MA 321 (Optimization)

End-semester Examination

Time: 2 pm - 5 pm | 26th November, 2022 | Maxm marks: 40

Notations:

For $\mathbf{x}^* \in Fea(P)$, $D_{\mathbf{x}^*}$ is the set of all feasible directions at $\mathbf{x}^*$.

$I_{\mathbf{x}^*} = \{i \in \{1, \ldots, m\} : g_i(\mathbf{x}^*) = 0\}$, $G_0(\mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \nabla g_i(\mathbf{x}^*)\mathbf{d} < 0$ for all $i \in I_{\mathbf{x}^*}\}$.

$F_0(\mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \nabla f(\mathbf{x}^*)\mathbf{d} < 0\}$, where f is the objective function.

Convention: $G_0(\mathbf{x}^*) = \mathbb{R}^n$ for interior points of $Fea(P)$,

$\nabla f(\mathbf{x}^*), \nabla g_i(\mathbf{x}^*)$'s are row vectors, $\mathbf{d}$ is a column vector.

No credit will be given for answers given without any justification

1. Consider the following problem (P):
   Minimize $-x_2^2 + x_1 - \frac{4}{3}x_2^3 + x_1x_2^2$
   subject to
   $x_2^2 - x_1 \leq 0.$
   $-x_1 \leq 0.$
   $-x_2 \leq 0.$

   $-(x_2^2 - x_1) + x_2^2(-\frac{4x_2}{3} + x_1)$ $\quad \geq 0$ $\quad x_1 < \frac{4x_2}{3}$, $\quad x_1 \geq x_2^2$ $\quad [0, -1]^T d < 0$ $\quad d_2 > 0$

   (a) Give $D_{\mathbf{x}^*}$, where $\mathbf{x}^* = [1,1]^T$. $d_1 > 2d_2$

   (b) If possible find an $\mathbf{x}^* \in Fea(P)$ (not an interior point) at which $G_0(\mathbf{x}^*) = D_{\mathbf{x}^*}$ and an $\mathbf{x}^* \in Fea(P)$ at which $G_0(\mathbf{x}^*) \neq D_{\mathbf{x}^*}$. $[1,1]^T$, $[1,0]^T$

   (c) Check whether $\mathbf{x}^* = [1,0]^T$ satisfies the first order necessary conditions for a local minimum. If not, then give a feasible direction $\mathbf{d}$ at $\mathbf{x}^*$ (if possible), such that $f(\mathbf{x}^* + t\mathbf{d}) < f(\mathbf{x}^*)$ for $t > 0$, sufficiently small. $[-1, 0]^T$

   (d) If possible give an $\mathbf{x}^* \in Fea(P)$ such that $\nabla f(\mathbf{x}^*)\mathbf{d} > 0$ for all feasible directions $\mathbf{d}$ at $\mathbf{x}^*$, but $\mathbf{x}^*$ is not a local minimum. $[6, 0]^T$

   (e) Does there exist an $\mathbf{x}^* \in Fea(P)$ which satisfies the second order necessary conditions for a local minimum but is not a local minimum? $[0,0]^T$

   (f) If possible find an $\mathbf{x}^* \in Fea(P)$ such that, $F_0(\mathbf{x}^*) \cap G_0(\mathbf{x}^*) = \phi$, but $F_0(\mathbf{x}^*) \cap D_{\mathbf{x}^*} \neq \phi$.

   (g) If possible find an $\mathbf{x}^* \in Fea(P)$ such that, $F_0(\mathbf{x}^*) \cap G_0(\mathbf{x}^*) = \phi$, but $\mathbf{x}^*$ is not a local minimum. $(0,0)^T$

   (h) Is $\mathbf{x}^* = [1,1]^T$ a KKT point? Yes

   At $(1,1)$, $G_0 = D_{x^*} = \{d \in \mathbb{R}^2 : d_1 \geq 2d_2\}$

   $f([1,1]^T) = -1/3$

   $[1,0]$, $G_0 : d_2 > 0$, $D : d_2 \geq 0$, $\nabla f = [1, 0]$

   [18]

2. Consider the following problem (P):
   Minimize $ax_2^2 + bx_1^2x_2 + cx_1^3$ $\quad (a, b, c$ real)
   subject to
   $g_i(\mathbf{x}) \leq 0$ for $i = 1, 2, \ldots, 5$.
   where $g_i(\mathbf{x})$ are such that $Fea(P)$ is a polyhedral set with five corner points given by, $(1,1), (3,1), (3,2), (1,2), (2,4)$.

$d_1 \not< 0$ $\quad$ $2d_1 - 4d_2 < 0$

$d_1 < 2d_2$

(a) If possible choose a, b, c (not all zeros) such that every KKT point is a local minimizer. If this is possible, then for that choice of a, b, c give all the local minimizers. $1, 0, 1$ $[1, 1]^T$

(b) If $\mathbf{x}^* \in Fea(P)$ is such that $F_0(\mathbf{x}^*) \cap G_0(\mathbf{x}^*) = \phi$ and $G_0(\mathbf{x}^*) \neq \phi$, then is it true that $\mathbf{x}^*$ is a KKT point?

(c) If possible choose a, b, c such that Fea(P) has no FJ point. Not possible

(d) If possible choose a, b, c such that Fea(P) has a local minimizer which is not a KKT point.

(e) If possible choose a, b, c and an $\mathbf{x}^* \in Fea(P)$ which is an FJ point but does not satisfy the second order necessary conditions for a local minimum.

[14]

3. Consider the following problem (P):
Minimize $x_1^2 + 2x_2^2 + x_1 x_2$
subject to
$x_2 + x_1 - 2 = 0.$

(a) Check whether there exists an $\mathbf{x}^* \in Fea(P)$ which is an FJ point (here FJ conditions are with respect to equality constraints). $\left[\frac{3}{2}, \frac{1}{2}\right]^T$

(b) Find all optimal solutions of (P)(if there exists one). $7/2$ [5]

4. Consider the following transportation problem (P) with c_{ij}'s, a_i's (40,20,30,20) and d_j's (60,10,40) as given below:

4	2	1	40
5	4	3	20
4	2	9	30
1	5	9	20
60	10	40	

(a) Check whether the initial BFS $\mathbf{x}_0$ with basic cells No
$B = \{(1,1), (2,1), (2,2), (3,2), (3,3), (4,3)\}$, is optimal for (P) (by taking $u_2 = 0$, where u_2 is the dual variable corresponding to the second supply constraint). If it is not optimal, then by choosing the entering and the leaving variable, get a better (with respect to the value of objective function) BFS of (P) in the next iteration (You do NOT have to find an optimal solution). $\theta = 10$

(b) Is it true that given any collection of three cells in the above array there will be a BFS of the above problem containing those three cells as basic cells? False [3 + 3]

$x_2 - 2 = -2x_1 + 6$
$-2x_1 - x_2 + 8$

*** END***

$\frac{x_2 - 2}{x_1 - 3} = \frac{4-2}{2-1} = -2$

$(2-x_2)^2 + 2x_2^2 + 2x_2 - x_2^2$

$x_2/2 = 2x_1 - \cancel{2}$

$4 + \cancel{x_2^2} - 4x_2 + 2x_2^2 - \cancel{x_2^2} + 2x_2$

$x_2 = 2x_1$ $x_2 = 1$

$= 2x_2^2 - 2x_2 + 4$

$1 \ngtr 6$

$2x_1 - x_2 \leq 0$

$x_2 = 1/2$