Shape of $\mu - \sigma^2$ locus

We have

$$\begin{aligned} \mu &= a\mu_1 + (1-a)\,\mu_2 &(1)\\ \sigma^2 &= a^2\sigma_1^2 + (1-a)^2\,\sigma_2^2 + 2a\,(1-a)\,\rho\sigma_1\sigma_2 &(2) \end{aligned}$$

After solving for a and $1-a$ from the first equation, and putting the values in (2), we can write it in the following quadratic form

$$A\mu^2 + B\sigma\mu + C\sigma^2 + D\mu + E\sigma + F = 0$$

where (after some tedious algebra)

$$\begin{aligned} A &= \frac{1}{(\mu_1-\mu_2)^2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)\\ B &= 0\\ C &= -1\\ D &= -2\frac{\mu_1}{(\mu_1-\mu_2)^2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)\\ E &= F = 0 \end{aligned}$$

For the determinant

$$\begin{aligned} \Delta &= \det\begin{bmatrix} A & \frac{B}{2}\\ \frac{B}{2} & C \end{bmatrix}\\ &= \det\begin{bmatrix} \frac{1}{(\mu_1-\mu_2)^2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right) & 0\\ 0 & -1 \end{bmatrix}\\ &= -\frac{1}{(\mu_1-\mu_2)^2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right) < 0 \end{aligned}$$

Hence, the function defined by (1) and (2) is a hyperbola.