1.1 Assuming that the set $\{1, \cos(x), \sin(x), \sin(2x), \cos(2x)\}$ is linearly independent, find the Wronskian of the functions $1, \cos(x), \sin(x), \sin(2x), \cos(2x)$ and show that it is identically zero.

Wronskian being determinant of $\begin{bmatrix} 1 & \cos x & \sin x & \sin 2x & \cos 2x \\ 0 & -\sin x & \cos x & 2\cos 2x & -2\sin 2x \\ 0 & -\cos x & -\sin x & -4\sin 2x & -4\cos 2x \\ 0 & \sin x & -\cos x & -8\cos 2x & 8\sin 2x \\ 0 & \cos x & \sin x & 16\sin 2x & 16\cos 2x \end{bmatrix}$ is zero because $\det \begin{bmatrix} \cos x & \sin x & \sin 2x & \cos 2x \\ -\sin x & \cos x & 2\cos 2x & -2\sin 2x \\ -\cos x & -\sin x & -4\sin 2x & -4\cos 2x \\ \sin x & -\cos x & -8\cos 2x & 8\sin 2x \end{bmatrix} = 0$

1.2 Let $\mathcal{F}$ be function space of every R-valued function continuous on the interval $[a, b]$ and let $\mathcal{F}_0$ be the set of functions in $\mathcal{F}$ such that $f(a) = f(b) = 0$. Show that $\mathcal{F}_0$ is a linear subspace of $\mathcal{F}$ and that $\mathcal{F}$ is not finite dimensional.

Let $f, g \in \mathcal{F}_0$ and $\alpha \in \mathbb{R}$. Then $f(a) = f(b) = 0$ and $g(a) = g(b) = 0$. Hence $(f+g)(a) = f(a) + g(a) = 0 + 0 = 0$ and $(f+g)(b) = f(b) + g(b) = 0 + 0 = 0$. Also $(\alpha f)(a) = \alpha f(a) = \alpha \cdot 0 = 0$ and $(\alpha f)(b) = \alpha f(b) = \alpha \cdot 0 = 0$. Thus $f+g, \alpha f \in \mathcal{F}_0$. Hence $\mathcal{F}_0$ is closed under addition and scalar multiplication. Thus $\mathcal{F}_0$ is a linear subspace of $\mathcal{F}$.

Let $f_n(x) = x^n(x-a)(x-b)$ for $n \in \mathbb{N}$. Then $f_n \in \mathcal{F}_0$ for all $n \in \mathbb{N}$. Suppose $\{f_1, f_2, \ldots, f_k\}$ is a finite set. Let $M = \max\{|f_1(x)|, |f_2(x)|, \ldots, |f_k(x)| : x \in [a, b]\}$. Then $|f_{k+1}(x)| = |x^{k+1}(x-a)(x-b)| \leq M$ for all $x \in [a, b]$. Hence $\{f_1, f_2, \ldots, f_k\}$ cannot be a basis for $\mathcal{F}$. Thus $\mathcal{F}$ is not finite dimensional.

1.3 Show that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ satisfies the recurrence relation $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Using the recurrence relation $F_n = F_{n-1} + F_{n-2}$ and the $\sum_{n=0}^{\infty} x^n$ $\frac{1}{1-x}$ $\frac{x}{1-x-x^2}$

$$\frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$x = A(1-\beta x) + B(1-\alpha x)$$

$$x = A - A\beta x + B - B\alpha x$$

$$A + B = 0, \quad -A\beta - B\alpha = 1$$

$$A = -B, \quad -A\beta + A\alpha = 1$$

$$A(\alpha - \beta) = 1$$

$$A = \frac{1}{\alpha - \beta}, \quad B = -\frac{1}{\alpha - \beta}$$

$$\frac{x}{1-x-x^2} = \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)$$

$$= \frac{1}{\alpha-\beta}\left(\sum_{n=0}^{\infty} \alpha^n x^n - \sum_{n=0}^{\infty} \beta^n x^n\right)$$

$$= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n$$

$$= \sum_{n=0}^{\infty} F_n x^n$$

$$F_n = \frac{1}{\alpha-\beta}(\alpha^n - \beta^n)$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

1.4 Show that the Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1}(x))$ satisfy the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

$$T_n(x) = \cos(n\theta), \quad x = \cos\theta$$

$$T_{n-1}(x) = \cos((n-1)\theta)$$

$$T_{n-2}(x) = \cos((n-2)\theta)$$

$$2xT_{n-1}(x) - T_{n-2}(x) = 2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta)$$

$$= \cos(\theta + (n-1)\theta) + \cos(\theta - (n-1)\theta) - \cos((n-2)\theta)$$

$$= \cos(n\theta) + \cos((n-2)\theta) - \cos((n-2)\theta)$$

$$= \cos(n\theta)$$

$$= T_n(x)$$

1.5 Show that $\{\sin(x), \sin(2x), \sin(3x), \ldots, \sin(nx)\}$ is a linearly independent set of functions in $C[0, \pi]$.

Let $\alpha_1 \sin(x) + \alpha_2 \sin(2x) + \cdots + \alpha_n \sin(nx) = 0$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. Then $\int_0^{\pi} \sin(kx)\sin(jx)\,dx = 0$ for $k \neq j$ and $\int_0^{\pi} \sin^2(kx)\,dx = \frac{\pi}{2}$ for $k = j$. Hence $\int_0^{\pi} \sum_{k=1}^{n} \alpha_k \sin(kx)\sin(jx)\,dx = \frac{\pi}{2}\alpha_j = 0$ for $j = 1, 2, \ldots, n$. Thus $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

1.6 Consider the $n \times n$ Hilbert matrix H_n with entries $h_{ij} = \frac{1}{i+j-1}$. Compute the determinant of H_n for $n = 1, 2, \ldots, 10$ and plot the determinant versus n on a semi-log plot. Comment on the behavior of the determinant.

1.7 Consider the $n \times n$ Vandermonde matrix V_n with entries $v_{ij} = x_i^{j-1}$ for $i, j = 1, 2, \ldots, n$. Compute the determinant of V_n for $n = 1, 2, \ldots, 10$ and plot the determinant versus n on a semi-log plot. Comment on the behavior of the determinant.

1.8 Consider the $n \times n$ Toeplitz matrix T_n with entries $t_{ij} = \frac{1}{1+|i-j|}$. Compute the determinant of T_n for $n = 1, 2, \ldots, 10$ and plot the determinant versus n on a semi-log plot. Comment on the behavior of the determinant.

1.9 Consider the $n \times n$ Cauchy matrix C_n with entries $c_{ij} = \frac{1}{x_i - y_j}$ where $x_i = i$ and $y_j = j + \frac{1}{2}$. Compute the determinant of C_n for $n = 1, 2, \ldots, 10$ and plot the determinant versus n on a semi-log plot. Comment on the behavior of the determinant.

1.10 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.11 Consider the $n \times n$ Toeplitz matrix T_n with entries $t_{ij} = \frac{1}{1+|i-j|}$. Compute the eigenvalues of T_n for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.12 Consider the $n \times n$ Cauchy matrix C_n with entries $c_{ij} = \frac{1}{x_i - y_j}$ where $x_i = i$ and $y_j = j + \frac{1}{2}$. Compute the eigenvalues of C_n for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.13 Consider the $n \times n$ Hilbert matrix H_n with entries $h_{ij} = \frac{1}{i+j-1}$. Compute the eigenvalues of H_n for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.14 Consider the $n \times n$ Vandermonde matrix V_n with entries $v_{ij} = x_i^{j-1}$. Compute the eigenvalues of V_n for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.15 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.16 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.17 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.18 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.19 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.

1.20 Let A be an $n \times n$ matrix with entries $a_{ij} = \cos(i + j)$. Compute the eigenvalues of A for $n = 1, 2, \ldots, 10$ and plot the eigenvalues versus n on a semi-log plot. Comment on the behavior of the eigenvalues.