Transportation problem

- There are m supply stations $S_1, ..., S_m$ for product **Q**.
- There are n destination stations $D_1, D_2, ..., D_n$ where **Q** is transported.
- c_{ij} is the cost of transportation of unit amount of **Q** from S_i to D_j.
- a_i is the amount of **Q** available at S_i.
- d_j is the demand of **Q** at D_j.
- To find x_{ij}, $i = 1, 2, .., m$, $j = 1, 2, ..., n$, where x_{ij} is the amount of **Q** to be transported from S_i to D_j such that the demand at each D_j is met and the cost of transportation is minimum.
- The problem is given by:
  $Min \sum_{i,j} c_{ij}x_{ij}$
  subject to
  $\sum_{j=1}^{n} x_{ij} \leq a_i$, $i = 1, 2, ..., m$
  $\sum_{i=1}^{m} x_{ij} \geq d_j$, $j = 1, 2, ..., n$,
  $x_{ij} \geq 0$ for $i = 1, 2, .., m, j = 1, 2, .., n$.

- It is clear that for the transportation problem to be **feasible** $\sum_i a_i \geq \sum_j d_j$.
- A transportation problem is called **balanced** if $\sum_i a_i = \sum_j d_j$.
- In that case all the inequalities in the constraints should hold as equalities.
- A balanced transportation problem is given by,
  $Min \sum_{i,j} c_{ij} x_{ij}$
  subject to
  $\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m$
  $\sum_{i=1}^{m} x_{ij} = d_j,\ j = 1, 2, ..., n,$
  $x_{ij} \geq 0$ for $i = 1, 2, .., m, j = 1, 2, .., n$.
- Since $\sum_i a_i = \sum_j d_j$
  if $\mathbf{x} = (x_{ij})_{mn \times 1}$ satisfies any $(m + n - 1)$ equations then it automatically satisfies all the $(m + n)$ equations.

- The constraints are of the form
  $A\mathbf{x} = \mathbf{b}$,
  where

$$A_{(m+n)\times mn} = \left[\begin{array}{cccccc}
\overbrace{111..11}^{n} & \mathbf{0_n} & \mathbf{0_n} & \ldots & . & \mathbf{0_n} \\
\mathbf{0_n} & \overbrace{111..11}^{n} & \mathbf{0_n} & \ldots & . & \mathbf{0_n} \\
\mathbf{0_n} & \mathbf{0_n} & \overbrace{111..11}^{n} & \mathbf{0_n} & . & \mathbf{0_n} \\
. & . & . & . & \ldots & . \\
\mathbf{0_n} & . & . & \ldots & \mathbf{0_n} & \overbrace{111..11}^{n} \\
\overbrace{100...0} & \overbrace{100...0} & . & . & \ldots & \overbrace{100...0} \\
\overbrace{010...0} & \overbrace{010...0} & . & . & \ldots & \overbrace{010...0} \\
. & . & . & \ldots & \ldots & . \\
\overbrace{000..01} & \overbrace{000..01} & . & . & \ldots & \overbrace{000..01}
\end{array}\right]$$

  $\mathbf{b} = [a_1, a_2, .., a_m, d_1, d_2, ..., d_n]^T$ (the $\mathbf{0_n}$'s are row vectors with n components).
- rank(A)$= m + n - 1$.

- Remove the last equation (any equation) from the $(m+n)$ equations.
- In the new $A\mathbf{x} = \mathbf{b}$, the dimension of A is $(m+n-1) \times mn$ and $\mathbf{b} = [a_1, a_2, .., a_m, d_1, d_2, ..., d_{n-1}]^T$.
- Any BFS of this problem will have $m+n-1$ **basic variables** and the order of any **basis matrix**, **B** is $(m+n-1) \times (m+n-1)$.
- **Theorem 1 :** Let B be a basis matrix then:
  1. There exists a row of **B** with exactly one nonzero entry (which is a 1).
  2. The sub matrix obtained by deleting the corresponding row and column (containing that nonzero entry) from B will again be nonsingular and will have a row with a single nonzero entry.
- Such matrices (such as B) are called triangular matrices, and because of this special structure of B it is easy to solve system of equations of the form $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ (which will give a basic solution of the transportation problem).

- If B is a square sub matrix of A with properties 1 and 2 of theorem 1, then $|B| = \pm 1$
- If D is any nonsingular sub matrix of A then D will again have the same structure as **B**.
- If D is any square submatrix of A, then $\mid D \mid$ is either **0,1** or **-1**.
- If the i th row of **B** has a **single nonzero entry** at the j th column, then one should start by assigning the value $x_{ij} = b_i$ (where b_i is either a_i or d_j).
  Then **remove** the ith row and the j th column from **B** which gives the matrix B_1 .
  Solve the system $B_1\mathbf{x}' = \mathbf{b}'$,where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by removing x_{ij} and $\mathbf{b}'$ is obtained from $\mathbf{b}$ by removing b_i and changing the j th component from b_j to $b_j - b_i$.
  Proceeding in this way one can solve the system of equations $B\mathbf{x} = \mathbf{b}$.

- In any **basic solution** the **basic variables** takes values of the form, $\sum_i \gamma_i b_i$, where the γ_i's are either **0,1** or **-1**.
- In any **BFS** $\mathbf{x}$ of the transportation problem with supplies a_i, $i = 1, 2, ..., m$ and demand d_j, $j = 1, 2, ..., n$ the basic variables taking values of the form, $x_{ij} = \sum_i \alpha_i a_i + \sum_j \beta_j d_j$ where the α_i's and β_j's take values **0,1** or **-1**.
- **Transportation Array:** The mn variables x_{ij} can be arranged in an $m \times n$ array known as the $m \times n$ **transportation array**.
- In a transportation array each cell corresponds to a variable, that is the (i, j)th cell corresponds to variable x_{ij}.
- The m **rows** correspond to the m **supply constraints**, hence the sum of the values of the variables in **row** i is given by a_i.
- Similarly the n **columns** correspond to the n **demand constraints** and the sum of the values of the variables in **column** j is given by d_j.

- **Definition 1:** A subset of **cells** of the transportation array is said to be linearly independent if the set of column vectors in the matrix A corresponding to the variables associated with the cells are linearly independent. Otherwise they are said to be linearly dependent.
- **Definition 2:** A subset of (m+n-1) **cells** of the transportation array is said to be a basic set if they are linearly independent. The cells in a basic set are called basic cells.
- **Remark 2:** Note that a basic set corresponds to a basic solution of the transportation problem, where the variables corresponding to the basic cells are basic variables and the rest are nonbasic variables.
- **Remark 3:** Let $\mathcal{B}$ be a basic set of cells. If we consider the **submatrix** of $A_{(m+n-1)\times mn}$ obtained by taking the **columns** corresponding to the variables associated with the basic set $\mathcal{B}$, then the submatrix will be a basis matrix, a square nonsingular matrix of dimension $m+n-1$.

- There exists a **row** of B with exactly **one** nonzero entry.
- Since we are now solving $Bx_B = [a_1, .., a_m, d_1, .., d_{n-1}]^T$ and each row of B corresponds to a constraint (supply or demand).
- There exists a constraint which has **exactly one** basic variable.
- Since each **row** and **column** of the transportation array corresponds to a **constraint**, there exists a row or column of the transportation array which has **exactly one** cell from the basic set $\mathcal{B}$.
  If row i contains a single nonzero entry at (i, j) th position, then the submatrix obtained from B after **deleting** the i th row and the j th column from B again has the same property.

- If $\mathcal{B}$ is a **basic** set of cells and if the **row or column** having a single basic cell is struck off from the transportation array, then in the **reduced** (or remaining) array there will again be a **row** or **column** with a **single** basic cell.
- Since every row and column of the array has at least one basic cell, one can continue this process (of striking off a row or column) till **all** the rows and columns of the transportation array are **struck off** (or deleted).
- **Example 1:** Consider the transportation problem with a_i and d_j as given below:
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	$j=1$	2	3	4	5	6	a_i
$i=1$							7
2							17
3							5
4							24
d_j	15	10	9	3	8	8	

- Let us first start with cell $(2,3)$ is a basic cell and then try to construct a BFS of the above problem.
- Since the minimum of a_2 and d_3 is $d_3 = 9$, we take $x_{23} = 9$. Delete the third column and change a_2 from 17 to $a_2' = 17 - 9 = 8$.
- In the reduced array choose a basic cell say $(2,4)$. Take $x_{24} = 3$ since $3 = min\{d_4 = 3, a_2' = 8\}$. Proceeding in this way we get the following BFS.

	$j=1$	2	3	4	5	6	a_i
$i=1$		[7]					7
2			[9]	[3]	[5]		17
3					[3]	[2]	5
4	[15]	[3]				[6]	24
d_j	15	10	9	3	8	8	

θ -loops

- A collection of cells of the transportation array is said to form a θ- loop if it satisfies the following conditions.
  1. Nonempty.
  2. Every row and column of the transportation array either has 0 or 2 cells from this collection.
  3. No proper subset of this collection satisfies both property **1** and property **2**.
  Consider the following examples.

	1	2	3	4
1	○			○
2	○	○		
3		○	○	
4			○	○

	1	2	3	4
1	○			○
2	○	○		
3		○		
4				

	1	2	3	4
1	○	○		
2	○	○		
3			○	○
4			○	○

- In the second and third example, the marked cells do not form a θ loop of the 4×4 transportation array, since it violates properties 2 and 3, respectively.
- The first one however is a θ loop.

- **Theorem 4:** The cells in a θ loop are linearly dependent.
- **Theorem 5 :** If $\triangle$ is a nonempty collection of cells of the transportation array which contains no θ loop then it satisfies,
  1. There exists a row or column of the array with **exactly** one cell from $\triangle$.
  2. Every nonempty subset of $\triangle$ should satisfy property 1.

  Theorem 6 : If $\triangle \neq \phi$ is a collection of cells of the transportation array which contains no θ loop, then $\triangle$ is **linearly independent**.

  Corollary 6: So from the previous theorems we can conclude that a subset of cells $\triangle$ of the transportation array is **linearly independent** if and only if it contains **no** θ loop.

- **Theorem 7:** If $\mathcal{B}$ is a collection of $m+n-1$ basic cells of the transportation array and $(p,q) \notin \mathcal{B}$, then $\mathcal{B} \cup \{(p,q)\}$ contains one and only one θ-loop and this loop includes the cell (p,q).
- **How to get the optimal solution from a given basic feasible solution?**
  Let $\mathbf{x} = (x_{ij})$ be the initial BFS.
- The dual of the transportation problem is given by
  Max$\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$
  subject to,
  $u_i + v_j \leq c_{ij}$ for all $i = 1, .., m$, $j = 1, .., n$.

- **Step 1:** For the basic cells corresponding to **x** $c_{ij} = u_i + v_j$.
- Solve this set of $m + n - 1$ equations for u_i and v_j.
- Since there are $m + n - 1$ equations and $m + n$, u_i, v_j's we can fix the value of any one of the variables and solve for the others.
- Since any one of the $(m + n)$ equations of the transportation problem can be removed, one can take the corresponding variable of the dual say $v_n = 0$ and can consider that variable as absent from the equations $c_{ij} = u_i + v_j$.
- This set of equations is obtained from $\mathbf{y}^T B = \mathbf{c}_B^T$, where $\mathbf{y}^T = [u_1, ..., u_m, v_1, ..., v_{n-1}]$.
- We have $m + n - 1$ equations and $m + n - 1$ unknowns, which can be easily solved.

- **Step 2:** Check if this $\mathbf{y}$ is feasible for the dual, that is if $u_i + v_j \leq c_{ij}$ for all the non basic cells. If yes, then stop.
- The corresponding BFS is then **optimal** for the primal.
- If not, then go to Step 3.
- **Step 3:** Find the θ loop in $\mathcal{B} \cup \{(p, q)\}$, where the cell (p, q) is such that
  $c_{p,q} - u_p - v_q = min\{c_{ij} - u_i - v_j : c_{ij} - u_i - v_j < 0\}$.
- **Step 4:** Assign value $+\theta$ to cell (p, q) and alternately assign $+\theta$ and $-\theta$ to all the cells in the θ- loop, so that sum of the allocations ($+\theta$ and $-\theta$ allocations) in each row and column add up to zero.
- Take
  $+\theta = min\{x_{ij} \in \theta\text{-loop} \;:\; \text{cell } (i, j) \text{ is assigned value} - \theta\}$.
  Find the new BFS say $\mathbf{x}'$ where x'_{ij} is either equal to x_{ij}, $x_{ij} + \theta$ or $x_{ij} - \theta$.

- Now (p, q) is a basic cell.
- If $x_{rs} = min\{x_{ij} \in \theta\text{-loop} \ : (i, j) \text{ is assigned value } -\theta\}$, then the variable x_{rs} becomes a nonbasic variable in $\mathbf{x}'$.
- If there is a **tie** for **this minimum value**, choose any **one** amongst them as the leaving variable (or cell) arbitrarily such that you again have $(m + n - 1)$ basic cells in the next iteration.
- **Step 5:** Go to Step 1.
- If x_{pq} is a nonbasic variable in a BFS and if the column corresponding to this variable in the corresponding simplex table be denoted by $B^{-1}\tilde{\mathbf{a}}_{p,q} = \mathbf{u_{pq}}$, then the $\mathbf{k}$ th component of this column, $u_{\mathbf{k},pq} = -1, 1,$ or 0 depending on whether the $\mathbf{k}$ th basic variable gets the allocation θ, $-\theta$ or is not included in the θ-loop containing the cell (p, q) in $\mathcal{B} \cup \{(p, q)\}$.

- If (p, q) is the **entering variable** of the new basis then according to the minimum ratio rule given by the simplex algorithm, the **leaving variable** is (r, s) if $x_{rs} = min\{x_{ij} \in \theta\text{-loop} : \text{ cell } (i, j) \text{ is assigned value } -\theta\}$.
- **Example:** Consider the following transportation problem (P) with c_{ij}'s, a_i's (40,30,30) and d_j's (30,50,20) as given below:

2	5	1	40
1	4	5	30
1	5	3	30
30	50	20	

- Check whether the initial basic feasible solution $\mathbf{x}_0$ with basic cells $\mathcal{B} = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 2)\}$, is optimal for (P) (by taking $v_2 = 0$, where v_2 is the dual variable corresponding to the second demand constraint).
  Also find the optimal solution.

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by
  $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- On solving we get, $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$).
- Check that
  $c_{13} - u_1 - v_3 = 1 - 5 - 1 = -5, c_{21} - u_2 - v_1 = 1 - 4 - (-3) = 0,$
  $c_{31} - u_3 - v_1 = 1 - 5 - (-3) = -1,$
  $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$, which is indicated in the

	0	0	-5	40
	0	0	0	30
	-1	0	-3	30
	30	50	20	

- Since all the $c_{ij} - u_i - v_j$ values are not non negative, the above table is not optimal.
- The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1, 3)$, so this will be the entering variable of the new BFS.
- The unique θ- loop in $\mathcal{B} \cup (1, 3)$ is given by $\{(1, 2), (2, 2), (2, 3), (1, 3)\}$.
- Since $(1, 3)$ is the entering variable, so if we give $+\theta$ allocation to cell $(1, 3)$ (or value of $x_{13} = +\theta$) then $x_{12} = 10 - \theta$, $x_{22} = 10 + \theta$, $x_{23} = 20 - \theta$.

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,
- $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
  we get
- $v_1 = 2, v_2 = 0, v_3 = 1, u_2 = 4, u_3 = 5.$
- $c_{21} - u_2 - v_1 = 1-4-2 = -5, c_{12} - u_1 - v_2 = 5-0-0 = 5,$
  $c_{31} - u_3 - v_1 = 1-5-2 = -6,$
  $c_{33} - u_3 - v_3 = 3-5-1 = -3.$
  The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with
  $\mathcal{B} = \{(1,1),(1,3),(2,3),(2,2),(3,2)\}$

-

	0	5	0	40
	-5	0	0	30
	-6	0	-3	30
	30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(2,2),(2,3),(1,3),(1,1)\}$.
- $(3,1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3,1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.
- So $\theta = 10$.
- The entering variable for the new BFS is $x_{31} = 10$ and x_{23} is the leaving variable.
- The values of the basic variables in the new BFS is given by $x_{11} = 20, x_{13} = 20, x_{22} = 30, x_{31} = 10, x_{32} = 20$.
- The basic set of cells is given by $\mathcal{B} = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$.

- We take $u_1 = 0$, then by solving the 5 equations given below:

  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$

  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$

  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$

  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$

  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- Check that $v_1 = 2, v_2 = 6, v_3 = 1, u_2 = -2, u_3 = -1$.
- Check that $c_{23} - u_2 - v_3 = 5 - (-2) - 1 = 6$, $c_{21} - u_2 - v_1 = 1 - (-2) - 2 = 1$, $c_{12} - u_1 - v_2 = 5 - 0 - 6 = -1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.
- The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with $\mathcal{B} = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$.

	0	-1	0	40
	1	0	6	30
	0	0	3	30
	30	50	20	

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.
- Take $\theta = 20$.
  Any one of x_{11} or x_{32} can be the leaving variable.
- Let x_{32} leave the basis.
- If we take $u_1 = 0$, then by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$.
  we get

- $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1.$
- $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0, c_{32} - u_3 - v_2 = 5 - (-1) - 5 = 1, c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3.$
- Since $c_{ij} - u_i - v_j \geq 0$ for all i, j,
  the above BFS is optimal and the optimal value is given by:
  $c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{22}x_{22} + c_{31}x_{31} =$
  $2 \times 0 + 5 \times 20 + 1 \times 20 + 4 \times 30 + 1 \times 30 = 270.$