Topic 4
Economics of Information

VITO CORLEONE

And in a month from now, this – Hollywood bigshot's gonna give you what you want.

JOHNNY

It's too late, they start shooting in a week.

VITO CORLEONE

I'm gonna make him an offer he can't refuse.

`Godfather,1972)`

1 Introduction

Remember the story regarding the wise king Solomon. One day, two women came to his court. Both claimed that a child (a son, who was highly prized at that time) belongs to her, accusing each other of stealing the child. Solomon proposed to cut off the child in two parts and divide each part to the women. One of the women instantly said: "Give it to her, let the child live". Solomon decided that she is the true mother.

So far in this course we have not encountered King Solomon's problem. We have dealt with situations under uncertainty. These are about social situations where every agent experiences the same degree of uncertainty about the world. On the other hand, Economics of Information deals with social situations where agents might have different degrees of information about the world or about each other. For example, in case of tax evasion, the government knows less about my income than me. I know less about the students' ability than the students. The students know less about my grading pattern (before the midterm, at least) than myself. When an insurance company offers a contract, it knows less about the buyers' type than the buyer. The banks know less about the credit worthiness of the borrower than the borrower himself. When we elect a new representative, we almost know nothing about his/her ability of ruling. Such mutual lack of knowledge pervades our social life.

Of course, society *still* functions, because various social institutions have evolved since the day of antiquity to take care of such situations. Within the formal discipline of Economics, the necessary tools to analyze the institutions could be developed only during the 1970's.

2 Typology

Economics of Information deals with interaction of different groups of individual. The most basic group is that of a (or many) principal (s) and one (or many) agent(s). The principal wants the agent to perform some tasks. For example

- An insurance company wants the buyer to take good care of himself or his property.
- The government expects that everybody should honestly report their income and pay taxes.
- Voters expect that the representative will serve their interest.
- The representative, after doing good work, expects that voters will vote for him.

And this list can be expanded *ad infinitum*.

Note that asymmetric information can bite both pre and post contract. For example,an insurance company should sell the contract to a healthy person, but it is only I who know if all my health parameters are under control. On the other hand, after purchasing the insurance, even a healthy person can take recourse to unhealthy lifestyle. If such a person makes a claim for insurance, I, as the insurance company, do not know whether he/she is sick due to an accident or whether due to the fact that he/she did not take adequate care for him/herself. In literature these are referred to the problems of "*adverse selection*" and "*moral hazard*".[1]

Adverse selection is typically solved by two ways. First, the uninformed party (principal) can devise a menu of contracts in such a way that the informed party will take the contract meant for him/her, thus automatically revealing his/her type. This method of solution is called "mechanism design". Second, the informed party (agent), can send certain credible signals about his/her type with the best hope that such signals will reveal his/her type. Such solutions are called signaling. For moral hazard, the contracts must be provided in a way that post contractual deviations from the "*good path*" do not happen.

In what follows, we will provide a concrete example of each solution, related to a typical social/economic situation. From the results, we would try to extract the prediction of the general theory.

3 Screening

In *monopolistic screening*, we have one principal and multiple agents. It is the onus of the principal to develop a screening mechanism (a menu of contracts) so that agents reveal their type by taking the contract. As usual, we will develop the theory by using example(s) and application(s).

3.1 Non Linear Pricing

We have all experienced the fact that shopkeepers sometime offer special discount/other offers. If you buy a good in bulk, you pay less. Restaurants offer reduced price during weekdays. If you are a student/senior citizen, you

[1]If you want to be more discering, adverse selection is one of the problems of *hidden information* (pre contractual problems), while moral hazard is one of the problems associated with *hidden action* (post contractual problem). But, in many books, these are treated at par.

are likely to get some offers at lower price. Such *price discrimination* means that different units of goods have different unit price, e.g. if I buy 20 units, the package price is RS 40, but if I buy 100 units, the package price is Rs 150. The tuple (T, Q) is a contract offered by the seller, i.e. to buy Q units, you need to pay a total amount of T, the unit price being $\frac{T}{Q}$. So obviously, the poor will buy the Rs 5 shampoo sachet, while the rich will buy the 500 Rs "Big Saver" bottle. Question is, why price discrimination is done and how to do it?

Let us give the moral of the story first, which we will develop in the subsequent analysis: price discrimination is done to tap different markets: i.e. people whose willingness to pay is less and people whose willingness to pay is high. The question is how to do it?

3.2 Monopoly: A Very Short Introduction

It is clear that price discrimination cannot be done by competitive market, as firms must offer only one price. Therefore, one needs to know how non competitive (or imperfect) markets operate. The easiest way is to move to the other spectrum of the market landscape: just as perfect competition has infinite number of sellers, a monopoly has only one seller.

Like a perfectly competitive firm, the monopolist also maximize the profit

$$\begin{aligned} \pi &= pq - c(q) \\ &= D(q)q - c(q) \end{aligned}$$

Here, $p = D(q)$ is the demand curve

The first order condition gives

$$D(q) + qD'(q) = c'(q)$$

The left hand side is marginal revenue: the infinitesimal increase in revenue as q increases marginally; the right hand side is marginal cost (for a perfectly competitive firm, marginal revenue is just $p = D(q)$). As $D'(q) < 0$, it is clear that monopolists' marginal revenue curve, as a function of q, is below the demand (or average revenue) curve.

Thus monopolist produces where marginal revenue (something lower than price) equals marginal cost. Therefore, the monopolist ends up producing less and charging more than the competitive market

This can be shown in the following diagram.

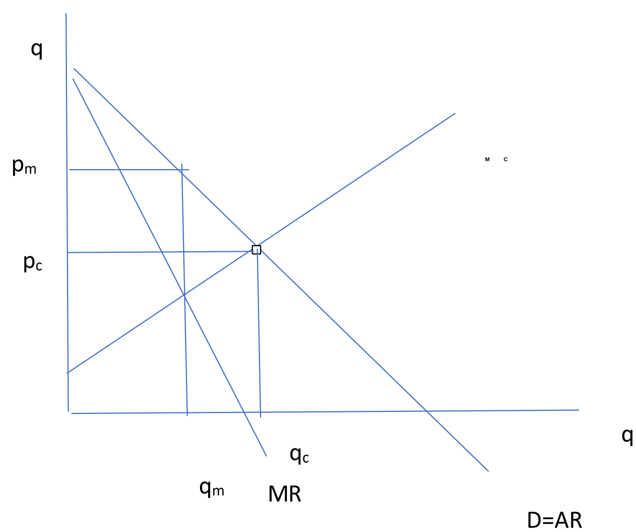

Figure 1: Monopoly vis-a-vis Competitive Firm: Price and Output

However, a more serious problem in monopoly (again, vis-a-vis perfect competition) is that it entails lower welfare (sum of consumer and producer surplus). To make this point clear we assume that the marginal cost curve is flat, e.g. the cost function is $C = kq$. I also omit the MR curve to clear the clutter a bit[2]

[2] You do not have the luxury during, say, exams.

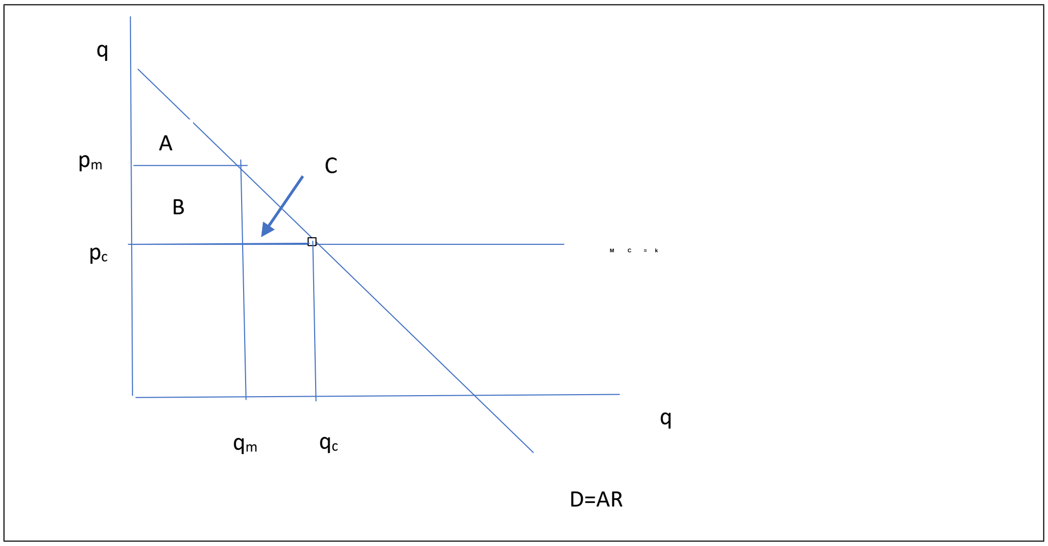

Figure 2: Welfare Loss under Monopoly

Suppose a monopolist produces by competitive rule. Then price and quantities are p_c and q_c. Consumer surplus is $CS = A+B+C$ and producer surplus is $PS = 0$. Hence welfare is $W = A + B + C$. Now suppose she behaves as a monopolist and the price-quantity configuration is (p_m, q_m). Here, $CS = A$ and $PS = B$, and hence $W = A + B$. One can see that C is forever lost from the society. This is known as deadweight loss (DWL) of monopoly.

Notice that the monopolist still follows a linear pricing rule: q_m units are produced and each unit is priced at p_m.

3.2.1 A Discriminating Monopolist

Assume that monopolist can sell each unit at the highest price, for example, the first unit goes to the highest bidder, the second unit goes to the highest bidder and so on and so forth. Given that the monopolist can correctly identify the demand curve, the monopolist extracts the willingness to pay for each marginal unit. The monopolist stops production only when the marginal willingness to pay (height of the demand curve) equals marginal cost: exactly at the competitive outcome. This is efficient in the sense that there is no deadweight loss.

However, as far as the consumers are concerned, the good story ends here.

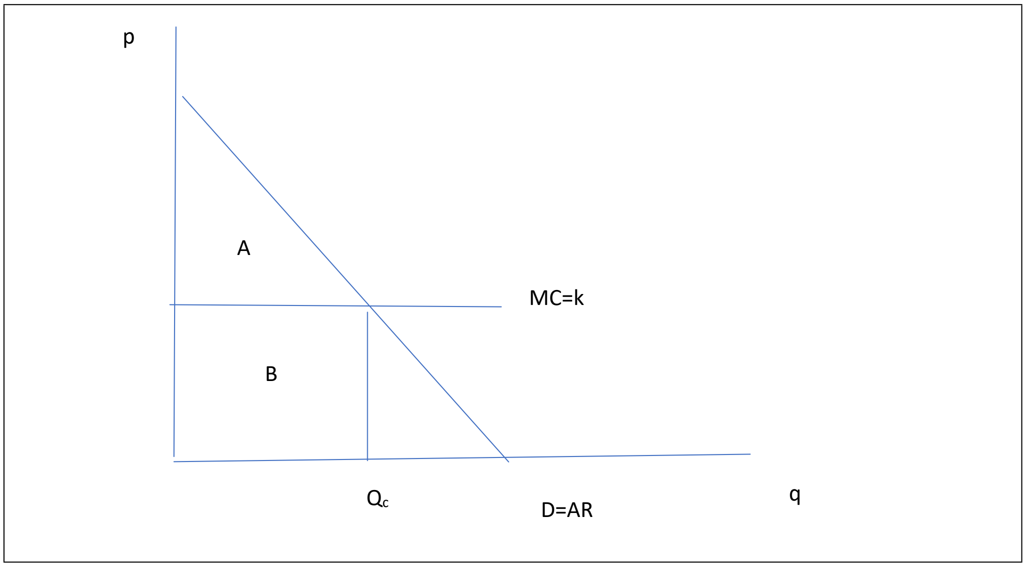

Figure 3: First Degree Price Discrimination

The monopolist produces up to Q_c. For this, the markets' total WTP is the area of the trapezium $(A+B)$. Monopolists' total cost is area $B\,(= kQ_c)$. Hence the rest (area A) becomes producers surplus, while consumer surplus is zero.

One way of operationalize such first-degree (or perfect) price discrimination is to offer a block price contract. In it, the monopolist offers a block price of $W(= A+B)$ in exchange of quantity Q_c. If there are multiple (identifiable) groups, each group should be offered the first best quantity, and the block price would be according to their demand, such that for all, monopolist is able to extract full consumer surplus.

Example 1 *Assume that there are two groups of the society. The groups have the following demand curves* $p^1 = 16 - q$ *and* $p^2 = 12 - q$.

a) The monopolist have a cost curve $C = 4q$. If the monopolist perfectly identifies each demand curve, what must be the block pricing contracts B_1 and B_2?

b) If the monopolist cannot identify the types (he knows number of types, the demand curves, and even the distribution of types. But he does not know who is who), show that type 2 people will never gain by posing as type 1 people, but type 1 people will gain by posing as type 2 people.

3.3 A Formal Model

The contracts (or the series of contracts) must satisfy the following conditions

a) People must participate in the market. That is, the offer should not turn away people.

b) It must maximize monopolists' profit.

c) The different menus, such as (T_i, Q_i) should be such that different groups will choose the best menu, and, by taking the offer, they should also reveal their own types.

Let there be two types, each indicated by θ_l, θ_h with $\theta_h > \theta_l$ Each consume quantity q and pay a transfer T to the monopolist. The group specific utility function is $u(q, T, \theta) = \theta v(q) - T$, here $v' > 0$,$v'' < 0$. From here, we can derive the slope of the indifference curve of a typical consumer, i.e. $\dfrac{dT}{dq} = \theta v'(q) > 0$, while $\frac{d^2T}{dq^2} = \theta v'' < 0$. Given $\theta_h > \theta_l$, the high type indifference curve cuts that of the low type from below (steeper) and only once (assumption: single crossing property: SCP). The direction of increasing preferences is to the South-East (low T and higher q)

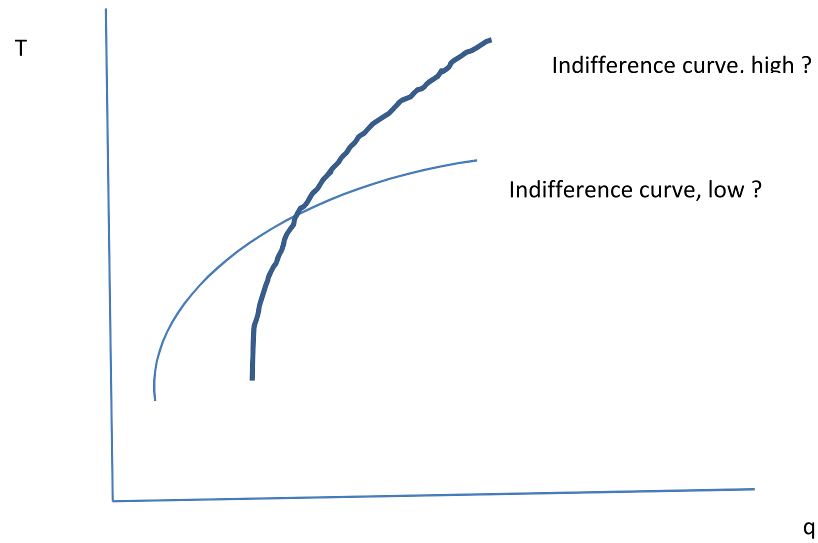

Figure 4: Single Crossing

Producers, on the other hand, are risk neutral, captured by a linear profit function $\pi = T - cq$. Thus, we have isoprofit lines (with constant slope $-c$) and the most favoured ones lie towards the north-west (less q, more T). So the interests of the principal and agent(s) are diametrically opposite.

3.3.1 The First Best

The first best case appears when the producer knows the type, i.e. θ. In this case, he/she will choose (T_j, q_j) in such a way that profit garnered from each type, $\pi_j = T_j - cq_j$ is maximum. Since he/she will aim for the highest transfer, he knows that each consumer is willing to give up $T_j = \theta_j v(q_j)$ for the bundle (the other alternative is to buy nothing and get a reservation utility of 0). Thus $\pi_j = \theta_j v(q_j) - cq_j$. Maximizing with respect to q_j, one gets $\theta_j v'(q_j) = c$. Suppose $\theta_h > \theta_l$. Then $v'(q_h) < v'(q_l) \rightarrow q_h > q_l$ and $T_h > T_l$. So, ideally, the monopolist should extract all consumer surplus from each type, produce less for low type and more for the high type. The situation is depicted in the following figure.

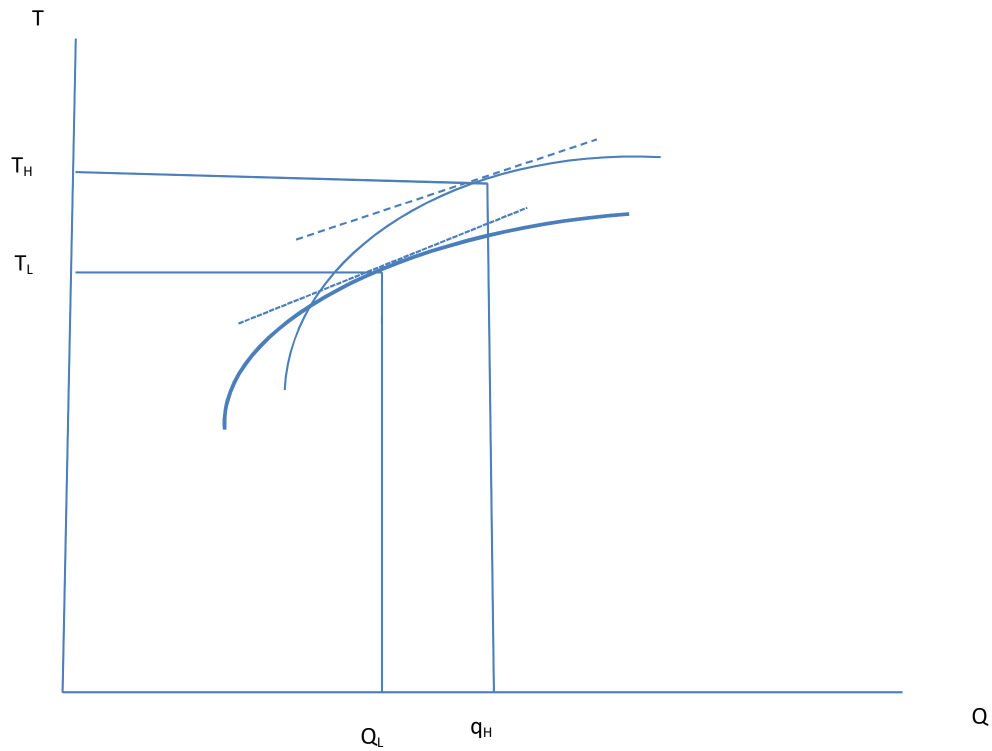

Figure 5: The First Best Outcome

The following figure provides the notional price schedule from the first best outcome. Notice that the monopolist, in effect, offer three bundles (T_h^*, q_h^*), (T_L^*, q_l^*) and $(0, 0)$ since not buying anything is also an option. At each point, the slope of the $(T - q)$ line is given by c. To accommodate the points it must be the case that there is a discontinuity somewhere between q_l and q_h.

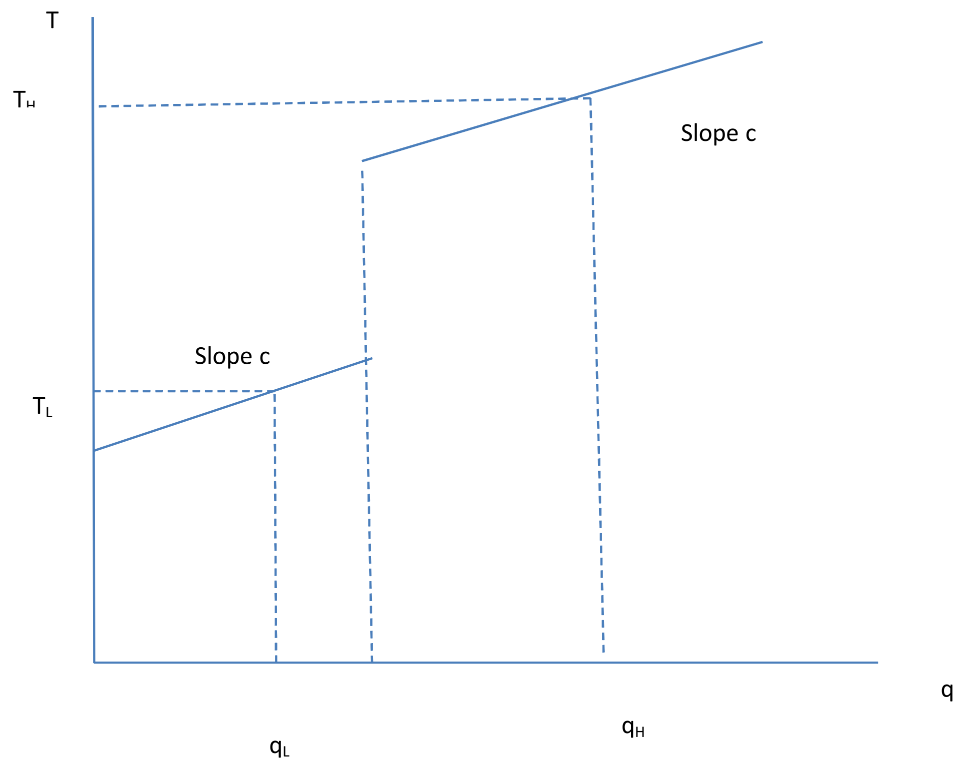

Figure 5A: Revenue-vs-Quantity in First Best

For both consumers, the marginal price $\left(\frac{\Delta T}{\Delta q}\right)$ is equal to c (that is why the price schedule is called linear). However, the high type consumer pays more per unit (as $\frac{T}{q}$) is higher for her. This also appeals to our innate sense of justice: if you value a good enough, you must pay higher amount per unit consumed.

3.3.2 Problem with First Best

Suppose now θ is unknown. Can the monopolist still offer the same menu of contracts? The answer is no. Note that, since the welfare of the agents are increasing in south-east direction, the high type consumer can move down and take the low types' contract. He will earn more surplus in a sense that for the same q_j his benefit is higher (i.e. $\theta_h v(q_L) > T_L$) The surplus ($\Delta = \theta_h v(q_L) - T_L$) is called the information rent accruing to the mimic (high type in this case) . The situation is depicted in the following figure. The monopolist's ex post profit goes down.

Figure 6: See Handout/Class Lecture

3.3.3 Solution

Let us work through the mathematics first and then we can come back to the intuition. If the monopolist cannot figure out who is who, but can figure out the probability distribution of the types (say λ and$1-\lambda$). The monopolist want to devise a contract (T_j, q_j) such that the following three criteria should be met

- For each group of customers, the contract should at least provide the reservation utility (participation is guaranteed).
- Each group of customer should take their designated contracts (sorting).
- Monopolists' (expected) profit is maximized.

The monopolists' maximization is, therefore

$$\max \lambda \left(T_H - cq_H\right) + (1-\lambda)\left(T_L - cq_L\right)$$

such that

$$\theta_i V\left(q_i\right) - T_i \geq 0$$

and

$$\theta_i V(q_i) - T_i \geq \theta_i V\left(q_j\right) - T_j$$

The first set of constraints (participation constraint or PC) ensures that each type of consumer must receive at least their reservation utility (normalized to 0). The second set of constraints (incentive compatible constraint, or ICC) ensures that no type gains by mimicking the other type.

So there are four constraints altogether, and there would be four corresponding Lagrangian multipliers. However, we can investigate, *a priory*, the nature of the solution which obeys all the constraints. We devote ourselves to this task now.

First, observe that from the ICC of both types, we can write

$$\theta_h(V(q_h) - V(q_l)) \leq T_h - T_l \leq \theta_l (V(q_h) - V(q_l))$$

Given $\theta_h > \theta_l$, the only way the above inequality is satisfied is through $q_h \geq q_l$. We will assume that $q_h > q_l$.

Second, ICC and PC of type h implies

$$\begin{aligned} T_h &\leq \theta_h V(q_h) \\ T_h &\leq \theta_h V(q_h) - \theta_h V(q_l) + T_l \end{aligned}$$

Since T_h is to be maximized by monopolist, it will exactly equal *one of those expressions* on the right, the other inequality will be strict (hence ceases to be a constraint).

We have $\theta_h V(q_h) - T_h \geq \theta_h V(q_l) - T_l > \theta_l V(q_l) - T_l \geq 0 \rightarrow PC_h$ holds with strict inequality$\rightarrow$ ICC_h holds with equality (thus the high type is indifferent between revealing his/her own type and mimicking.

Similarly, for type l either of the following two holds with strict inequality.

$$\begin{aligned} T_l &\leq \theta_l V(q_l) \\ T_l &\leq \theta_l V(q_l) - \theta_l V(q_h) + T_h \end{aligned}$$

Suppose ICC_l holds with strict equality. Thus $T_h - T_l = \theta_l [V(q_h) - V(q_l)] = \theta_h [V(q_h) - V(q_l)]$, since ICC_h holds. This equality will hold only if $q_h = q_l$, but $q_h > q_l$ (already shown). Hence ICC_l holds with strict inequality, and PC_l holds with equality. The l type never tells a lie (in equilibrium). Hence ICC_l ceases to be a constraint.

Incorporating all these observations, Lagrangian of the monopolist is

$$\mathcal{L} = \lambda (T_l - cq_l) + (1-\lambda)(T_h - cq_h) + \mu_1 (\theta_l V(q_l) - T_l) + \mu_2 (T_h - \theta_h V(q_h) + \theta_h V(q_l) - T_l)$$

FOC's are

$$\begin{aligned} \lambda - \mu_1 - \mu_2 &= 0 \\ 1 - \lambda + \mu_2 &= 0 \\ -\lambda c + \mu_1 \theta_l V'(q_l) + \mu_2 \theta_h V'(q_l) &= 0 \\ -(1-\lambda) c - \mu_2 \theta_h V'(q_h) &= 0 \end{aligned}$$

From the second and fourth equations,

$$\theta_h V'(q_h) = c$$

From here, one can figure out q_h^*.

From the first and third equations,

$$\theta_l V'(q_l) = \frac{c}{1 - \frac{1-\lambda}{\lambda} \frac{\theta_h - \theta_l}{\theta_l}} > c$$

This equation determines q_l^*.

The corresponding payments are obtained from the binding equations $T_l^* = \theta_l V(q_l^*), T_h^* = \theta_h V(q_h^*) - \theta_h V(q_l^*) + T_l^*$

1. The high type earns some surplus $(\Delta^* = \theta_h V(q_h^*) - T_h^*)$.

2. The low type customer earns no surplus.

3. The quantity for the high type is such that for the type, marginal benefit to the customer equals the marginal benefit of production. Thus, the quantity produced for type H is socially optimal.

4. Quantity for low type is such that the marginal benefit exceeds the marginal cost. Thus, the production is lower than the socially optimal amount.

5. By choosing the bundles designed for them, both types reveal their identity.

6. The low type never mimics. The high type is indifferent between revealing OR not revealing.

Figure 7 Equilibrium in Diagram: See Handout

It is clear from the diagram that both contracts can be read off from the high types' indifference curve, i.e. $T_i - \theta_h v(q_i) = R^*$. Note that the slope of the curve, i.e. the marginal payment is positive: $\frac{dT_i}{dq_i} > 0$. However, the marginal payment is decreasing with q as $\frac{d}{dq_i}\left(\frac{dT_i}{dq_i}\right) < 0$. Here, L type faces a higher marginal price ($> c$) than the H type ($= c$). It is also evident that there is volume discount. In case we had a continuum of types, so that each quantity is associated with a particular type, we would have a revenue schedule which starts from $(0,0)$ and then becomes a concave function of q.

3.4 Competitive Screening: Insurance Market[3]

Let us review back the model of insurance. Suppose that the initial wealth of an agent is W_0. If an accident occurs and he does not buy insurance, his income is $(W_0 - d)$, where d is the damage cost. Insurance companies offer a contract (α, I) where α is the premium, while I is the indemnity in case of accident. Thus his incomes are $W_1 = W_0 - \alpha$ in the good state of the world and $W_2 = W_0 - d + I - \alpha$ in the bad state of the world. Let his utility be $u(W)$, and let the probability of accident be p. His expected utility is $(1-p)\,U(W_1) + pU(W_2)$ if he buys the insurance. In the $W_1 - W_2$ plane, the slope of the indifference curves are given by $\frac{dW_2}{dW_1} = -\dfrac{1-p}{p}\frac{U'(W_1)}{U'(W_2)}$.

[3] The model is due to Rothschild and Stiglitz.

As already noted, the indifference curves are convex to the origin if he is risk averse. Any movement towards up or right implies better contracts (for him). Let his reservation utility be $\bar{U} = (1-p)U(W_0) + pU(W_0 - d)$, i.e. when he does not buy any insurance. This is known as autarky situation. The firms are risk neutral. Their expected profit, given any contract (α, d), is $(1-p)\alpha + p(\alpha - I)$. Let the expected profit be equal to zero, due to competition in the market. In the $W_2 - W_1$ plane, the isoprofit relation (actuarially fair line) describes a straight line with slope $-\frac{1-p}{p}$. Fixing any value of α, lower values of I implies higher profit. So lower the isoprofit curve, the higher the profit. Also, any contract below the zero isoprofit curve yields positive profit, while those above are associated with negative profit.

Suppose the market has only one type of consumer with accident probability (or risk) p. In this case, we will observe a homogeneous risk pool. An equilibrium contract must have the following properties:

a) It must not make negative profits (break even condition).

b) If any other contract is offered (and taken), it must not break even. In other words, *there cannot be any other contract favored by both consumers and firms.*

We know that the (interior) optimum must occur where the slope of the isoprofit curve equals that of the indifference curve, i.e. $\frac{1-p}{p}\frac{U'(W_1)}{U'(W_2)} = \frac{1-p}{p} \rightarrow U'(W_1) = U'(W_2) \rightarrow W_1 = W_2$. In other words, the contracts must be such that the agent's utility before and after the accident are same. Such an insurance policy is known as full insurance.[4]. The 45 degree line is the full insurance line.

Figure 8 (Full Insurance: One Type): See From Handout

Now suppose there are two types of agents, one with low risk and another high risk. The riskiness is measured by accident probabilities $p_h > p_l$.

[4]This is a standard result with risk neutral principals and risk averse agents. It is obvious that α_1 and α_2 depend on both π and d. **Homework:** Express the contract in terms of π and d.

The slope of the indifference curves are $\dfrac{1-p_h}{p_h}\frac{U'(W_1)}{U'(W_2)}$ and $\dfrac{1-p_l}{p_l}\frac{U'(W_1)}{U'(W_2)}$ respectively. Since $p_h > p_l \to \dfrac{1-p_h}{p_h} < \dfrac{1-p_l}{p_l}$ and the agents are otherwise similar, the indifference curve of the high risk type must be flatter than that of the low risk type at any given $W_1 - W_2$ pair. This is the familiar single crossing property.

As long as the insurance firm knows the identity of the agents (i.e. I have too many punches in my license to indicate that I am a bad driver and thus must belong to the high risk category), it is not a problem for them. They offer the high and the low risk individuals different contracts (C_H and C_L): both are fully insured (on two actuarially fair lines, each with slope $\frac{1-p_i}{p_i}$, $i = H, L$). The high risk consumer pays higher premium and gets lower compensation than the low risk agent. She likes the contract of the low risk agent. The fact that the company knows her identity prevents her from taking the other contract.

Figure 9 (Two types: Full Insurance): See from Handouts

Of course, problem starts if the firm (principal) does not know the characteristics of the agent. If the agent continues to offer the two contracts, all 'bad' types will take C_L. The firm will make a loss and go out of business. There can be two potential solutions: to offer each individual the same contract (pooling contract) or to offer them different contracts (separating contract) *in equilibrium*.

Let us consider a set of separating contracts, (C_H, S_L).

Figure 10: Separating Equilibrium: See from Handout

Notice that the high type gets full insurance, while the low types do not. Notice that S_L is at the intersection (X) of U_H and U_L. Thus the high risk agents do not have any incentive to mimic low risk agents. Any contract higher than S_L will attract them. If the firm chooses any policy lower than

S_L, it will not be chosen by either type. Moreover, the firm breaks even, since the policies are on the respective fair odds lines.

It is easy to show that no pooling contract exists. Let λ be the proportion of low risk people in the population and $(1-\lambda)$ that of high risk people. Then $m = \lambda p_L + (1-\lambda)p_H$ is the average or aggregate risk. $\frac{1-m}{m}$ is the slope of market (or aggregate) fair-odds line. Any contract on this line must earn a profit of zero. Let C be such a contract (see slide 4). The indifference curves must intersect at a point like C.

Figure 11 : Pooling Equilibrium: See from Handout

But C violates the second condition of equilibrium. Suppose there is a contract at C'. This will certainly be preferred by low risk types, and they will buy it. If they do, the firm offering C' makes positive profit (since it is below the fair-odds line for the low risk type). Thus, the original contract at C is not an equilibrium. This is true for all pooling equilibrium. Therefore, no pooling equilibrium exists.

Rothschild and Stiglitz showed that there *may* not exist any separating equilibrium either.[5] An upshot is, there will not exist any equilibrium whatsoever. The market will be constantly moving from a separating equilibrium to a pooling equilibrium and vice versa.

4 Signaling

As we have discussed, the agents can also take actions prior to the contract. These actions, called signals, are observed by principal. The principal deduces the type from the signals, and then offer the contract contingent on (deduced) types.

Which signals are credible? There are a few conditions. For example, suppose you apply for a new job. An essential precondition is that you

[5] See the handout

must come up with good recommendations. Now these recommendations (signals) must be verifiable. That is, your future employer must be able to call your present employer and ensure that he/she indeed wrote the recommendation. Second, the document/signal must be credible before a third party,e.g. court. If you produce a false document, there must be a punishment. Third, acquiring the signal (i.e. a good recommendation) must be costly and the cost must differ across the types (e.g. everybody sends a good recommendation then probably there is not much in the recommendation).

One of the first attempts to analyze signaling in the context of economics was done by Michael Spence (1974). We turn to this pioneering model now.

4.0.1 Educational Signaling

We have a situation where firms compete for workers. Workers differ by productivity. Let θ_H be the productivity (say, value generated in production in 1 hour of work) of high productivity worker and θ_L is that of low productivity worker, $\theta_H > \theta_L$. Their fraction in total population is given by $\lambda, 1-\lambda$. The productivity are private knowledge to workers. If productivity are perfectly observable, firms would like to offer an wage $w_j = k\theta_j$ where $j = H, L$ and $k > 0$. Suppose workers can undertake education (in years) as a signaling device. If e be the number of years in education, then cost of acquiring education for type $\theta : c(e, \theta) = \frac{e}{\theta}$. Utility of workers $U(w, e, \theta) = w - \frac{e}{\theta} \rightarrow$high productivity workers have less steep indifference curves. Note that, the role of education is not enhanced productivity (that is, you cannot manipulate θ by e), but a signal of high productivity.

The situation we are looking for is a *separating equilibrium*. In a separating equilibrium, different types acquire *different* educational levels and, are offered different wages contingent on observed educational level. The followings are the constituent blocks for a separating equilibrium.

1. Agent's action e.

2. Belief of the principal (regarding types) contingent on observing e.

3. Principal's action (wage level offered contingent on belief)

Beliefs and actions must be consistent with each other

As opposed to a separating equilibrium, in a pooling equilibrium, each type acquire the same level of education and an uniform wage is offered.

Let us specify if the requirements more formally

1. Agents' action e_H and e_L, where $e_H > e_L$

2. Belief of the principal $prob\,(\theta = \theta_i | e = e_i) = 1$, where $i = H, L$

3. Action of the principal: $w(e = e_H) = k\theta_H$, $.w\,(e \neq e_H) = k\theta_L$

We need to see if any e can support this kind of equilibrium.

In equilibrium, the following ICC's must hold.

Given the belief and action of the employer, each type must acquire θ_H, θ_L such that

$$\begin{aligned} k\theta_H - \frac{e_H}{\theta_H} &\geq k\theta_L - \frac{e_L}{\theta_H} \\ k\theta_L - \frac{e_L}{\theta_L} &\geq k\theta_H - \frac{e_H}{\theta_L} \end{aligned}$$

Note that, for any $e \neq e_H$, $w = k\theta_L$. So the best thing for low type is to acquire the minimum education: i.e. $e_L^* = 0$

Putting the value in the ICC1 as well as ICC2, we get

$$0 < k\theta_L\,(\theta_H - \theta_L) \leq e_H \leq k\theta_H\,(\theta_H - \theta_L)$$

In fact, there is multiple equilibrium (many values of e_H^* will support the structure of a separating equilibrium). For example, if $k = 2, \theta_L = 1, \theta_H = 2$, then $0 < 2 < e_H^* < 4$ will satisfy the separating equilibrium. Equilibrium refinement suggests that the high type should choose the minimum education: $e_H^* = k\theta_L\,(\theta_H - \theta_L)$, because education is costly. Note that, this is still high (ideally, high type should have chosen a small $e_H = \delta > 0$).

Next, we turn to the possibility of a pooling equilibrium in which both types acquire identically (low) level of education, $e_L = e_H = 0$ Firms cannot distinguish between types. Since the firms cannot extract the signal, so they just pay $\bar{w} = k * \bar{\theta}$ where $\bar{\theta} = \lambda\theta_H + (1 - \lambda)\,\theta_L$. Low type will prefer this since $\bar{w} > k\theta_L$

High type will not choose any education if

$$
\begin{aligned}
k\theta_H - \frac{k\theta_L\,(\theta_H - \theta_L)}{\theta_H} &\leq k * \bar{\theta} \\
&\to \theta_H - \frac{\theta_L\,(\theta_H - \theta_L)}{\theta_H} \leq \bar{\theta}
\end{aligned}
$$

I leave a homework here:

Exercise 2 *Show that the above condition is satisfied if* $\lambda \geq 1 - \frac{\theta_L}{\theta_H}$.

5 Hidden Action

Suppose the agents' type is known, and a contract is offered. However, post contract, the agents have no incentive to exert effort to obtain 'good' outcome. A worker may become lazy. A careful driver may become reckless (with the car) because it is insured. Contracts must be designed in such a way that agents provide highest effort.

This ceases to be a problem if effort is observable or there is an one-to-one relation between effort and outcome. However, note that project outcome is stochastic, with high efforts may lead to low production or vice versa. Thus, project outcomes are imperfect proxies for (unobserved) effort levels

5.1 Observable Effort

We will keep things as simple as possible, and assume that the agent can provide only two levels of effort, $e \in \{0, 1\}$.Only two project outcomes are possible, with $q_H > q_L$. If $e = 1$ (resp. $e = 0$), then the project outcome is q_H with probability π_1 (resp. π_0), and q_L with the complementary probability; such that $\pi_1 > \pi_0$. Thus higher effort is consistent with higher probability of success. Since wage is conditional on effort, wage offered to the agent is w_H if outcome is q_H and w_L if outcome is q_L. The principal earns a value of $S_a = S(q_a)$, (where $a = H, L$). The principal is risk neutral. His/her expected utility is given by $\pi_i (S_H - w_H) + (1 - \pi_i)(S_L - w_L)$ (for $i = 1, 0$) where w_a is state contingent wage rate. Notice that $\pi_1 (S_H - w_H) + (1 - \pi_1)(S_L - w_L) \geq \pi_0 (S_H - w_H) + (1 - \pi_0)(S_L - w_L)$, so the principal prefers $e = 1$ rather than $e = 0$ if

$$\begin{aligned} \pi_1 (S_H - w_H) + (1 - \pi_1)(S_L - w_L) &\geq \pi_0 (S_H - w_H) + (1 - \pi_0)(S_L - w_L) \\ &\rightarrow (\pi_1 - \pi_0)\left[(S_H - w_H) - (S_L - w_L)\right] \geq 0 \end{aligned}$$

which is satisfied if net profit from the good state of the world, $(S_H - w_H)$ is higher than that of the bad state $(S_L - w_L)$. We will make this assumption.

From the agents' side, let the effort cost from high effort be $\psi(1) = \psi$ and that from low effort $\psi(0) = 0$. Consequently, in order to induce him/her to exert effort $e = 1$, wages offered should be such that the following ICC is satisfied.

$$\pi_1 u(w_H) + (1 - \pi_1) u(w_L) - \psi \geq \pi_0 u(w_H) + (1 - \pi_0) u(w_L)$$

Assuming reservation utility to be $\bar{U}$, the agent should have at least as much as possible from $e = 1$, i.e. the following PC must be satisfied.

$$\pi_1 u(w_H) + (1 - \pi_1) u(w_L) - \psi \geq \bar{U}$$

Since the principal does not want to implement $e = 0$, it will not consider a separate PC for low effort, i.e.

$$\pi_0 u(w_H) + (1 - \pi_0) u(w_L) \geq \bar{U}$$

is not considered.

Once a contract is accepted, one cannot renege upon it. This can be enforced by a third party (e.g. a court). If effort is observable, the principal can always get $e = 1$. Then only the participation constraint is binding (because the principal will lower w_i to make it just binding. Hence the corresponding Lagrangian (to be maximized by choosing appropriate w_H, w_L) is

$$\mathcal{L} = \pi_1 (S_H - w_H) + (1 - \pi_1)(S_L - w_L) + \lambda\{\pi_1 u(w_H) + (1 - \pi_1) u(w_L) - \psi - \bar{U}\}$$

The FOC yields

$$\begin{aligned} u'(w_H) &= u'(w_L) = \frac{1}{\lambda} \\ &\rightarrow w_H = w_L = w^* \end{aligned}$$

In other words, a fixed wage contract! Thus the agent is insured against any fluctuation in output.

Principal goes for this kind of contract if

$$\pi_1 S_H + (1 - \pi_1) S_L - w^* > \pi_0 S_H + (1 - \pi_0) S_L$$

That is, profit from enforcing the first best contract is higher than offering no wage.

In order to obtain the wage rate, note that we can write $u_i = u(w_i) \rightarrow w_i = u^{-1}(u_i) = h(u_i)$. The h function is convex.

From the (binding) PC

$$u(w^*) = \psi + \bar{U} \rightarrow w^* = h\left(\psi + \bar{U}\right)$$

5.2 Unobservable Effort

Of course, the problem arises because efforts are unobservable (and cannot be contracted upon). So the principal does not know whether the low output due to state of nature or due to low effort. Hence the principal must offer sufficient incentive for applying higher effort. Principal's problem now is

$$\max_{w_H, w_L} \pi_1 (S_H - w_H) + (1 - \pi_1)(S_L - w_L)$$

such that

$$\begin{aligned} \pi_1 u(w_H) + (1 - \pi_1) u(w_L) - \psi &\geq \pi_0 u(w_H) + (1 - \pi_0) u(w_L) \\ \pi_1 u(w_H) + (1 - \pi_1) u(w_L) - \psi &\geq \bar{U} \end{aligned}$$

From the first constraint,

$$\begin{aligned} (\pi_1 - \pi_0) * u(w_H) - (\pi_1 - \pi_0) * u(w_L) &\geq \psi \\ &\rightarrow (\pi_1 - \pi_0) * [u(w_H) - u(w_L)] \geq \psi \\ &\rightarrow u(w_H) - u(w_L) \geq \frac{\psi}{\pi_1 - \pi_0} > 0 \\ &\rightarrow w_H > w_L \end{aligned}$$

Thus, in equilibrium, we cannot have fixed wages as in the first best case.

Hence the Lagrangian is[6]

[6] Here, as well as in hidden information problem, we run into a technical difficulty while setting up the Lagrangian. We are trying to maximimise the Lagrangian. Hence, the shape of the Lagrangian must be concave, so that the first order conditions identify a maximum. The objective function is linear in w_a. Since $u(.)$, a concave function, appears on both sides of ICC, it is not immediately clear if *difference of two concave functions* is concave (if $f(x)$ is concave, then $-f(x)$ is convex). As a result, Principal's Lagrangian may turn out to be convex. We gloss over this difficulty, here, as well as earlier. However, the technically correct way is to use the convex h functions. Hence the Lagrangian is

$$\begin{aligned} \Lambda &= \pi_1 (S_H - h(u_H)) + (1 - \pi_1)(S_L - h(u_L)) \\ &\quad + \lambda_1 [\pi_1 u_H + (1 - \pi_1) u_L - \psi - \pi_0 u_H - (1 - \pi_0) u_L] \\ &\quad + \lambda_2 [\pi_1 u_H + (1 - \pi_1) u_L - \psi] \end{aligned}$$

Objective function is now concave and constraints are linear. Hence Λ is concave in u_H, u_L. We can think of the contract in terms of (u_H, u_L) instead of (w_H, w_L)

$$
\begin{aligned}
\Lambda &= \pi_1 \left(S_H - w_H\right) + \left(1-\pi_1\right)\left(S_L - w_L\right)) \\
&\quad +\lambda_1 \left[\pi_1 u\left(w_H\right) + \left(1-\pi_1\right) u\left(w_L\right) - \psi - \pi_0 u\left(w_H\right) - \left(1-\pi_0\right) u\left(w_L\right)\right] \\
&\quad +\lambda_2 \left[\pi_1 u\left(w_H\right) + \left(1-\pi_1\right) u\left(w_L\right) - \psi - \bar{U}\right]
\end{aligned}
$$

The FOC's with respect to w_i are

$$
\begin{aligned}
-\pi_1 + \lambda_1 \left[\pi_1 u'\left(w_H\right) - \pi_0 u'\left(w_H\right)\right] + \lambda_2 \pi_1 u'\left(w_H\right) &= 0 \\
-\left(1-\pi_1\right) + \lambda_1 \left[\left(1-\pi_1\right) u'\left(w_L\right) - \left(1-\pi_0\right) u'\left(w_L\right)\right] + \lambda_2 \left(1-\pi_1\right) u'\left(w_L\right) &= 0
\end{aligned}
$$

FOC's can be rewritten as

$$
\begin{aligned}
-\pi_1 + \lambda_1 \Delta * u'\left(w_H\right) + \lambda_2 \pi_1 u'\left(w_H\right) &= 0 \\
-\left(1-\pi_1\right) - \lambda_1 \Delta * u'\left(w_L\right) + \lambda_2 \left(1-\pi_1\right) u'\left(w_L\right) &= 0
\end{aligned}
$$

Here, $\Delta = \pi_1 - \pi_0 > 0$

From the first equation,

$$
\begin{aligned}
\lambda_2 \pi_1 u'\left(w_H\right) &= \pi_1 - \lambda_1 \Delta * u'\left(w_H\right) \\
&\rightarrow \quad \lambda_2 = \frac{1}{u'\left(w_H\right)} - \frac{\lambda_1 \Delta}{\pi_1}
\end{aligned}
$$

Similarly, from the second equation,

$$
\lambda_2 = \frac{1}{u'\left(w_L\right)} + \frac{\lambda_1 \Delta}{1-\pi_1}
$$

Eliminating λ_2,

$$
\begin{aligned}
\frac{1}{u'(w_L)} + \frac{\lambda_1 \Delta}{1-\pi_1} &= \frac{1}{u'(w_H)} - \frac{\lambda_1 \Delta}{\pi_1} \\
&\rightarrow \frac{\lambda_1 \Delta}{1-\pi_1} + \frac{\lambda_1 \Delta}{\pi_1} = \frac{1}{u'(w_H)} - \frac{1}{u'(w_L)} \\
&\rightarrow \lambda_1 \Delta \left[\frac{1}{(1-\pi_1)\pi_1}\right] = \frac{1}{u'(w_H)} - \frac{1}{u'(w_L)} \\
&\rightarrow \lambda_1 = \frac{\pi_1(1-\pi_1)}{\Delta}\left(\frac{1}{u'(w_H)} - \frac{1}{u'(w_L)}\right)
\end{aligned}
$$

Note that $w_H > w_L \rightarrow u(w_H) > u(w_L) \rightarrow u'(w_H) < u'(w_L) \rightarrow \frac{1}{u'(w_H)} > \frac{1}{u'(w_L)}$. Hence $\lambda_1 > 0$. By Kuhn-Tucker conditions, the corresponding constraint (ICC) must hold with equality, i.e.

$$\pi_1 u(w_H) + (1-\pi_1) u(w_L) - \psi = \pi_0 u(w_H) + (1-\pi_0) u(w_L)$$

Try to prove the following.

Exercise 3 *Show that*

$$\lambda_2 = \frac{1-\pi_1}{u'(w_L)} + \frac{\pi_1}{u'(w_H)} > 0$$

Hence argue that PC must hold with equality as well.

As a final step, let us denote $u_i = u(w_i)$. Therefore, the binding constraints can be written as[7]

$$
\begin{aligned}
\pi_1 u_H + (1-\pi_1) u_L - \psi &= \pi_0 u_H + (1-\pi_0) u_L \\
\pi_1 u_H + (1-\pi_1) u_L - \psi &= \bar{U}
\end{aligned}
$$

In the following exercise, you are asked to solve explicitly for the utility.

[7] Since w_H, w_L are "leakages" from the principal's profit, he/she will tinker with these until and unless the equations hold with equality.

Exercise 4 *Show that*

$$\begin{aligned} u_H &= \psi + \bar{U} + (1-\pi_1)\frac{\psi}{\Delta} \\ u_L &= \psi + \bar{U} - \pi_1\frac{\psi}{\Delta} \end{aligned}$$

Using the wage equations,

$$\begin{aligned} \tilde{w}_H &= h\left(\psi + \bar{U} + (1-\pi_1)\frac{\psi}{\Delta}\right) \\ \tilde{w}_L &= h\left(\psi + \bar{U} - \frac{\pi_1}{\Delta}\psi\right) \end{aligned}$$

Since production is risky, one has to provide the agent a payment which is more than the first best case ($w^* = h(\psi + \bar{U})$) in order to make him participate and offer a high effort. The agent also shares the risk. The difference between $\tilde{w}_H$ and w^* is called risk premium.[8]

Do the agents care if effort is observable or not? In the *observable effort* case, agents' expected utility is

$$\begin{aligned} & \pi_1\left(\psi + \bar{U}\right) + (1-\pi_1)\left(\psi + \bar{U}\right) \\ = \;& \psi + \bar{U} \end{aligned}$$

When effort is unobservable, agents' expected utility is

$$\begin{aligned} & \pi_1\left(\psi + \bar{U} + (1-\pi_1)\frac{\psi}{\Delta}\right) + (1-\pi_1)\left(\psi + \bar{U} - \pi_1\frac{\psi}{\Delta}\right) \\ = \;& \psi + \bar{U} \end{aligned}$$

Thus agents' (expected) return from both contract is same, but the second situation is risky: there is no full insurance.

What about the principal? He/she is risk neutral, and will prefer a least cost outcome. His/her cost in the first best world is

$$\begin{aligned} C^{FB} &= \pi_1 w^* + (1-\pi_1) w^* \\ &= \pi_1 h\left(\psi + \bar{U}\right) + (1-\pi_1) h(\psi + \bar{U}) \\ &= h\left(\psi + \bar{U}\right) \end{aligned}$$

[8]Not to be confused with Pratt's definition of risk premium.

If effort is unobservable, the expected cost is

$$\begin{aligned}
C^{SB} &= \pi_1 h\left(\psi+\bar{U}+(1-\pi_1)\frac{\psi}{\Delta}\right)+(1-\pi_1)\,h\left(\psi+\bar{U}-\pi_1\frac{\psi}{\Delta}\right)\\
&> h\left(\pi_1\left[\psi+\bar{U}+(1-\pi_1)\frac{\psi}{\Delta}\right]+(1-\pi_1)\left[\psi+\bar{U}-\pi_1\frac{\psi}{\Delta}\right]\right)\\
&= h\left(\psi+\bar{U}\right)
\end{aligned}$$

The second line comes form Jensen's inequality regarding convex (h) functions. Hence, in the second best world, expected cost (of the principal) to induce the agent to provide high effort is higher than the first best case.

6 Moral Hazard in Insurance Market

We go back to the canonical model of insurance market. Without any problem of information, the insurance company maximizes

$$E\pi = p\,(\alpha - I) + (1-p)\,\alpha$$

such that the agent participates

$$pu\,(W-\alpha-d+I)+(1-p)\,u\,(W-\alpha) \geq pu\,(W-d)+(1-p)\,u\,(W)$$

The solution to this, if insurance companies operate in a competitive market was discussed in topic 3A. This is the benchmark case of full insurance.

{Figure 12: The Benchmark Case}

Now suppose the probability of accident is p^h, but it can be lowered to p^L if the agent makes an effort. There is a cost of effort associated with p^L. Let the cost of effort be a fixed amount c (e.g. 10 rounds of jogging across the IITG field). Given that the slope of the indifference curve is $\frac{1-p^i}{p_i}\frac{u'(w_1)}{u'(w_2)}$, it is clear that "care-inclusive" indifference curve" is steeper than "no-effort" indifference curve.

In order to induce the agents to provide an effort, the insurance firms must offer them α, I such that

$$p^L u(W_2) + \left(1 - p^L\right) u(W_1) - c \geq p^h u(W_2) + \left(1 - p^h\right) u(W_1)$$

This can be written as

$$u(W_1) - u(W_2) \geq \frac{c}{p^h - p^L} > 0$$

Thus, the ICC implies that $W_1 > W_2$ i.e. there is no full insurance. To fix ideas, let $u(.) = \ln(.)$. Then one can derive a line $W_2 = \frac{1}{k} W_1 \left(k = \exp\left(\frac{c}{p^h - p^L}\right)\right)$ such that if $W_2 > \frac{1}{k} W_1$, the agent do not take care, and if $W_2 \leq \frac{1}{k} W_1$ the agent takes care.

{Figure 13: Effort and No Effort Zone}

Since the probability of accident change abruptly at the line, the slope of the indifference curve as well as zero-profit line change as well. While indifference curves become kinked at the border, the zero profit lines become discontinuous.

{Figure 14 and 15: Indifference Curve and Zero Profit Lines}

If c is relatively low compared to $p_h - p_L$, that is, accident probability can be lowered with lower cost, then the optimal contract is exactly at the kink. There could be a local optimum with full insurance, but the global optimum (favored by both agents and principals) involves non-full insurance and makes sure that the agent provides an effort.

{Figure 16: Global Optima: No Full Insurance}

On the other hand, if c is relatively high compared to $p_h - p_L$, that is, accident probability can be lowered with high cost. In this case, the "border" is fa away from the 45° line. Here, the global optimal contract would involve full insurance and no effort.

{Figure 17: Global Optima: Full Insurance, No Effort}

7 Auctions

I include auctions here as a special case of hidden information. The following can be thought of as an element of an auction.

Assume that there is a single good to be auctioned. The good may (a coal block) or may not (William Shakespeare's toothbrush) have any intrinsic economic value. There are n bidders for the good, each with valuation v_i. The valuation , however, is private knowledge to the bidders. We assume that each valuation is uncorrelated in the sense that $cov(v_i, v_j) = 0$.[9] What is observed is the bidding of each bidder, b_i.Normally we expect that b_i to be equal to v_i, but there is no reason to be so.

To fix ideas, let v_i be uniformly distributed within $[0, 1]$. However, bidding can be $b_i \in [0, \infty)$. The object (prize) goes to the highest bidder.

The two questions are

a) How would b_i relate to v_i (or, more formally, what is the nature of the bidding function)?

b) How much does the auctioneer get out of the auction?

The latter question is important for, e.g. a government auctioning off a coal block or a nationalized airlines to private bidders.

[9] The valuation may depend on one's own signal about the good's value. Valuations are uncorrelated if signals are not related. On the other hand, example of correlated valuations is value of an oil field to be auctioned.

7.1 First Price, Sealed Bid Auction

Under first price sealed bid auction, the bidders propose a sealed bid (so that other bidders do not observe what you bid) to auctioneer. One can bid only once. The sealed bids are opened at the end and the good (the "prize") goes to the highest bidder.

First we will think of equilibrium with linear strategies and two bidders. Assume each person's (only two person) bidding strategy is $b_i = a_i + c_i v_i$ $(i = 1, 2)$. Then person 1's (risk neutral) welfare is

- $$\begin{aligned} EU_i &= (v_1 - b_1) * prob\,(b_1 > b_2) \ \ (\text{if } b_1 > b_2) \\ &= \frac{(v_1 - b_1)}{2} * prob\,(b_1 = b_2) \ \ (\text{if } b_1 = b_2) \\ &= 0, \text{ otherwise} \end{aligned}$$

Note that by the proposed strategies, $prob\,(b_1 > b_2) = prob\,(b_1 > a_2 + c_2 v_2) = prob\left(v_2 < \dfrac{b_1 - a_2}{c_2}\right) = \dfrac{b_1 - a_2}{c_2}$.The last equality comes from the fact that v_2 is uniformly distributed between 0 and 1..Similarly, $prob\,(b_1 = a_2 + c_2 v_2) = prob\left(v_2 = \frac{b_1 - a_2}{c_2}\right) = 0$, assuming probability is continuous. So agent 1 maximizes $(v_1 - b_1) * \frac{b_1 - a_2}{c_2}$ with respect to b_1. The optimal value of b_1 is given by

$$b_1 = \frac{a_2}{2} + \frac{v_1}{2}$$

and similarly for player 2,

$$b_2 = \frac{a_1}{2} + \frac{v_2}{2}$$

Comparing with the proposed policies, it must be the case that $c_1 = c_2 = \frac{1}{2}$ and $a_1 = a_2 = 0$. Thus, linear strategy is $b_i = \dfrac{v_i}{2}$: each person bids less than valuation. This is known as underbidding.

Let us extend the analysis to symmetric, general bidding function Now we are looking for a $b(v_i)$ (the only reasonable restriction is $b' > 0$), i.e. $b_i\,() = b_j\,() = b\,()$. Discounting the fact of equal bids, and noting that

$v_i = b^{-1}(b_i)$ the expected payoff of player 1 is $(v_1 - b_1) * b^{-1}(b_1)$. The first order condition is

$$-b^{-1}(b_1) + (v_1 - b_1)\frac{d\left(b^{-1}(b_1)\right)}{db_1} = 0$$

Since $b_1 = b(v_1)$, $b^{-1}b(v_1) = v_1$ and $\frac{dv_1}{db_1} = \frac{1}{b'(v_1)}$, the FOC implies gives a differential equation

$$v_1 b'(v_1) + b_1(v_1) = v_1$$

solving which we get

$$v_1 b(v_1) = \frac{v_1^2}{2} + k_1$$

Here, k_1 is a constant of integration. When $v_1 = 0$ a reasonable restriction is $b(0) = 0$, then $k_1 = 0 \rightarrow b(v_1) = \frac{v_1}{2}$. This is the same bidding function as above.

Let us generalize the case to n bidders, maintaining all the assumptions as above.

Claim 5 *In equilibrium, the symmetric bidding function is* $b(v_i) = \frac{n-1}{n}v_i$

The line of proof is pretty simple. fix everybody else's ($j = 2, 3, ...n$ persons) bidding to be $b_j = \frac{n-1}{n}v_j$. Then,

$$\begin{aligned} & prob\left(b_1 > \frac{n-1}{n}v_j\right) \\ = \; & prob\left(v_j < \frac{nb_1}{n-1}\right) \\ = \; & \left(\frac{nb_1}{n-1}\right)^{n-1} \end{aligned}$$

The equality of the third line comes from the fact that all v_j are independent, and therefore

$$\begin{aligned} & prob\left(v_j < \frac{nb_1}{n-1}\right) \\ = \; & prob\left(v_2 < \frac{nb_1}{n-1}\right) * prob\left(v_3 < \frac{nb_1}{n-1}\right) * ...... * prob\left(v_n < \frac{nb_1}{n-1}\right) \end{aligned}$$

Bidder 1 then maximizes $EU_1 - (v_1 - b_1)\left(\frac{nb_1}{n-1}\right)^{n-1}$ with respect to b_1

Exercise 6 *Show that $b_1 = \frac{n-1}{n}v_1$ is the solution.*[10]

Thus, under the first-price-sealed-bid auction, everybody underbids (tells a lie, if we assume that bid reflects valuation). However, as n increases, the extent of underbidding goes down. If $n \to \infty$, $b_i = v_i$,: underbidding vanishes if participants are large enough.

The second question is auctioneer's revenue. In a first price, sealed bid auction, the auctioneer's payoff is $b_{\max} = \frac{n-1}{n}v_{\max}$. But notice that $v_{\max}$ is a random variable. Hence $b_{\max}$ is also a random variable. Thus, at best, we can predict $E(b_{\max})$.

Let x be a random variable that gives the highest valuation, i.e. $x = \max(v_1, v_2, ...v_n)$. In one realization of the x, probability that it is the highest bid is x^{n-1}(i.e. all other $n-1$ quotes are lower than x). But one can choose the winner (out of n bidders) in ${}^nP_1 = n$ mutually exclusive ways. So the probability distribution of x (the highest bid) is nx^{n-1}. Therefore, the expected maximum valuation is given by $\int_0^1 x * nx^{n-1}dx$. We must have, then,

$$\begin{aligned} E(b_{\max}) &= E\left(\frac{n-1}{n}v_{\max}\right) \\ &= \frac{n-1}{n}\int_0^1 nx^n dx \\ &= \frac{n-1}{n+1} \end{aligned}$$

As $n \to \infty$, the auctioneer expects to earn an amount of money which is equal to the highest possible valuation (1).

[10] Those who have done a prior course, will realize that this is how one checks for Nash equilibrium in game theory. Actually, in chapter 4, we are doing social interacations , and whatever we have done can be recast into game theoretic language.

7.2 Second Price, Sealed Bid Auctions

In first price auction, the bidders always under-report their valuation ($b_i < v_i$). So their types (that is, v_i) cannot be immediately known from their bids.[11] The question is, therefore, can we design an auction in which people will voluntarily reveal their type? This question is identical to the "mechanism design" problem that we had studies before. Second price auction will seem a bid mad: let the highest bidder win the prize, but the amount that (s)he pays is equal to the second highest bidder's bid![12]

The result is very intuitive.

Claim 7 *Regardless of the probability distributions, it is always a equilibrium to bid own valuation, $b_i = v_i$.*

So the payoffs of the highest bidder are

$$\begin{aligned} EU_i &= v_i - \tilde{b} \text{ if } b_i > \tilde{b} = \max_{j \neq i} b_j \\ &= 0, \text{ otherwise} \end{aligned}$$

Suppose $v_i = b_i > \tilde{b}$. In this case, agent i gets the prize and his payoff is $\left(v_i - \tilde{b}\right) > 0$. If he underbids $b_i < v_i$, he is not worse off. But if he underbids too much such that $b_i < \tilde{b}$, then his payoff is zero. If he overbids, his payoff is not altered, and hence there is no incentive either.

Next, assume that $v_i = b_i < \tilde{b}$. He does not get the prize. If he underbids, his payoff is still zero, so there is no incentive. If he overbids, his payoff either remains the same or becomes negative when $b_i > \tilde{b}$.

So truth telling is weakly dominating.

[11] You may reply that since we know the optimal bidding function, it must be the case that $v_i = \frac{n}{n-1} b_i$. How can I counter this argument?

[12] William Vickrey had this idea is 1961 (at that time nobody paid any heed), for which he was awarded a Nobel Prize in 1996 (Vickrey died three days later after announcement: he was 82!)

Auctioneer's expected revenue is $E\left(\tilde{b}\right) = E\left(v_2^{\max}\right)$ where $v_2^{\max}$ is the second highest quote. As before, let x denote the random variable representing the second highest quote. Probability (maintaining the assumption that v_i is uniformly distributed between $[0,1]$) that it is the second highest quote is $x^{n-2}(1-x)m$.One can choose 2 (highest and 2nd highest valuation) agents out of n in ${}^nP_2 = n\,(n-1)$ ways. Thus the probability that x is the second highest bid is $n\,(n-1)\,x^{n-2}(1-x)dx$. The expectation is

$$\begin{aligned} E(v_2^{\max}) &= \int_0^1 n(n-1)x^{n-1}(1-x)dx \\ &= \frac{n-1}{n+1} \end{aligned}$$

the above analysis leads us to the Revenue Equivalence Theorem

Proposition 8 *If (a) Bidders are risk neutral, and (b) Bidders' valuations are drawn independently from the same continuous and increasing cumulative distribution, then the second price auction and the first price auction yield equal average revenue to the auctioneer.*

Intuition for this result is in the first price auction, people underbid, but the highest bidder pays his bid. In the second price auction, people say the truth, but the auctioneer does not receive the highest bid.