Lecture Notes on Non-deterministic Finite automata with ϵ-transitions

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Contents

1 Introduction

We study NFAs with ϵ transitions (NFA/e) and their equivalence with NFAs.

2 Definition

2.1 NFA/e

Definition 2.1 (NFA/e)**.** Let Σ be a finite alphabet not containing the symbol ϵ. An NFA/e over Σ is an NFA whose alphabet is $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.

2.2 ϵ transition

An ϵ transition is a relation between two states q and q' such that $q \xrightarrow[E]{\epsilon} q'$.

3 Reachability in NFA/e

We define several reachability relations:

3.1 ϵ reachability

Definition 3.1 (ϵ-reachability)**.** $\xrightarrow[E]{\epsilon *}\!\!\!\!\rightarrow$ is the reflexive transitive closure of the ϵ transition relation $\xrightarrow[E]{\epsilon}$. In other words, $q \xrightarrow[E]{\epsilon *}\!\!\!\!\rightarrow q'$ iff q can reach q' in zero or more ϵ transitions.

3.2 ϵ closure

Definition 3.2. Given a state q, the ϵ closure of q

$$\epsilon\text{-}clo = \{q' \in Q_E \mid q \xrightarrow[E]{\epsilon *}\!\!\!\!\rightarrow q'\}$$

The ϵ closure of q is the set of all states that are ϵ reachable from q.

3.3 Macro transition via ϵ's

The next definition relates two states that are separated by a single alphabet symbol interspersed with any number of ϵ transitions.

Definition 3.3.

$$\frac{q_1 \xrightarrow[E]{\epsilon *}\!\!\!\!\rightarrow q_2 \qquad q_2 \xrightarrow[E]{a} q_3 \qquad q_3 \xrightarrow[E]{\epsilon *}\!\!\!\!\rightarrow q_4}{q_1 \xrightarrow[E*]{a} q_4} \quad \Sigma$$

$\xrightarrow[E*]{}\!\!\!\!\rightarrow$ denotes the transitive closure of $\xrightarrow[E*]{}$ as is defined by the rules below:

$$\frac{q \xrightarrow[E*]{a} q'}{q \xrightarrow[E*]{a}\!\!\!\!\rightarrow q'} \qquad \Sigma$$

$$\frac{q \xrightarrow[E*]{a} q'' \quad q'' \xrightarrow[E*]{w}\!\!\!\!\rightarrow q'}{q \xrightarrow[E*]{aw}\!\!\!\!\rightarrow q'} \qquad TRANS$$

4 Understanding the different relations

- $q \xrightarrow[E]{a} q'$: q transits to q' on a in E, where $a \in \Sigma$. (non-ϵ transitions)
- $q \xrightarrow[E]{\epsilon} q'$: q transits to q' on ϵ in E (ϵ transitions)
- $q \xrightarrow[E]{\epsilon*}\!\!\!\!\rightarrow q'$: q reaches q' in E consuming zero or more ϵ's along the way. (ϵ reachability)
- $q \xrightarrow[E*]{a} q'$: q reaches q' consuming zero or more ϵ's and the symbol a along the way.
- $q \xrightarrow[E*]{a}\!\!\!\!\rightarrow q'$: The transitive closure of $\xrightarrow[E*]{a}$.

5 Acceptance

An NFA/e E accepts w iff

1. $w = \epsilon$ and $i_E \xrightarrow[E]{\epsilon*}\!\!\!\!\rightarrow q$ for some $q \in F_E$.
2. $w \neq \epsilon$ and $i_E \xrightarrow[E*]{w}\!\!\!\!\rightarrow q$ for some $q \in F_E$.

6 ϵ-elimination

Given an NFA/e E over Σ, the NFA N obtained from E by ϵ elimination is defined as follows:

$$N = (Q_N, i_N, \Sigma, \xrightarrow[N]{}, F_N)$$

where

- $Q_N = Q_E$
- $i_N = i_E$
- $q \xrightarrow[N]{a} q' \stackrel{\mathsf{def}}{=} q \xrightarrow[E*]{a} q'$, and
- $F_N = \begin{cases} F_E \cup \{i_E\} & \text{if } \epsilon\text{-}clo(i_E) \cap F_E \neq \emptyset \\ F_E & \text{otherwise} \end{cases}$

The set of final states of N includes all of F_E plus the initial state i_E, provided a final state of E is ϵ reachable from i_E. Note also that N does not have any ϵ transitions.

7 Relating NFA/e reachability with NFA reachability

Let N denote the NFA obtained by ϵ elimination from an NFA/e E.

Lemma 7.1 (E*-reachability vs. N-reachability)**.** . *If $w \neq \epsilon$, and $q \xrightarrow[E*]{w}\!\!\!\!\rightarrow q'$, then $q \xrightarrow[N]{w}\!\!\!\!\rightarrow q'$*

Proof:
By induction on the derivation of $\xrightarrow[E*]{}\!\!\!\!\rightarrow$.
Base case:

1. $q \xrightarrow[E*]{w}\!\!\!\!\rightarrow q'$ using the Σ rule on $\xrightarrow[E*]{}\!\!\!\!\rightarrow$ — Given.
2. $w = a$ for some $a \in \Sigma$ — from 1 and Σ inversion on $\xrightarrow[E*]{}\!\!\!\!\rightarrow$
3. $q \xrightarrow[E*]{a} q'$ — from 1, 2 and Σ inversion on $\xrightarrow[E*]{}\!\!\!\!\rightarrow$
4. $q \xrightarrow[N]{a} q'$ — from 3 and construction of N.

Inductive case:

1. $q \xrightarrow[E*]{w}\!\!\!\!\rightarrow q'$ using the *TRANS* rule on $\xrightarrow[E*]{}\!\!\!\!\rightarrow$ Given.
2. $w = aw'$ for some $a \in \Sigma$, $w' \neq \epsilon$ from 1 using TRANS inversion
3. $q \xrightarrow[E*]{a} q''$ and $q'' \xrightarrow[E*]{w'}\!\!\!\!\rightarrow q'$ for some $q'' \in Q_E$ from 1, 2 using TRANS inversion on $\xrightarrow[E*]{}\!\!\!\!\rightarrow$.
4. $q \xrightarrow[E*]{a} q''$ from 3 and AND-elim$_1$
5. $q \xrightarrow[N]{a} q''$ from 4 and construction of N
6. $q \xrightarrow[E*]{w'}\!\!\!\!\rightarrow q''$ from 3 and AND-elim$_2$
7. $q \xrightarrow[N]{w'}\!\!\!\!\rightarrow q''$ from 6 and the Induction Hypothesis
8. $q \xrightarrow[N]{aw'}\!\!\!\!\rightarrow q'$ from 5, 7 and TRANS on $\xrightarrow[N]{}\!\!\!\!\rightarrow$.
9. $q \xrightarrow[N]{w}\!\!\!\!\rightarrow q'$ from 8 and 2

QED. $\square$

7.1 E-acceptance implies N-acceptance

Lemma 7.1 directly implies that acceptance by E implies acceptance by N.

Lemma 7.2 (E-acceptance implies N-acceptance)**.** *Let E accept w. Then N accepts w.*

Proof:

- Case $w = \epsilon$:

  1. $i_E \xrightarrow[E]{\epsilon*}\!\!\!\!\rightarrow q$ for some $q \in F_E$ by defn of E-acceptance
  2. $i_E \xrightarrow[E]{\epsilon*}\!\!\!\!\rightarrow q$ From 1 using AND-elim$_1$
  3. $q \in \epsilon\text{-}clo(i_E)$ From 2 and defn of ϵ closure
  4. $q \in F_E$ From 1 using AND-elim$_2$
  5. $\epsilon\text{-}clo(i_E) \cap F_E \neq \emptyset$ From 3, 4 using set theory

6. i_E is a final state in N From 5 and the construction of F_N

- Case $w \neq \epsilon$

1. $w \neq \epsilon$ Given
2. $i_E \xrightarrow[E]{w}\!\!\twoheadrightarrow q$ for some $q \in F_E$ by defn of E-acceptance
3. $i_E \xrightarrow[N]{w}\!\!\twoheadrightarrow q$ From 2 and Lemma 7.1
4. $i_N = i_E$ construction of N
5. $q \in F_E$ From 2 using AND-elim$_2$
6. $F_E \subseteq F_N$ by construction of N
7. $q \in F_N$ From 5, 6, using set theory
8. $i_E \xrightarrow[N]{w}\!\!\twoheadrightarrow q$ From 2 using AND-elim$_1$
9. $i_N \xrightarrow[N]{w}\!\!\twoheadrightarrow q$ Rewriting 8 using 4
10. N accepts w From 7, 9 using defn of N-acceptance

QED □

7.2 N reachability implies E*-reachability

Lemma 7.3 (N-reachability vs. E*-reachability)**.** . *If $w \neq \epsilon$, and $q \xrightarrow[N]{w}\!\!\twoheadrightarrow q'$, then $q \xrightarrow[E*]{w}\!\!\twoheadrightarrow q'$*

Proof. Left as an exercise. □

7.3 N-acceptance implies E-acceptance

Lemma 7.4 (N-acceptance implies E-acceptance)**.** *Let N be the NFA constructed by ϵ-elimination of and NFA/e E. Let N accept w. Then E accepts w.*

Proof. Left as an exercise. □