AE584 Midterm Exam

Due Date: November 6, 2024

Instructions

i) For this midterm, you are not allowed to discuss the exam with anyone except the GSI and the Instructor, and you cannot use any materials other than what is posted on Canvas. Therefore, all detailed work must be your own. No use of solutions from prior offerings of this course is allowed.

ii) Your work must be neat and professional in appearance. You may type it in Word or LaTeX if you wish. Use a ruler to draw all lines and diagrams. No crossouts of any kind may appear anywhere.

iii) Put a box around your final answer to help the grader.

iv) Your submission should include a report PDF file that includes a copy-paste of all your MATLAB code and your MATLAB `.m` files. All files should be bundled together into a ZIP file and uploaded to Canvas.

v) Label your report file as: `LastnameAE584Midterm.pdf`

vi) Label your ZIP file as: `LastnameAE584Midterm_CodeFiles.zip`

vii) The midterm involves many steps, so be sure to start soon and leave enough time to do careful work.

viii) Read through each problem before you start doing any work to make sure you understand what is being asked. If anything is not clear, please email the GSI, and clarifications will be posted. I do not want anyone to lose points due to any questions not being clear.

ix) Note: Canvas will not accept late uploads. If you miss the deadline, your score will be zero. Therefore, I suggest you upload a partial version a few hours before the deadline to make sure you have something in case you miss the deadline.

x) You will have to use `fminunc` and `ode45` in this midterm. Examples have been posted on the course Canvas page.

xi) There are six problems. The first four problems are worth 20 points; the last two problems are worth 10 points.

Problem 1

Let radian (rad) be the unit of angle. Let $\mathcal{F}_A = \{\hat{\imath}_A, \hat{\jmath}_A, \hat{k}_A\}$ be a frame such that $\hat{\imath}_A$ points toward the East, $\hat{\jmath}_A$ points toward the North, and $\hat{k}_A = -\hat{\imath}_A \times \hat{\jmath}_A$. Let w be a point, L_1, L_2, and L_3 be points representing the locations of 3 lighthouses, and let P be a point that represents your position. For all $i \in \{1, 2, 3\}$, define the true bearing angles from the lighthouses to your position relative to a star in the North direction as

$$\theta_i \triangleq \theta_{\vec{r}_i|\hat{\jmath}_A|\hat{k}_A} \tag{1}$$

where

$$\vec{r}_i \triangleq \vec{r}_{L_i|P}$$

Assume that the bearing measurements have a bias β_i and are corrupted by zero-mean Gaussian noise $n_i \sim \mathcal{N}(0, \sigma^2)$, where $\sigma > 0$. The noisy bearing measurements from the lighthouses to your position are given by

$$\hat{\theta}_i = \theta_i + \beta_i + n_i. \tag{2}$$

Furthermore, for all distinct $i, j \in \{1, 2, 3\}$, let $\hat{P}_{i,j}$ be a point that represents the estimated position of P obtained from the noisy bearing measurements $\hat{\theta}_i$ and $\hat{\theta}_j$. For a single run, the points $\hat{P}_{1,2}$, $\hat{P}_{1,3}$, and $\hat{P}_{2,3}$ form a triangle with centroid $\hat{c}$.

Suppose that the positions are given by

$$\mathbf{r}_{L_1|w}\Big|_A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ m}, \tag{3}$$

$$\mathbf{r}_{L_2|w}\Big|_A = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \text{ m}, \tag{4}$$

$$\mathbf{r}_{L_3|w}\Big|_A = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \text{ m}, \tag{5}$$

$$\mathbf{r}_{P|w}\Big|_A = \begin{bmatrix} 2.5 \\ 2 \\ 0 \end{bmatrix} \text{ m}. \tag{6}$$

Generally, the biases are unknown constants, but here, for simplicity, let $\beta_1 = 0.05$ rad, $\beta_2 = -0.03$ rad, and $\beta_3 = 0.04$ rad.

a) Determine the true bearing angles θ_1, θ_2, and θ_3.

b) For each $\sigma \in \{0.1, 0.2, \ldots, 0.9, 1\}$ rad, estimate the position $\hat{P}_{1,2}$, $\hat{P}_{1,3}$, and $\hat{P}_{2,3}$. Run the estimation algorithm 100,000 times and determine:

- The probability that P is located inside the triangle formed by $\hat{P}_{1,2}$, $\hat{P}_{1,3}$, and $\hat{P}_{2,3}$.
- The average distance from P to the centroid $\hat{c}$ (use the 2-norm).

c) In a figure, plot the percentage of times that P was inside the triangle versus σ.

d) In another figure, plot the average distance of P to $\hat{c}$ versus σ.

e) Discuss any trends observed in these figures and provide an explanation.

Hints:

- To obtain the position from bearings, you can use the formulas from Eq. 4 Problem 1 in Homework 3.
- To determine whether a point is inside a triangle, you can use the algorithm given in `https://www.geeksforgeeks.org/check-whether-a-given-point-lies-inside-a-triangle-or-not/`. To calculate the area of a triangle, you may use Heron's formula.
- In the previous method, instead of the equality comparison, obtain the absolute value of the difference between the left- and right-hand sides of the equality and determine whether this value is lower that $\varepsilon = 10^{-6}$.
- The structure of the code after determining θ_1, θ_2, and θ_3 should be similar to the following snippet:

```
for jj = 1:10 % For each sigma value
    % Do some setup stuff (if needed)
    for ii = 1:100000
        % Calculate 3 estimated positions based on noisy bearing measurments
        % Determine whether actual position is inside the triangle
        % Determine distance from triangle centroid to actual position
        % Save these results somewhere for use outside current loop
    end
    % Use data obtained in inner loop to calculate average distance from
    % triangle centroid to actual position and the percentage of times the
    % actual position was inside the triangle formed by the estimated positions.
end
```

Problem 2

Let w be a point on the Sun with zero inertial acceleration, let F_A be an inertial frame, let Astronomical Unit (AU) be the unit of length, and let radian (rad) be the unit of angle. Let L_1 and L_2 be points representing the locations of Mars and the Sun, respectively, such that

$$\vec{r}_{L_1|w}|_A = \begin{bmatrix} 1.52 \\ 0 \\ 0 \end{bmatrix} \text{ AU}, \quad \vec{r}_{L_2|w}|_A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ AU}. \tag{4}$$

Let the unit vectors $\hat{s}_1, \hat{s}_2$ be such that

$$\hat{s}_1|_A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{s}_2|_A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

For all $k \in \{0, \ldots, 50\}$, $i \in \{1, 2\}$, let P_k denote the location of a spacecraft at step k, let $\psi_{2,\hat{s}_i,k}$ be the bearing measurement of the spacecraft location P_k from point L_2 relative to a star in the direction $\hat{s}_i$, and let $\theta_{L_1/L_2,k}$ be the subtended angle between the points L_1, L_2 from the spacecraft location P_k. Suppose that

$$\vec{r}_{P_0|w}|_A = \begin{bmatrix} 0.52 \\ 0 \\ -1 \end{bmatrix} \text{ AU},$$

and that, for all $k \in \{1, \ldots, 50\}$, the bearings and subtended angles are given in the file `AE584_Midterm_P2.mat`, such that $\psi_{2,\hat{s}_1,k} = \text{bearingL2St1}(k)$, $\psi_{2,\hat{s}_2,k} = \text{bearingL2St2}(k)$, and $\theta_{L_1/L_2,k} = \text{subAngL1L2}(k)$.

For all $k \in \{1, \ldots, 50\}$, determine $\vec{r}_{P_k|w}|_A$ from the given bearings and subtended angles. Then, for all $k \in \{0, \ldots, 50\}$, plot each component of $\vec{r}_{P_k|w}|_A$ versus step k in a 3-by-1 subfigure. Finally, in a 3D plot, use `scatter3` to place points on the locations of the Sun and Mars in yellow and red, respectively, and plot the 3D trajectory of the spacecraft.

HINT: In order to determine the current position, you can use the previous position as the initial guess.

Problem 3

Let $\mathcal{F}_I$ be an inertial frame, and let the Earth frame $\mathcal{F}_E$ be obtained by applying a 3-1-3 sequence of Euler-angle rotations to $\mathcal{F}_I$, where Φ_E, Θ_E, and Ψ_E denote the precession, nutation, and spin angles, respectively. The frames $\mathcal{F}_E$ and $\mathcal{F}_I$ are thus related by

$$\mathcal{F}_I \xrightarrow[3]{\Phi_E} \mathcal{F}_{E'} \xrightarrow[1]{\Theta_E} \mathcal{F}_{E''} \xrightarrow[3]{\Psi_E} \mathcal{F}_E, \tag{7}$$

where the rotations are about the Z, X', and Z'' axes, respectively.

Furthermore, let $\mathcal{F}_B$ be a body-fixed frame obtained by applying a 3-2-1 sequence of Euler-angle rotations to $\mathcal{F}_E$, where Φ, Θ, and Ψ denote the bank, elevation, and azimuth angles, respectively. The frames $\mathcal{F}_B$ and $\mathcal{F}_E$ are thus related by

$$\mathcal{F}_E \xrightarrow[3]{\Psi} \mathcal{F}_{B'} \xrightarrow[2]{\Theta} \mathcal{F}_{B''} \xrightarrow[1]{\Phi} \mathcal{F}_B, \tag{8}$$

where the rotations are about the Z, Y', and X'' axes, respectively.

Assume that, for all $t \geq 0$,

$$\vec{\omega}_{B|I}\Big|_B(t) = \begin{bmatrix} \cos 2t \\ \cos 2t \\ 0.025t \end{bmatrix} \text{ rad/s}, \tag{9}$$

and that $O_{B/I}(0) = I_3$, where I_3 is the 3×3 identity matrix. Furthermore, assume that, for all $t \geq 0$,

$$\dot{\Psi}_E(t) = \sin 0.05t, \tag{10}$$

$$\dot{\Theta}_E(t) = 0.3 \cos 0.01t, \tag{11}$$

$$\dot{\Phi}_E(t) = 0.5 \sin 0.01t, \tag{12}$$

and that $\Psi_E(0) = 0$ rad, $\Theta_E(0) = \pi/6$ rad, and $\Phi_E(0) = 0$ rad.

For all $t \in \{0, 0.01, \ldots, 9.99, 10\}$ s, determine $O_{B|E}(t)$. Plot all components of $O_{B|E}$ versus time in a 3×3 figure grid using the `subplot` function. Also, plot Ψ, Θ, and Φ versus time in a 3×1 figure grid using the `subplot` function.

Problem 4

Let $g = 9.80665$ m/s^2 be the acceleration due to gravity, and let $\alpha = \pi/4$ rad. Suppose that a 3-axis accelerometer and a 3-axis rate gyro are attached to a quadcopter following an inclined, circular trajectory. Let $\mathcal{F}_A$ be an inertial frame, let $\mathcal{F}_B$ be a frame fixed to the quadcopter, and suppose that the axes of both the rate gyro and the accelerometer are aligned with $\mathcal{F}_B$. Let c be the center of mass of the quadcopter, let w be a point with zero inertial acceleration, and let $\vec{r}_{c|w}(t)$ and $O_{B|A}(t)$ be the position vector of the quadcopter center of mass and the orientation matrix of $\mathcal{F}_B$ relative to $\mathcal{F}_A$ at time t, respectively.

Furthermore, suppose that, for all $t \in [0, 20]$ s, the measurements from the sensors are given by

$$\vec{\omega}_{B|A}\Big|_B(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ rad/s,} \tag{13}$$

$$\vec{a}_{c|w|A}\Big|_B(t) + \vec{g}\Big|_B(t) = \vec{a}_{c|w|A}\Big|_B(t) + O_{B|A}(t) \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = \begin{bmatrix} -1 - g\sin\alpha\sin t \\ -g\sin\alpha\cos t \\ -g\cos\alpha \end{bmatrix} \text{ m/s}^2, \tag{14}$$

and that

$$\vec{r}_{c|w}\Big|_A(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ m,} \tag{15}$$

$$\overset{A\cdot}{\vec{r}}_{c|w}\Big|_A(0) = \begin{bmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{bmatrix} \text{ m/s,} \tag{16}$$

$$O_{B|A}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}. \tag{17}$$

For all $t \in \{0, 0.01, \ldots, 19.99, 20\}$ s, determine $\vec{r}_{c|w}(t)$ and $O_{B|A}(t)$. In a figure, plot all components of $O_{B|A}$ versus time in a 3×3 figure grid using the `subplot` function. In another figure, plot the 3 components of $\vec{r}_{c|w}\Big|_A$ versus time in a 3×1 figure grid using the `subplot` function. Finally, in a 3D figure, plot the 3D trajectory of the center of mass of the quadcopter.

Hint: Use a single `ode45` call to integrate the rate-gyro and accelerometer measurements.

Problem 5

You are given a random vector $\mathbf{X}$ representing the state estimation error, with mean $\mathbb{E}[\mathbf{X}] = \mathbf{0}$ and covariance matrix $\mathbf{P}$. A linear transformation is applied:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}, \tag{18}$$

where $\mathbf{A}$ is a known matrix and $\mathbf{b}$ is a known vector.

a) **Expected Value and Covariance of $\mathbf{Y}$**:

Derive expressions for the expected value $\mathbb{E}[\mathbf{Y}]$ and the covariance $\mathrm{Cov}[\mathbf{Y}]$ in terms of $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{P}$.

b) **Numerical Computation**:

Given:

$$\mathbf{A} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad T = 1,$$

compute $\mathbb{E}[\mathbf{Y}]$ and $\mathrm{Cov}[\mathbf{Y}]$.

Problem 6

Consider a one-dimensional discrete-time system where the state evolves according to:

$$x_{k+1} = x_k + w_k, \tag{19}$$

where x_k represents the state at time step k, and w_k is zero-mean Gaussian process noise with variance $Q = \mathrm{Var}(w_k) = 1$.

At each time step, a noisy measurement of the state is obtained as:

$$y_k = x_k + v_k, \tag{20}$$

where y_k is the measurement at time step k, and v_k is zero-mean Gaussian measurement noise with variance $R = \mathrm{Var}(v_k) = 2$.

a) **Kalman Filter Equations**:

Derive the Kalman Filter prediction and update equations for this system, given the state transition and measurement models described above.

b) **Simulation**:

Simulate the system for 101-time steps, from $k = 0$ to $k = 100$, using the following initial conditions: an initial state estimate $\hat{x}_0 = 0$ and an initial error covariance $P_0 = 1$. Generate one sample (also referred to as a realization) of the process noise w_k and measurement noise v_k, and compute the state estimates $\hat{x}_k$ at each time step using the Kalman filter equations.

c) **Plotting**:

Plot the true state x_k, the noisy measurements y_k, and the Kalman filter estimates $\hat{x}_k$ as functions of the time step k.