Notations:
$\mathbf{x}, \mathbf{d}, \mathbf{b}$, etc, that is characters in boldface represent (column) vectors.
$\tilde{\mathbf{a}}_i$, the ith column of A.

Simplex Algorithm

From now on we will consider linear programming problems of the type (P*):

Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $\qquad (*)$
where $rank(A) = m$.
Let us now denote the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ by $Fea(LPP)$.
Note that if we are given a problem of the type:
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
we add some variables and we can convert the system of constraints to
$A_{m\times n}\mathbf{x} + \mathbf{s}_{m\times 1} = [A : I]\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{s} \geq \mathbf{0}$.
So it is now of the form of problem (P*).
If suppose we are given a problem of the type (P**):
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{k\times n}\mathbf{x} = \mathbf{b}_{k\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
where $k > rank(A) = m$.
Let (P**) have atleast one feasible solution and WLOG let $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ be a set of m LI rows of A ($rank(A) = m$).
Let $\mathbf{x}_0 \in \{\mathbf{x} \in \mathbb{R}^n : A_{k\times n}\mathbf{x} = \mathbf{b}_{k\times 1}, \quad \mathbf{x} \geq \mathbf{0}\}$.
Then for all $i = 1, \ldots, k$,
$\mathbf{a}_i^T = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T$ for some real u_{ji}'s, $i = 1, \ldots, k$, $j = 1, \ldots, m$, $\qquad (**)$
hence $\mathbf{a}_i^T\mathbf{x}_0 = b_i = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T\mathbf{x}_0 = \sum_{j=1}^m u_{ji}b_j$, which implies $b_i = \sum_{j=1}^m u_{ji}b_j$, for all $i = 1, \ldots, k$. $\qquad (***)$
Hence from (**) and (***) it follows that for any $\mathbf{x}$
$\mathbf{a}_i^T\mathbf{x} = b_i$, for $i = 1, \ldots, m \iff \mathbf{a}_i^T\mathbf{x} = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T\mathbf{x} = \sum_{j=1}^m u_{ji}b_j = b_i$, for all $i = 1, \ldots, k$.
Hence if the system $A_{k\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$ is consistent(that is it has at least one solution), then we can throw away $k - m$ rows of A and the corresponding components of $\mathbf{b}$ in (P**) to get an equivalent (having the same set of solutions) system of equations of the form (*).
Hence an LPP of the form (P**) can again be converted to a problem of the type (P*), by throwing away some of the constraints, such that the feasible region of the changed LPP remains same.
Note: Here it is important to point out that if we have a problem (P) of the form:
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{k\times n}\mathbf{x} \leq \mathbf{b}_{k\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
where $k > rank(A) = m$.
Then if we throw away constraints (corresponding to LD rows of A) from the above set of constraints, then we may get a different feasible region.

An $\mathbf{x} \in Fea(LPP)$ is called a **basic feasible solution** (**BFS**) of the LPP if the columns of the matrix A corresponding to the nonzero components of $\mathbf{x}$ are LI.
An $\mathbf{x}$ satisfying the system $A\mathbf{x} = \mathbf{b}$, and the condition that the columns corresponding to the nonzero components are LI, is called a **basic solution** of the LPP. So a basic solution may not be a nonnegative vector, hence need not be a feasible solution of the LPP.
So a basic feasible solution of a LPP of the form (1), can have at most m strictly positive components (since $rank(A) = m$).

A basic feasible solution is called a **non degenerate** BFS if it has exactly m positive components, otherwise it is said to be a **degenerate** BFS.
If $\mathbf{x}$ is a **non degenerate** BFS, then the columns of A corresponding the nonzero (positive) components of $\mathbf{x}$ form a basis of $\mathbb{R}^m$. WLOG, let the positive components of $\mathbf{x}$ be $x_1, \ldots, x_m$, then $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$,

forms a basis of $\mathbb{R}^m$, where $\tilde{\mathbf{a}}_i$ is the ith column of A.
The $m \times m$ matrix, $B_{m\times m} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$ formed with columns $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$, of A is called the **basis matrix** corresponding to $\mathbf{x}$.
The variables $x_1, \ldots, x_m$ are called the **basic variables**, and $x_{m+1} = x_{m+2} = \ldots = x_n = 0$, are called the **non basic variables** of the BFS $\mathbf{x}$.

If $\mathbf{x}$ is a **degenerate** BFS, then consider the columns of A corresponding to the nonzero components of $\mathbf{x}$. WLOG let them be $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, $k < m$.
Then add $(m-k)$ LI columns of A, such that these $(m-k)$ columns of A together with $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, form a set of m LI vectors from the columns of A (you can always do that), hence a basis of $\mathbb{R}^m$.
Let as before the matrix $B_{m\times m}$ formed with these m columns, (WLOG let it be $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$) be called a basis matrix corresponding to $\mathbf{x}$.
The components $x_1, \ldots, x_m$ of $\mathbf{x}$ are called as before, **basic variables** corresponding to $\mathbf{x}$ and the basis matrix B. The components $x_{m+1} = \ldots = x_n = 0$ are the **nonbasic variables**.

Note that depending on the choice of the rest of the $(m-k)$ columns which are added, the same $\mathbf{x}$ will correspond to different basis matrices B, and hence will have different basic and nonbasic variables.
Suppose if $\mathbf{x}$ is a BFS such that $x_{m+1} = \ldots = x_n = 0$, then since $\mathbf{x}$ is a feasible solution of the LPP, $A\mathbf{x} = \mathbf{b}$, hence we get $\sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$, which implies $B\mathbf{x}_B = \mathbf{b}$ or $\mathbf{x}_B = B^{-1}\mathbf{b}$,
where $\mathbf{x}_B$ are the components of $\mathbf{x}$ corresponding to the basic variables.
Hence a BFS $\mathbf{x}$ is of the form, $\mathbf{x} = \begin{bmatrix} B^{-1}\mathbf{b}_{m\times 1} \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix}$.

Hence in the context of the diet problem, that is if we consider the above problem to be the diet problem (with $\geq$ inequalities changed to equalities in the original problem), then $\mathbf{x}$ gives the quantities of the various food products F_j, $j = 1, \ldots, n$ which are to be included in the diet. The food products $F_1, \ldots, F_m$, which correspond to the basic variables $x_1, x_2, \ldots, x_m$ of $\mathbf{x}$, are the ones which are included in the diet in the quantities x_j, the other food products corresponding to the nonbasic variables (they have zero values) are not to be consumed, hence not included in the diet.

Let us assume that (P*) has atleast one feasible solution and let $\mathbf{x}$ be a BFS (degenerate or non degenerate) of the LPP (P*) (the existence of a BFS will be justified later).
WLOG let $B = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$ be a (or the) basis matrix corresponding to $\mathbf{x}$.
Then note that for all $k = 1, 2, \ldots, n$,

$$\tilde{\mathbf{a}}_k = \sum_{i=1}^m u_{ik}\tilde{\mathbf{a}}_i = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} \qquad (*)$$

for some real u_{ik}'s, $i = 1, \ldots, m$, $k = 1, \ldots, n$,

which implies $\begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = B^{-1}\tilde{\mathbf{a}}_k$ for all $k = 1, 2, \ldots, n$.

In the context of the diet problem the above equations (*), mean that in order to obtain the same amount of nutrient as unit amount of F_k, $k = 1, \ldots, n$, one needs to consume
(u_{1k}) amount of F_1+ (u_{2k}) amount of $F_2 + \ldots + (u_{mk})$ amount of F_m.
Hence the value of unit amount of F_k, if we include only $F_i, i = 1, \ldots, m$ in the diet (which has to be bought from the market) comes out to be
$u_{1k}c_1 + u_{2k}c_2 + \ldots + u_{mk}c_m$, which we denote by z_k.

So $z_k = \sum_{i=1}^m u_{ik}c_i = [c_1, \ldots, c_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}}_k$.

Note that the cost of the objective function corresponding to this BFS is given by
$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$,
where, $\mathbf{c}_B^T$ are the components of the vector $\mathbf{c}^T$ which correspond to the basic variables.

Also for all $k = 1, \ldots, m$, note that $z_k = c_k$.

Now the simplex table corresponding to the BFS $\mathbf{x}$ will be given by

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$\ldots$	$c_m - z_m = 0$	$\ldots$	$c_s - z_s$	$\ldots$	$c_k - z_k$	$\ldots$	$c_n - z_n$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$\ldots$	$B^{-1}\tilde{\mathbf{a}}_m$	$\ldots$	$B^{-1}\tilde{\mathbf{a}}_s$	$\ldots$	$B^{-1}\tilde{\mathbf{a}}_k$	$\ldots$	$B^{-1}\tilde{\mathbf{a}}_n$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	1	0	$\ldots$	0	$\ldots$	u_{1s}	$\ldots$	u_{1k}	$\ldots$	u_{1n}	x_1
$\tilde{\mathbf{a}}_2$	0	1	$\ldots$	0	$\ldots$	u_{2s}	$\ldots$	u_{2k}	$\ldots$	u_{2n}	x_2
$\vdots$	0	0	$\ldots$	0	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_r$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	u_{rs}	$\ldots$	u_{rk}	$\ldots$	u_{rn}	x_r
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_m$	0	0	$\ldots$	1	$\ldots$	u_{ms}	$\ldots$	u_{mk}	$\ldots$	u_{mn}	x_m

Case 1: $c_k - z_k < 0$ for atleast one k, $k = m+1, \ldots, n$.
So if $c_k - z_k < 0$ for some $k = m+1, \ldots, n$, then the market price of unit amount of F_k given by c_k is less than the value of unit amount of F_k (as obtained by consuming only $F_1, \ldots, F_m$) given by z_k, hence it might be sensible (profitable) to include this F_k in the diet.
Let $c_s - z_s = min\{c_k - z_k : c_k - z_k < 0, \quad k = m+1, \ldots, n\}$.
Simplex algorithm says that, then include x_s in the diet.
If there exists, s, l, such that for both s, l,
$c_s - z_s = c_l - z_l = min\{c_k - z_k : c_k - z_k < 0, \quad k = m+1, \ldots, n\}$,
then include any **one** of these food products in the diet.
Let $\mathbf{x}'$ be the new solution to be obtained.
Note that (refer to (*)) u_{is} amount of F_i, $i = 1, \ldots, m$, is used in the constitution of unit amount of F_s, which is now included in the diet in an amount say x'_s. Hence the same corresponding amount of the F_i's, $i = 1, \ldots, m$, given by $u_{is}x'_s$ need not be consumed any more.
Hence the components of $\mathbf{x}'$ are given by
$x'_i = x_i - u_{is}x'_s$ for $i = 1, \ldots, m$,
$x'_s \geq 0$ and
$x'_i = 0$ for $i = m+1, \ldots, n$, $i \neq s$.
Then note that if we want $\mathbf{x}'$ to be a feasible solution of the LPP, then we have to choose $x'_s \geq 0$ such that $x'_i \geq 0$ for all $i = 1, \ldots, m$.
Note that if $u_{is} \leq 0$, for some $i = 1, \ldots, m$, then for all $x'_s \geq 0$, $x'_i \geq 0$ for that i.
So in order that $x'_i \geq 0$, for all $i = 1, \ldots, m$,
we have to choose $x'_s \geq 0$, such that $x'_s \leq \frac{x_i}{u_{is}}$ for which $u_{is} > 0$ (if at all there is one such i).

Case 1a: For some (atleast one) $i = 1, \ldots, m$, $u_{is} > 0$ (where s is as defined in Case 1).
Let $\frac{x_r}{u_{rs}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$.
(Note that there could exist r, t such that for both $r, t \in \{m+1, \ldots, n\}$
$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$).

Here $\frac{x_r}{u_{rs}}$ is called the **minimum ratio**.
Then in order that $\mathbf{x}' \geq \mathbf{0}$, $x'_s \leq \frac{x_r}{u_{rs}}$.
Then it can be easily checked that $A\mathbf{x}' = \mathbf{b}$ or $\mathbf{x}' \in Fea(LPP)$, where $\mathbf{x}'$ is as defined in Case 1.
To see this, note that $A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x'_i + \tilde{\mathbf{a}}_s x'_s$, which implies
$A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is} x'_s + \tilde{\mathbf{a}}_s x'_s = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i + x'_s(\tilde{\mathbf{a}}_s - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is}) = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$.

Also $\mathbf{c}^T\mathbf{x}' = \sum_{i=1}^m c_i x'_i + c_s x'_s$, which implies
$\mathbf{c}^T\mathbf{x}' = \sum_{i=1}^m c_i x_i - \sum_{i=1}^m c_i u_{is} x'_s + c_s x'_s = \sum_{i=1}^m c_i x_i + x'_s(c_s - \sum_{i=1}^m c_i u_{is}) = \mathbf{c}^T\mathbf{x} + x'_s(c_s - z_s) \leq \mathbf{c}^T\mathbf{x}$.
Note that $\mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x}$ if $x'_s > 0$, which is possible only if the **minimum ratio** $\frac{x_r}{u_{rs}} > 0$.
In order to reduce cost (or the value of the objective function) as much as possible, x'_s is given its maximum possible value, which is equal to $\frac{x_r}{u_{rs}}$.
Then note that $x'_r = x_r - u_{rs}\frac{x_r}{u_{rs}} = 0$.
The variable x_s is called the **entering variable** for the new basis (choose any one if there are more than

one choice for the **entering variable**), and the variable x_r is called a **leaving variable**.
If there exists r, t such that for both $r, t \in \{m+1, \ldots, n\}$
$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$,
then take any **one** of r, t as the **leaving variable**.
So $\mathbf{x}' \in Fea(LPP)$ again has at most m nonzero components, and we can easily check that $\mathbf{x}'$ is a basic feasible solution of the LPP.
To prove this we need to show that the set of columns $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}, \tilde{\mathbf{a}_s}\}$ of A is LI and hence forms a basis of $\mathbb{R}^m$.
If suppose not, then since the collection $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}\}$ is LI, it implies that $\tilde{\mathbf{a}_s}$ can be written as a linear combination of the $(m-1)$ columns, $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}\}$, but this would imply that $u_{rs} = 0$, which is a contradiction.
Hence $\mathbf{x}'$ is an improved (with respect to cost or the value of the objective function) BFS as compared to the BFS $\mathbf{x}$.
Let us denote the new basis matrix corresponding to $\mathbf{x}'$ as $B' = [\tilde{\mathbf{a}_1} \ldots \tilde{\mathbf{a}_{r-1}} \tilde{\mathbf{a}_{r+1}} \ldots \tilde{\mathbf{a}_m} \tilde{\mathbf{a}_s}]$.

The simplex table corresponding to the BFS $\mathbf{x}'$ will be obtained again by expressing each of the vectors $\tilde{\mathbf{a}_i}$, $i = 1, \ldots, n$, and $\mathbf{b}$ in terms of the new set of basis vectors $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}, \tilde{\mathbf{a}_s}\}$.
Since $\tilde{\mathbf{a}_s} = \sum_{i=1, i\neq r}^{m} u_{is}\tilde{\mathbf{a}_i} + u_{rs}\tilde{\mathbf{a}_r}$,
so $\tilde{\mathbf{a}_r} = \frac{\tilde{\mathbf{a}_s}}{u_{rs}} - \sum_{i=1, i\neq r}^{m} \frac{u_{is}}{u_{rs}}\tilde{\mathbf{a}_i}$. $\qquad (*)$
Also since for any $k = 1, \ldots, n$,
$\tilde{\mathbf{a}_k} = \sum_{i=1}^{m} u_{ik}\tilde{\mathbf{a}_i} = \sum_{i=1, i\neq r}^{m} u_{ik}\tilde{\mathbf{a}_i} + \tilde{\mathbf{a}_r} u_{rk}$. $\qquad (**)$
By substituting the expression for $\tilde{\mathbf{a}_r}$ given in $(*)$ in the equation $(**)$, we get
$\tilde{\mathbf{a}_k} = \sum_{i=1, i\neq r}^{m} u_{ik}\tilde{\mathbf{a}_i} + \tilde{\mathbf{a}_r} u_{rk} = \sum_{i=1, i\neq r}^{m} (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk})\tilde{\mathbf{a}_i} + \tilde{\mathbf{a}_s}\frac{u_{rk}}{u_{rs}}$.

Let us denote the new coefficients corresponding to $\tilde{\mathbf{a}_i}$, in the above expression as u'_{ik},
for $i = 1, \ldots, r-1, r+1, m, s$, and $k = 1, \ldots, n$.
Hence for all $k = 1, \ldots, n$,
$u'_{ik} = u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}$ for $i = 1, \ldots, r-1, r+1, m$
and $u'_{sk} = \frac{u_{rk}}{u_{rs}}$.

Note that the coefficients corresponding to $\mathbf{b}$, when $\mathbf{b}$ is expressed as a linear combination of the new basis of $\mathbb{R}^m$, (which was already obtained and the philosophy behind explained) can be recalculated as follows:
Recall that $\mathbf{b} = \sum_{i=1, i\neq r}^{m} \tilde{\mathbf{a}_i} x_i + \tilde{\mathbf{a}_r} x_r$.
Now again substituting the expression for $\tilde{\mathbf{a}_i}$ given in $(*)$ in the above equation we get
$\mathbf{b} = \sum_{i=1, i\neq r}^{m} \tilde{\mathbf{a}_i} x_i + x_r(\frac{\tilde{\mathbf{a}_s}}{u_{rs}} - \sum_{i=1, i\neq r}^{m} \frac{u_{is}}{u_{rs}}\tilde{\mathbf{a}_i})$.
On simplifying the above expression we get
$\mathbf{b} = \sum_{i=1, i\neq r}^{m} (x_i - \frac{u_{is}}{u_{rs}} x_r)\tilde{\mathbf{a}_i} + \tilde{\mathbf{a}_s}(\frac{x_r}{u_{rs}})$.
Hence Let us denote the new values of z_k now corresponding to $\mathbf{x}'$, by z'_k. Then we get
$z'_k = \sum_{i=1, i\neq r}^{m} u'_{ik} c_i + u'_{sk} c_s$
$= \sum_{i=1, i\neq r}^{m} (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}) c_i + \frac{u_{rk}}{u_{rs}} c_s = \sum_{i=1, i\neq r}^{m} u_{ik} c_i - \sum_{i=1, i\neq r}^{m} (\frac{u_{is}}{u_{rs}} u_{rk}) c_i + \frac{u_{rk}}{u_{rs}} c_s$
$= (\sum_{i=1}^{m} u_{ik} c_i - u_{rk} c_r) - \frac{u_{rk}}{u_{rs}} (\sum_{i=1}^{m} u_{is} c_i - u_{rs} c_r) + \frac{u_{rk}}{u_{rs}} c_s$
$= z_k - u_{rk} c_r - \frac{u_{rk}}{u_{rs}} z_s - \frac{u_{rk}}{u_{rs}} (-u_{rs} c_r) + \frac{u_{rk}}{u_{rs}} c_s = z_k + \frac{(c_s - z_s)}{u_{rs}} u_{rk}$.

From this we get the new values of $c_k - z'_k$ as
$c_k - z'_k = (c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$.

Hence the simplex table corresponding to the new BFS $\mathbf{x}'$ is given by

	$c_1 - z_1'$	$\dots$	$c_m - z_m'$	$\dots$	$c_s - z_s'$	$\dots$	$c_k - z_k'$	$\dots$	$c_n - z_n'$	
	0	$\dots$	0	$\dots$	$c_s - z_s - \frac{(c_s - z_s)}{u_{rs}} u_{rs} = 0$	$\dots$	$(c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$	$\dots$	$\dots$	
	$B'^{-1}\tilde{\mathbf{a}}_1$	$\dots$	$B'^{-1}\tilde{\mathbf{a}}_m$	$\dots$	$B'^{-1}\tilde{\mathbf{a}}_s$	$\dots$	$B'^{-1}\tilde{\mathbf{a}}_k$	$\dots$	$B'^{-1}\tilde{\mathbf{a}}_n$	$B'^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	1	$\dots$	0	$\dots$	$u_{1s} - \frac{u_{1s}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{1k} - \frac{u_{1s}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_1 - \frac{u_{1s}}{u_{rs}} x_r$
$\tilde{\mathbf{a}}_2$	0	$\dots$	0	$\dots$	$u_{2s} - \frac{u_{2s}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{2k} - \frac{u_{2s}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_2 - \frac{u_{2s}}{u_{rs}} x_r$
$\vdots$	0	$\dots$	0	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_r$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\frac{u_{rs}}{u_{rs}} = 1$	$\dots$	$\frac{u_{rk}}{u_{rs}}$	$\dots$	$\frac{u_{rn}}{u_{rs}}$	$\frac{x_r}{u_{rs}}$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_m$	0	$\dots$	1	$\dots$	$u_{ms} - \frac{u_{ms}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{mk} - \frac{u_{ms}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_m - \frac{u_{ms}}{u_{rs}} x_r$

The entry u_{rs} of the previous table which is made 1 (by dividing) in this table is called the **pivot element**.

Case 1b: For all $i = 1, \dots, m$, $u_{is} \leq 0$ (where s is as defined in **Case 1**).
In that case $\mathbf{x}' \geq \mathbf{0}$, for all $x_s' \geq 0$, hence $\mathbf{x}' \in Fea(LPP)$ for all values of $x_s' \geq 0$.
Also since $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + x_s'(c_s - z_s)$, and $(c_s - z_s) < 0$, so given any real number M, by taking x_s' sufficiently large we can make $\mathbf{c}^T\mathbf{x}'$ smaller than M, hence in this case the LPP has unbounded solution, and does not have an optimal solution.

Also note that $\mathbf{x}' = \mathbf{x} + x_s' \begin{bmatrix} -u_{1s} \\ \vdots \\ -u_{ms} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{x} + x_s' \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$,

where the 1 in the above vector occurs at the s th position.

Let us call this vector $\begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ as $\mathbf{d}$, then since $\mathbf{x} + x_s'\mathbf{d} \in Fea(LPP)$ for all $x_s' \geq 0$, from definition, $\mathbf{d}$ should be a direction of $Fea(LPP)$.

Result 1: The set of all directions of $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, rank(A) = m\}$, is given by
$D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq 0, A_{m\times n}\mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}\}$.
Proof: Exercise.

Result 2: If for some basis matrix B and a column $\tilde{\mathbf{a}}_s$ of A, $B^{-1}\tilde{\mathbf{a}}_s \leq \mathbf{0}$ then

$$\mathbf{d} = \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is an extreme direction of $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where the entry 1 in the above vector is at the s th position.

Proof: That it is a direction of S, follows from the direction of a set.
It can also be checked that

$$A\mathbf{d} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}_m} | \tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [B | \tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$= -\tilde{\mathbf{a}_s} + \tilde{\mathbf{a}_s} = \mathbf{0}$.
To check that $\mathbf{d}$ is an extreme direction, let there exist directions $\mathbf{d}_1, \mathbf{d}_2$ of S and $\alpha_1, \alpha_2 > 0$ such that $\mathbf{d} = \alpha_1 \mathbf{d}_1 + \alpha_2 \mathbf{d}_2$.
Since $\alpha_1, \alpha_2 > 0$ and $\mathbf{d}_1, \mathbf{d}_2 \geq 0$

$$\mathbf{d}_1 = \begin{bmatrix} \mathbf{c}_{m\times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

for some $\mathbf{c}_{m\times 1} \geq \mathbf{0}$ and $u \geq 0$, where u is at the s th position.

But form result 1, $A\mathbf{d}_1 = [B : \tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} \mathbf{c}_{m\times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$

$\Rightarrow B\mathbf{c} = -u\tilde{\mathbf{a}_s}$.
Hence $\mathbf{c} = -uB^{-1}\tilde{\mathbf{a}_s}$ or $\mathbf{d}_1 = u\mathbf{d}$, and $u > 0$.
Similarly we get $\mathbf{d}_2 = v\mathbf{d}$ for some $v > 0$.
Hence $\mathbf{d}$ is an extreme direction.
Alternatively note that this $\mathbf{d}$ lies on $(n-1)$ LI hyperplanes (check this) of the $m+n$ hyperplanes defining D.
It lies on m LI hyperplanes given by $A\mathbf{d} = \mathbf{0}$ and also on the $(n-m-1)$ LI hyperplanes given by $d_i = 0$ for $i = m+1, \ldots, n$, $i \neq s$.
It is easy to see that the $(n-1)$ hyperplanes are LI.
(Hint: The normals corresponding to these $(n-1)$ hyperplanes written together as a matrix with $(n-1)$ rows look somewhat like

$$\begin{bmatrix} B & \tilde{\mathbf{a}_{m+1}} & \ldots & \tilde{\mathbf{a}_s} & \ldots & \tilde{\mathbf{a}_n} \\ \mathbf{0} & 1 & \ldots & 0 & \ldots & 0 \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 0 \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}.$$

Note that the column below the vector $\tilde{\mathbf{a}_s}$ in the above matrix is the zero column, since $d_s \neq 0$.)

Since the LPP has unbounded solution, so there should exist at least one extreme direction $\mathbf{d}$ of the LPP such that $\mathbf{c}^T\mathbf{d} < 0$.
Check that if $\mathbf{d}$ is as defined above then

$$\mathbf{c}^T\mathbf{d} = [\mathbf{c}_B^T \mid c_{m+1} \dots c_s \dots c_n] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_s} + c_s = c_s - z_s < 0.$$

Case 2:(**Optimality Condition**) $c_k - z_k \geq 0$ for all $k = 1, \dots, n$.

Then to show that $\mathbf{x}$ is optimal for the LPP.
Among the various ways to check this, one way is produce a feasible solution of the dual of this problem say $\mathbf{y}$ such that $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$. Yet another way (which is actually same as the previous) is to produce a feasible solution $\mathbf{y}$ of the dual and to show that this $\mathbf{y}$ satisfies the complementary slackness property with this BFS $\mathbf{x}$ of the primal.
Since $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$, and since this has to be equal to $\mathbf{b}^T\mathbf{y} = \mathbf{y}^T\mathbf{b}$ for some feasible solution $\mathbf{y}$ of the dual, so we can start by checking if $\mathbf{y}_0^T = \mathbf{c}_B^T B^{-1}$ gives a feasible solution of the dual.

Recall that the problem (LPP) is given by,

Min $\mathbf{c}^T\mathbf{x}$
subject to
$A_{m\times n}\mathbf{x} = \mathbf{b}$,
$\mathbf{x} \geq \mathbf{0}$.

The dual of the above problem is given by,

Max $\mathbf{b}^T\mathbf{y}$
subject to
$A_{n\times m}^T\mathbf{y} \leq \mathbf{c}$.

Proof : In order to see this note that,
if $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$ then
$\mathbf{a}_i^T\mathbf{x} \geq b_i$ for $i = 1, \dots, m$ (*)
and $-\mathbf{a}_i^T\mathbf{x} \geq -b_i$ for $i = 1, \dots, m$ (**),
where $\mathbf{a}_i^T$ is the i th row of the matrix A.
The above inequalities can be written as
$\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$.

Hence the LPP (primal problem) can be written as:
Min $\mathbf{c}^T\mathbf{x}$
subject to
$\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$, $\mathbf{x} \geq \mathbf{0}$.
Since each constraint of the primal corresponds to a variable of the dual, let u_i correspond to the ith constraint of the type $(*)$, $i = 1, \dots, n$, and let w_i correspond to the ith constraint of the type $(**)$, $i = 1, \dots, n$.

Then the dual of the above LPP problem is given as follows:
Given an LPP
Max $\mathbf{b}^T\mathbf{u} - \mathbf{b}^T\mathbf{w}$
subject to

$[A^T \;\; -A^T] \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = A^T\mathbf{u} - A^T\mathbf{w} \leq \mathbf{c}$
$\mathbf{u} \geq \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$.

Hence if we let $\mathbf{y} = \mathbf{u} - \mathbf{w}$ then the above dual reduces to
Max $\mathbf{b}^T\mathbf{y}$
subject to
$A^T\mathbf{y} \leq \mathbf{c}$,
where $\mathbf{y}$ is unrestricted in sign.

Case 2 (continued):
If $c_k - z_k \geq 0$ for all $k = 1, \ldots, n$, then $z_k \leq c_k$ for all $k = 1, \ldots, n$.
Then note that $z_k = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_k} = \mathbf{y}_0^T\tilde{\mathbf{a}_k} \leq c_k$ for all $k = 1, \ldots, n$, or $\mathbf{y}_0$ satisfies the condition: $A^T\mathbf{y}_0 \leq \mathbf{c}$, which implies that $\mathbf{y}_0$ is a feasible solution of the dual satisfying $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}_0$, hence $\mathbf{x}$ and $\mathbf{y}_0$ are optimal solutions of the LPP (primal problem) and its dual, respectively.

Also note that if $(A^T\mathbf{y}_0)_k = \mathbf{y}_0^T\tilde{\mathbf{a}_k} = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_k} = z_k < c_k$, the x_k has to be a nonbasic variables, then $x_k = 0$. Hence $\mathbf{y}_0 \in Fea(D)$ satisfies the complementary slackness property with this $\mathbf{x}$ as expected.
(The complementary slackness condition for this LPP and its dual reduces to only one condition:
$x_i = 0$, whenever $(A^T\mathbf{y})_k < c_k$.)

Remark: Note that if instead the LPP would have been a maximization problem as given below:
Max $\mathbf{c}^T\mathbf{x}$
subject to
$A_{m\times n}\mathbf{x} = \mathbf{b}$,
$\mathbf{x} \geq \mathbf{0}$, $rank(A) = m$,
then **Case 1** and **Case 2** conditions would change accordingly, as given below:

Case 1: $c_k - z_k > 0$ for at least one k, $k = m+1, \ldots, n$.
And the condition for the **entering variable** becomes,
s th variable will enter the basis if $c_s - z_s = max\{c_k - z_k : c_k - z_k > 0, \;\; k = m+1, \ldots, n\}$.

Case 2: (**Optimality condition (sufficient condition)**) $c_k - z_k \leq 0$ for all $k = 1, \ldots, n$.
Note that the above optimality condition is sufficient but not necessary for an optimal solution.

Exercise: Give an example of a Linear Programming Problem for which an optimal table does not satisfy the sufficient condition for optimality (given above).

It remains to justify that a LPP with nonempty feasible region has at least one BFS.