Plan

- Dual of a Primal
- Fundamental theorem of Duality
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Dual of a Primal

- Given an LPP (P) (*)
  Max $\mathbf{c}^T\mathbf{x}$
  subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$.
- $Fea(P) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
- The **Dual** of this problem is given by (D) (**)
  Min $\mathbf{b}^T\mathbf{y}$
  subject to $A^T_{n\times m}\mathbf{y} \geq \mathbf{c}_{n\times 1}$, $\mathbf{y} \geq \mathbf{0}$.
- $Fea(D) = \{\mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$.
- The LPP(*) is called a **Primal** problem.

- **Theorem 1:** If $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.
- **Corollary 1:** If both the Primal and the Dual have feasible solutions then both have optimal solutions.
- **Corollary 2:** Let $\mathbf{x}_0 \in Fea(P)$ and $\mathbf{y}_0 \in Fea(D)$, be such that $\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{y}_0$, then $\mathbf{x}_0$ and $\mathbf{y}_0$ are optimal for the Primal and the Dual, respectively.

- **Example 1:** Consider the problem (P)
  Max $5x_1 + 2x_2$ subject to
  $3x_1 + 2x_2 \leq 6$
  $x_1 + 2x_2 \leq 4$
  $x_1 \geq 0, x_2 \geq 0.$
- The dual of this problem is given by (D),
  Min $6y_1 + 4y_2$
  subject to
  $3y_1 + y_2 \geq 5$
  $2y_1 + 2y_2 \geq 2,$
  $y_1 \geq 0, y_2 \geq 0.$
- $\mathbf{x} = [2, 0]^T$ is a feasible solution of (P) and $\mathbf{y} = [\frac{5}{3}, 0]^T$ is a feasible solution of (D)
  such that $\mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} = 10$, where $\mathbf{c} = [5, 2]^T$ and $\mathbf{b} = [6, 4]^T$.
- Hence $\mathbf{x} = [2, 0]^T$, is optimal for the **Primal (P)** and $\mathbf{y} = [1, 1]^T$ is optimal for the **Dual (D)**.

- **Theorem 2: (Fundamental Theorem of Duality)**
  If **both** the Primal (P) and the Dual (D) of the Primal (P) have feasible solutions then **both** have optimal solutions and the **optimal value** of (P) and (D) equal (that is Min $\mathbf{b}^T\mathbf{y} =$ Max $\mathbf{c}^T\mathbf{x}$). If one of them (the Primal or the Dual) does not have a feasible solution then the other does not have an optimal solution.
- To complete the proof it is enough to show that
- when both (P) and (D) have feasible solutions then their optimal values are equal.
- if any one of (P) or (D) does not have a feasible solution then the other does not have an optimal solution.
- **Theorem 3:** The Dual of the Dual (D) (of the Primal (P)) is the Primal (P).

- **Example 2:** Consider (P)
  Min $-x + 2y$
  subject to
  $x + 2y \geq 1$
  $-x + y \leq 1$
  $x \geq 0, y \geq 0.$
- (P) **does not** have an optimal solution although it has a feasible solution.
  The dual (D) of (P) is given by
  Max $u - v$
  subject to
  $u + v \leq -1$
  $2u - v \leq 2$,
  $u \geq 0, v \geq 0.$
- Clearly (D) **does not** have any feasible solution, since the first constraint cannot be satisfied by any non negative u and v.

- **Example 3:** The LPP (P)
  Max $-x + 2y$
  subject to
  $x + 2y \leq 1$
  $-x + y \geq 1,$
  $x, y \geq 0.$
- (P) **does not** have a feasible solution, hence **does not** have an optimal solution.
- The dual of (P) is given by (D)
  Min $u - v$
  subject to
  $$u + v \geq -1$$
  $$2u - v \geq 2$$
  $$u \geq 0, v \geq 0.$$
  Check that (D) has a feasible solution but **no** optimal solution.

- **Definition:** A nonempty set $T \subseteq \mathbb{R}^n$ is said to be a **cone** if $\mathbf{x} \in T$ implies $\lambda \mathbf{x} \in T$ for all $\lambda \geq 0$.
- So a cone always contains the origin.
- Also a **cone** T is said to be a **convex cone** if it is also a convex subset of $\mathbb{R}^n$.
- **Exercise:** If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are all vectors in $\mathbb{R}^n$ then check that $T = \{\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_k \mathbf{a}_k : \alpha_i \geq 0 \text{ for all } i = 1, \ldots, k\}$ is a **convex cone**.
- $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are called **generators** of the convex cone T.

Fundamental Theorem of Duality (revisited):

- **Theorem 4:** (**Farka's Lemma**) Exactly **one** of the following two systems has a solution.

$$A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$\mathbf{y}^T A \geq \mathbf{0}_{1\times n}, \quad \mathbf{y}^T\mathbf{b} < 0 \qquad (2)$$

- **Corollary 4:** Exactly **one** of the following two systems has a solution.

$$A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$\mathbf{y}^T A \geq \mathbf{0}, \quad \mathbf{y}^T\mathbf{b} < 0, \qquad \mathbf{y} \geq \mathbf{0} \qquad (2)$$

- Since $Fea(D) = \{\mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} \geq \mathbf{c}, \ \ \mathbf{y} \geq \mathbf{0}\}$, the set of all directions of $Fea(D)$ is given by
- $D_D = \{\mathbf{d} \in \mathbb{R}^m : A^T\mathbf{d} \geq \mathbf{0}, \ \ \mathbf{d} \geq \mathbf{0}\}$
- The **dual** (D) has an optimal solution if and only if either $Fea(D)$ is bounded or if not then $\mathbf{b}^T\mathbf{d} \geq 0$ for all $\mathbf{d} \in D_D$.
- From Corollary 4 we get if the Primal does **not** have a feasible solution then the dual does not have an optimal solution.
- From the **fundamental theorem of duality** we get that if any one of Primal or Dual has an optimal solution then the other also has an optimal solution.

- To complete the proof of Fundamental Theorem of Duality we need to show that if $Fea(P) \neq \phi$ and $Fea(D) \neq \phi$, then the following system (1) has a solution:

$$A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$$

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

- If $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$, hence system (1) has a solution $\Leftrightarrow$ system $(1)''$ has a solution:
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$$A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)''$$

$$A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$$

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$$

- System $(1)''$ can be written as:

$$\begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \mathbf{0}_{m\times 1} \end{bmatrix}$$

- By **corollary 4**, exactly one of the two systems, $(1'')$ (given above) and $(2)''$ (given below) has a solution.

$$\begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times m} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix} < 0 \quad \begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \geq \mathbf{0}_1$$

- We show in the proof that given $Fea(P) \neq \phi$ and $Fea(D) \neq \phi$, system $(2)''$ does not have a solution, hence system $(1'')$ has solution.
- **Theorem 5 (Complementary Slackness Theorem):** Let $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for the primal and the dual respectively if and only if

  $x_j = 0$ whenever $(A^T\mathbf{y})_j > c_j,\ j = 1, 2, .., n$ **(1)**

  and

  $y_i = 0$ whenever $(A\mathbf{x})_i < b_i,\ i = 1, 2, ..., m.$ **(1^*)**

- **Example 1:** Consider the following primal problem (P):
  Max $4x_1 + 4x_2 + 2x_3$
  subject to
  $2x_1 + 3x_2 + 4x_3 \leq 10 \qquad (i)$
  $2x_1 + x_2 + 3x_3 \leq 4 \qquad (ii)$
  $x_1, x_2, x_3 \geq 0$
  The dual (D) of the above problem is given by:
  Min $10y_1 + 4y_2$
  subject to
  $2y_1 + 2y_2 \geq 4 \qquad (i)'$
  $3y_1 + y_2 \geq 4 \qquad (ii)'$
  $4y_1 + 3y_2 \geq 2 \qquad (iii)'$
  $y_1, y_2 \geq 0$