CS 561 Artificial Intelligence
Lecture # 4-5

Reasoning with uncertainty

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Outline

- Belief Networks
  - Structure and inference

Bayesian networks

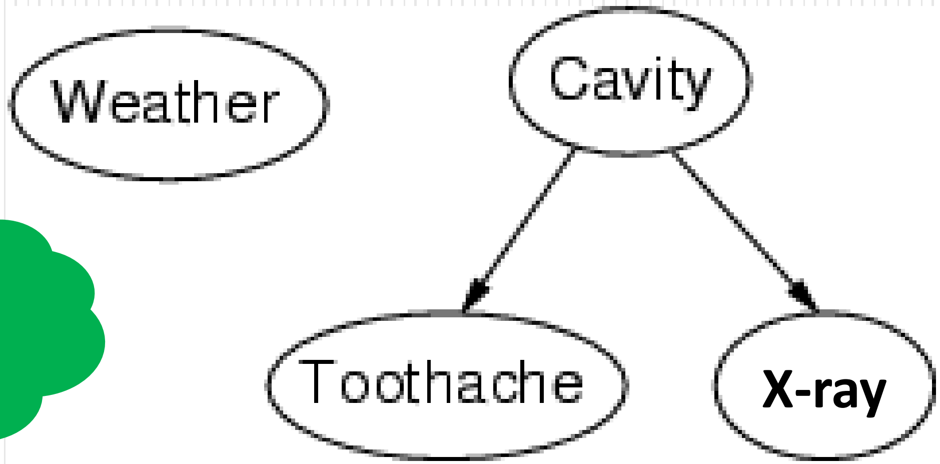
- Representing knowledge in uncertain domain
- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions.
- Syntax:
  - a set of nodes, one node per random variable
  - a directed, acyclic graph (link ≈ "directly influences")
  - a conditional distribution for each node given its parents:

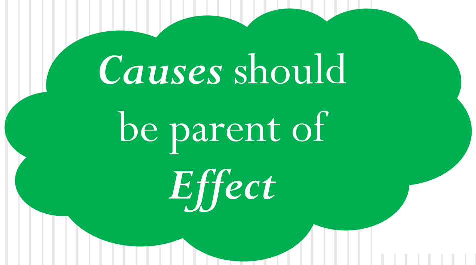
$$\mathbf{P}(X_i \mid Parents(X_i))$$

- In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:





- *Weather* is independent of the other variables
- *Toothache* and *X-raySpot* are conditionally independent given *Cavity*

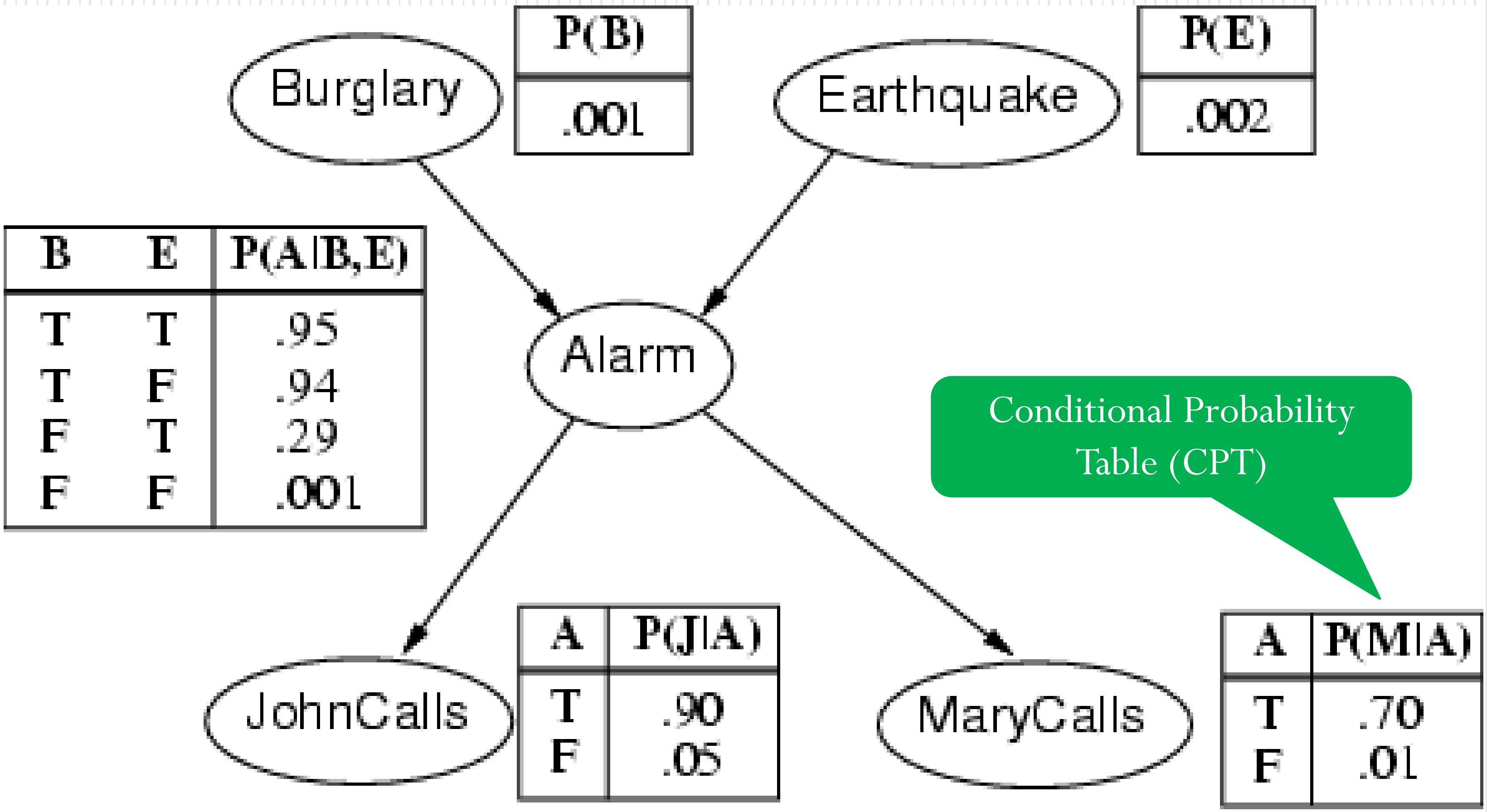
Example

- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: ***Burglary***, ***Earthquake***, ***Alarm***, ***JohnCalls***, ***MaryCalls***

- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

Example contd.



P(B)
.001

P(E)
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B	E	P(A|B,E)
T	T	.95
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F	T	.29
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A	P(J|A)
T	.90
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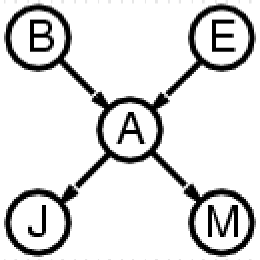
Compactness

- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values

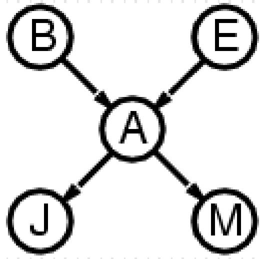


- Each row requires one number p for $X_i = true$ (the number for $X_i = false$ is just $1\text{-}p$)
- If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers
- i.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution
- For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5\text{-}1 = 31$)

Semantics

The full joint distribution is defined as the product of the local conditional distributions that are associated with the nodes of the network:

$$\mathbf{P}(X_1, \ldots, X_n) = \pi_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i))$$



e.g., $\mathbf{P}(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$= \mathbf{P}(j \mid a)\, \mathbf{P}(m \mid a)\, \mathbf{P}(a \mid \neg b, \neg e)\, \mathbf{P}(\neg b)\, \mathbf{P}(\neg e)$

$= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998$

≈ 0.00063

Constructing Bayesian networks

- The joint distribution $P(X_1 = x_1, \ldots, X_n = x_n)$ can be given in terms of conditional probability using product rule:

$$P(x_1, \ldots, x_n) = P(x_n \mid x_{n-1}, ..x_1)\, P(x_{n-1}, \ldots, x_1)$$

repeating the process, reducing each conjunctive probability to a conditional probability and a smaller conjunction

$$P(x_1, \ldots, x_n) = P(x_n \mid x_{n-1}, ..x_1)\, P(x_{n-1} \mid x_{n-2} \ldots, x_1) \ldots P(x_2 \mid x_1)\, P(x_1)$$

$$P(x_1, \ldots, x_n) = \pi^{n}_{i=1} P(x_i \mid x_{i-1} \ldots, x_1)$$

Chain Rule

This specification of joint distribution is equivalent to

$$\mathbf{P}(X_1, \ldots, X_n) = \pi_{i=1} \mathbf{P}(X_i \mid Parents(X_i))$$

provided $Parents(X_i) \subseteq \{X_{i-1}, \ldots, X_1\}$

Take care of the node ordering while constructing the network.

Constructing Bayesian networks

- Determine the set of variables, choose an ordering of variables $X_1, \ldots, X_n$ (if causes precede effects, this will result in compact network)
- For $i = 1$ to n
  - add X_i to the network
  - select parents from $X_1, \ldots, X_{i-1}$ such that

$$\boldsymbol{P}(X_i \mid Parents(X_i)) = \boldsymbol{P}(X_i \mid X_1, \ldots X_{i-1})$$

this choice of parents guarantees:

$$\boldsymbol{P}(X_1, \ldots, X_n) = \pi_{i=1}^{n} \boldsymbol{P}(X_i \mid X_1, \ldots, X_{i-1}) \text{ (chain rule)}$$

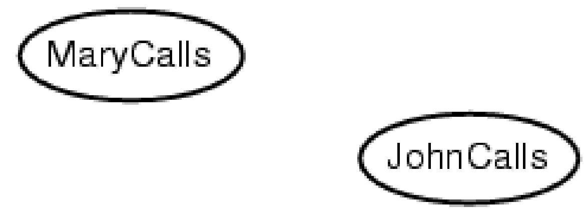
$$= \pi_{i=1}^{n} \boldsymbol{P}(X_i \mid Parents(X_i)) \text{ (by construction)}$$

  - for each parent insert a link from parent to X_i
  - CPTs: write down the conditional probability table, ${}_1\boldsymbol{P}(X_i \mid Parents(X_i))$

Example

- Suppose we choose the ordering M, J, A, B, E
-

Wrong ordering??



$\mathbf{P}(J \mid M) = \mathbf{P}(J)$?

Example

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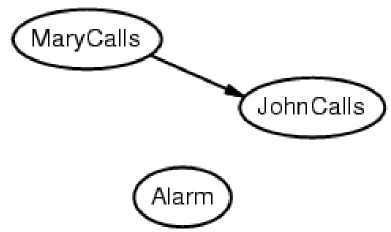

$\mathbf{P}(J \mid M) = \mathbf{P}(J)$?

No

$\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$?

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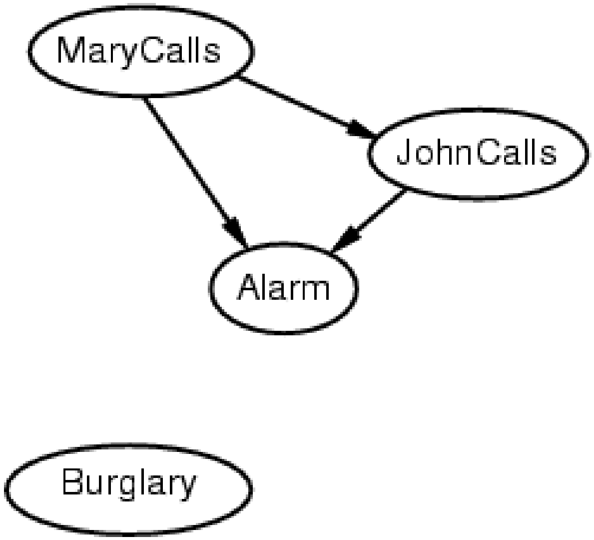

$\mathbf{P}(J \mid M) = \mathbf{P}(J)$? **No**

$\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$? **No**

$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B \mid A)$?

$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B)$?

Example

- Suppose we choose the ordering M, J, A, B, E
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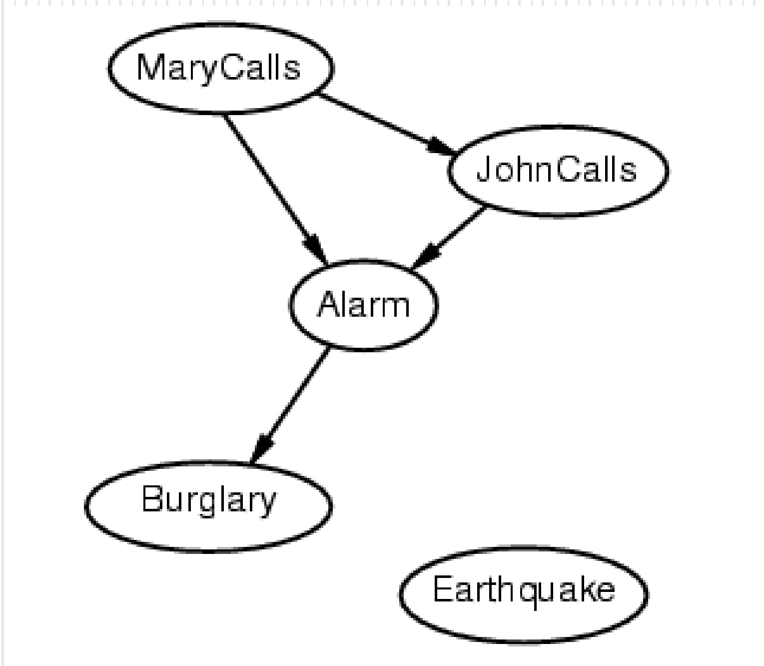

$\mathbf{P}(J \mid M) = \mathbf{P}(J)$?

No

$\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$? **No**

$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B \mid A)$? **Yes**

$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B)$? **No**

$\mathbf{P}(E \mid B, A, J, M) = \mathbf{P}(E \mid A)$?

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Example

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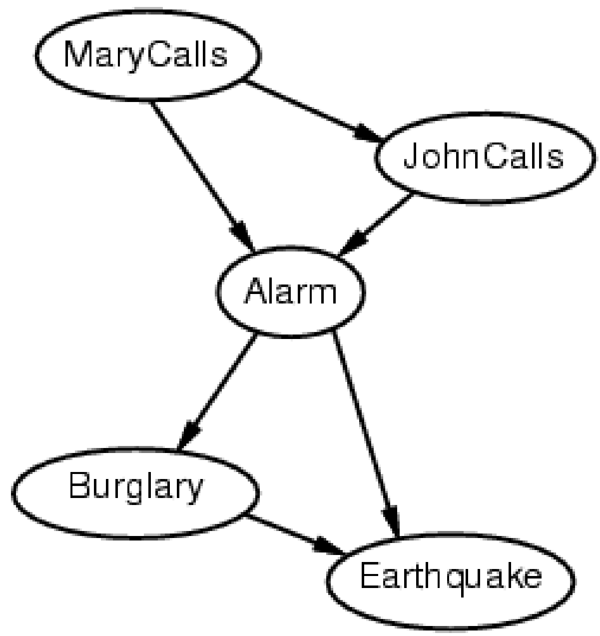

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$\boldsymbol{P}(B \mid A, J, M) = \boldsymbol{P}(B \mid A)$? **Yes**

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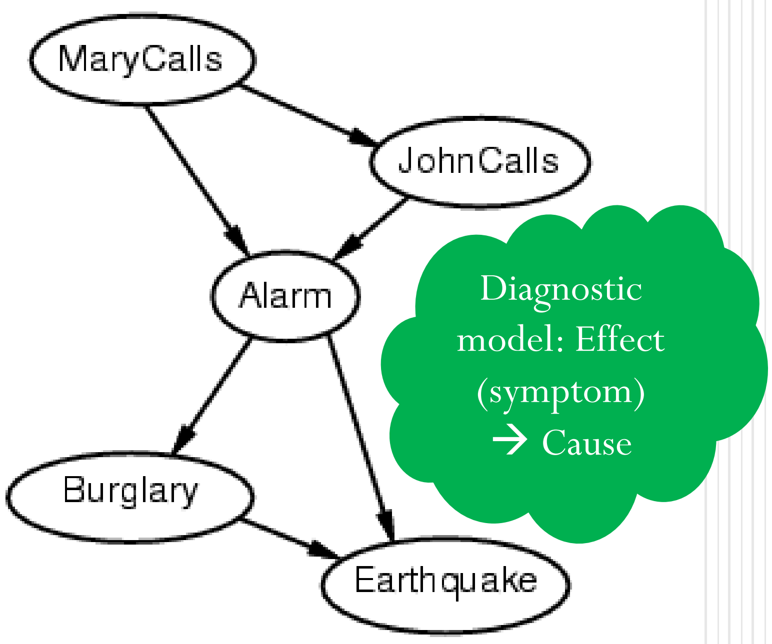
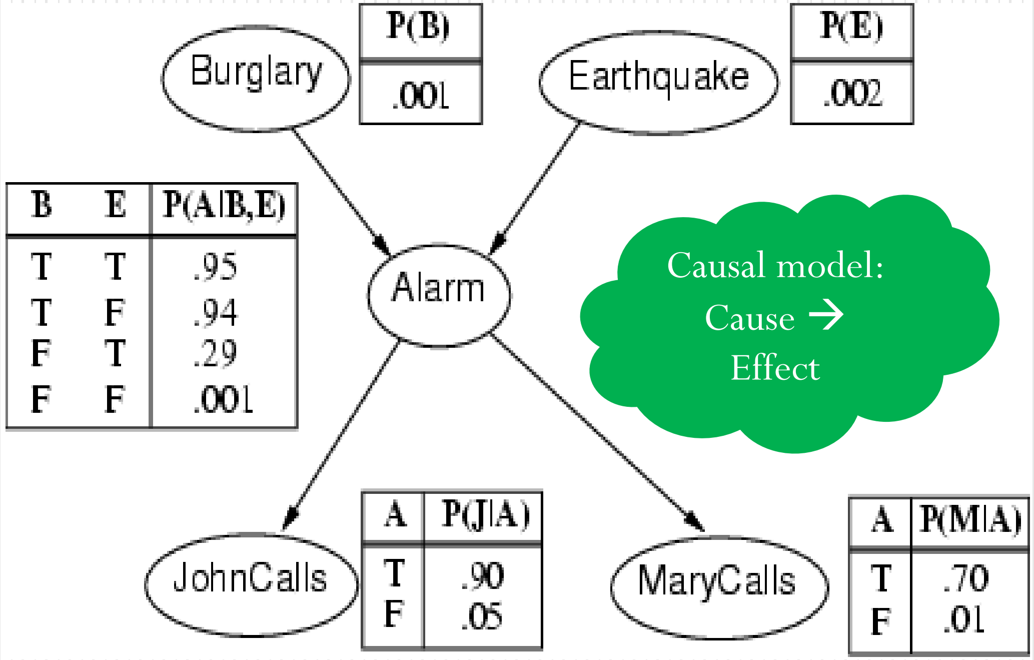
$\boldsymbol{P}(E \mid B, A, J, M) = \boldsymbol{P}(E \mid A)$? **No**

$\boldsymbol{P}(E \mid B, A, J, M) = \boldsymbol{P}(E \mid A, B)$? **Yes**

Example contd.



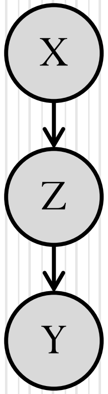


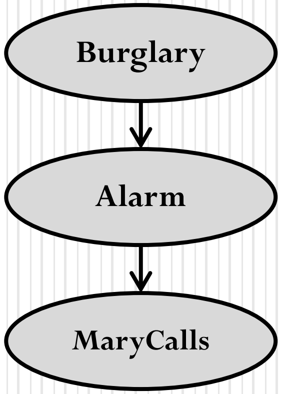
- Resulting network has two more links, requires three more probabilities to be specified: Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed
- Deciding conditional independence is hard in noncausal directions

Conditional Independence

- Can we find all the independences of a BN by inspecting its structure (from the graph)?
- Let us first see a three-node network where variables X and Y are connected via third variable Z in four different ways and we will try to understand when an observation regarding a variable X can possibly change our beliefs about Y, in the presence of evidence variable Z.

  - Forward serial connection (Causal trail - active iff Z is not observed)

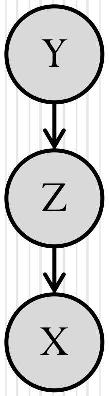




- When Z is not instantiated (its truth value is not known variable is not observed) X can influence Y via Z (having observed X will tell something about Y).
- When Z is instantiated then X cannot influence Y (if we observe Z then knowing about X will not tell anything new about Y).

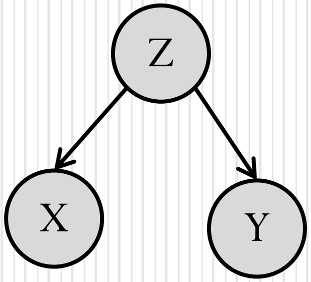
Conditional Independence

- Backward serial connection (evidential trail- active iff Z is not observed)

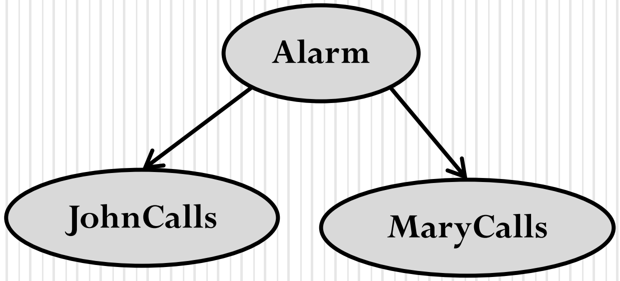


- When Z is not instantiated Y can influence X via Z (knowing about Y will tell something about X).
- When Z is instantiated then Y cannot influence X (if we observe Z then knowing about Y will not tell anything new about X).

- Diverging connection (Common cause- active iff Z is not observed)

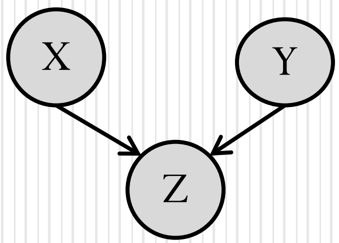


- Similar to previous two cases: X can influence Y via Z if and only if Z is not observed.
- In other words, if we know Z (or observe Z), then knowing about X will not give us any additional information about Y.

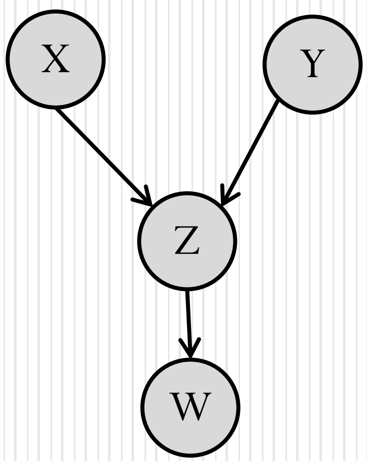


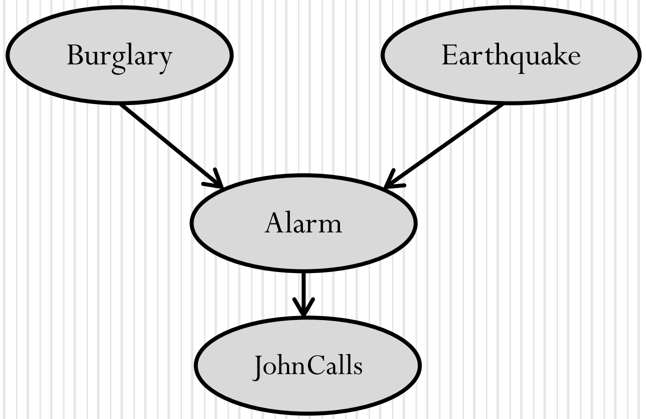
Conditional Independence

- Converging connection (Common effect- active iff either Z or one of Z's descendants is observed)



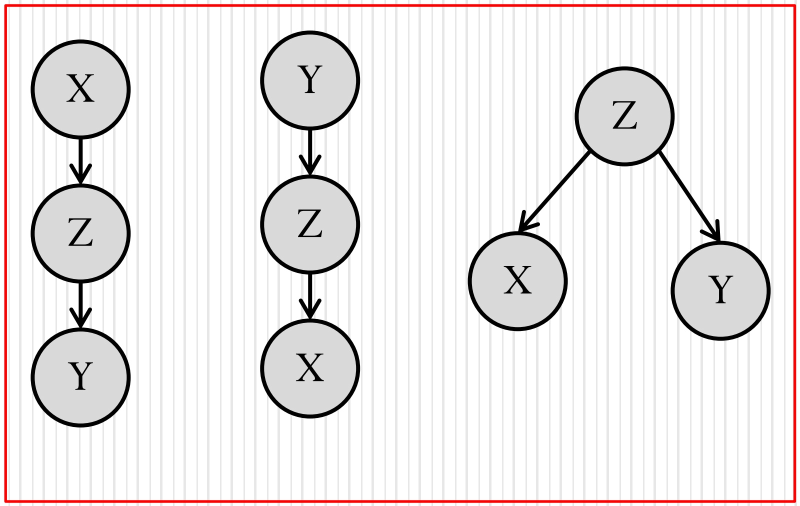
- X can influence Y only if Z or descendant of Z is instantiated.
- Without observing Z, knowing X does not tell anything about Y.
- When either node Z is instantiated, or one of its descendants is, then we know something about whether Z, and in that case information does propagate through from X to Y.

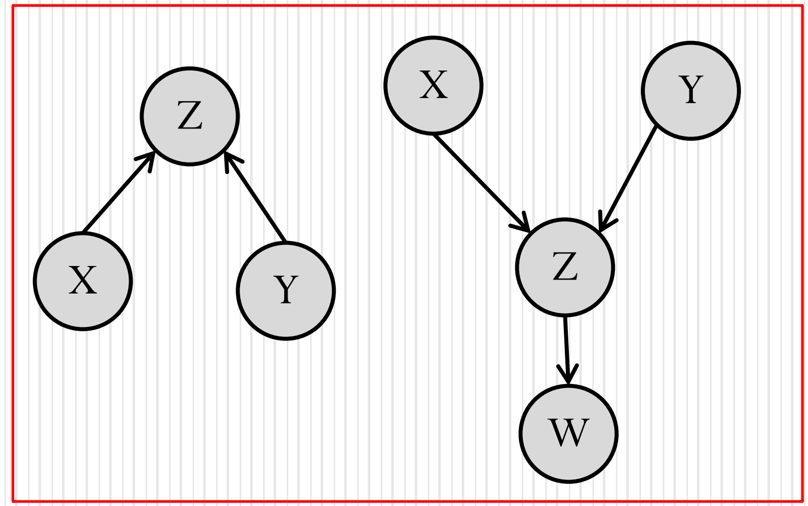




Conditional Independence

- Serial connections and diverging connections are essentially the same.





- **General case:** Considering longer trail $X_1 \rightleftharpoons \dots \rightleftharpoons X_n$, for influence to “flow” from X_1 to X_n, it needs to flow through every single node on the trail.
- When multiple trails are there between two nodes then one node can influence another if there is any trail along which influence can flow.

Conditional Independence

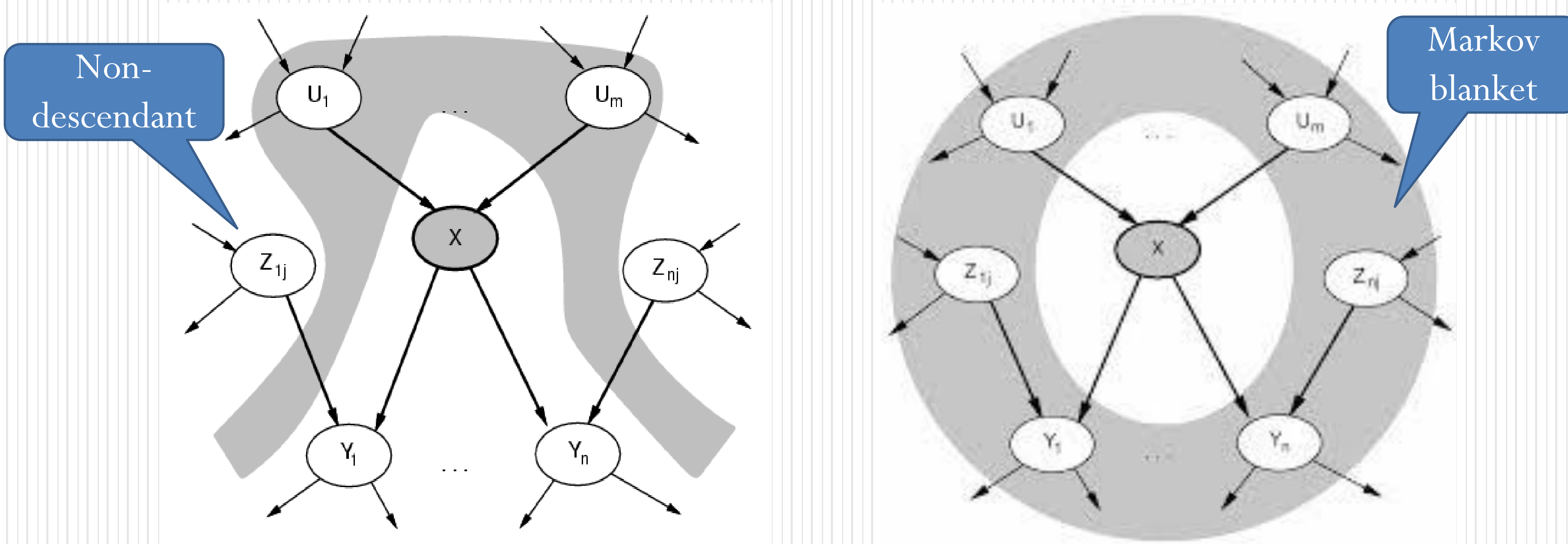
- d-separation (d-dependence): provides a notion of separation between nodes in a directed graph.
- Variables X and Y are d-separated iff for every trail between them, there is an intermediate variable Z such that either
  - Z is in a serial or diverging connection and Z is known (observed).
  - Z is in converging connection and neither Z not any of Z's descendants are known.
- Two variable X and Y are d-connected if they are not d-separated.
- If variables X and Y are d-separated by Z then, X and Y are conditionally independent given Z.

- **Definition:** Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be three sets of nodes in G (BN structure). We say that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are d-separated given $\boldsymbol{Z}$, denoted $dsep(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{Z})$, if there is no active trail between any node $X \in \boldsymbol{X}$ and $Y \in \boldsymbol{Y}$ given $\boldsymbol{Z}$.
- Let $I(G)$ denote the set of independencies that correspond to d-separation:

$$I(G) = \{\boldsymbol{X} \perp \boldsymbol{Y}|\boldsymbol{Z}): dsep(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{Z})\}$$

This set is also called the set of global **Markov independencies**.

Conditional Independence relations



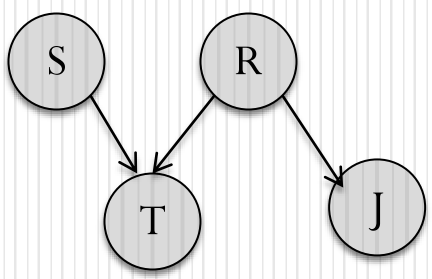
- Each node is conditionally independent of its non-descendants, given its parents.
- Each node is conditionally independent of all others given its Markov blanket: parents+children+children's parents

Conditional Independence

- **Example:** One morning Tracey leaves her house and realise that her grass is wet. Is it due to overnight rain or did she forget to turn off the sprinkler last night?
- Next she notices that the grass of her neighbour, Jack, is also wet.
- This **explains away** to some extent the possibility that her sprinkler was left on, and she concludes therefore that it is probably been raining (it decreases her belief that the sprinkler is on).
- Using the following four propositional random variables, construct the BN and determine if S is d-separated from J when T is known.
  - R: Rain $\in \{0,1\}$ (Rain = 1 means that it has been raining, and 0 otherwise)
  - S: Sprinkler $\in \{0,1\}$
  - J: Jack's grass wet $\in \{0,1\}$
  - T: Tracy's Grass wet

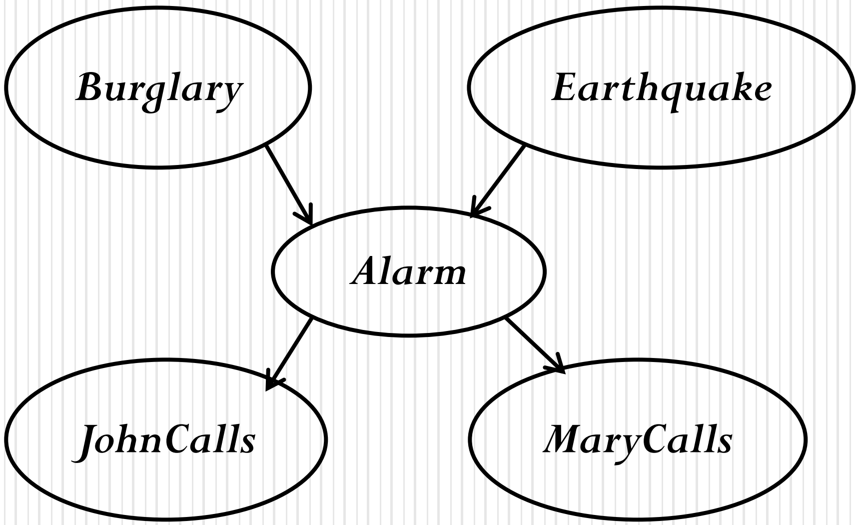
Conditional Independence

- Four propositional random variables are:
  - R: Rain ∈ {0,1} (Rain = 1 means that it has been raining, and 0 otherwise)
  - S: Sprinkler ∈ {0,1}
  - J: Jack's grass wet ∈ {0,1}
  - T: Tracy's Grass wet



- The trail between S and J: S-T-R-J
- S-T-R converging connection and T is known so influence flows from S to R.
- T-R-J diverging connection and R is not known so influence flows from T to J.
- So, S and J are not d-separated given T

Conditional Independence relations



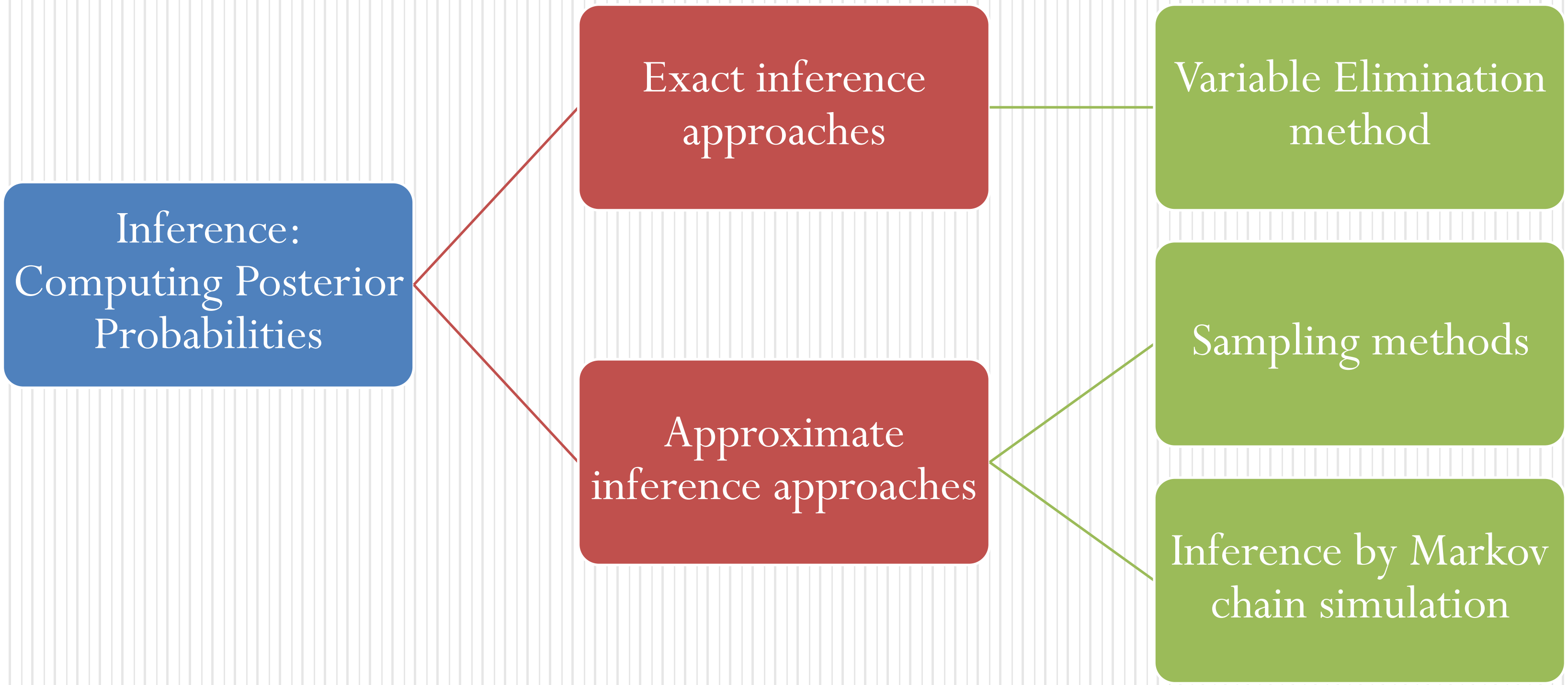
$$P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$$
$$= P(j \mid m, a, \neg b, \neg e)\ P(m, a, \neg b, \neg e)$$
$$= P(j \mid m, a, \neg b, \neg e)\ P(m, a, \neg b, \neg e)$$
$$= P(j \mid a)\ P(m \mid a, \neg b, \neg e)\ P(a, \neg b, \neg e)$$
$$= P(j \mid a)\ P(m \mid a)\ P(a \mid \neg b, \neg e),\ P(\neg b, \neg e)$$
$$= P(j \mid a)\ P(m \mid a)\ P(a \mid \neg b, \neg e),\ P(\neg b \mid \neg e),\ P(\neg e)$$
$$= P(j \mid a)\ P(m \mid a)\ P(a \mid \neg b, \neg e),\ P(\neg b),\ P(\neg e)$$
$$= \pi_{i=1}^{n}\ \mathbf{P}\ (X_i \mid Parents(X_i))$$

- The node *JohnCalls* independent of *Burglary*, *Earthquake*, and *MaryCalls* given the value of *Alarm*.
- The node *Burglary* is independent of *JohnCalls* and *MaryCalls*, given *Alarm* and *Earthquake*.

Inference in Bayesian Networks

- Given a Bayesian network, what queries one might ask?
  - Simple query: compute posterior probability i.e. $P(X_i \mid E=e)$
- X denotes query variable, E is set of evidence variables, E_1, …E_m, and e is particular observed event, Y denotes non-query, non-evidence variables (called hidden variables).
- Example: $P(Burglary \mid JohnCalls = true, MaryCalls = true)$
  - $X = Bruglary, E = \{JohnCalls, MaryCalls\}, Y = \{Alarm, Earthquake\}$

Inference in Bayesian Networks



Inference by Enumeration

- $\boldsymbol{P}(X \mid \boldsymbol{e}) = \alpha\, \boldsymbol{P}(X, \boldsymbol{e}) = \alpha \sum_{\mathrm{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$: Sum over the variables not involved in the query.

$\boldsymbol{P}(\mathrm{B} \mid \mathrm{j,m}) = \alpha\, \boldsymbol{P}(B, j, m) = \alpha \sum \sum \boldsymbol{P}(B, j, m, e, a)$

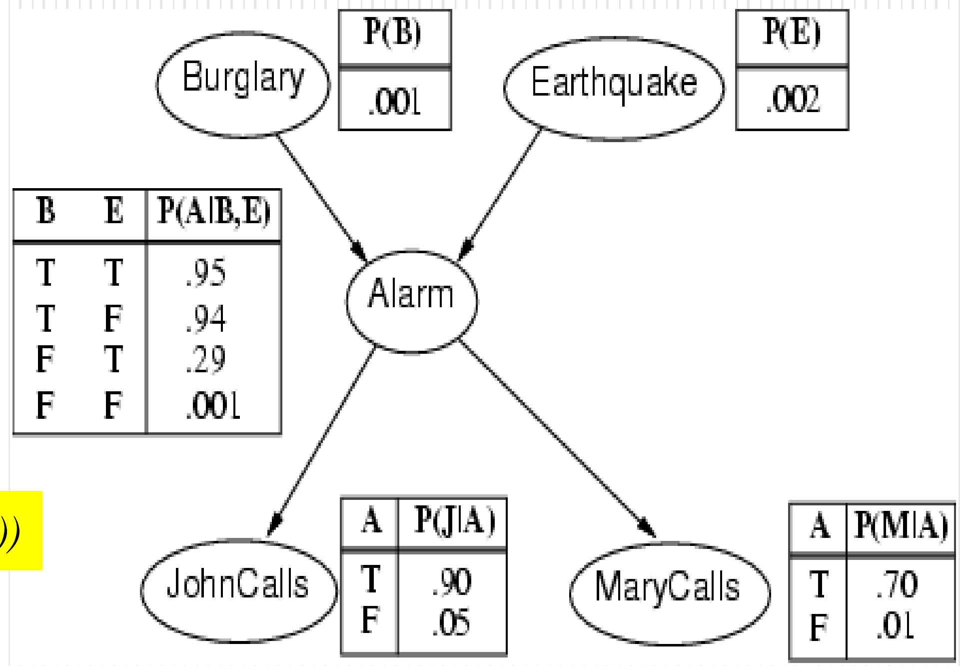
Using Bayesian network semantics we can get the expression in terms of CPTs.

Let us write this for *Burglary= true*

$P(\mathrm{b} \mid \mathrm{j,m}) =$

$\alpha \sum_{\mathrm{e}} \sum_{\mathrm{a}} P(b)\, P(e)\, P(a \mid b,e)\, P(j \mid a)\, P(m \mid a)$

$= \alpha\, P(b) \sum_{\mathrm{e}} P(e) \sum_{\mathrm{a}} P(a \mid b,e)\, P(j \mid a)\, P(m \mid a)$



P(B)
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B	E	P(A|B,E)
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F	T	.29
F	F	.001

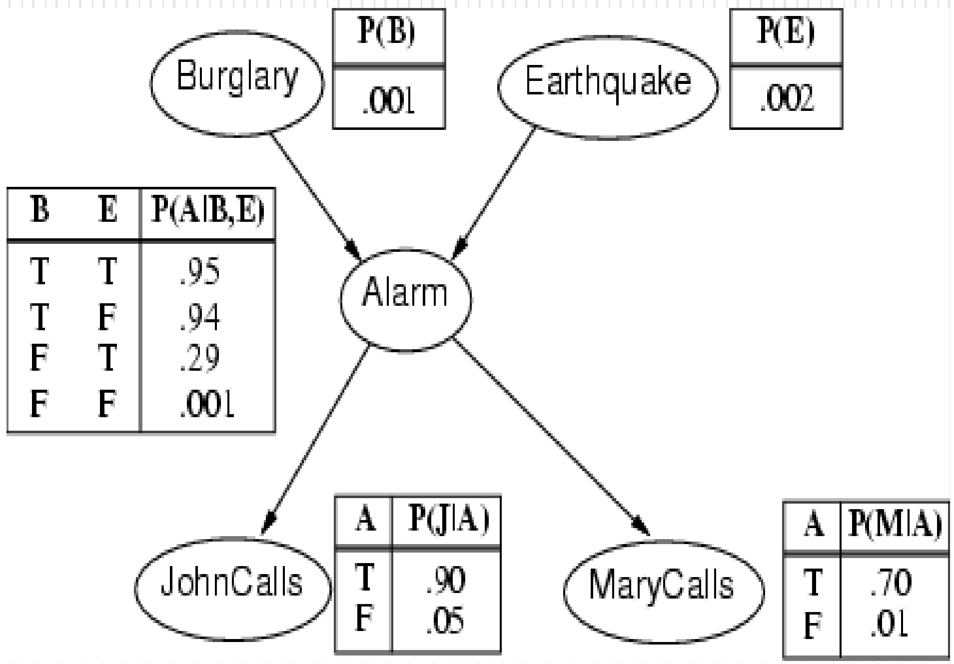
A	P(J|A)
T	.90
F	.05

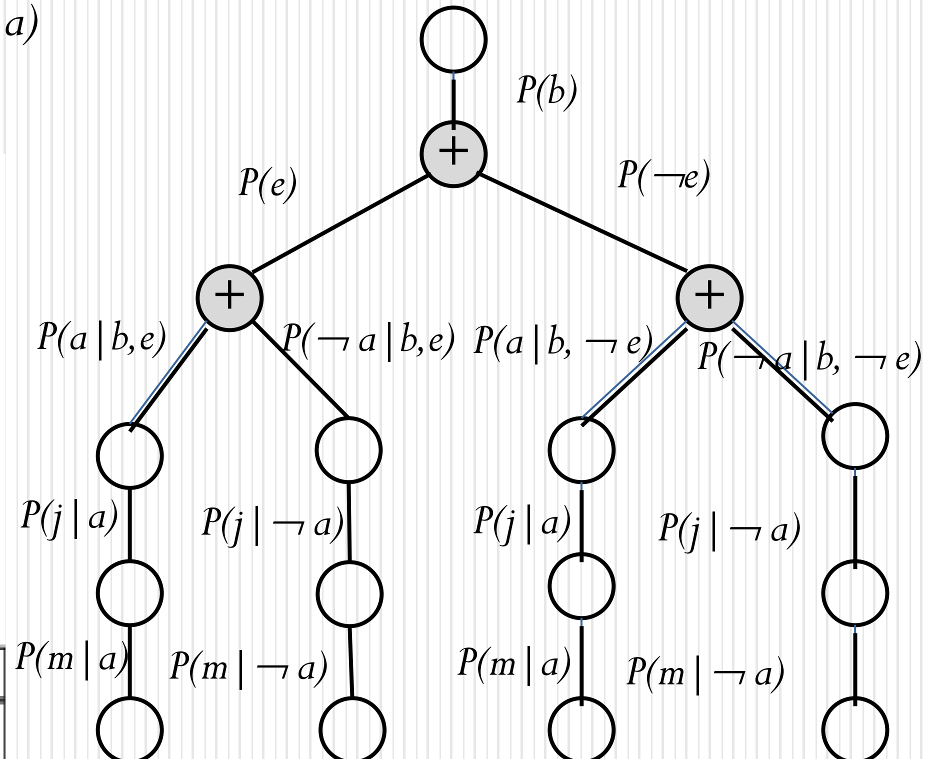
A	P(M|A)
T	.70
F	.01

We know: $\boldsymbol{P}(X_1, \ldots, X_n) = \pi_{i=1} \boldsymbol{P}(X_i \mid Parents(X_i))$

Inference by Enumeration

$$P(b \mid j,m) = \alpha\, P(b) \sum_{e} P(e) \sum_{a} P(a \mid b,e)\, P(j \mid a)\, P(m \mid a)$$





Proceed top down, multiplying values along each path and summing at the "+" nodes. Note repetition of the paths for *j* and *m* (repeated computation). For large networks takes long time.

Inference by Variable elimination

- Eliminates repeated calculations
- Variable elimination:
  - expression is evaluated from right-to-left order
  - intermediate results are stored
  - summations over each variable are done only for those portions of the expression that depend on the variable.

$$\mathbf{P}(B \mid j,m) = \alpha\, \mathbf{P}(B) \sum_{e} P(e) \sum_{a} P(a \mid B,e)\, P(j \mid a)\, P(m \mid a)$$

$$f_1(B) \quad f_2(E) \quad f_3(A,B,E) \quad f_4(A) \quad f_5(A) \leftarrow \quad \textit{factor}$$

$$= \alpha\, f_1(B) \times \sum_{e} f_2(E) \times \sum_{a} f_3(A,B,E) \times f_4(A) \times f_5(A)$$

$$Eg.\ f_4(A) = \begin{pmatrix} P(j|a) \\ P(j|\sim a) \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.05 \end{pmatrix}$$

- Each factor is a matrix indexed by the values of its arguments
- $\times$ operator is point-wise product operation

Inference by variable elimination

Sum out variables (right-to-left) from point-wise products of factors to produce new factors, eventually yielding a factor that is the solution.

$$\boldsymbol{P}(B \mid j,m) = \alpha\, f_1(B) \times \sum f_2(E) \times \sum f_3(A,B,E) \times f_4(A) \times f_5(A)$$

- sum out A from the product f_3, f_4, and f_5 giving f_6

$$f_6(B,E) = \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$
$$= (f_3(a,B,E) \times f_4(a) \times f_5(a)) + (f_3(\sim a,B,E) \times f_4(\sim a) \times f_5(\sim a))$$

Now we are left with expression

$$P(B \mid j,m) = \alpha\, f_1(B) \times \sum_e f_2(E) \times f_6(B,E)$$

- sum out E from the product of f_2 and f_6

$$f_7(B) = \sum_e f_2(E) \times f_6(B,E) = f_2(e) \times f_6(B,e) + f_2(\sim e) \times f_6(B,\sim e)$$

Now the expression becomes

$$P(B \mid j,m) = \alpha\, f_1(B) \times f_7(B)$$

Inference by variable elimination

- Basic operations
  - Point-wise product
    - two factors f_1 and f_2 yields a new factor f whose variables are the union of the variables in f_1 and f_2
    - f's elements are given by product of the corresponding elements in the two factors
    - Example:
      - given two factors $f_1(A,B)$ and $f_2(B,C)$, the pointwise product $f_1 \times f_2 = f_3$ (A,B,C) has 2^{1+1+1} entries (table in next slide)
  - Summing out variable
    - It is done by adding up the submatrices formed by fixing the variable to each of its value in turn.

Inference by variable elimination

A	B	$f_1(A,B)$	B	C	$f_2(B,C)$	A	B	C	$f_3(A,B,C)$
T	T	0.3	T	T	0.2	T	T	T	$0.3 \times 0.2 = 0.06$
T	F	0.7	T	F	0.8	T	T	F	0.3 x 0.8 = 0.24
F	T	0.9	F	T	0.6	T	F	T	0.7 x 0.6 = 0.42
F	F	0.1	F	F	0.4	T	F	F	0.7 x 0.4 = 0.28
						F	T	T	0.9 x 0.2 = 0.18
						F	T	F	0.9 x 0.8 = 0.72
						F	F	T	0.1 x 0.6 = 0.06
						F	F	F	0.1 x 0.4 = 0.04

- To sum out A from $f_3(A,B,C)$, we write
- $f(B,C) = \sum_a f_3(A,B,C) = f_3(a,B,C) + f_3(\sim a,B,C)$

$$= \begin{pmatrix} .06 & .24 \\ .42 & .28 \end{pmatrix} + \begin{pmatrix} .18 & .72 \\ .06 & .04 \end{pmatrix} = \begin{pmatrix} .24 & .96 \\ .48 & .32 \end{pmatrix}$$

What did we discuss in L4-L5?

- How Bayesian Networks can be used to represent knowledge under uncertainty?
- How to construct a Bayesian network and how to infer from it ?