Practice problems-7

Notations: $S = Fea(P) = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, \text{ for } i = 1, 2, .., m\}$,
$D_{\mathbf{x}^*} = \{\mathbf{d} \in \mathbb{R}^n : g_i(\mathbf{x}^* + t\mathbf{d}) \leq 0, \text{ for all } i = 1, 2, .., m, \text{and for all } 0 \leq t \leq c_d, c_d > 0\}$,
$F_{0,x^*} = \{\mathbf{d} \in \mathbb{R}^n : \nabla f(\mathbf{x}^*)\mathbf{d} < 0\}$, $I_{\mathbf{x}^*} = \{i \in \{1, \ldots, m\} : g_i(\mathbf{x}^*) = 0\}$, $G_{0,\mathbf{x}^*} = \{\mathbf{d} \in \mathbb{R}^n : \nabla g_i(\mathbf{x}^*)\mathbf{d} < 0 \text{ for all } i \in I\}$.

1. Consider the problem of minimizing
   $f(x_1, x_2) = -x_1x_2 + x_1^2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}$ over R^2.
   (a) Write the first order necessary optimality condition for this problem. Is this condition also sufficient for optimality?
   (b) Is $[0, 0]^T$ a local minimum point for this problem? If not find a direction $\mathbf{d}$ along which the function f will decrease.

2. Consider the following problem:
   Minimize $4x_1^2 - x_2^2 + 8x_1x_2$
   subject to
   $2x_1 + x_1^2 - x_2 \geq 0$
   $x_1 \geq 0,\ x_2 \geq 0.$
   (a) Does $[0, 0]^T$ satisfy the first and the second order necessary conditions for a local minimum?
   (b) Find all points $\mathbf{x}^*$ in the feasible region at which $G_{0,\mathbf{x}^*} = \phi$.

3. If $\mathbf{x}^* \in S$ is such that for all $\mathbf{d} \in D_{\mathbf{x}^*} (\neq \phi)$, $\nabla f(\mathbf{x}^*)\mathbf{d} > 0$ then does it imply that $\mathbf{x}^*$ is a local minimizer of f? If yes justify, if no then give an example to justify your claim.

4. Consider the following problem:
   Minimize $-x_1^2 - 4x_1x_2 - x_2^2$
   subject to
   $x_2^2 + x_1^2 = 1.$
   (a) If possible find an FJ point which is not a KKT point.
   (b) If possible find a KKT point which is not an optimal point.
   (c) Are the first and the second order necessary conditions for a local minimum satisfied at the KKT point/s?
   (d) If the objective function is changed to $2x_1^2 - x_1x_2 + x_2^2 - x_2$ and if $S = Fea(P) = \{(x_1, x_2) : x_2^2 + x_1^2 \leq 1\}$ then find all optimum solutions to this problem.

5. Minimize $(x_1 - 2)^2 + (x_2 - 3)^2$
   subject to
   $x_1^2 + x_2^2 \leq 5.$
   $2x_1 + x_2 \leq 4.$
   $-x_1 \leq 0$
   $-x_2 \leq 0.$
   (a) Check whether all the g_i's and f are convex functions.

(b) Check that $\mathbf{x}^* = [1, 2]^T$ is a KKT point (done in class).

(c) Is $\mathbf{x}^*$ a global minimum of f over this feasible region?

(d) Does there exist an $\mathbf{x}^* \in Fea(P)$ such that $G_{0,\mathbf{x}^*} = D_{\mathbf{x}^*}$?

(e) Does there exist an $\mathbf{x}^* \in Fea(P)$ such that $G_{0,\mathbf{x}^*} \neq D_{\mathbf{x}^*}$?

6. For a nonlinear programming problem (P) of the form,

$$\begin{aligned}&\text{Minimize } f(\mathbf{x})\\&\text{subject to } g_i(\mathbf{x}) \leq \mathbf{0}, \text{ for } i = 1, \ldots, m, \quad \mathbf{x} \in \mathbb{R}^n,\end{aligned}$$

where all the g_i's and f are continuously differentiable throughout $\mathbb{R}^n$, check the correctness of the following statements with proper justification.

(a) If $\mathbf{x}^* \in Fea(P)$ is a KKT point of the above problem then $-\nabla f(\mathbf{x}^*)$ lies in the cone generated by $\nabla g_i(\mathbf{x}^*), i \in I$, where I gives the indices of the binding constraints (given by g_i's) at $\mathbf{x}^*$.

(b) If $g_2 = -g_3$ then all points $\mathbf{x} \in Fea(P)$ are FJ (Fritz John) points.

(c) If $\nabla f(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in Fea(P)$ then $\mathbf{x}^*$ is a KKT point.

(d) If $\mathbf{x}^*$ is an FJ point and there is a solution to the FJ conditions at $\mathbf{x}^*$ with $u_0 = 0$ (u_0 is the coefficient of $\nabla f(\mathbf{x}^*)$ in the FJ conditions), then $\mathbf{x}^*$ is not a KKT point.

(e) If $\mathbf{x}^*$ is a KKT point then it is also an FJ point.

(f) If $\mathbf{x}^*$ is an FJ point and $\nabla g_i(\mathbf{x}^*)$'s are LD, then $G_{0,\mathbf{x}^*} = \phi$ (or in other words there exists a solution to the FJ conditions with $u_0 = 0$).

(g) If $\nabla g_i(\mathbf{x}^*)$'s are LI, then $G_{0,\mathbf{x}^*} \neq \phi$.

(h) If $G_{0,\mathbf{x}^*} = \phi$ then $\nabla g_i(\mathbf{x}^*)$'s are LD.

(i) If $\mathbf{x}^*$ is not an interior point of the feasible region S of (P) and $F_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is a local minimizer.

(j) If all the g_i's and f are convex functions and $\mathbf{x}^*$ is such that $F_{0,\mathbf{x}^*} \cap G_{0,\mathbf{x}^*} = \phi$, then $\mathbf{x}^*$ is a global minimum of f in S.

(k) If $\mathbf{x}^*$ is a local minima of (P) with $F_{0,\mathbf{x}^*} \neq \phi$ but $G_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is not a KKT point.

(l) For all $\mathbf{x}^* \in S$, $G_{0,\mathbf{x}^*} = D_{\mathbf{x}^*}$.

(m) There exists a (P) such that $G_{0,\mathbf{x}^*} \neq D_{\mathbf{x}^*}$, for all $\mathbf{x}^* \in S$.

(n) There exists a (P) with $Fea(P) \neq \phi$ such that $G_{0,\mathbf{x}^*} = \phi$ for all $\mathbf{x}^* \in S$.

(o) There exists a (P) with $Fea(P) \neq \phi$ such that $D_{\mathbf{x}^*} = \phi$ for all $\mathbf{x}^* \in S$.

7. Give examples of nonconstant functions on $\mathbb{R}^n$ which are both convex and concave and those which are neither convex nor concave.

8. For a linear programming problem (P) of the form, Minimize $\mathbf{c}^T\mathbf{x}$ subject to $A_{m \times n}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, find the KKT conditions.

9. Consider the linear programming problem.
   Minimize $2x_1 - 3x_2$
   subject to $x_1 + 2x_2 \leq 3$
   $2x_1 + 3x_2 \leq 5$
   $x_1 \geq 0, x_2 \geq 0.$

   Find the KKT conditions for this problem at a local minimum point of this problem. Solve the KKT conditions for u_i's. Hence find an optimal solution of the dual.

10. Minimize $-x_1$
    subject to
    $-(1 - x_1)^3 + x_2 = 0$
    $-(1 - x_1)^3 - x_2 = 0.$
    Check whether $[1, 0]^T$ is a KKT point of the above problem. How many feasible points does this problem have? What is your conclusion?