Lecture Notes on non-deterministic finite automata

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Contents

1 Introduction

We consider non-deterministic finite state automata (NFAs) and their relation to DFAs.

2 Definition

Definition 2.1 (NFA)**.** An NFA N is a tuple

$$N = (Q_N, i_N, \Sigma_N, \delta_N, F_N)$$

where
1. Q_N is a finite set of states,
2 $i_N \in Q_N$ is the /initial/ state,
3. Σ_N is a finite alphabet,
4. $\delta_N : Q_N \times \Sigma_N \to 2^{Q_N}$ is the transition relation. We write $\xrightarrow[N]{}$ to denote the relation δ_N and $q \xrightarrow[N]{a} q'$ just when $q' \in \delta_N(q, a)$.
5. $F_N \subseteq Q_N$ is the set of /final/ states.

Note that this definition is different (more restricted than) Sipser 3rd Ed. You should read the text to understand the difference.

2.1 NFA reachability

We wish to generalize a one-step transition relation to a multi-step reachability relation.
We define the relation $q \overset{w}{\underset{N}{\twoheadrightarrow}} q'$, read "the state q reaches the state q' on (consuming) the nonempty string w" to be the transitive closure of the transition relation $\xrightarrow[N]{}$.

$$\frac{q \xrightarrow[N]{a} q'}{q \overset{a}{\underset{N}{\twoheadrightarrow}} q'}\ \Sigma$$

$$\frac{q \xrightarrow[N]{a} q'' \qquad q'' \overset{w}{\underset{N}{\twoheadrightarrow}} q'}{q \overset{aw}{\underset{N}{\twoheadrightarrow}} q'}\ \mathit{TRANS}$$

The first (Σ) rule says that if q can transit to q' on a letter a, then q can reach q' on the singleton string a.
The second (TRANS) rule says that if q transits to q'' on a and q'' reaches q' on w, then q reaches q' on aw.

2.2 How to read inference rules

The top part (antecedent) of the rule corresponds to the premise ('if' part). The bottom part (consequent) corresponds to the conclusion ('then' part). A rule may have zero or more antecedents. It has one consequent. Usually, we label rules to refer to them later.

Exercise 2.2. Show that the definition of acceptance via reachability and that given in the book coincide.

2.3 Derivations

A derivation is a sequence of judgements. A judgement is either an assumptions (given as part of the problem) or a consequent of one of the inference rules. We write, next to the consequent, the references to the antecedents from which it follows, and the rule used to make infer the judgement. The last judgement is called the *result* of the derivation.

2.4 Example derivation using inference rules

Inference rules may be chained to obtain derivations.
Suppose we have an NFA containing transitions $q_1 \xrightarrow[N]{a} q_2$ and $q_2 \xrightarrow[N]{b} q_3$, we can make the following derivation:

1. $q_1 \xrightarrow[N]{a} q_2$ given.
2. $q_2 \xrightarrow[N]{b} q_3$ given.
3. $q_2 \xrightarrow[N]{b}\!\!\!\!\rightarrow q_3$ From 1 using Σ
4. $q_1 \xrightarrow[N]{ab}\!\!\!\!\rightarrow q_3$ From 1,3 using Σ

2.5 Inversion Lemmas

Inversion lemmas help us guess the rule and the antecedents of the last inference of a derivation by examining the pattern of the consequent.

Lemma 2.3 (Σ inversion)**.** *Let N be an NFA, and let $q, q' \in Q_N$ and $a \in \Sigma_N$. If $q \xrightarrow[N]{a}\!\!\!\!\rightarrow q'$ then it follows that $q \xrightarrow[N]{a} q'$.*

Proof. The only way we can arrive at a judgement $q \xrightarrow[N]{a} q'$ is by employing the Σ rule. By simple pattern matching, we can conclude that the $q \xrightarrow[N]{a} q'$. $\square$

Lemma 2.4 (TRANS inversion)**.**

Let N be an NFA, and let $q, q' \in Q_N$ and let w be a nonempty word in Σ_N^* such that $|w| > 1$. If $q \xrightarrow[N]{w} q'$ is the result of the derivation, then it follows that

1. $w = aw'$ for some $a \in \Sigma_N$ and $w' \in \Sigma^*$ such that $w \neq \epsilon$.
2. There exists $q'' \in Q_N$
3. $q \xrightarrow[N]{a} q''$ is a derivation
4. $q'' \xrightarrow[N]{w'} q'$ is a derivation
5. The judgement $q \xrightarrow[N]{w} q'$ is inferred by the application of the *TRANS* rule with the antecedents coming from the judgements in steps 3 and 4 above.

Proof. The proof is by "pattern matching". Suppose we indeed have a derivation of $q \xrightarrow[N]{w} q'$. Then it must the consequence of some rule. Since $|w| > 1$, it follows that the rule in question cannot be the Σ rule, since that would imply that $|w| = 1$. Therefore, the rule used is the *TRANS* rule. From this, all the assertions 1-5 follow by simple pattern matching the rule with the given consequent. $\square$

[Induction on Derivations] Note that each derivation is a tree, with the leaves as assumptions as root as the result of the derivation.
This allows us to do induction on derivations (like induction on trees).
[Correspondence between rule and string length] Note that in the derivation rules for $\xrightarrow[N]{}$, in the Σ rule, the string used is of length 1, and in the *TRANS* rule, the string in the consequent is of length greater than 1. Thus we can guess the rule being used by simply looking at the length of the string in the consequent.
This observation essentially means that we can either do induction on derivations, or induction on the length of w in the judgement $q \xrightarrow[N]{w} q'$.

2.6 NFA Acceptance and Language recognition

An NFA N *accepts* w iff

1. either $w = \epsilon$ and $i_N \in F$,
2. or $w \neq \epsilon$ and $i_N \overset{w}{\underset{N}{\twoheadrightarrow}} q$ for some $q \in F_N$.

The language *recognized* by N, is

$$L(N) = \{w \in \Sigma^* \mid N \text{ accepts } w\}$$

3 DFA induced by an NFA

Let $N = (Q_N, i_N, \Sigma_N, \underset{N}{\longrightarrow}, F_N)$ be an NFA. The DFA *induced* by N is

$$D = (Q_D, i_D, \Sigma_D, \delta_D, F_D)$$

where

1. $Q_D \overset{\mathsf{def}}{=} 2^Q_N$
2. $i_D \overset{\mathsf{def}}{=} \{i_N\}$
3. $\Sigma_D = \Sigma_N = \Sigma$
4. $\delta_D(R, a) \overset{\mathsf{def}}{=} \bigcup_{r \in R} \delta_N(r, a)$, where $R \in Q_D, r \in Q_N, a \in \Sigma$.
5. $F_D \overset{\mathsf{def}}{=} \{R \in Q_D \mid R \cap F_N \neq \emptyset\}$

3.1 Relating transitions in N with the transitions in D

3.1.1 N transitions to D transitions

Lemma 3.1 (N transitions to D transitions)**.** *If* $r \in R$ *and* $r \overset{a}{\underset{N}{\longrightarrow}} r'$*, then there is an* R' *such that* $R \overset{a}{\underset{D}{\longrightarrow}} R'$ *and* $r' \in R'$*.*

Proof. Just take

$$R' = \delta_D(R, a) = \{r' \in Q_N \mid \exists r \in R \wedge r \overset{a}{\underset{N}{\longrightarrow}} r'\}$$

□

Here is a way of visualising the lemma via a "commutative diagram":

$$\begin{tikzcd} r \arrow[r, "a", "N"'] & r' \\ R \arrow[u, "\ni"] \arrow[r, dashed, "a", "D"'] & \mathbf{R}' \arrow[u, dashed, "\ni"] \end{tikzcd}$$

1. The solid arrows denote the relations between objects assumed in the lemma.
2. The bold face $\mathbf{R}'$ is notation for the object whose existence is asserted by the lemma.
3. The dashed arrows assert the existence of relationships asserted by the lemma.
4. The notation $\ni$, (for set containment) is the inverse of $\in$ (set membership) : $r \in R$ iff $R \ni r$.

3.1.2 D transitions to N transitions

Conversely,

Lemma 3.2 (D transitions to N transitions)**.** *If $r' \in R'$ and $R \xrightarrow[D]{a} R'$, then there is an $r \in R$ such that $r \xrightarrow[N]{a} r'$.*

Again visualizing the statement as a commutative diagram helps:

$$\begin{tikzcd} \mathbf{r} \arrow[r, dashed, "a", "N"'] & r' \\ R \arrow[u, dashed, "\ni"] \arrow[r, "a", "D"'] & R' \arrow[u, "\ni"] \end{tikzcd}$$

Proof. Follows from the definition δ_D. □

3.2 Relating reachability in N with the reachability in D

We want to formalise the intuition that by composing single transitions in N (respectively D), we can reason about relation between reachability in N and reachability in D.
In terms of diagrams, we want to relate the sequences

$$r_0 \xrightarrow[N]{a_1} r_1 \xrightarrow[N]{a_2} \dots \xrightarrow[N]{a_n} r_n$$

and

$$R_0 \xrightarrow[D]{a_1} R_1 \xrightarrow[D]{a_2} \dots \xrightarrow[D]{a_n} R_n$$

3.2.1 N reachability to D reachability

Lemma 3.3 (N reachability to D reachability)**.** *Let $w \neq \epsilon$. If $r \in R$ and $r \overset{w}{\underset{N}{\twoheadrightarrow}} r'$ for some $r' \in Q_N$, then there exists an R' such that*

1. $R \overset{w}{\underset{D}{\twoheadrightarrow}} R'$, and

2. $r' \in R'$.

Again, diagrammatically

$$\begin{tikzcd}
r \arrow[r, two heads, "a", "N"'] & r' \\
R \arrow[u, "\ni"] \arrow[r, dashed, two heads, "a", "D"'] & \mathbf{R}' \arrow[u, dashed, "\ni"']
\end{tikzcd}$$

Proof. Exercise (was done in class). □

3.3 $L(N)$ is included in $L(D)$

Lemma 3.4 ($L(N) \subseteq L(D)$)**.** *Let $w \in L(N)$. Then $w \in L(D)$.*

Proof. □

a) Case $w = \epsilon$:

1. N accepts ϵ	given
2. $i_N \in F$	from 1 and defn of NFA acceptance
3. $i_D = \{i_N\}$	from the defn of D induced by N.
4. $i_D \in F_D$	from 2, 3, defn of F_D
5. D accepts ϵ	from 4, defn of acceptance.

b) Case $w \neq \epsilon$:

1.	$w \neq \epsilon$	Given
2.	N accepts w	Given
3.	$\exists\ q$: $i_N \overset{w}{\underset{N}{\twoheadrightarrow}} q$ and $q \in F_N$	from 2 using defn of acceptance
4.	$i_D = \{i_N\}$	from the construction of D
5.	$i_N \in i_D$	from 4
6.	$i_N \overset{w}{\underset{N}{\twoheadrightarrow}} q$	from 3 using AND-elim$_1$
7.	$\exists R' \in Q_D : i_D \overset{w}{\underset{D}{\twoheadrightarrow}} R'$ and $q \in R'$	from 5, 6 using Lemma 3.3
8.	$q \in R'$	from 7 using AND-elim$_1$
9.	$q \in F_N$	from 3 using AND-elim$_2$
10.	$R' \in F_D$	from 8, 9 using defn of F_D
11.	$i_D \overset{w}{\underset{D}{\twoheadrightarrow}} R'$	from 7 using AND-elim$_1$
12.	D accepts w	from 10, 11, and definition of acceptance.

QED

3.3.1 D-reachability to N-reachability

Lemma 3.5 (D-reachability to N-reachability)**.** *Let $w \neq \epsilon$. If $R' \in Q_D$ and $r' \in R'$ and $R \overset{w}{\underset{N}{\twoheadrightarrow}} R'$ for some $R \in Q_D$, then there exists an $r \in Q_N$ such that*
1. $r \overset{w}{\underset{N}{\twoheadrightarrow}} r'$, and
2. $r \in R$.

Proof:
By induction on derivation of $\underset{N}{\twoheadrightarrow}$:

- Base case:

1.	$R \overset{w}{\underset{D}{\twoheadrightarrow}} R'$ via the Σ rule	Given
2.	$w = a$ for some $a \in \Sigma$	from 1 using Σ inversion on $\underset{D}{\twoheadrightarrow}$

3. $R \xrightarrow[D]{a} R'$ from 1,2 using Σ inversion.
4. $r' \in R'$ Given
5. $\exists r \in R$ such that $r \xrightarrow[N]{a} r'$ from 2, 3 using Lemma 3.2
6. $r \xrightarrow[N]{a}\!\!\!\!\rightarrow r'$ from 4 using Σ inversion on $\xrightarrow[N]{}\!\!\!\!\rightarrow$.

- Inductive case:

1. $R \xrightarrow[D]{w}\!\!\!\!\rightarrow R'$ via the *TRANS* rule Given
2. $w = aw'$ and $w' \neq \epsilon$ from 1 using *TRANS* inversion on $\xrightarrow[D]{}\!\!\!\!\rightarrow$
3. $r' \in R'$ Given
4. $R \xrightarrow[D]{a} R''$ and $R'' \xrightarrow[D]{w'}\!\!\!\!\rightarrow R'$ from 1, 2 using *TRANS* inversion on $\xrightarrow[D]{}\!\!\!\!\rightarrow$.
5. $R'' \xrightarrow[D]{w'}\!\!\!\!\rightarrow R'$ from 4 using AND-elim$_2$ [1]
6. $\exists r'' \in R''$ and $r'' \xrightarrow[N]{w'}\!\!\!\!\rightarrow r'$ from 3, 5 and Induction Hypothesis
7. $R \xrightarrow[D]{a} R''$ from 4 using AND-elim$_1$
8. $r'' \in R''$ from 6 using AND-elim$_1$
9. $\exists r \in R$ and $r \xrightarrow[N]{a} r''$ from 7, 8 using Lemma 3.2
10. $r \xrightarrow[N]{a} r''$ from 9 using AND-elim$_2$
11. $r'' \xrightarrow[N]{w'}\!\!\!\!\rightarrow r'$ from 6 using AND-elim$_2$
12. $r \xrightarrow[N]{aw'}\!\!\!\!\rightarrow r'$ from 10, 11 using TRANS rule of $\xrightarrow[N]{}\!\!\!\!\rightarrow$
13. $r \xrightarrow[N]{w}\!\!\!\!\rightarrow r'$ from 2, 12 substituting w for aw'.

□

3.4 $L(D)$ is included in $L(N)$

Lemma 3.6 ($L(D) \subseteq L(N)$)**.** *Let $w \in L(D)$. Then $w \in L(N)$.*

Proof. Left as an exercise. □

[1] AND-elim$_1$ lets us infer A from $A \wedge B$. AND-elim$_2$ lets us infer B from $A \wedge B$.

4 Equivalence of NFA and DFA

The main theorem relating NFAs and DFAs is that of language equivalence.

Theorem 4.1. *For any NFA N there is a DFA D such that $L(N) = L(D)$.*

Proof. Consider the DFA D induced by N. The result follows from Lemmas3.4 and 3.6. $\square$