NOTE: For this homework, Problem 1 is optional.

Problem 1 (0.5 pt BONUS):

(a) Let $L \in \mathbb{R}^{n\times n}$ be the Laplacian of a connected undirected graph with no self-loops. Show that the induced ∞−norm of e^{-L} is unique, that is, vector $v = \mathbf{1}_n$ is the unique solution of

$$\max_{\|v\|_\infty = 1} \left\| e^{-L} v \right\|_\infty .$$

(b) Consider the dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Show that for a zero-order hold on the input, that is, $u(t) \equiv u_k$ for $t \in [k\Delta t, (k+1)\Delta t)$, $k = 1, 2, \ldots,$ the discrete-time system is

$$x_{k+1} = A_d x_k + B_d u_k$$

where $x_k = x(k\Delta t)$,

$$A_d = e^{A\Delta t}, \quad B_d = \int_{\tau=0}^{\Delta t} e^{A\tau} B d\tau.$$

Further, show that, in fact,

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\Delta t}.$$

Problem 2 (6pts):

Consider the system shown below where a DC motor is used to drive a slender spotlight (which may be thought of as a pendulum).

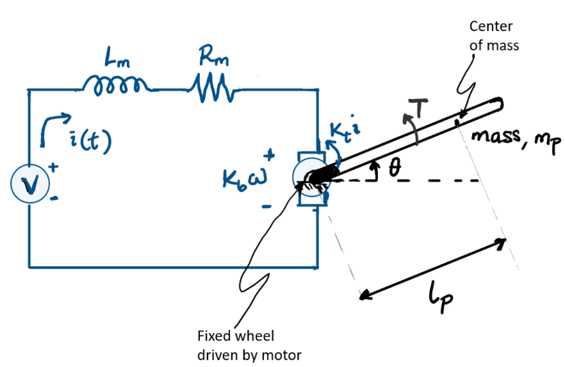

Figure 1: Motor driving slender spotlight (pendulum) of length l_p and mass m_p.

Our goal is to design a feedback controller that is able to drive the spotlight back-and-forth from the horizontal to the vertical, that is, from $\theta = 0\ rad$ to $\theta = \pi/2\ rad$. Our measurable output is angle θ and the input to the system is voltage V. The parameters of the system are given in attached file `hwk3_p2.m`. Assume that:

- the torque $T(t)$ is counter-clockwise positive when a positive current $i(t)$ flows through the motor;
- angles are measured counter-clockwise positive from the positive $x-$axis, as shown in the figure;
- a viscous friction torque $T_f(t) = -B_m\dot{\theta}(t)$ opposes motion; and
- the motor electric circuit has negligible inductance.

(a) Find a state space set of equations for this system. (You may refresh modeling of DC motors circuits and find other related excellent resources at the CTMS website).

(b) Find the equilibrium points and comment on the stability of the points. Perturb the (original nonlinear) system about the equilibrium points and simulate what happens to the angular displacement and velocity over $10s$. How does the system behavior change if the electric circuit is open? Compare plots of the angular displacement vs time and velocity vs time for both closed and open circuits.

(c) What is the impulse response function of the system at the equilibrium points?

(d) Now, design a feedback controller that will stabilize the system about $\theta = pi/4\ rad$ with an overshoot of less than 15% and a settling time less than 0.2 s.

(e) Verify that your controller will enable the spotlight track the following angular displacement reference

$$r(t) = \begin{cases} \pi/2, & 0 \le t < 5s, \\ 0, & 5 \le t < 10s, \\ \pi/2, & 10 \le t < 15s, \\ 0, & 15 \le t < 20s, \\ \pi/2, & 20 \le k < 25s; \end{cases}$$

with a maximum input voltage of $9V$ available. Using a `subplot` or a *double y-axes* figure, contrast a plot of the controlled input voltage against time with a plot of the angular displacement against time. Comment on your results.

Problem 3 (4pts):

Consider the model of an active suspension control system for a quarter car on the following MATLAB webpage:

https://www.mathworks.com/help/robust/gs/active-suspension-control-design.html

Using a zero-order hold with some sampling time T_s =0.01s, the suspension system dynamics in discrete time can be expressed as

$$x_{k+1} = A_d x_k + B_d u_k + F_d d_k,$$

where $x \in \mathbb{R}^4$ is the state of the system as described in the webpage above, the control input $u \in \mathbb{R}$ is the actuator force f_s in the website, while the external input $d \in \mathbb{R}$ is the road disturbance (profile) r in the website. A_d, B_d, F_d are matrices of appropriate dimensions.

Suppose the quarter car is moving at a constant horizontal velocity of 10m/s and has a preview of the road profile up to a half second away (resolved in 0.01s intervals). Our goal is to design a model predictive control (MPC) scheme for the suspension system which when given the current state of the suspension, and the preview of the road, will help reject any disturbance in its preview (e.g., a speed bump). Specifically, at any time step j, we wish to minimize the objective

$$J = \frac{1}{2} \sum_{k=j}^{j+N-1} x_{k+1}^T Q x_{k+1} + \rho u_k^2,$$

where $N = 50$ (corresponding to a half second). The parameters Q, and ρ are provided in the script `hwk3_p3.m` attached to this assignment.

The function `activeSuspSim.p` simulates the system for 0.01s. The function takes the current state x_k of the suspension system, the quarter car's horizontal position s_k (only in multiples of 0.1m) and the control input u_k; and outputs state x_{k+1} and position s_{k+1} after 0.01s as well as the half-second-long road profile $[d_k, ..., d_{k+N-1}]$.

(a) Write a function `MPCCtrl.m` that takes in the current state x_k of the suspension system and the road profile preview $[d_k, ..., d_{k+N-1}]$, to give the predictive control input u_k for the system at k. Assuming that the system is initially at $x_0 = \mathbf{0}$ when $s_0 = 0$, simulate the system with the predictive controller for 3s. Use the `subplot` command to plot (i) the MPC state (displacement/travel) trajectories (both wheel and body displacements on one subplot, distinguished by a legend), (ii) the MPC state (velocity) trajectories (both wheel and body velocities on one subplot, distinguished by a legend), and (iii) the corresponding MPC input. Do not use inbuilt toolbox functions for this part.
(As an example, the script `hwk3_p3.m` simulates the system for 3s with the control input derived from the erroneous function `MPCCtrl.m`. Your goal is to edit `MPCCtrl.m` to have the correct function that minimizes the objective J. The plot templates have also been created for you in the script. Run the script to observe what it does).

NOTE: Your report should explain your approach for solving this problem.

(b) Repeat the above simulation with an LQR controller, that has no access to the disturbance information. You may use MATLAB's `dlqr` function or any other function for this part. Use the `subplot` command to plot a comparison of the MPC, LQR control and no control ($u \equiv 0$) scenarios for (i) body displacement (i) body velocity, (iii) body acceleration, and (iv) the corresponding input. Properly label your figures.
(Again, as an example, the script `hwk3_p3.m` simulates the system with the LQR control input derived from the erroneous function `LQRCtrl.m`. Edit `LQRCtrl.m` appropriately. The plot templates have also been created for you in the script).

(c) Comment on the plots in part (b). How does the active suspension system compare with a passive suspension? How does the MPC controller compare with the LQR control?