Floor and ceiling functions[1]

- $\lfloor x \rfloor$ or $[x]$, *floor of* x, is the greatest integer less than or equal to x

  $\lceil x \rceil$, *ceiling of* x, is the least integer greater than or equal to x

  both are monotonically increasing functions

- $\lfloor x \rfloor = n$ iff $n \leq x < n+1$

  $\lceil x \rceil = n$ iff $n-1 < x \leq n$

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- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

- $x < n$ iff $\lfloor x \rfloor < n$

  $n < x$ iff $n < \lceil x \rceil$

  $x \leq n$ iff $\lceil x \rceil \leq n$

  $n \leq x$ iff $n \leq \lfloor x \rfloor$

- $\lfloor -x \rfloor = -\lceil x \rceil$

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- $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

  $\lceil x + n \rceil = \lceil x \rceil + n$

- for any integer n, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$

- $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$

  $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$

- for an integer x and a positive integer y,

[1] Prepared by R. Inkulu. http://www.iitg.ac.in/rinkulu/

$x \mod y = x - y\lfloor \frac{x}{y} \rfloor$

- $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$, $\lceil \sqrt{x} \rceil = \lceil \sqrt{\lceil x \rceil} \rceil$
- for integers $m > 0$ and n,

  $\lfloor \frac{x+n}{m} \rfloor = \lfloor \frac{\lfloor x \rfloor + n}{m} \rfloor$,

  $\lceil \frac{x+n}{m} \rceil = \lceil \frac{\lceil x \rceil + n}{m} \rceil$

  for positive integer a_j,

  $\lfloor \ldots \lfloor \lfloor x/a_1 \rfloor / a_2 \rfloor \ldots / a_k \rfloor = \lfloor \frac{x}{a_1 a_2 \ldots a_k} \rfloor$,

  $\lceil \ldots \lceil \lceil x/a_1 \rceil / a_2 \rceil \ldots / a_k \rceil = \lceil \frac{x}{a_1 a_2 \ldots a_k} \rceil$

- converting floors to ceilings, and vice versa -

  $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor = \lfloor \frac{n-1}{m} \rfloor + 1$

  $\lfloor \frac{n}{m} \rfloor = \lceil \frac{n-m+1}{m} \rceil = \lceil \frac{n+1}{m} \rceil - 1$

- for integer $b \geq 2$ and $x \geq 1$,

  $\lfloor \lg_b x \rfloor = \lfloor \lg_b \lfloor x \rfloor \rfloor$

  $\lceil \lg_b x \rceil = \lceil \lg_b \lceil x \rceil \rceil$

  $k = \lfloor \lg_b x \rfloor$ iff $b^k \leq x < b^{k+1}$

  $k = \lceil \lg_b x \rceil$ iff $b^{k-1} \leq x < b^k$

- $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- for any real numbers c, x, and $y \neq 0$,

  $x \mod y = x - y\lfloor x/y \rfloor$

  $x = \lfloor x \rfloor + (x \mod 1)$

  $c(x \mod y) = (cx) \mod (cy)$

- for all real x and integer m, $\lfloor mx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{m} \rfloor + \ldots + \lfloor x + \frac{m-1}{m} \rfloor$

- $\sum_{k=0}^{n-1} \lfloor \sqrt{k} \rfloor = n - \frac{1}{3}(\lfloor \sqrt{n} \rfloor)^3 - \frac{1}{2}(\lfloor \sqrt{n} \rfloor)^2 - \frac{1}{6}\lfloor \sqrt{n} \rfloor$

  $\sum_{k=0}^{m-1} \lfloor \frac{nk+x}{m} \rfloor = \sum_{k=0}^{n-1} \lfloor \frac{mk+x}{n} \rfloor$