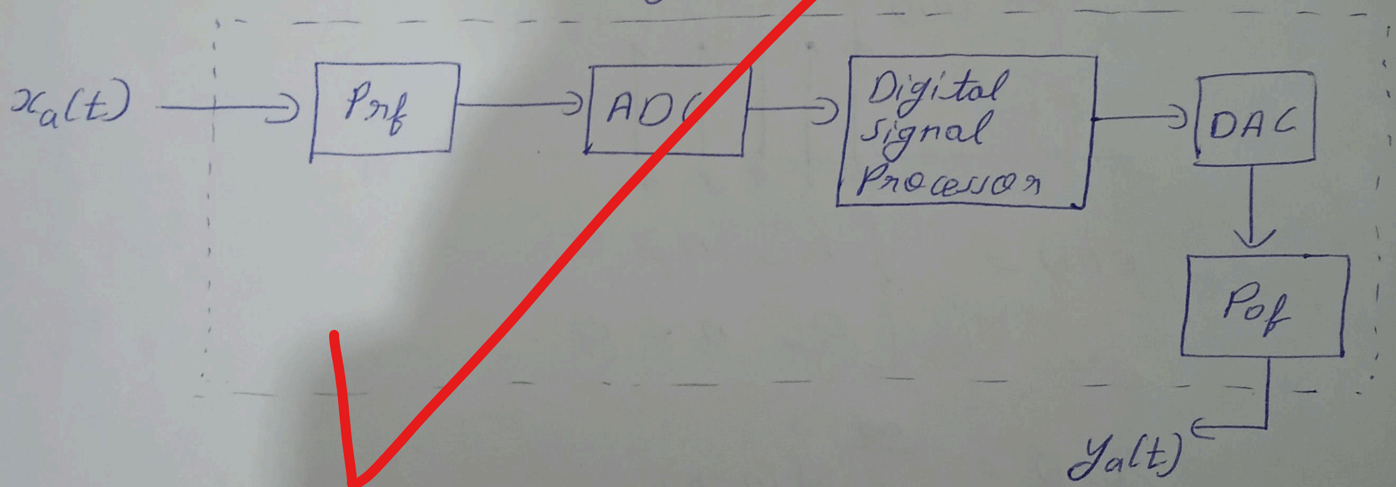
Putting $n = 1$, $u(1) = 1$

Extraction ~~of~~ and enhancing useful information from a mixture of conflicting information is known as signal processing.

Analog Signal Processing -



Digital Signal Processing -



Prf - Pre-Filter or Anti-aliasing filter - Conditions the signal to prevent aliasing.

Output of DAC is staircase and it is the first stage towards analog.

Pof - ~~Ou~~ Post-Filter smoothens the signal to give analog signal.

Advantages of DSP -

1.) DSP is easy to develop, test and software is portable.

2.) DSP operations are temperature independent, so it is useful for highly stable systems.

3.) In analog, new circuit has to be designed for different applications while in DSP only reloading of some program and by changing some resistor, we can use DSP.

For very high signal processing applications, we don't use DSP.

Discrete time signals -

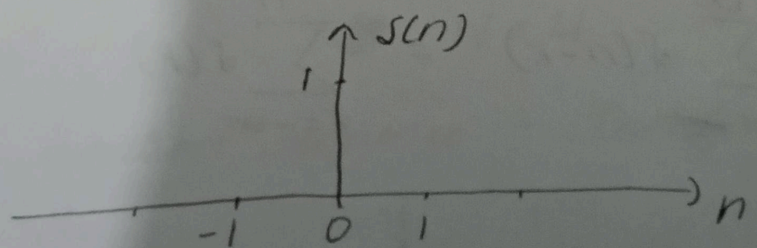
$$x(n) = \{x(n)\}, \quad -\infty < n < \infty$$

$$= \{\ldots x(-2), x(-1), \underset{\uparrow}{x(0)}, x(1), x(2) \ldots\}$$

Types of sequences -

Unit sample sequence -

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

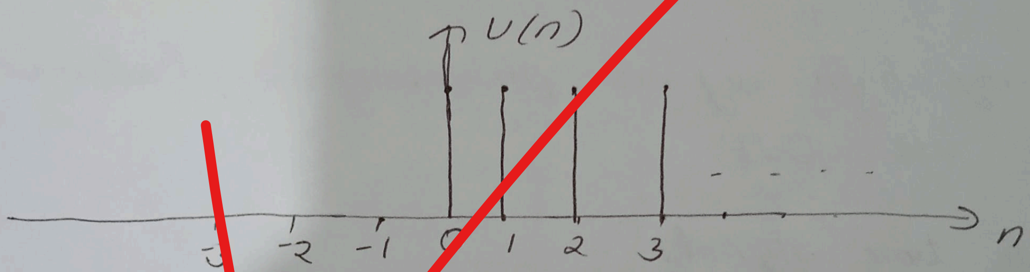


$$\delta(n) = \{\ldots\ 0, 0, \underset{\uparrow}{1}, 0, 0\ \ldots\}$$

$$\delta(n-n_0) = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

Unit step sequence –

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u(n) = \{\ldots\ 0, 0, \underset{\uparrow}{1}, 1, 1, \ldots\}$$

$$u(n-n_0) = \begin{cases} 1, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$$

By multiplying by u(n), we can make a system causal.

$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \sum_{k=-\infty}^{n} \delta(k)$$

Real exponential ~~signals~~ sequence -

$$x(n) = Aa^n$$

where A and a are real

Complex exponential sequence -

$$x(n) = Aa^n$$

where A and a are complex.

Sinusoidal sequence -

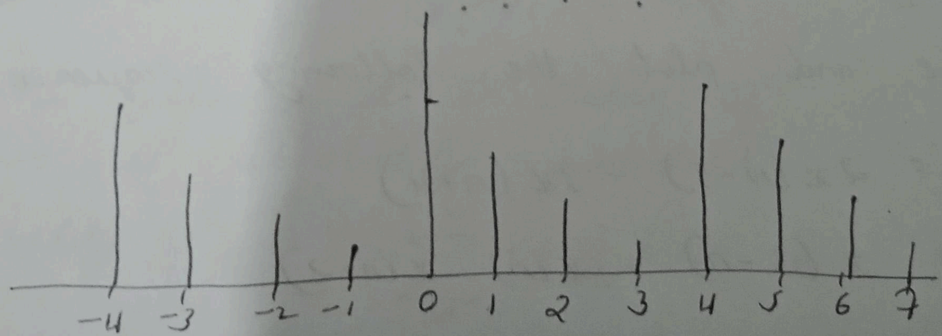
$$x(n) = A\cos(\alpha n + b)$$

Random sequence - Cannot be represented by any mathematical expression.

Periodic sequence - ~~Repeats after~~ A sequence is periodic if it satisfies -

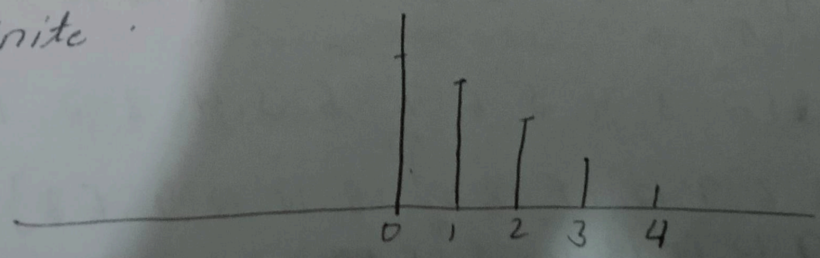
$$x(n) = x(n+N) \quad \forall n$$

Smallest integer N which satisfies this expression is known as fundamental period.



$N = 4$

Finite duration sequence - Duration of sequence is finite.



$N = 5$

5 point sequence

Operation on sequence -

1.) Signal addition - Sample-by-sample addition takes place.

$$\{x_1(n)\} + \{x_2(n)\} = \{x_1(n) + x_2(n)\}$$

2.) Signal multiplication -

$$\{x_1(n)\}\{x_2(n)\} = \{x_1(n)\,x_2(n)\}$$

3.) Scaling - $y(n) = \alpha\, x(n)$

$$\alpha\{x(n)\} = \{\alpha\, x(n)\}$$

Each sample is scaled by same amount.

4.) Folding - Each sample is flipped around $n = 0$.

Energy $= \sum x(n)\, x^*(n)$

Ex - $x(n) = \{1, 2, \underset{\uparrow}{3}, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1\}$.

?

Determine and plot the following sequence.

a.) $x_1(n) = 2x(n-5) - 3x(n+4)$

b.) $x_2(n) = x(3-n) + x(n) \cdot x(n-2)$

a.) $x(n-5) = \{\underset{\uparrow}{-}\ -\ -\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 6\ 5\ 4\ 3\ 2\ 1\}$

$2x(n-5) = \{\underset{\uparrow}{-}\ -\ -\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 12\ 10\ 8\ 6\ 4\ 2\}$

$x(n+4) = \{1\ 2\ 3\ 4\ 5\ 6\ \underset{\uparrow}{7}\ 6\ 5\ 4\ 3\ 2\ 1\}$

$3x(n+4) = \{3\ 6\ 9\ 12\ 15\ 18\ \underset{\uparrow}{21}\ 18\ 15\ 12\ 9\ 6\ 3\}$

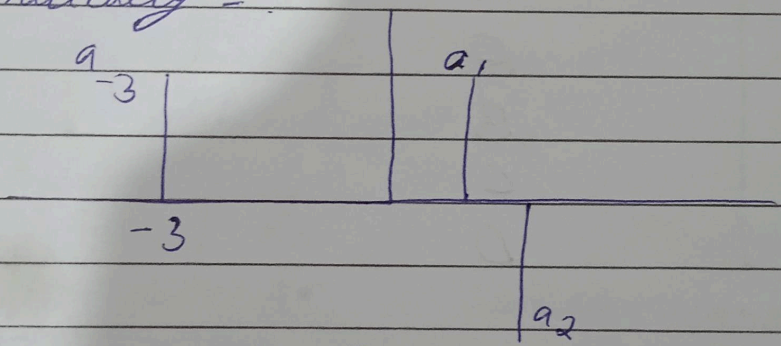
$x_1(n) = \{3\ 9\ 12\ 15\ 18\ \underset{\uparrow}{21}\ 18\ 15\ \cancel{4}\ 2\ 3\ 1\ 2\ 3\ 4\ 2\}$

8) High Reliability - and ease of maintenance - It works without failure for 20-30 years, very less maintenance required.

9.) Low cost - Fibre optics are usually made of silicon and polymers whose cost is less and silica is readily available on Earth.

DSP

How arbitrary sequence can be expressed mathematically -:



$$u(n) = a_1 \delta(n-1) + a_2 \delta(n-2) + a_3 \delta(n+3)$$

$\Rightarrow$ sum of scaled delayed unit samples.

$$\therefore x(n) = \sum_{K=-\infty}^{\infty} x(K) \delta(n-K)$$

Linear Shift Invariant System -

Discrete System - Defined by unique operator T[ ] which will map input sequence x(n) into output sequence y(n).

PAGE No
DATE

Write MATLAB program to plot $x(n) = a^n$

i) $0 \leq a \leq 1$
ii) $-1 \leq a \leq 0$
iii) $a < -1$
iv) $a > 1$

$$x(n) \longrightarrow \boxed{T[\ ]} \longrightarrow y(n) = T[x(n)]$$

Linear System $\Rightarrow$ $T[a x_1(n) + b x_2(n)]$
$= a\,T[x_1(n)] + b\,T[x_2(n)]$
$= a y_1(n) + b y_2(n)$

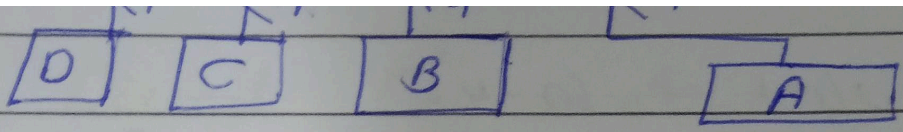
Now, $y(n) = T[x(n)]$
$$= T\left[\sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)\right]$$
$$= \sum_{K=-\infty}^{\infty} x(K)\,T[\delta(n-K)]$$
$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h_K(n)$$

the shift in input would also be reflected in the output by the same margin

Shift invariant - If $T[x(n)] = y(n)$
$\Rightarrow T[x(n-K)] = y(n-K)$

$$\Rightarrow y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K) \quad \text{(convolution sum)}$$
$$y(n) = x(n) * h(n) \quad \Rightarrow \text{(Linear convolution)}$$

go with diagonal sum method taught by rajiv sir
and length of the sequence is N1+N2-1



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Arithmetic microoperations –

Add Micro op. – $R_3 \leftarrow R_1 + R_2$
Subtract Microop. – $R_3 \leftarrow R_1 - R_2$
Increment Microop. – $R_1 \leftarrow R_1 + 1$
Decrement Microop. – $R_1 \leftarrow R_1 - 1$

Arithmetic unit –

$$A \rightarrow A_{n-1} \cdots A_1 A_0$$

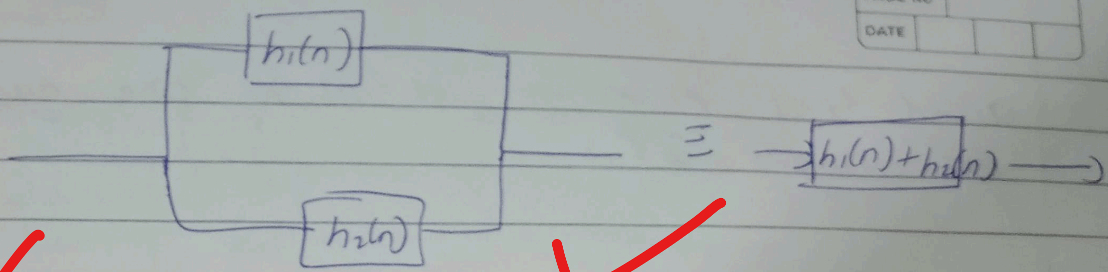
$$B \rightarrow B_{n-1} \cdots B_1 B_0$$

DSP

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) * h(n)$$
$$= h(n) * x(n)$$

$$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow y(n)$$

$$\equiv \quad x(n) \rightarrow \boxed{h_1(n) * h_2(n)} \rightarrow y(n)$$



Q. $h(n) = (0.5)^n u(n)$. Input is $x(n) = u(n) - u(n-4)$
Find response $y(n)$.

$\boxed{\begin{array}{l} x(n)=1 \text{ ,for } x=0,1,2,3 \\ 0\text{,elsewhere} \end{array}}$

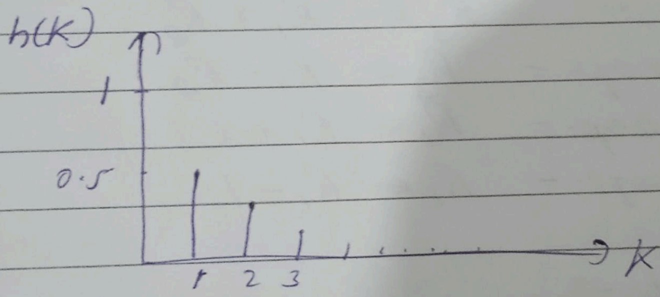
$x(n) = u(n) - u(n-4)$

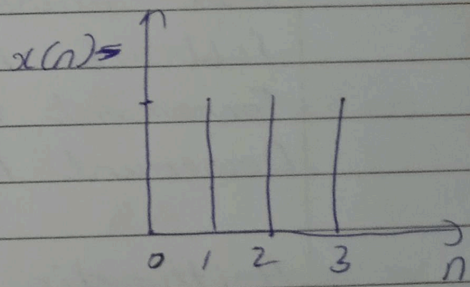
its ans is y(n)=(1/2)^n * U(n) + (1/2)^n-1 * U(n-1) + (1/2)^n-2 * U(n-2) + (1/2)^n-3 * U(n-3)

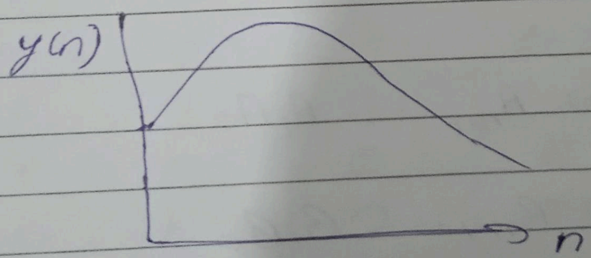
$y(n) = x(n) * h(n)$

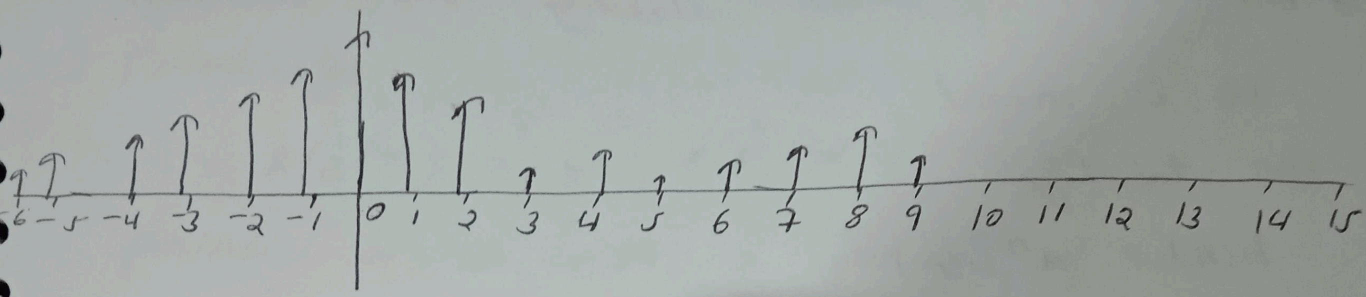
$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$

y(n) has to be in terms of n and hence got the name y(n)









b.) $x_2(n) = x(3-n) + x(n)\,x(n-2)$

$x(3-n) = \{$

In other copy -

System Properties of LTI System -

1) Stability - BIBO Stability - A system is said to be BIBO stable, if for every bounded input, it produces a bounded output.

$$|x(n)| < \infty \Rightarrow |y(n)| < \infty$$

For LTI system, it is stable if its impulse response is absolutely summable.

$$\text{Stability } S \Rightarrow \sum_{-\infty}^{\infty} |h(n)| < \infty$$

2.) Causality - ~~If~~ ~~outp~~ Output does not depend upon future values. If output at any instance $n = n_0$, depends on input at $n \leq n_0$.

For LTI system, $\boxed{\text{impulse response}}$ $h(n) = 0$ for $n < 0$.

impulse response is being used for LTI systems for classification purposes

Q. $h(n) = a^n u(n)$. Check stability and causality

It is causal.

For stability

$h(n) = a^n u(n)$

$$S = \sum_{-\infty}^{\infty} h(n) = \sum_{0}^{\infty} |a^n| = \frac{1}{1-|a|}$$

converge for $|a| < 1$

$\therefore$ stable for $|a| < 1$

Q. $h(n) = a^n u(n+2)$

it can be proved via

h(-1),h(-2) is not equal to 0

OR

h(n) depens on future values

Not causal

For stability - $h(n) = a^n u(n+2)$

$$S = \sum_{-\infty}^{\infty} a^n u(n+2)$$

$$a^{-2} + a^{-1} + \sum_{0}^{\infty} |a^n|$$

same condition as above.

DTFT is equivalent to the normal fourier transform of any continuous signa;.

DTFT - Discrete Time Fourier Transform.

DTFT of any absolutely summable $x(n)$ is given by

it can be calculated for any absolutely summable signal

$$X(e^{j\omega}) = F[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

DFS=summation(0,N-1)x(n)*e(-j*2*pi*k*n/N)
DFT=same as DFS
But the difference is DFT is for a finite length sequence while DFS is for any periodic sequence,
DTFS and DFS are similar other than the fact that the coefficients in DTFS are being multiplied with N

Properties -:

bz cosine and sine are periodic with 2*pi

1.) Periodicity -

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

periodic in ω with a period of 2π.

2.) Symmetry - For real valued $x(n)$, $X(e^{j\omega})$ is conjugate symmetric.

so that x*(n)=x(n)

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

$$Re[X(e^{-j\omega})] = Re[X(e^{j\omega})] \quad \text{(even symmetric)}$$

$$Im[X(e^{-j\omega})] = Im[X(e^{j\omega})] \quad \text{(odd symmetric)}$$

$$|X(e^{-j\omega})| = |X(e^{j\omega})| \quad \text{(even symmetric)}$$

$$\angle X(e^{-j\omega}) = -\angle X(e^{j\omega}) \quad \text{(odd symmetric)}$$

$$\Rightarrow \quad \omega \in [0, \pi]$$

3.) Linearity - $F[\alpha x_1(n) + \beta x_2(n)] =$

$$\alpha F[x_1(n)] + \beta F[x_2(n)]$$

$$= \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

$$\forall \; \alpha, \beta, x_1(n) \text{ and } x_2(n)$$

it is same as that of z-transform ,i.e. parameter is being raised to the shifted power,i.e. e^(jw) is being raised to -k similarly z is being raised to -n in case of z transform.

4.) Time Shifting

$$F[x(n-k)] = X(e^{j\omega})\,e^{-j\omega k}$$

5.) Frequency Shifting -

$$F[x(n)e^{j\omega_0 n}] = X(e^{j(\omega-\omega_0)})$$

same as z-transform

6.) Folding - $F[x(-n)] = X(e^{-j\omega})$

7.) Conjugation -

$$F[x^*(n)] = X^*(e^{-j\omega})$$

$\downarrow$

Folding and conjugation

8.) Convolution -

$$F[x_1(n) * x_2(n)] = F[x_1(n)]\, F[x_2(n)]$$

$$= X_1(e^{j\omega})\, X_2(e^{j\omega})$$

9.) Multiplication - Dual of Convolution

$$F[x_1(n)\, x_2(n)] = F[x_1(n)] \overset{\text{Periodic convolution}}{\circledast} F[x_2(n)]$$

$$= \frac{1}{2\pi} \int X_1(e^{j\omega})\, X_2(e^{j(\omega - z)})\, dz$$

10.) Energy - Energy of sequence $x(n)$ is

$$E_x = \sum_{-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Parseval's Theorem - $= \int_0^{\pi} \frac{|X(e^{j\omega})|^2}{\pi} d\omega$

For proof of this use,
1. |x(n)|^2=x(n) * x*(n)
2.x*(n)=1/2pi*integral(-pi,pi)*X*(e^(jw))*e^(-jwn)dw

Frequency domain representation of LTI system -

1) Response to a complex exponential $e^{j\omega_0 n}$

The output is simple multiplication of DTFT of impulse response signal(also known as frequency response)at frequency wo with e^(jwon).

$$e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Output $\Rightarrow$ $e^{j\omega_0 n} * h(n) = y(n)$

$$y(n) = \sum_{-\infty}^{\infty} h(K)\, e^{j\omega_0 (n-K)}$$

$$= \left[\sum_{-\infty}^{\infty} h(K)\, e^{-j\omega_0 K}\right] e^{j\omega_0 n}$$

$$= F[h(n)]_{\omega=\omega_0}\, e^{j\omega_0 n} = H(e^{j\omega_0})\, e^{j\omega_0 n}$$

2) Frequency Response - DTFT of impulse response is frequency response.

$$H(e^{j\omega}) = \sum_{-\infty}^{\infty} h(n)\, e^{-j\omega n} = H(j\omega)$$

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega_0})\, e^{j\omega_0 n}$$

3) Response to sinusoidal input -

$$x(n) = A\cos(\omega_0 n + \theta_0) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = A|H(e^{j\omega_0})| \cos(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0}))$$

$$y(n) = A|H(e^{j\omega_0})| \cos(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0}))$$

$$= y_{ss}(n) \quad [\text{Steady State Response}]$$

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Follow the method of z-transform given in proakis page no. 208

Q An LTI system is specified by difference equation

$$y(n) = 0.8y(n-1) + x(n)$$

i.e. the response corresponding to sinusoidal input

Calculate. steady state response $y_{ss}(n)$ to input $x(n) = \cos(0.05\pi n)u(n)$

$$y(n) = 0.8y(n-1) + x(n)$$

$$x(n) = \cos(0.05\pi n)(n)$$

$$\cancel{h(n) = \frac{0.8y(n-1)}{x(n)} + 1}$$

??

$$y(n) = 0.8[y(n) - \delta(n)] + x(n)$$

$$0.2y(n) = -0.8\delta(n) + x(n)$$

$$\frac{y(n)}{x(n)} = -4\delta(n) + 5$$

$$h(n) = 5 - \frac{4\delta(n)}{x(n)}$$

??

$$H(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}}$$

$$x(n) = \cos(0.05\pi n)u(n)$$

$$f = \frac{0.05}{2} = 0.025 \qquad \theta = 0$$

$$\omega_0 = 0.05\pi \qquad \theta_0 = 0$$

$$H(e^{j\omega_0}) = \cancel{1} \; \frac{1}{1-0.8e^{-j0.05\pi}}$$

$$= 4.09e^{-j0.5377}$$

$$= 4.09\cos(0.05\pi n - 0.5377)$$

$$= 4.09 \, (\cos(0.05n - 0.5377)$$

HW

Q. $h(n) = (0.9)^n u(n)$

$x(n) = 0.1\, u(n)$

Find $y_{ss}(n)$

yss(n)=1
and yss(z)=1/1-z^(-1)

$= 10$

PW(physics wallah)

Paley Wiener's Criterion – It gives the frequency domain equivalence of causality condition in time domain. According to Paley Weiner, the necessary and sufficient condition for an amplitude response to be realizable is that –

$$\int_{-\infty}^{\infty} \frac{|\ln|H(j\omega)||}{1+\omega^2}\, d\omega < \infty$$

1. Requirements on amplitude response to be realizable:
For a range of frequency, $H(j\omega) \neq 0$, for any realizable network.

2. The amplitude response should not fall to zero faster than the exponential order.

Q. Using Paley Weiner, show that "$|H(j\omega)| = e^{-\omega^2}$" is not a suitable amplitude response for a causal LTI system.

$$H(j\omega) = e^{-\omega^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^2}\, d\omega = \int_{-\infty}^{\infty}\left(1 - \frac{1}{1+\omega^2}\right) d\omega$$

$$= \left[\omega - \tan^{-1}\omega\right]_{-\infty}^{\infty}$$

$$= \infty \quad \text{(not suitable)}$$

Q $|H(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$ given in S. Sallivaham

$$\int_{-\infty}^{\infty} \frac{\left|\ln \frac{1}{\sqrt{1+\omega^2}}\right|}{1+\omega^2}\, d\omega$$

$$\frac{-1}{2}\int_{-\infty}^{\infty} \frac{\ln(1+\omega^2)}{(1+\omega^2)}\, d\omega$$

now put w=tanx

and then use integral(0,pi/2)log(cosx)=-1*pi/2*log2

ans=0.945
and hence realisable

Z - Transform -

Shortcoming of DTFT -

As it exists for absolutely summable signals only

1.) For $u(n)$ and ~~at~~ $n u(n)$, does not exist

2.) For changing and transient input, does not exist.

The Z-transform of any sequence $x(n)$ is given by

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The set of value of z for which $X(z)$ exists is known as region of convergence (ROC)

$$R_{x_-} < |z| < R_{x_+}$$

where R_{x_-} and R_{x_+} are some positive numbers.

Comments -:

1.) $z = |z| e^{j\omega}$

2.) $\because$ ROC is defined for magnitude condition,

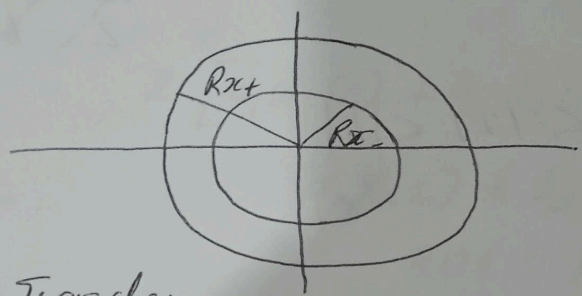
$\therefore$ ROC is always

Rx_- min $= 0$ $\qquad$ Rx_+ max $= \infty$

3.) If $Rx_+ < Rx_-$, ROC will not exist.

4.) If $|z| = 1 \Rightarrow z = e^{j\omega}$

$\therefore$ z-transform will become DTFT (when ROC contains the unit circle)



Finding Z-Transform -

1.) $x_1(n) = a^n u(n)$ $\qquad$ $0 < |a| < \infty$

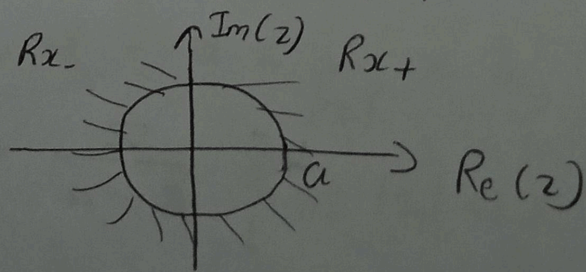
This is positive sequence -

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1 - az^{-1}}, \text{ if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z-a}, \quad |z| > |a|$$

$ROC_1 : |a| < |z| < \infty$



2.) $x_2(n) = -b^n u(-n-1)$, $0 < |b| < \infty$ (negative sequence)

$$X_2(z) = \sum_{n=-1}^{-\infty} \cdot b^n z^{-n}$$

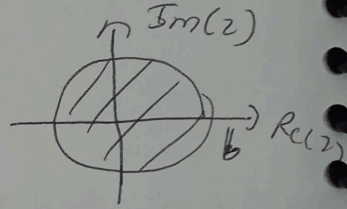
(for negative sequence we use - sign to show it)

$$= \sum_{n=-1}^{-\infty} \left(\frac{b}{z}\right)^n = \frac{+1}{1 - bz^{-1}}$$

$$= \cancel{\frac{z}{z-b}} \quad \frac{z}{z-b}$$

ROC_2 : $0 \leq |z| < |b|$

R_{x_-} $\qquad$ R_{x_+}



If $a = b$, then $X_1(z) = X_2(z)$

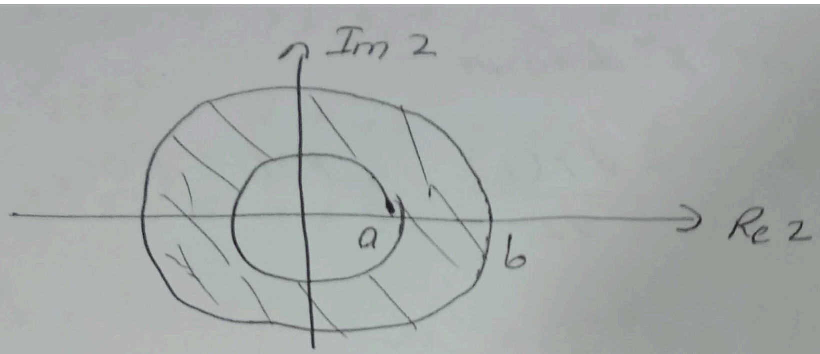
but $ROC_1 \neq ROC_2$

ROC is the only distinguishing feature for z-transform.

3.) $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$
$(b > a)$

$$X_3(z) = \left\{ \frac{z}{z-a}, ROC_1 : |z| > |a| \right\} + \left\{ \frac{z}{z-b}, ROC_2 : |z| < |b| \right\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}, \quad ROC_3 : ROC_1 \cap ROC_2$$



If $b < a$, ROC is null space and z-transform will not exist.

Properties -:

1.) Linearity - holds for DTFT too $Z[a_1 x_1(n) + a_2 x_2(n)] =$

$$a_1 X(z) + a_2 X_2(z),$$

$$ROC: ROC_{x_1} \cap ROC_{x_2}$$

it is similar to DTFT ,there e^(jw) was being raised to shifted power.

2.) Time Shifting - $Z[x(n-n_0)]$

$$= z^{-n_0} X(z) \quad , \quad ROC: ROC_x$$

3.) Multiplication ~~of~~ by exponential sequence -:

$$Z[a^n x(n)] = X\left(\frac{z}{a}\right) \; , \; ROC: \boxed{ROC_x \text{ scaled by } |a|}$$

4.) Folding (Time reversal) - again it is similar to DTFT

$$Z[x(-n)] = X\left(\frac{1}{z}\right) \quad \boxed{ROC : \text{Inverted } ROC_x}$$

let $ROC_x : a < |z| < b$

ROC of inverted (reflected) signal : $a < \frac{1}{|z|} < b$

$$\frac{1}{b} < |z| < \frac{1}{a}$$

5.) Differentiation in z-domain -

$$Z[n\,x(n)] = -z\frac{dX(z)}{dz} \quad , \quad ROC : ROC_x$$

6.) Convolution -

$$Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$$

$$ROC : ROC_{x_1} \cap ROC_{x_2}$$

7.) Cross correlation -

$$x_1(n) \longrightarrow X_1(z)$$

$$x_2(n) \longrightarrow X_2(z)$$

$$r_{x_1,x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)$$

$$\boxed{r_{x_1,x_2}(l) \longleftrightarrow R_{x_1,x_2}(z) = X_1(z)X_2(z^{-1})}$$

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Q Obtain cross correlation $r_{x_1,x_2}(l)$ of the following sequence -:

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 1\}$$

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\,x_2(n-l)$$

$$= \sum_{n=0}^{3} x_1(n)\,x_2(n-l)$$

$$= x_1(0)\,x_2(-l) + x_1(1)\,x_2(1-l)$$

$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

$X_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$

$X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$

$R_{x_1 x_2}(z) = (4 + 3z^{-1} + 2z^{-2} + z^{-3})(4 + 3z + 2z^2 + z^3)$

$= z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}$

$r_{x_1, x_2}(l) = z^{-1}[R_{x_1, x_2}(z)]$

$= [1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16]$

Some important results :-

just remember z-tranform of U(n)==-1*U(-(n-1))

Sequence	Transform	ROC
$\delta(n)$	1	$\forall z$
$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$ right sided sequence
~~u(n)~~ $-u(-n-1)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert < 1$ left sided sequence
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < b$
$[a^n \sin \omega_0 n] u(n)$	$\frac{(a \sin \omega_0) z^{-1}}{1-(2a\cos\omega_0)z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

$[a^n \cos\omega_0 n]u(n)$ $\quad \dfrac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}-a^2z^{-2}}$ $\quad |z|>|a|$

$na^n u(n)$ $\quad \dfrac{az^{-1}}{(1-az^{-1})^2}$ $\quad |z|>|a|$

$-nb^n u(-n-1)$ $\quad \dfrac{bz^{-1}}{(1-bz^{-1})^2}$ $\quad |z|<|b|$

Q1. Determine $x(n) = (n-2)(0.5)^{n-2}\cos\left[\frac{\pi}{3}(n-2)\right] u(n-2)$

differentiation property Folding property

Q2 $x(n) = \left[n\left(-\frac{1}{2}\right)^n u(n)\right] * \left(\frac{1}{4}\right)^{-n} u(-n)$

first calculate X(z) & then find its inverse z-transform

2) $x(n) = \underbrace{\left[n\left(\frac{-1}{2}\right)^n u(n)\right]}_{w(n)} * \underbrace{\left(\frac{1}{4}\right)^{-n} u(-n)}_{y(n)}$ $\quad X(z) = W(z)\,Y(z)$

$\left(-\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \dfrac{z}{z+\frac{1}{2}}$ $\quad$ ROC : $|z| > \frac{1}{2}$

Using differentiation property :-

$w(n) = n\left(\frac{-1}{2}\right)^n u(n) = -z\frac{d}{dz}\left(\frac{z}{z+\frac{1}{2}}\right)$

ROC : $|z| > \frac{1}{2}$

$= \dfrac{-z \times 1/2}{\left(z+\frac{1}{2}\right)^2}$ $\quad$ ROC : $|z| > \frac{1}{2}$

$$y(n) = \left(\frac{1}{4}\right)^{-n} u(-n)$$

$$\left(\frac{1}{4}\right)^{n} u(n) = \frac{z}{z - \frac{1}{4}} \qquad ROC: |z| > \frac{1}{4}$$

$$\left(\frac{1}{4}\right)^{-n} u(-n) = \frac{1/z}{\frac{1}{z} - \frac{1}{4}} \qquad ROC: \left|\frac{1}{z}\right| > \frac{1}{4}$$

$$Y(z) = \frac{-4}{z-4} \qquad ROC: |z| < 4$$

$$\therefore X(z) = W(z) Y(z)$$

$$= \frac{-z \times \frac{1}{2}}{\left(z + \frac{1}{2}\right)^2} \times \frac{-4}{z-4} \qquad ROC: \frac{1}{2} < |z| < 4$$

$$= \frac{2z}{(z-4)\left(z+\frac{1}{2}\right)^2} \qquad ROC: \frac{1}{2} < |z| < 4$$

b) $$x(n) = (n-2)(0.5)^{n-2} \cos\left[\frac{\pi}{3}(n-2)\right] u(n-2)$$

$$= (n-2)\left[\frac{1 - 0.5 \times \frac{\sqrt{3}}{2} z^{-1}}{1 - \frac{\sqrt{3}}{2} z^{-1} - 0.25 z^{-2}}\right]$$

$$= (n-2)\left[\frac{\left(z - \frac{\sqrt{3}}{4}\right)/z}{\left(z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}\right)/z^2}\right]$$

$$= (n-2)\left[\frac{z^2 - \sqrt{3}z/4}{z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}}\right]$$

$$= -z \frac{d}{dz}\left[\frac{z^2 - \frac{\sqrt{3}z}{4}}{z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}}\right]$$

$$= -z\left[\frac{\left(2z - \frac{\sqrt{3}}{4}\right)\left(z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}\right) - \left(2z - \frac{\sqrt{3}}{2}\right)\left(z^2 - \frac{\sqrt{3}z}{4}\right)}{\left(z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}\right)^2}\right]$$

$$= -z\left[\frac{\cancel{2z^3} - \sqrt{3}z^2 - \frac{z}{2} - \frac{\sqrt{3}}{4}z^2 + \cancel{\frac{3}{8}z} + \frac{\sqrt{3}}{16} - \cancel{2z^3} + \frac{2\sqrt{3}}{4}z^2 + \frac{\sqrt{3}}{2}z^2 - \cancel{\frac{3}{8}z}}{\left(z^2 - \frac{\sqrt{3}z}{2} - \frac{1}{4}\right)^2}\right]$$

$$= -z\left[\frac{-\frac{\sqrt{3}}{4}z^2 - \frac{z}{2} + \frac{\sqrt{3}}{16}}{\left(z^2 - \frac{\sqrt{3}}{2}z - \frac{1}{4}\right)^2}\right] = \frac{\frac{\sqrt{3}}{4}z^3 + \frac{z^2}{2} + \frac{\sqrt{3}z}{16}}{\left(z^2 - \frac{\sqrt{3}}{2}z - \frac{1}{4}\right)^2}$$

[In the end multiply with z^-2 to deal with n-2(i.e. time-domain shift)]

Inverse Z - Transform

While taking inverse z-transform ,the outcome would vary as per the constraint on ROC ,i.e. whether we are allowed to choose Right sided sequence or left sided sequence.

1) Partial Fraction Method -

so first evaluate normal answer and then put U(n) or -1*U(-n-1) as per the demand.

$$X(z) = \frac{z}{3z^2 - 4z + 1} = \frac{z}{(3z-1)(z-1)}$$

$(3z+1)$
$(z+1)$
$3z^2$ ...

$$\frac{X(z)}{z} = \frac{1}{(3z-1)(z-1)} = \cancel{\frac{A}{(3z+1)} + \frac{B}{(z+1)}}$$

$\cancel{A = -2}$
$\cancel{B = \frac{2}{3}}$

$$= \frac{1}{3(z-1)\left(z-\frac{1}{3}\right)}$$

$-\frac{1}{3}$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{\left(z-\frac{1}{3}\right)}$$

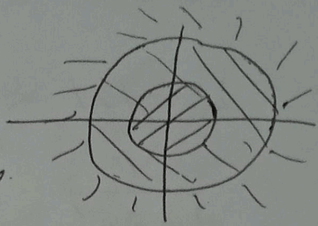
$$A_1 = f(z)(z-1)\Big|_{z=1/3} = \frac{1}{2}$$

$$A_2 = -1/2$$

$$\frac{X(z)}{z} = \frac{1/2}{(z-1)} + \frac{(-1/2)}{(z-1/3)}$$

$$\therefore X(z) = \frac{z/2}{(z-1)} + \frac{-z/2}{(z-1/3)}$$

There will be three ROCs



ROC_1 $\therefore$ $|z| > 1$, $x(n)$ is causal

$$x(n) = \frac{1}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

$ROC_2 \Rightarrow$ $|z| < \frac{1}{3}$, $x(n)$ is anti causal

$$x(n) = -\frac{1}{2}u(-n-1) + \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$$

$ROC_3 \Rightarrow$ $\frac{1}{3} < |z| < 1$

$|z| > \frac{1}{3}$ $\Rightarrow$ causal

$|z| < 1$ $\Rightarrow$ anti causal

$$x(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

Q. $$x(z) = \frac{z+2}{2z^2-7z+3}$$ $\cancel{(2z^2-6)(z}$

2) Long Division Method • (Power Series) -

it is applicable for single-sided sequence only

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Q. $$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$ (find power of z^{-1} right sided, causal)

$$1-az^{-1} \overline{\big)\, 1 \qquad} \; (1 + az^{-1} + a^2z^{-2} \ldots$$

$$\begin{array}{l} 1 \\ \underline{-1 \mp az^{-1}} \\ az^{-1} \\ \underline{-az^{-1} \mp a^2z^{-2}} \\ a^2z^{-2} \end{array}$$

$$\Rightarrow \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

$$x(n) = a^n u(n)$$

Now if $|z| < |a|$ (find powers of z)

(left-sided anti-causal)

$$\therefore \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$-a+z \overline{) z} \quad (-a^{-1}z - a^{-2}z^2$$

$$\underline{-z \mp a^{-1}z^2}$$

$$a^{-1}z^2$$

$$a^{-1}z^2 - a^{-2}z^3$$

$$\therefore \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 + \cdots$$

$$\therefore x(n) = -a^n u(-n-1)$$

Q. $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-2}+\frac{1}{8}z^{-1}}$

i) $|z| > \frac{1}{2}$

ii) $|z| < \frac{1}{4}$

iii) $\frac{1}{4} < |z| < \frac{1}{2}$

System Properties -

1) Causality - ~~At speed time~~ discreet time LTI system is causal if and only if :

a) ROC is exterior of circle outside of outermost pole.

b) $X(z) = \dfrac{A(z)}{B(z)}$, order of $A(z)$ can't be greater than order of $B(z)$

2.) Stability - LTI system is stable if and only if ROC of system function $H(z)$ includes the unit circle $|z| = 1$

i.e.the outernmost pole should lie inside the unit circle(this property is being used to prove the stability of filter)

Q. $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} = \frac{2z^2 - \frac{5z}{2}}{z^2 - \frac{5z}{2} + 1}$

i) $|z| > 2$, $H(z)$ causal, not stable.

ii) $|z| < \frac{1}{2}$, $H(z)$ neither causal, not stable

iii) $\frac{1}{2} < |z| < 2$, $H(z)$ not causal, stable

Q Solve $y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n)$ $\quad x \geq 0$

and $x(n) = \left(\frac{1}{4}\right)^n u(n)$

subject to $y(-1) = 4$ and $y(-2) = 10$

* One-sided z-transform -

$$z^{+}[x(n)] = z[x(n)u(n)]$$

$$z^{+}[x(n-k)] = z[x(n-k)u(n)]$$

$$= \sum_{n=0}^{\infty} x(n-k)z^{-n}$$

let $n-k = m \quad \therefore n = m+k$

$$\therefore \sum_{m=-k}^{\infty} x(m)z^{-(m+k)}$$

$$= \sum_{m=-k}^{-1} x(m)z^{-(m+k)} + \left[\sum_{m=0}^{\infty} x(m)z^{-m}\right]z^{-k}$$

$$\therefore \boxed{z^{+}[x(n-k)] = x(-1)z^{1-k} + x(-2)z^{2-k} \ldots \cancel{x(-n)z}\ x(-k) + z^{-k}X(z)}$$

Now, $Y^+(z) = \dfrac{2 - \frac{9}{4}z^{-1} + \frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})(1-\frac{1}{4}z^{-1})}$

$\therefore Y^+(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} + \dfrac{2/3}{1-z^{-1}} + \dfrac{1/3}{1-\frac{1}{4}z^{-1}}$

$y(n) = \left[\left(\frac{1}{2}\right)^n + \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n)$

Form of solution ⇒

Homogenous and Particular part

↓ System poles
$\left[\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u(n)$

↓ Input poles
$\frac{1}{3}\left(\frac{1}{4}\right)^n u(n)$

this property is being used to solve the steady state reponse Q. given above.

* Transient and ~~stable~~ steady state response

↓ poles which are inside the unit circle
$\left[\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n)$

↓ poles onto the unit circle
$\frac{2}{3}(u(n))$

If poles are outside unit circle, response is unbounded

In other copy →

T.P. $\tilde{x}(n)\tilde{y}(n) \overset{DFS}{\longleftrightarrow} \frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}(l)\tilde{Y}(K-l)$

RHS → $IDFS\left[\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}(l)\tilde{Y}(K-l)\right]$

DSP - It means that DTFT was in terms of e(^jw) while Z-transform was in terms of z so exact value was not there

DFT - DTFT and Z are not numerically computable transform because they don't give exact values.

DFT - numerically computable

Fast DFT → FFT (Fast FT)

Fast DFT<=>FFT

DFS - Discrete Fourier Series - Consider a periodic sequence, $\tilde{x}(n)$ with fundamental period N. ⇒ $\tilde{x}(n) = \tilde{x}(n + kN)$, for any integer value of k.

it exists for a periodic signal only.

$$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(K)e^{j\frac{2\pi}{N}Kn}, \quad n = 0, \pm 1, \pm 2 \ldots$$

while N =2*pi for DTFT and integral range was -inf to +inf.

where $\{\tilde{X}(K), \ k = 0, \pm 1, \pm 2\}$ are called the discrete fourier series coefficient given by

$$\tilde{X}(K) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}Kn}, \quad n = 0, \pm 1, \pm 2, \ldots$$

complex valued

Also $\boxed{\tilde{X}(K+N) = \tilde{X}(K)}$ and its corresponding sequence is also periodic

$$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

take care of negative sign

$$\tilde{X}(K) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{nK}, \quad K = 0, \pm 1, \pm 2 \ldots$$

$$\tilde{x}(n) = \boxed{\frac{1}{N}} \sum_{K=0}^{N-1} \tilde{X}(K) W_N^{-nK}, \quad n = 0, \pm 1, \pm 2 \ldots$$

while it was 1/2pi for DTFT

it is similar to z-transform or DTFT ,here that function parameter is e^(jwok/N) so it would be raised to the shifted power ,while they were z and e^(jw) in the former cases.

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Properties :-

$$\tilde{x}_1(n) \xrightarrow[\text{coeff}]{DFS} \tilde{X}_1(k) \quad N$$

$$N \quad \tilde{x}_2(n) \xrightarrow[\text{coff}]{DFS} \tilde{X}_2(k) \quad N$$

1.) Linearity - If two sequences $\tilde{x}_1(n)$ and $\tilde{x}_2(n)$ with $FP = N$

$$\therefore \tilde{x}_3(n) = a\tilde{x}_1(n) + b\tilde{x}_2(n)$$

$$\therefore \tilde{X}_3(k) = a\tilde{X}_1(k) + b\tilde{X}_2(k)$$

2.) Shift of sequence - If $\tilde{x}_1(n) \xrightarrow[\text{coefficient}]{DFS} \tilde{X}_1(k)$

it implies change in x(n) to x(n+m) while the power of exponential term would be left untouched

Then, $\tilde{x}(n+m) \xrightarrow[\text{coeff}]{DFS} e^{j\frac{2\pi}{N}km}\tilde{X}(k)$

Proof

$$\therefore \tilde{x}(n+m) = \cancel{\frac{1}{N+m}} \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi}{N}kn}$$

let $n+m = p$

$$= \frac{1}{N}\sum_{p=m}^{N+m-1} \tilde{X}(p)\, e^{-j\frac{2\pi}{N}k(p-m)}$$

$$= \frac{1}{N}\sum_{p=0}^{N-1} \tilde{X}(p)\, e^{-j\frac{2\pi}{N}kp} \times e^{j\frac{2\pi}{N}km}$$

$$= \cancel{\tilde{x}(n)\, e^{j\frac{2\pi}{N}km}} \quad \tilde{X}(k)\, e^{j\frac{2\pi}{N}km}$$

(proved)

it means change in power of exponential term by replacing z with the supplied parameter while x(n) would be left untouched.

3.) If $\tilde{x}(n) \longrightarrow \tilde{X}(k)$

Then, $e^{-j\frac{2\pi}{N}ln}\tilde{x}(n) = \tilde{X}(k+l)$

similar to DTFT.

4.) Periodic convolution - $\tilde{x}_1(n)$, $\tilde{x}_2(n)$ $FP = N$ with DFS $\tilde{X}_1(k)$, $\tilde{X}_2(k)$ then we find $\tilde{x}_3(n)$ whose DFS is $\tilde{X}_1(k) \cdot \tilde{X}_2(k)$ then $\tilde{x}_3(n)$ is periodic convolution.

Done in my notebook

summation range is NOT from -inf to +inf

Periodic convolution $\tilde{x}_3(n)$ is given by :-

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\,\tilde{x}_2(n-m) \quad , n = 0,1,2 \dots N-1$$

$$= \sum_{m=0}^{N-1} \tilde{x}_2(m)\,\tilde{x}_1(n-m)$$

T.P DFS

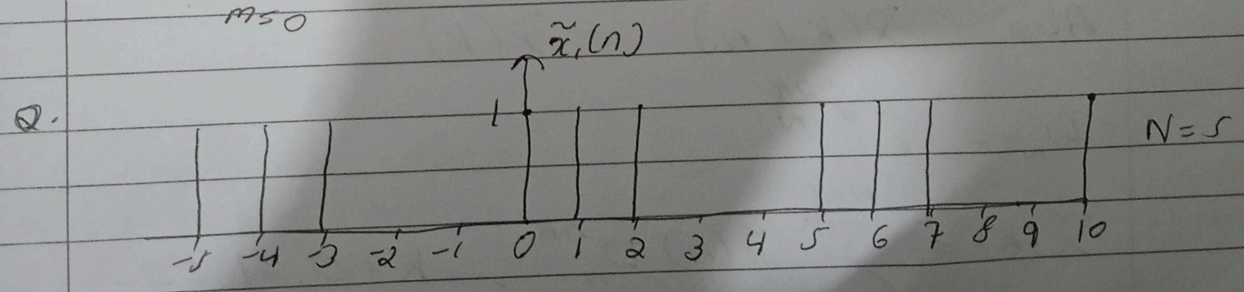
$$LHS \Rightarrow \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} \tilde{x}_1(m)\,\tilde{x}_2(n-m) \right] e^{-j\frac{2\pi}{N}kn}$$

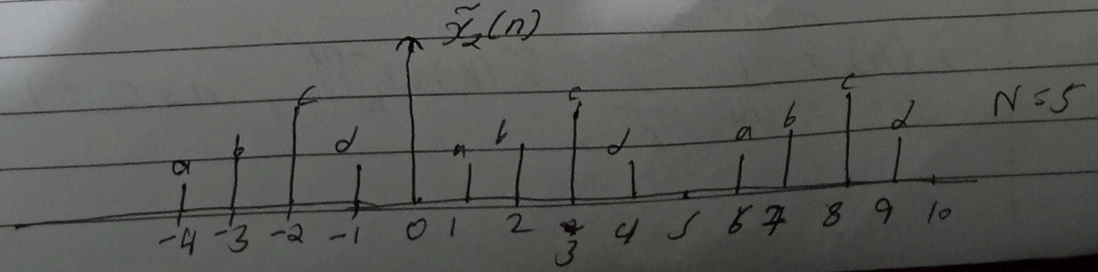
$$= \sum_{m=0}^{N-1} \tilde{x}_1(m) \left[\sum_{n=0}^{N-1} \tilde{x}_2(n-m)\, e^{-j\frac{2\pi}{N}kn} \right]$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1(m)\, e^{-j\frac{2\pi}{N}km}\, \tilde{X}_2(k) = \tilde{X}_1(k)\,\tilde{X}_2(k)$$

$= RHS$

Q.





it implies x(n)*y(n)=(1/N)*[convolution of X(l) and Y(l)] while periodic convolution implies that convolution of x(n) and y(n) =X(l)*Y(l)

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Determine and plot the ~~con~~ periodic convolution of these sequences

S:) $\tilde{x}(n)\,\tilde{y}(n) \xleftrightarrow{DFS} \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}(l)\,\tilde{Y}(k-l)$

Prove HW

Done

steps

1.first write Y(k-l) in its expanded form
2.equate y term with y(n) and include 1/N in that.
3.the l-power term would cancel out the l-power term in
expansion of X(l)

are inside the unit circle | the unit circle | If p... are ou... unit ci... response i... unbound...

$\left[\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n)$ | $\frac{2}{3}(u(n))$

In other copy $\Rightarrow$

T.P. $\tilde{x}(n)\tilde{y}(n) \xleftrightarrow{DFS} \frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}(l)\tilde{Y}(k-l)$

RHS $\Rightarrow$ IDFS $\left[\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}(l)\tilde{Y}(k-l)\right]$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left\{\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}(l)\,\tilde{Y}(k-l)\right]e^{+j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}(l)\left[\frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}(k-l)\,e^{j\frac{2\pi}{N}kn}\right]$$

$$= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}(l)\,e^{j\frac{2\pi}{N}kn}\cdot\tilde{y}(n)$$

$$= \tilde{x}(n)\,\tilde{y}(n)$$

Q. $\tilde{x}^*(n) \xrightarrow{DFS} \tilde{X}^*(-k)$

complex conjugate of $\tilde{x}(n)$

$$DFS[\tilde{x}^*(n)] = \sum_{n=0}^{N-1}\tilde{x}^*(n)\,e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1}\tilde{x}^*(n)\left[e^{-j\frac{2\pi}{N}(-k)n}\right]^*$$

$$= \tilde{X}^*(-k)$$

Q. $\tilde{x}^*(-n) \xrightarrow{DFS} \tilde{X}^*(k)$ time reversal property

$$DFS[\tilde{x}^*(-n)] = \sum_{n=0}^{N-1}\tilde{x}^*(-n)\left[e^{-j\frac{2\pi}{N}k(-n)}\right]^*$$

$$= \tilde{X}^*(k)$$

A function f(a) is conjugate symmetric if f(-a) = f*(a) while a function f(a) is conjugate antisymmetric if f(-a)=-f*(a).

* $Re[\tilde{x}(n)] \xrightarrow{DFS} \tilde{X}_e(k) = \boxed{(X(k)+X^*(-k))/2;}$

conjugate symmetric part of $\tilde{X}(k)$

* $Im[\tilde{x}(n)] \xrightarrow{DFS} \tilde{X}_o(k) = (X(k)-X^*(-k))/2;$

conjugate anti-symmetric part of $\tilde{X}(k)$

~~FP~~ ~~periodic~~ Re[

Any sequence can be expressed as sum of conjugate symmetric and conjugate anti-symmetric

$$\tilde{x}(n) = \tilde{x}_e(n) + \tilde{x}_o(n)$$

where, $\tilde{x}_e(n) = \frac{1}{2}\left[\tilde{x}(n) + \tilde{x}^*(-n)\right]$

$$\tilde{x}_o(n) = \frac{1}{2}\left[\tilde{x}(n) - x^*(-n)\right]$$

Any sequence can also be expressed as

$$\underbrace{\tilde{x}(n)}\ = \underbrace{\tilde{x}_R(n)}_{\text{Real}} + j\underbrace{\tilde{x}_I(n)}_{\text{Imaginary}}$$

$$\therefore \tilde{x}^*(n) = \tilde{x}_R(n) - j\tilde{x}_I(n)$$

$$\therefore \tilde{x}_R(n) = \frac{1}{2}\left[\tilde{x}(n) + \tilde{x}^*(n)\right]$$

$$DFS[\tilde{x}_R(n)] = DFS\left[\frac{\tilde{x}(n) + \tilde{x}^*(n)}{2}\right]$$

$$= \cancel{DFS} \frac{\tilde{X}(K) + \tilde{X}^*(-K)}{2}$$

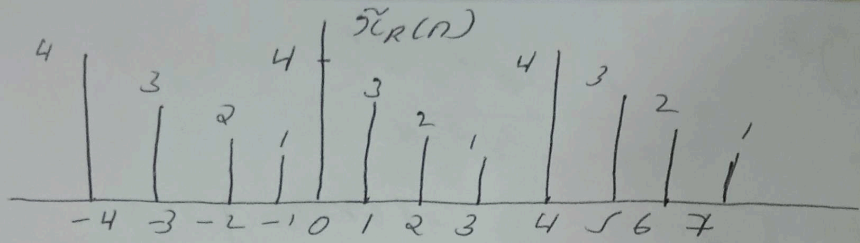
$$= \tilde{X}_e(K)$$

$$DFS[\tilde{x}_I(n)] = DFS\left[\frac{\tilde{x}(n) - \tilde{x}^*(n)}{2j}\right]$$

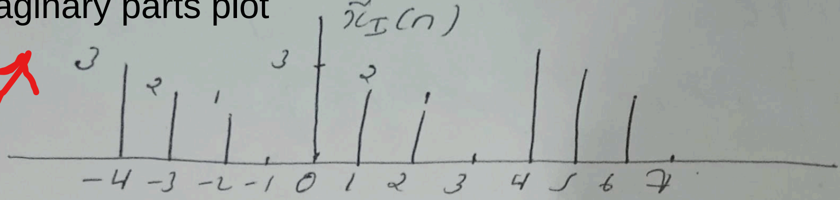
$$= \frac{\tilde{X}(K) - \tilde{X}^*(-K)}{2j} = \tilde{X}_o(K)$$

?

Q.



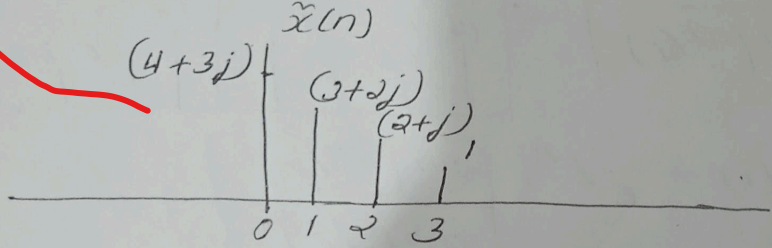
real and imaginary parts plot



?

Determine and plot conjugate symmetric and anti-symmetric part.

$$\tilde{x}(n) = \tilde{x}_R(n) + j\,\tilde{x}_I(n)$$

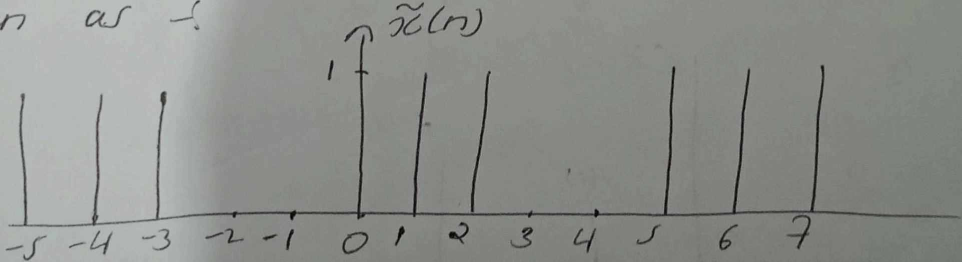


$$\tilde{x}_e(n) = \{4, (2+j), 2, (2-j)\}$$

$$\tilde{x}_o(n) = \{3j, (1+j), j, (-1+j)\}$$

for different values of k

Q. Determine fourier series coefficient $|\tilde{X}(K)|$ of $\tilde{x}(n)$ given as :-



$$\cancel{\tilde{X}(K)}\ \tilde{x}(n) = \{\underset{\uparrow}{1}, 1, 1, 0, 0\}$$

$$\tilde{X}(k) = \sum_{n=0}^{4} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= 1 + e^{-j\frac{2\pi}{5}k} + e^{-j\frac{2\pi}{5}k\cdot 2}$$

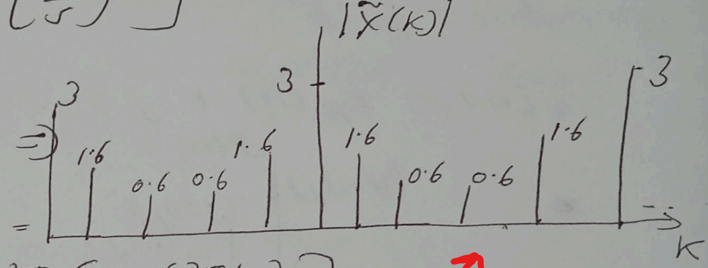
$$= \frac{1-\left(e^{-j\frac{2\pi k}{5}}\right)^3}{1-e^{-j\frac{2\pi k}{5}}} = \frac{1-e^{-j\frac{6\pi k}{5}}}{1-e^{-j\frac{2\pi k}{5}}}$$

$$= \frac{e^{-j\frac{6k\pi}{10}}\left[e^{j6k\pi/10} - e^{-j6k\pi/10}\right]}{e^{-j\frac{2k\pi}{10}}\left[e^{j2k\pi/10} - e^{-j2k\pi/10}\right]}$$

$$\therefore \tilde{X}(k) = e^{-j\frac{2\pi k}{5}}\left[\frac{\sin\left(\frac{3\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\right]$$

Being modulus it is being kept above x-axis and negative sign is being considered in shifting by pi units.

$$|\tilde{X}(k)| = \left|\frac{\sin\left(\frac{3\pi k}{5}\right)}{\sin\left(\pi k/5\right)}\right| \Rightarrow$$



At $k=1$, $\tilde{X}(1) = e^{-j\frac{2\pi}{5}}\left[\frac{\sin(3\pi/5)}{\sin(\pi/5)}\right]$ → if +, phase $= -\frac{2\pi}{5}$

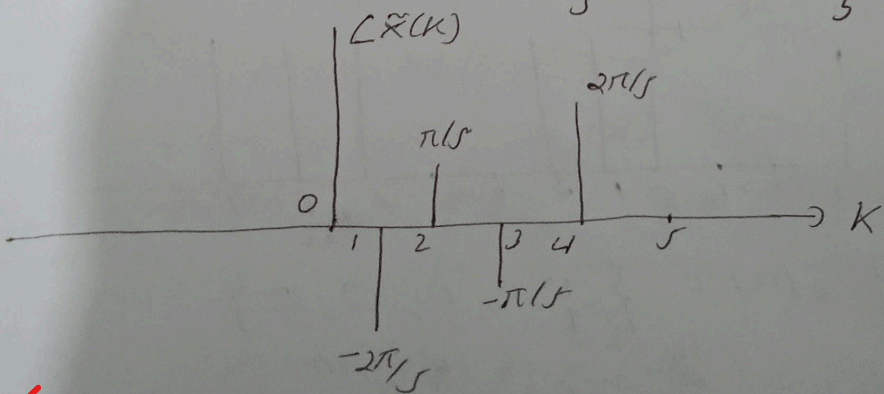
if −, shift it by π

if amplitude part comes out to be negative then add pi to compensate for the negative sign

$\therefore$ phase $= -\frac{2\pi}{5}$

At $k=2$, $\tilde{X}(2) = e^{-j\frac{4\pi}{5}}\left[\frac{\sin(6\pi/5)}{\sin(2\pi/5)}\right]$

phase $\Rightarrow -\frac{4\pi}{5} + \pi = \frac{\pi}{5}$



HW

Q. Determine and plot DFS of

$$\tilde{x}(n) = 1, \quad 0 \le n \le 4$$
$$= 0, \quad 5 \le n \le 9$$

$$\tilde{X}(k) = \sum_{n=0}^{9} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn}$$

$$= 1 + e^{-j\frac{2\pi k}{9}} + e^{-j\frac{4\pi k}{9}} + e^{-j\frac{6\pi k}{9}} + e^{-j\frac{8\pi k}{9}}$$

$$= \frac{1 - \left(e^{-j\frac{2\pi k}{9}}\right)^{5}}{1 - e^{-j2\pi k/9}}$$

$$= \frac{1 - e^{-j\frac{16\pi k}{10}}}{1 - e^{-j2\pi k/9}} = \frac{e^{-j\frac{16\pi k}{10}}\left[e^{j\frac{16\pi k}{10}} - e^{-j\frac{16\pi k}{10}}\right]}{e^{-j\frac{2\pi k}{10}}\left[e^{j\frac{2\pi k}{10}} - e^{-j\frac{2\pi k}{10}}\right]}$$

$$e^{-j\frac{7\pi k}{5}}\left[\frac{\sin\left(\frac{8\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\right]$$

$$|\tilde{X}(k)| = \left|\frac{\sin\left(\frac{8\pi k}{5}\right)}{\sin\left(\pi k/5\right)}\right|$$

DFT – Consider a finite duration sequence $x(n)$ of length $n \Rightarrow x(n)$ has N samples over $0 \le n \le N-1$. The corresponding periodic sequence of period N for which $x(n)$ is one period is given by

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n+rN)$$

$$\tilde{x}(n) = x(n \text{ modulo } N)$$

$$= x((n))_N \quad \text{(notation)}$$

The finite duration sequence $x(n)$ is obtained from $\tilde{x}(n)$ by extracting one period

i.e. $$x(n) = \begin{cases} \tilde{x}(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

If we define rectangular sequence $R_N(n)$

$$R_N(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \boxed{x(n) = \tilde{x}(n) R_N(n)}$$

$$\tilde{x}(n) \longrightarrow \tilde{X}(K)$$

$$x(n) \longrightarrow X(K)$$

$X(K)$ and $\tilde{X}(K)$ are related as

$$\tilde{X}(K) = X((K))_N \quad [X(K \text{ modulo } N)]$$

$$\therefore \boxed{X(K) = \tilde{X}(K) R_N(K)}$$

We prove $\tilde{X}(K)$ and ~~$X(K)$~~ $\tilde{x}(n)$ are related as

$$\tilde{X}(K) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{Kn}, \quad \boxed{K = 0, \pm 1, \pm 2} \quad \text{— (2a)}$$

$$\tilde{x}(n) = \frac{1}{N} \sum_{K=0}^{N-1} \tilde{X}(K) W_N^{-nK}, \quad n = 0, \pm 1, \pm 2 \quad \text{— (2b)}$$

$$X(K) = \begin{cases} \sum_{n=0}^{N-1} x(n) W_N^{Kn}, & \boxed{0 \le K \le N-1} \\ 0, & \boxed{\text{otherwise}} \end{cases} \quad \text{— (3a)}$$

$$x(n) = \begin{cases} \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-nK}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{— (3b)}$$

the difference between the 2 is in fact that DFS is defined for periodic sequence only while DFS is for finite length sequence

Properties of DFT -

1.) Linearity - Two finite duration sequences $x_1(n)$ and $x_2(n)$ of ~~length~~ N duration

$$x_3(n) = a x_1(n) + b x_2(n)$$

$$X_3(K) = a X_1(K) + b X_2(K)$$

If duration is different then x_3 duration is maximum of ~~both~~ all.

To avoid data loss,linear shift is being avoided and periodic shift is being used.

2.) Circular Shift - If a finite duration sequence is shifted in either direction, the result will no longer be in 0, N-1 so we make it periodic and shift it by N. This is periodic shift and if we truncate it in its duration, it is called circular shift.

$x(n) \longrightarrow$ finite duration, $0 \leq n \leq N-1$

$\downarrow$

$\tilde{x}(n)$

$$\tilde{x}(n+m) = x((n+m))_N$$

$\hookrightarrow$ converted to N periodic sequence

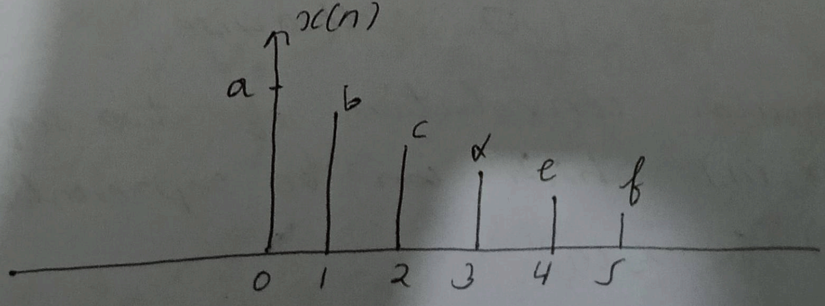
$$\tilde{x}(n+m)R_N(n) = x((n+m))_N R_N(n) \longrightarrow \text{circular shift}$$

(truncating it in its duration)

proof in my notebook

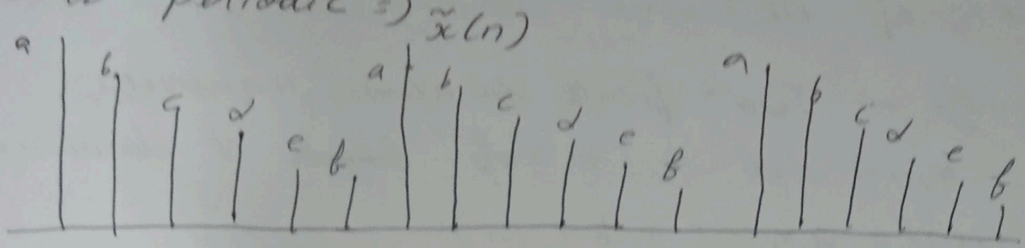
$$\boxed{DFT[x((n+m))_N R_N(n)] = W_N^{-Km} X(K)}$$ Prove

Q.

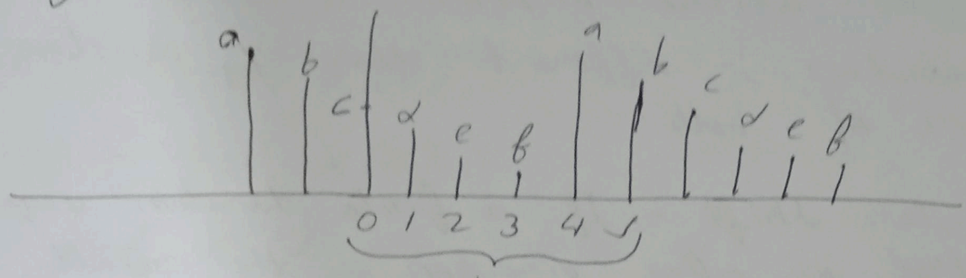


Find $x((n+2))_N R_N(n)$

Make it periodic $\Rightarrow \tilde{x}(n)$



Shift by 2



answer (circular shift)

3) Circular convolution - Two finite duration sequences

$x_1(n)$ and $x_2(n)$ (N duration) with DFT

$\bar{X}_1(K)$ and $\bar{X}_2(K)$. We need to determine

$x_3(n)$ such that its DFT $X_3(K)$ is $X_1(K) \cdot X_2(K)$

then $x_3(n)$ is circular convolution.

$$x_3(n) = \left[\sum_{m=0}^{N-1} \tilde{x}_1(n)\, \tilde{x}_2(n-m)\right] R_N(n)$$

$$= \left[\sum_{m=0}^{N-1} x_1((n))_N\, x_2((n-m))_N\right] R_N(n)$$

where $R_N(n) = \begin{cases} 1 & , n = 0, 1, 2 \ldots (N-1) \\ 0 & , \text{otherwise} \end{cases}$

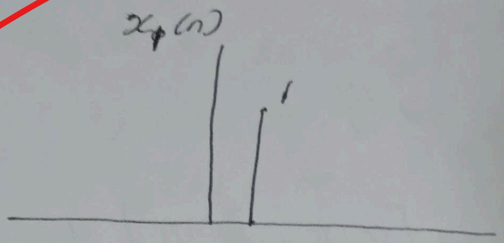
N point circular convolution of two sequences $x_1(n)$ and $x_2(n)$ which can be represented by

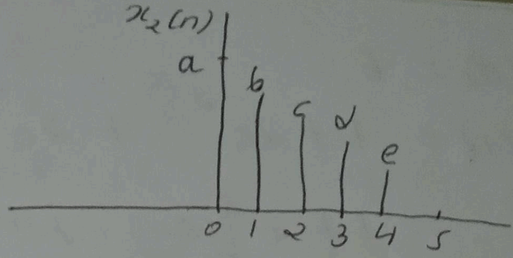
$$x_1(n) \, Ⓝ \, x_2(n)$$

DFT $\Rightarrow$

$$\text{DFT}\left[x_1(n) \, Ⓝ \, x_2(n)\right] = X_1(K) \cdot X_2(K)$$

Q.





Determine and plot circular convolution of following sequence

Stockham's Method / DFT - IDFT Method - Find $X_1(k)$ and $X_2(k)$. Multiply and find IDFT.

(frequency domain approach) (IDFT approach)

Q. Compute the circular convolution of two sequences given by

first evaluate X1(k) and X2(k) and then multiply term-by-term and then finally take its IDFT and donot forget 1/N while taking inverse

$x_1(n) = [1, 2, 2, 1]$ $\qquad$ $x_2(n) = [2, 1, 1, 2]$

$$X_1(K) = \sum_{n=0}^{3} x_1(n) e^{-j\frac{2\pi}{N}kn} \qquad N = 4$$

$$= 1 + 2e^{-j\frac{4\pi}{3}k} + 2e^{-j\frac{6\pi k}{3}} + e^{-j\frac{8\pi k}{3}}$$

$$X_2(K) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$= 2 + e^{-j\frac{4\pi k}{3}} + e^{-j\frac{6\pi k}{3}} + 2e^{-j\frac{8\pi k}{3}}$$

$$X_1(K)X_2(K) = 2 + e^{-j\frac{4\pi}{3}k} + e^{-j\frac{6\pi k}{3}} + 2e^{-j\frac{8\pi k}{3}} + 4e^{-j\frac{4\pi k}{3}} + 2e^{-j\frac{8\pi k}{3}} + 2e^{-j\frac{10\pi k}{3}} + 4e^{-j\frac{12\pi k}{3}} + 4e^{-j\frac{6\pi k}{3}} + 2e^{-j10\pi k/3} + 2e^{-j\frac{12\pi k}{3}} + 4e^{-j}$$

$$X_1(K) = 1 + 2e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

$$X_2(K) = 2 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

$x_1(0) = 6$

$x_1(1) = 1 + 2e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}}$

$= 1 + 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 2\left[\cos\pi - j\sin\pi\right] + \left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 1 + 2(-j) + 2(-1) + j$

$= -1 - j$

$x_1(2) = 1 + -2 + 2(1) + (-1)$

$= 0$

$x_1(3) = -1 + j$

$\therefore x_1(K) = \{6, -1-j, 0, -1+j\}$

$x_2(0) = 6$

$x_2(1) = 2 + e^{-j\pi/2} + e^{-j\pi} + 2e^{-j3\pi/2}$

$= 2 - j - 1 + 2j$

$= 1 + j$

$x_2(2) = 2 - 1 + 1 - 2 = 0$

$x_2(3) = 1 - j$

$x_2(K) = \{6, 1+j, 0, 1-j\}$

$1 + -1 + 2j$

$X_3(K) = \{36, -2j, 0, 2j\}$

$x_3(n) = \boxed{\frac{1}{N}} \sum_{K=0}^{3} X_3(K)\, e^{+j\frac{2\pi}{N}Kn}$

$= \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j\frac{2\pi nk}{4}}$

$= \frac{1}{4}\left[36 + (-2j)e^{j\pi n/2} + 2je^{j3\pi n/2}\right]$

$$4x_3(n) = 36 - 2j\,e^{+j\pi n/2} + 2j\,e^{j3\pi n/2}$$

$x_3(0) = 9$ $\qquad x_3(2) = 9$

$x_3(1) = 10$ $\qquad x_3(3) = 8$

First length should be made same to evaluate circular convolution.

Q. Determine the circular convolution of

$x_1(n) = \{\underset{\uparrow}{1}, 2, 3\}$ $\qquad x_2(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

To make duration same we add 0 to $x_1(n)$ known as zero-padding $\therefore x_1(n) = \{1, 2, 3, 0\}$

Q. $x_1(n) = \delta\{n-1\}$ $\xrightarrow{\text{too good}}$ $\{\underset{\uparrow}{0}, 1, 0, 0, 0\}$

$x_2(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$

Q. Compute the DFT of each of following finite length sequences of length N.

a.) $x(n) = \delta(n)$

b.) $x(n) = \delta(n - n_0)$, $0 < n_0 < N$

c.) $x(n) = a^n$, $0 \le n \le N-1$

a.) $\delta(n) \Rightarrow 1$

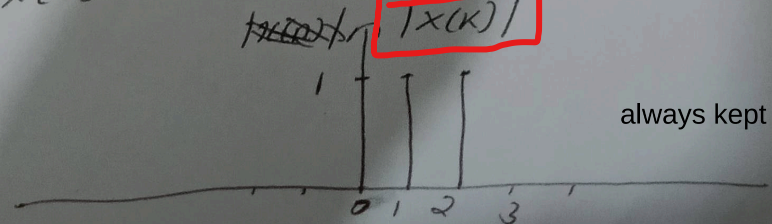
b.) $\delta(n - n_0) \Rightarrow \sum_{n=0}^{N-1} \delta(n - n_0)\, e^{-j\frac{2\pi}{N}kn}$

$$= e^{-j\frac{2\pi k n_0}{N}}$$

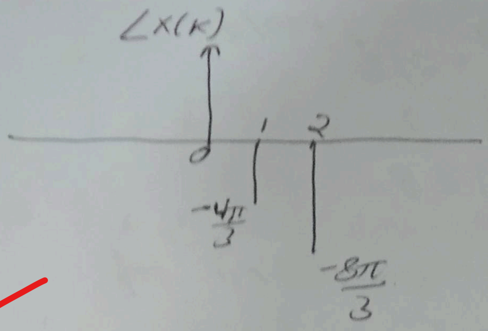
Take, $n_0 = 2$, $N = 3$

$$\therefore X(K) = e^{-j2\pi k \times \frac{2}{3}} = e^{-j4\pi k/3}$$

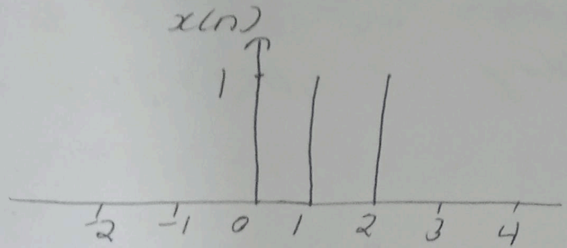
$X(0) = 1$ $\qquad X(1) = e^{-j\frac{4\pi}{3}}$ $\qquad X(2) = e^{-j\frac{8\pi}{3}}$



always kept abpve x-axis



Q. Determine and plot DFT of following sequence -



$N = 3$

$$X(K) = \sum_{n=0}^{2} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= 1 + e^{-j\frac{2\pi}{3}K} + e^{-j\frac{4\pi}{3}K}$$

$X(0) = 1$

$X(1) = 1 + e^{-j\frac{2\pi}{3}} + e^{-j\frac{4\pi}{3}}$

$X(2) = 1 + e^{-j4\pi/3} + e^{-j8\pi/3}$

$$X(1) = 1 + \left[\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right] + \left[\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right]$$

$$= 1 + \left[\frac{-1}{2} + j\frac{\sqrt{3}}{2}\right] + [$$