Practice problems 3

1. Write the dual of the following problems:

(a) Maximize
$$x_1+x_2$$
 subject to $-x_1+x_2+x_3\geq 1$
$$x_1+2x_2+3x_3=2$$

$$x_1\geq 0, x_2\geq 0, x_3\geq 0.$$

(b) Maximize
$$x_1 + x_2 + x_3$$
 subject to $-x_1 + x_2 + x_3 \le 2$
$$x_1 + 2x_2 + 3x_3 \ge 2$$

$$x_1 \ge 0, x_2 \ge 0.$$

(c) Maximize
$$x_1 - x_2 + x_3$$
 subject to $-x_1 + x_2 + x_3 \le 2$ $x_1 + 2x_2 + 3x_3 = 2$.

Hint: Write x_i as $x_i = z_i - w_i$, where $z_i, w_i \ge 0$.

Check whether the above problems have optimal solutions by using the dual(you need not find the optimal solutions).

- 2. Find an infeasible (that has no feasible solution) primal problem which also has an infeasible dual.
- 3. (a) Using similar results done in class show that exactly one of the following two systems has a solution.

$$\mathbf{x} \ge \mathbf{0}, A\mathbf{x} > \mathbf{0}$$
 (all components of $A\mathbf{x}$ is positive). (1) $\mathbf{y} \ge \mathbf{0}, \mathbf{y} \ne \mathbf{0}, A^T\mathbf{y} \le \mathbf{0}.$

Hence can the feasible region of both a primal problem (in standard form) and its dual be bounded?

Hint: For the first part use Farka's lemma and convert the system (2) to

$$\begin{bmatrix} A^T & I \\ \mathbf{e}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \mathbf{y} \ge \mathbf{0} \text{ and } \mathbf{z} \ge \mathbf{0}$$
 (2)

Then by Farka's lemma exactly one of the following two systems (1)' and (2)' has a solution,

where (1)' is given by:

$$\begin{bmatrix} A & \mathbf{e} \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ a \end{bmatrix} \ge \mathbf{0}, \begin{bmatrix} \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ a \end{bmatrix} < 0$$

Show that (1) is equivalent to the system (1) above.

For the second part, answer is **No**.

- (b) Can both the feasible regions be unbounded? **Hint:** Answer is Yes.
- 4. Consider a LPP of the type:

Maximize
$$\mathbf{c}^T \mathbf{x}$$

subject to
$$A_{m \times n} \mathbf{x} \leq \mathbf{b}$$
.

Write the dual of the above problem such that it (the dual) has only n constraints (leaving out the non negativity constraints).

5. Given a LPP of the type:

Minimize
$$\mathbf{c}^T \mathbf{x}$$

subject to
$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$
.

- (a) Write the dual of the above problem.
- (b) Obtain the complementary slackness conditions of the above problem.
- 6. Using the complementary slackness conditions obtain the optimal solution of the following problem and show that it is unique.

Minimize
$$x_1 + 3x_2 + x_3$$

subject to
$$3x_1 + x_2 \ge 6$$

$$x_1 + x_2 - x_3 \ge 2$$

$$x_1 + x_3 \ge 2$$

$$x_1 > 0, x_2 > 0, x_3 > 0.$$

Does the dual have a unique solution? Obtain feasible solutions \mathbf{x} and \mathbf{y} of the primal(P) and the dual(D) respectively, which does **not** mutually satisfy the complementary slackness condition.

7. Consider the problem

Minimize
$$\mathbf{c}^T \mathbf{x}$$

subject to
$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$

where A has m rows and n columns. Suppose an optimal solution exists such that only $x_1, x_2, ..., x_k$ are positive and $x_j = 0$ for j = k + 1, k + 2, ..., n. Now change the vector **b** to **b**'. Prove that if there exists a $\mathbf{z} \geq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{b}'$ and $z_j = 0$ for j = k + 1, k + 2, ..., n, then this **z** is optimal for the changed problem.