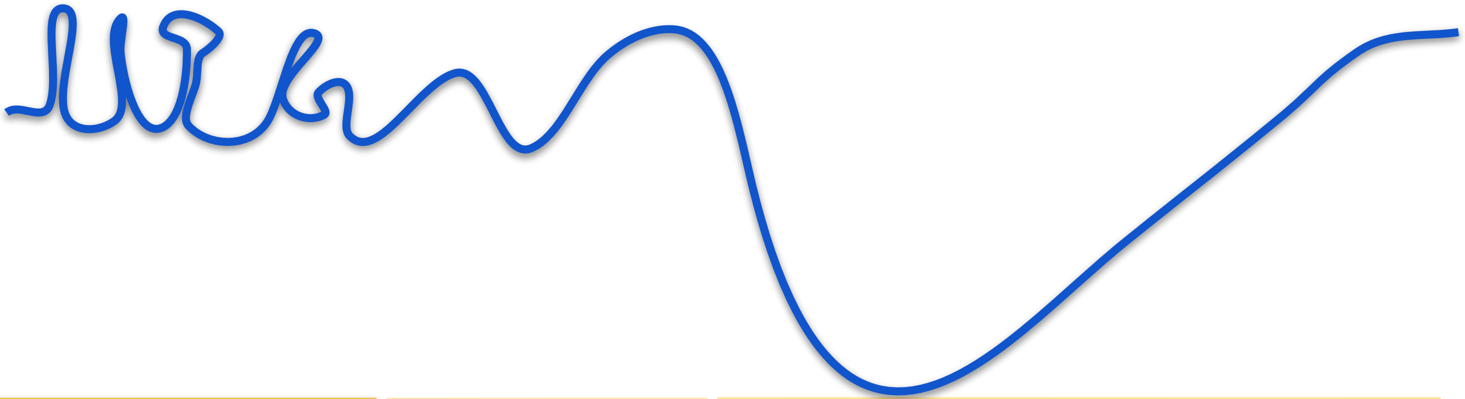
Computing with Signals



DA 623

Jan - May 2024

IIT Guwahati

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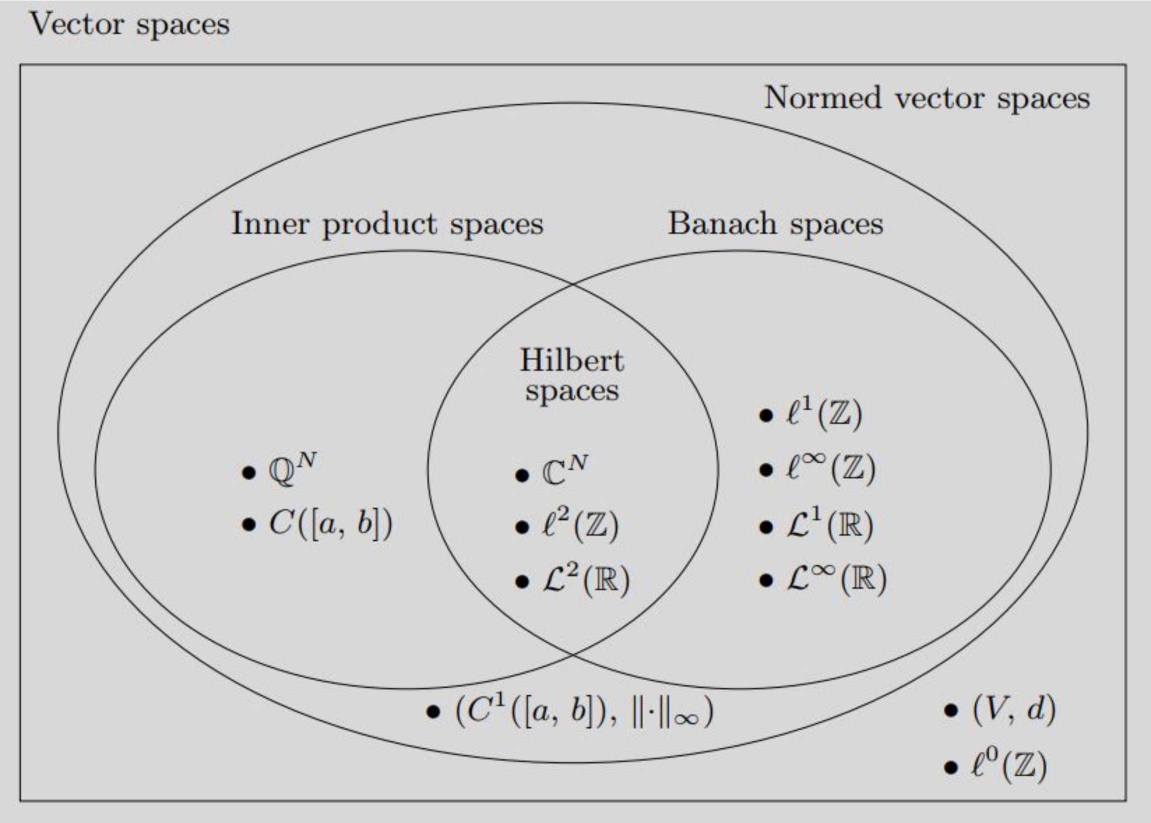
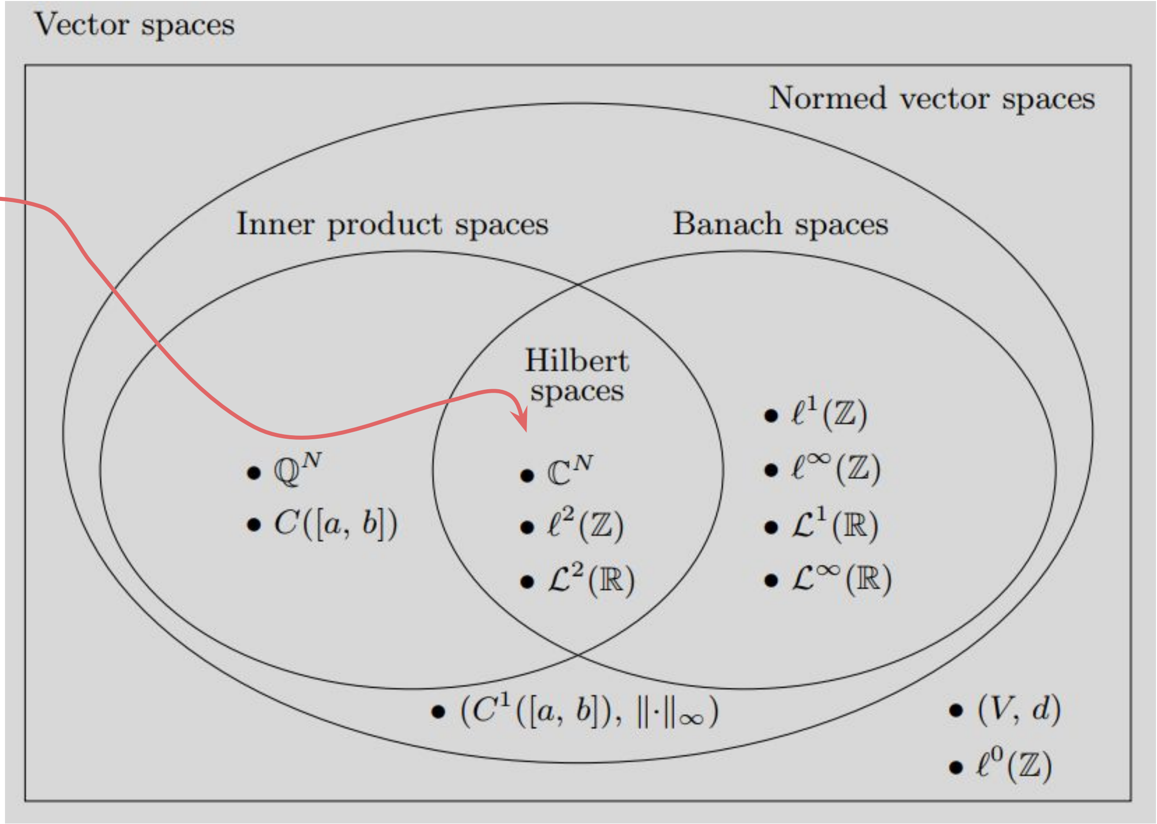
Lecture-07

Vector Space review



Hilbert Space

To ease our analysis we will enforce that our signal/data resides (mostly)



Operators in Hilbert Space

DEFINITION 2.17 (LINEAR OPERATOR) A function $A : H_0 \to H_1$ is called a *linear operator* from H_0 to H_1 when, for all x, y in H_0 and α in $\mathbb{C}$ (or $\mathbb{R}$), the following hold:

(i) *Additivity:* $A(x + y) = Ax + Ay$.

(ii) *Scalability:* $A(\alpha x) = \alpha(Ax)$.

Operators in Hilbert Space

DEFINITION 2.19 (INVERSE) A bounded linear operator $A : H_0 \to H_1$ is called *invertible* if there exists a bounded linear operator $B : H_1 \to H_0$ such that

$$BAx = x, \quad \text{for every } x \text{ in } H_0, \quad \text{and} \tag{2.46a}$$

$$ABy = y, \quad \text{for every } y \text{ in } H_1. \tag{2.46b}$$

Operators in Hilbert Space

DEFINITION 2.22 (UNITARY OPERATOR) A bounded linear operator $A : H_0 \to H_1$ is called *unitary* when

(i) it is *invertible*; and

(ii) it *preserves inner products*,

$$\langle Ax, Ay \rangle_{H_1} = \langle x, y \rangle_{H_0} \qquad \text{for every } x, y \text{ in } H_0. \tag{2.55}$$

Operators in Hilbert Space

DEFINITION 2.24 (EIGENVECTOR OF A LINEAR OPERATOR) An *eigenvector* of a linear operator $A : H \to H$ is a nonzero vector $v \in H$ such that

$$Av = \lambda v, \tag{2.58}$$

for some $\lambda \in \mathbb{C}$. The constant λ is called the corresponding *eigenvalue* and (λ, v) is called an *eigenpair*.

Approximations

Most of the linear operators we will encounter in this course are (orthogonal) projection operators

What is the approximation problem?

$$\widehat{x} = \arg\min_{s \in S} \|x - s\|$$

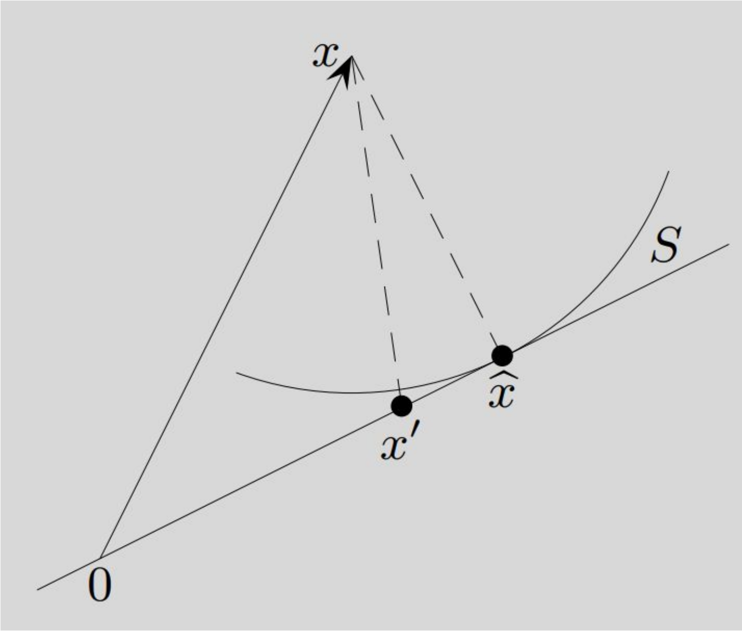
resides in a subspace S

resides in H

Approximations

Most of the linear operators we will encounter in this course are (orthogonal) projection operators

Visualizing in 2-D



What is the approximation problem?

$$\widehat{x} = \arg\min_{s \in S} ||x - s||$$

resides in a subspace *S*

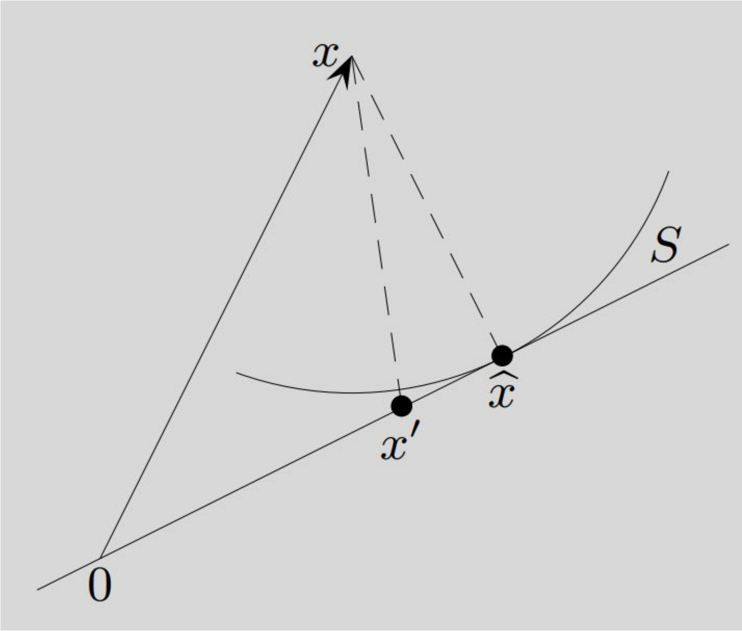
resides in *H*

Most commonly the Hilbert norm used here is the 2-norm.
Least squares approximation.

Approximations

Most of the linear operators we will encounter in this course are (orthogonal) projection operators

Visualizing in 2-D



What is the approximation problem?

$$\widehat{x} = \arg\min_{s \in S} ||x - s||$$

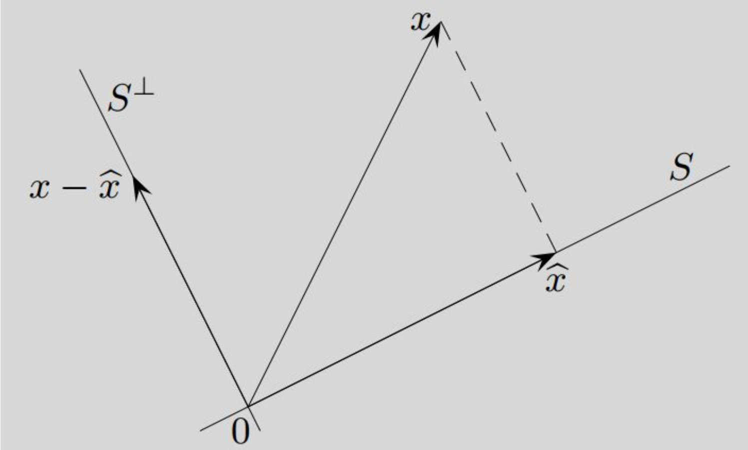
$$x - \widehat{x} \perp S$$

(residual)

Approximations

Most of the linear operators we will encounter in this course are (orthogonal) projection operators

The best approximation of $x \in H$ within a closed subspace S is uniquely determined by $x - \widehat{x} \perp S$. The solution generates an orthogonal decomposition of x into $\widehat{x} \in S$ and $x - \widehat{x} \in S^{\perp}$.



Approximations

Consider the function $x(t) = \cos(\frac{3\pi}{2}t)$ in the Hilbert space $\mathcal{L}^2([0,1])$. Find the degree-1 polynomial closest to x. Use ideas from orthogonal projection.

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$$a_0 + a_1 t$$

Defining the approximation problem?

$$\min_{a_0, a_1} \int_0^1 \left| \cos\left(\frac{3}{2}\pi t\right) - (a_0 + a_1 t) \right|^2 dt$$

Approximations

Consider the function $x(t) = \cos(\frac{3\pi}{2}t)$ in the Hilbert space $\mathcal{L}^2([0,1])$. Find the degree-1 polynomial closest to x. Use ideas from orthogonal projection.

$$x(t) - \widehat{x}(t) \;=\; \cos\left(\frac{3}{2}\pi t\right) - (a_0 + a_1 t)$$

should be orthogonal to

the entire subspace of degree-1 polynomials

$$0 = \langle x(t) - \widehat{x}(t),\, 1\rangle = \int_0^1 \left(\cos\left(\frac{3}{2}\pi t\right) - (a_0 + a_1 t)\right) \cdot 1\, dt = -\frac{2}{3\pi} - a_0 - \frac{1}{2}a_1,$$

$$0 = \langle x(t) - \widehat{x}(t),\, t\rangle = \int_0^1 \left(\cos\left(\frac{3}{2}\pi t\right) - (a_0 + a_1 t)\right) \cdot t\, dt = \frac{4 + 6\pi}{9\pi^2} - \frac{1}{2}a_0 - \frac{1}{3}a_1.$$

$$a_0 \;=\; \frac{8 + 4\pi}{3\pi^2}, \qquad a_1 \;=\; -\frac{16 + 12\pi}{3\pi^2}.$$

Approximations

Consider the function $x(t) = \cos(\frac{3\pi}{2}t)$ in the Hilbert space $\mathcal{L}^2([0,1])$. Find the degree-1 polynomial closest to x. Use ideas from orthogonal projection.

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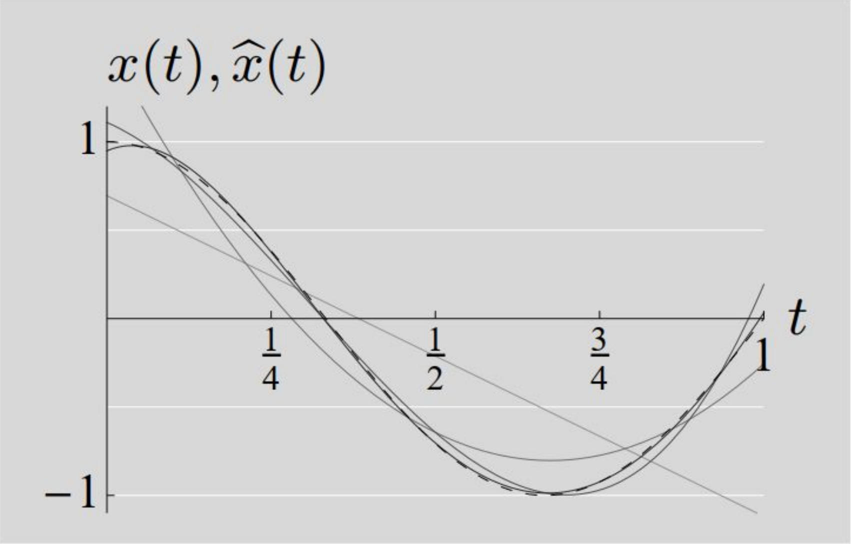

(b) $K = 1, 2, 3, 4.$

Bases

DEFINITION 2.34 (BASIS) The set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is finite or countably infinite, is called a *basis* for a normed vector space V when

(i) it is *complete* in V, meaning that, for any $x \in V$, there is a sequence $\alpha \in \mathbb{C}^{\mathcal{K}}$ such that
$$x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k; \tag{2.87}$$
and

(ii) for any $x \in V$, the sequence α that satisfies (2.87) is unique.

Riesz Bases

DEFINITION 2.35 (RIESZ BASIS) The set of vectors $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$, where $\mathcal{K}$ is finite or countably infinite, is called a *Riesz basis* for a Hilbert space H when

(i) it is a *basis* for H; and

(ii) there exist *stability constants* $\lambda_{\min}$ and $\lambda_{\max}$ satisfying $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that, for any x in H, the expansion of x with respect to the basis Φ, $x = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k$, satisfies

$$\lambda_{\min} \|x\|^2 \leq \sum_{k \in \mathcal{K}} |\alpha_k|^2 \leq \lambda_{\max} \|x\|^2. \tag{2.89}$$

The largest such $\lambda_{\min}$ and smallest such $\lambda_{\max}$ are called *optimal stability constants* for Φ.

Synthesis operator

DEFINITION 2.36 (BASIS SYNTHESIS OPERATOR) Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space H, the *synthesis operator* associated with it is

$$\Phi : \ell^2(\mathcal{K}) \to H, \qquad \text{with} \qquad \Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k. \tag{2.90}$$

Synthesis operator

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Analysis operator

DEFINITION 2.37 (BASIS ANALYSIS OPERATOR) Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for a Hilbert space H, the *analysis operator* associated with it is

$$\Phi^* : H \to \ell^2(\mathcal{K}), \qquad \text{with} \qquad (\Phi^* x)_k = \langle x, \varphi_k\rangle, \quad k \in \mathcal{K}. \tag{2.91}$$

Analysis - Synthesis

THEOREM 2.39 (ORTHONORMAL BASIS EXPANSIONS) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for a Hilbert space H. The unique expansion with respect to Φ of any x in H has expansion coefficients

$$\alpha_k = \langle x, \varphi_k \rangle \qquad \text{for } k \in \mathcal{K}, \quad \text{or,} \tag{2.93a}$$

$$\alpha = \Phi^* x. \tag{2.93b}$$

Synthesis with these coefficients yields

$$x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k \rangle \varphi_k \tag{2.94a}$$

$$= \Phi\alpha = \Phi\Phi^* x. \tag{2.94b}$$

Parseval equalities

THEOREM 2.40 (PARSEVAL EQUALITIES) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for a Hilbert space H. Expansion with coefficients (2.93) satisfies the *Parseval equality*,

$$\begin{aligned} \|x\|^2 &= \sum_{k\in\mathcal{K}} |\langle x, \varphi_k\rangle|^2 && (2.96a) \\ &= \|\Phi^* x\|^2 = \|\alpha\|^2, && (2.96b) \end{aligned}$$

and the *generalized Parseval equality*,

$$\begin{aligned} \langle x, y\rangle &= \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \langle y, \varphi_k\rangle^* && (2.97a) \\ &= \langle \Phi^* x, \Phi^* y\rangle = \langle \alpha, \beta\rangle. && (2.97b) \end{aligned}$$

Approximation of functions on finite intervals by polynomials

Let x be a real-valued function in $\mathcal{L}^2([a, b])$. An approximation $\widehat{x}$ that minimizes

$$\|x - \widehat{x}\|_2^2 = \int_a^b (x(t) - \widehat{x}(t))^2 \, dt$$

orthonormal basis

$$\boxed{p_K(t) = \sum_{k=0}^{K} \langle x, \varphi_k \rangle \varphi_k(t)}$$

analysis

synthesis

Legendre polynomials

$$L_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k}(t^2-1)^k, \qquad k \in \mathbb{N}, \quad \text{are orthogonal on } [-1,1]$$

$$\begin{aligned} L_0(t) &= 1, & L_3(t) &= \tfrac{1}{2}(5t^3-3t), \\ L_1(t) &= t, & L_4(t) &= \tfrac{1}{8}(35t^4-30t^2+3), \\ L_2(t) &= \tfrac{1}{2}(3t^2-1), & L_5(t) &= \tfrac{1}{8}(63t^5-70t^3+15t). \end{aligned}$$