Choice Under Uncertainty: Part B

That's right! We demand rigidly defined areas of doubt and uncertainty!

(The philosopher Vroomfondel in Hitchhikers' Guide to the Galaxy, Douglas Adams)

1 Introduction

In this section, I introduce another set of examples of choice under uncertainty. As we have mentioned earlier, these examples are drawn from "non-financial" realm. But you should be able to notice that same operating principles will be applied here as well, albeit under different guises.

2 Agricultural Contract in a Developing Economy: Sharecropping

Rarely a landlord tills his/her own land. It is mostly done by a tenant, who transfers a "fee" to the landlord. There are three major forms of landlord tenant relationship. First, the tenant can pay a fixed rent to the landlord (fixed rent tenancy). Second, the landlord can pay the tenant a fixed wage (treating the tenant as worker). Third, the product from the land can be divided between landlord and tenant (i.e. tenant gets s fraction and the landlord retains $1-s$ fraction) The last is called sharecropping contract, which have puzzled economists for a long time.

2.1 Marshallian Inefficiency

It is easy to see why. Suppose, given land, the output of any agricultural activity depends on effort (e). Output is related to labor buy the relation $Y = F(e)$, with $F' > 0$ and $F'' < 0$. There is a cost of effort, borne entirely by the tenant, $c(e)$, with $c' > 0$, $c'' \geq 0$. An agricultural contract can be written as landlord's income $(1-s)Y + R$. Here, $R > 0$ means fixed rent and $R < 0$ means fixed wage. Similarly, tenant's income is $sY - R$. If $s \neq 0$ and $R = 0$, the contract is pure sharecropping.

In the "first best" case, when the landlord tills his/her own field, first order condition is $F'(e) = c'(e)$, that is, the marginal productivity of labor must be equal to the marginal cost of labor. This maximizes the social surplus $F - c$, which is the output of the society net of cost. The first best scenario can be achieved by maximizing the joint utility of the landlord and tenant.

Now suppose there is a sharecropping contract (*with or without fixed terms*). The tenant's (who supplies effort) income is $sF(e) - R - c(e)$. The first order condition says $sF'(e) = c'(e)$. Since $s < 1$, the optimal e is less than the case where $s = 1$, i..e. the first best case. It is understandable why. Since the tenant keeps only a part of his output as reward, he will supply less effort. Consequently, the landlord can put $s = 1$, and charge an $R > 0$ in such a way that the landlord's income remains same under both scenarios. In itself, sharecropping is associated with lower social surplus and lower income for both parties. This is known as Marshallian inefficiency of sharecropping.[1]

The question, then, is the following. Why do landlords and tenants actively seek sharecropping contract even if is arguably inferior to other types of contract? As it turns out, the answer can be given in multiple ways. What follows is the first take to the question.

2.2 Risk Sharing

The intuition that eluded Marshall was the following: Agriculture is a risky business, given the long gestation period (three to six months) and occurrence of pest, droughts or floods. If the parties settle for pure wage (rental) contract $(s = 0, R < 0)$ then the risk of production is borne solely by the

[1] Named after Alfred Marshall, a Victorioan economist (who is also responsible for supply and demand curves, consumer surplus, producer surplus etc.). He was comparing the English system (fixed rent) with French system (metayage: sharecropping).

landlord (tenant). In places where insurance markets are missing, sharecropping plays a role of limited insurance by sharing the risk. If crops are successful, the payment to both landlord and tenant go up. If crops fail, both parties agree to share lower produce (and income).

We want to formalize the intuition. Since risk aversion plays a crucial role, we propose the following simple utilities

$$\begin{aligned} EU_T &= Ey_T - \frac{\beta_T}{2} var(y_T) - ce^2 \\ EU_L &= Ey_L - \frac{\beta_L}{2} var(y_L) \end{aligned}$$

So the tenant derives utility from consumption, but there is a cost of effort.

We propose the following simple[2] production rule, $y = e + \varepsilon$. Here, ε is a random variable with mean 0 and variance σ^2 (with appropriate restrictions on supports such that y is always positive). Thus, $E(y) = e$ and $var(y) = \sigma^2$.Landlords' income is $y_L = (1-s)\,y + R$. So $Ey_L = (1-s)\,e + R$ and $var(y_L) = (1-s)^2\,\sigma^2$

Thus, expected utility of the landlord is

$$\begin{aligned} EU_L &= E\left[y_L\right] - \frac{\beta_L}{2} var\left(y_L\right) \\ &= (1-s)\,e + R - \frac{\beta_L}{2}\,(1-s)^2\,\sigma^2 \end{aligned}$$

Similarly,

$$EU_T = se - R - \frac{\beta_T}{2} s^2 \sigma^2 - ce^2$$

To focus on risk sharing aspect, we assume that the landlord can monitor e, the effort supplied by the tenant. So the landlord's problem is to choose (s, e, R) in such a way that EU_L is maximized, given $EU_T = \Lambda$. Here, Λ can be thought as the reservation utility of the tenant. If the landlord leaves too

[2] A more general form could be $y = f(e) + \varepsilon$, but that will not change the basic message.

little EU_T for the tenant, the tenant can leave agriculture altogether and move to another job. The utility from the outside option is captured in Λ.

The relevant Lagrangian is

$$L = (1-s)\,e + R - \frac{\beta_L}{2}(1-s)^2\sigma^2 + \lambda\left(\Lambda - se + R + \frac{\beta_T}{2}s^2\sigma^2 + ce^2\right)$$

FOC's are

$$\begin{aligned}
e &: (1-s) - \lambda s + 2ce = 0 \\
s &: -e + \beta_L(1-s)\sigma^2 + \lambda\left(-e + \beta_T s\sigma^2\right) = 0 \\
R &: 1 + \lambda = 0 \\
\lambda &: \Lambda - se + R + \frac{\beta_T}{2}s^2\sigma^2 + ce^2 = 0
\end{aligned}$$

From the third equation, $\lambda = -1$. Thus, from the first equation, we have $e^* = \frac{1}{2c}$, and from the second equation, $s^* = \frac{\beta_L}{\beta_L + \beta_T}$. Once these values are determined, the value of R^* can be calculated from the last equation.

Note the following

1. Since the landlord can monitor (and enforce) effort, the level of effort is first best (i.e. maximizes $y - c(e)$).

2. The share crucially depends on the attitude towards risk. Suppose $\beta_L = 0$, i.e. the landlord is risk neutral. Then $s = 0$, i.e. all produce is grabbed by landlord. For tenants to survive, it must be the case that $R < 0$, i.e. the contract must be fixed wage. If $\beta_T = 0$, then the contract is pure rental contract.

3. For all intermediate values of $\beta_T, \beta_L \neq 0$, $0 < s < 1$, i..e. a share-cropping contract will emerge. Whether there will be a corresponding $R \neq 0$ is determined by the outside option of the tenant.

4. Note that $\frac{\partial s^*}{\partial \beta_T} < 0$. So if the tenant becomes more risk averse (than the landlord), then he/she gets less share (why?).

5. The equation does not specify, *a priori*, the value of R. Note that landlord's utility is highest if it reduces tenant to earn the least utility. Therefore, once s and e are determined, it will choose an R such that $EU_T(s^*, e^*, R^*) = \Lambda$. Whether $R^* > 0$ (fixed rent), $R^* < 0$ (fixed wage) or $R^* = 0$ (pure share cropping without rent or wage component) depends on, among other things, the bargaining power of the two parties and Λ.

One main problem here is the existence of sharecropping ($0 < s^* < 1$) depends only on the risk attitude of both parties. More specifically, it is not possible that one (or both) of them is (are) risk neutral. We will come back to this deficiency later.

3 Crime and Punishment

It was Gary Becker, a University of Chicago Economist first thought about applying tools of Economics to analyze criminal activities. He was trying to find a parking place for his car in a snowstorm, and all the nearby ones seemed booked. If he parked his car in a non-designated place, it might get a parking ticket or get towed away by the police. But police does not patrol the street regularly, and so, with some *probability*, he might escape punishment.

Now the script has all the elements of writing a model of expected utility maximization. We will now pursue one, and one which is very closely related to Indian Economy.

3.1 Tax Evasion

At the beginning, we make a point regarding what we will do here. *Tax evasion* is the failure to pay tax, while *tax avoidance* is the reorganization of economic activity in order to lower tax payment. Tax avoidance is legal but tax evasion is not, although the distinction between those are sometimes blurred. It is an important consideration for tax policy.

In what follows, following Becker, The simplest model of the evasion decision considers it to be a gamble. Tax evasion occurs if a taxpayer declares less than his/her true income (or overstates deductions). They may do so without being detected. However, There is also a chance that they may be caught. When they are caught, a punishment is inflicted: usually a fine but sometimes imprisonment. A taxpayer has to balance these gains and losses taking account of the chance of being caught and the level of the punishment

Let us now put more structure for the story. Let Y be the true income and X is the declared income. The taxpayer pays a tax on declared income. If t is the rate of tax, the amount of tax paid is tX. If he/she is not caught by the tax authorities, income is given by $Y_{nc} = Y - tX$ and the amount of evaded tax is $t\,(Y - X)$. If he(she) is honest, then $X = Y$, and the take home income is $Y\,(1 - t)$

When the taxpayer is caught evading all income, we can think of two punishment regimes. In one (harsh punishment), all income is taxed and a fine at rate F is levied on the tax that has been evaded. In the other (mellow punishment), only an extra tax is paid on income which is being evaded. Thus, for the harsh punishment regime, the income level when caught is $Y_c = [1-t]Y - Ft[Y - X]$. For the punishment to be credible, we need $(1-t)\,Y > [1-t]Y - Ft[Y-X] \Rightarrow F > 0$. On the other hand, with the "mellow" punishment regime, the income level when caught is $Y_c = Y - tX - Ft[Y-X]$ For this to be credible, we need $(1-t)\,Y > Y - tX - Ft[Y-X] \rightarrow F > 1$. Both punishment strategies lead to qualitatively identical outcomes.

Assume punishment strategy 1. A rational individual maximizes $EU = pu(Y_c) + (1-p)\,u\,(Y_{nc})$ with respect to X. Here, $u' > 0, u'' < 0$. Assuming no corner solution, $Y < X < 0$, the FOC is

$$t\left[pu'\,(Y_c)\,F - (1-p)\,u'\,(Y_{nc})\right] = 0$$

The SOC is

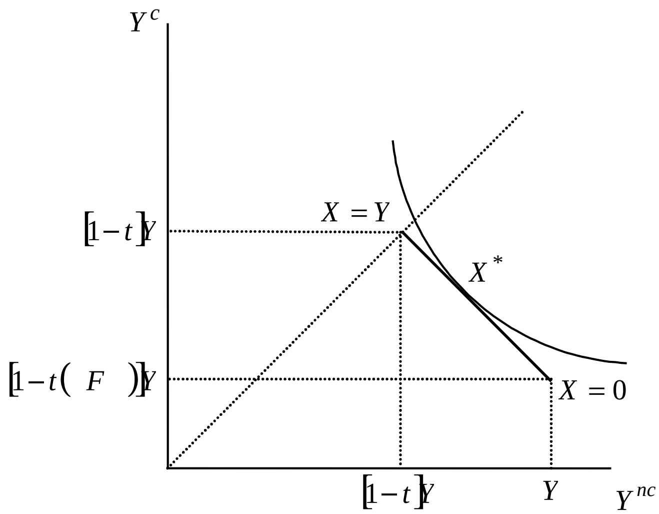
$$t^2 \left[pu''(Y_c) F^2 + (1-p) u''(Y_{nc})\right] < 0$$

if the person is risk averse. However, the above equation determines the extent of evasion, but is silent on when do people evade.

Reverting to the state-space diagram, the slope of the indifference curve is $\frac{dY_c}{dY_{nc}} = -\frac{1-p}{p}\left(\frac{u'(Y_{nc})}{u'(Y_c)}\right)$. If there is full disclosure ($X = Y$), we must have $Y_{nc} = Y_c = (1-t) Y$. That is, *on the* full disclosure line (a 45° line through origin), the slope of the indifference curve must be $-\frac{1-p}{p}$. If $X = 0$ (no disclosure), income of the taxpayer is $Y_{nc} = Y$, and $Y_c = (1 - t - Ft) Y$. We can assume values of F and t such that $1 > t(1+F)$, such that negative income is ruled out. As X varies from Y to 0, the line[3] connecting these income levels is the constraint for the taxpayer. Slope of the line is $-F < 0$. Evasion occurs ($X^* < Y$) if the (magnitude of) the slope of the indifference curve on the 45° is more than slope of the constraint, that is $\frac{1-p}{p} > F$. This simplifies into $p(1+F) < 1 \rightarrow$ either p OR (or both) F is (are) too low. These two are symptoms of weak tax administration.

This can be summarized in the following diagram

[3] In the $Y_c - Y_{nc}$ plane, the change in Y and Y_{nc} comes solely through change in X. Since the marginal effect of X on either is constant, $\frac{\Delta Y}{\Delta Y_{nc}}$ will also be constant, and hence it will be a straight line.



The amount of optimal evasion can be worked out from the first order condition. This is given by $E = Y - X^*$, and the tax shortfall is $t(Y - X^*)$

3.2 Comparative Statics

Now that we know what determines (a) decision to evade and (b) optimal evasion, we can perform some comparative statics results. We have four parameters: p, F, Y and t. Questions that we are trying to answer are: do stricter institutions reduce evasion? Do rich people evade more? Does increasing tax rates increase compliance?

3.2.1 Higher Probability of Audit

If p is increased, the slope of the indifference curve at the 45° line falls. Flatter indifference curve (on 45° line) implies the tangency will be achieved closer to full disclosure point. Higher p increases compliance, as expected. This is also evident from the derivative. Notice that, from our comparative statics methods (topic 3A), $sign\left(\frac{dX}{dp}\right) = sign\,(EU_{Xp})$. Looking back at the FOC we have

$$EU_{Xp} = t\left[u'(Y_c) + u'(Y_{nc})\right] > 0$$

3.2.2 Increased Fine Rate

An increase in F will make the 'budget line' rotate to left (steeper) through the point of full disclosure (on 45° line). Slope of indifference curves do not change. New tangency will occur to the north-east of initial tangency. Thus X increases (moving towards full disclosure). As expected, higher punishment increases disclosure. Again, going back to the FOC

$$EU_{XF} = t\left[pu''(Y_c) * \{-t(Y - X)\}\right] > 0$$

3.2.3 Income

Suppose income increases from Y_1 to Y_2. The budget constraint moves out in a parallel fashion, the end points being $((1-t)Y_2, (1-t)Y_2)$ and $[Y_2, (1-t-Ft)Y_2]$. The level of evasion, $E = Y - X$ also changes. Whether X will increase or decrease or stay same, comes from the risk attitude. For evasion to increase, one must have $\dfrac{dX}{dY} < 1$

Suppose agents exhibit decreasing risk aversion, which means indifference curves become flatter. On the new budget line, one can surmise that X_2 is closer to Y_2. However, the graph does not provide a ready answer to whether or not evasion itself increases.

Given

$$EU_{XY} = t\left[pu''(Y_c) * (1 - t - Ft) - \big(1-p)u''(Y_{nc})\right]$$

It is not immediately clear whether X is increasing (honesty) or decreasing (more evasion). However, one can multiply and divide the above equation by a positive number, $u'(Y_c)$. Then we have (disregard the positive multiplicative term now)

$$\begin{aligned} EU_{XY} &= t\left[pF\frac{u''(Y_c)}{u'(Y_c)} * (1-t-Ft) - (1-p)\frac{u''(Y_{nc})}{u'(Y_c)}\right] \\ &= t\left[pF\frac{u''(Y_c)}{u'(Y_c)} * (1-t-Ft) - (1-p)\frac{u''(Y_{nc})}{\frac{1-p}{pF}u'(Y_{nc})}\right] \\ &= t\left[-pF(1-t-Ft)R_A(Y_c) + pFR_A(Y_{nc})\right] \\ &= tpF\left[R_A(Y_{nc}) - (1-t-Ft)R_A(Y_c)\right] \end{aligned}$$

The second equality follows from the FOC. Here, R_A is the Arrow-Pratt measure of absolute risk aversion.

Exercise 3.1 *Show that*

$$-SOC = tpF\left[tFR_A(Y_c) + tR_A(Y_{nc})\right]$$

Now suppose we take the difference between denominator and numerator. Then we have

$$\begin{aligned} & -SOC - (EU_{XY}) \\ = \; & [tFR_A(Y_c) + tR_A(Y_{nc})] - [R_A(Y_{nc}) - (1-t-Ft)R_A(Y_c)] \\ = \; & (1-t) * [R_A(Y_c) - R_A(Y_{nc})] \end{aligned}$$

Thus, if there is constant absolute risk aversion, $\frac{dX}{dY} = 1$, and hence evasion stays constant. If we have decreasing absolute risk aversion, $Y_c < Y_{nc} \rightarrow R_A(Y_c) > R_A(Y_{nc})$. Hence the denominator is more than the numerator$\rightarrow \frac{dX}{dY} < 1$, so that evasion increases. Thus the extent of evasion increases with income.[4] If there is IARA, $R_A(Y_c) < R_A(Y_{nc})$ and hence $\dfrac{dE}{dY} < 0$.

This result is analogous to topic 3(a): we demonstrated that if an agent shows decreasing risk aversion, he/she will invest more in risky asset if income increases (risk is a normal commodity). Here, too, one can assume that the agent is splitting his/her income into two parts: declared (X) and

[4] Malya-Modi syndrome

undeclared (E) such that $Y = X + E$. In both states of the world. the declared income undergoes a capital loss: it becomes $X(1-t)$. If the world is in good state, and he is not caught, his income is $Y_{nc} = Y - tX = Y - X + X - tX = X(1-t) + E$. However, if he gets caught, his income becomes

$$\begin{aligned} Y_c &= Y(1-t-Ft) + FtX \\ &= X(1-t) - X(1-t) + Y(1-t) - Ft(Y-X) \\ &= X(1-t) + E(1-t-Ft) \end{aligned}$$

That is, there is stochastic return on his risky asset, E.

3.2.4 Tax Rate

With higher t, the constraint shifts inwards in a parallel fashion. This is a mirror image of increased Y. Thus, invoking the result from the previous section, we can say that with DARA, *tax compliance increases with increased t.* This result has generated some controversy (assuming decreasing risk aversion is the most common behavior), as empirics fail to match the theory.

To some extent, the problem can be rectified by using a 'moral cost' of evasion. Assume that each agent suffers a psychological cost that is proportional to evasion (because he/she is participating in an illegal act). Utility function now, in both states of the world is

$$u(Y) - \phi * (Y - X)$$

The parameter ϕ can range from 0 ('utterly shameless') to a large positive number.

The expression for expected utility is now

$$pu((1-t-Ft)Y + FtX) + (1-p)u(Y - tX) - \phi(Y - X)$$

The first order condition (with respect to X) is

$$\phi = t * p * u'(Y_{nc}) - Ft * (1-p) * u'(Y_c)$$

The left hand side is the marginal (psychological) cost of evasion, and the right hand side is net marginal benefit of evasion.

If a person does nor evade, $(X = Y)$, the net marginal benefit from evading one more \$ is

$$\begin{aligned} & tp * u'((1-t)Y) - Ft(1-p) * u'((1-t)Y) \\ = \; & (tp - Ft(1-p)) * u'((1-t)Y) \\ = \; & k_0(t, p, F, Y, u) \end{aligned}$$

This is a constant since tax rate, probabilities, fine rate, income and form of utility function are unchanged.

Therefore the evasion decision (from no evasion) is following

$$\begin{aligned} E &> 0 \text{ if } \phi < k_0 \\ &= 0 \text{ if } \phi > k_0 \end{aligned}$$

For the agent indifferent between evading and not-evading, we must have $\phi = k_0$.

Remark 3.1 *If ϕ follows a distribution (within supports $[0, \phi_{\max}]$) with CDF F, the proportion of population that evades is $F(k_0) = \int_0^{k_0} f(\phi)\, d\phi$, and the proportion of the population that does not evade is $1 - F(k_0) = \int_{k_0}^{\phi_{\max}} f(\phi)\, d\phi$.*

Let us see what happens to k_0 as t increases. Differentiating k_0 with respect to t

$$(p - F(1-p)) * u'((1-t)Y) - t(p - F(1-p))\, u''() * Y$$

Noting $p - F(1-p) > 0$, the expression is positive. Hence, more people evade as tax rates increase.

Observe that there are two (opposite) effects working at the same time. First, with increasing t, if anybody is already evading, evasion falls if the person shows DARA. However, with increasing t, more person (who were not evading before), takes recourse to evasion. The end result depends on the relative strengths of these two opposing factors.

Exercise 3.2 *How does k_0 depend on p, F and Y?*

4 Consumption and Savings

Consumption/savings are vital for macroeconomic performance of a country. Note that, savings is always a virtue for an individual (as the advice is drummed by people from older generation), but the answer for national (or global) economy is not so straightforward. Higher (than usual) savings lead to lower consumption, and hence lower expectation about the future and lower investment. Keynes, the father of modern macroeconomics, termed this phenomenon the "paradox of thrift".

In what follows, we will assume two periods. A consumer earns income y_t in period t. In the first period, income is risk free. Second period income is risky. In presence of uncertainty in the next period, consumer might save more in the first period. It would appear that in presence of risk aversion, consumers will be ready to save more. However, the idea is not correct: risk aversion is related to contemporary risk, *not* future risk.

4.1 The Model

Let c_t be consumption (expressed in \$) in period $t\,(=1,2)$. Assume the utility function to be $U(c_1, c_2) = u(c_1) + \beta u(c_2)$. So we have additive structure of utility and future utility is discounted $(0 < \beta < 1)$. Period 1 budget constraint is $c_1 + s_1 = y_1$.Period 2 budget constraint is $c_2 = y_2 + (1+r)\,s_1$, where y_t is income, r is the interest rate and $s_1 = y_1 - c_1$ is the initial period savings. If there is no uncertainty, consumer simply maximizes

$$\max_{c_1, c_2} u\left(c_1\right) + \beta u\left(c_2\right)$$

Subject to the budget constraint that $c_2 = y_2 + (1+r)(y_1 - c_1) \Leftrightarrow c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$. The FOC is

$$u'(c_1) = \beta(1+r)u'(y_2 + (y_1 - c_1)(1+r))$$

The SOC is

$$u''(c_1) + \beta(1+r)^2 u''(c_2) < 0$$

which is guaranteed if $u'' < 0$. An increase in either β (consumer is more patient) or r (current consumption is costlier), c_1 decreases, thereby reducing savings. However, there is also a positive income effect (of increased interest rate) that would increase c_1. Therefore, the effect of r on c_1 (and hence savings) is ambiguous.

Uncertainty can affect of future income in two ways. First, the distribution of income can be such that mean return increases (the new distribution FOSD the initial, degenerate distribution). For example, y_2 can be y_h with probability p, y_l with probability $1-p$, such that $p(y_h) + (1-p)y_l > y_2$. The agent maximizes expected utility $u(c_1) + \beta E u(c_2)$. The FOC is

$$u'(c_1) = \beta(1+r)Eu'(c_2)$$

The SOC is

$$u''(c_1) + \beta(1+r)^2 Eu''(c_2) < 0$$

The SOC holds if $u_i'' < 0$, i.e. risk aversion guarantees SOC. If the shift in income is such that p increases, the stochastic income dominates the static income in a FOSD sense. *Ceteris paribus*, this implies the RHS of the equation to fall. Therefore, the LHS of the equation should fall as well, i.e. c_1 should increase. Thus, if the future prospects are unambiguously better, agents save less (or may be borrow more) in anticipation of a bright future.

Now suppose the income in period 2 be stochastic in a mean preserving way, that is the stochastic income is $\hat{y}_2 = y_2 + \varepsilon$, where $E(\varepsilon) = 0$. Thus, future income becomes more riskier than now, without altering the mean.

The FOC now is (assuming the normalization that $\beta(1+r)=1$)

$$u'(c_1) = Eu'(y_2 + \varepsilon + (y_1 - c_1)(1+r))$$

The second order condition is

$$u''(c_{1\cdot}) + (1+r)\, Eu''(y_2 + \varepsilon + (y_1 - c_1)(1+r)) < 0$$

This is, as we have seen before, guaranteed by assumption of risk aversion.

Do savings increase? Not necessarily. Note that $u'(y_2 + (y_1 - c_1)(1+r)) = u'(y_2 + E(\varepsilon) + (y_1 - c_1)(1+r)) < Eu'(y_2 + \varepsilon + (y_1 - c_1)(1+r))$, if u' is convex[5] (that is $u''' > 0$). In that case, compared to the "no-shock" regime, the RHS increases, and therefore the LHS increases as well. Therefore the value of c_1 falls, which implies the agent must be saving more compared to no-shock case. If the above criteria is not fulfilled (marginal utility is concave), then savings will fall in anticipation of the shock.

One can also look at the result from a equivalent, but different angle. To fix ideas, let $\beta(1+r) = 1$, WLOG. Then the equalities can be written as,respectively

$$u'(c_1) = u'(c_2) \quad \text{(a)}$$
$$u'(c_1) = Eu'(c_2) \quad \text{(b)}$$

Let us interpret the sure case also as a degenerate lottery (say F) with probabilities $(1,0)$. Thus, the second situation involves (say lottery G) increasing in riskiness in the sense of SOSD. We know that (topic 2), iff $F(SOSD)G$, then for any concave (convex) function f, $\int f dF > (<) \int f dG$. Looking at convex u' (hence $u''' > 0$) thus Eu' increases $\rightarrow u'(c_1)$ increases$\rightarrow$ c_1 must fall and hence s_1 must rise.

Such savings are known as precautionary savings (for "rainy days"). What we show here is that such savings do depend on the sign of u'''. For

[5] Jensen's inequality.

this reason, agents for whom $u''' > 0$ are termed as "prudent".[6]

5 Critique of Expected Utility

The structure of expected utility is elegant and widely applicable. However, there are some problems with it. Here, we offer some well known paradoxes and critiques.

5.1 Allais Paradox

The Allais paradox was proposed by Maurice Allais, apparently in one of the seminar on expected utility by von Neumann. In essence, he prepared a questionnaire, to be asked during the seminar. It involves choice between two lotteries

A: 100% chance of 1 million$.

B: 10% chance of 5 million, 89% chance of 1 million, 1% chance of 0

Now compare this with the following lotteries

C: 11% chance of 1 million, and 89% chance of nothing

D: 10% chance of 5 million and a 90% chance of nothing.

In various experiments, about 65% of individuals[7] chose A over B and D over C. Why is this paradoxical?

Choosing A over B implies, in a vNM framework

$$v(1) > .1 * v(5) + .89 * v(1) + .01 * v(0)$$

where v is a standard Bernoulli utility function.

Choice of C over D implies

$$.1 * v(5) + .9 * v(0) > .11 * v(1) + .89 * v(0)$$

[6] One can define coefficient of absolute prudence as $P_A = -\frac{u'''}{u''}$, coefficient of relative prudence as $P_R = -\frac{u'''}{u''} y$ etc.

[7] Read Munier (*Journal of Economic Perspective,* 1991, "Nobel Laureate: The Many Other Allais Paradoxes"), for a non technical analysis.

If one works with the inequalities, then these two cannot hold at the same time (I am leaving this as a homework).

Allais proposed that decision makers evaluate lotteries differently when a lottery is very close to certainty. Lottery A is certain, and risk aversion implies one wants something with certainty. This preference for certainty gets diluted when choice is between C and D. Thus, not only different people have different risk preferences, but a single individual may also have changing perception of risk/return.

5.2 Ellsberg Paradox

Daniel Ellsberg, in 1961, proposed an experiment which apparently violated axioms of expected utility. Let us consider an urn with 300 marbles. 100 are red, the rest are either blue or green. One takes out one marble from the urn. We have the following two gambles:

A: $1000 if the marble is red, 0 otherwise.

B: 1000$ if the marble is blue, 0 otherwise.

As opposed to that, two other gambles are

C: 1000$ is the ball is not red.

D: 1000$ if the ball is not blue.

It is common for people to choose A over B and C over D. Let us define our Bernoulli utility functions such that $v(0) = 0$. Thus, the first gamble implies

$$p(R) * v(1000) > p(B) * v(1000)$$

here, $p(i), i = B, R$ reflects the probability of having a marble of certain color.

The second choice implies

$$p(\sim R) * v(1000) > p(\sim B) * v(1000)$$

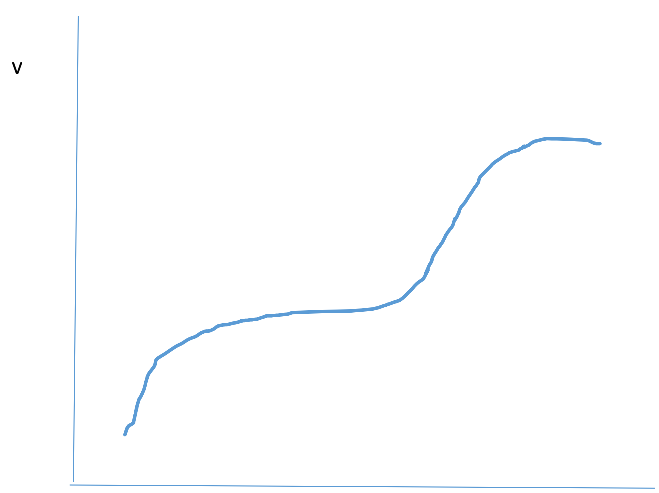
There is a contradiction since for any event A, $p(A) = 1 - p(\sim A)$

Since the probability of blue/red ball is not given, people tend to bias their decision towards available information, however incomplete. This is known as ambiguity aversion (rather than risk aversion).

However, regarding the importance of these paradoxes, the jury is still out. Some economists (but not all) tend to argue that these paradoxes are more like "optical illusions" (or suffer from framing effect, as psychologists say).

5.3 Slope of Bernoulli Utility

Other criticisms pick on the slope of Bernoulli utility. For example, the Friedman-Savage utility function has the following form



In other words, curvature and attitudes towards risk depend on the amount of wealth. At very low and very high income levels (or wealth levels), people exhibit risk aversion. There is an intermediate range where agents exhibit risk loving behavior. Such utility functions were proposed to explain the observed phenomena that why both insurance and gambling businesses flourish in a society. Assuming everybody has similar utility functions, different people belong to different economic classes.

5.4 Loss Aversion

The idea of loss aversion is one of the established critique of expected utility hypothesis. It was proposed by two psychologists, Daniel Kahneman and Amos Tversky. The starting point is that we do not care so much about our present wealth or income *per se*, but what matters is gain and loss from the status quo. In our value function, we treat gain and losses differently. A 10$ loss in wealth is given more importance than a 10$ gain. If x is gain/loss, there is a standard utility function $v(x)$ defined over x. Kahneman and Tversky have proposed the following utility function

$$\begin{aligned} v(x) &= x^{\alpha} \text{ if } x \geq 0 \\ &= -\lambda\left(-x\right)^{\beta} \text{ if } x < 0 \end{aligned}$$

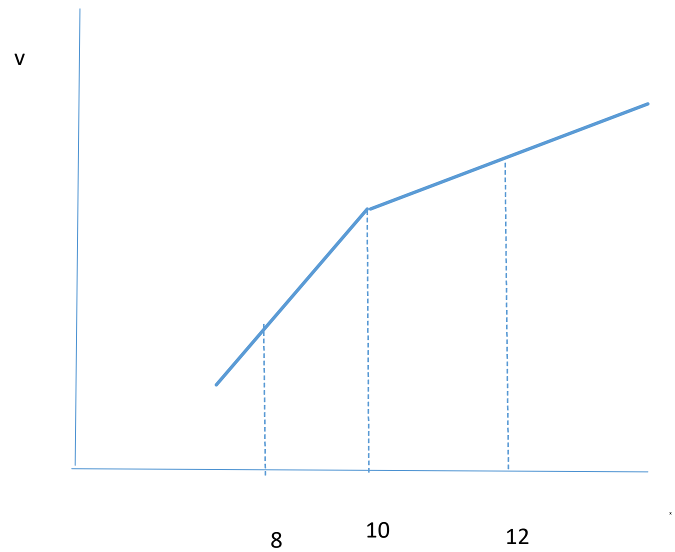
Here, $\lambda > 1$ is called the "*loss aversion*" parameter. If we put $\alpha = \beta = .5$, and put $\lambda = 2$, the theory predicts the following utility function, with a kink at the point $x = 0$.

We will illustrate the difference with a series of examples.

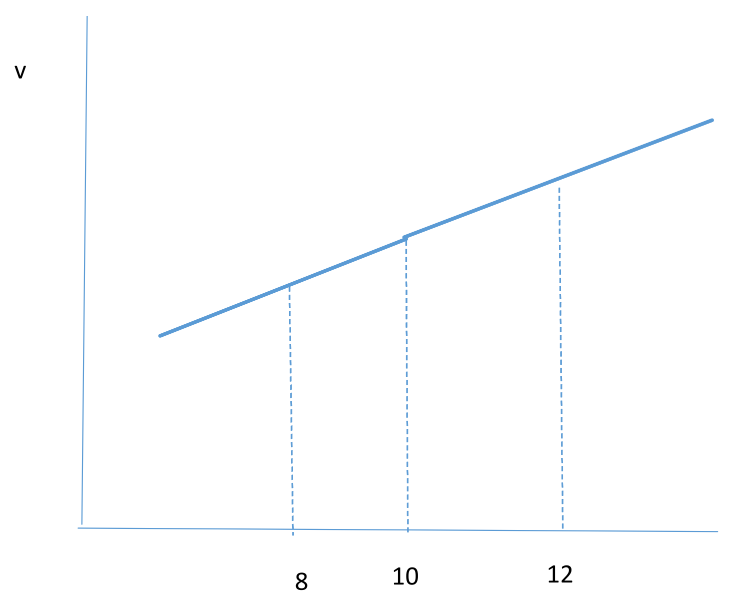
5.4.1 Example 1

Suppose initially we have 10$. A lottery gives us $(8, 12)$ with equal probabilities..Assume that these amounts are "small" to permit linearization of utility function.[8] K-T paradigm suggests that, with 10$ as reference point, the linearized preference has the following graph

[8] If you think they are not "small", you may think about ± 0.5 $



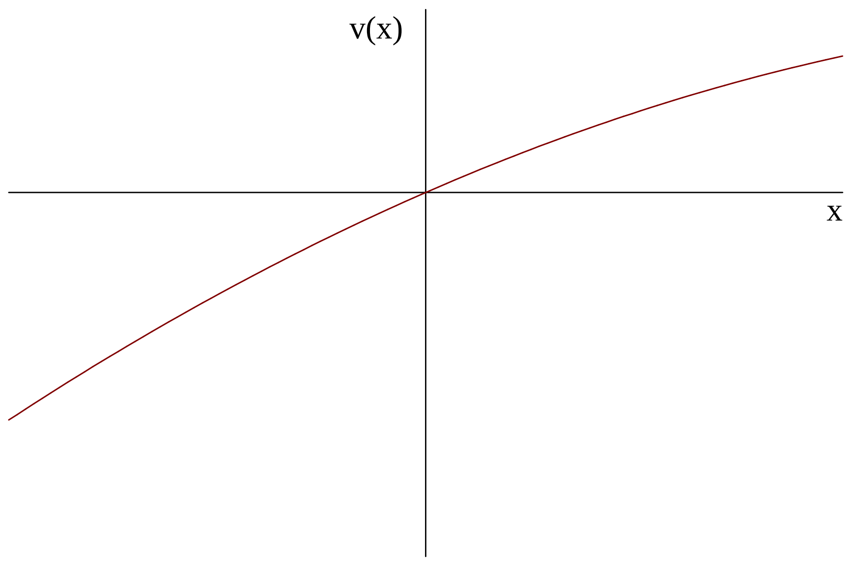
Compare this with a (linearized) traditional Bernoulli utility function:



In the latter version, the gains and losses are almost evened out.

5.4.2 Example 2

The idea here is to extend example 1 to the case when we cannot linearize. Assume we have 10$ in pocket and a lottery offers us $(10.2, 9.8)$ with equal probabilities. Thus, we have .2$ gain and .2$ loss. If we use the gain and loss in a traditional utility function, we need to define the status quo $(10, v(10))$ as $(0, 0)$. We redefine the lottery as $(.2, -.2)$ with equal probabilities. In that case, the graph of the relevant Bernoulli utility function (showing risk aversion) can be drawn as[9]



If we do not have loss aversion, then, with the above example, $v\,(CE) = 0.5\left(.2 - \frac{1}{2}\,(.2)^2\right) + 0.5(-.2 - \frac{1}{2}\,(-.2)^2) = -0.02, \rightarrow CE = -.0198.$[10] To the risk averse agent, the lottery implies a loss of $-.0198$$ from *status quo*.

But suppose we are considering loss aversion. Let the utility function be

[9] $v\,(x) = x - 0.5x^2$ with $-0.5 \leq x \leq 0.5$

[10] Solution to $x - \frac{x^2}{2} = -.02$.

$$
\begin{aligned}
v(x) &= x - \frac{x^2}{2} \text{ if } x \geq 0 \\
&= -2\left((-x) - \frac{(-x)^2}{2}\right) \text{ if } x < 0
\end{aligned}
$$

Notice that the second line becomes $v\left(x\right) = 2x + x^2 \rightarrow v''(x) = 2$. So the utility is convex.

Given the same lottery defined over gain and loss, $v\left(cE\right) = 0.5\left(.2 - \frac{1}{2}\left(.2\right)^2\right) - 2 * 0.5((.2) - \frac{1}{2}\left(.2\right)^2) = -0.09$. Since this is negative, we need to find out a value of x such that $-2\left((-x) - \frac{(-x)^2}{2}\right) = -.09$. Assuming $x > -.5$, the solution is $CE = -.046$. Thus, to the agent, the lottery is equivalent to a constant loss of $-.046$\$ under each state. Under loss aversion, the agent will be more disinclined to take the same gamble.

5.4.3 Example 3

Suppose we offer the following lotteries to an agent exhibiting loss aversion (utility function is described above). A gain of \$0.5 for sure, or $\{.3, .7; .5, .5\}$ With sure gain, the agent's utility is $(.5) - \frac{.5^2}{2} = 0.375$, while with the lottery, it is $0.5 * \left((.3) - \frac{.3^2}{2}\right) + 0.5 * \left((.7) - \frac{.7^2}{2}\right) = 0.355$. So he will settle for the sure gain (it SOSD the lottery, and the agent's utility is concave in gain).

On the other hand, if one offers him a loss of \$ (.5) for sure, or $\{-.3, -.7; .5, .5\}$, he will take the lottery. To see this, his utility with sure loss is $-2\left((.5) - \frac{(.5)^2}{2}\right) = -0.75$. On the other hand, with stochastic loss, the expected utility is $-2 * \left[0.5 * ((.3) - \frac{1}{2}\left(.3\right)^2) + 0.5 * ((.7) - \frac{1}{2}\left(.7\right)^2)\right] = -0.455$. Hence he will prefer the gamble (again, the sure loss SOSD the lottery, and his Bernoulli utility is *convex in loss*).

6 Epilogue

To a large extent, notwithstanding criticisms, the expected utility framework has remained the main workhorse of analysis of decisions made under uncertainty. However, the problems which we have discussed so far is that how one individual deals with a given uncertainty. In real life, people interact with each other and they are subject to different degrees of uncertainty, because they have access to different information. For example, I will know more about the mid term/end term exams or grading strategies ("soft"-vs- "hard"), but very limited information about the students ("brilliant brilliant "-vs- "moderately brilliant"). So my action needs to take account of the fact, just as your preparation for the exam is. This, however, takes us to the next topic: the Economics of Information.