- **Result 1:** $\mathbf{x}$ is a BFS of $A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}$ if and only if $\mathbf{x}$ is an extreme point of $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x}_{m\times n} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\}$.
- **Result 2:** Let $S' = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{O}\}$ and let
$$S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}, \mathbf{s} \geq \mathbf{O} \right\}.$$
Then $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ is an extreme point of S if and only if $\mathbf{x}$ is an extreme point of S'.
Result 3: If
$$S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}, \mathbf{s} \geq \mathbf{O} \right\} \neq \phi,$$
then it has at least one BFS.

- **Result 4:** If $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\} \neq \phi$ then it has atleast one red BFS.
- **Result 5:** $\mathbf{d}_0$ is an extreme direction of $S' = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{O}\}$ if and only if $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}'_0 \end{bmatrix}$ is an extreme direction of
$$S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}, \mathbf{s} \geq \mathbf{O} \right\}$$
where $\mathbf{d}'_0 \geq \mathbf{O}$ is such that $A\mathbf{d}_0 + \mathbf{d}'_0 = \mathbf{O}$.
- In simplex method in any iteration we move from one extreme point to an adjacent extreme point if the current BFS is nondegenerate.
- It remains at the same extreme point if the current BFS is degenerate.

Sensitivity Analysis:

- Consider the problem (**P**),
  Min $\mathbf{c}^T\mathbf{x}$
  subject to
  $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}$.
  Let $\mathbf{x}_0$ be an optimal solution of this problem. WLOG let $B = [\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_m]$ be a basis matrix corresponding to $\mathbf{x}_0$, hence a set of basic variables of $\mathbf{x}_0$ are $x_1, \ldots, x_m$.
- **Changing the cost vector c:**
- If the new $c'_j - z'_j \geq 0$ for all $j = 1, \ldots, n$, then $\mathbf{x}_0$ will again be optimal for the new problem.
- If not, then simplex algorithm can be used to get an optimal solution for the new problem or to conclude that the new problem has no optimal solution.

Changing the vector b:

- If the vector **b** is changed to **b**$'$, and if the basic solution $\mathbf{x}_0' = [(B^{-1}\mathbf{b}')^T, \mathbf{O}_{1\times(n-m)}]^T$ corresponding to $\mathbf{x}_0 = [(B^{-1}\mathbf{b})^T, \mathbf{O}_{1\times(n-m)}]^T$, is such that $B^{-1}\mathbf{b}' \geq \mathbf{O}$, then $\mathbf{x}_0'$ is optimal for the new problem.
- If $B^{-1}\mathbf{b}' \ngeq \mathbf{O}$, then the basic solution $\mathbf{x}_0'$ is not feasible for the changed problem.
- $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ belongs to Fea(D) (since $c_j - z_j \geq 0$, for all $j = 1, \ldots, n$).
- Since $\mathbf{y}^T$ satisfies $\mathbf{y}^T\tilde{\mathbf{a}}_j = z_j = c_j$ for $j = 1, \ldots, m$, it lies on m LI hyperplanes defining $Fea(D)$, so **y** is an extreme point of $Fea(D)$.
- The **dual simplex algorithm** can be used to either get an **optimal solution** of the new problem or to conclude that the new problem does not have a **feasible solution**.

- **Example :** Consider the LPP given by
  Max $-x_1 + 2x_2$
  subject to
  $-x_1 + x_2 \leq 1,$
  $x_1 + x_2 \leq 7,$
  $x_1 + 3x_2 \leq 15,$
  $x_1, x_2 \geq 0.$
  Check that the optimal solution for the above problem is given by $[3, 4]^T$.
  If we convert the above problem to a problem with equality constraints by adding (slack) variables, then it becomes
- Max $-x_1 + 2x_2$
  subject to
  $-x_1 + x_2 + s_1 = 1,$
  $x_1 + x_2 + s_2 = 7,$
  $x_1 + 3x_2 + s_3 = 15,$
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$

- The optimal BFS $[3, 4, 0, 0, 0]^T$ is degenerate and corresponds to three different basis matrix, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_2]$ and $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$.
  The tables corresponding to these three bases are given by
-

$c_j - z_j$	0	0	0	$\frac{5}{2}$	$-\frac{3}{2}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			0	$\frac{3}{2}$	$-\frac{1}{2}$	3
$\tilde{\mathbf{a}}_2$			0	$-\frac{1}{2}$	$\frac{1}{2}$	4
$\mathbf{s}_1$			1	2	-1	0

- Note that not all the $c_j - z_j$ values in the above table are nonpositive, but the above BFS is still optimal.
- So the optimality condition, $c_j - z_j \geq 0$ for all j, is a **sufficient condition** but **not** a **necessary condition** for the corresponding BFS to be optimal.

-

$c_j - z_j$	0	0	$-\frac{5}{4}$	0	$-\frac{1}{4}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			$-\frac{3}{4}$	0	$\frac{1}{4}$	3
$\tilde{\mathbf{a}}_2$			$\frac{1}{4}$	0	$\frac{1}{4}$	4
$\mathbf{s}_2$			$\frac{1}{2}$	1	$-\frac{1}{2}$	0

-

$c_j - z_j$	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			$-\frac{1}{2}$	$\frac{1}{2}$	0	3
$\tilde{\mathbf{a}}_2$			$\frac{1}{2}$	$\frac{1}{2}$	0	4
$\mathbf{s}_3$			-1	-2	1	0

- The dual of the above problem is given by
  Min $y_1 + 7y_2 + 15y_3$
  subject to
  $-y_1 + y_2 + y_3 \geq -1,$
  $y_1 + y_2 + 3y_3 \geq 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$
- The optimal solutions of the Dual obtained from the optimal tables are given by:
  $\mathbf{y}^T = \mathbf{c}_B^T B^{-1} = \mathbf{c}_B^T B^{-1}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = [z_{s_1}, z_{s_2}, z_{s_3}]$
  where z_{s_i} is the z_j value corresponding to the slack variable s_i.
- $[\frac{5}{4}, 0, \frac{1}{4}]^T$ and $[\frac{3}{2}, \frac{1}{2}, 0]^T$, are both optimal solutions of the Dual as well as extreme points of $Fea(D)$.
- The dual has infinitely many optimal solutions.

- If we convert the dual problem into a problem with equality constraints by adding (surplus) variables then we get
- Min $y_1 + 7y_2 + 15y_3$
  subject to
  $-y_1 + y_2 + y_3 - s_1' = 1,$
  $y_1 + y_2 + 3y_3 - s_2' = 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s_1' \geq 0, s_2' \geq 0.$
- The BFS corresponding to the extreme point $[\frac{5}{4}, 0, \frac{1}{4}]^T$ of the Dual will have basic variables as y_1, y_3.
- The BFS corresponding to the extreme point $[\frac{3}{2}, \frac{1}{2}, 0]^T$ have basic variables as y_1, y_2.

- The table corresponding to these basic feasible solutions will be given by (check this)

$c_j - z_j$	0	0	0	3	4	
	$B^{-1}\tilde{\mathbf{a}}'_1$	$B^{-1}\tilde{\mathbf{a}}'_2$	$B^{-1}\tilde{\mathbf{a}}'_3$	$B^{-1}(-\mathbf{e}_1)$	$B^{-1}(-\mathbf{e}_2)$	B^{-1}
$\tilde{\mathbf{a}}'_1$					$-\frac{1}{2}$	$\frac{3}{2}$
$\tilde{\mathbf{a}}'_2$					$-\frac{1}{2}$	$\frac{1}{2}$

-

$c_j - z_j$	0	0	0	3	4	
	$B^{-1}\tilde{\mathbf{a}}'_1$	$B^{-1}\tilde{\mathbf{a}}'_2$	$B^{-1}\tilde{\mathbf{a}}'_3$	$B^{-1}(-\mathbf{e}_1)$	$B^{-1}(-\mathbf{e}_2)$	B^{-1}
$\tilde{\mathbf{a}}'_1$					$-\frac{1}{4}$	$\frac{5}{4}$
$\tilde{\mathbf{a}}'_3$					$-\frac{1}{4}$	$\frac{1}{4}$

Here $\tilde{\mathbf{a}}'_i$ gives the columns corresponding to the variables $y_i, i = 1, 2, 3$ in the dual constraints, when the constraints are written in the greater than equal to form.

- Suppose if the RHS of the primal problem corresponding to the third constraint is changed from 15 to 14, then which basis among the three mentioned above, will correspond to the new optimal solution?
- The new problem is given as following:
  Max $-x_1 + 2x_2$
  subject to
  $-x_1 + x_2 + s_1 = 1$,
  $x_1 + x_2 + s_2 = 7$,
  $x_1 + 3x_2 + s_3 = 14$,
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$
- Note that the BFS corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, should have $s_2 = s_3 = 0$ hence should lie at the intersection of the two lines
  $x_1 + x_2 = 7$, and $x_1 + 3x_2 = 14$
  and is given by $x_1 = \frac{7}{2}, x_2 = \frac{7}{2}$. Hence $s_1 = 1$.
- The corresponding BFS is $[\frac{7}{2}, \frac{7}{2}, 1, 0, 0]^T$, which is a nondegenerate BFS.

- Since all the $c_j - z_j$ values are not ≤ 0, so by entering s_2 into the basis one can get a BFS with better value of the objective function.
  Hence this BFS is not optimal.
- The BFS corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_2]$, should have $s_1 = s_3 = 0$, should lie at the intersection of the two lines
  $-x_1 + x_2 = 1$, and $x_1 + 3x_2 = 14$
  and is given by $x_1 = \frac{11}{4}, x_2 = \frac{15}{4}$. Hence $s_2 = \frac{1}{2}$.
- The corresponding BFS is $[\frac{11}{4}, \frac{15}{4}, 0, \frac{1}{2}, 0]^T$, a nondegenerate BFS.
- Since $c_j - z_j \leq 0$ for all $j = 1, \ldots, n$, so this BFS is optimal solution for the new problem.

- The basic solution corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$, should have $s_1 = s_2 = 0$ hence should lie at the intersection of the two lines
  $-x_1 + x_2 = 1$, and $x_1 + x_2 = 7$
  and is given by $x_1 = 3, x_2 = 4$ which implies $s_3 = -1$.
- The corresponding basic solution is $[3, 4, 0, 0, -1]^T$ which is not feasible for the new problem.
- Will the new dual now have a unique optimal solution or will it again have infinitely many optimal solutions?

Dual Simplex Algorithm :

- Consider the following LP problem (P) :
  Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{m\times n}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{O}$.
- Let $\mathbf{x}_0$ be a basic solution of (P) corresponding to a basis matrix B, such that $B^{-1}\mathbf{b} \ngeq \mathbf{O}$.
- The c_j's are such that all $c_j - z_j$ values in the simplex table corresponding to $\mathbf{x}_0$ are non negative.
- Then $\mathbf{y}$ given by $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ is feasible for the Dual.
- In this case the **Dual Simplex Method** is used to get an optimal solution of (P) or to conclude that (P) **does not** have a feasible solution.

- Take x_r to be the leaving basic variable if $(B^{-1}\mathbf{b})_r = min\{(B^{-1}\mathbf{b})_j : (B^{-1}\mathbf{b})_j < 0\}$.
- **Case 1:** $u_{rj} \geq 0$ for all $j = 1, 2, ..., n$.
  Then the primal (P) **does not** have a feasible solution.
- $(-B^{-1})_{r.}$ is a direction for $Fea(D)$.
- If we denote $(-B^{-1})_{r.}$ as $\mathbf{d}_0^T$,
  $-(B^{-1}\mathbf{b})_r = \mathbf{d}_0^T\mathbf{b} > 0$.
- Since the **Dual** is a maximization problem it follows that the Dual **does not** have an optimal solution.
- Hence the primal problem (P) **does not** have a feasible solution.

- **Case 2 :** $u_{rj} < 0$ for atleast one $j = 1, 2, ..., n$.
- Then x_s is the entering variable if
  $\frac{|c_s - z_s|}{|u_{rs}|} = min_j\{\frac{|c_j - z_j|}{|u_{rj}|} : u_{rj} < 0\}$.
  So the pivot element is u_{rs}.
- The table is updated by performing the necessary elementary row operations.
- The new column corresponding to $\tilde{\mathbf{a}}_s$ in the table is the r th column of I_m and $c_s - z_s = 0$.
- New $c_j - z_j' = c_j - z_j - \frac{c_s - z_s}{u_{rs}} u_{rj}$.

- If $\mathbf{x}'$ is the new basic solution of the primal (P) and $\mathbf{y}'$ be the corresponding feasible solution of the Dual, then $\mathbf{b}^T\mathbf{y}' = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + (c_s - z_s)\frac{x_r}{u_{rs}} \geq \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.
- Since the Dual is a maximization problem so $\mathbf{y}'$ is a better solution with respect to cost (or the objective function) than $\mathbf{y}$.
- If the new basic solution $\mathbf{x}'$ is non negative, then $\mathbf{x}'$ is an optimal BFS of (P).
- If not repeat the procedure till you get a BFS and hence an optimal solution of (P) or conclude that (P) has no feasible solution.

- In case (P) is **Max $\mathbf{c}^T\mathbf{x}$**
  subject to $A_{m\times n}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{O}$.
  Then Dual of (P) is given by
  Min $\mathbf{b}^T\mathbf{x}$
  subject to $A^T_{n\times m}\mathbf{y} \geq \mathbf{c}$.
- **Case 1':** $u_{rj} \geq 0$ for all $j = 1, 2, ..., n$.
  Then the primal (P) **does not** have a feasible solution.
- $(B^{-1})_{r.}$ is a direction for $Fea(D)$.
- If we denote $(B^{-1})_{r.}$ as $\mathbf{d}_0^T$,
  $(B^{-1}\mathbf{b})_r = \mathbf{d}_0^T\mathbf{b} < 0$.
- Since the **Dual** is a minimization problem it follows that the Dual **does not** have an optimal solution.
- Hence the primal problem (P) **does not** have a feasible solution.

- **Case 2' :** $u_{rj} < 0$ for atleast one $j = 1, 2, ..., n$.
- Then x_s is the entering variable if $\frac{c_s - z_s}{u_{rs}} = min_j\{\frac{c_j - z_j}{u_{rj}} : u_{rj} < 0\}$.
  So the pivot element is u_{rs}.
- The table is updated by performing the necessary elementary row operations.
- The new column corresponding to $\tilde{\mathbf{a}}_s$ in the table is the r th column of I_m and $c_s - z_s = 0$.

- If $\mathbf{x}'$ is the new basic solution of the primal (P) and $\mathbf{y}'$ be the corresponding feasible solution of the Dual, then $\mathbf{b}^T\mathbf{y}' = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + (c_s - z_s)\frac{x_r}{u_{rs}} \leq \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.
- Since the Dual is a minimization problem so $\mathbf{y}'$ is a better solution with respect to cost (or the objective function) than $\mathbf{y}$.
- If the new basic solution $\mathbf{x}'$ is non negative, then $\mathbf{x}'$ is an optimal BFS of (P).
- If not repeat the procedure till you get a BFS and hence an optimal solution of (P) or conclude that (P) has no feasible solution.

- Max $-3x_1 + 2x_2$
  subject to
  $-x_1 + x_2 + s_1 = 1$,
  $x_1 + x_2 + s_2 = 7$,
  $x_1 + 3x_2 + s_3 = 15$,
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$
  The table corresponding to the basic solution with $x_1 = -1$ and $x_2 = 0$ is given by
-

$c_j - z_j$		-1	-3			
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$		-1	-1			-1
$\mathbf{s}_2$		2	1			8
$\mathbf{s}_3$		4	1			16

- According to the Dual Simplex method, x_1 is the leaving variable, since $(B^{-1}\mathbf{b})_1 < 0$ and $(B^{-1}\mathbf{b})_2 > 0$, $(B^{-1}\mathbf{b})_3 > 0$.
- Also since in the row corresponding to the leaving variable, only two entries are negative, u_{12}, and u_{13} and $\frac{c_2-z_2}{u_{12}} = 1 < \frac{c_3-z_3}{u_{13}} = 3$, hence the entering variable is x_2 and the **pivot** is u_{12}, where u_{ij}'s have their usual meaning. After doing the necessary elementary row operations we get the following table :

$c_j - z_j$	-1	0	-2			
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_2$	-1	1	1			1
$\mathbf{s}_2$	2	0	-1			6
$\mathbf{s}_3$	4	0	-3			12

Hence an optimal solution of the primal is given by, $x_2 = 1$ and $x_1 = 0$.

The optimal solution of the Dual is given by,

$y_1 = 2, y_2 = 0, y_3 = 0$,

where the Dual is given by:

Min $y_1 + 7y_2 + 15y_3$

subject to

$-y_1 + y_2 + y_3 \geq -3$,

$y_1 + y_2 + 3y_3 \geq 2$,

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.

Introduction of a new variable

- Consider the problem (**P**),
  Min $\mathbf{c}^T\mathbf{x}$
  subject to
  $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}.$
  Let $\mathbf{x}_0$ be an optimal BFS of this problem.
- A new variable x_{n+1} is added to the LPP given above.
- A column is added to the matrix A call it $\tilde{\mathbf{a}}_{n+1}$ and a component c_{n+1} is added to the cost vector $\mathbf{c}$, which is the cost associated with the new variable x_{n+1}.
- Since $\mathbf{x}_0$ is an optimal solution for **P**
  $\mathbf{x}'_0 = [\mathbf{x}_0^T, 0]^T$ is a feasible solution for the new problem, given by:
- Min $[\mathbf{c}, c_{n+1}]^T\mathbf{x}_{(n+1)\times 1}$
  subject to
  $[A : \tilde{\mathbf{a}}_{n+1}]\mathbf{x}_{(n+1)\times 1} = \mathbf{b}, \quad \mathbf{x}_{(n+1)\times 1} \geq \mathbf{O}.$

- To check whether $\mathbf{x}_0'$ is optimal for the new problem, add a new column to the optimal table and calculate $c_{n+1} - z_{n+1}$.
- If $c_{n+1} - z_{n+1} \geq 0$, then $\mathbf{x}_0'$ is optimal for the new problem.
- If **not**, then enter $\tilde{\mathbf{a}}_{n+1}$ in the basis and use the **simplex algorithm** to get an optimal solution or to conclude that the problem **does not** have an optimal solution.
- Adding a variable to **P** results in adding a constraint to the dual, which might result in the feasible region of the dual to become an empty set, in that case the problem **P** will **not** have an optimal solution.

Introduction of a new constraint

- Addition of a new constraint makes the feasible region of **P** smaller.
- Let us assume that the new constraint added is of the form $\mathbf{a}_{m+1}{}^T\mathbf{x} \leq b_{m+1}$, where $\mathbf{a}_{m+1}{}^T$ is a row vector.
- **Case 1:** $\mathbf{x}_0$ satisfies $\mathbf{a}_{m+1}{}^T\mathbf{x} \leq b_{m+1}$.
  Then $\mathbf{x}_0$ is optimal for the changed problem also.
- **Case 2:** $\mathbf{x}_0$ does not satisfy the new constraint, that is
  $\mathbf{a}_{m+1}{}^T\mathbf{x}_0 > b_{m+1}$,
  or $\quad \mathbf{a}^T_{B,m+1}B^{-1}\mathbf{b} > b_{m+1}$,
  where $\mathbf{a}^T_{B,m+1}$ and $\mathbf{a}^T_{N,m+1}$ are the components of $\mathbf{a}^T_{m+1}$ corresponding to the basic variables and nonbasic variables of $\mathbf{x}_0$, respectively.

- The new constraint can be written as $\mathbf{a}_{m+1}{}^T\mathbf{x} + s_{m+1} = b_{m+1}$, where $s_{m+1} \geq 0$.
- If B denotes the basis matrix corresponding to $\mathbf{x}_0$, then $\begin{bmatrix} B & \mathbf{O} \\ \mathbf{a}_{B,m+1}^T & 1 \end{bmatrix}$ is the new basis matrix after the new constraint is added, the new added **basic variable** being s_{m+1}.
- The **inverse** of this new basis matrix is given by $\begin{bmatrix} B^{-1} & \mathbf{O} \\ -\mathbf{a}_{B,m+1}^T B^{-1} & 1 \end{bmatrix}$.

- The new row added to the (main) simplex table will be of the form
$$[-\mathbf{a}_{B,m+1}^T B^{-1}, 1]\begin{bmatrix} B & N & \mathbf{O} \\ \mathbf{a}_{B,m+1}^T & \mathbf{a}_{N,m+1}^T & 1 \end{bmatrix} =$$
$$[\mathbf{O}, -\mathbf{a}_{B,m+1}^T B^{-1} N + \mathbf{a}_{N,m+1}^T, 1]$$
- The new RHS becomes
$$\begin{bmatrix} B^{-1} & \mathbf{O} \\ -\mathbf{a}_{B,m+1}^T B^{-1} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{b} \\ b_{m+1} \end{bmatrix} = \begin{bmatrix} B^{-1}\mathbf{b} \\ b_{m+1} - \mathbf{a}_{B,m+1}^T B^{-1}\mathbf{b} \end{bmatrix}$$
- If $\mathbf{x}_0$ does not satisfy the newly added constraint then the $(m+1)$ th entry of the RHS will be **strictly** less than zero.
- Since the cost associated with s_{m+1} is equal to zero, all the $c_j - z_j$ entries are non negative.
- Then the **Dual Simplex method** can be used to obtain an optimal solution of the new **P** or conclude that the new problem is infeasible.

Changing entries in the coefficient matrix A:

- **Case 1:** The column corresponding to a **nonbasic variable** say $\tilde{\mathbf{a}}_j$ of the optimal solution is changed to $\tilde{\mathbf{a}}'_j$.
- The column corresponding to that variable in the optimal simplex table $B^{-1}\tilde{\mathbf{a}}_j$ is changed to $B^{-1}\tilde{\mathbf{a}}'_j$.
- If the new $c_j - z'_j$ value satisfies the **optimality condition**, then the previous optimal solution will be optimal for the new problem.
- If **not**, then use the **Simplex method** to obtain the new **optimal solution** or to conclude that the primal does **not** have an optimal solution.

Changing entries in the coefficient matrix A:

- **Case 2:** If the column corresponding to the j th **basic variable** say $\tilde{\mathbf{a}}_j$ is changed to $\tilde{\mathbf{a}}'_j$.
- Then treat this case as the case when a variable (say a variable x_{n+1}) is added to the problem with column $\tilde{\mathbf{a}}'_j$ and cost c_j.
- Add a column $B^{-1}\tilde{\mathbf{a}}'_j$ and the corresponding $c_{n+1} - z_{n+1}$ value in the simplex table.
- If $u_{j,(n+1)} = 0$, then it implies that $\{\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_{j-1}, \tilde{\mathbf{a}}'_j, \tilde{\mathbf{a}}_{j+1}, \ldots, \tilde{\mathbf{a}}_m\}$ is LD and we have to find a initial BFS for the changed problem.
- If $u_{j,(n+1)} \neq 0$, then pivot on this element and make x_{n+1} enter the basis and x_j leave the basis.

- Perform the necessary elementary row operations to make the (n+1) th column as the j **th column** of I and the $c_{n+1} - z_{n+1}$ value equal to 0.
- Delete the column corresponding to $\tilde{\mathbf{a}}_j$ from all subsequent calculations, and the variable x_{n+1} is now indexed as x_j.
- The necessary calculations mentioned above might disturb the optimality as well as the feasibility of the optimal table, hence again an initial BFS of the new problem has to be found.
- If however the new $c_j - z_j$ values are all **nonnegative** and all the RHS entries remain **non negative**, then the table is an **optimal table** and the corresponding BFS is optimal for the new problem.

- If the new table has all RHS entries nonnegative, but atleast one of the $c_j - z_j$ values is negative then use the **Simplex method** to obtain the new optimal solution, or to conclude that the new problem **does not** have an optimal solution.
- If all the $c_j - z_j$ values are nonnegative but atleast one of the RHS entries in the simplex table is negative then the **Dual Simplex method** to obtain the new optimal solution or to conclude that the new problem has **no** feasible solution.

Artificial Variable Method to find an initial BFS of an LPP:

Big- M method:

- Consider the problem **P**:
  Min $\mathbf{c}^T\mathbf{x}$
  subject to
  $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}$.
- Assume $\mathbf{b} \geq \mathbf{O}$.
- Add artificial variables $w_1, \ldots, w_m$ with costs M (M large).
- We get an initial BFS to the following problem **LP(M)** given by,
  Min $\mathbf{c}^T\mathbf{x} + [M, \ldots, M]\mathbf{w}$
  subject to
  $$A\mathbf{x} + \mathbf{w} = [A : I]\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}, \mathbf{w} \geq \mathbf{O},$$
  where $\mathbf{w} = [w_1, \ldots, w_m]^T$ and the initial BFS is $[\mathbf{O}_{1\times n}, w_1, \ldots, w_m]^T$.
- For any $\mathbf{x}$ feasible for **P**, $[\mathbf{x}^T, \mathbf{O}_{1\times m}]^T$ is feasible for **LP(M)**.

- **Case 1: LP(M)** has an optimal solution.
- **Case 1a:** $[\mathbf{x}_*{}^T, \mathbf{O}_{1\times m}]^T$ is an **optimal** BFS of **LP(M)** with all the artificial variables as nonbasic variables.

  Then $\mathbf{x}_*$ is an **optimal** BFS for **P**.
- **Case 1b:** $[\mathbf{x}_*{}^T, \mathbf{w}_*{}^T]^T$ is optimal for **LP(M)** and $\mathbf{w}_* \neq \mathbf{O}$.

  Then **(P) does not** have a **feasible solution**.
- **Case 2: LP(M) does not** have an **optimal** solution.

- **Case 2a:** In some iteration (or simplex table), there exists a k such that $c_k - z_k < 0$, the corresponding column $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{O}$ and $\mathbf{w} = \mathbf{O}$ in the corresponding BFS for **LP(M)**.
- Then there exists a **direction** $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}$ of *Fea*(**LP(M)**) such that
  $\mathbf{c}^T\mathbf{d}_1 + [M, \ldots, M]\mathbf{d}_2 < 0$.
  This implies that $\mathbf{d}_1$ is a **direction** of *Fea*(**P**) and $\mathbf{c}^T\mathbf{d}_1 < 0$, hence **P** does **not** have an **optimal** solution but **(P)** has a **feasible** solution.
- **Case 2b:** In some iteration (or simplex table), $c_k - z_k = min\{c_j - z_j : c_j - z_j < 0\}$, $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{O}$ and $\mathbf{w} \neq \mathbf{O}$ in the corresponding BFS for **LP(M)**.
  Then **(P)** does **not** have a **feasible** solution.
- The above conclusion of **Case 2b** that **P** is **infeasible** may not be true if $c_k - z_k$ is **not** the **most negative** among the $c_j - z_j$ values.

- **Case 2c:** In some iteration (or simplex table), there exists a k such that $c_k - z_k < 0$, the corresponding column $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{O}$ and $\mathbf{w} \neq \mathbf{O}$ in the corresponding BFS for **LP(M)**.
  Then combining **Case 2(a)** and **Case 2(b)** we can say **P** does **not** have an **optimal** solution (may be **infeasible**, **may not** be **infeasible**)
- **Remark 1:** If at some stage of solving the artificial variables **leave** the basis, then the corresponding BFS is a BFS of **(P)**.
  Then delete all columns corresponding to the artificial variables and continue.
- **Remark 2:** If $[\mathbf{x}^T, \mathbf{O}]$ is optimal for **LP(M)**, but one or more artificial variables remain in the basis for the optimal BFS at zero value. Then $\mathbf{x}$ will be optimal for **P**.
- To get an optimal BFS for **P**, pivot appropriately on a non zero entry in the rows corresponding to the basic artificial variables and make one of the variables x_j **enter** the basis in place of each of the artificial variables.

- **Example 1 :** Consider the problem **(P)**,
  Minimize $x_1 - x_2$
  subject to
  $2x_1 + x_2 \geq 4$
  $x_1 - x_2 \leq 1$
  $x_1 \geq 0,\ x_2 \geq 0.$
- By adding variables we get the following problem,
  Minimize $x_1 - x_2$
  subject to
  $2x_1 + x_2 - s_1 = 4$
  $x_1 - x_2 + s_2 = 1$
  $x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.$
- If we consider the initial basic solution with basic variables s_1 and s_2, then the $c_j - z_j$ values are not ≥ 0 for all j.
  We do **not** have a **feasible solution** of the dual of **(P)**.
- The (**Big-M**) is used which provides an initial BFS for Simplex algorithm.

- Consider the modified problem
  Minimize $x_1 - x_2 + Mw$
  subject to
  $2x_1 + x_2 - s_1 + w = 4$
  $x_1 - x_2 + s_2 = 1$
  $x_1, x_2, s_1, s_2, w \geq 0.$
  Here w is called the artificial variable and cost M associated with it is very large.
- The initial table corresponding to the basic variables w and s_2 is given below.

$c_j - z_j$	$1 - 2M$	$-M - 1$	M	0	0	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
w	2	1	-1	0	1	4
s_2	1	-1	0	1	0	1

- s_2 will be the **leaving variable** and x_1 will be the **entering variable** for the next table.

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$c_j - z_j$	0	$-3M$	M	$2M-1$	0	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
w	0	3	-1	-2	1	2
x_1	1	-1	0	1	0	1

- Now the artificial variable w will be the **leaving variable** and x_2 will be the **entering variable**.
-

$c_j - z_j$	0	0	0	-1	M	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
x_2	0	1				$\frac{2}{3}$
x_1	1	0				$1+\frac{2}{3}$

- Continue with the BFS corresponding to the basic vectors x_1 and x_2 and drop the column corresponding to the artificial variable w from all future calculations.
  Use simplex method to get the optimal basic feasible solution.