# Switching Algebra

Dr. Chandan Karfa

CSE, IIT Guwahati

- Source:

  Chapter 3 of Z. Kohavi and N. Jha, Switching and Finite Automata Theory, 3rd Ed., Cambridge University Press, 2010

# Introduction

- The analysis and design of combinational switching circuits.
- A *combinational switching circuit* is that its outputs are functions of only the present circuit inputs.
- Switching algebra is the existence of a two-valued switching variable that can take either of two distinct values, 0 and 1
- if $x$ is a switching variable then
  - $x$ != 0 if and only if $x$ = 1,
  - $x$ != 1 if and only if $x$ = 0.

# *Switching algebra*

- A *switching algebra* is an algebraic system consisting of the set *{*0, 1*}*, two binary operations called OR and AND, denoted by the symbols + and · respectively, and one unary operation called NOT, denoted by a prime also known as complement and a set of postulates.

| AND | | | OR | | | NOT | |
|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $x \cdot y$ | $x$ | $y$ | $x + y$ | $x$ | $x'$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | | |
| 1 | 1 | 1 | 1 | 1 | 1 | | |

# Applications in Combinational Circuits

- Combinational circuit minimization/simplification
- Equivalence of Combinational circuits
- Conversion of one form to another

# Basic properties of *switching algebra*

$$x + x = x, \quad x \cdot x = x \qquad (idempotency).$$

$$x + y = y + x, \quad x \cdot y = y \cdot x \qquad (commutativity).$$

$$(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad (associativity).$$

$$x + x' = 1, \quad x \cdot x' = 0 \qquad (complementation).$$

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad x + y \cdot z = (x + y) \cdot (x + z) \qquad (distributivity).$$

- Switching algebra is known as **the *principle of duality*.**
- One statement can be obtained from the other by interchanging the OR and AND operations and replacing the constants 0 and 1 by 1 and 0, respectively.

# Switching expressions and its Simplification

- A *switching expression* we mean the combination of a finite number of switching variables and constants (0, 1) by means of switching operations (+, ·, and )
- any switching constant or variable is
- a switching expression, and if $T1$ and $T2$ are switching expressions then so are $T1.T2$, $T1 + T2$, and $T1T2$.

# Simplification Rules

- *absorption law* of switching algebra

$$x + xy = x, \qquad x(x + y) = x \qquad (\textit{absorption}).$$

- Simplification

$$x + x'y = x + y,$$
$$x(x' + y) = xy.$$

- Consensus theorem

$$xy + x'z + yz = xy + x'z, \qquad (x + y)(x' + z)(y + z) = (x + y)(x' + z) \qquad (\textit{consensus theorem}).$$

# Simplification Example

$$
\begin{aligned}
x'y'z + yz + xz &= z(x'y' + y + x) \\
&= z(x' + y + x) \\
&= z(y + 1) \\
&= z1 \\
&= z.
\end{aligned}
$$

# De Morgan's theorems

- Governing complementation operations

$$(x')' = x \qquad (involution).$$

- De Morgan's theorems for two variables are

$$(x + y)' = x' \cdot y',$$
$$(x \cdot y)' = x' + y'.$$

| $x$ | $y$ | $x'$ | $y'$ | $x + y$ | $(x + y)'$ | $x'y'$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 |

- *General De Morgan's theorem:* The complement of any expression can be obtained by replacing each variable and element with its complement and, at the same time, interchanging the OR and AND operations

$$[f(x_1, x_2, \ldots, x_n, 0, 1, +, \cdot)]' = f(x_1', x_2', \ldots, x_n', 1, 0, \cdot, +).$$

# Switching functions

- A *switching function f* ($x1$, $x2$, . . . , $xn$) is a correspondence that associates an element of the algebra with each of the $2^n$combinations of variables $x1$, $x2$, . . . , $xn$.
- This correspondence is best specified by means of a truth table.

$x'z + xz' + x'y'$

| $x$ | $y$ | $z$ | $T$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

# Switching functions

- **Complement of a function**: If a function $f(x1, x2, \ldots, xn)$ is specified by means of a truth table, its complement is obtained by complementing each entry in the column headed $f$. New functions that are equal to the sum $f + g$ and the product $fg$ are obtained by adding or multiplying the corresponding entries in the $f$ and $g$ columns.

| $x$ | $y$ | $z$ | $f$ | $g$ | $f'$ | $f+g$ | $fg$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |

# Canonical forms: Sum of Products

- How to represent a function uniquely?
- Minterm: A product term that contains each of the *n* variables as factors in either complemented or uncomplemented form is called a *minterm*
- *$2^n$ possible minterms f*
- Sum of products (SoP): The sum of all minterms derived from those rows for which the value of the function is 1.
- The SoP is the canonical representation of function

| Decimal code | $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

# Canonical forms: Product of Sums

- A sum term that contains each of the *n* variables in either a complemented or an uncomplemented form is called a *maxterm*.
- An expression formed of the product of all maxterms for which the function takes on the value 0 is called a *canonical product of sums* or *conjunctive normal* expression
- In each maxterm, a variable *xi* appears in uncomplemented form if it has the value 0 in the corresponding row in the truth table, and it appears in complemented form if it has the value 1.

| Decimal code | $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

# Canonical forms

| Decimal code | $x$ | $y$ | $z$ | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 |

Sum of Products

$$f(x, y, z) = x'y'z' + x'yz' + x'yz + xyz' + xyz.$$

$$f(x, y, z) = \sum(0, 2, 3, 6, 7)$$

Product of Sums

$$f(x, y, z) = (x + y + z')(x' + y + z)(x' + y + z').$$

$$f(x, y, z) = \prod(1, 4, 5),$$

# Converting to Canonical Form using Shannon's Expansion Theorem

- One way of obtaining the canonical forms of any switching function is by means of *Shannon's expansion theorem repeatedly.*

$$f(x_1, x_2, \ldots, x_n) = x_1 \cdot f(1, x_2, \ldots, x_n) + x_1' \cdot f(0, x_2, \ldots, x_n)$$

$$f(x_1, x_2, \ldots, x_n) = [x_1 + f(0, x_2, \ldots, x_n)] \cdot [x_1' + f(1, x_2, \ldots, x_n)].$$

- Now apply the expansion theorem with respect to variable *x*2 to each of the two terms

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = {} & x_1 x_2 f(1, 1, x_3, \ldots, x_n) + x_1 x_2' f(1, 0, x_3, \ldots, x_n) \\ & + x_1' x_2 f(0, 1, x_3, \ldots, x_n) + x_1' x_2' f(0, 0, x_3, \ldots, x_n). \end{aligned}$$

- The expansion of the function about the remaining variables yields the disjunctive normal form (SoP)

# Converting to SoP form

- A simpler and faster procedure for obtaining the canonical sum-of-products form of a switching function.
  - Examine each term; if it is a minterm, retain it, and continue to the next term.
  - In each product that is not a minterm, check the variables that do not occur, for each $x_i$ that does not occur, multiply the product by $(x_i + x'_i)$.
  - Multiply out all products and eliminate redundant terms.

**Example** Determine the canonical sum-of-products form for $T(x, y, z) = x'y + z' + xyz$. Applying rules 1–3, we obtain

$$\begin{aligned} T &= x'y + z' + xyz \\ &= x'y(z + z') + (x + x')(y + y')z' + xyz \\ &= x'yz + x'yz' + xyz' + xy'z' + x'yz' + x'y'z' + xyz \\ &= x'yz + x'yz' + xyz' + xy'z' + x'y'z' + xyz. \end{aligned}$$

# How to convert SoP to PoS and vice versa?

- Option 1: Using Truth Table method
- Option 2: Use involution theorem $(x')' = x$

**Example** Find the canonical product-of-sums form for the function

$$T(x, y, z) = x'y'z' + x'y'z + x'yz + xyz + xy'z + xy'z'.$$

Using the involution theorem,

$$T = (T')' = [(x'y'z' + x'y'z + x'yz + xyz + xy'z + xy'z')']'.$$

The complement $T'$ consists of those minterms that are not contained in the expression for $T$, i.e.,

$$\begin{aligned} T &= [x'yz' + xyz']' \\ &= (x + y' + z)(x' + y' + z). \end{aligned}$$

# Functional properties

- Two switching functions are **equivalent** if and only if their canonical sum of products forms are identical.
- A factor $a_i$ is set to 1 (0) if the corresponding minterm is (is not) contained in the canonical form of the function.

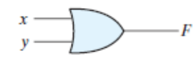
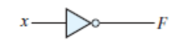
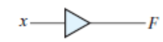
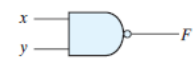
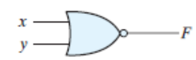
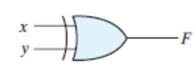
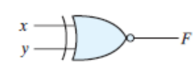
$$f(x_1, x_2, \ldots, x_n) = a_0 x_1' x_2' \cdots x_n' + a_1 x_1' x_2' \cdots x_n + \cdots + a_r x_1 x_2 \cdots x_n.$$

- There are $2^n$ coefficients, each of which can have two values, 0 and 1. Hence, thus *there exist* 2^($2^n$) *switching functions of n variables.*
- *Functions with 2 variables are of our interests*

$$f(x, y) = a_0 x' y' + a_1 x' y + a_2 x y' + a_3 x y.$$

**Table 3.6** List of switching functions $f(x, y)$ of two variables, $x$ and $y$

| $a_3$ | $a_2$ | $a_1$ | $a_0$ | $f(x, y)$ | Name of function | Symbol |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $0$ | Inconsistency | |
| 0 | 0 | 0 | 1 | $x'y'$ | NOR | $x \downarrow y$[a] |
| 0 | 0 | 1 | 0 | $x'y$ | | |
| 0 | 0 | 1 | 1 | $x'$ | NOT | $x'$ |
| 0 | 1 | 0 | 0 | $xy'$ | | |
| 0 | 1 | 0 | 1 | $y'$ | | |
| 0 | 1 | 1 | 0 | $x'y + xy'$ | EXCLUSIVE-OR (modulo-2 addition) | $x \oplus y$ |
| 0 | 1 | 1 | 1 | $x' + y'$ | NAND | $x \mid y$[b] |
| 1 | 0 | 0 | 0 | $xy$ | AND | $x \cdot y$ |
| 1 | 0 | 0 | 1 | $xy + x'y'$ | Equivalence | $x \equiv y$ |
| 1 | 0 | 1 | 0 | $y$ | | |
| 1 | 0 | 1 | 1 | $x' + y$ | Implication | $x \rightarrow y$ |
| 1 | 1 | 0 | 0 | $x$ | | |
| 1 | 1 | 0 | 1 | $x + y'$ | Implication | $y \rightarrow x$ |
| 1 | 1 | 1 | 0 | $x + y$ | OR | $x + y$ |
| 1 | 1 | 1 | 1 | $1$ | Tautology | |

| Name | Graphic symbol | Algebraic function | Truth table |
|---|---|---|---|
| AND |  | $F = x \cdot y$ | x y F<br>0 0 0<br>0 1 0<br>1 0 0<br>1 1 1 |
| OR |  | $F = x + y$ | x y F<br>0 0 0<br>0 1 1<br>1 0 1<br>1 1 1 |
| Inverter |  | $F = x'$ | x F<br>0 1<br>1 0 |
| Buffer |  | $F = x$ | x F<br>0 0<br>1 1 |
| NAND |  | $F = (xy)'$ | x y F<br>0 0 1<br>0 1 1<br>1 0 1<br>1 1 0 |
| NOR |  | $F = (x + y)'$ | x y F<br>0 0 1<br>0 1 0<br>1 0 0<br>1 1 0 |
| Exclusive-OR (XOR) |  | $F = xy' + x'y = x \oplus y$ | x y F<br>0 0 0<br>0 1 1<br>1 0 1<br>1 1 0 |
| Exclusive-NOR or equivalence |  | $F = xy + x'y' = (x \oplus y)'$ | x y F<br>0 0 1<br>0 1 0<br>1 0 0<br>1 1 1 |

# XOR

- It assigns value 1 to two arguments if and only if they have complementary values; that is, $A \oplus B = 1$ if either $A$ or $B$ is 1 but not when both $A$ and $B$ are 1.

$$
\begin{aligned}
A \oplus B &= B \oplus A && (\textit{commutativity}),\\
(A \oplus B) \oplus C &= A \oplus (B \oplus C) && \\
&= A \oplus B \oplus C && (\textit{associativity}),\\
(AB) \oplus (AC) &= A(B \oplus C) && (\textit{distributivity}).
\end{aligned}
$$

$$
\text{if } A \oplus B = C \text{ then } \begin{cases} A \oplus C = B, \\ B \oplus C = A, \\ A \oplus B \oplus C = 0. \end{cases}
$$

# Functionally complete operations

- A set of operations is said to be *functionally complete* (or *universal*) if and only if every switching function can be expressed entirely by means of operations from this set.
- Every switching function can be expressed in a canonical sum-of-products form, where each expression consists of a finite number of switching variables
- *{+, ·, ’}* is clearly functionally complete.
- Other functionally complete sets: *{+, ’}, {., ‘}, {NAND}, {NOR}*

**Example** Prove that the NOR operation is functionally complete.

A common method for proving the completeness of an operation is to show that it is capable of generating each operation of a set that is already known to be functionally complete, for example, $\{+, '\}$ or $\{\cdot, '\}$.

Since $x \downarrow y = x'y'$ (see Table 3.6), then

$$x \downarrow x = x'x' = x',$$
$$(x \downarrow y) \downarrow (x \downarrow y) = (x'y')' = x + y.$$

# Boolean algebras

- A Boolean algebra *B* is a set of elements *a*, *b*, *c, . . . ,* together with two binary operations, + and ·, that satisfy the **idempotent, commutative, absorption, and associative laws and are mutually distributive.**
- *B* contains two bounds, 0 and 1, which are the **least and greatest elements**, respectively; B **is closed under** + and . .
- The A set *S* is closed with respect to a binary operator if, for every pair of elements of *S*, the binary operator specifies a rule for obtaining a unique element of *S*.
- B has a unary operation of complementation that assigns to every element its complement. *The **complement a of any element a in B is unique**.*

# Basic properties of *switching algebra*

$$x + x = x, \quad x \cdot x = x \qquad (idempotency).$$

$$x + y = y + x, \quad x \cdot y = y \cdot x \qquad (commutativity).$$

$$(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad (associativity).$$

$$x + x' = 1, \quad x \cdot x' = 0 \qquad (complementation).$$

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad x + y \cdot z = (x + y) \cdot (x + z) \qquad (distributivity).$$

- Switching algebra is known as **the *principle of duality*.**
- One statement can be obtained from the other by interchanging the OR and AND operations and replacing the constants 0 and 1 by 1 and 0, respectively.

Relation between switching algebra and Boolean algebra?

# Boolean algebras

- In 1854, George Boole developed an algebraic system now called *Boolean algebra*
- In 1938, Claude E. Shannon introduced a two-valued Boolean algebra called *switching algebra*
- The **switching algebra is two-valued Boolean algebra.**
  - **Closed under + and . . And each element has unique complement element.**

| $+$ | $0$ | $1$ | $\cdot$ | $0$ | $1$ | |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0' = 1$ |
| $1$ | $1$ | $1$ | $1$ | $0$ | $1$ | $1' = 0$ |