Plan

- Directions of Fea(LPP)
- Extreme directions of Fea(LPP)
- Representation Theorem for Fea(LPP)
- Necessary and sufficient conditions for existence of optimal solutions
- Optimal solutions in atleast one corner point

- $Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
- **Observation 5:** Suppose if a LPP has an unbounded feasible region, then there exists a vector $\mathbf{d} \neq \mathbf{0}$ such that starting from any point of the feasible region if you move in the positive direction of **d**, then you will always remain in the feasible region.

  That is for any $\mathbf{x} \in Fea(LPP)$, $\mathbf{x} + \alpha\mathbf{d} \in Fea(LPP)$ for all $\alpha \geq 0$.

  Then $\mathbf{d} \neq \mathbf{0}$ is called a **direction** of $S = Fea(LPP)$. Throughout our discussion, **d** will denote a column vector given by $\mathbf{d} = [d_1, ..., d_n]^T$.
- **Definition:** Given a non empty convex set S, $S \subset \mathbb{R}^n$, $\mathbf{d} \neq \mathbf{0}$ is called a **direction** of S if for all $\mathbf{x} \in S$, $\mathbf{x} + \alpha\mathbf{d} \in S$ for all $\alpha \geq 0$.
- If **d** is a direction of a convex set S, then for all $\gamma > 0$, $\mathbf{x} + \alpha\mathbf{d} = \mathbf{x} + (\frac{\alpha}{\gamma})\gamma\mathbf{d} \in S$ for all $\alpha > 0$, $\Rightarrow \gamma\mathbf{d}$ is again a direction for all $\gamma > 0$.

- Two directions $\mathbf{d}_1, \mathbf{d}_2$ of S are said to be distinct if $\mathbf{d}_1 \neq \gamma \mathbf{d}_2$ for any $\gamma > 0$
(or equivalently $\mathbf{d}_2 \neq \beta \mathbf{d}_1$ for any $\beta > 0$).
- **Example 2: (revisited)** Consider the problem,
Min $-x + 2y$
subject to
$x + 2y \geq 1$
$-x + y \leq 1$,
$x \geq 0, y \geq 0.$
Note that $\mathbf{d}_1 = [1, 1]^T, \mathbf{d}_1' = [2, 2]^T$,..
are all equal as directions of $Fea(LPP)$.
Similarly $\mathbf{d}_2 = [1, 0]^T, \mathbf{d}_2' = [2, 0]^T$,..
are all equal as directions of $Fea(LPP)$.
Whereas $\mathbf{d}_1 = [1, 1]^T, \mathbf{d}_2 = [1, 0]^T$ give two distinct directions.

- **Result:** The set of all directions of $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is given by $D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq \mathbf{0}, \quad A_{m\times n}\mathbf{d} \leq \mathbf{0}, \quad \mathbf{d} \geq \mathbf{0}\}$ or by $D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq \mathbf{0}, \quad \mathbf{a}_i^T\mathbf{d} \leq 0, \text{ for all } i = 1, 2, \ldots, m, \quad \mathbf{d} \geq \mathbf{0}\}$.
- **Remark:** Note that the set of all directions of $S = Fea(LPP)$ is a convex set.

  So if $\mathbf{d}_1$, $\mathbf{d}_2$ are two directions of S, then $\mathbf{d} = \alpha\mathbf{d}_1 + \beta\mathbf{d}_2$ will again be a direction of S, for any α, β non negative(as long as both α, β are not equal to zero, or $\alpha + \beta \neq 0$).

- **Definition:** A direction $\mathbf{d}$ of S is called an **extreme direction** of S, if it **cannot** be written as a positive linear combination of two distinct directions of S, that is, if $\mathbf{d}$ an **extreme direction** of S and

  $\mathbf{d} = \alpha \mathbf{d}_1 + \beta \mathbf{d}_2$, for $\alpha, \beta > 0$ and $\mathbf{d}_1, \mathbf{d}_2 \in D$
  then $\mathbf{d}_1 = \gamma \mathbf{d}_2$ for some $\gamma > 0$.
- If D denotes the set of all directions of S (which will be the empty set if S is bounded) then $D' = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, A\mathbf{d} \leq \mathbf{0}, \sum_i d_i = 1\}$ is a set of all distinct directions of S.
- Also each $\mathbf{d} \in D$ is of the form $\mathbf{d} = \alpha \mathbf{d}'$ for some $\mathbf{d}' \in D'$ where $\alpha = \sum_i d_i (> 0)$.
- D' can be written as

  $$D' = \left\{ \mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, \begin{bmatrix} & A & \\ 1 & 1, .., & 1 \\ -1 & -1, .., & -1 \end{bmatrix} \mathbf{d} \leq \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \right\}.$$

  The set D' now looks exactly like the feasible region of an LPP.

- If D' is non empty then D' has at least one extreme point (why?).
- **Result:** $\mathbf{d}$ is an extreme direction of S if and only if $\mathbf{d}' = \frac{\mathbf{d}}{\sum_i d_i}$ is an extreme point of D'.
- **Remark:** Hence the number of distinct extreme directions of S is finite (why?).
- Also since D' is like the set, $Fea(LPP) = S$, if $D' \neq \phi$, then D' must have atleast one extreme point.
- Hence if $Fea(LPP) = S$ is unbounded then S must have atleast one extreme direction.

- The extreme directions of S which are extreme points of D' (after suitable normalization) will lie on n LI hyperplanes defining D'.
- Since $\mathbf{d} \in \mathbb{R}^n$, $\mathbf{d} \neq \mathbf{0}$ cannot be orthogonal to n LI vectors, so $\mathbf{d}$ cannot lie on n LI hyperplanes of the $(m+n)$ hyperplanes given by,
  $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{d} = \mathbf{0}\}$ for $i = 1, 2, \ldots, m$, and
  $\{\mathbf{d} \in \mathbb{R}^n : -\mathbf{e}_j^T\mathbf{d} = \mathbf{0}\}$ for $j = 1, 2, \ldots, n$.
- If $\mathbf{d} \in D'$, is an extreme direction of S then it should lie on $(n-1)$ LI hyperplanes of the above mentioned $(m+n)$ hyperplanes, and the hyperplane
  $\{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$
  gives a collection of n LI hyperplanes, on which $\mathbf{d}$ lies.

- Any $\mathbf{d} \in D$, which lies on $(n-1)$ LI hyperplanes out of the $(m+n)$ hyperplanes given by
  $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{a}_i^T \mathbf{d} = \mathbf{0}\}$ for $i = 1, 2, \ldots, m$, and
  $\{\mathbf{d} \in \mathbb{R}^n : -\mathbf{e}_j^T \mathbf{d} = \mathbf{0}\}$ for $j = 1, 2, \ldots, n$,
  is an extreme direction of S.
- **Exercise:** Check that if a $\mathbf{d} \in D$ lies on $(n-1)$ LI hyperplanes (out of the $(m+n)$ defining hyperplanes of D) given by $\{H_1, \ldots, H_{n-1}\}$, then $\{H, H_1, \ldots, H_{n-1}\}$ is LI where $H = \{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$.

- **Example 2: (revisited)** Consider the problem,
  Min $-x+2y$
  subject to
  $x+2y \geq 1$
  $-x+y \leq 1,$
  $x \geq 0, y \geq 0.$
  Note that here the set of all directions of S is given by
- $D = \{\mathbf{d} \in \mathbb{R}^2 : [-1,-2]\mathbf{d} \leq 0, [-1,1]\mathbf{d} \leq 0, \mathbf{d} \geq \mathbf{0}\}$.
  Also if $\mathbf{d} \in D$ is an extreme direction of S then it has to lie on exactly one of the hyperplanes given by
- **(i)** $\{\mathbf{d} \in \mathbb{R}^2 : [-1,-2]\mathbf{d} = 0\}$,
- **(ii)** $\{\mathbf{d} \in \mathbb{R}^2 : [-1,1]\mathbf{d} = 0\}$,
- **(iii)** $\{\mathbf{d} \in \mathbb{R}^2 : d_1 = 0\}$,
- **(iv)** $\{\mathbf{d} \in \mathbb{R}^2 : d_2 = 0\}$.

- Note that there exists no $\mathbf{d} \geq \mathbf{0}, \mathbf{d} \neq \mathbf{0}$ such that $[-1, -2]\mathbf{d} = 0$.
- Also if $\mathbf{d} \geq \mathbf{0}, \mathbf{d} \neq \mathbf{0}$, satisfies the condition $d_1 = 0$, then $[-1, 1]\mathbf{d} \leq 0$ cannot be satisfied, hence such a **d** does not belong to D.
- Hence if $\mathbf{d} \in D$, is an extreme direction of S then it lies on either the hyperplane
- $\{\mathbf{d} \in \mathbb{R}^2 : [-1, 1]\mathbf{d} = 0\}$, or in $\{\mathbf{d} \in \mathbb{R}^2 : d_2 = 0\}$.
- Check that $\mathbf{d}' = [1, 1]^T$ and any **positive scalar multiple** of $\mathbf{d}'$, and $\mathbf{d}'' = [1, 0]^T$ and any positive scalar multiple of $\mathbf{d}''$, are the only extreme directions of the above $S = Fea(LPP)$.

- **Theorem:**
  If $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty, then S has atleast one extreme point.
- **Remark:** Note that the above result is not necessarily true for all polyhedral sets.
  For example take any single half space, or say a straight line in $\mathbb{R}^n$, which are polyhedral sets, but does not have any extreme point.
- The theorem works for $Fea(LPP)$ because of the non negativity constraints, that is because $Fea(LPP)$ is given a supply of n LI hyperplanes, among the (m+n) defining hyperplanes of S.

- **Exercise:** Can you find a nonempty polyhedral set S, $S \subset \mathbb{R}^3$ which has two defining hyperplanes but does not have any extreme point.
- **Exercise:** Can you find a nonempty polyhedral set S, $S \subset \mathbb{R}^3$ which has three LI defining hyperplanes (not necessarily the nonnegativity constraints) but does not have any extreme point.
- **Definition:** Given S, a nonempty subset of $\mathbb{R}^n$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S$, $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$, is called a convex combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, where $0 \le \lambda_i \le 1$ for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$.

- All possible convex combinations of two distinct points gives a straight line segment with those two points as boundary points.
- All possible convex combinations of three non colinear points gives a triangle with those points as corner points.
- All possible convex combinations of four points no three of which are colinear gives a quadrilateral.
- **Result:** Given $\phi \neq S \subset \mathbb{R}^n$, S is a convex set if and only if for all $k \in \mathbb{N}$,
  the convex combination of any k elements of S is again an element of S.

- **Theorem: (Representation Theorem)** If $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty and if $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ are the extreme points of S and $\mathbf{d}_1, \mathbf{d}_2, .., \mathbf{d}_r$ are the distinct extreme directions of S (the set of directions is empty if S is bounded) then $\mathbf{x} \in S$ if and only if
  $\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i + \sum_{j=1}^{r} \mu_j \mathbf{d}_j$
  where $0 \leq \lambda_i \leq 1$ for all $i = 1, 2, \ldots, k$, $\sum_i \lambda_i = 1$, and $\mu_j \geq 0$, for all $j = 1, 2, \ldots, r$.
- That is, $\mathbf{x} \in S \Leftrightarrow \mathbf{x}$ can be written as a convex combination of the extreme points of S plus a non negative linear combination of the extreme directions of S.

- **Observation 6:** If $S = Fea(LPP)$ is a nonempty bounded set then any $\mathbf{x} \in S$ can be written as a convex combination of the extreme points of S.
- **Observation 7:** Since D', the set of distinct directions of S (if it is nonempty) is a bounded set ($\mathbf{d} \geq \mathbf{0}$ and $\sum_{i=1}^{n} d_i = 1$),
  so any $\mathbf{d} \in D'$ can be written as a convex combination of the extreme points of D'.
- So any direction $\mathbf{d} \in D$ of S can be written as a nonnegative linear combination of the extreme directions of S.

- **Observation 8:** Note that if there exists a $\mathbf{d} \in D$ such that $\mathbf{c}^T\mathbf{d} < 0$ then the LPP(*)

  ((*) Min $\mathbf{c}^T\mathbf{x}$, subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$)

  does not have an optimal solution.
  Since for any given $\mathbf{x} \in S$, $\mathbf{c}^T(\mathbf{x} + \alpha\mathbf{d}) = \mathbf{c}^T\mathbf{x} + \alpha\mathbf{c}^T\mathbf{d}$ can be made smaller than any real M, by choosing $\alpha > 0$ sufficiently large.

- **Exercise:** If $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all extreme directions $\mathbf{d}_j$ of the nonempty and unbounded feasible region S of a LPP, then does it imply that $\mathbf{c}^T\mathbf{d} \geq 0$ for all directions $\mathbf{d} \in D$, of the feasible region S?

  Ans is **yes**.

- **Observation 9:** From the representation theorem of S we can see that if $S \neq \phi$ and $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all $j = 1, 2, .., r$, then LPP(*) has an optimal solution, and atleast one optimal solution is attained at an extreme point of S.

- **Observation 10:** From the representation theorem of S we can also see that if $S = Fea(LPP)$ is nonempty and bounded then the LPP(*) has an optimal solution and the optimal value is attained in atleast one extreme point.

  From the above observations we can conclude the following:
- **Conclusion 1:** If $S = Fea(LPP) \neq \phi$, then the LPP (*) has an optimal solution if and only if one of the following is true:
- **(i)** $S = Fea(LPP)$ is bounded (also seen before by using extreme value theorem)
  (ii) $S = Fea(LPP)$ is unbounded and $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all extreme directions $\mathbf{d}_j$ of the feasible region S.

- **Conclusion 2:** If LPP (*) has an optimal solution then there exists an extreme point of the feasible region S, which is an optimal solution.
- **Exercise:** Give an example of a **nonlinear** function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}$ is a closed and bounded polyhedral subset of $\mathbb{R}$, (what are these sets?) such that f has a minimum value in S but the minimum value is not attained at any extreme point of S.
- **Conclusion 3:** If $S = Fea(LPP)$ is nonempty, and there exists an $M \in \mathbb{R}$ such that for all $\mathbf{x} \in S$, $\mathbf{c}^T\mathbf{x} \geq M$, then the LPP (*) has an optimal solution.

- To understand the significance of the previous result solve the following problems.
- **Exercise:** Give an example of a **linear** function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}$ is not a polyhedral subset of $\mathbb{R}$, such that $f(x) \geq 1$ but f **does not** have a minimum value in S.
- **Exercise:** Give an example of a **nonlinear** function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}$ is a polyhedral subset of $\mathbb{R}$, such that $f(x) \geq 1$ but f **does not** have a minimum value in S.

- We can come to similar conclusions if we consider a linear programming problem, LPP(**) as

  (**)Max $\mathbf{c}^T\mathbf{x}$
  subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
- **Conclusion 1a:** If $S = Fea(LPP) \neq \phi$, then the LPP (**) has an optimal solution if and only if one of the following is true:

  (i) $S = Fea(LPP)$ is bounded
  (ii) $S = Fea(LPP)$ is unbounded and $\mathbf{c}^T\mathbf{d}_j \leq 0$ for all extreme directions $\mathbf{d}_j$ of the feasible region S.
- **Conclusion 2a:** If a LPP (**) has an optimal solution then there exists an extreme point of the feasible region S, which is an optimal solution.
- **Conclusion 3a:** If $S = Fea(LPP)$ is nonempty, and there exists an $M \in \mathbb{R}$ such that for all $\mathbf{x} \in S$, $\mathbf{c}^T\mathbf{x} \leq M$, then the LPP (**) has an optimal solution.