A few special numbers [1]

* The recurrence relation for *Fibonacci numbers* is,

$$f_i = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ f_{i-1} + f_{i-2} & \text{if } i \geq 2. \end{cases}$$

The ordinary generating function of this sequence is $G(x) = f_0 x^0 + f_1 x^1 + \ldots$.

Since $f_i + f_{i+1} = f_{i+2}$ for $i \geq 2$,
$G(x) - xG(x) - x^2 G(x) = x$, i.e., $G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$ for $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The $\frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*, denoted with ϕ. Naturally, $\frac{1-\sqrt{5}}{2}$ is known as the *conjugate of the golden ratio*, denoted with $\widehat{\phi}$.

Specifically, $\frac{x}{(1-\phi x)(1-\widehat{\phi} x)} = \frac{1}{\sqrt{5}}(\frac{1}{1-\phi x} - \frac{1}{1-\widehat{\phi} x}) = \frac{1}{\sqrt{5}}((1+(\phi x)+(\phi x)^2+\ldots)-(1+(\widehat{\phi} x)+(\widehat{\phi} x)^2+\ldots))$.
Therefore, $f_n = [x^n]G(x) = \frac{1}{\sqrt{5}}(\phi^n - \widehat{\phi}^n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$.

* The recurrence relation for *Catalan numbers* is,

$$C_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} & \text{if } n \geq 2. \end{cases}$$

The ordinary generating function of this sequence is $G(x) = C_0 x^0 + C_1 x^1 + C_2 x^2 + \ldots$.

Then, $G(x) - x$
$= C_2 x^2 + C_3 x^3 + \ldots$
$= \sum_{n \geq 2} C_n x^n$
$= \sum_{n \geq 2} \sum_{i=1}^{n-1} C_i C_{n-i} x^n$
$= \sum_{n \geq 2} \sum_{i=1}^{n-1} C_i x^i C_{n-i} x^{n-i}$
$= (\sum_{i \geq 1} C_i x^i)(\sum_{j \geq 1} C_j x^j)$
$= (G(x))^2$.

Therefore, $G(x)^2 - G(x) + x = 0$, i.e., $G(x) = \frac{1 \pm \sqrt{1-4x}}{2}$.

[1] Prepared by R. Inkulu , with the help of note taken by Sawinder Kaur (TA) in class.

If $G(x) = \frac{1+\sqrt{1-4x}}{2}$, $G(0) = 1$; however, $C_0 = 0$. Hence, $G(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$.

Using extended binomial theorem, $G(x) = \frac{1}{2} - \frac{1}{2}(\sum_{k \geq 0} \binom{1/2}{k}(-4x)^k)$.

Therefore, $[x^n]G(x) = \frac{-1}{2}\binom{1/2}{n}(-4)^n$.

But, $\binom{1/2}{n}$
$= \frac{(1/2)(1/2-1)(1/2-2)...(1/2-(n-1))}{n!}$
$= \frac{(1/2)(-1/2)(-3/2)...(-(\frac{2n-3}{2}))}{n!}$
$= \frac{1}{2^n}\frac{(-1)^n(1)(3)...(2n-3)}{n!}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{(2)(4)...(2n-2)}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{2^{n-1}(n-1)!}$
$= \frac{2}{4^n}(-1)^{n-1}\frac{(2n-2)!}{n!(n-1)!}$
$= \frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}$.

Hence, $[x^n]G(x) = \frac{-1}{2}\frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}(-1)^n 4^n = \frac{(-1)^{2n}}{n}\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

If the recurrence is defined as,

$$C_n = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} & \text{if } n \geq 2. \end{cases}$$

substituting $n+1$ for n in the above, $C_n = [x^n]G(x) = \frac{1}{n+1}\binom{2n}{n}$.

* If $f(k)$ is a positive monotonically increasing continuous function (refer to Fig. 1), then

- $\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$
- $f(m) + \int_{m}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq f(n) + \int_{m}^{n} f(x)dx$

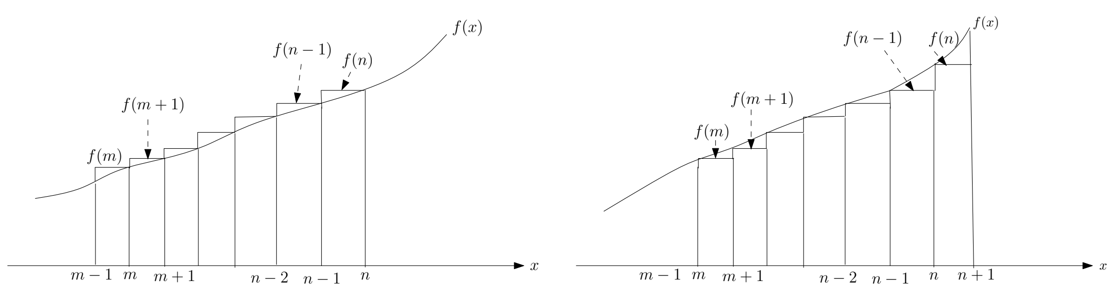

Figure 1: Approximating a summation with an integral.

Noting that $\lg(k)$ is a monotonically increasing function,
$n \ln(n) - n + 1 \leq \sum_{i=1}^{n} \ln(i) \leq n \ln(n) - n + 1 + \ln(n)$. ——— (1)

Hence, $\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$.

* From (1), we know $\lg(n!) - n \ln(n) + n \leq 1 + \ln(n)$.
We claim there exists a positive constant α such that $(\ln(n!) - n \ln n + n - \frac{1}{2} \ln n) \approx \alpha$.
Then, $e^{\alpha} \approx e^{(\ln(n!) - (n+\frac{1}{2}) \ln n + n)} = \frac{n! e^n}{n^{n+1/2}}$. ——- (2a)
That is, $n! \approx e^{\alpha} n^{n+\frac{1}{2}} e^{-n}$. —— (2b)

From Wallis' inequality, when n is asymptotically large, $\frac{(2)(4)(6)...(2n)}{(1)(3)(5)...(2n-1)\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$
$\Rightarrow \frac{(2^n n!)^2}{(2n)!} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$.

Substituting (2b), $\frac{2^{2n} e^{2\alpha} n^{2n+1} e^{-2n}}{e^{\alpha} (2n)^{2n+\frac{1}{2}} e^{-2n}} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}} \Rightarrow e^{\alpha} \approx \sqrt{2\pi}$.

Substituting (2a), $e^{\alpha} = \frac{n! e^n}{n^{n+1/2}} \approx \sqrt{2\pi} \Rightarrow n! \approx \sqrt{2\pi n} (\frac{n}{e})^n$.

This is known as the *Stirling's approximation of* $n!$.

* The recurrence relation for *Harmonic numbers* is,

$$H_n = \begin{cases} 1 & \text{if } n = 1, \\ a_{n-1} + \frac{1}{n} & \text{if } n \geq 2. \end{cases}$$

If $f(k)$ is a positive monotonically decreasing continuous function, then

- $\int_{m}^{n+1} f(x) dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x) dx$
- $f(n) + \int_{m}^{n} f(x) dx \leq \sum_{k=m}^{n} f(k) \leq f(m) + \int_{m}^{n} f(x) dx$ —— (3)

Noting that $1/k$ is a monotonically decreasing function, from (3), $\frac{1}{n} + \ln(n) \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \ln(n)$.

* The *sum of first* n *Harmonic numbers* is, $\sum_{k=1}^{n} H_k$
$= \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j}$
$= 1 + (1 + \frac{1}{2}) + (1 + \frac{1}{2} + \frac{1}{3}) + \ldots + (1 + \frac{1}{2} + \ldots + \frac{1}{n})$
$= n + \frac{1}{2}(n-1) + \frac{1}{3}(n-2) + \ldots + \frac{1}{n}(n-(n-1))$

$= (\sum_{j=1}^{n} \frac{n-j+1}{j}) = (\sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{j})$
$= n(1 + \frac{1}{2} + \ldots + \frac{1}{n}) - (\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n})$
$= n(1 + \frac{1}{2} + \ldots + \frac{1}{n}) - (\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n}) - H_n + H_n$
$= (n+1)H_n - n.$

* The r^{th} *Stirling number of second kind of* n is the number of ways to distribute n labeled balls into r unlabeled bins with no bin left empty, denoted by $\left\{ n \atop r \right\}$. The recurrence relation is, $\left\{ n+1 \atop r \right\} = r\left\{ n \atop r \right\} + \left\{ n \atop r-1 \right\}$ for $r > 0$, with initial conditions $\left\{ 0 \atop 0 \right\} = 1, \left\{ k \atop 0 \right\} = \left\{ 0 \atop k \right\} = 0$ for $k > 0$. Further, $\left\{ n \atop r \right\} = \frac{1}{r!}\cdot$ (number of onto functions from a set with n elements to a set with r elements) $= \frac{1}{r!} \sum_{i=0}^{r-1} (-1)^i C(r,i)(r-i)^n$.