Convention: Throughout this discussion a feasible direction $\mathbf{d}$ at a point is by definition taken to be a nonzero vector, although there is no significant harm even if assumed otherwise. Sometimes I may have forgotten to explicitly write it.
Notation: $\nabla^2 f(\mathbf{x}) = H(\mathbf{x})$ is the Hessian matrix of f at $\mathbf{x}$.
$\nabla f(\mathbf{x})$ is the gradient **row** vector (or the vector of partial derivatives written as a row vector).

Nonlinear Programming

Let f be a real valued function defined on $\Omega \subseteq \mathbb{R}^n$.
The problem is to minimize or maximize $f(\mathbf{x})$ subject to $\mathbf{x} \in \Omega$, where f need not be a linear function.
Throughout the discussion we will assume that $\Omega \subseteq \mathbb{R}^n$, for some n.
Definition 1: A point (or an element) $\mathbf{x}^* \in \Omega$ is called a local minimum (maximum) of f if there exists an $\epsilon > 0$, such that
$\mathbf{x} \in \Omega$ and $\|\mathbf{x} - \mathbf{x}^*\| < \epsilon$ implies $f(\mathbf{x}^*) \leq f(\mathbf{x})$ $(f(\mathbf{x}^*) \geq f(\mathbf{x}))$.
Definition 2: A point $\mathbf{x}^* \in \Omega$ is called a global minimum (maximum) of f if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ $(f(\mathbf{x}^*) \geq f(\mathbf{x}))$ for all $\mathbf{x} \in \Omega$.

Definition 3: A vector $\mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$ is said to be a feasible direction at $\mathbf{x}^* \in \Omega$, if there exists a $c > 0$ such that for all t, $0 \leq t \leq c$, $\mathbf{x}^* + t\mathbf{d} \in \Omega$.

Example 1: Let $\Omega = \{[x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}$.
At $[0, 0]^T$ if $\mathbf{d} = [d_1, d_2]^T$ is a feasible direction then $d_1 \geq 0$ and $d_2 \geq 0$.
At $[0, \frac{1}{2}]^T$ if $\mathbf{d}$ is a feasible direction then $d_1 \geq 0$ but d_2 can be any real number.
At $[\frac{1}{2}, 0]^T$ if $\mathbf{d}$ is a feasible direction then $d_2 \geq 0$ but d_1 can be any real number.
At $[\frac{1}{2}, \frac{1}{2}]^T$ any $\mathbf{d} \in \mathbb{R}^2$ will be a feasible direction.

Remark 1: If $\mathbf{x}^*$ is an interior point of Ω then any $\mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$ is a feasible direction at $\mathbf{x}^*$.

First order necessary conditions for a point to be a local minimum

The results obtained in this section is based on **first order approximation** of the function f near the local minimum point $\mathbf{x}^*$.
Throughout this discussion we will assume $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{x}^*, \mathbf{d}$ are elements of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Theorem 1: Let $f : \Omega \to \mathbb{R}$ be a **continuously differentiable function** (that is, the first order partial derivatives of f exists and are continuous as functions from Ω to $\mathbb{R}$). If $\mathbf{x}^*$ is a **local minimum point** then for any **feasible direction** $\mathbf{d}$ at $\mathbf{x}^*$,
$\nabla f(\mathbf{x}^*)\mathbf{d} \geq 0$,
where $\nabla f(\mathbf{x}^*)$, the gradient vector of f at $\mathbf{x}^*$ is written as a row vector (the components of $\nabla f(\mathbf{x}^*)$ are the first order partial derivatives of f at $\mathbf{x}^*$) and $\mathbf{d} \in \mathbb{R}^n$ is a column vector.

Proof: Let $\mathbf{x}^*$ be a local minimum and let $\mathbf{d}$ be a feasible direction at $\mathbf{x}^*$.
Let $g(t) = f(\mathbf{x}(t))$, where $\mathbf{x}(t) = \mathbf{x}^* + t\mathbf{d}$.
Since f is continuously differentiable throughout Ω and $\mathbf{d}$ is a **feasible direction** at $\mathbf{x}^*$
$\lim_{h \to 0} \frac{f(\mathbf{x}^* + h\mathbf{d}) - f(\mathbf{x}^*)}{h}$ exists.
Since $g(h) - g(0) = f(\mathbf{x}^* + h\mathbf{d}) - f(\mathbf{x}^*)$, $\lim_{h \to 0} \frac{g(h) - g(0)}{h}$ also exists and
$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(\mathbf{x}^* + h\mathbf{d}) - f(\mathbf{x}^*)}{h} = \nabla f(\mathbf{x}^*)\mathbf{d}$.
Since $g'(0)$ exists and we look at the first order Taylor's approximation of g around $t = 0$,
$g(t) = g(0) + tg'(0) + o(t)$, where $o(t)$ is a function of t such that $lim_{t \to 0} \frac{o(t)}{t} = 0$. (**)

If we take $0 < t \le c$, then (**) gives,
$f(\mathbf{x}^* + t\mathbf{d}) = f(\mathbf{x}^*) + t\nabla f(\mathbf{x}^*)\mathbf{d} + o(t)$.
Since for t sufficiently small,
$|t\nabla f(\mathbf{x}^*)\mathbf{d}| \ge |o(t)|$ (provided $\nabla f(\mathbf{x}^*)\mathbf{d} \ne 0$),
if $\nabla f(\mathbf{x}^*)\mathbf{d} < 0$, $f(\mathbf{x}^* + t\mathbf{d}) < f(\mathbf{x}^*)$ for all $t > 0$ sufficiently small, which contradicts that $\mathbf{x}^*$ minimizes f locally.

Theorem 2 : Let $f : \Omega \to \mathbb{R}$ be a **continuously differentiable** function. Let $\mathbf{x}^*$ be an **interior point** of Ω. If $\mathbf{x}^*$ is a **local minimum point** of f then
$\nabla f(\mathbf{x}^*) = \mathbf{0}$
Proof: Follows from **Theorem 1**, by taking $\mathbf{d} = -(\nabla f(\mathbf{x}^*))^T$. Since $\mathbf{x}^*$ is an interior point we know that every $\mathbf{d} \in \mathbb{R}^n, \mathbf{d} \ne \mathbf{0}$ is a feasible direction at $\mathbf{x}^*$.

Second order necessary conditions for a point to be a local minimum

The following conditions are obtained by considering second order approximation of the function f near the local minimum point $\mathbf{x}^*$.
Theorem 3: Let $f : \Omega \to \mathbb{R}$ be **twice continuously differentiable** (that is all the second order partial derivatives of f (given by $\frac{\partial^2 f}{\partial x_j \partial x_i}$) exists and are continuous as functions from Ω to $\mathbb{R}$).

If $\mathbf{x}^*$ is a **local minimum** of f then for any **feasible direction d** at $\mathbf{x}^*$
1. $\nabla f(\mathbf{x}^*)\mathbf{d} \ge 0$.
2. If $\nabla f(\mathbf{x}^*)\mathbf{d} = 0$, then $\mathbf{d}^T \nabla^2 f(\mathbf{x}^*)\mathbf{d} \ge 0$.

Note: The matrix $\nabla^2 f$ (also denoted by H) is called the **Hessian matrix** of f,
$(\nabla^2 f(\mathbf{x}^*))_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(x^*) = \left[\frac{\partial^2 f}{\partial x_j \partial x_i}\right]_{x^*}$. Since f **twice continuously differentiable** the **Hessian matrix** $\nabla^2 f$ is a symmetric matrix for all $\mathbf{x} \in \Omega$.

Proof: Let $\mathbf{d}$ be a **feasible direction** at $\mathbf{x}^*$. That $\mathbf{x}^*$ satisfies condition 1 is already shown in Theorem 1.
As before, take $g(t) = f(\mathbf{x}(t))$, where $\mathbf{x}(t) = \mathbf{x}^* + t\mathbf{d}$.
Since f is **twice continuously differentiable** $g''(o)$ exists. Also the **second order Taylor's approximation** of g around $t = 0$ gives,
$g(t) = g(0) + tg'(0) + \frac{t^2}{2!}g''(0) + o(t^2)$, (**)
Since $g'(t) = (\nabla f(\mathbf{x}^* + t\mathbf{d}))\mathbf{d} = \sum_i (\frac{\partial f}{\partial x_i})(\mathbf{x}^* + t\mathbf{d})d_i$,
$g'(t) = \sum_i h_i(t)d_i$,
where $h_i(t) = \frac{\partial f}{\partial x_i}(\mathbf{x}^* + t\mathbf{d})$.

Hence $g''(t) = \sum_i h_i'(t)d_i$,
where $h_i'(t) = ((\nabla \frac{\partial f}{\partial x_i})(\mathbf{x}^* + t\mathbf{d}))\mathbf{d} = \sum_j \frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})(\mathbf{x}^* + t\mathbf{d})d_j = \sum_j (\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}^* + t\mathbf{d}))d_j$.
Hence $g''(t) = \sum_i h_i'(t)d_i = \sum_i (\sum_j (\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}^* + t\mathbf{d}))d_j)d_i$,
Hence $g''(0) = \sum_i (\sum_j (\frac{\partial^2 f}{\partial x_j \partial x_i}(x^*))d_j)d_i = \mathbf{d}^T \nabla^2 f(\mathbf{x}^*)\mathbf{d}$,
where $\nabla^2 f(\mathbf{x}^*)(= H(\mathbf{x}^*))$ is an $n \times n$ matrix whose (i, j) th entry is given by $\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}^*)$.
Note that because we have assumed f to be **twice continuously differentiable** the matrix $\nabla^2 f(\mathbf{x}^*)$ is a symmetric matrix.
Again from (**) we get,
$f(\mathbf{x}^* + t\mathbf{d}) = f(\mathbf{x}^*) + t\nabla f(\mathbf{x}^*)\mathbf{d} + \frac{t^2}{2}\mathbf{d}^T \nabla^2 f(\mathbf{x}^*)\mathbf{d} + o(t^2)$.
Since for sufficiently small t,

$|\frac{t^2}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d}| \geq |o(t^2)|$, (provided $\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} \neq 0$)
hence if $\nabla f(\mathbf{x}^*)\mathbf{d} = 0$ and $\mathbf{x}^*$ is a **local minimum** then it should satisfy the condition
$\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} \geq 0$ if $\nabla f(\mathbf{x}^*)\mathbf{d} = 0$.

Theorem 4: Let $f : \Omega \to \mathbb{R}$ be a twice continuously differentiable function and let $\mathbf{x}^*$ be an interior point of Ω. If $\mathbf{x}^*$ is a local minimum of f then
1. $\nabla f(\mathbf{x}^*) = \mathbf{0}$.
2. $\nabla^2 f(\mathbf{x}^*)$ is positive semidefinite(defined later).

Proof: Follows from the previous theorem and the fact that for an **interior point** of Ω, every nonzero vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction.
Hence from **Theorem 3**, $\nabla f(\mathbf{x}^*)(-\nabla f(\mathbf{x}^*))^T \geq 0$, which implies $\nabla f(\mathbf{x}^*) = \mathbf{0}$.
Applying the second order conclusion of **Theorem 3** , we get $\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} \geq 0$ for all $\mathbf{d} \in \mathbb{R}^n$.

Definition: A real symmetric matrix A is said to be positive semidefinite (negative semidefinite) if $\mathbf{x}^T A\mathbf{x} \geq 0$ ($\mathbf{x}^T A\mathbf{x} \leq 0$) for all $\mathbf{x} \in \mathbb{R}^n$.

Definition: A real symmetric matrix A is said to be positive definite (negative definite) if $\mathbf{x}^T A\mathbf{x} > 0$ ($\mathbf{x}^T A\mathbf{x} < 0$) for all **nonzero** vectors $\mathbf{x} \in \mathbb{R}^n$.

Remark: Note that in general a matrix satisfying the condition $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ need not be symmetric for example $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Theorem: If A is a symmetric, $n \times n$, real matrix then the following statements are equivalent:

1. A is positive semidefinite.
2. All eigenvalues of A are nonnegative.
3. All principal minors of A are nonnegative.

Definition: λ is called an eigenvalue of an $n \times n$ matrix A if there exists an $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$ (that is atleast one component of $\mathbf{x}$ is nonzero) such that $A\mathbf{x} = \lambda\mathbf{x}$.
For example the $\mathbf{0}$ matrix has all n eigenvalues equal to 0, the identity matrix I_n has all n eigenvalues equal to 1 and for an upper triangular matrix the diagonal entries are its eigenvalues.

Definition: If A is an $n \times n$ matrix and $\alpha \subseteq \{1, \ldots, n\}$, $\beta \subseteq \{1, \ldots, n\}$ then $A[\alpha, \beta]$ is the (sub)matrix obtained from A by deleting all the rows of A which do not belong to α and by deleting all the columns of A which do not belong to β.
If $\alpha = \beta$ then $A[\alpha, \alpha]$ is called a principal submatrix of A and $det A[\alpha, \alpha]$ is called a principal minor of A.
For example if $\alpha = \beta = \{i\}$ where $i \in \{1, \ldots, n\}$ then $A[\alpha, \alpha] = [a_{ii}]$ and $det A[\alpha, \alpha] = a_{ii}$, the i th diagonal entry.
If $\alpha = \beta = \{i, j\}$ where $i, j \in \{1, \ldots, n\}$ and $i < j$ then $A[\alpha, \alpha] = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$.
If $\alpha = \{1, \ldots, n\}$ then $A[\alpha, \alpha] = A$ and $det A[\alpha, \alpha] = det(A)$.

Remark: A nonsingular (nonzero determinant) positive semidefinite matrix is positive definite.

In the following examples there is a slight abuse of notation. Instead of writing $f([x_1, x_2]^T)$, to avoid cumbersome notation, I have written it as $f(x_1, x_2)$.

Example 1 revisited: Consider the following problem:
Minimize $f(x_1, x_2) = x_1^3 - x_1^2 x_2 + 2x_2^2$
subject to $x_1 \geq 0$, $x_2 \geq 0$, hence $\Omega = \{[x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}$.
To search for points which might be local minimum points of f we initially search for points in Ω which satisfy first order necessary conditions for a local minimum. The points which obviously satisfy first order necessary conditions for a local minimum are the points at which the gradient vector vanishes.
Hence we try to find $\mathbf{x}$ at which $\nabla f(\mathbf{x}) = 0$.
$\nabla f(\mathbf{x}) = (3x_1^2 - 2x_1x_2, -x_1^2 + 4x_2) = [0, 0]$ has two solutions $x_1 = 0, x_2 = 0$ and $x_1 = 6, x_2 = 9$.
Here $[6, 9]^T$ an interior point of the feasible region $\Omega = \{[x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}$ satisfies the first order necessary conditions for a local minimum.
But since $\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 6x_1 - 2x_2 & -2x_1 \\ -2x_1 & 4 \end{pmatrix}$.
At $\mathbf{x}^* = [6, 9]^T$, $\quad \nabla^2 f(\mathbf{x}^*) = \begin{pmatrix} 18 & -12 \\ -12 & 4 \end{pmatrix}$ is not positive semidefinite.
Hence $\mathbf{x}^* = [6, 9]^T$ is not a local minimum point of f.
Hence the first order necessary conditions are **necessary** but **not sufficient** for a point to be a local minimum.

At $\mathbf{x}^* = [0, 0]^T$ a $\mathbf{d} \neq \mathbf{0}$ is a feasible direction if and only if $d_1 \geq 0$ and $d_2 \geq 0$.
Since $\nabla f(\mathbf{x}^*) = \mathbf{0}$, $\quad \nabla f(\mathbf{x}^*)\mathbf{d} = 0$ for all $\mathbf{d}$.
Since $\nabla^2 f(\mathbf{x}^*) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$, $\quad \mathbf{d}^T \nabla^2 f(\mathbf{x}^*)\mathbf{d} = 4d_1^2 \geq 0$ for all $\mathbf{d}$.
Since $[0, 0]^T$ satisfies both the first and second order necessary conditions, we can only say that $\mathbf{x}^* = [0, 0]^T$ can be a candidate for local minimum, but since these are only necessary conditions we cannot conclude from previous calculations that $[0, 0]^T$ is indeed a local minimum of f in Ω.
Exercise: Check that $[0, 0]^T$ is a local minimum point of f in **Example 1**.
Solution: Note that $f(x_1, x_2) = x_1^3 - x_1^2 x_2 + 2x_2^2 = x_1^2(x_1 - x_2) + 2x_2^2$ can take values < 0 only when either $x_2 = 0$ and $x_1 < 0$ or when $x_2 \neq 0$ and $x_2 > x_1$. But $[-1, 0]^T$ is not a feasible direction at $\mathbf{x}^* = [0, 0]^T$ for $\Omega = \{[x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}$.
For $x_2 \neq 0$ and $x_2 > x_1$ sufficiently small, that is $|x_1|, |x_2| < 1$, clearly $x_1^2(x_1 - x_2) + 2x_2^2 \geq 0$. Hence $[0, 0]^T$ is a local minimum of f.
It is quite clear that for any f, of the form $f(x_1, x_2) = x_1^3 - x_1^2 x_2 + cx_2^2, c > 0$, $\mathbf{x}^* = [0, 0]^T$ will be a local minimum point of f for the domain given in **Example 1**.

However if we take $c = 0$ in the above expression, then $f(x_1, x_2) = x_1^3 - x_1^2 x_2$, then one can check that $\nabla^2 f(\mathbf{x}^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, is a positive semidefinite matrix, but $\mathbf{x}^* = [0, 0]^T$ is not a local optimum point.
Hence the **first order** and the **second order** necessary conditions are **necessary** but **not sufficient** for a point $\mathbf{x}^*$ to be a **local minimum**.

Exercise: Will $[0, 0]^T$ be a local minimum point of f given in **Example 1** if $\Omega = \mathbb{R}^2$?

Example 2 : Consider the following problem:
Minimize $f(x_1, x_2) = x_1^2 - x_1 + x_2 + x_1 x_2$
subject to $x_1 \geq 0$, $x_2 \geq 0$.
f is a twice continuously differentiable function.
At $[\frac{1}{2}, 0]^T$, $\frac{\partial f}{\partial x_1} = 2x_1 - 1 + x_2 = 0$

$\frac{\partial f}{\partial x_2} = 1 + x_1 = \frac{3}{2}$.
If $\mathbf{d}$ is a feasible direction at $[\frac{1}{2}, 0]^T$, then d_2 has to be nonnegative.
Hence $\nabla f(\mathbf{x})|_{[\frac{1}{2},0]^T}\mathbf{d} = \frac{3}{2}d_2 \geq 0$ for any feasible direction $\mathbf{d}$.
Hence the first order necessary conditions for $\mathbf{x}^* = [\frac{1}{2}, 0]^T$ to be a locally minimum point is satisfied.
Also if $\nabla f(\mathbf{x}^*)\mathbf{d} = 0$, then $d_2 = 0$, and for all such $\mathbf{d}$
$\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} = 2d_1^2 \geq 0$.
Hence the second order necessary conditions for $\mathbf{x}^*$ to be locally minimum is also satisfied.
One can easily check that f has a global minimum at $x_1 = \frac{1}{2}, x_2 = 0$.

Sufficient conditions for a local minima :

Theorem 4 : Let $f : \Omega \to \mathbb{R}$ be a **twice continuously differentiable** function. Let $\mathbf{x}^*$ be an interior point of Ω. If $\mathbf{x}^*$ satisfies the following conditions
1. $\nabla f(\mathbf{x}^*) = \mathbf{0}$
2. $\nabla^2 f(\mathbf{x}^*)$ is **positive definite**,

then $\mathbf{x}^*$ is a **local minimum point** of f.

Proof: Since $\mathbf{x}^*$ is an interior point of Ω, if $\nabla f(\mathbf{x}^*) = \mathbf{0}$, then by Taylor's second order approximation formula for f near $\mathbf{x}^*$, we get
$f(\mathbf{x}^* + t\mathbf{d}) = f(\mathbf{x}^*) + \frac{t^2}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} + o(t^2)$, for all $\mathbf{d} \in \mathbb{R}^n$ and all $t > 0$ sufficiently small.
For t small, $|\frac{t^2}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d}| > |o(t^2)|$ (if $\mathbf{d} \neq \mathbf{0}$).
Hence $f(\mathbf{x}^* + t\mathbf{d}) > f(\mathbf{x}^*)$ for all t sufficiently small. The above proof is not enough to conclude that $\mathbf{x}^*$ is a **local minimum point** of f.
The actual proof is given below but you may skip it if you find it difficult.
Optional reading:
$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \frac{1}{2}\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} + o(\| \mathbf{d} \|^2)$.
Since $\nabla^2 f(\mathbf{x}^*)$ is real, symmetric (By spectral thoerem for real symmetric matrices),
there exists a real matrix U, orthogonal (that is, $U^TU = UU^T = I$) such that

$$\nabla^2 f(\mathbf{x}^*) = U^T \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} U, \text{ where } \lambda_i\text{'s are the eigenvalues of } \nabla^2 f(\mathbf{x}^*).$$

$$\Rightarrow \quad \mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} = \mathbf{y}^T \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \mathbf{y}, \text{ where } \mathbf{y} = U\mathbf{d}.$$

$\Rightarrow \quad \mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} = \sum_i \lambda_i y_i^2$, where y_i s the i th component of $\mathbf{y}$. (multiply and check).

$\Rightarrow \quad \mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} \geq \lambda_{min}\sum_i y_i^2 = \lambda_{min}\mathbf{y}^T\mathbf{y} = \lambda_{min}\mathbf{d}^T\mathbf{d}$
($\mathbf{y}^T\mathbf{y} = \mathbf{d}^TU^TU\mathbf{d} = \mathbf{d}^T\mathbf{d} =\| \mathbf{d} \|^2$ since $U^TU = I$).
Hence $f(\mathbf{x}^* + \mathbf{d}) \geq f(\mathbf{x}^*) + \lambda_{min} \| \mathbf{d} \|^2 + o(\| \mathbf{d} \|^2)$.

For $\| \mathbf{d} \| > 0$ sufficiently small we have
$| \lambda_{min} \| \mathbf{d} \|^2 | > | o(\| \mathbf{d} \|^2) |$ hence $f(\mathbf{x}^* + \mathbf{d}) > f(\mathbf{x}^*)$ for all $\| \mathbf{d} \| > 0$, sufficiently small.

Remark: Since **maximizing** f is same as **minimizing** $-f$, all the previous theorems have corresponding analogues for a **maximization problem** with some obvious changes. For example $\leq$ conditions in the results are replaced by $\geq$ conditions and with positive semidefinite (or positive definite) matrices in the results are appropriately replaced by negative semidefinite matrices (or negative definite matrices).

Definition 4: A real valued function f defined on a **convex set** $\Omega \subseteq \mathbb{R}^n$ is said to be a **convex function** on Ω if for all $\mathbf{x}, \mathbf{y} \in \Omega$ and all $0 \leq \alpha \leq 1$,
$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$.

Definition 5: An $f : \Omega \rightarrow \mathbb{R}$ is said to be a concave function if $-f$ is a convex function.

Theorem 1 : If f is a **convex function** on Ω (a convex set), then the set $S = \{\mathbf{x} \in \Omega : f(\mathbf{x}) \leq c\}$ is a convex set (for all real c).
Proof: Exercise.

Theorem 2: Let f be a **continuously differentiable** function defined on a convex set, $\Omega \subseteq \mathbb{R}^n$, then f is **convex** on Ω if and only if $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \Omega$.

Proof: Let $f : \Omega \rightarrow \mathbb{R}$ be a convex function. Then for all $\mathbf{x}, \mathbf{y} \in \Omega$ and all $0 \leq \alpha \leq 1$,
$f(\alpha\mathbf{y} + (1-\alpha)\mathbf{x}) \leq \alpha f(\mathbf{y}) + (1-\alpha) f(\mathbf{x})$.

For all $\alpha > 0$, sufficiently small,
$\frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x})) - f(\mathbf{x})}{\alpha} \leq f(\mathbf{y}) - f(\mathbf{x})$
Letting $\alpha \rightarrow 0$ we get
$\nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \leq f(\mathbf{y}) - f(\mathbf{x})$.

To show the converse,
let $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \Omega$. (**)
Fix $\mathbf{x}, \mathbf{y} \in \Omega$, and let $\mathbf{z}$ be a point in between and on the straight line segment joining $\mathbf{x}$ and $\mathbf{y}$.
That is $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ for some $0 \leq \alpha \leq 1$. From (**) we get
$f(\mathbf{x}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})(\mathbf{x} - \mathbf{z})$ and
$f(\mathbf{y}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})(\mathbf{y} - \mathbf{z})$.
By multiplying the first equation by α , the second by $(1-\alpha)$ and adding the two equations we get,
$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})(\alpha\mathbf{x} + (1-\alpha)\mathbf{y} - \mathbf{z})$.
Since $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ we get
$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \geq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})$.
But the above condition is difficult to verify. The following result gives an easy way to the convexity of a function provided that the function satisfies certain conditions.

Theorem 3 : Let f be a **twice continuously differentiable** function on a convex set Ω (let Ω be such that it has atleast one interior point), then f is convex on Ω if and only if for all $\mathbf{x} \in \Omega$, $\nabla^2 f(\mathbf{x})$ is **positive semidefinite**.
Proof: For those interested in the proof, refer to Luenberger(Page 196, Third Edition) or Bazaara Sherali Shetty (page 92, Second edition).

Revisiting Example 1 : Let $f(x_1, x_2) = x_1^3 - x_1^2 x_2 + 2x_2^2$ be defined on
$\Omega = \{[x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}$.
Since $\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 6x_1 - 2x_2 & -2x_1 \\ -2x_1 & 4 \end{pmatrix}$, at $x_1 = 1, x_2 = 3$, $\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 0 & -2 \\ -2 & 4 \end{pmatrix}$
is clearly not positive semidefinite. Hence f is not a convex function on Ω.

Theorem 4: Let $f : \Omega \to \mathbb{R}$ be a **continuously differentiable** function. If f is **convex** on Ω, then $\mathbf{x}^*$ is a **global minimum** of f if and only if for all **feasible directions d** at $\mathbf{x}^*$ $\nabla f(\mathbf{x}^*)\mathbf{d} \geq 0$.

Proof: Since the **only if** part is already shown before, we have to only show the **if** part.
Let $\mathbf{x}^* \in \Omega$ satisfy $\nabla f(\mathbf{x}^*)\mathbf{d} \geq 0$ for all feasible $\mathbf{d}$ at $\mathbf{x}^*$.
Since f is convex $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \Omega$. (1)
Hence for all $\mathbf{y} \in \Omega$, $f(\mathbf{y}) \geq f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*)$.
Since Ω is convex, $\mathbf{x} = \alpha\mathbf{y} + (1-\alpha)\mathbf{x}^* = \mathbf{x}^* + \alpha(\mathbf{y} - \mathbf{x}^*)$ belongs to Ω, for all $0 \leq \alpha \leq 1$, hence $(\mathbf{y} - \mathbf{x}^*)$ is a feasible direction at $\mathbf{x}^*$ and $\nabla f(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \geq 0$.
Hence $f(\mathbf{y}) \geq f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \geq f(\mathbf{x}^*)$.
Since $\mathbf{y} \in \Omega$ was arbitrary, $\mathbf{x}^*$ is a global minimum of f.

Corollary 4: Let $f : \Omega \to R$ be a **continuously differentiable** function. If f is **convex** on Ω and $\mathbf{x}^*$ an **interior point** of Ω, then $\mathbf{x}^*$ is a **global minimum** for f if and only if $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

Proof: Follows from the previous result.

That the above result is not necessarily true if f is not convex as you have already seen in **Example 1**.

Remark : Since **minimizing** f is same as **maximizing** $-f$, all the previous theorems for **minimizing a convex function** have corresponding analogues for **maximizing** a concave function.

Theorem 5: Let $f : \Omega \to \mathbb{R}$ be a **convex function**, then the following statements are true.

1. Let S be the collection of all $\mathbf{x}$'s where f attains its **minimum value** (that is, the set of all optimal solutions of f for a minimization problem). Then S is a **convex** set, or in other words the set $S = \{\mathbf{x} : f(\mathbf{x}) \leq f(\mathbf{y}) \text{ for all } \mathbf{y} \in \Omega\}$, is a **convex set**.
2. If $\mathbf{x}^*$ is a **local minimum** point of f then it is also a **global minimum** point of f.

Proof:

1. If f does not have a minimum then the above result is vacuously true.
   If f takes a minimum value then let $a = min_{\mathbf{x} \in \Omega}\{f(\mathbf{x})\}$.
   Since f is a convex function, $S_1 = \{\mathbf{x} : f(\mathbf{x}) \leq a\}$ is convex, by Theorem 1.
   Note that $S_1 = S$.

2. To show that a local minimum of f is a global minimum of f.
   If not, then let $\mathbf{x}^*$ be a local minimum point of f and let there exist a $\mathbf{y} \in \Omega$ such that $f(\mathbf{y}) < f(\mathbf{x}^*)$.
   Join $\mathbf{x}^*$ and $\mathbf{y}$ by a straight line.
   Since Ω is a convex set, the straight line segment joining $\mathbf{x}^*$ and $\mathbf{y}$ lies entirely in Ω.
   Since f is a convex function, for all $0 < \alpha \leq 1$,
   $f((1-\alpha)\mathbf{x}^* + \alpha\mathbf{y}) \leq (1-\alpha)f(\mathbf{x}^*) + \alpha f(\mathbf{y}) < f(\mathbf{x}^*)$.
   This contradicts that $\mathbf{x}^*$ is a local minimum point of f.

Remark: A natural question would be whether the conclusions of Theorem 5 holds good when **maximizing** a **convex** function. The answer however is a **NO**.

Take $f(x) = x^2$, $-1 \leq x \leq 2$.
From previous discussions however it is clear that the above result is true if you are **maximizing** $-f$ or **maximizing** a **concave** function.

Remark: We had seen while **minimizing** or **maximizing** a **linear** function over a **polyhedral** set, the extremum was attained in at least one **extreme point**. An extreme point of a polyhedral set, (more generally convex set) is one which cannot be written as a strict convex combination of two distinct points of that set).
But in the problem of **minimizing** a **convex** function over a **polyhedral** set, the only minimum may be attained at an **interior point** of Ω.
For example take $f(x) = x^2$, $-1 \leq x \leq 1$, the only minimum of this function on the interval $[-1, 1]$ which is a convex set is attained st $x = 0$ which is an interior point of $[-1, 1]$.
However when maximizing a convex function over a closed and bounded convex set the maximum is attained at an extreme point as the following theorem will illustrate.

Theorem 6: Let f be a **convex** function defined on a **closed** and **bounded convex** set Ω (so it has atleast one extreme point), then there exists an **extreme point** of Ω, where f takes its **maximum** value. (In case you are interested to know the proof refer to Luenberger page 198, Third Edition).