DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
MA 321 (Optimization)
Mid-semester Examination

Time: 2 pm - 4 pm September 22, 2022 **Maximum marks:** 30

Notation: $\mathbf{a}_k{}^T$ denotes the k-th row of A and $\tilde{\mathbf{a}}_{\mathbf{k}}$ denotes the k-th column of A.

1. For a linear programming problem (P) of the form,

$$\text{Maximize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } \mathbf{A}_{2\times 3}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

where $\tilde{\mathbf{a}}_1 = [1,2]^T, \tilde{\mathbf{a}}_2 = [2,-2]^T, \tilde{\mathbf{a}}_3 = [3,3]^T$, $\mathbf{b} = [4,2]^T$, $\mathbf{c} = [1,2,4]^T$.

(a) Give all the entries of the simplex table for the BFS corresponding to the basis $\{\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2\}$.

Solution:

$c_j - z_j$	0	0	1	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{b}$
x_1	1	0	2	2
x_2	0	1	$\frac{1}{2}$	1

(b) Find an optimal solution of (P) by using the simplex algorithm. Hence give an optimal solution of the Dual of (P).

Solution: In the previous table x_3 is the entering and x_1 is the leaving variable. The optimal table is given by:

$c_j - z_j$	$-\frac{1}{2}$	0	0	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{b}$
x_3	$\frac{1}{2}$	0	1	1
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$

An optimal solution of (P) is $x_1 = 0, x_2 = \frac{1}{2}, x_3 = 1$.

An optimal solution of the Dual is given by

$\mathbf{c}_B^T B^{-1} = [4,2]^T \frac{1}{12}\begin{bmatrix} 3 & -3 \\ 2 & 2 \end{bmatrix} = [\frac{7}{6}, \frac{1}{6}]$. **[3.5 + 3.5 + 2]**

2. Consider the following linear programming problem (P) given below:

$$\text{Maximize } 2x_2 + 3x_3$$
$$\begin{aligned} \text{subject to} \quad & x_1 - x_2 - x_3 \geq a \\ & 4x_1 + x_2 + 3x_3 \leq b \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

(a) Write the dual of (P).

Solution: The Primal can be rewritten as:

$$\text{Maximize } 2x_2 + 3x_3$$
$$\text{subject to} \quad -x_1 + x_2 + x_3 \leq -a$$

$$4x_1 + x_2 + 3x_3 \le b$$
$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

The dual is given by:

$$\begin{aligned} \text{Minimize} \quad & -ay_1 + by_2 \\ \text{subject to} \quad & -y_1 + 4y_2 \ge 0 \\ & y_1 + y_2 \ge 2 \\ & y_1 + 3y_2 \ge 3 \\ & y_1 \ge 0, y_2 \ge 0. \end{aligned}$$

(b) Is the feasible region of the Dual unbounded? If yes, give a direction of Fea(D). If no, justify.
Solution: The feasible region of the Dual is unbounded with $[0,1]^T$ as a direction of Fea(D).

(c) If $[0, r, s]^T$ is an optimal solution of (P) (where $r \neq 0, s \neq 0$), then if possible give an optimal solution of the Dual of (P). Also, if possible give the number of optimal solutions (P) has, with proper justification.
Solution: Since $r, s \neq 0$, by complementary slackness any optimal solution $\mathbf{y}$ of the Dual will satisfy $y_1 + y_2 = 2$ and $y_1 + 3y_2 = 3$, solving which we get the unique optimal solution of the Dual, $y_1 = \frac{3}{2}$ and $y_2 = \frac{1}{2}$.
Since $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{2}$ is optimal for the Dual, by complementary slackness any optimal solution $\mathbf{x}$ of primal (P) must satisfy

$$-x_1 + x_2 + x_3 = -a$$

$$4x_1 + x_2 + 3x_3 = b$$

.

Since $-y_1 + 4y_2 > 0$ for the optimal solution $y_1 = \frac{3}{2}$ and $y_2 = \frac{1}{2}$ of the Dual, for any optimal solution of the primal, $x_1 = 0$.
But since $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{2}$ is optimal for the Dual, by complementary slackness

$$x_2 + x_3 = -a$$

$$x_2 + 3x_3 = b$$

for any optimal solution of the Primal. The above system has a unique solution, hence (P) also has a unique optimal solution.

$[\mathbf{2+2+4}]$

3. Consider a linear programming problem (P) of the form,

$$\begin{aligned} &\text{Minimize } \mathbf{c}^T\mathbf{x}, \quad (\mathbf{c} \neq \mathbf{0}) \\ &\text{subject to } A_{3\times 2}\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

such that $[0,2]^T, [4,0]^T$ and $[p,q]^T$ (where $p > 0, q > 0$) are the **only** extreme points of $Fea(P)$ and $[1,2]^T$, $[0,1]^T$ are extreme directions of $Fea(P)$. Let the extreme point $[p,q]^T$ lie on the hyperplanes corresponding to $\mathbf{a}_1{}^T$, $\mathbf{a}_2{}^T$. Add variables s_i, $i = 1,2,3$, (s_i corresponding to $\mathbf{a}_i{}^T$) to convert all the inequality constraints (except the non negativity constraints) to equality constraints.

(a) Draw the region $Fea(P)$ satisfying the above conditions.
Solution: There are two types of feasible region possible:
(i) The extreme direction is given by the constraint $s_3 = 0$.
(ii) The extreme direction is given by either the constraint $s_1 = 0$ or $s_2 = 0$.

(b) Give the basic and non basic variables of the BFS of (P) corresponding to $[0,2]^T$.
Solution:
(i) If Fea(P) is as in **(i)** then basic variables are x_2, s_3 and **one** of s_1, s_2 (depending on your picture). The rest, x_1 and **one** of s_1, s_2 are non basic variables .
(ii) If Fea(P) is as in **(ii)** then basic variables are x_2, s_1 and s_2. The rest, x_1 and s_3 are non basic variables.

(c) If possible give the signs (either +, or ≤ 0) of all the entries possible in the non basic columns (not the $c_j - z_j$ values) of the simplex table corresponding to $[p,q]^T$. Give brief justification.
Solution:

(i)

	$B^{-1}\tilde{\mathbf{e}}_1$	$B^{-1}\tilde{\mathbf{e}}_2$
x_1	$-$	$+$
x_2	$+$	$-$
s_3	$+$	$-$

(ii)

	$B^{-1}\tilde{\mathbf{e}}_1$	$B^{-1}\tilde{\mathbf{e}}_2$
x_1	$+$	$-$
x_2	$+$	$-$
s_3	$+$	$-$

Depending on the choice of s_1, s_2 the columns may get interchanged.

(d) If $[0,2]^T$ is the optimal solution of (P) then in the corresponding optimal simplex table of the Dual of (P) will there be any non negative row (not the $c_j - z_j$ row) ? If yes, then it is the row corresponding to which variable of the Dual? If no, justify.
Solution:
(i) In the simplex table corresponding to $[0,2]^T$ there will be a non positive column in $B^{-1}s_1$ (or $B^{-1}s_2$) because of the extreme direction $[0,1]^T$.
The corresponding optimal BFS of the dual will have basic variables s'_1 and **one** of y_1 or y_2.
The non negative row will correspond to y_1 or y_2.

(ii) In the simplex table corresponding to $[0,2]^T$ there will be a non positive column in $B^{-1}s_3$ because of the extreme direction $[0,1]^T$.
The corresponding optimal BFS of the dual will have basic variables s'_1 and y_3.
The non negative row will correspond to y_3.

(e) If $[4,0]^T$ is the optimal solution of (P) and b_3 (the third component of $\mathbf{b}$) is changed to $b_3 + t$ (everything else remaining same as (P)), then if possible give a value of t ($t > 0$), such that the optimal solution of the new (P) has basic variables x_2, s_1, s_2. If not, justify.

(f) If b_3 is changed to $b_3 + t$ ($0 < t < 1$) (everything else remaining same as (P)) then can there exist an optimal solution of the Dual of the new (P) with $y_1 > 0, y_2 > 0, y_3 > 0$?

(g) If $[p,q]^T$ is the optimal solution for (P) and $\mathbf{c}$ (the cost vector) is changed to $\mathbf{c}'$ (everything else remaining same as (P)) such that $[p,q]^T$ is no longer optimal for the new (P), then can $[0,2]^T$ be the unique optimal solution for the new (P)? Justify. **Solution:** True statement but $\mathbf{c}'$ needs to be given.
$[\mathbf{1}+\mathbf{2}+\mathbf{4}+\mathbf{2}+\mathbf{3}+\mathbf{3}+\mathbf{1}]$

The parts (d), (e), (f), (g) in the above question are independent, that is, conditions assumed for one part may not be valid for the other parts

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