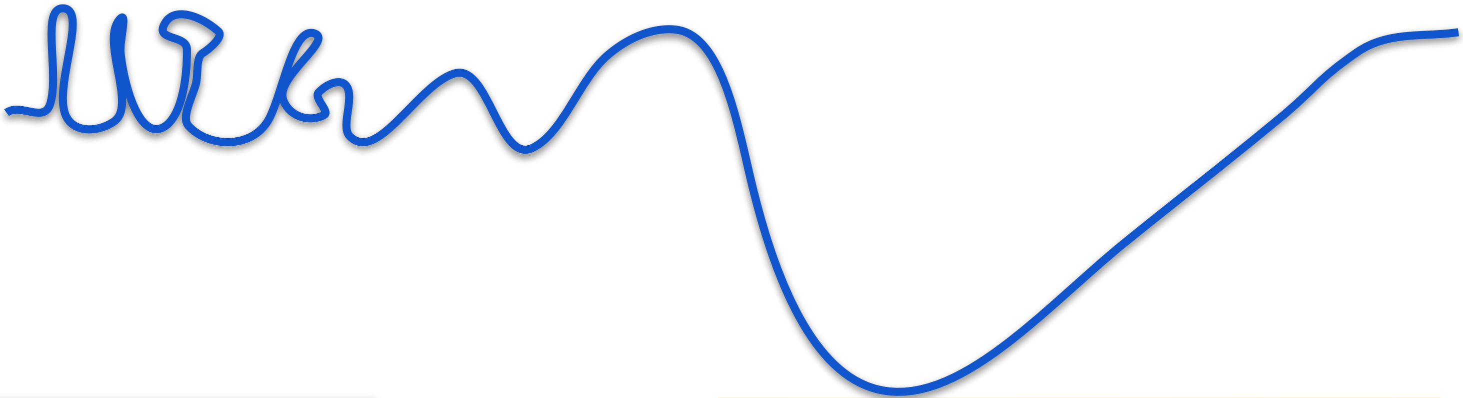
Computing with Signals



DA 623

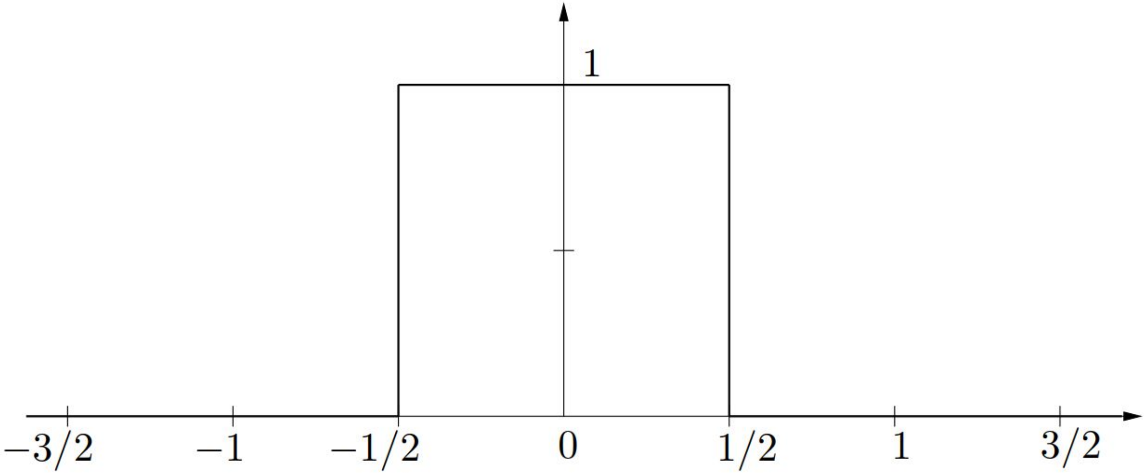
Jan - May 2024

IIT Guwahati

Instructors: Neeraj Sharma

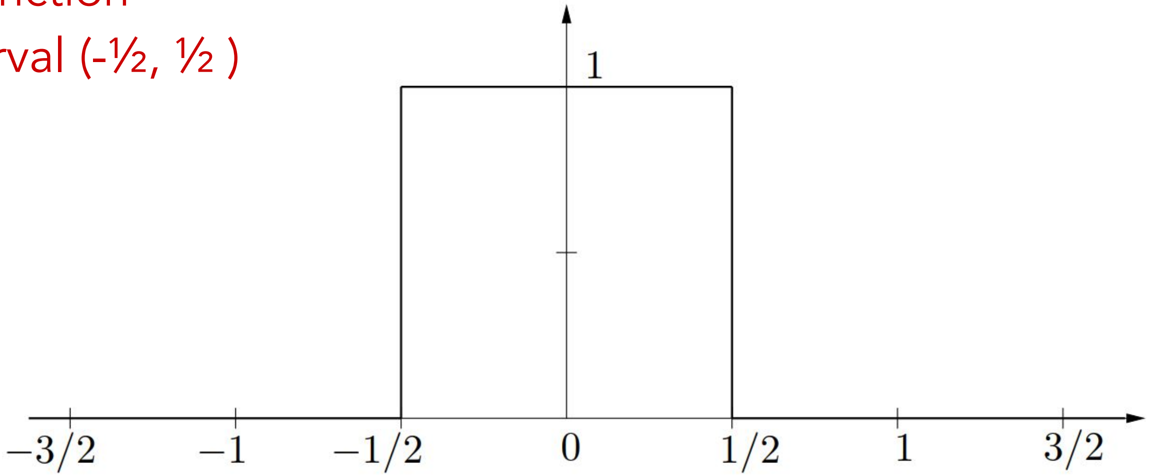
Lecture-09

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$



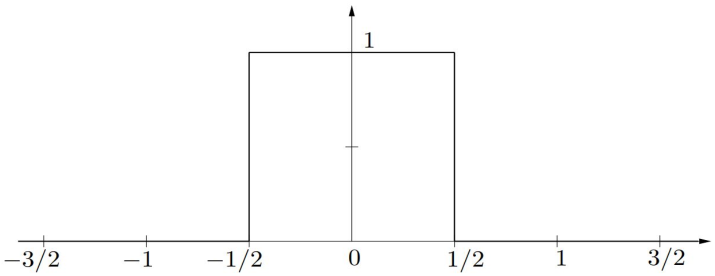
$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$

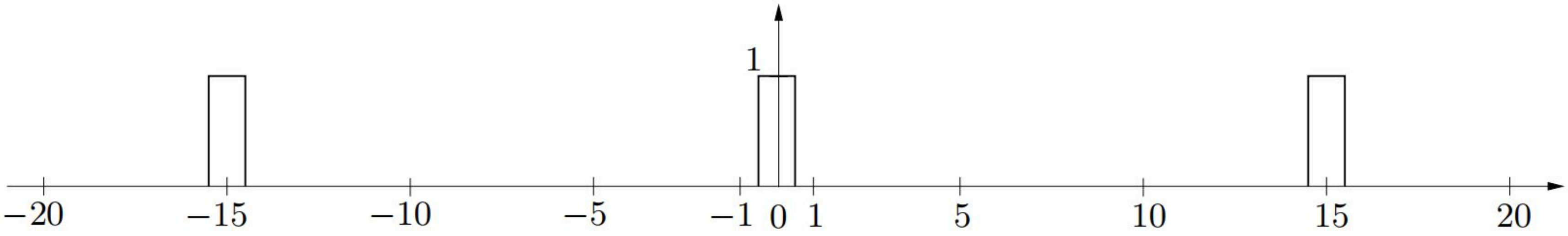
- Top hat function
- rect()
- Indicator function for the interval (-½, ½)

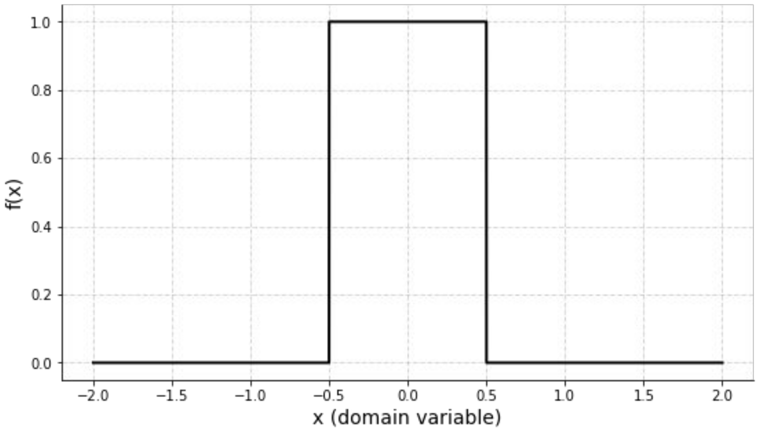


- rect(t) is not periodic
- It does not have Fourier series
- Let's create another function by periodically repeating rect(t) with a long period

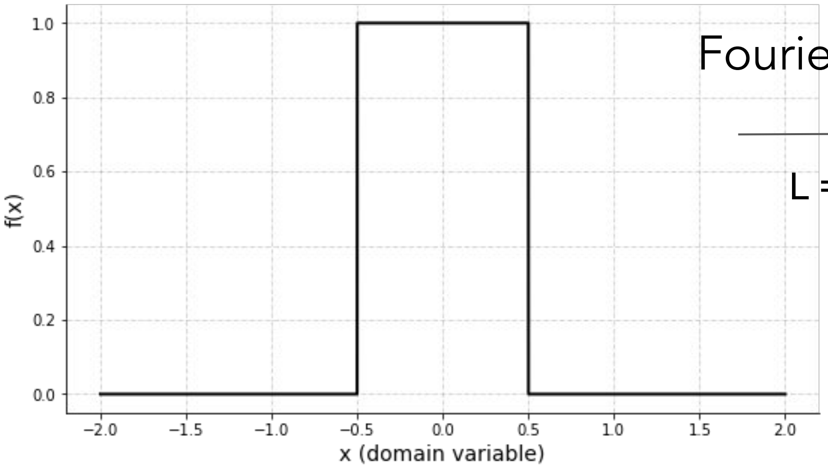
$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$





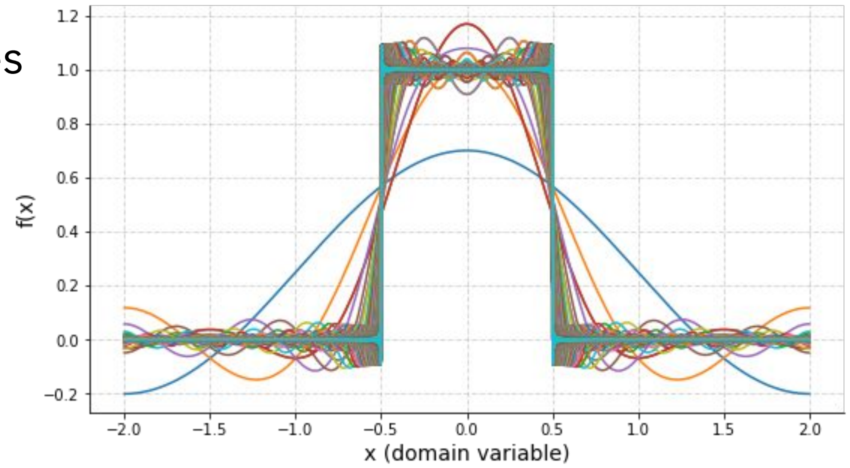


1.0
0.8
0.6
0.4
0.2
0.0
f(x)
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x (domain variable)



Fourier Series

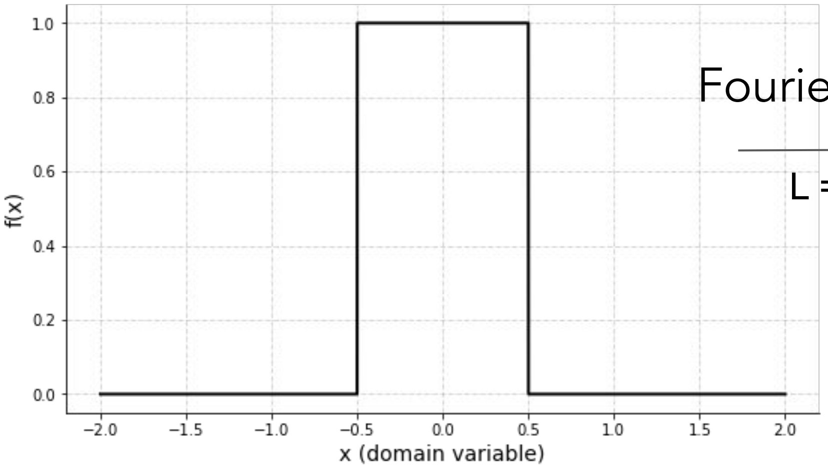
L = 4



Colors indicate approximation with increasing m

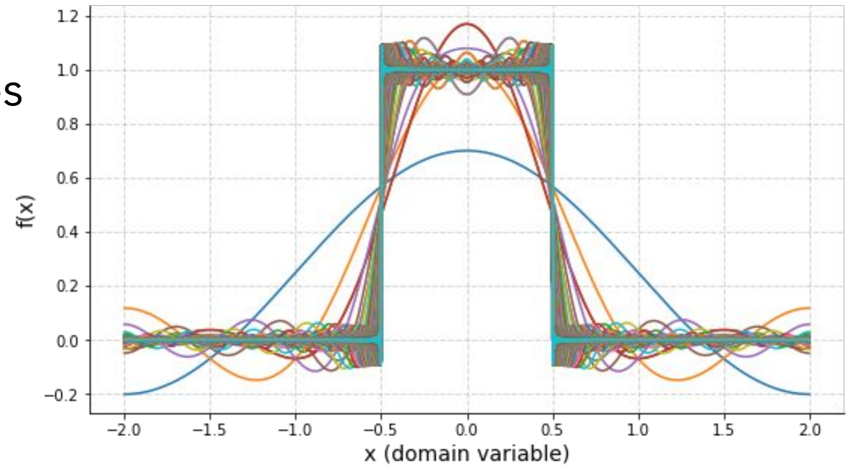
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) dx \\ a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{m\pi x}{L}) dx, \text{ and} \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{m\pi x}{L}) dx. \end{aligned}$$



Fourier Series

L = 2



Colors indicate approximation with increasing m

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

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Did you understand the Fourier series?
10:37 pm
Haven't watched yet...is it available on Netflix?
10:38 pm
Thank you 🙏
10:40 pm

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

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Fourier Series

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Using

Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi}+1=0$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

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Using Euler formula

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j\pi m x/L}$$

$$c_0 = a_0/2$$

$$c_m = \frac{(a_m - jb_m)}{2}, \text{ for m>0}$$

$$c_{-m} = \frac{(a_m + jb_m)}{2}, \text{ for m<0}$$

Exponential representation

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j\pi mx/L}$$

$$c_0 = a_0/2$$

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$$c_{-m} = \frac{(a_m + jb_m)}{2}, \text{ for m<0}$$

Equivalently,

with T = 2L

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kx}{T}}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi kx}{T}} f(x)dx$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j\pi m x/L}$$

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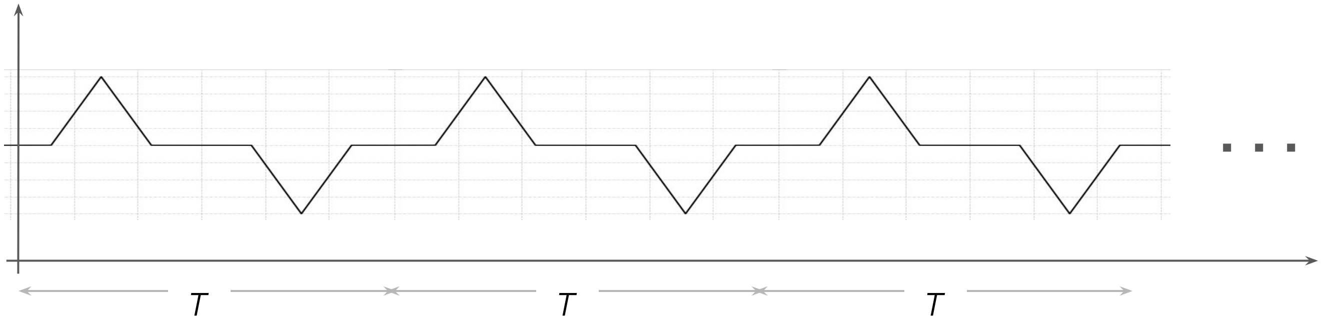
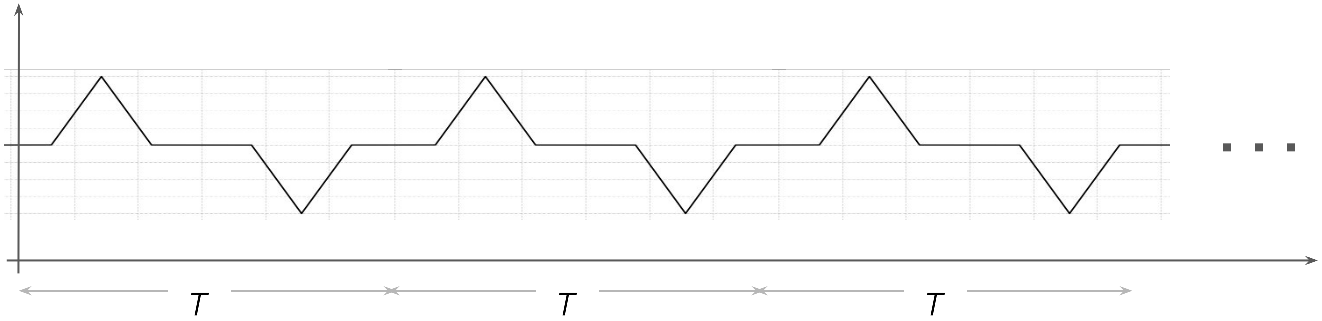
Notion of frequency: $\dfrac{k}{T}$



- For a signal with period *T* the Fourier series representation has the following form.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$



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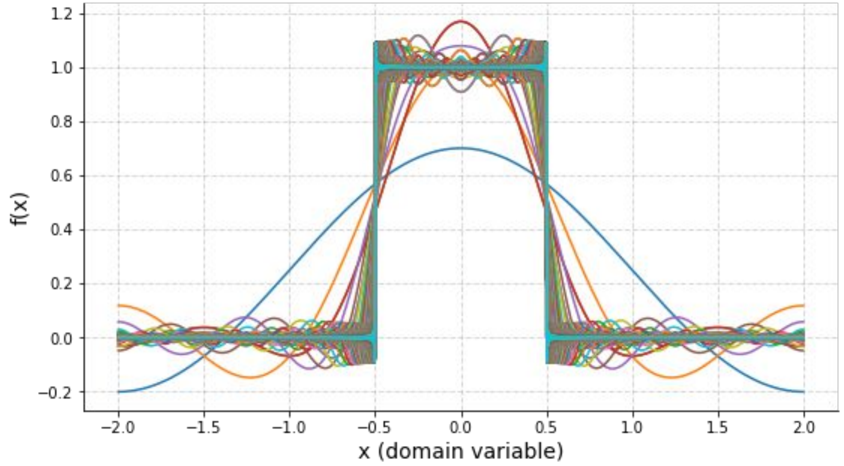
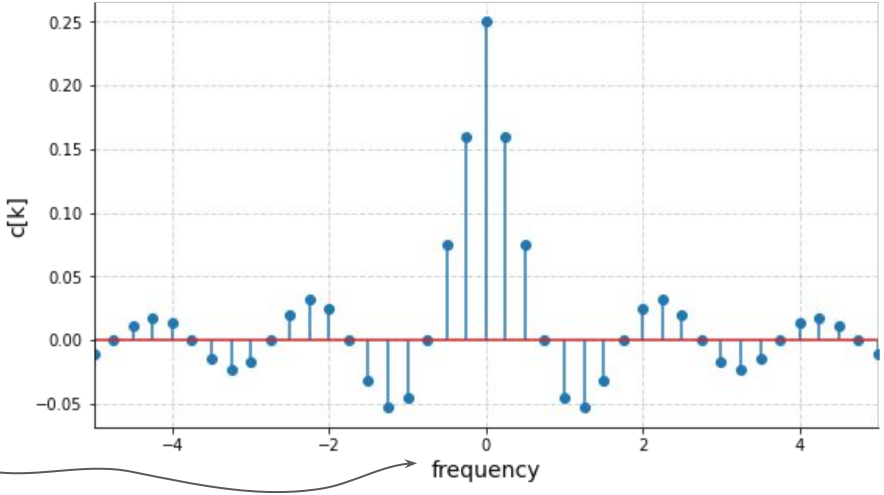
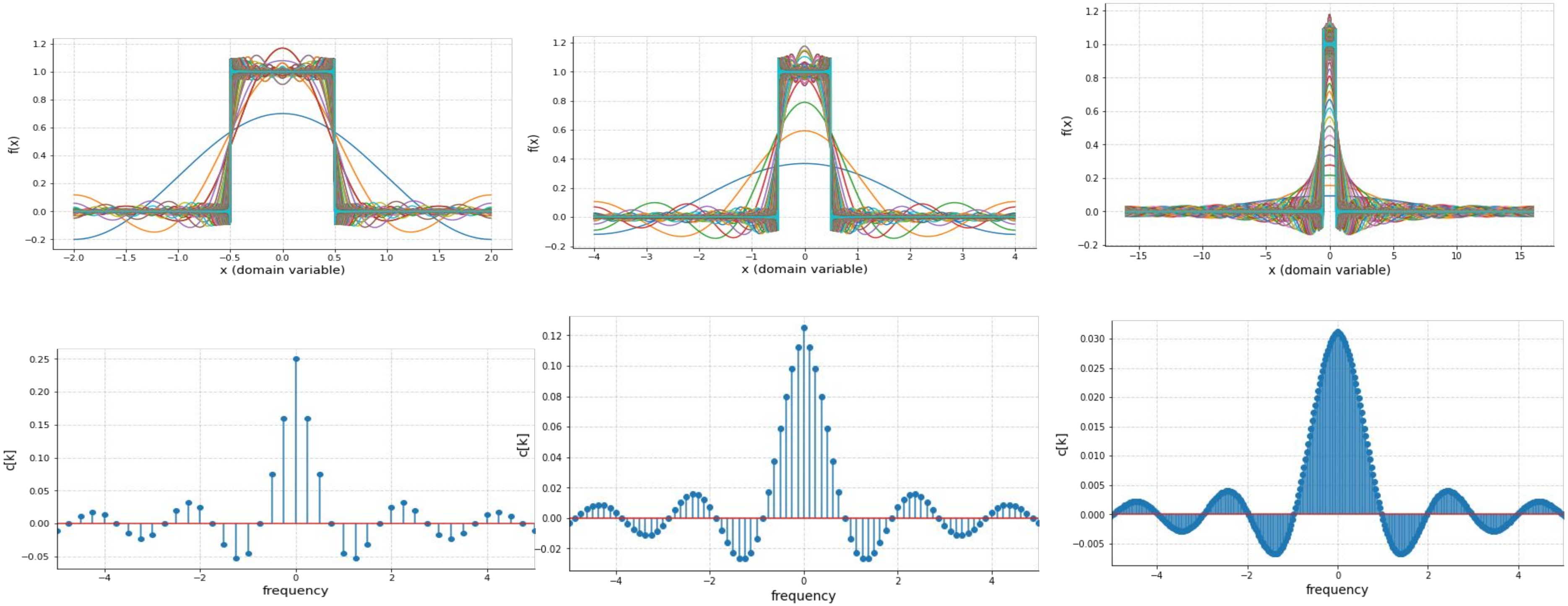




Notion of frequency: $\dfrac{k}{T}$

T = 4 T = 16 T = 32



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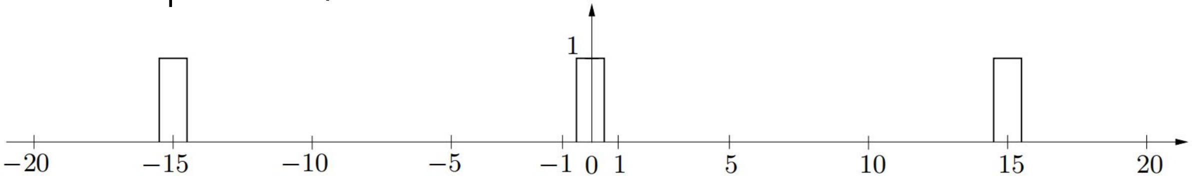
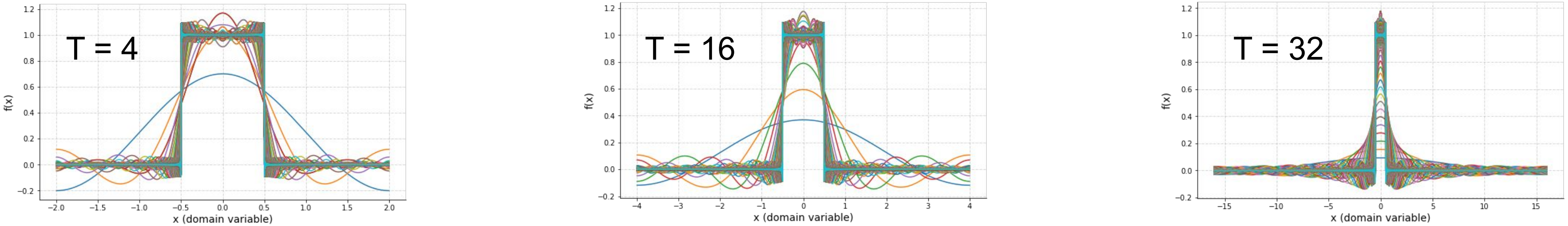
$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi k x}{T}} f(x)\,dx$$

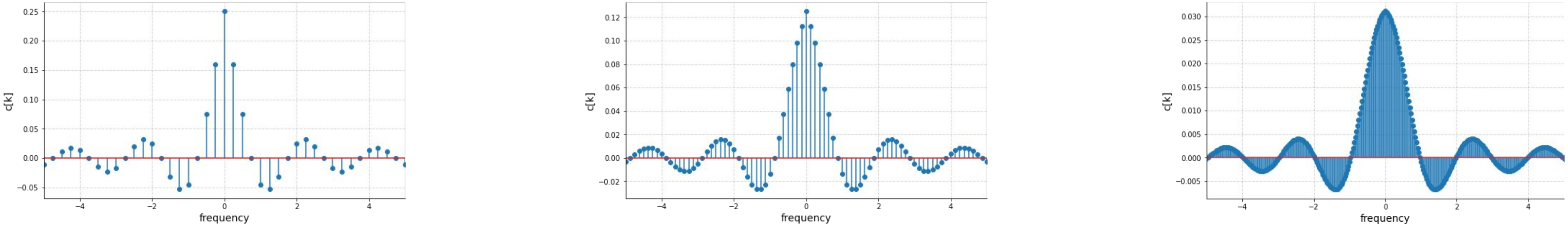
- For a rect(t) signal, periodized with a period T,

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$



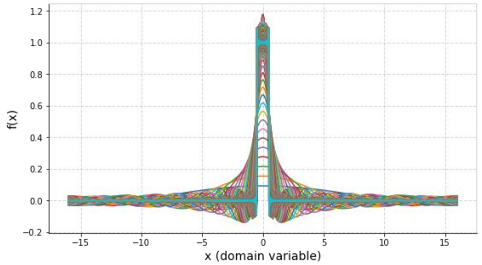


Fourier series coefficients: The c[k] representation starts getting crowded (or denser) as T increases



Computing Fourier series of rect(t) function after periodization with period *T*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$



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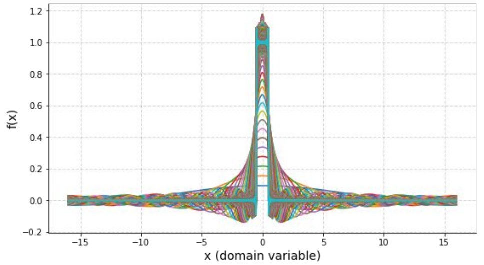
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T}\Pi(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T}\cdot 1\,dt$$



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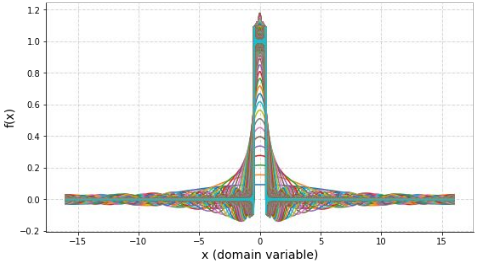
$$= \frac{1}{T}\left[\frac{1}{-2\pi i n/T}e^{-2\pi i n t/T}\right]_{t=-1/2}^{t=1/2} = \frac{1}{2\pi i n}\left(e^{\pi i n/T} - e^{-\pi i n/T}\right)$$



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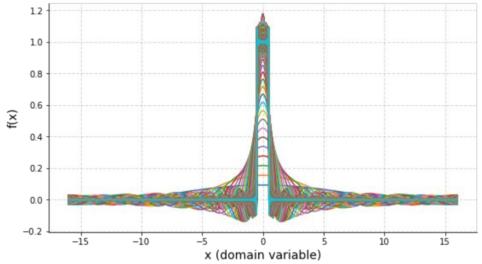
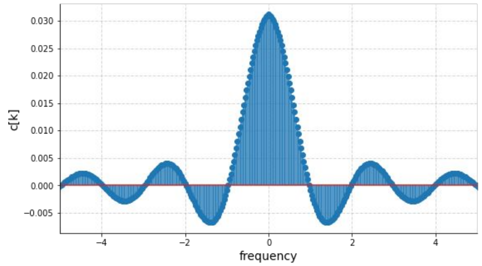
$$= \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$





Computing Fourier series of rect(t) function after periodization with period *T*

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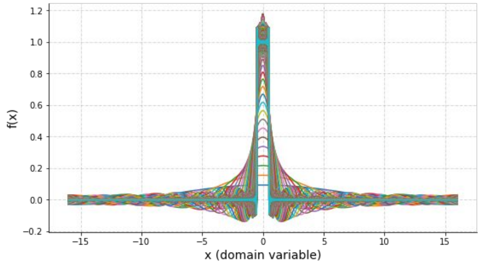
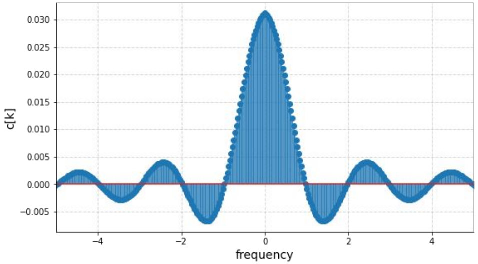
$$= \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$

$$\text{(Transform of periodized }\Pi)\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$





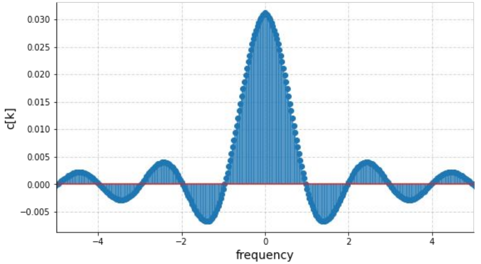
Computing Fourier series of rect(t) function after periodization with period *T*

$$\text{(Transform of periodized } \Pi\text{)}\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$

$$\text{(Scaled transform of periodized } \Pi\text{)}\left(\frac{n}{T}\right) = T\,\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) = \frac{\sin(\pi n/T)}{\pi n/T}$$

notion of frequency (s) = n/T

$$\text{(Scaled transform of periodized } \Pi\text{)}(s) = \frac{\sin \pi s}{\pi s}$$



- As T tends to infinity, *s* represents a continuous variable
- Have obtained a representation of an aperiodic signal rect(t)
- Fourier transform is born!

Computing Fourier representation of any function

Let's consider any *f(t)* in general

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T} f(t)\, dt$$

$$\begin{aligned} |c_n| &= \frac{1}{T}\left|\int_{-1/2}^{1/2} e^{-2\pi i n t/T} f(t)\, dt\right| \\ &\le \frac{1}{T}\int_{-1/2}^{1/2} |e^{-2\pi i n t/T}|\, |f(t)|\, dt = \frac{1}{T}\int_{-1/2}^{1/2} |f(t)|\, dt = \frac{A}{T} \end{aligned}$$

$$A = \int_{-1/2}^{1/2} |f(t)|\, dt\,,$$

Computing Fourier representation of any function

Let's consider any *f(t)* in general

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T} f(t)\,dt$$

$$(\text{Scaled transform of periodized } f)\left(\frac{n}{T}\right) = Tc_n = \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt$$

In the limit as $T \to \infty$ we replace n/T by s and consider

$$\boxed{\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt}$$

- Fourier Series

The spectrum of a periodic signal is a discrete set of frequencies, possibly an infinite set (when there's a corner) but always a discrete set.

$$\boxed{c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\, dt}$$

- ## Fourier Series

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- ## Fourier Transform

The spectrum of a aperiodic signal is a continuum of frequencies, or a continuous spectrum.

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt$$

Fourier Transform

produces a continuous spectrum, or a continuum of frequencies.

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

Signal $f(t)$ → Fourier Transform → Continuous representation (or spectrum) $\hat{f}(s)$

- Transforms signal from t-domain to s-domain
- Domain are inversely related
- s~n/*T* (with *T* tending to infinity)

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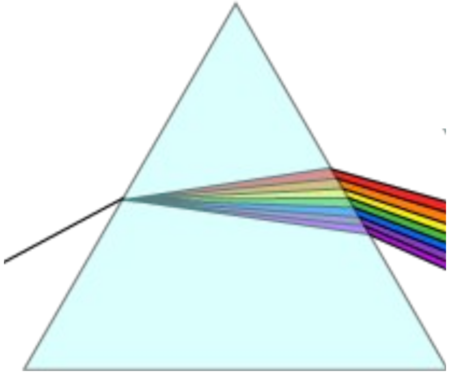

- Transforms signal from t-domain to s-domain
- Domain are inversely related
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Fourier Transform

Can we get back $\hat{f}(s)$ from $f(t)$?

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t / T}$$

Fourier Transform

Can we get back $\hat{f}(s)$ from $f(t)$?

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

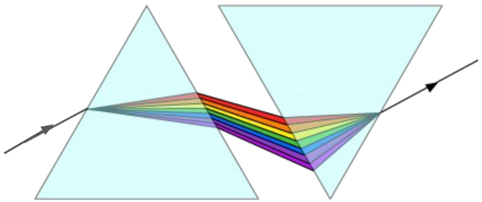
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt = \frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi i n t/T} f(t)\, dt$$

$$= \frac{1}{T}\hat{f}\left(\frac{n}{T}\right) = \frac{1}{T}\hat{f}(s_n)$$

$$\boxed{\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n)e^{2\pi i s_n t} \\ &= \sum_{n=-\infty}^{\infty} \hat{f}(s_n)e^{2\pi i s_n t}\Delta s \approx \int_{-\infty}^{\infty} \hat{f}(s)e^{2\pi i s t}\, ds \end{aligned}}$$

Yes!

Fourier Transform and Inverse Transform



$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$f(t)$
Signal

Fourier Transform

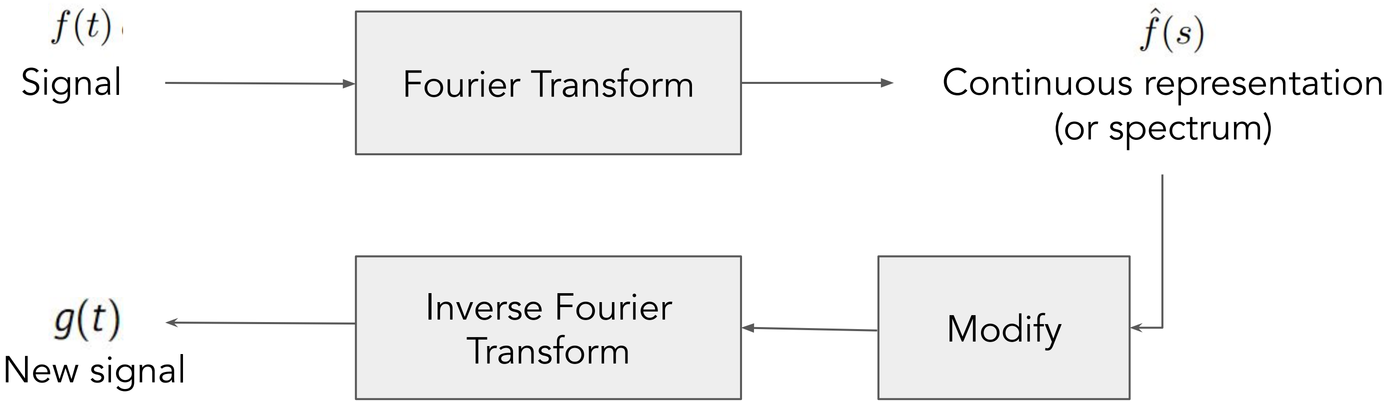
$\hat{f}(s)$
Continuous representation
(or spectrum)

Inverse Fourier
Transform

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\, ds$$

Modifying the Spectrum

Can we modify the spectrum and get a new (and more useful) signal?



Modifying the spectrum - But how? Is there a "nice" method?

$$\mathcal{F}[f] := \hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} dt$$

This will be our new notation for representing Fourier transform

Modifying the spectrum - But how? Is there a “nice” method?

$$\mathcal{F}[f] := \hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}dt$$

This will be our new notation for representing Fourier transform

Few ways to modify a signal

$$\boxed{\begin{aligned} m(t) &= f(t) + g(t) \\ \mathcal{F}[m] &= \mathcal{F}[f(t) + g(t)] = \mathcal{F}[f] + \mathcal{F}[g] \end{aligned}}$$

adding another signal

$$\boxed{\begin{aligned} m(t) &= \alpha f(t) \\ \mathcal{F}[m] &= \alpha \mathcal{F}[f] \end{aligned}}$$

scaling with a constant

- Few ways to modify a signal

$$\boxed{\begin{aligned} m(t) &= f(t) + g(t) \\ \mathcal{F}[m] &= \mathcal{F}[f(t) + g(t)] = \mathcal{F}[f] + \mathcal{F}[g] \end{aligned}}$$

adding another signal

$$\boxed{\begin{aligned} m(t) &= \alpha f(t) \\ \mathcal{F}[m] &= \alpha \mathcal{F}[f] \end{aligned}}$$

scaling with a constant

- How about multiplying two spectrums?

- Few ways to modify a signal

$$\boxed{\begin{aligned} m(t) &= f(t) + g(t) \\ \mathcal{F}[m] &= \mathcal{F}[f(t) + g(t)] = \mathcal{F}[f] + \mathcal{F}[g] \end{aligned}}$$

adding another signal

$$\boxed{\begin{aligned} m(t) &= \alpha f(t) \\ \mathcal{F}[m] &= \alpha \mathcal{F}[f] \end{aligned}}$$

scaling with a constant

- How about multiplying two spectrums?
  - Use case: spectrum of one signal can be weighted using spectrum of another signal

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi st} g(t)dt \int_{-\infty}^{\infty} e^{-i2\pi sx} f(x)dx$$

  - What is the resulting time domain operation?

- Multiplying two spectrums

$$
\begin{aligned}
\mathcal{F}[g]\mathcal{F}[f] &= \int_{-\infty}^{\infty} e^{-i2\pi st} g(t)dt \int_{-\infty}^{\infty} e^{-i2\pi sx} f(x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi s(t+x)} g(t)f(x)dtdx \\
(t+x) &= u \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi su} g(u-x)f(x)dudx \\
&= \int_{-\infty}^{\infty} e^{-i2\pi su} \underbrace{\left(\int_{-\infty}^{\infty} g(u-x)f(x)dx\right)}_{h(u)} du \\
\mathcal{F}[g]\mathcal{F}[f] &= \int_{-\infty}^{\infty} e^{-i2\pi su} h(u)du \\
&= \mathcal{F}[h]
\end{aligned}
$$

We refer to this fondly by
"f is convolved with g"

- Multiplying two spectrums

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi su} h(u)du$$

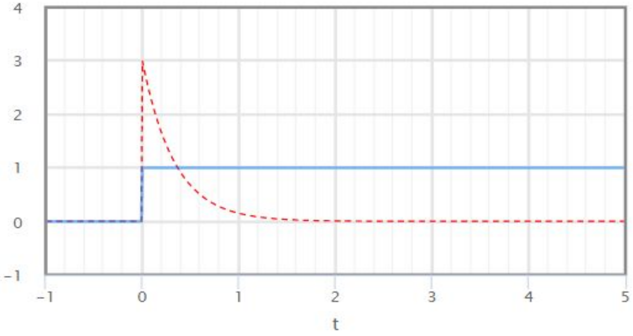
Equivalent operation in time domain

$$\underbrace{\int_{-\infty}^{\infty} g(u-x)f(x)dx}_{h(u)}$$

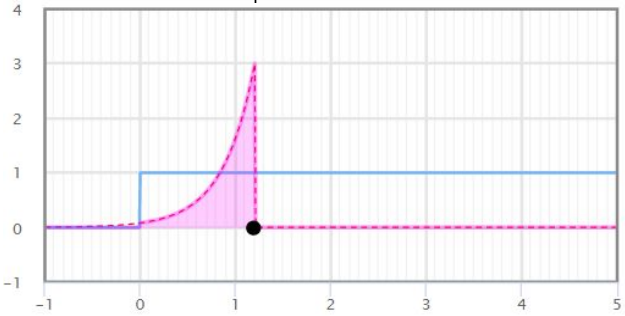
Convolution is born!

Example: https://lpsa.swarthmore.edu/Convolution/CI.html

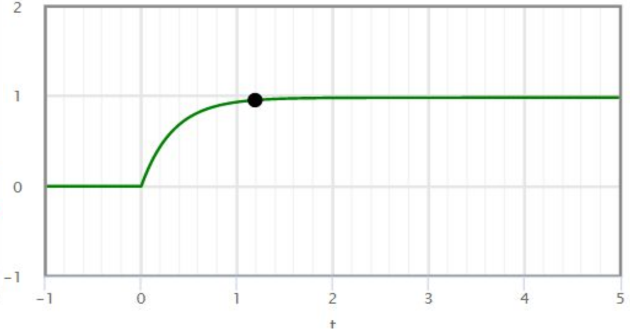
Signals



Step 1: Flip one of them,
Step 2: translate inside



Step 3: Product and Integrate,
Step 4: Go to step 2



- Multiplying two spectrums

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi su} h(u) du$$

Equivalent operation in time domain

$$\underbrace{\int_{-\infty}^{\infty} g(u-x) f(x) dx}_{h(u)}$$

- Convolution is a linear operation - nothing fancy but cleaver
- Multiplying two spectrums helps to weight the spectrum of one signal using another
- This weighting operation is useful for:
    - Spectrum enhancement
    - Noise removal
    - Feature extraction

Example: https://dspillustrations.com/pages/posts/misc/convolution-examples-and-the-convolution-integral.html

- Summary

Signal → Model *or* Representation → Processing

- Convolution

- Polynomial series representation
- Fourier series representation
- Fourier transform representation

Thank you