Practice problems 1

Notation:

$\mathbb{N}$: Set of natural numbers.
Boldface letters, for example $\mathbf{x}, \mathbf{x}_i, \mathbf{a}_i, \mathbf{d}, \mathbf{d}_i, \mathbf{b}, \mathbf{y}_i$, etc, are column vectors, whereas x_i's are scalars.
$Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$
$= \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{a}_i^T\mathbf{x} \leq b_i \text{ for all } i = 1, \ldots, m, \ -\mathbf{e}_j^T\mathbf{x} \leq 0 \text{ for all } j = 1, \ldots, n\}$,
where $\mathbf{e}_j$ is the j th column of the Identity matrix I.

$\tilde{\mathbf{a}}_i = \mathbf{a}_i$ for $i = 1, 2, \ldots, m$
$= -\mathbf{e}_{i-m}$ for $i = m+1, m+2, \ldots, m+n$.
$\tilde{b_i} = b_i$ for $i = 1, 2, \ldots, m$
$= 0$ for $i = m+1, m+2, \ldots, m+n$.
Definition: Two column vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal to each other if $\mathbf{a}^T\mathbf{b} = 0$.

1. A furniture company manufactures four models of desks. Each desk is first constructed in the carpentry shop and is next sent to a finishing shop where it is varnished waxed and polished. The number of man hours required in each shop is as shown below.

	Desk 1 (hrs)	Desk 2 (hrs)	Desk 3 (hrs)	Desk 4 (hrs)	Available (hrs)
carpentry shop	4	9	7	10	6000
Finishing shop	1	1	3	40	4000

The profit from the sale of each item is as follows.

	Desk 1	Desk 2	Desk 3	Desk 4
Profit in Rs	12	20	80	40

Assuming that raw materials and supplies are available in adequate supply and all desks produced can be sold, the desk company wants to determine the quantities to make of each type of product which will maximize profit. Formulate this as a linear programming problem (you need not solve the LPP).

2. A furniture manufacturer has 3600 bd.ft of walnut, 4300 bd.ft of Maple, and 6550 bd.ft of Oak in stock. He can produce three types of products using this with input requirements as given below:

	Wood needed	(bd.ft /unit)		Revenue per unit
	Walnut	Maple	Oak	
Table	10	50	100	Rs 1000
Desk	10	30	40	Rs 500
Chair	80	5	0	Rs 100

.

A table is always sold in combination with 4 chairs; and a desk is always sold in combination with one chair. But a chair can be sold independently. Formulate the problem of finding how many of tables, chairs and desks he is going to produce to maximize his revenue, as a linear programming problem (ignoring the integer requirement of the variables).

3. A manufacturer of plastics is planning to blend a new product from four chemical compounds. These compounds are mainly composed of three elements, A, B and C. The composition and unit cost of these chemicals are given below:

Chemical compound	1	2	3	4
Percentage of A	30	20	40	20
Percentage of B	20	60	30	40
Percentage of C	40	15	25	30
Cost in rupees per Kilogram	20	30	20	15

The new product consists of 20 percent of element A, at least 30 percent of element B and at least 20 percent of element C. Due to possible side effects of compounds 1 and 2, they must not exceed 30 percent and 40 percent of the content of the new product. Formulate this as a LPP if the objective is to find the least cost of blending.

4. Consider the following problem (P):

$$\text{Max } x_1 - x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 - x_3 &\leq 2 \\ 2x_1 + 3x_2 - 5x_3 &= 2 \\ 3x_1 + 4x_2 - 3x_3 &\geq 0 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

Write the above problem in the form,
Min $\mathbf{c}^T \mathbf{x}$
subject to $A_{4\times 3}\ \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and call the new problem as (P').
Check that (P) and (P') have the same optimal solution set.
If 20 is the optimal value of (P), then what is the optimal value of (P')?

5. Consider the following problem (P):

$$\text{Max } x_1 + 4$$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 &\leq 8 \\ x_1 + x_2 &\geq 2. \end{aligned}$$

Write the above problem in the form,
Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{2\times 4}\ \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and call the new problem as (P'), such that (P) and (P') have the same optimal solution set.
Does the feasible region of (P) (that is the collection of all elements of $\mathbb{R}^2$ which satisfies the two constraints) have any extreme point? Are the defining hyperplanes of (P) LI?

6. Let (P) be a LPP with $(m+n)$ constraints, and let $\mathbf{x}_0 + \alpha\mathbf{d}$ lie in both H_1 and H_2, for all $\alpha \geq 0$, where $\mathbf{d} \in \mathbb{R}^n$ and H_1, H_2 are two defining hyperplanes of $Fea(P)$. Does it imply that $\mathbf{x}_0 + \alpha\mathbf{d}$ also lies in H_3 for all $\alpha \geq 0$, where H_1, H_2, H_3 are LD?

7. If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^2$, such that no two of them lie on the same line passing through the origin, then sketch the regions given by:

(a) $W_0 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda\mathbf{x}_2, \lambda \in \mathbb{R}\}$.

(b) $W_1 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda\mathbf{x}_2, \lambda \geq 0\}$.

(c) $W_2 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda\mathbf{x}_2, 0 \leq \lambda \leq 1\}$.

(d) $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{x}_1 + \lambda\mathbf{x}_2, \lambda \in \mathbb{R}\}$.

(e) $S_{-2} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{x}_1 + \lambda\mathbf{x}_2, \lambda \geq 0\}$.

(f) $S_{-1} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{x}_1 + \lambda\mathbf{x}_2, 0 \leq \lambda \leq 1\}$.

(g) $S_0 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, \lambda_1, \lambda_2 \in \mathbb{R}\}$.

(h) $S_1 = \{\mathbf{x} : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}$.

(i) $S_2 = \{\mathbf{x} : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq 1\}$.

(j) $S_3 = \{\mathbf{x} : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, \lambda_1, \lambda_2 \geq 0\}$.

(k) $S_4 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \frac{1}{2}\mathbf{x}_3, \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = \frac{1}{2}\}$.

(l) $S_5 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \frac{1}{2}\mathbf{x}_3, \lambda_1, \lambda_2 \geq 0\}$.

(m) $S_6 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \frac{1}{2}\mathbf{x}_3, \lambda_1, \lambda_2 \in \mathbb{R}\}$.

(n) $S_7 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$.

(o) $S_8 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3, \lambda_1, \lambda_2, \lambda_3 \geq 0\}$.

(p) $T_{-1} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3, \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 + \lambda_3 \leq 1\}$.

(q) $T = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3, \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1\}$.

(r) $T_1 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \alpha(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3) + (1-\alpha)\mathbf{x}_4, \quad 0 \leq \alpha \leq 1, \quad \lambda_1, \lambda_2, \lambda_3 \geq 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1\}$.

(s) $T_2 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 + \lambda_4\mathbf{x}_4, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0, \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1\}$.

(t) $T_3 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \alpha(\lambda_1\mathbf{x}_1 + (1-\lambda_1)\mathbf{x}_2) + (1-\alpha)(\lambda_2\mathbf{x}_3 + (1-\lambda_2)\mathbf{x}_4), \quad 0 \leq \alpha \leq 1, \quad \lambda_1, \lambda_2 \geq 0\}$.

(u) $T_4 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \alpha(\lambda_1\mathbf{x}_1 + (1-\lambda_1)\mathbf{x}_2) + (1-\alpha)(\lambda_2\mathbf{x}_3 + (1-\lambda_2)\mathbf{x}_4), \quad 0 \leq \alpha \leq 1, \quad 0 \leq \lambda_1, \lambda_2 \leq 1\}$.

Are the sets T_1, T_2, T_3 equal?

8. Given $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$, $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_k\mathbf{x}_k$ is called a **convex combination** of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.
Show that if $S \subseteq \mathbb{R}^n$ is a convex set and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in S$, then any **convex combination** of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ belongs to S.
In general show that if $S \subseteq \mathbb{R}^n$ is a convex set, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S$, $(k \in \mathbb{N})$, then any convex combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ belongs to S.
(**Hint:** Use induction.)

9. (a) Give an example of a convex set say $S \subset \mathbb{R}^2$ such that $\mathbf{x}_1, \mathbf{x}_2 \in S$, but $T = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2, 0 \leq \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 \leq 1\}$ is not a subset of S.

(b) Let $S \subset \mathbb{R}^2$ be a convex set such that $\mathbf{0} \in S$, and let $\mathbf{x}_1, \mathbf{x}_2$ and T be as defined in part(a), then is $T = S$?

10. If possible give an example of a LPP in three variables such that $Fea(LPP) \subset \mathbb{R}^3$ has exactly 3 corner points (extreme points).

11. If possible give an example of a LPP in three variables such that $Fea(LPP) \subset \mathbb{R}^3$ has exactly 5 corner points (extreme points).

12. If possible give an example of a LPP in three variables and three constraints (other than the nonnegativity constraints) such that $Fea(LPP) \subset \mathbb{R}^3$ has exactly 8 corner points (extreme points).

13. If possible find a nonempty polyhedral subset S of $\mathbb{R}^3$, such that S has three **LI** defining hyperplanes (not necessarily the nonnegativity constraints) but does not have an extreme point.

14. Consider the following problem:

$$\text{Max } -x_1 + 2x_2$$

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & -2x_1 + x_2 \geq 2 \\ & x_2 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Check that the above problem does not have an optimal solution.

(b) Check that if $\mathbf{x}_0 \in Fea(LPP)$ then $\mathbf{x}_0 + \alpha[0,1]^T \in Fea(LPP)$ for all $\alpha > 0$, and $\mathbf{c}^T\mathbf{x}_0 < \mathbf{c}^T(\mathbf{x}_0 + \alpha[0,1]^T)$ and for all $\alpha > 0$.

(c) Can you give one more vector other than the vector $[0,1]^T$, say $\mathbf{d}_1$ which satisfies the same conditions as $[0,1]^T$ (as given in part (b)) at $\mathbf{x}_0$?

(d) Give an example of a set of two linearly dependent hyperplanes say H_1, H_2 defining the feasible region and an $\mathbf{x}_0$ such that $\mathbf{x}_0 \in H_1$ but $\mathbf{x}_0$ does not belong to H_2.

(e) Find a point in $Fea(LPP)$ which does not lie in any of the hyperplanes defining the feasible region.

(f) Find a point in $Fea(LPP)$ which lies in exactly one of the hyperplanes defining the feasible region.

(g) If possible give an example of an $\mathbf{x}_1 \in \mathbb{R}^2$ such that $\mathbf{x}_1$ lies in exactly one linearly independent hyperplanes defining $Fea(LPP)$ but does not belong to $Fea(LPP)$.

(h) If possible give an example of an $\mathbf{x}_1 \in \mathbb{R}^2$ such that $\mathbf{x}_1$ lies in two linearly independent hyperplanes defining $Fea(LPP)$ but is not a corner point of $Fea(LPP)$.

(i) By changing the objective function **only** in the above LPP, give a LPP which has a unique optimal solution. Give the optimal solution and the optimal value in this case.

(j) By changing the objective function **only** in the above LPP, give a LPP which has infinitely many optimal solutions. How many optimal solutions are also extreme points of Fea(LPP)? Give the optimal solution and the optimal value in this case.

(k) By changing the second constraint **only** in the above LPP, give a LPP which always has an optimal solution for any objective function. Then find the optimal solution and the optimal value for the given objective function.

15. If the feasible region for a linear programming problem (P) given below as
Min $\mathbf{c}^T\mathbf{x}$
subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
is unbounded, then is it always possible to choose a $\mathbf{c}$ (feasible region remaining same as given in (P)) such that the LPP does not have an optimal solution? Also is it possible to choose a $\mathbf{c} \neq \mathbf{0}$ (feasible region remaining same as (P)) such that the LPP has an optimal solution?

16. If Min $\mathbf{c}^T\mathbf{x}_{2\times1}$, $\mathbf{c} \neq \mathbf{0}$
subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
and Max $\mathbf{c}^T\mathbf{x}_{2\times1}$, $\mathbf{c} \neq \mathbf{0}$
subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
both have optimal solution/s then can the common feasible region of the above problems be unbounded?

17. If Min $\mathbf{c}^T\mathbf{x}_{2\times1}$, $\mathbf{c} \neq \mathbf{0}$
subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
and Max $(\mathbf{c}^T + [1,2])\mathbf{x}_{2\times1}$, $\mathbf{c} \neq \mathbf{0}$
subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
both have optimal solution/s then can the common feasible region of the above problems be unbounded?

18. Let the linear programming problem (LPP)
Min $\mathbf{c}^T$ $\mathbf{x}$
subject to $A_{3\times2}$ $\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
has $\mathbf{x}_0$ as an optimal solution.

(a) If $\mathbf{b}' = \mathbf{b} + [1,2,0]^T$($[1,2,0]^T$ is a column vector with components 1, 2 and 0) then will the above LPP with $\mathbf{b}$ replaced by $\mathbf{b}'$ again have an optimal solution?

(b) If $\mathbf{b}' = \mathbf{b} + [-1,0,0]^T$($[-1,0,0]^T$ is a column vector with components -1, 0 and 0) and the above LPP with $\mathbf{b}$ replaced by $\mathbf{b}'$ (everything else remaining same) again has $\mathbf{x}_0$ as a feasible solution then will $\mathbf{x}_0$ also be an optimal solution of the changed LPP?

19. Consider the following problem:

$$\text{Max } x_1 + x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 - x_3 &\leq 2 \\ 2x_1 + 3x_2 - 2x_3 &\geq 2 \\ 3x_1 + 4x_2 - 3x_3 &\geq 0 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) Check that the hyper planes corresponding to the first three constraints are linearly dependent.

(b) Check that $\tilde{x} = [1, 1, 1]^T$ is a feasible solution of the LPP.

(c) Does $\tilde{x} = [1, 1, 1]^T$ lies in any of the hyperplanes defining $Fea(LPP)$?

(d) Find a $\mathbf{d} \neq \mathbf{0}$ such that $\tilde{x} + \alpha\mathbf{d}, \tilde{x} - \alpha\mathbf{d} \in Fea(LPP)$, for all $\alpha > 0$, sufficiently small, where $\tilde{x}$ is as given in part(b).

(e) Is this $\mathbf{d}$ necessarily a direction of Fea(LPP)?

(f) Find a $\mathbf{d}_0 \neq \mathbf{0}$ such that $\mathbf{y}_0 = \tilde{x} - \alpha_0\mathbf{d}_0$, where $\alpha_0 = max\{\alpha > 0 : \tilde{x} - \alpha\mathbf{d}_0 \in Fea(LPP)\}$, lies in exactly one LI hyperplane defining Fea(LPP).

(g) If possible find a $\mathbf{d}_1 \neq \mathbf{0}$ such that $\mathbf{y}_1 = \tilde{x} - \alpha_1 d_1$, where $\alpha_1 = max\{\alpha > 0 : \tilde{x} - \alpha\mathbf{d}_1 \in Fea(LPP)\}$, lies in exactly two LI hyperplane defining Fea(LPP).

(h) If possible find a $\mathbf{d}_2 \neq \mathbf{0}$ such that $\mathbf{y}_2 = \tilde{x} - \alpha_2\mathbf{d}_2$,where $\alpha_2 = max\{\alpha > 0 : \tilde{x} - \alpha\mathbf{d}_2 \in Fea(LPP)\}$, lies in exactly three LI hyperplane defining Fea(LPP).

(i) Find an extreme (or corner point) point of Fea(LPP).

(j) How many extreme points (corner points) does this feasible region have?

(k) Does this problem have an optimal solution?

20. **The Transportation Problem.** Let there be m supply stations $S_1, ..., S_m$ for a particular product and n destination stations $D_1, D_2, ..., D_n$ where the product is to be transported. Let c_{ij} be the cost of unit amount of the product from S_i to D_j. Let s_i be the available amount of the product at S_i and let d_j be the demand at D_j. The problem is to find x_{ij}, $i = 1, 2, .., m$, $j = 1, 2, ..., n$, where x_{ij} is the amount of the product to be transported from S_i to D_j such that the cost of transportation is minimum. The problem is given by
Min $\sum_{i,j} c_{ij}x_{ij}$
subject to $\sum_{j=1}^{n} x_{ij} \leq s_i$, $i = 1, 2, ..., m$
$\sum_{i=1}^{m} x_{ij} \geq d_j, j = 1, 2, ..., n$, $x_{ij} \geq 0$ for $i = 1, 2, .., m, j = 1, 2, .., n$.
Consider a transportation problem with 2 supply stations and 5 destinations such the maximum capacity of the two supply stations S_1 and S_2 are given by 30 units and 40 units respectively. The demands at each of the 5 destinations are 10 units each.

(a) Obtain a feasible solution for this problem. Without trying to calculate the optimal solution, justify that this problem will always have an optimal solution.

(b) By changing **only** the demands at the five supply stations of the above problem, find a transportation problem with no feasible solution.

(c) Hence find a necessary and sufficient condition for the transportation problem with m supply stations and n destination stations to have an optimal solution.