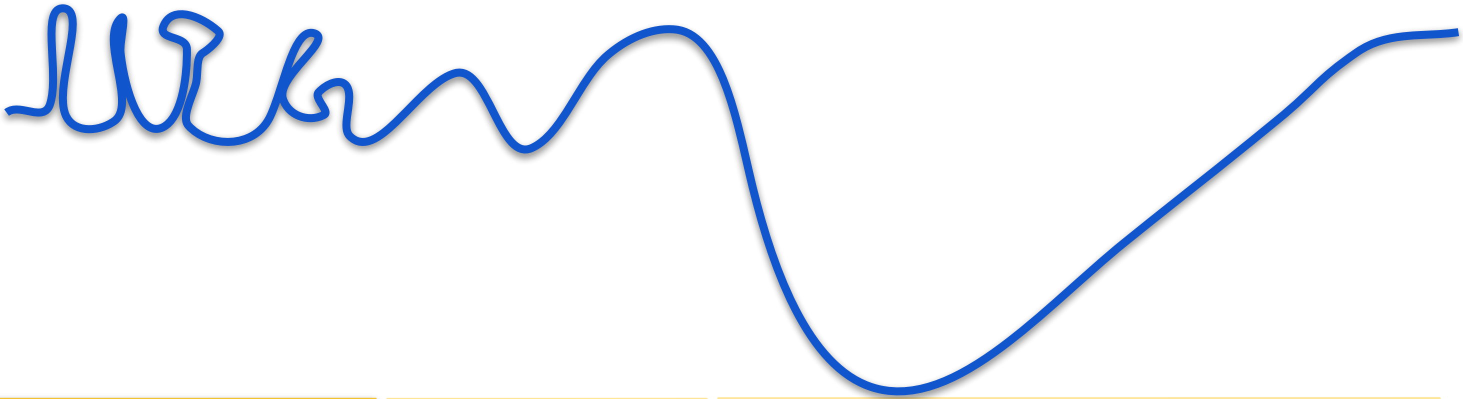
Computing with Signals



DA 623

Jan - May 2024

IIT Guwahati

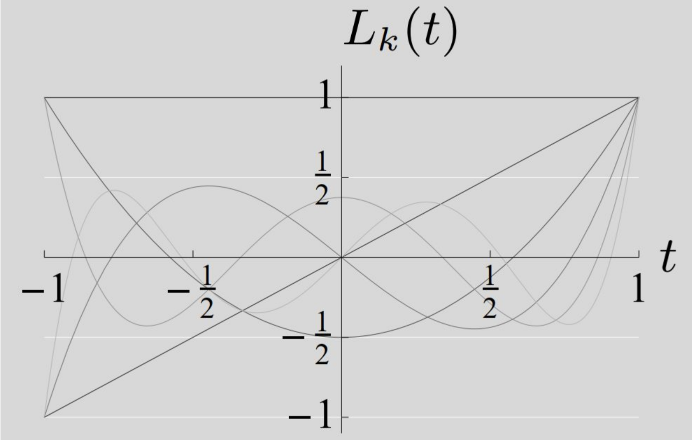
Instructors: Neeraj Sharma

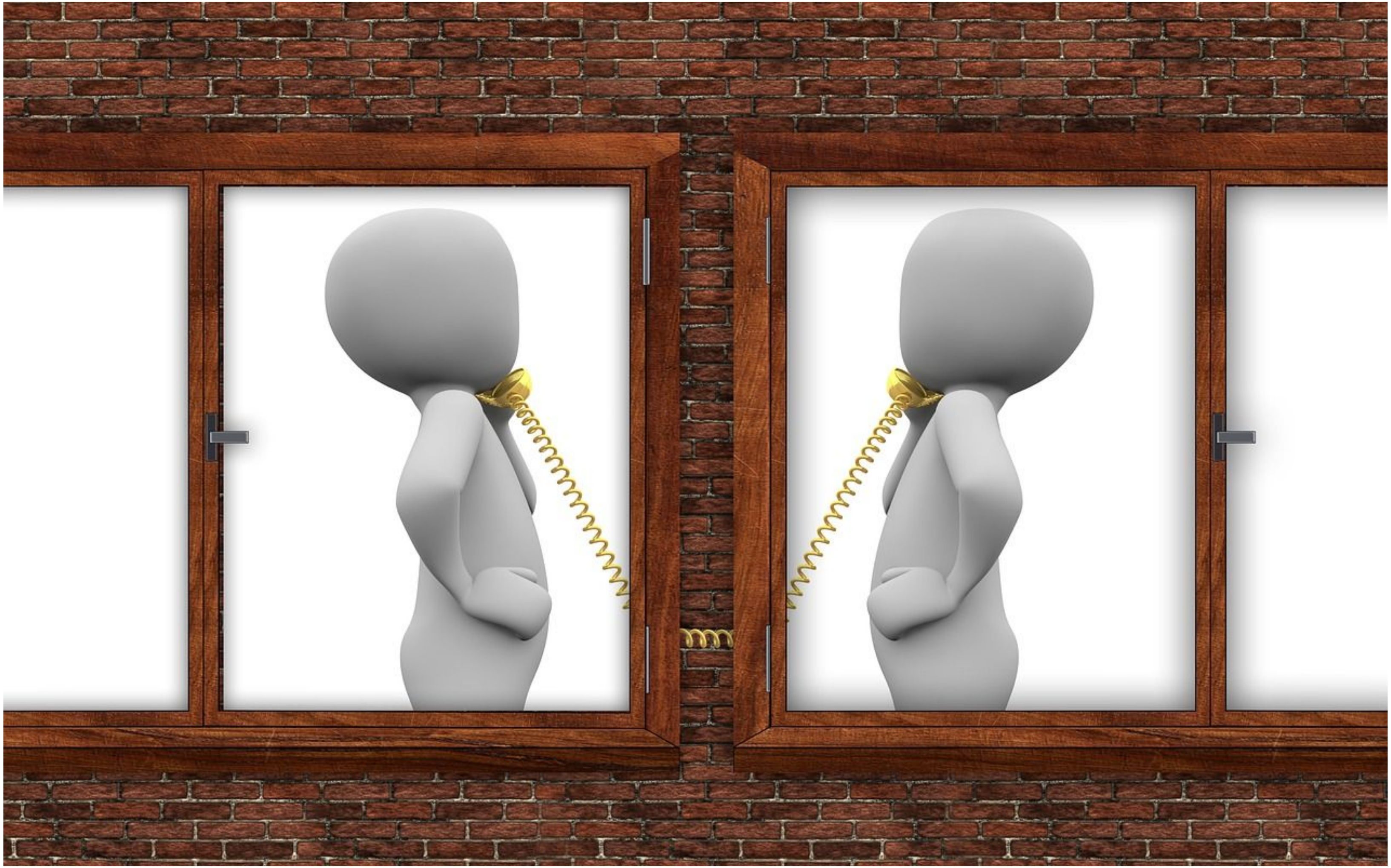
Lecture-08

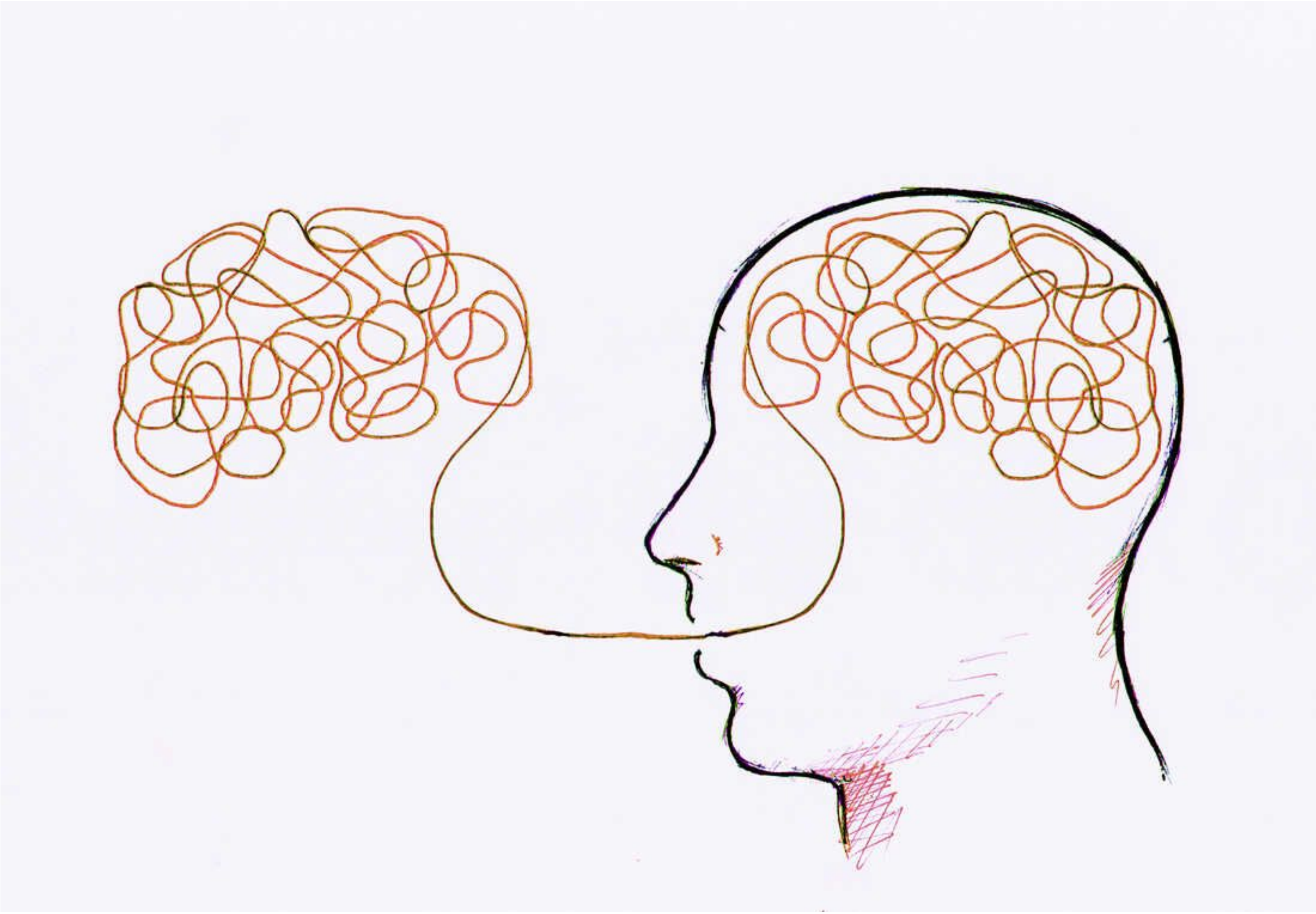
Legendre polynomials

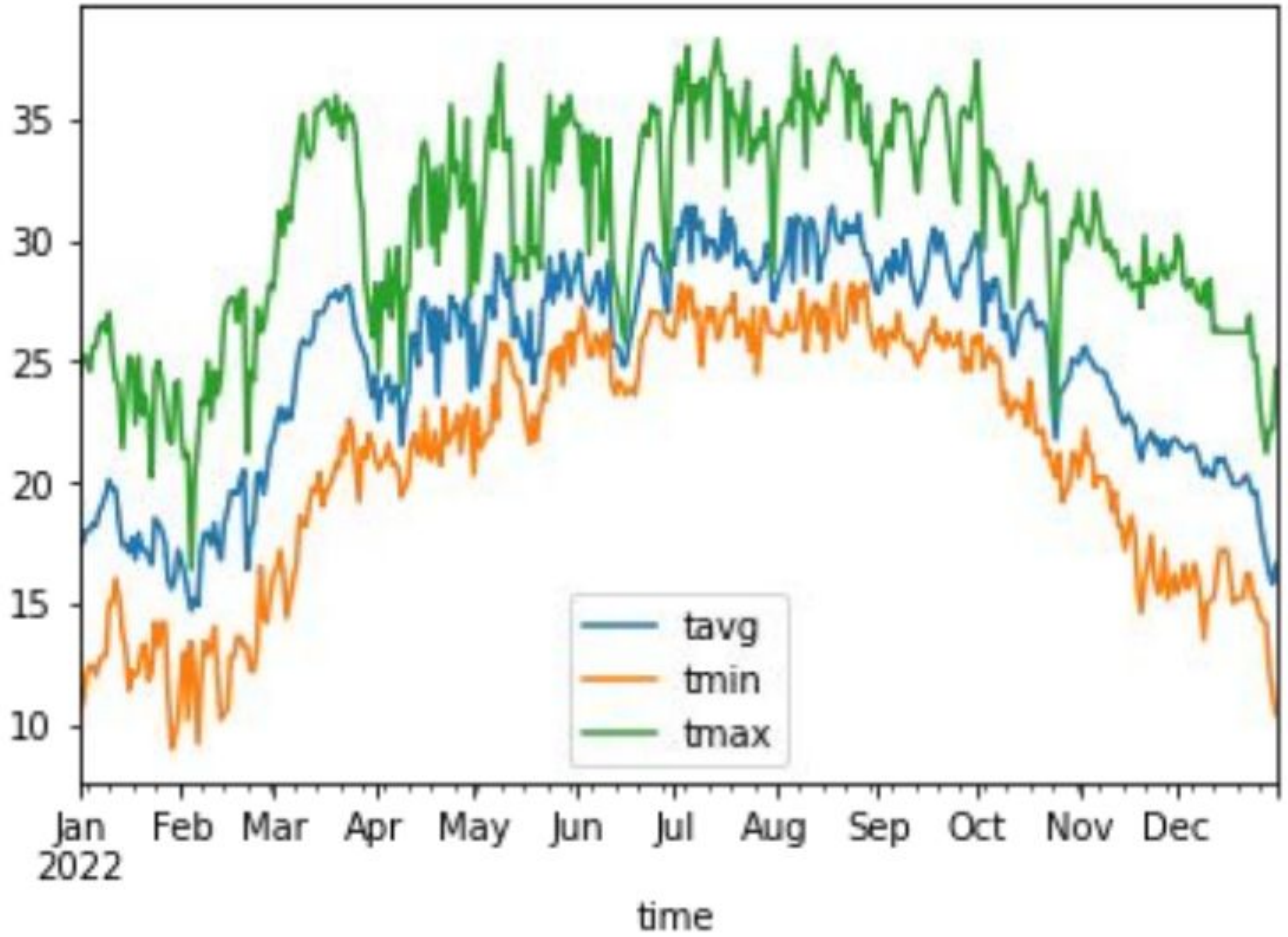
$$L_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k}(t^2 - 1)^k, \qquad k \in \mathbb{N}, \quad \text{are orthogonal on } [-1, 1]$$

$$\boxed{\begin{aligned} L_0(t) &= 1, & L_3(t) &= \tfrac{1}{2}(5t^3 - 3t), \\ L_1(t) &= t, & L_4(t) &= \tfrac{1}{8}(35t^4 - 30t^2 + 3), \\ L_2(t) &= \tfrac{1}{2}(3t^2 - 1), & L_5(t) &= \tfrac{1}{8}(63t^5 - 70t^3 + 15t). \end{aligned}}$$





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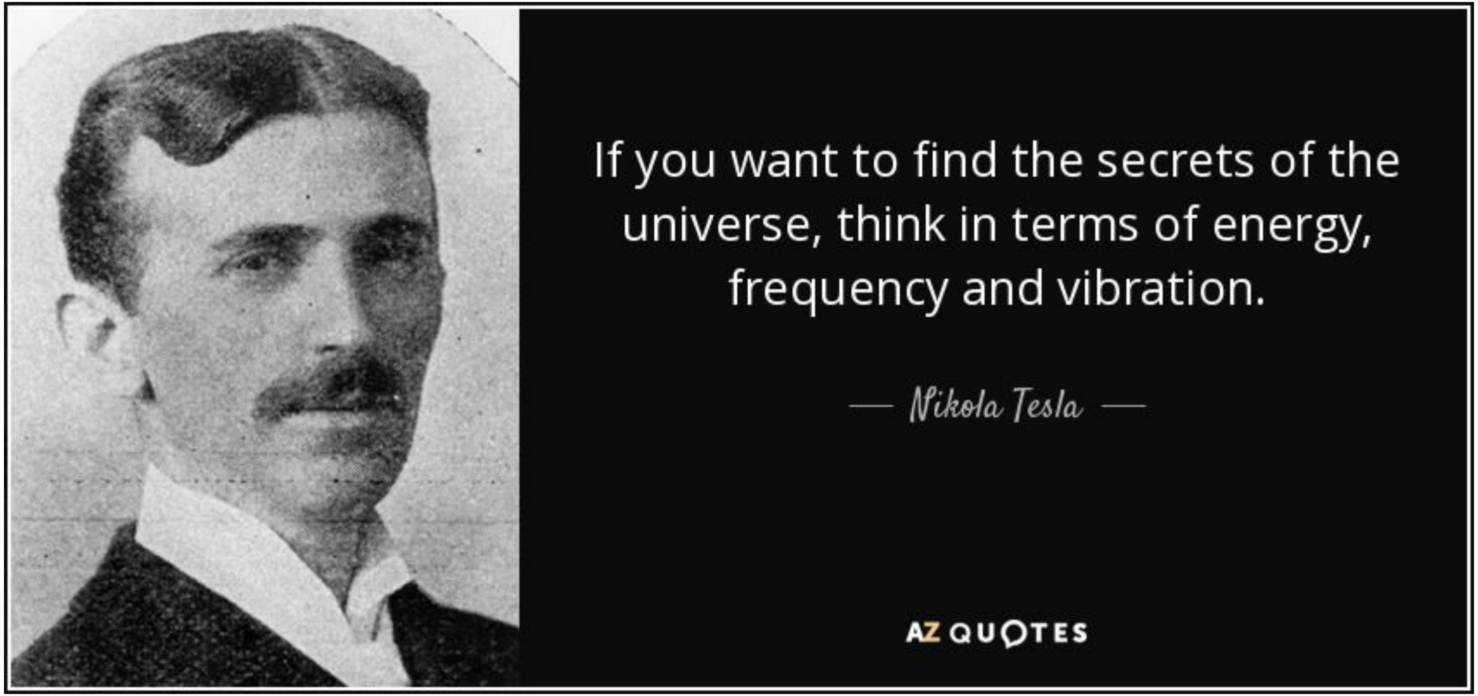
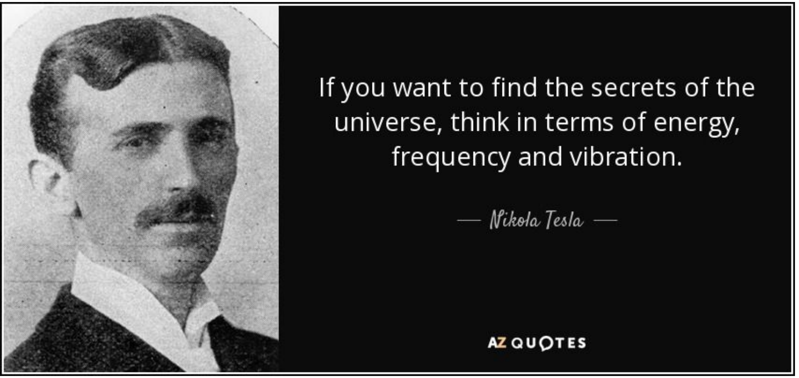
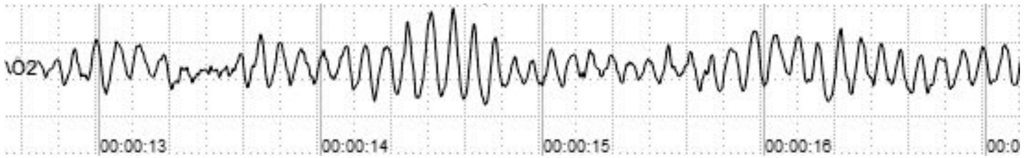
Do periodic signal exist in real-life?



Do periodic signal exist in real-life?

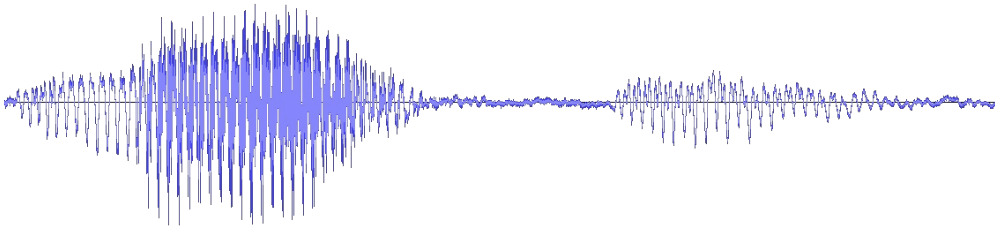


EEG signal (correlate of electrical activity in the brain)



Stock market fluctuations

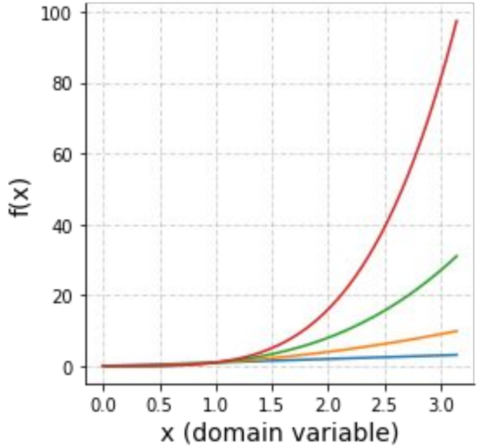
Air pressure associated with spoken speech utterance



… and a lot more!

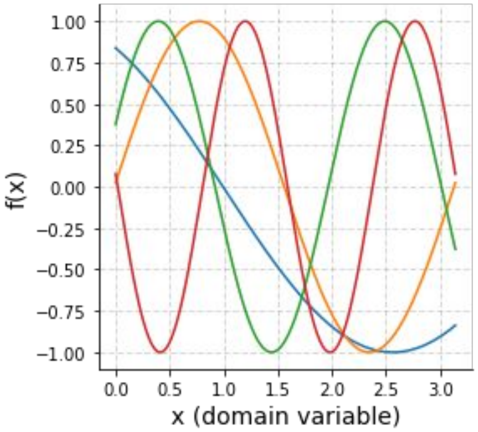
- Polynomial representation

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= \sum_{m=0}^{\infty} a_m x^m$$



- Fourier series representation

$$f(x) = \sum_{m=0}^{\infty} A_m \sin\left(\frac{\pi m x}{L} + \phi_n\right)$$



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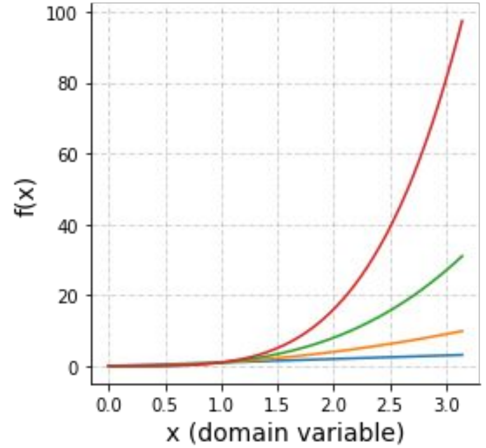

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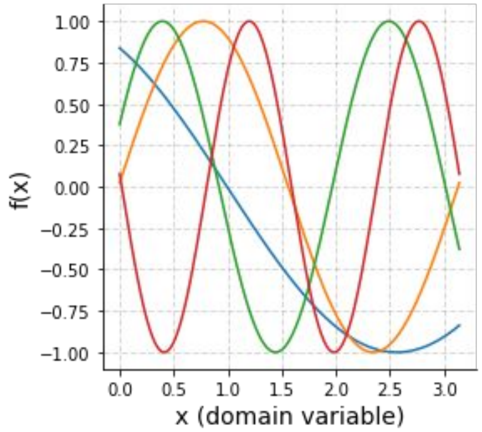

Focus of this lecture

- Fourier series representation

$$f(x) = \sum_{m=0}^{\infty} A_m \sin\left(\frac{\pi m x}{L} + \phi_n\right)$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{\pi m x}{L}\right) + b_m \sin\left(\frac{\pi m x}{L}\right)$$

sin(A + B) = sin(A).cos(B) + cos(A).sin(B)

A function can be written as sum of scaled cosine() and sine() functions

Jean-Baptiste Joseph Fourier
French Mathematician & Physicist
(1768 - 1830)

https://www3.nd.edu/~powers/ame.20231/fourier1878.pdf

THE

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ANALYTICAL THEORY OF HEAT

BY

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Jean Baptist JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

BY

ALEXANDER FREEMAN, M.A.,

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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LONDON: CAMBRIDGE WAREHOUSE, 17, PATERNOSTER ROW.
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Jean-Baptiste Joseph Fourier
French Mathematician & Physicist
(1768 - 1830)

On the Eiffel Tower, 72 names of French scientists, engineers, and mathematicians are engraved in recognition of their contributions.

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- It is a linear summation of cos(.) and sin(.), with no cross-terms
- It is sum of many (infinite) terms
- Each of the cos(.) and sin(.) term is periodic - $2L/m$
- Parameters of the sum are $\{a_o, a_m, b_m\}$

Fourier series representation

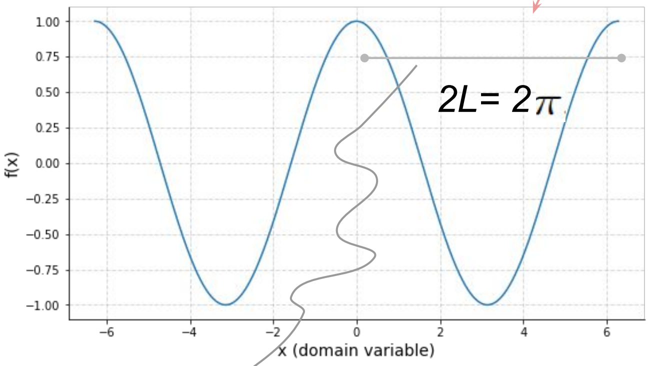
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Let's visualize the cosine term

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L})\right)$$

Example: Let L = π.

m=1



This function has a period of 2L

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

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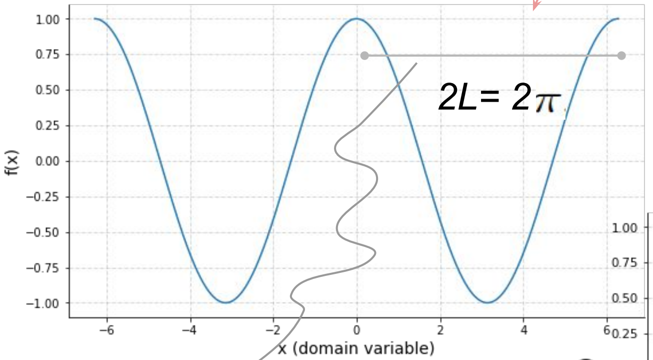
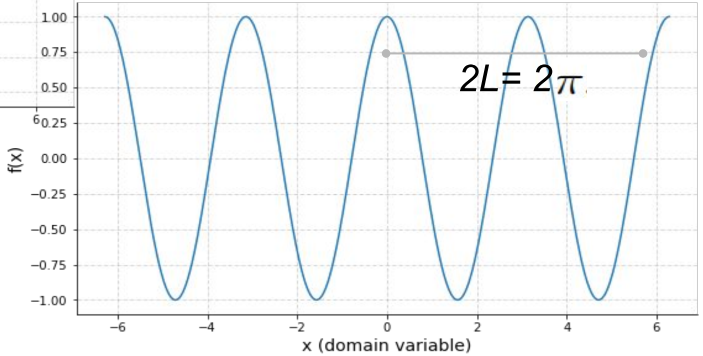


This function has a period of 2L

m=2



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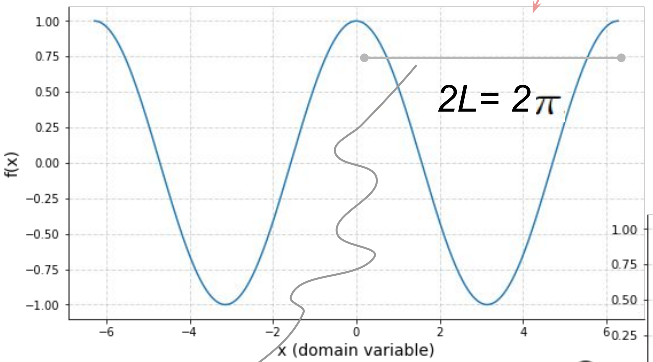

This function has a period of 2L

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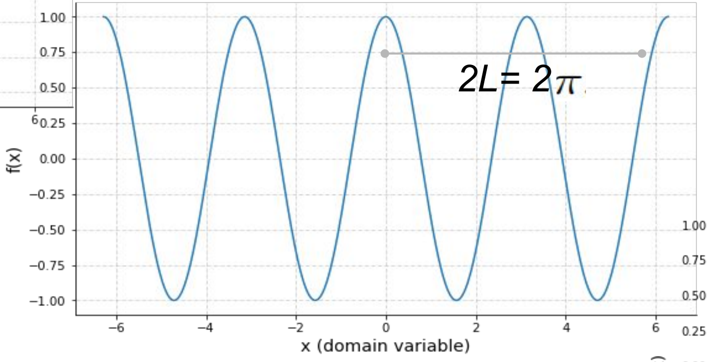
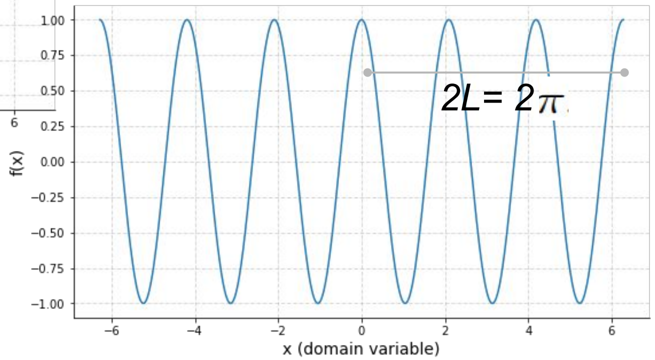


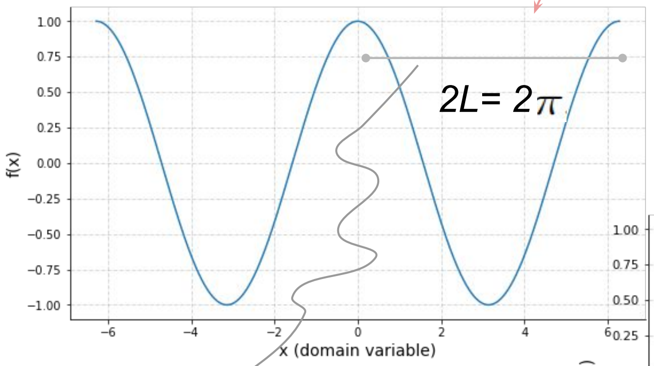
m=3



$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

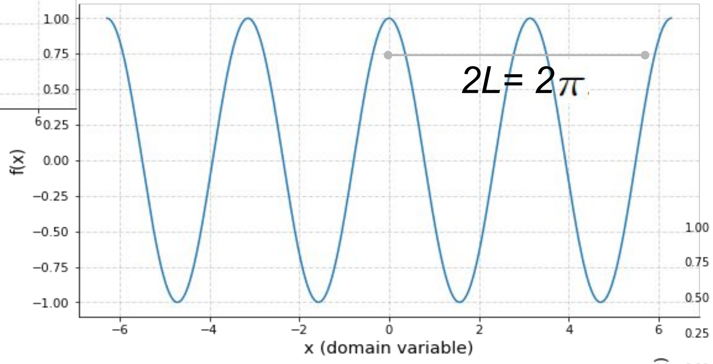
Example: Let L = π.

m=1

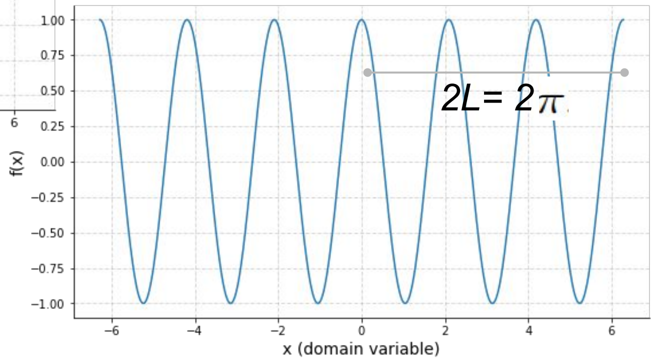


This function has a period of 2L

m=2

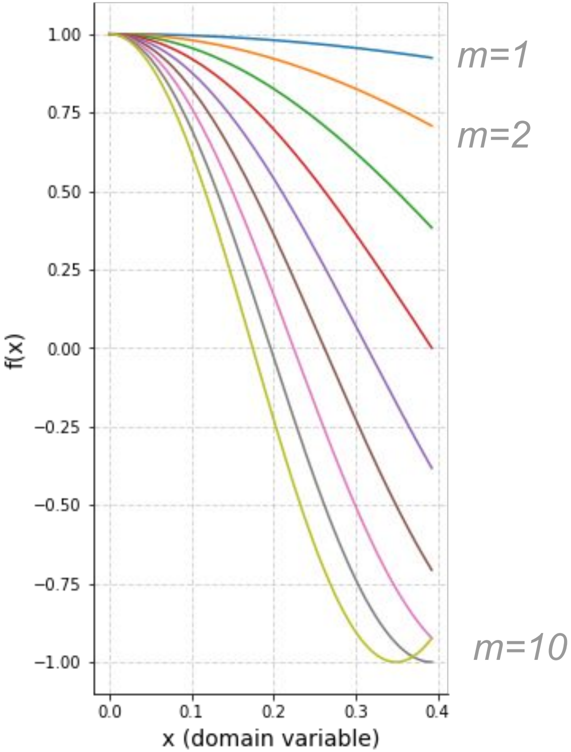


m=3



increasing m implies increasing cycles within 2L

$$\cos\left(\frac{m\pi x}{L}\right)$$



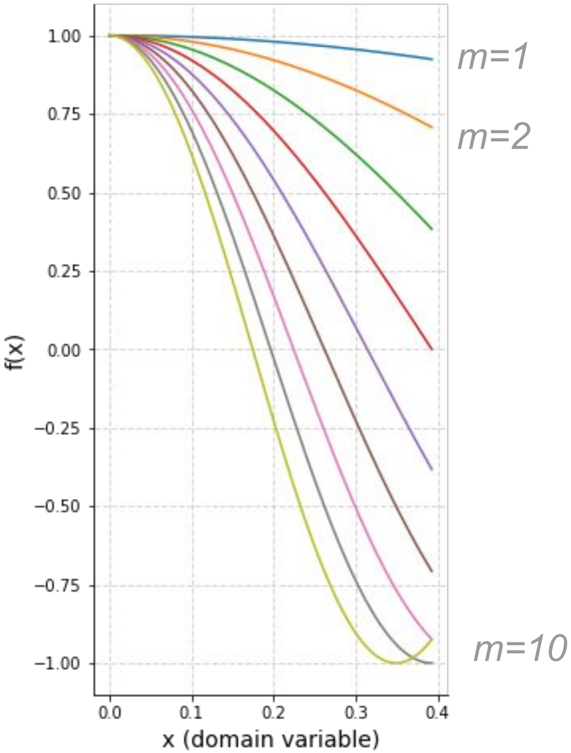
Spans only even functions of period 2L

$$\cos(\frac{m\pi x}{L})$$

+

$$\sin(\frac{m\pi x}{L})$$

Spans only odd functions of period 2L



Together they span ~ all functions of period 2L

Spans only even functions of period 2L

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

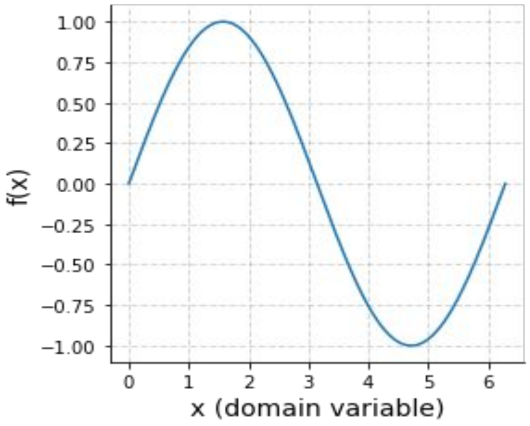
- Consider a 2L periodic signal f(x)
- How do we compute $\{a_o, a_m, b_m\}$ to represent f(x)?

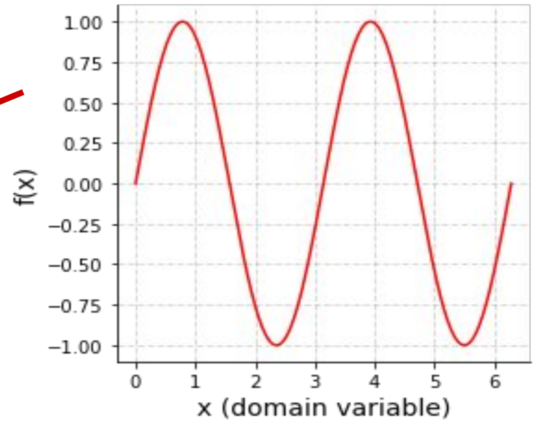
Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Consider a 2L periodic signal f(x)
- How do we compute $\{a_o, a_m, b_m\}$ to represent f(x) in the above form?

Let's first review some properties of sine and cosine functions. This will help us.





$$\int_0^{2L} f(x) = 0$$

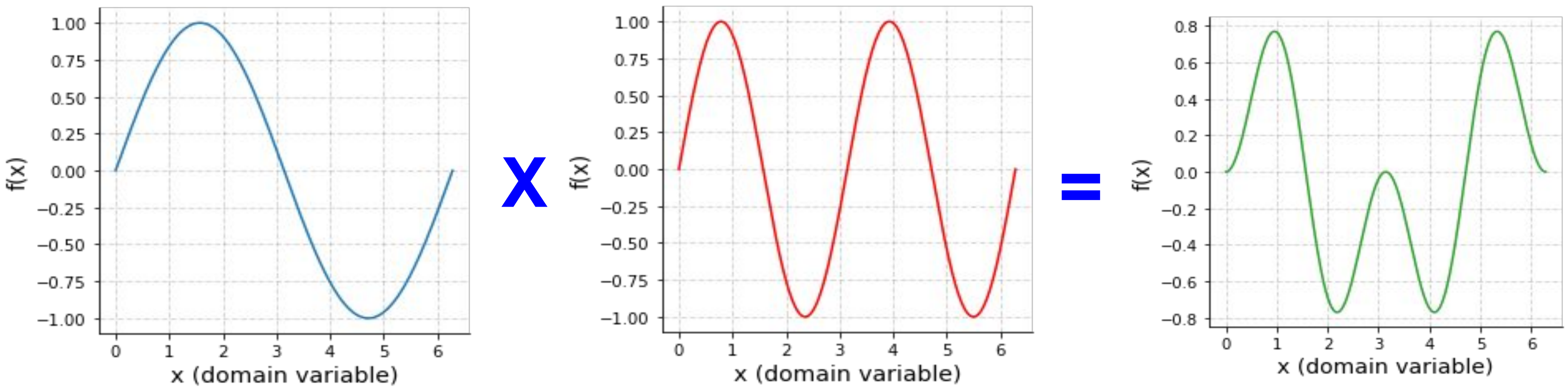
Integration of sin(.) function over a period (or its multiples) is 0.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Integration of sin(.) function over a period (or its multiples) is 0.
- The same holds for cosine(.) as well.
- Integrating both sides of the above equation, we get

$$\boxed{a_0 \quad = \quad \frac{1}{L} \int_{-L}^{L} f(x) dx}$$

Let's see a cross-term, that is multiplication of two sin(.)



$$\int_0^{2L} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) = 0$$

Integration of cross-terms over a period (or its multiples) is 0.

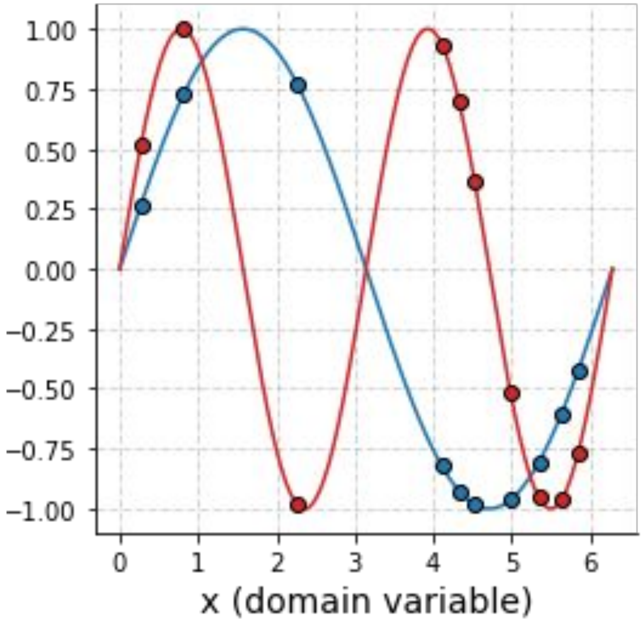
Integration of any two cross-terms over a period (or its multiples) is 0.

$$\int_0^{2L} \sin\left(\frac{m_1 \pi x}{L}\right) \sin\left(\frac{m_2 \pi x}{L}\right) = 0,\ m_1 \neq m_2$$

$$\int_0^{2L} \cos\left(\frac{m_1 \pi x}{L}\right) \cos\left(\frac{m_2 \pi x}{L}\right) = 0,$$

$$\int_0^{2L} \sin\left(\frac{m_1 \pi x}{L}\right) \cos\left(\frac{m_2 \pi x}{L}\right) = 0$$

The sin(.) and cosine(.) functions as used in Fourier series are orthogonal functions.

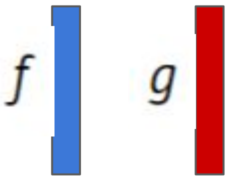


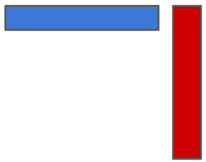
Familiar with **orthogonal vectors** in Euclidean spaces

Visualize the vectors as composed of values sampled from functions.

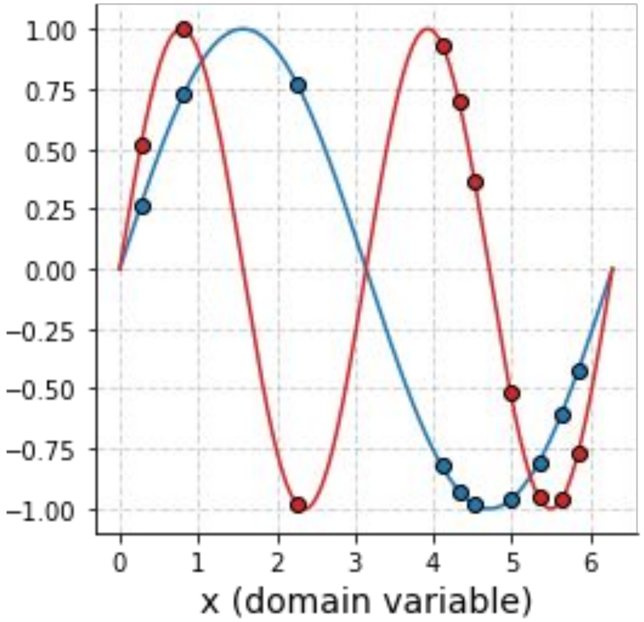
Then, orthogonality implies,

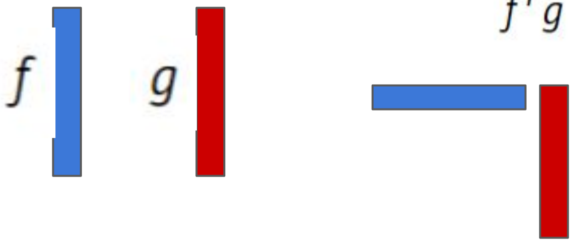
$$f^T g = \sum_{k=1}^{n} f[k]g[k] = 0$$





The sin(.) and cosine(.) functions as used in Fourier series are orthogonal functions.





$$f^T g = \sum_{k=1}^{n} f[k]g[k] = 0$$

Related by Riemann integral approximation.

$$< f, g > = \int_0^{2L} f(x)g(x)dx = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Multiplying both sides by corresponding sin() or cosine() term and integrating, we get

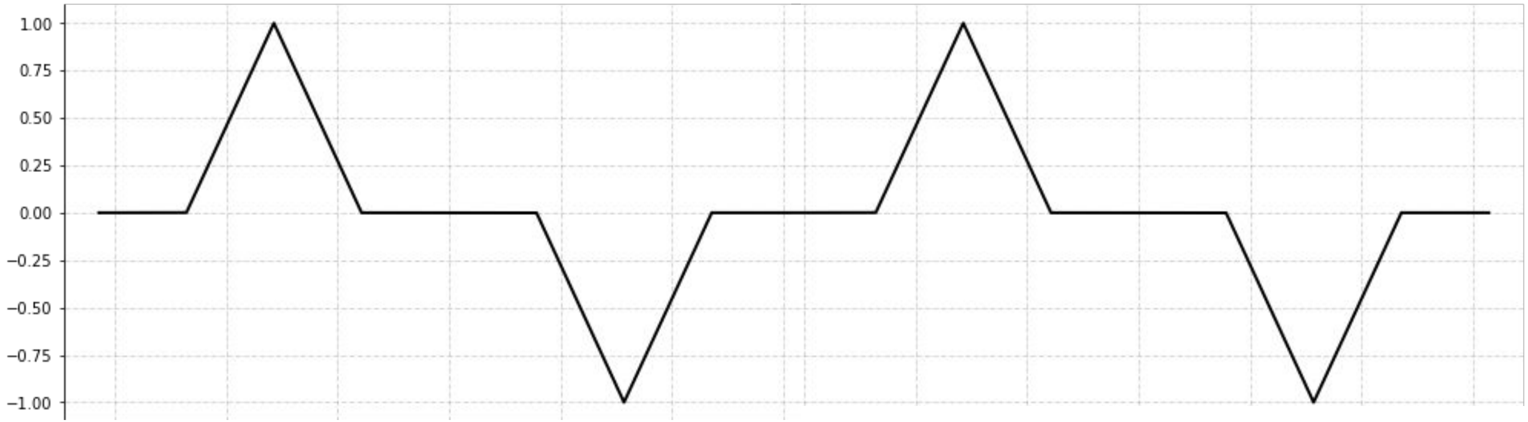
$$\boxed{\begin{aligned} a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{m\pi x}{L}) dx, \quad \text{and} \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{m\pi x}{L}) dx. \end{aligned}}$$

Summary,

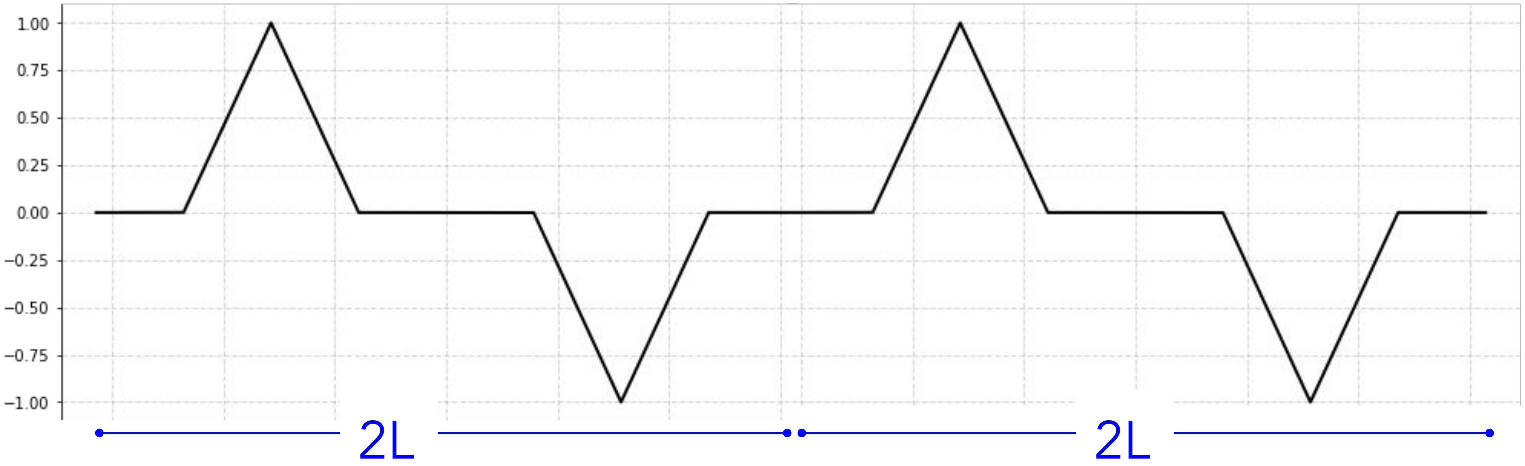
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{m\pi x}{L})dx, \text{ and} \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})dx. \end{aligned}$$

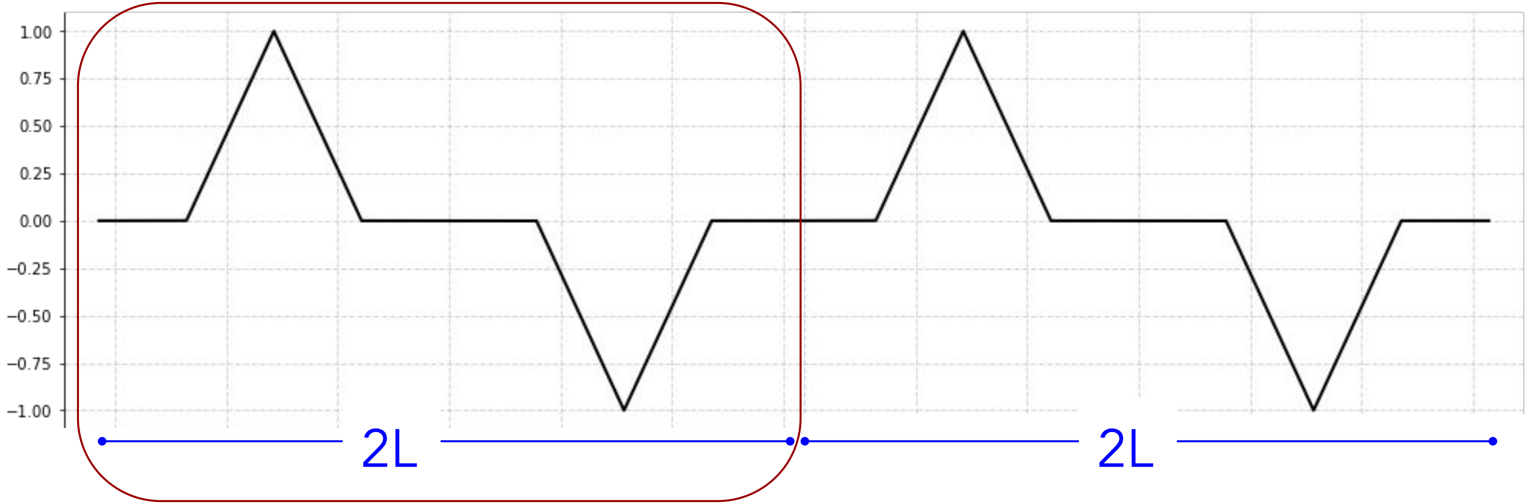
Consider the signal,



Consider the signal,



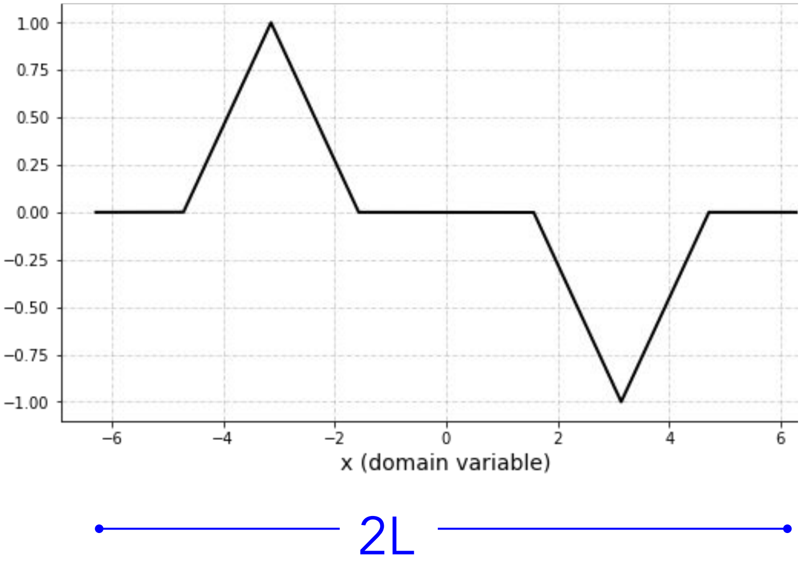
Consider the periodic signal,

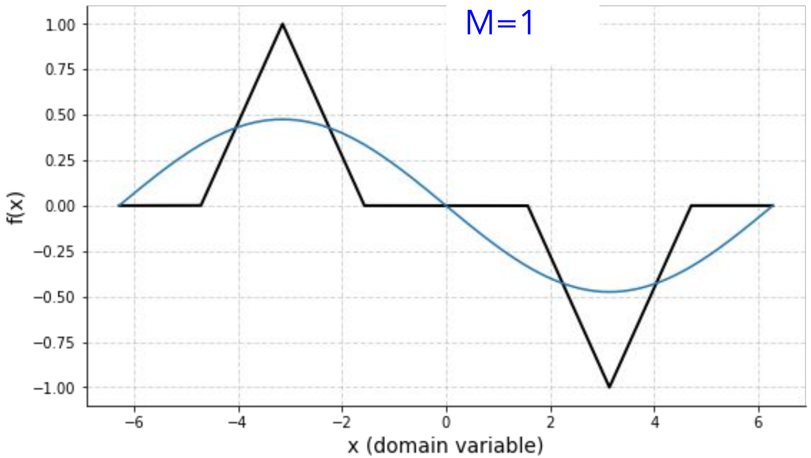


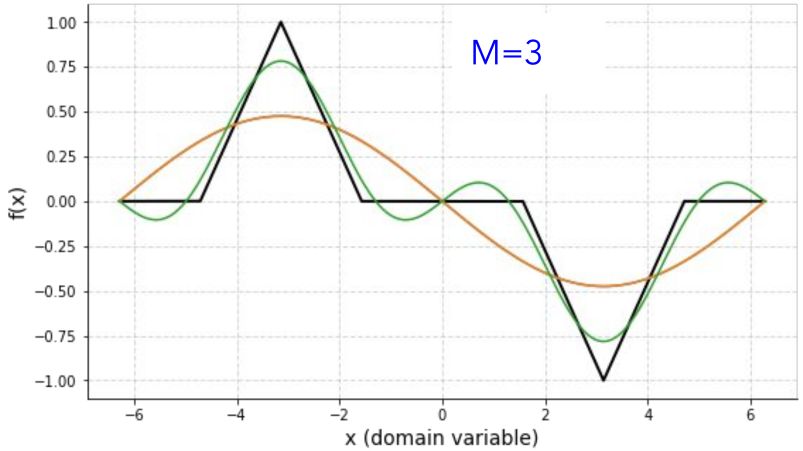
Can we express this signal using a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Consider the periodic signal,

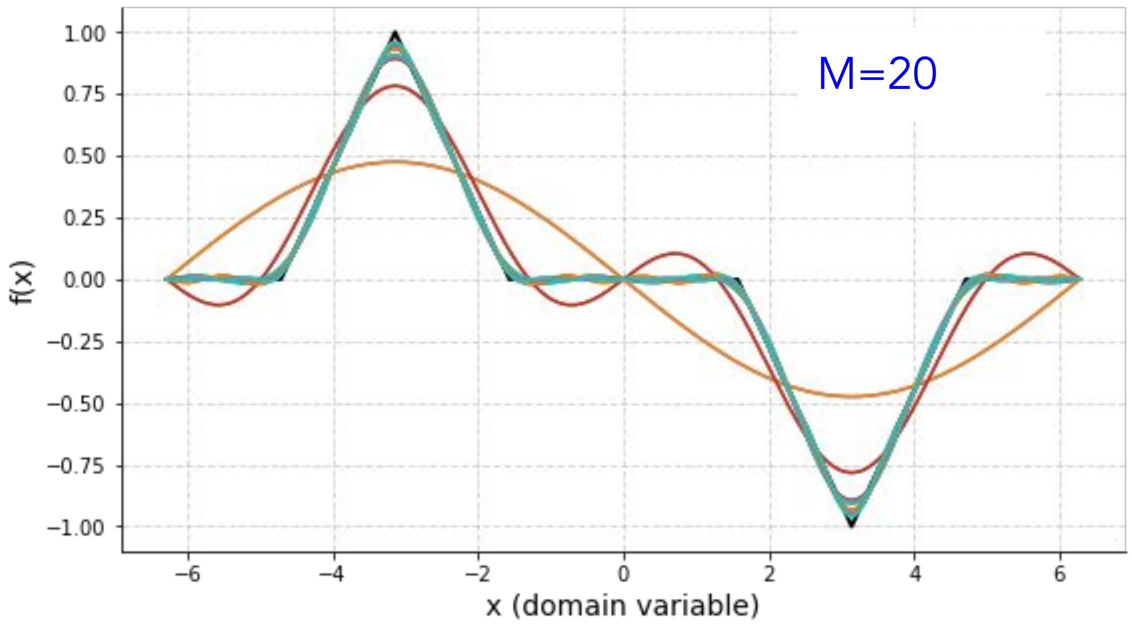


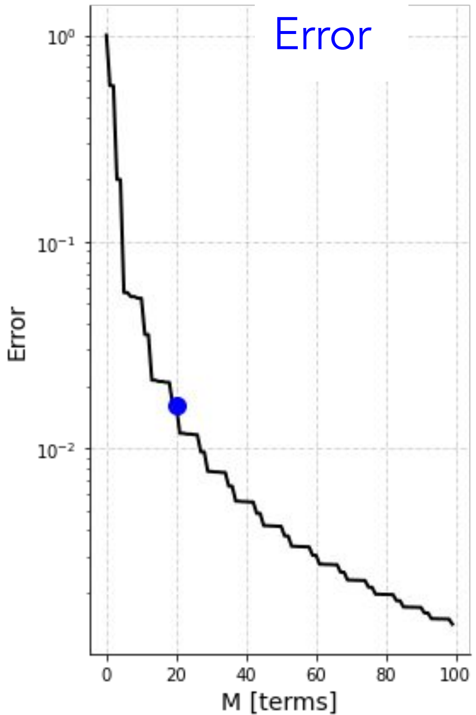




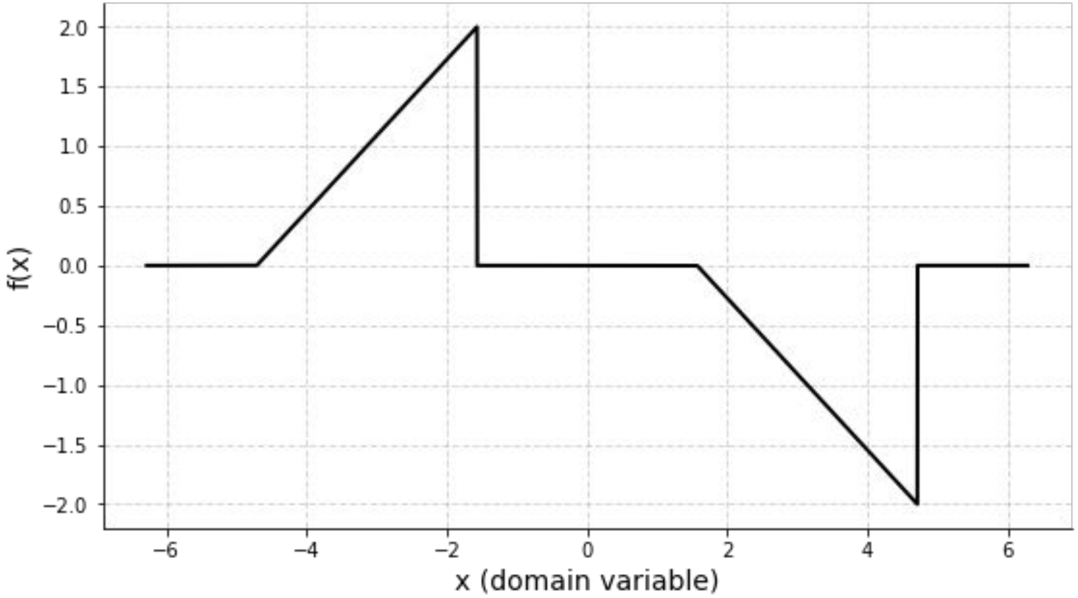
Fourier series approximation,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$



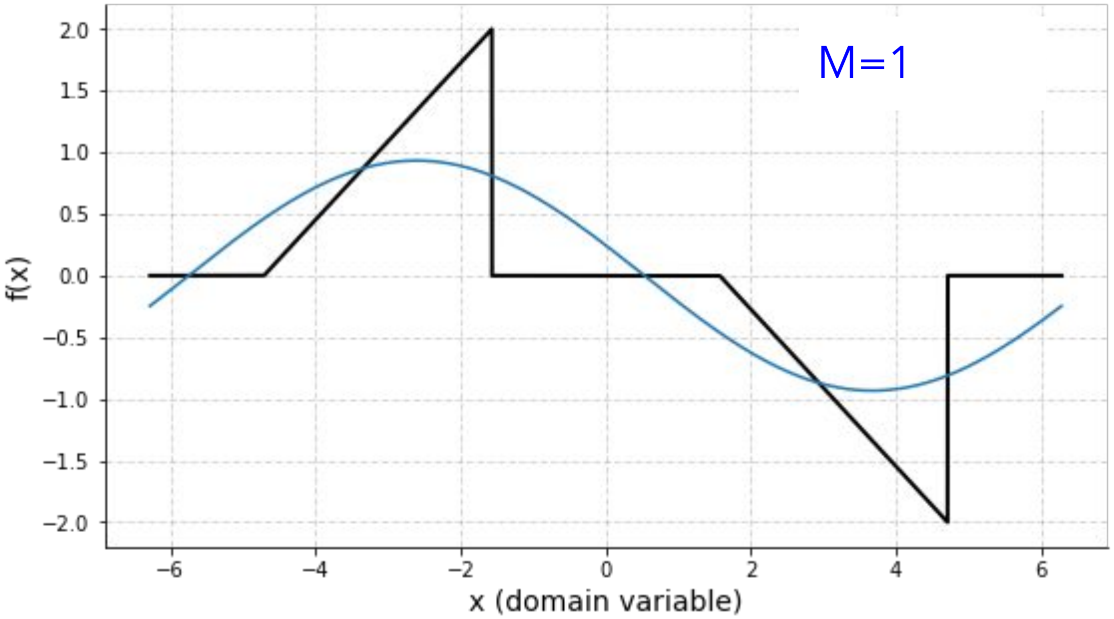


Another example,



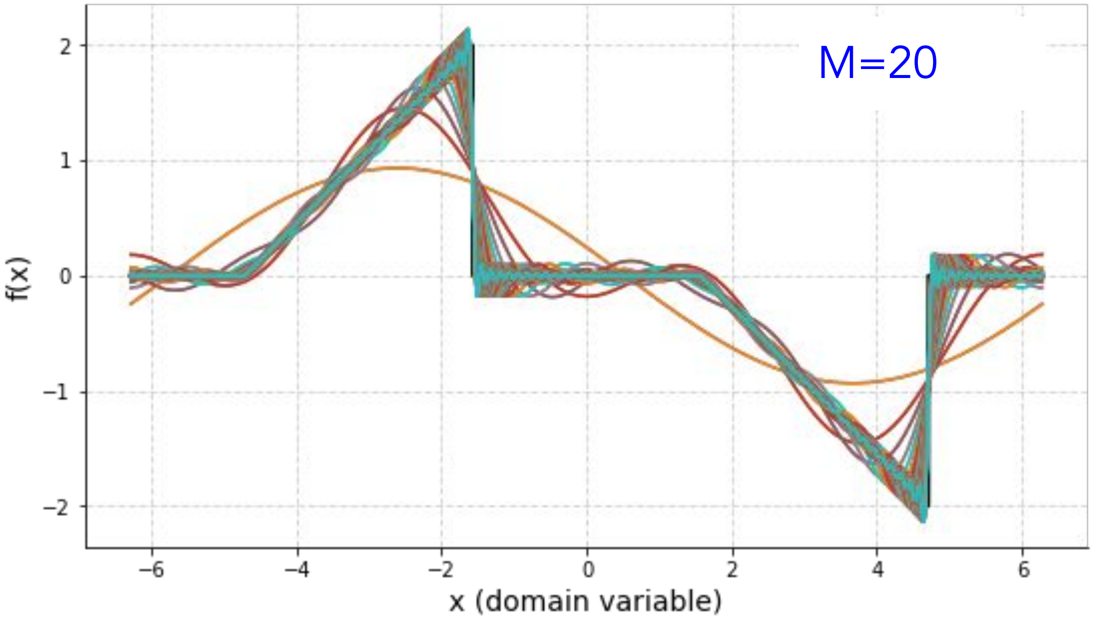
Fourier series approximation,

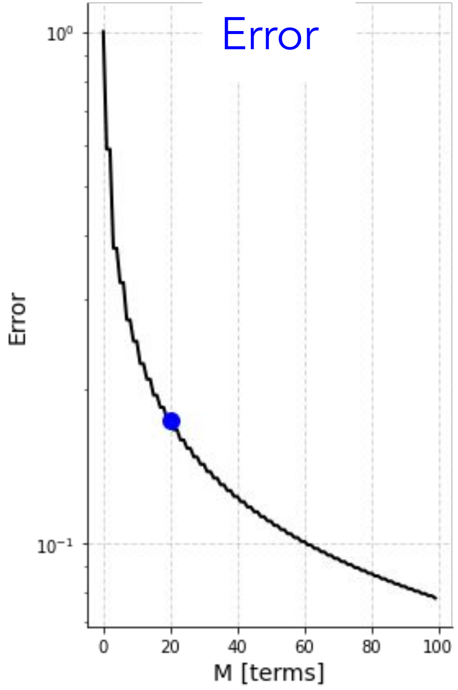
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$



Fourier series approximation,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$





Summary, Fourier series approximation,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Can model (or represent) a periodic signal
- Parameters of the model are $\{a_o, a_m, b_m\}$ and M
- Suitable if signal has oscillatory patterns (or fluctuations)