MA-321-Optimization

Quiz-2

Time: 6:00 pm - 7:15 pm 27th October, 2022 **Total marks:** 17

Notation: BFS: Basic Feasible Solution.
LI: Linearly Independent
$\tilde{\mathbf{a}}_i$: the i-th column of A.

Answers without proper justification will be awarded zero marks

1. For a linear programming problem **P** of the form,

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } \mathbf{A}_{2\times 3}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{c} = [-2, 2, -1]^T$, the following is the optimal table corresponding to the optimal extreme point $\mathbf{x}_0$ of Fea(**P**).

$c_j - z_j$						
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$				1		2
$\mathbf{e}_2$				-2		1

(a) If a variable x_4 is added to **P** (everything else remaining same as **P**) with $c_4 = -2$ and $\tilde{\mathbf{a}}_4 = [2, -3]^T$ then check whether $[\mathbf{x}_0^T, 0]^T$ is optimal for the new problem.
Solution: $B^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$B^{-1}\tilde{\mathbf{a}}_4 = [2, -7]^T$, $c_B^T = [-2, 0]^T$ hence $z_4 = c_B^T B^{-1}\tilde{\mathbf{a}}_4 = -4$.
Since $c_4 - z_4 = -2 - (-4) = 2$, and the added variable is non basic in the new table (with the added column) which remains optimal, hence the extreme point $[\mathbf{x}_0^T, 0]^T$ is optimal for the new **P**. [**2**]

(b) If a constraint $2x_1 - 3x_2 - x_3 \leq 3$ is added to **P** (everything else remaining same as **P**) then will the new **P** have an optimal solution? If yes, then will the optimal solution of the new **P** satisfy the newly added constraint as an equality?
Solution:
Let us call the new **P** as **P'**.

Since $B^{-1}\mathbf{b} = [2, 1]^T$, we get $\mathbf{b} = [2, 5]^T$. Hence $\mathbf{0}$ is a feasible solution of **P**. Since $\mathbf{0}$ also satisfies the constraint $2x_1 - 3x_2 - x_3 \leq 3$ hence the feasible region of the **P'** is nonempty.
Also since a constraint is added to Fea(**P**),
Fea(**P'**)$\subset$ Fea(**P**), and
$\forall \mathbf{x} \in$ Fea(**P'**) , $\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}_0$,
which implies that the objective function of **P'** is bounded below, and Fea(**P'**)$\neq \phi$ implies **P'** has an optimal solution. [**1.5**]

Note that the Dual of $\mathbf{P}$ is given by:

$$\begin{aligned} &\text{Maximize } -\mathbf{b}^T\mathbf{y} \\ &\text{subject to } -\mathbf{A}^T\mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

Let the optimal solution of Dual of $\mathbf{P}$ corresponding to $\mathbf{x}_0$ be $\mathbf{y}_0$.
Note that **adding a constraint** to $\mathbf{P}$ results in **adding a variable** to the Dual of $\mathbf{P}$, say y_3 variable is added to the Dual with cost $-b_3 = -3$ (obtained from the RHS of the newly added constraint).

If suppose the optimal solution of **P'** does **not** lie on the newly added constraint then $s_3 > 0$ in the optimal BFS, which implies (by complementary slackness theorem) that the corresponding newly added dual variable $y_3 = 0$ in any optimal BFS of the Dual of **P'**.
If $\mathbf{P}$ has a **unique** optimal solution and if $\mathbf{x}'$ is an optimal solution of **P'**, then $\mathbf{c}^T\mathbf{x}_0 < \mathbf{c}^T\mathbf{x}'$, since $\mathbf{x}'$ is feasible for $\mathbf{P}$ and $\mathbf{x}_0$ is the unique optimal solution of $\mathbf{P}$.
If $\mathbf{y}'$ is an optimal solution of the Dual of **P'**, then from above,
$\mathbf{b}^T\mathbf{y}_0 > \mathbf{b}'^T\mathbf{y}'$,   **(*)**
where $\mathbf{b}' = [\mathbf{b}^T, 3]^T$ (since $\mathbf{c}^T\mathbf{x}_0 = -\mathbf{b}^T\mathbf{y}_0$ and $\mathbf{c}^T\mathbf{x}' = -\mathbf{b}'^T\mathbf{y}'$).
If in the optimal solution $\mathbf{y}'$, $(\mathbf{y}')_3 = 0$, then $\mathbf{y}' = [y_1', y_2', 0]^T$,
and $\mathbf{b}'^T\mathbf{y}' = [\mathbf{b}^T, 3]^T\mathbf{y}' = \sum_{i=1}^{2} b_i(\mathbf{y}')_i$.   **(**)**

$[y_1', y_2', 0]^T$ is feasible (optimal) for the Dual of **P'** $\Rightarrow$ $[y_1', y_2']^T$ is feasible for Dual of $\mathbf{P}$
(in the previous Dual, the variable y_3 is added).
Also $\mathbf{y}_0$ is optimal for the Dual of $\mathbf{P}$ implies,
$-\sum_{i=1}^{2} b_i(\mathbf{y}')_i \leq -\sum_{i=1}^{2} b_i(\mathbf{y}_0)_i$, or $\sum_{i=1}^{2} b_i(\mathbf{y}')_i \geq \sum_{i=1}^{2} b_i(\mathbf{y}_0)_i$ **(***)**

But **(**)** and **(***)** contradicts **(*)**.
Hence $s_3 = 0$ in the optimal BFS of P.

Aliter:

The simplex table of the Basic solution corresponding to $\mathbf{x}_0$ after the addition of the constraint to $\mathbf{P}$ is given as following:

$c_j - z_j$							
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	1			1	0	0	2
$\mathbf{e}_2$	0			-2	1	0	1
$\mathbf{e}_3$	0			-2	0	1	-1

Since the cost associated with s_3 is equal to 0, the $c_j - z_j$ values remain the same (as the optimal table of $\mathbf{x}_0$) and nonnegative.

If we try to solve **P'** by the Dual simplex method, then in the first iteration $\mathbf{e}_3$ (or s_3) leaves the basis and since there is atleast one negative entry in that row in the main table (other than $(B^{-1}\mathbf{b})_3$) some nonbasic variable enters the basis.

So in the optimal table corresponding to $\mathbf{y}_0$ a column corresponding to y_3 is added, which is a nonbasic variable in the current BFS corresponding to $[\mathbf{y}_0^T, 0]^T$ (the basic variables corresponding to $[\mathbf{y}_0^T, 0]^T$ are s_2', s_3', y_1).

s_3 leaves the basic set of variables of **P'** implies the newly added variable y_3 which was nonbasic in $[\mathbf{y}_0^T, 0]^T$ enters as a basic variable of the Dual of **P'** in the next iteration, and the $c_j - z_j$ value corresponding to y_3 is $-(B^{-1}\mathbf{b}')_3 = 1$.
If **P** has a **unique** optimal solution then all the $c_j - z_j$ values corresponding to the **non basic variables** of the (given) optimal table are nonzero.
Hence the BFS of the Dual of **P'** corresponding to $[\mathbf{y}_0^T, 0]^T$ is **non degenerate**, which implies that y_3 enters the basis in the first iteration with a **strictly positive** value.
Hence if $\mathbf{y}'$ corresponds to the new BFS after the first iteration then
$-[\mathbf{b}^T, 3]\mathbf{y}' > -[\mathbf{b}^T, 3]\begin{bmatrix} \mathbf{y}_0 \\ 0 \end{bmatrix}$ **(**)**
(Recall the similar formula $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x}_0 + (c_s - z_s)x_s'$ where $x_s' = \frac{x_r}{u_{rs}}$).
Suppose the Dual for the **P'** has an optimal solution with $y_3 = 0$, let that be $\begin{bmatrix} \mathbf{y}'' \\ 0 \end{bmatrix}$.
Then $\mathbf{y}''$ is feasible for the Dual of **P**, and from **(**)**
$-[\mathbf{b}^T, 3]\begin{bmatrix} \mathbf{y}'' \\ 0 \end{bmatrix} > -[\mathbf{b}^T, 3]\begin{bmatrix} \mathbf{y}_0 \\ 0 \end{bmatrix}$
$\Rightarrow -\mathbf{b}^T\mathbf{y}'' > -\mathbf{b}^T\mathbf{y}_0$ which contradicts that $\mathbf{y}_0$ is optimal for the old Dual.
Hence for any optimal solution of Dual of **P'**,
$y_3 > 0$, which implies that for any optimal solution of **P'**, $s_3 = 0$.

However if **P** has **multiple** optimal solutions then every optimal solution of **P** may not lie on the added constraint but atleast one optimal solution will lie on the newly added constraint (in this case $[\mathbf{y}_0^T, 0]^T$ may be optimal for Dual of **P'**)
.

(c) If $\mathbf{b}$ is changed to $\mathbf{b}' = \mathbf{b} + [2, 2]^T$ (everything else remaining same as **P**), then is $\mathbf{x}_0$ optimal for the new **P**?
In the optimal solution of the Dual of the new **P** can y_2 (the Dual variable corresponding to the second constraint of new **P**) be equal to 0 ?
Solution:
Let us call the new **P** as **P'**.

If $\mathbf{b}$ is changed to $\mathbf{b}' = [4, 7]^T$ then $\mathbf{x}_0$ is **feasible** for **P**.
Also since $B^{-1}\tilde{\mathbf{a}}_1 = [1, -2]^T$, $B^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $\tilde{\mathbf{a}}_1 = [1, 2]^T$.
If we consider $\mathbf{x}' = [3, 0, 0]^T$, then $\mathbf{x}'$ satisfies $3[1, 2]^T \leq [4, 7]^T$ and $\mathbf{c}^T\mathbf{x}' = -6 < \mathbf{c}^T\mathbf{x}_0 = -4$, hence $\mathbf{x}_0$ is **not** optimal for **P'**. **[1.5]**

Those of you who have concluded the the **basic solution** corresponding to $\mathbf{x}_0$ is **not** feasible, hence **not** optimal for **P'** by checking that $B^{-1}\mathbf{b}' = [4, -1]^T \ngeq \mathbf{0}$, also got credit for their answers.

The table of the **basic solution** corresponding to $\mathbf{x}_0$ (that is with the same basic and nonbasic variables as $\mathbf{x}_0$) is given below:

$c_j - z_j$						
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\tilde{\mathbf{a}_3}$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}_1}$				1		2
$\mathbf{e}_2$				-2		-1

Since all the $c_j - z_j$ values remain nonnegative as in the optimal table so we can use the Dual simplex algorithm to solve the new problem. In the first iteration $\mathbf{e}_2$ leaves the basis and some non basic variable enters the basis.
So in the Dual's BFS corresponding to the above basic solution, corresponding to say $\mathbf{y}_0$, y_2 is **non basic** (its $c_j - z_j$ value is $-(B^{-1}\mathbf{b}')_2 = 1$), and it enters the basis in the next iteration of simplex.
So if the BFS corresponding to $\mathbf{y}_0$ is **non degenerate** and $\mathbf{y}'$ corresponds to the BFS of the dual of **P'** in the next iteration then
$-\mathbf{b}'^T\mathbf{y}_0 < -\mathbf{b}'^T\mathbf{y}'$ **(*)**
(again similar to $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x}_0 + (c_s - z_s)x'_s$ where $x'_s = \frac{x_r}{u_{rs}}$).

Hence for any optimal solution of Dual of **P'**, say $\mathbf{y}''$, then from **(*)**
$-\mathbf{b}'^T\mathbf{y}_0 < -\mathbf{b}'^T\mathbf{y}''$. **(**)**

Also since the feasible region of the Dual has not changed, $\mathbf{y}''$ is also feasible for Dual of **P**. If $\mathbf{y}''$ has $(\mathbf{y}'')_2 = 0$, then since $\mathbf{y}_0$ is optimal for the Dual of **P**
$-(\mathbf{y}_0)_1 b_1 = -\mathbf{b}^T\mathbf{y}_0 \geq -\mathbf{b}^T\mathbf{y}'' = -(\mathbf{y}'')_1 b_1$.
Since $b_1 > 0(=2)$, we get $(\mathbf{y}_0)_1 \leq (\mathbf{y}'')_1$. **(***)**
From **(**)** we get
$(\mathbf{y}_0)_1(b_1 + 2) > (\mathbf{y}'')_1(b_1 + 2) \Rightarrow (\mathbf{y}_0)_1 > (\mathbf{y}'')_1$,
which contradicts **(***)**.
So y_2 cannot be zero in any optimal solution of the Dual of **P'** in this case.

However if **P** has alternate optimal solutions or if the BFS corresponding to $\mathbf{y}_0$ is **degenerate** then y_2 may be zero in an optimal solution of the Dual of **P'**(infact $\mathbf{y}_0$ may remain optimal for the Dual of **P'**).

All parts in the above question are independent

$[\mathbf{2 + 4 + 4}]$

2. Consider the transportation problem **P** with supply a_i's, 40,30,50 (in this order) and demand d_j's, 30,50,20,20 (in this order).

   (a) If possible find a BFS of **P** such that $(1, 3)$ and $(3, 2)$ are basic cells.
   Solution: There will be numerous correct answers but all six basic cells must be

mentioned. **[2.5]**

In case one basic cell is missing then **only** [**1.5**] marks

In case two basic cells are missing then **only** [**1**] mark

Those who gave a **feasible solution** but not a BFS (that is the cells include a θ-loop) then **only** [**1**] mark.

(b) Is it true that any collection of five cells in the above array is LI?

Solution: False, since the collection of five cells may contain a θ-loop. You have to give an example. [**1**] mark

(c) If you consider **any** BFS of a balanced transportation problem with m supply and n destination constraints then in the corresponding simplex table will there always be a non negative row?

Solution:

Given any BFS there exists a row /column of the array with exactly one basic cell from the given basic set of cells **B**.

Let that cell be (i_0, j_0) and WLOG let the row i_0 of the transportation array has no other **basic cell** from the given set of basic cells **B**.

Consider a **non basic cell** say (p, q) (that is (p, q) is not in **B**), hence $p \neq i_0$.

Let $\tilde{\mathbf{a}_{p,q}} = \sum_{(i,j)\in B} \alpha_{i,j}\tilde{\mathbf{a}_{i,j}}$, **(*)**

where $\tilde{\mathbf{a}_{i,j}}$ is the column in the coefficient matrix A (of the transportation problem with feasible region $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$) corresponding to the variable x_{ij} (or cell (i, j)).

Since $p \neq i_0$, in the column $\tilde{\mathbf{a}_{p,q}}$ the entry in the row corresponding to the i_0 th supply constraint is equal to 0.

Since none of the other basic cells in **B** are in row i_0 of the array, hence each of the basic columns (other than that corresponding to (i_0, j_0)) has zero entry in the row corresponding to the i_0 th supply constraint, hence **(*)** implies, $\alpha_{i_0,j_0} = 0$.

Consider a **non basic cell** of the form (i_0, q) (here $q \neq j_0$).

If $\tilde{\mathbf{a}_{i_0,q}} = \sum_{(i,j)\in B} \alpha_{i,j}\tilde{\mathbf{a}_{i,j}}$, (**)

then since there is a 1 in $\tilde{\mathbf{a}_{i_0,q}}$ in the row corresponding to the i_0 th supply constraint and there is a 1 in column $\tilde{\mathbf{a}_{i_0,j_0}}$ in the same position(i_0 th supply constraint) and all other columns in **(**)** have 0 in that position, $\alpha_{i_0,j_0} = 1$.

Hence the row corresponding to the basic variable $x_{i_0 j_0}$ (or basic cell (i_0, j_0)) in the simplex table corresponding to this BFS will have entries equal to either 0 or 1.

Aliter: Alternatively using θ-loops the same can be proved. [**3.5**] [**7**]