Multi-level logic Minimization – Algebraic Method

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Text Book

- Chapter 6, Z. Kohavi and N. Jha, Switching and Finite Automata Theory, 3rd Ed., Cambridge University Press, 2010.

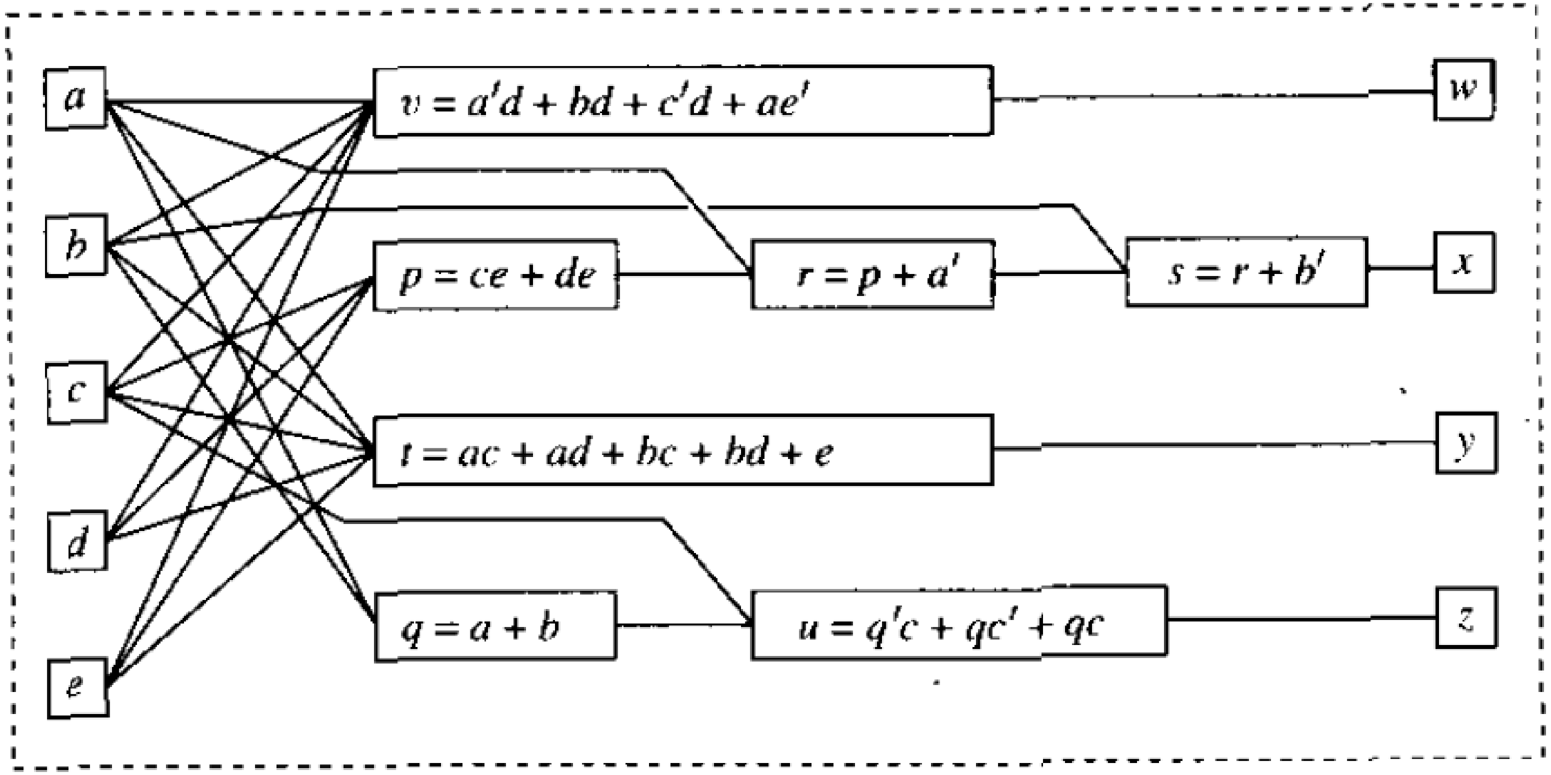
Multi-level Logic Optimization

- Input:
  - Set of Boolean expressions
- Objectives:
  - Find and eliminate common sub-expressions among all expressions
  - Factoring is the Key
- Output:
  - Set of optimized Boolean expressions

Q: How to perform factoring efficiently?

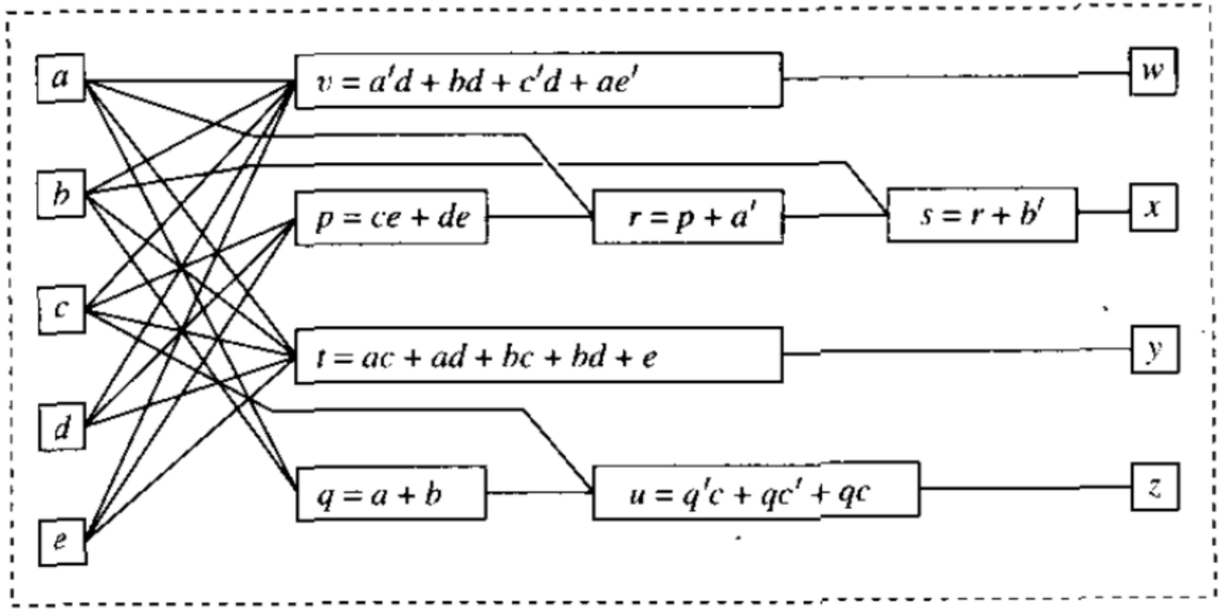
An Example

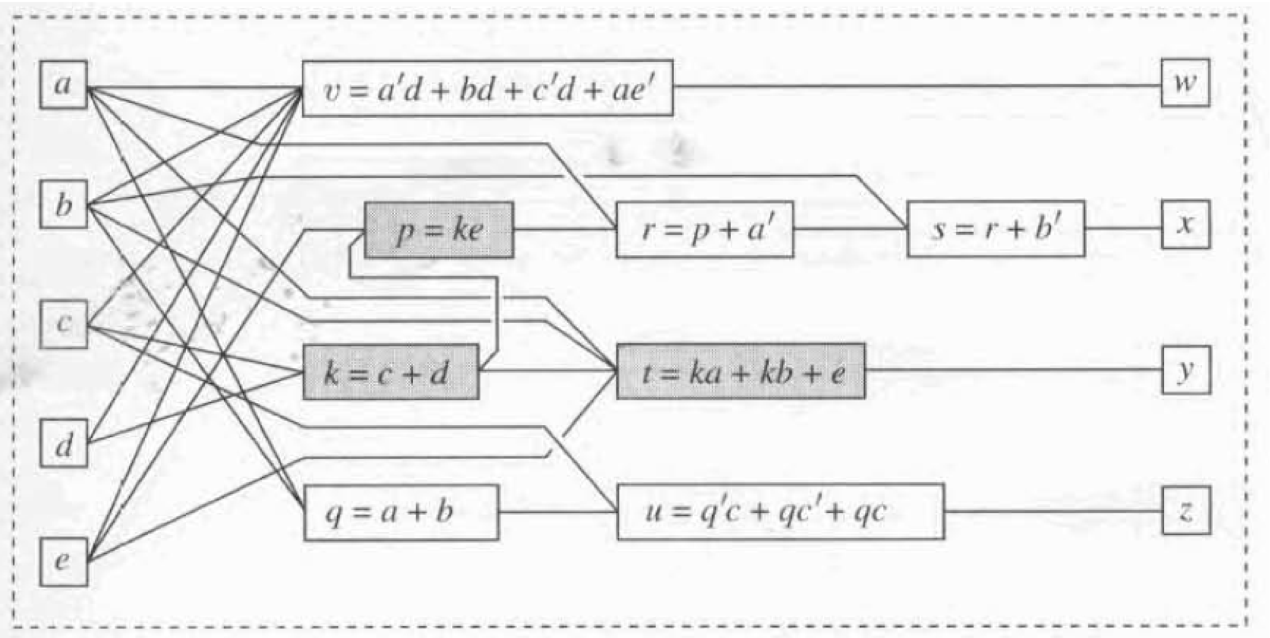
- *p = ce + de*
- *q = a+b*
- *r = p+a’*
- s = *r+b'*
- *t = ac+ad+bc+bd+e*
- *u = q'c+qc'+qc*
- v = *a'd + bd + c'd + ae'*
- *w=v*
- x=s
- y=t
- *z=u*



An Example

- p = (c + d) e
- t = (c + d) (a + b) + e





Multilevel Logic Optimization: The Algebraic Model

- Boolean functions are represented by algebraic expressions.
- Apply algebraic transformation.
- Apply rule of polynomial algebra.
  - Ex: Apply distributive law:
    - a.(b + c) = ab + ac
- Ignore specific feature of Boolean algebra.
  - a + (b.c) != (a + b) . (a + c)
  - No complement: e and e' are treated as two different variables.

– Why?

Algebraic Model

- By dropping some assumptions about Boolean algebra and represent as Polynomial, the logic network can be optimized by using general properties of polynomial algebra.
- Algebraic expressions modelling Boolean function are obtained by representing the function in sum of products form and making them minimal with respect to single-cube containment.
- Boolean expression ac + ad + bc + bd can be considered as algebraic expression
- aa + ac + ba + bc can not.
- aa’ + ac + ba’ + bc can not

Division

- Division plays an important role.
  - f = d.q + r ,
  - here d = Divisor, f = Dividend, q = Quotient, R = reminder
- Example:
  - f = ac + ad + bc + bd + e
  - Here, d = (a + b), q = c+d and r = e,
  - Because, f = (a+b) (c+d) + e

    = ac + ad + bc + bd + e

D F Q

R

F= DQ + R

Common Divisor/Quotient

- $f_1 = ce + de$
  $= e\,(c + d)$

- $f_2 = ac + ad + bc + bd + e$
  $= (c + d)\,.\,(a + b) + e$
- $(c + d)$ is common divisor here.
- Extract common divisor,
  - $f_3 = c + d$
  - $f_1 = e.\ f3$
  - $f_2 = f_3.\ (a + b) + e$

The Algebraic Method

- Objective: Look for common sub-expression.
- Question: Where to look for divisor for a function F ?
- Answer: In the kernel of f, K(f).

- K(f) is another set of two-level SoPs which are special, foundational structure of any function f, being interpreted in algebraic model.
- Kernels: A cube-free quotient k obtained by algebraically dividing F by a single cube C (co-kernel)

The Algebraic Method

- Question: How to find a kernel k ϵ k(F) ?
- Answer: Algebraically divide F by one of its co-kernel C.
- Kernels: A cube-free quotient k obtained by algebraically dividing F by a single cube C (co-kernel)

D | F | Q

R

F= DQ + R

C= Single cube,
abc,
ad,
efg

C | F | K

R

kernel k if multi-cube and cube-free

F= CK + R

What is cube-free ?

Kernel

- What is cube-free ?
  - You can not factor out a single cube (product term) from divider that leaves no reminder.
- Has no cube (product) that is a factor of the kernel expression
  - xy + xz = x (y+z) Not Cube free
  - wx + yz is cube free
  - xy + xy + w is Cube free no common cube can be extracted
- Do F/ single cube,
  - look at result if you can cross-out power cube in each term, not a kernel.

Algebraic kernels and Co-kernels

- The concept of kernels and co-kernels helps determine the common sub-expressions that can be extracted from switching expressions.
- If an expression cannot be factored by a cube (see Section 4.2), it is said to be cube-free.
- For example, (wx+yz) is cube-free. However (xy+xz) is not cube-free since it can be factored by x.
- For an expression to be cube-free, it must contain more than one cube.
- When an expression is divided by a cube, the result is a cube-free quotient then the quotient is called a **kernel** and the cube is the corresponding **co-kernel**.

Algebraic Kernels and Co-kernels cont'd...

- If a kernel has no kernel except itself, it is called a level-0 kernel.
- If a kernel has at least one kernel of level n−1 but no kernel of level nor greater except itself, it is called a level-n kernel. A co-kernel has the same level as its kernel.
- Consider the expression f = (uwz+uxz+vwz+vxz+yz+uv). Its kernels and co-kernels and their levels are shown in Table 6.1

Table 6.1 Kernels and their co-kernels

Level	Kernel	Co-kernel
0	$u + v$	wz, xz
0	$w + x$	uz, vz
1	$wz + xz + v$	u
1	$wz + xz + u$	v
1	$uw + ux + vw + vx + y$	z
2	$uwz + uxz + vwz + vxz + yz + uv$	1

Find all Kernels

- f = abc + abd + bcd
- Find all kernels

Divider cube d	F = d.Q + k	Is Q a kernel of F
1	(abc + abd + bcd) + 0	No, here cube b as factor.
a	a(bc + bd) + bcd	No. b is a factor
b	b (ac + ad + cd) + 0	Yes, kernel (ac + ad + cd) is cube-free
ab	ab (c + d) + bcd	Yes, kernel (c + d) is cube-free

Brayton and McMullen Theorem

- Kernels: Why are they important ?
  - If they are important, how do we actually compute them?

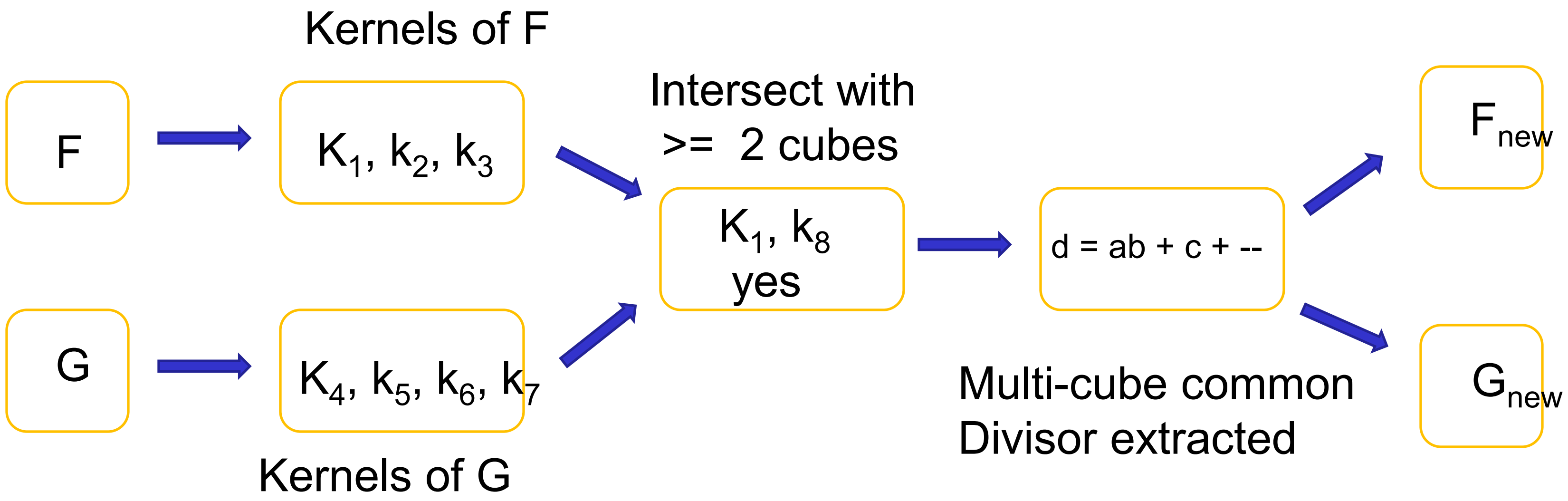
- Brayton and McMullen Theorem:
  - Expression 'F' and G have multi-cube common divisor d if and only if,
  1. There are kernels $K_1 \in K(F)$, $K_2 \in K(G)$ such that $d = k_1 \cap K_2$ (i.e. SOP form with common cubes exist)
  2. d is an expression with at least 2 cube in it.

Interpretation:

- To find common divisor of two expressions, the only place to look for is in the intersection of kernels.
- This intersection of kernels is the divisor, there is no other.

The Algebraic Method

1. Find kernels of F and G
2. Find kernels in intersections of K(F) and K(G)
3. Extract multi-cube common divisor D
4. Rewrite F and G using D



The Algebraic Method

F = cube1 * kernel1 + reminder1
G = cube2 * kernel2 + reminder2

⬇

F = cube1 * (X + Y + stuff1) + reminder1
G = cube2 * (X + Y + stuff2) + reminder2

⬇

F = cube1 * (X + Y) + ((cube1 * stuff1) + reminder1)
G = cube2 * (X + Y) + ((cube2 * stuff2) + reminder2)

⬇

Kernal1 ∩ kernel2 = [X + Y] = a multi-cube divisor of F and G
D = X + Y
F = cube1 * D + ((cube1 * stuff1) + reminder1)
G = cube2 * D + ((cube2 * stuff2) + reminder2)

Example

- F = ae + be + cde + ab
- G = ad + ae + be + bc

K(F)	Co-kernal	K(G)	Co-kernal
a + b + cd	e	a + b	e
b + c	a	d + e	a
a + e	b	d + e + c	b
ac + be + cde + ab	1	ab + ac + be + bc	1

(a + b + cd) ∩ (a + b) = (a + b)
So, this is workable multi-cube divisor of F and G.