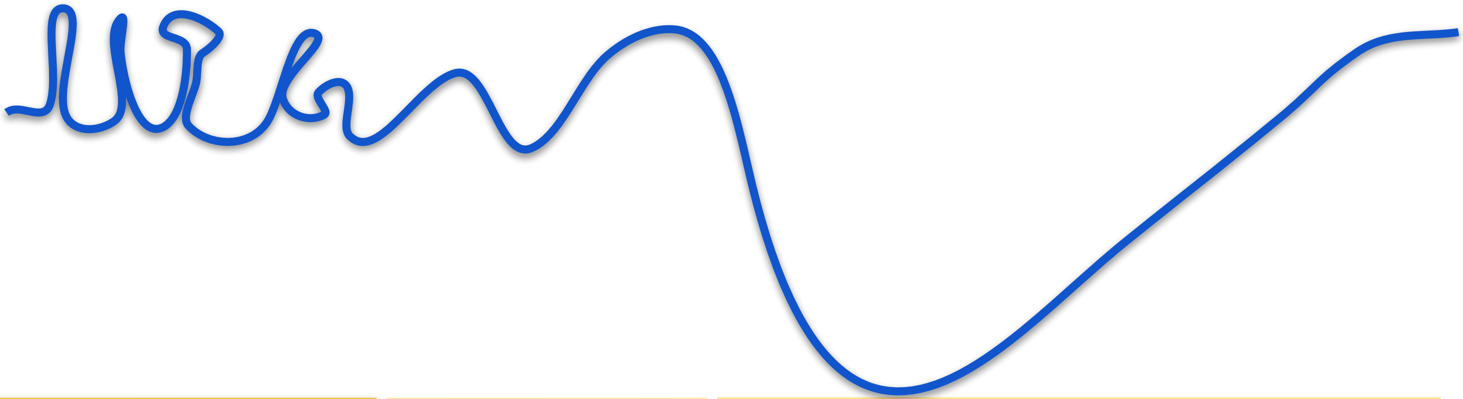
Computing with Signals



DA 623

Jan - May 2024

IIT Guwahati

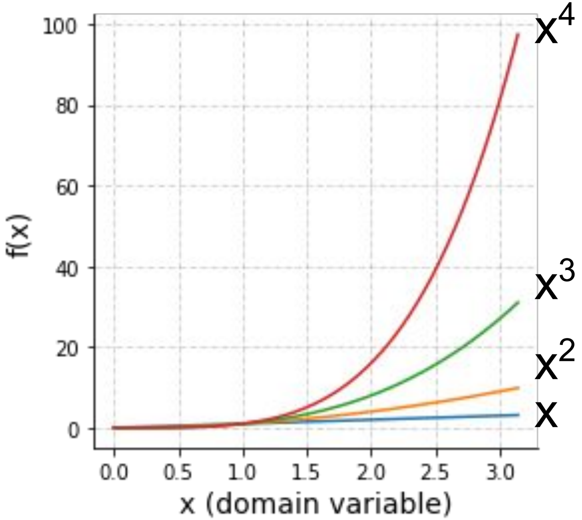
Instructors: Neeraj Sharma

Lecture-03

Signal ⇒ Model

or

Representation ⇒ Compute



Model

Taylor Series:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Model

Taylor Series:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Examples:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Model

Taylor Series:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Examples:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

More examples:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n} (-4)^n (1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

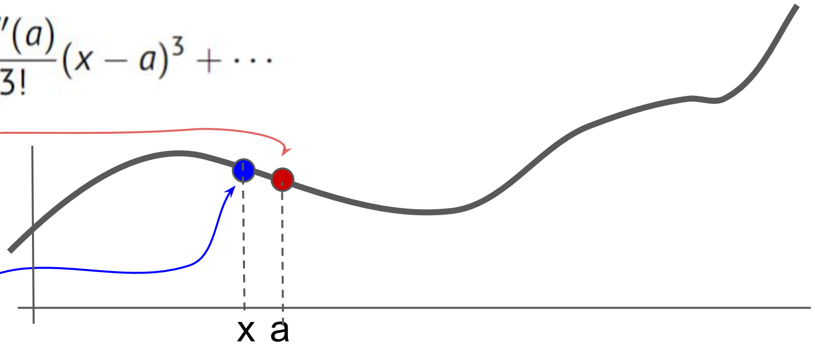
$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$$

Taylor Series

- Assumes the function is differentiable
- Works like a charm if you know the function (in closed form) apriori
- Approximation only in the neighborhood of the sampled point

Using in practice requires derivative information of the signal.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$



Can we use ideas from Taylor?

Signal ⇒ Model *or* Representation ⇒ Processing

Model

Polynomial Series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Model

Polynomial Series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

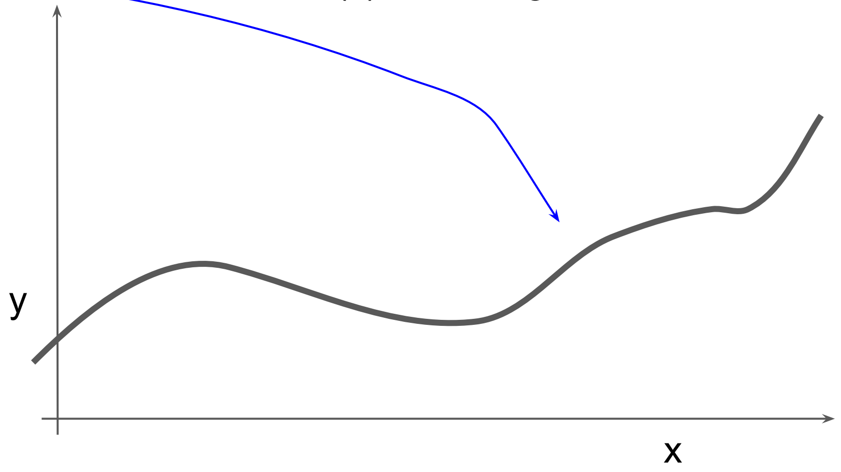
A univariate polynomial of degree n with real or complex coefficients has n complex roots, if counted with their multiplicities.

Model

Polynomial Series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Fit f(x) to this signal?



Model

Polynomial Series:

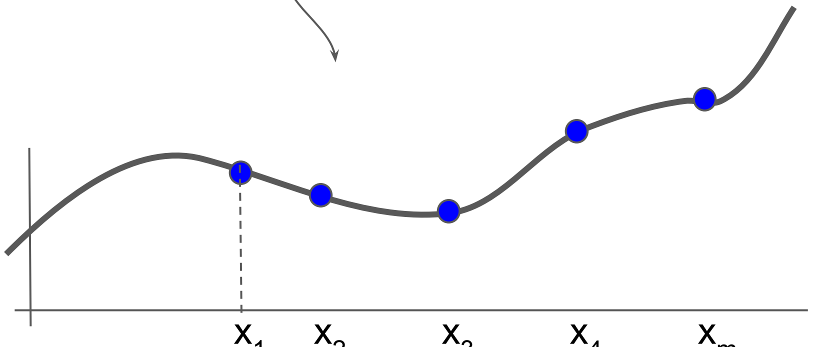
$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_3^3$$

$$y_2 = a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3$$

$$\vdots$$

$$y_m = a_0 + a_1x_m + a_2x_m^2 + a_3x_m^3$$



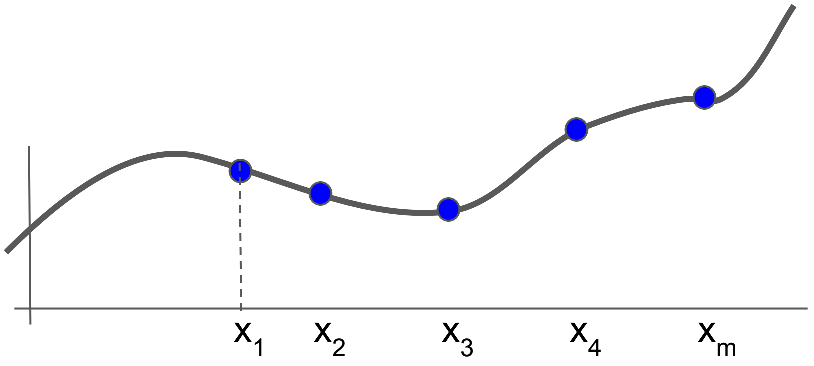
Model

Polynomial Series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdot$$

Step 1: Sample the signal at at least (n) points (why?). (n-1) is the degree of f(x).

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$



Model

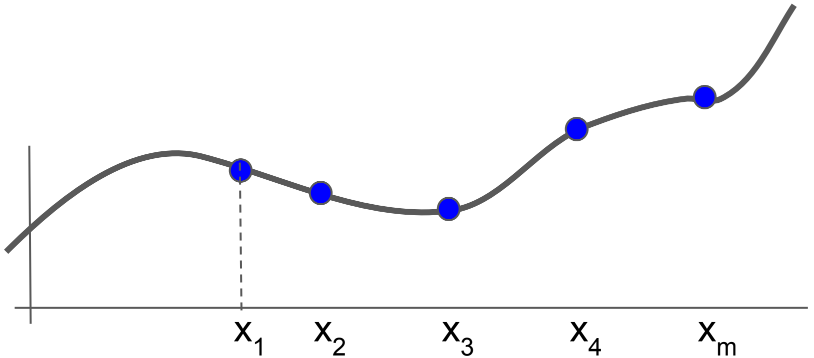
Polynomial Series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Step 1: Sample the signal at at least (n) points (why?). (n-1) is the degree of f(x).

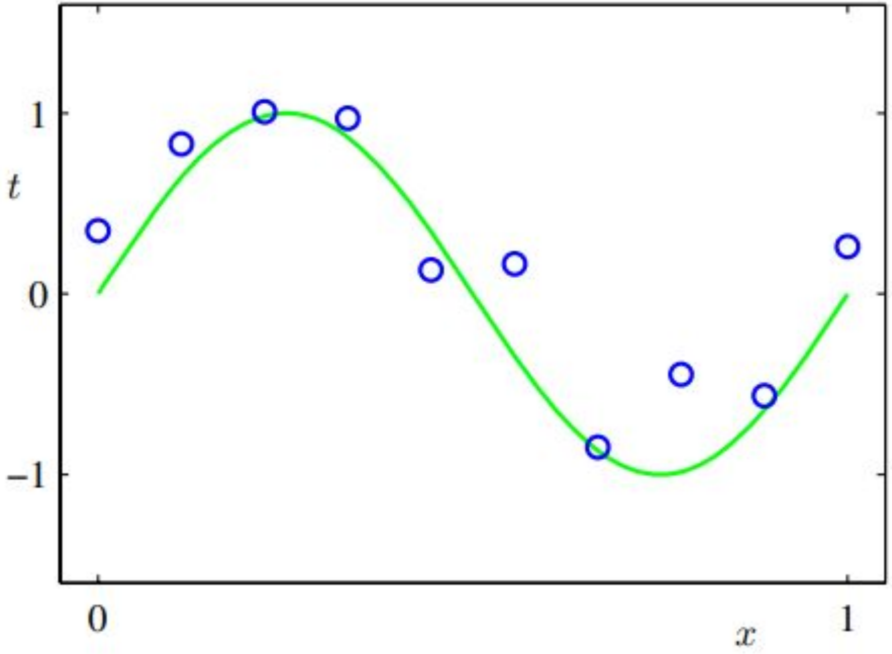
Step 2: Solve y = Xa

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$



Polynomial Curve Fitting

Data



Reference: Pattern Recognition and Machine Learning (Chapter 1), by C. Bishop.

Pattern Recognition and Machine Learning

Book by Christopher Bishop

Polynomial Curve Fitting

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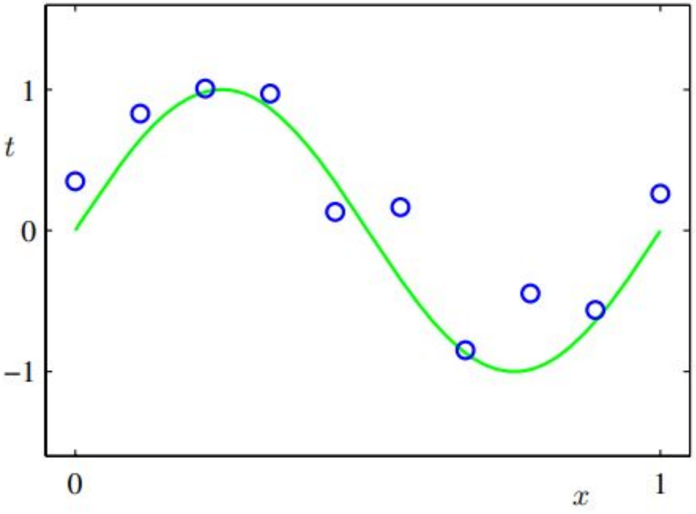

Model

$$y(x, \mathbf{w}) = w_0 + w_1 x + w_2 x^2 + \ldots + w_M x^M = \sum_{j=0}^{M} w_j x^j$$

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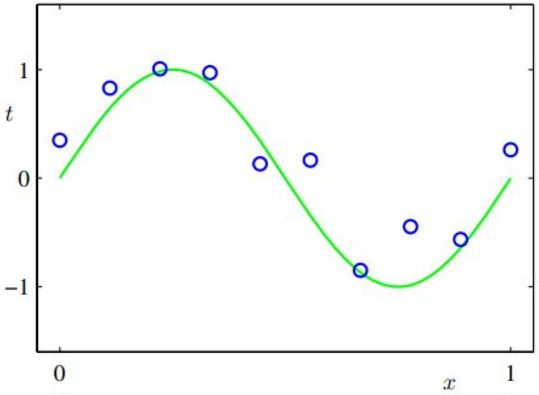

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Loss

$$E(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^{N} \{y(x_n, \mathbf{w}) - t_n\}^2$$

Polynomial Curve Fitting

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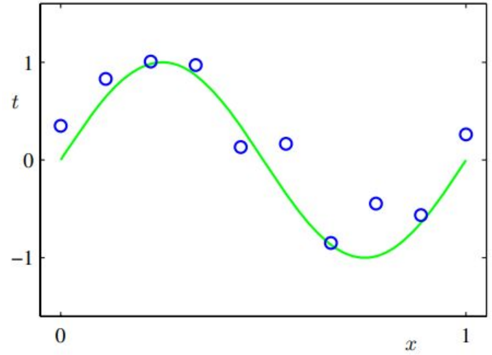

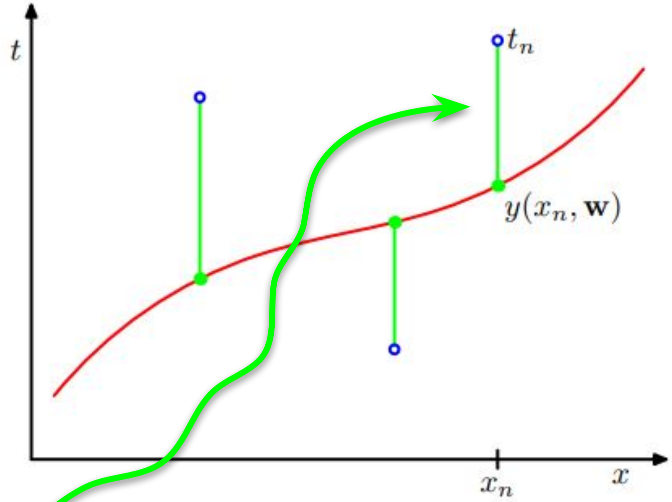

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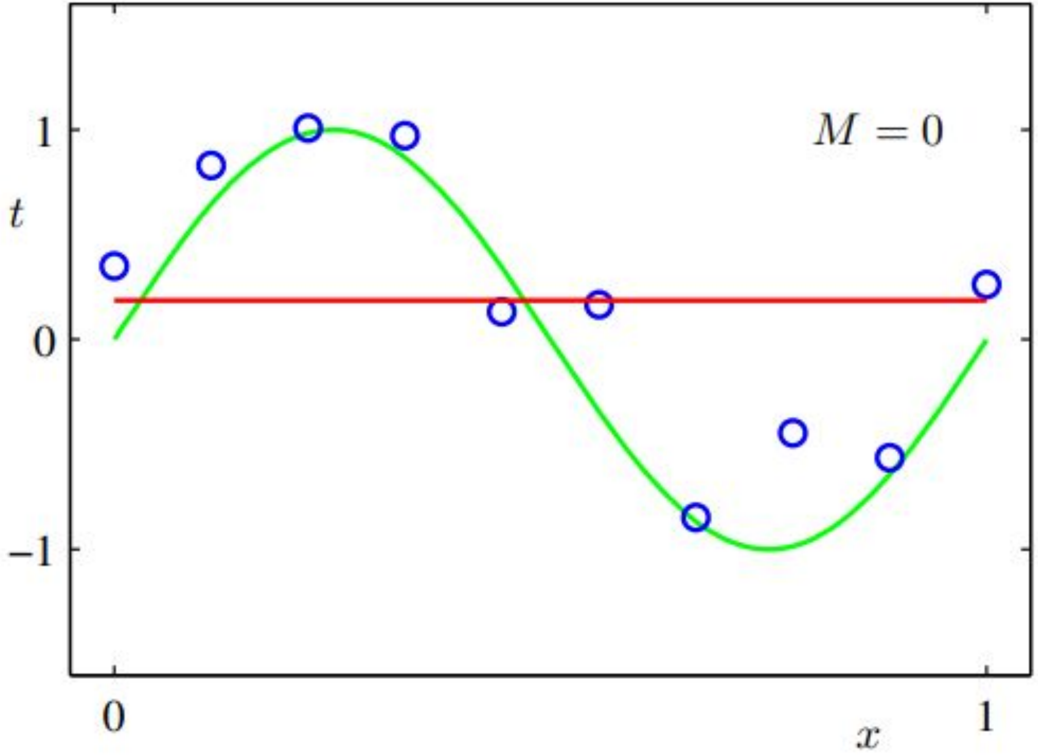
Impact of M



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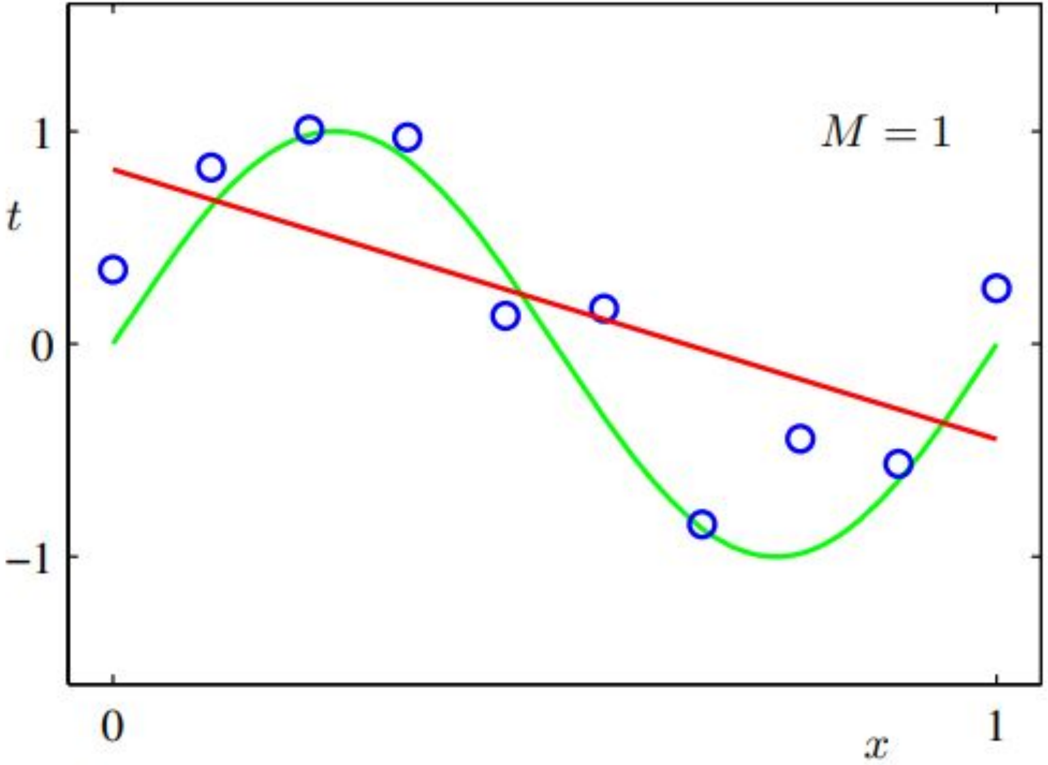

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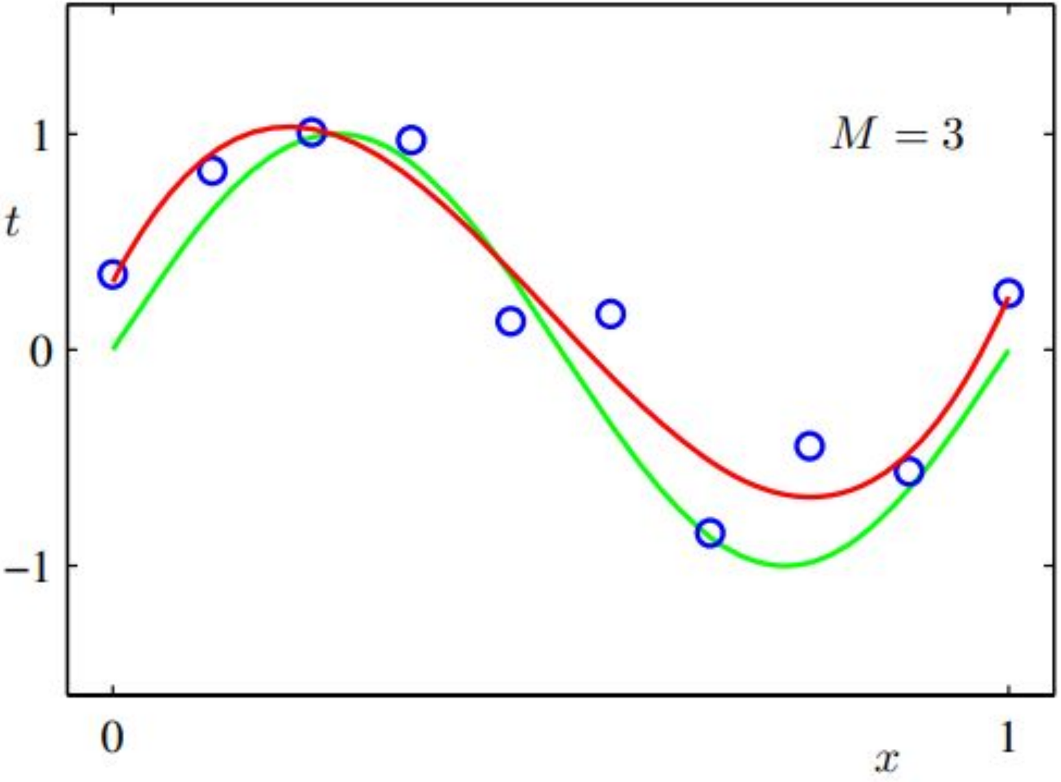

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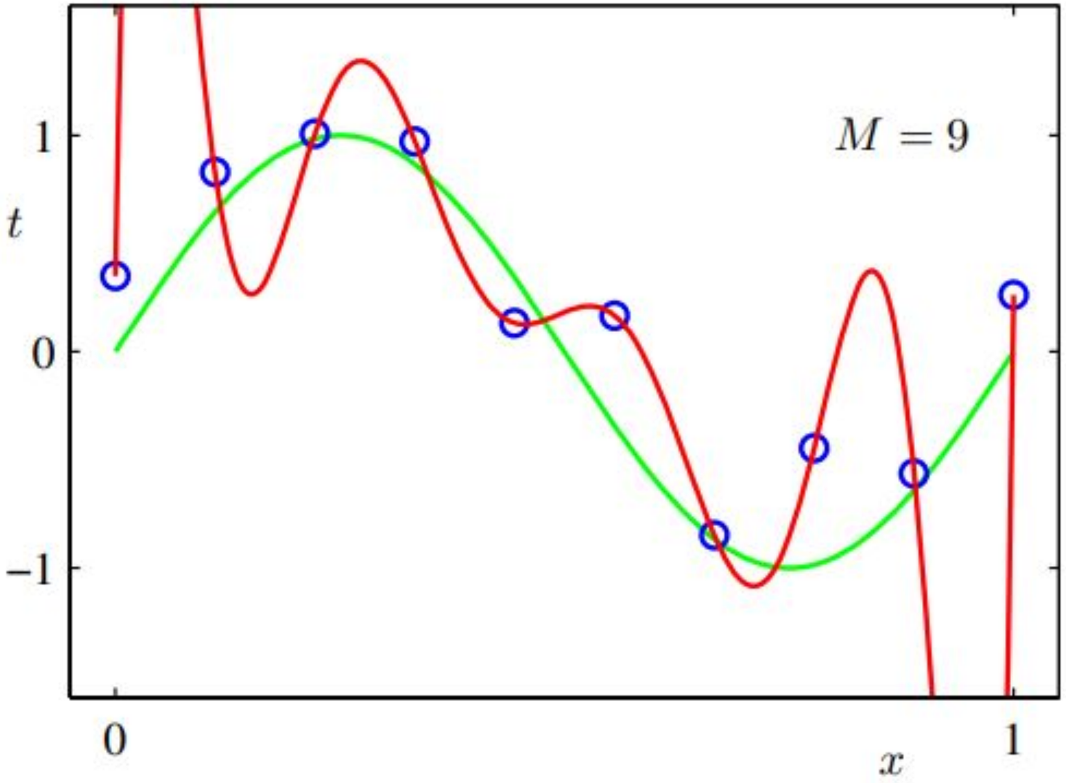

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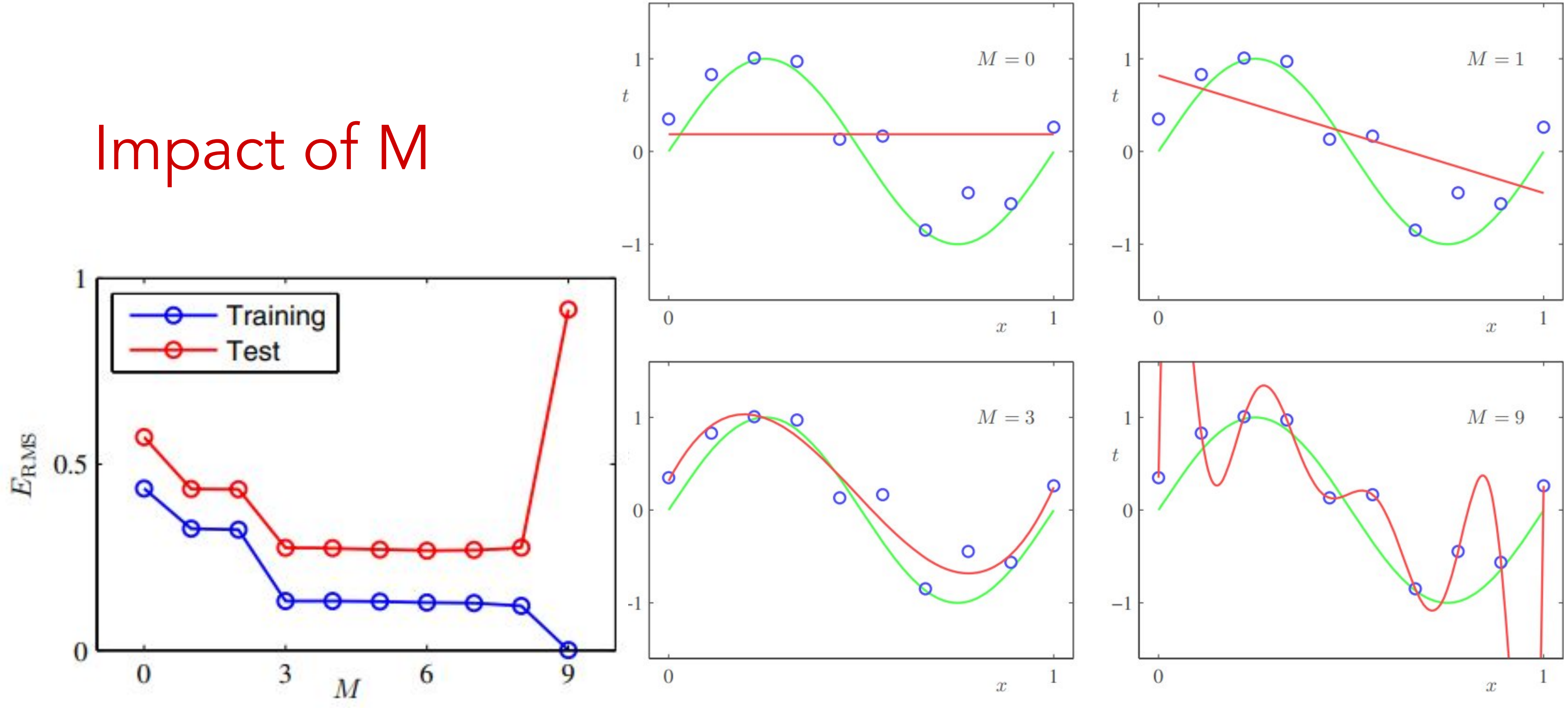

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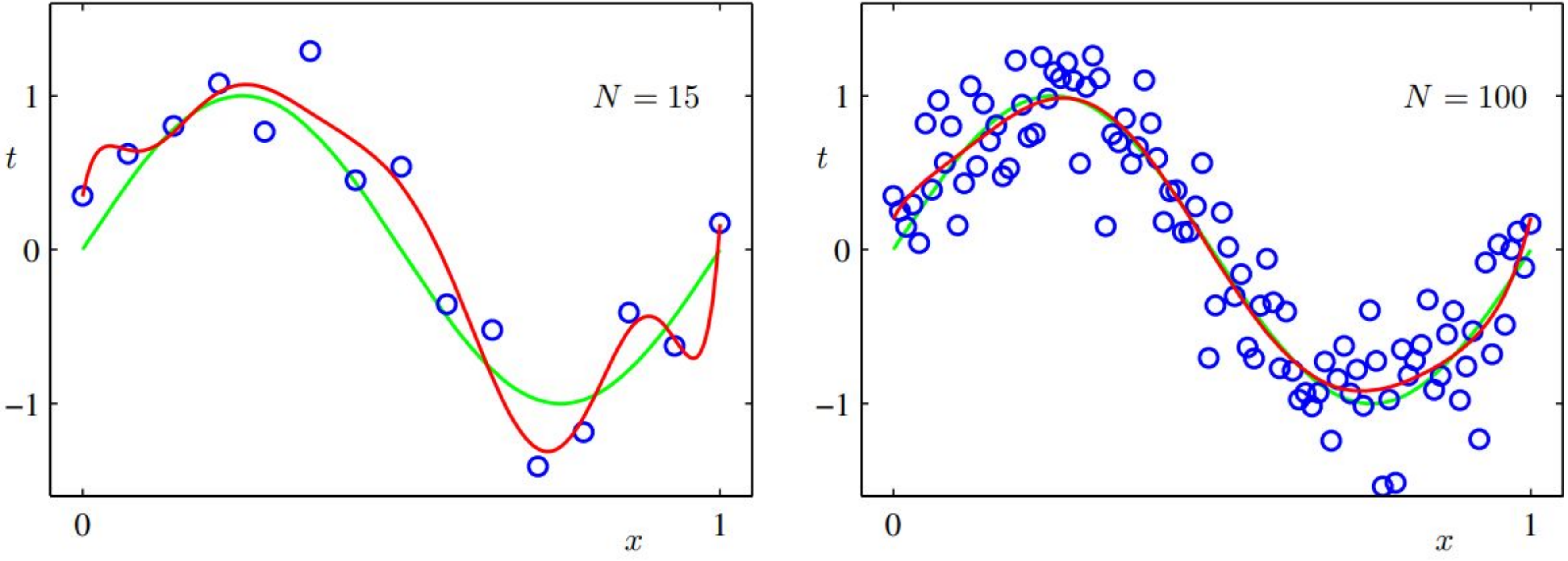
Same M, increasing data points (N)



Summary, Polynomial series approximation,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Weierstrass Approximation Theorem — Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - p\| < \varepsilon$.

- Parameters of the model are $\{a_o, a_1, \ldots, a_n\}$
- Estimating the parameters requires a regression approach

Model

$$y(x, \mathbf{w}) = w_0 + w_1 x + w_2 x^2 + \ldots + w_M x^M = \sum_{j=0}^{M} w_j x^j$$

	$M=0$	$M=1$	$M=6$	$M=9$
$w_0^\star$	0.19	0.82	0.31	0.35
$w_1^\star$		-1.27	7.99	232.37
$w_2^\star$			-25.43	-5321.83
$w_3^\star$			17.37	48568.31
$w_4^\star$				-231639.30
$w_5^\star$				640042.26
$w_6^\star$				-1061800.52
$w_7^\star$				1042400.18
$w_8^\star$				-557682.99
$w_9^\star$				125201.43

Model

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Regularized Loss

$$\widetilde{E}(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^{N} \{y(x_n, \mathbf{w}) - t_n\}^2 + \frac{\lambda}{2} \|\mathbf{w}\|^2$$

$$y(x, \mathbf{w}) = w_0 + w_1 x + w_2 x^2 + \ldots + w_M x^M = \sum_{j=0}^{M} w_j x^j$$

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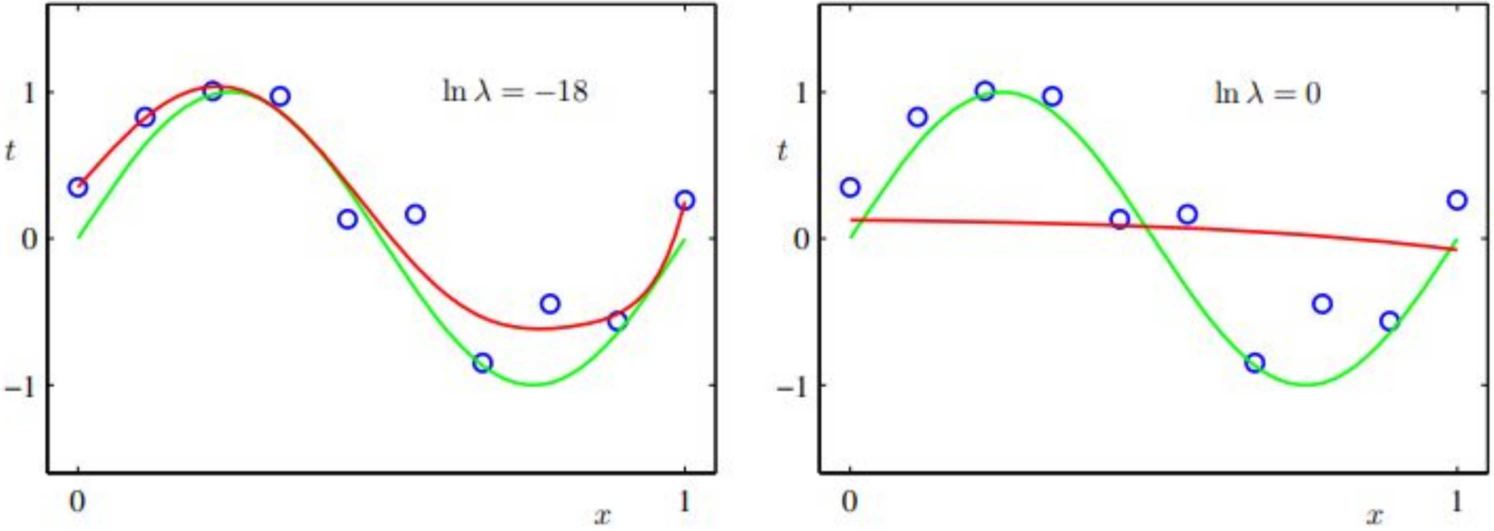


Tour into Vector Spaces

Vector Space review

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

Vectors

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

- Adding two vectors in the plane produces a third one also in the plane
- multiplying a vector by a real scalar produces a second vector also in the plane.

Operations on/with vectors

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

$$y = \begin{bmatrix} y_0 & y_1 \end{bmatrix}^\top$$

Inner Product and Norm

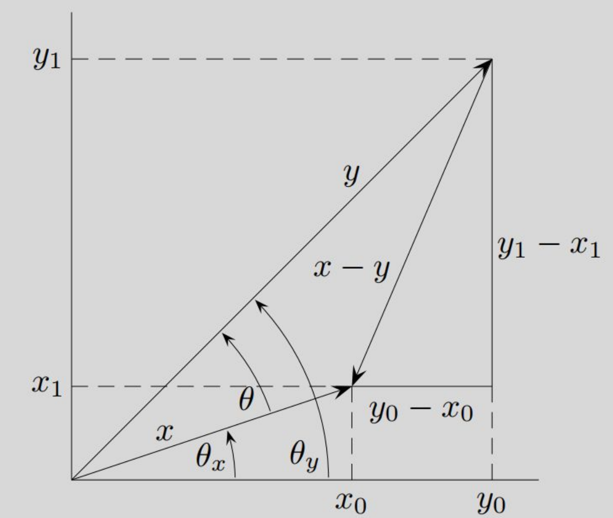
$$\langle x, y \rangle = x_0 y_0 + x_1 y_1$$

$$\langle x, x \rangle = x_0^2 + x_1^2$$

$$||x|| = \sqrt{\langle x, x \rangle} = \sqrt{x_0^2 + x_1^2}$$

Operations on/with vectors

Inner Product (alternate computation)



$$
\begin{aligned}
\langle x,\, y\rangle &= x_0 y_0 + x_1 y_1 \\
&= (\|x\| \cos\theta_x)(\|y\| \cos\theta_y) + (\|x\| \sin\theta_x)(\|y\| \sin\theta_y) \\
&= \|x\|\, \|y\| (\cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y) \\
&= \|x\|\, \|y\| \cos(\theta_x - \theta_y).
\end{aligned}
$$