Computing with Signals

DA 623

Jan - May 2024

IIT Guwahati

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Lecture-04

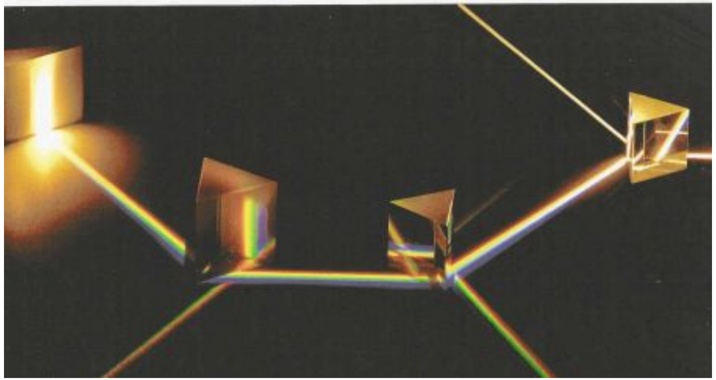
Vector Space review

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

Chap.1 and Chap. 2 from

Foundations of Signal Processing



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Vectors

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

- Adding two vectors in the plane produces a third one also in the plane
- multiplying a vector by a real scalar produces a second vector also in the plane.

Operations on/with vectors

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

- Adding two vectors in the plane produces a third one also in the plane
- multiplying a vector by a real scalar produces a second vector also in the plane.

Operations on/with vectors

Real plane as a vector space

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top$$

$$y = \begin{bmatrix} y_0 & y_1 \end{bmatrix}^\top$$

Inner Product and Norm

$$\langle x, y \rangle = x_0 y_0 + x_1 y_1$$

$$\langle x, x \rangle = x_0^2 + x_1^2$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_0^2 + x_1^2}$$

Operations on/with vectors

Inner Product (alternate computation)

$$
\begin{aligned}
\langle x,\, y\rangle &= x_0y_0 + x_1y_1 \\
&= (\|x\|\cos\theta_x)(\|y\|\cos\theta_y) + (\|x\|\sin\theta_x)(\|y\|\sin\theta_y) \\
&= \|x\|\,\|y\|(\cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y) \\
&= \|x\|\,\|y\|\cos(\theta_x - \theta_y).
\end{aligned}
$$

Operations on/with vectors

Inner Product (alternate computation)

$$\langle x, y \rangle = ||x|| \, ||y|| \cos(\theta_x - \theta_y)$$

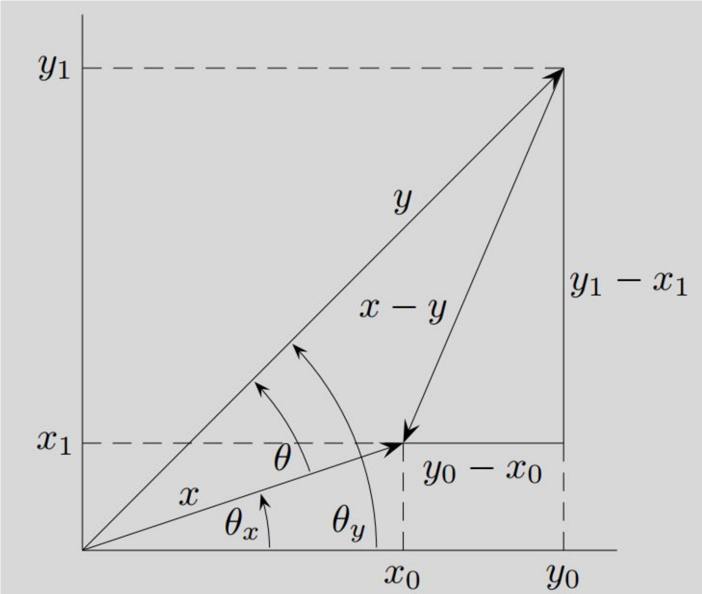
- norms
- similarity of their orientations

Operations on/with vectors

Inner Product (alternate computation)

$$\langle x,\, y\rangle \;=\; \|x\|\,\|y\|\cos(\theta_x - \theta_y)$$

- norms
- similarity of their orientations

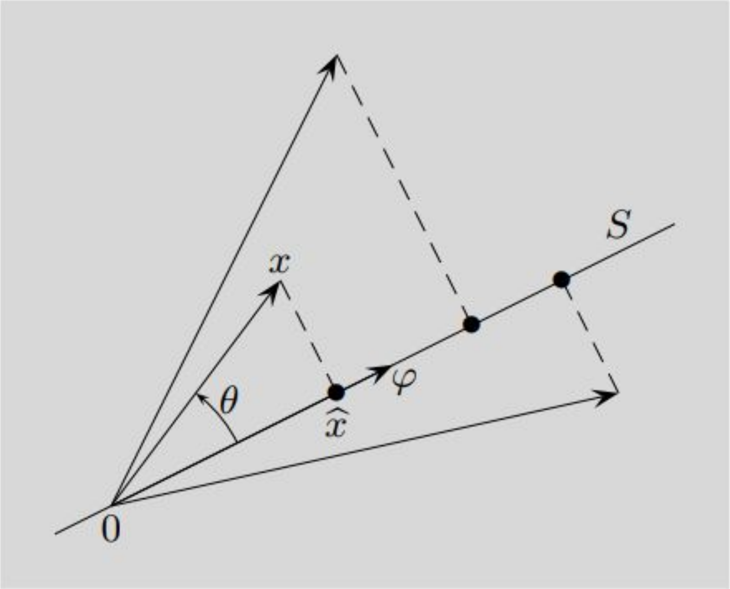


Distance between two vectors

$$d(x,y) \;=\; \|x-y\| \;=\; \sqrt{\langle x-y,\, x-y\rangle} \;=\; \sqrt{(x_0-y_0)^2+(x_1-y_1)^2}$$

Collection of vectors

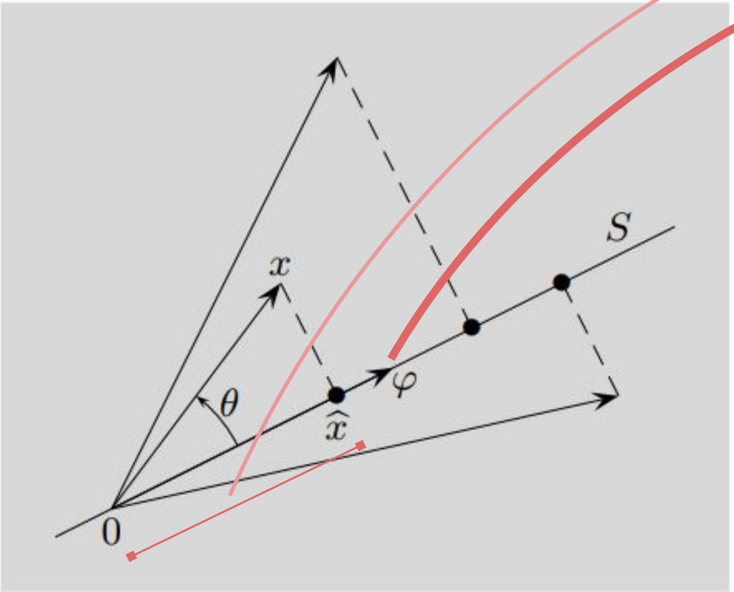
Subspaces and projections



(a) Orthogonal projections onto S.

Collection of vectors

Subspaces and projections



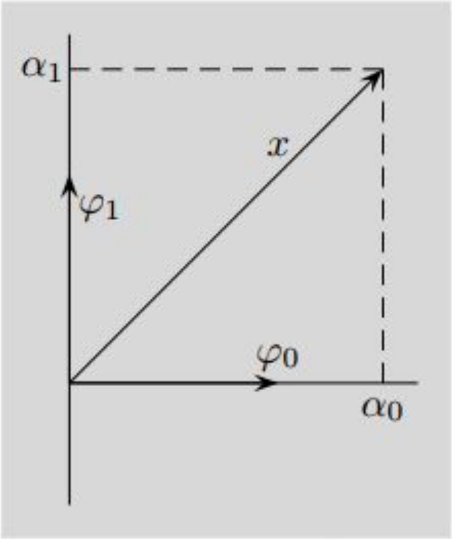
(a) Orthogonal projections onto S.

$$
\begin{aligned}
\widehat{x} &= \langle x, \varphi \rangle \varphi \\
&= (\|x\|\,\|\varphi\| \cos\theta)\varphi \\
&= (\|x\| \cos\theta)\varphi
\end{aligned}
$$

$(\ \|\varphi\| = 1\ )$

Representation

Bases and coordinates



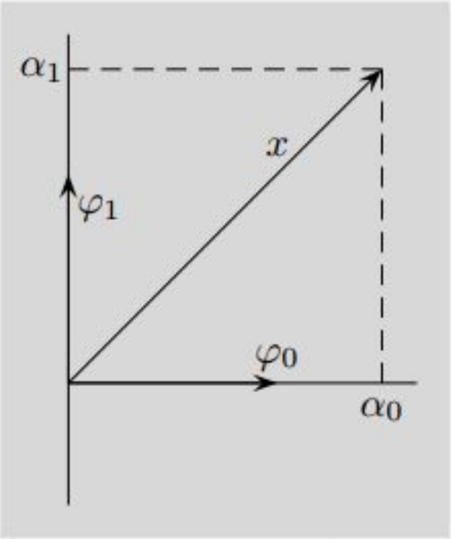
(a) Expansion with an orthonormal basis.

For orthogonal bases

$$x = \alpha_0\varphi_0 + \alpha_1\varphi_1$$

Representation

Bases and coordinates



(a) Expansion with an orthonormal basis.

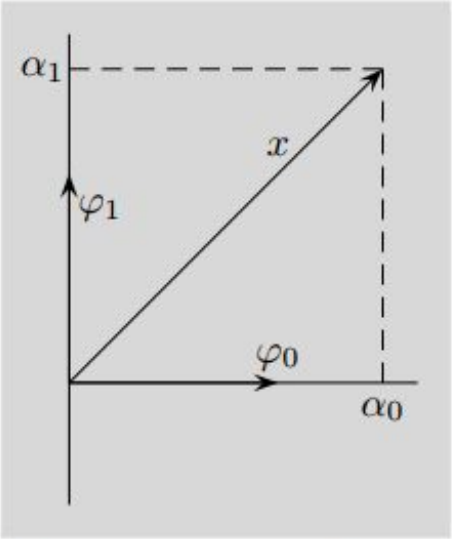
For orthogonal bases

$$x = \alpha_0 \varphi_0 + \alpha_1 \varphi_1$$

$$\alpha_0 = \langle x, \varphi_0 \rangle \qquad \text{and} \qquad \alpha_1 = \langle x, \varphi_1 \rangle$$

Representation

Bases and coordinates



(a) Expansion with an orthonormal basis.

For orthogonal bases

$$x = \alpha_0 \varphi_0 + \alpha_1 \varphi_1$$

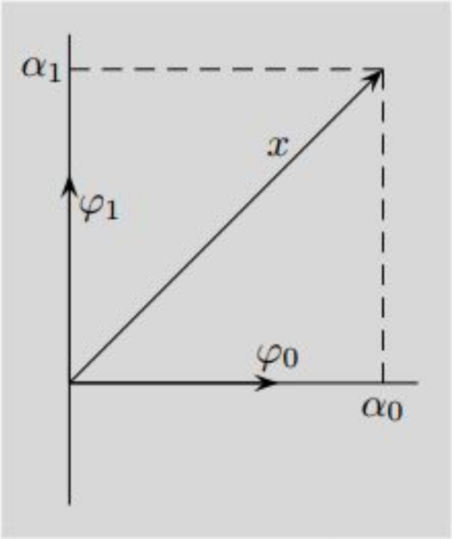
$$\alpha_0 = \langle x, \varphi_0 \rangle \qquad \text{and} \qquad \alpha_1 = \langle x, \varphi_1 \rangle$$

$$|\alpha_0|^2 + |\alpha_1|^2 = \|x\|^2$$

(length can be computed this way as well)

Representation

Bases and coordinates



(a) Expansion with an orthonormal basis.

For orthogonal bases

Synthesis or Representation:

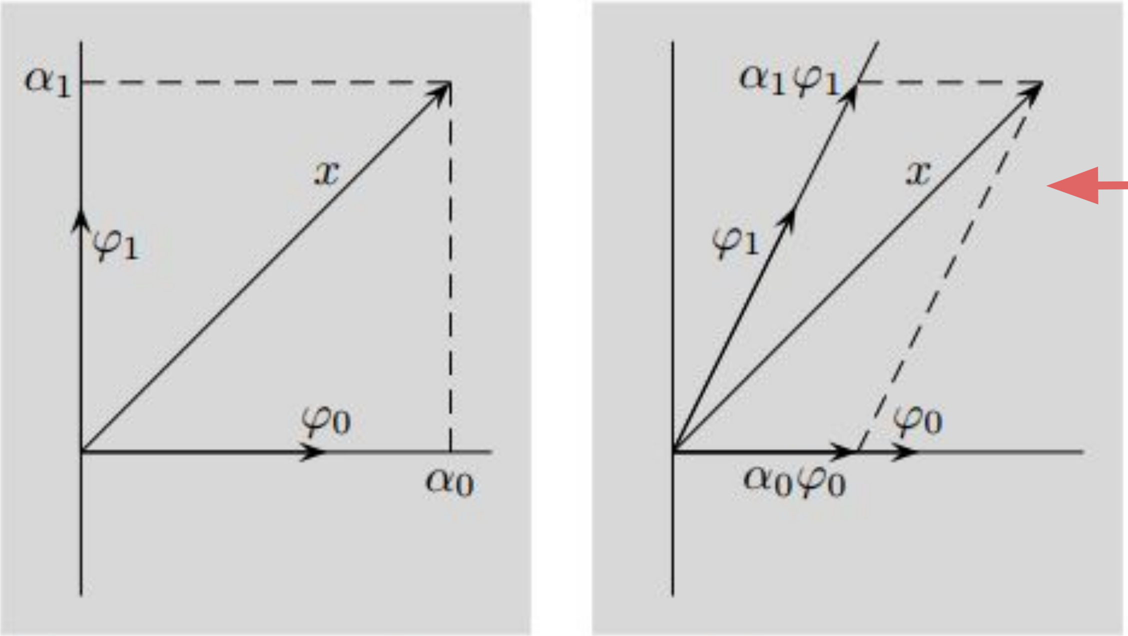
$$x = \alpha_0\varphi_0 + \alpha_1\varphi_1$$

Analysis:

$$\alpha_0 = \langle x,\ \varphi_0\rangle \qquad \text{and} \qquad \alpha_1 = \langle x,\ \varphi_1\rangle$$

Representation

Bases and coordinates



(a) Expansion with an orthonormal basis.

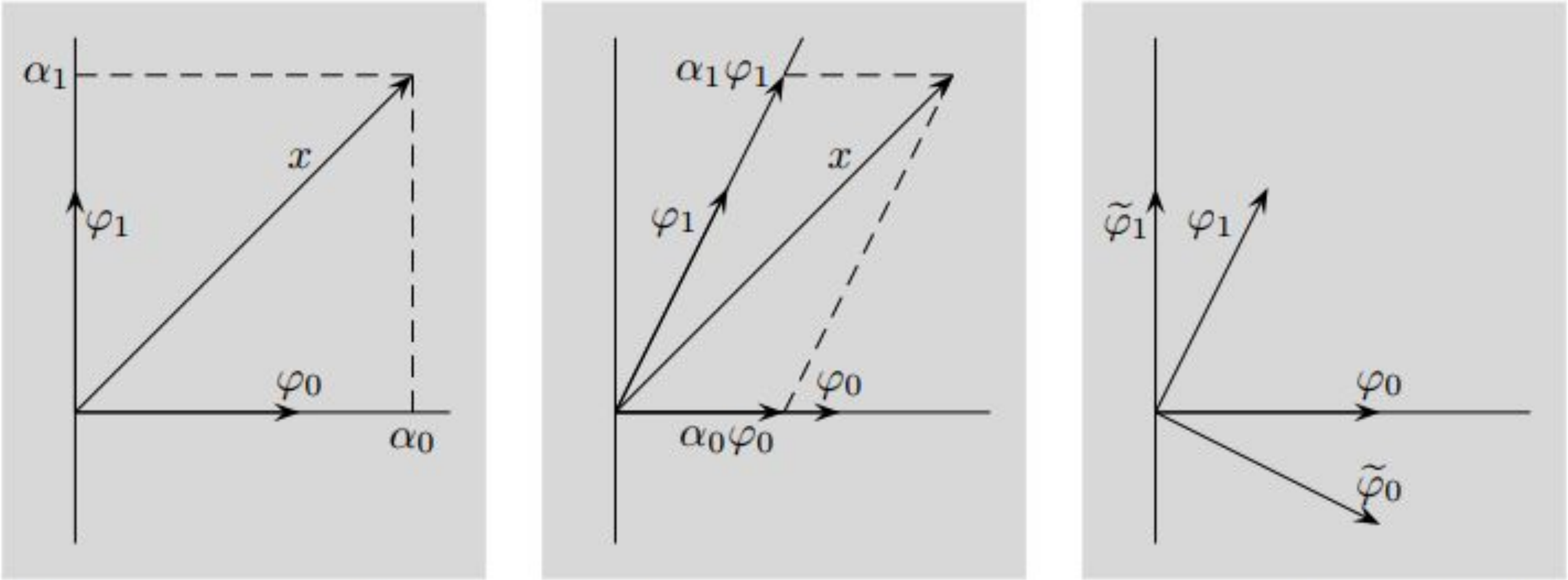
(b) Expansion with a nonorthogonal basis.

Non-orthogonal bases

(Analysis is not straightforward)

Representation

Bases and coordinates



(a) Expansion with an orthonormal basis.

(b) Expansion with a nonorthogonal basis.

(c) Basis $\{\varphi_0, \varphi_1\}$ and its dual $\{\widetilde{\varphi}_0, \widetilde{\varphi}_1\}$.

Definitions

Vector Space

DEFINITION 2.1 (VECTOR SPACE) A *vector space* over a field of scalars $\mathbb{C}$ (or $\mathbb{R}$) is a set of vectors, V, together with operations of vector addition and scalar multiplication. For any x, y, z in V and α, β in $\mathbb{C}$ (or $\mathbb{R}$), these operations must satisfy the following properties:

(i) *Commutativity:* $x + y = y + x$.

(ii) *Associativity:* $(x + y) + z = x + (y + z)$ and $(\alpha\beta)x = \alpha(\beta x)$.

(iii) *Distributivity:* $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$.

Furthermore, the following hold:

(iv) *Additive identity:* There exists an element $\mathbf{0}$ in V such that $x + \mathbf{0} = \mathbf{0} + x = x$ for every x in V.

(v) *Additive inverse:* For each x in V, there exists a unique element $-x$ in V such that $x + (-x) = (-x) + x = \mathbf{0}$.

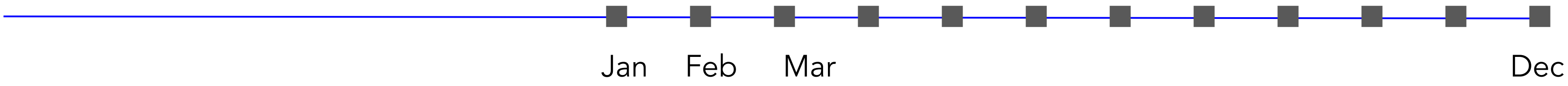
(vi) *Multiplicative identity:* For every x in V, $1 \cdot x = x$.

Examples

Vector Space

$\mathbb{C}^N$: Vector space of complex-valued finite-dimensional vectors

$$\mathbb{C}^N = \left\{ x = \begin{bmatrix} x_0 & x_1 & \ldots & x_{N-1} \end{bmatrix}^\top \,\middle|\, x_n \in \mathbb{C},\ n \in \{0, 1, \ldots, N-1\} \right\}$$



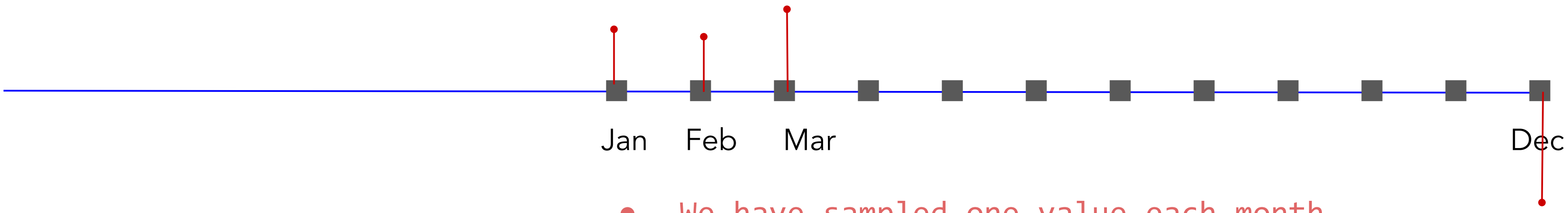
- We have sampled one value each month

Examples

Vector Space

$\mathbb{C}^N$: **Vector space of complex-valued finite-dimensional vectors**

$$\mathbb{C}^N = \left\{ x = \begin{bmatrix} x_0 & x_1 & \ldots & x_{N-1} \end{bmatrix}^\top \,\middle|\, x_n \in \mathbb{C},\ n \in \{0, 1, \ldots, N-1\} \right\}$$



- We have sampled one value each month
- Example, average temperature(real value)
- Each year we have a vector with N=12 elements

Examples

Vector Space

$\mathbb{C}^N$: **Vector space of complex-valued finite-dimensional vectors**

$$\mathbb{C}^N = \left\{ x = \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^\top \,\middle|\, x_n \in \mathbb{C},\ n \in \{0, 1, \dots, N-1\} \right\}$$

The two fundamental operations in a vector space

$$\boxed{\begin{aligned} x + y &= \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^\top + \begin{bmatrix} y_0 & y_1 & \dots & y_{N-1} \end{bmatrix}^\top \\ &= \begin{bmatrix} x_0 + y_0 & x_1 + y_1 & \dots & x_{N-1} + y_{N-1} \end{bmatrix}^\top, \\ \alpha x &= \alpha \begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix}^\top = \begin{bmatrix} \alpha x_0 & \alpha x_1 & \dots & \alpha x_{N-1} \end{bmatrix}^\top \end{aligned}}$$

Examples

Vector Space

$\mathbb{C}^{\mathbb{Z}}$: **Vector space of complex-valued sequences over** $\mathbb{Z}$

$$\mathbb{C}^{\mathbb{Z}} = \left\{ x = \begin{bmatrix} \cdots & x_{-1} & \boxed{x_0} & x_1 & \cdots \end{bmatrix}^{\top} \,\middle|\, x_n \in \mathbb{C},\ n \in \mathbb{Z} \right\}$$

- Negative indices helps to bring the notion of history

Examples

Vector Space

$\mathbb{C}^{\mathbb{R}}$: **Vector space of complex-valued functions over** $\mathbb{R}$

$$\mathbb{C}^{\mathbb{R}} = \{x \mid x(t) \in \mathbb{C},\ t \in \mathbb{R}\}$$

- Helps understand collection of functions (continuous) of a particular type as a vector space

Example

Vector Space

$\mathbb{C}^{\mathbb{R}}$: **Vector space of complex-valued functions over** $\mathbb{R}$

$$\mathbb{C}^{\mathbb{R}} = \{x \mid x(t) \in \mathbb{C},\ t \in \mathbb{R}\}$$

The two fundamental operations in a vector space

$$\boxed{\begin{aligned} (x+y)(t) &= x(t) + y(t), \\ (\alpha x)(t) &= \alpha x(t). \end{aligned}}$$

Example

Vector Space

$\mathbb{C}^{\mathbb{R}}$: **Vector space of complex-valued functions over** $\mathbb{R}$

$$\mathbb{C}^{\mathbb{R}} = \{x \mid x(t) \in \mathbb{C},\ t \in \mathbb{R}\}$$

EXAMPLE 2.1 (VECTOR SPACE OF POLYNOMIALS) Fix a positive integer N and consider the real-valued polynomials of degree at most $(N-1)$, $x(t) = \sum_{k=0}^{N-1} \alpha_k t^k$. These form a vector space over $\mathbb{R}$ under the natural addition and multiplication operations. Since each polynomial is specified by its coefficients, polynomials combine exactly like vectors in $\mathbb{R}^N$.

Definitions

Vector space $\longrightarrow$ Subspace

DEFINITION 2.2 (SUBSPACE) A nonempty subset S of a vector space V is a *subspace* when it is closed under the operations of vector addition and scalar multiplication:

(i) For all x and y in S, $x + y$ is in S.
(ii) For all x in S and α in $\mathbb{C}$ (or $\mathbb{R}$), αx is in S.

DEFINITION 2.3 (AFFINE SUBSPACE) A subset T of a vector space V is an *affine subspace* when there exist a vector $x \in V$ and a subspace $S \subset V$ such that any $t \in T$ can be written as $x + s$ for some $s \in S$.

Definitions

Vector space $\longrightarrow$ Subspace $\longrightarrow$ Span

DEFINITION 2.4 (SPAN) The *span* of a set of vectors S is the set of all finite linear combinations of vectors in S:

$$\operatorname{span}(S) = \left\{ \sum_{k=0}^{N-1} \alpha_k \varphi_k \;\middle|\; \alpha_k \in \mathbb{C} \text{ (or } \mathbb{R}\text{)},\ \varphi_k \in S, \text{ and } N \in \mathbb{N} \right\}.$$

Definitions

Vector space $\longrightarrow$ Subspace $\longrightarrow$ Linear Independence

DEFINITION 2.5 (LINEARLY INDEPENDENT SET) The set of vectors $\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$ is called *linearly independent* when $\sum_{k=0}^{N-1} \alpha_k \varphi_k = \mathbf{0}$ is true only if $\alpha_k = 0$ for all k. Otherwise, the set is linearly dependent. An infinite set of vectors is called linearly independent when every finite subset is linearly independent.

Example. Let $p_j(x) = x^j$ for $j = 0, 1, 2, \ldots, n$, where n is some positive integer. Then the polynomials $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ are linearly independent elements of the vector space consisting of all polynomials with real coefficients. Indeed if $c_0, c_1, \ldots, c_n$ are real numbers and if

$$c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x) = 0$$

then $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ is the zero polynomial, and therefore $c_j = 0$ for $j = 0, 1, 2, \ldots, n$.

Definitions

Vector space ⟶ Subspace ⟶ Linear Independence

Example. Let $p_j(x) = x^j$ for $j = 0, 1, 2, \ldots, n$, where n is some positive integer. Then the linearly independent polynomials $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ span the vector space consisting of all polynomials with real coefficients whose degree does not exceed n, since

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x)$$

for all polynomials $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ with real coefficients. We conclude that $1, x, x^2, \ldots, x^n$ is a basis of the vector space consisting of all polynomials with real coefficients whose degree does not exceed n.

Definitions

Vector space $\longrightarrow$ Subspace $\longrightarrow$ Linear Independence $\longrightarrow$ Dimension

DEFINITION 2.6 (DIMENSION) A vector space V is said to have *dimension* N when it contains a linearly independent set with N elements and every set with $N+1$ or more elements is linearly dependent. If no such finite N exists, the vector space is infinite-dimensional.

Definitions (notion of value derived from a pair of vectors)

Inner Product

DEFINITION 2.7 (INNER PRODUCT) An *inner product* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle$ defined on $V \times V$, with the following properties for any $x, y, z \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Distributivity:* $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) *Linearity in the first argument:* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) *Hermitian symmetry:* $\langle x, y \rangle^* = \langle y, x \rangle$.

(iv) *Positive definiteness:* $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

Definitions

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Example

$\langle x, y \rangle = x_0 y_0^* + 5 x_1 y_1^*$ ✓

$\langle x, y \rangle = x_0^* y_0 + x_1^* y_1$ ✗ (ii)

$\langle x, y \rangle = x_0 y_0^*$ ✗ (iii)

Definitions

Inner Product

Standard Inner Product

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x \quad \textbf{on } \mathbb{C}^N$$

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^* = y^* x \quad \textbf{on } \mathbb{C}^{\mathbb{Z}}$$

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt \quad \textbf{on } \mathbb{C}^{\mathbb{R}}$$

Definitions (notion of angle)

DEFINITION 2.8 (ORTHOGONALITY)

(i) Vectors x and y are said to be *orthogonal* when $\langle x, y\rangle = 0$, written as $x \perp y$.

(ii) A set of vectors S is called *orthogonal* when $x \perp y$ for every x and y in S such that $x \neq y$.

(iii) A set of vectors S is called *orthonormal* when it is orthogonal and $\langle x, x\rangle = 1$ for every x in S.

(iv) A vector x is said to be *orthogonal* to a set of vectors S when $x \perp s$ for all $s \in S$, written as $x \perp S$.

(v) Two sets S_0 and S_1 are said to be *orthogonal* when every vector s_0 in S_0 is orthogonal to the set S_1, written as $S_0 \perp S_1$.

(vi) Given a subspace S of a vector space V, the *orthogonal complement* of S, denoted $S^{\perp}$, is the set $\{x \in V \mid x \perp S\}$.

Definitions

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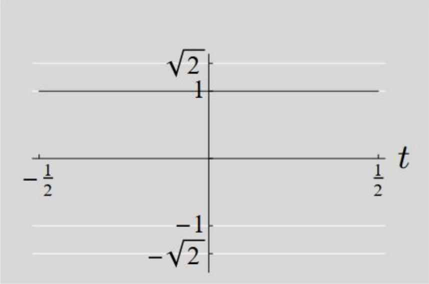
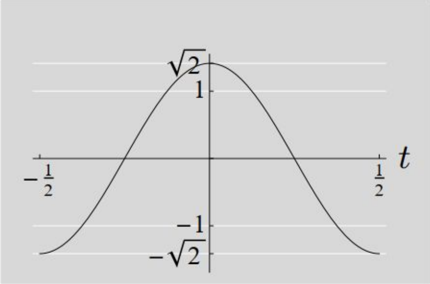
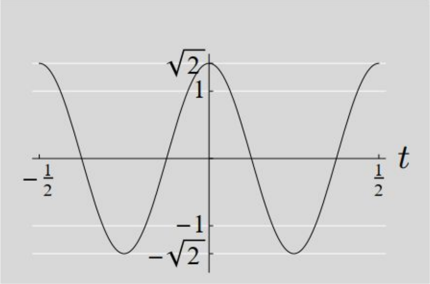


(a) $\varphi_0(t) = 1$.



(b) $\varphi_1(t) = \sqrt{2}\cos(2\pi t)$.



(c) $\varphi_2(t) = \sqrt{2}\cos(4\pi t)$.

Definitions (notion of length)

DEFINITION 2.9 (NORM) A *norm* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $\|\cdot\|$ defined on V, with the following properties for any $x, y \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Positive definiteness:* $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = \mathbf{0}$.

(ii) *Positive scalability:* $\|\alpha x\| = |\alpha| \, \|x\|$.

(iii) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$, with equality if and only if $y = \alpha x$.

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(iii) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$, with equality if and only if $y = \alpha x$.

Example

$$\|x\| = |x_0|^2 + 5|x_1|^2$$

✓

$$\|x\| = |x_0| + |x_1|$$

✓

$$\|x\| = |x_0|$$

✗ (i)

Definitions (notion of length)

Standard Norm

$$\|x\| = \sqrt{\langle x,\, x\rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2\right)^{1/2} \quad \textbf{on } \mathbb{C}^N$$

$$\|x\| = \sqrt{\langle x,\, x\rangle} = \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{1/2} \quad \textbf{on } \mathbb{C}^{\mathbb{Z}}$$

$$\|x\| = \sqrt{\langle x,\, x\rangle} = \left(\int_{-\infty}^{\infty} |x(t)|^2\, dt\right)^{1/2} \quad \textbf{on } \mathbb{C}^{\mathbb{R}}$$

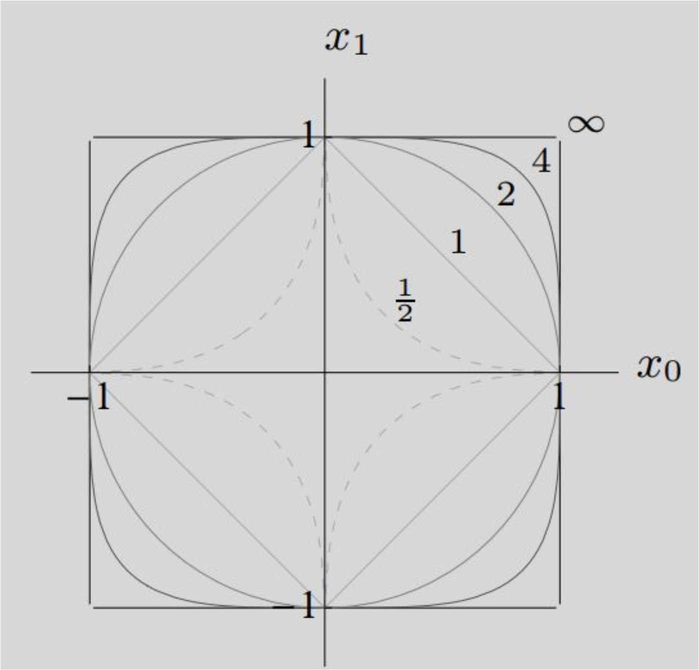
Definitions (notion of length)

More general p- norm

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p}$$ **on** $\mathbb{C}^N$

for $p \in [1, \infty)$

The unit ball contour under different p-norms



Definitions (notion of a computable vector space)

DEFINITION 2.13 (CONVERGENT SEQUENCE OF VECTORS) A sequence of vectors $x_0, x_1, \ldots$ in a normed vector space V is said to *converge* to $v \in V$ when $\lim_{k\to\infty} \|v - x_k\| = 0$. In other words, given any $\varepsilon > 0$, there exists a K_ε such that

$$\|v - x_k\| \; < \; \varepsilon \qquad \text{for all} \quad k > K_\varepsilon.$$

Definitions (notion of a computable vector space)

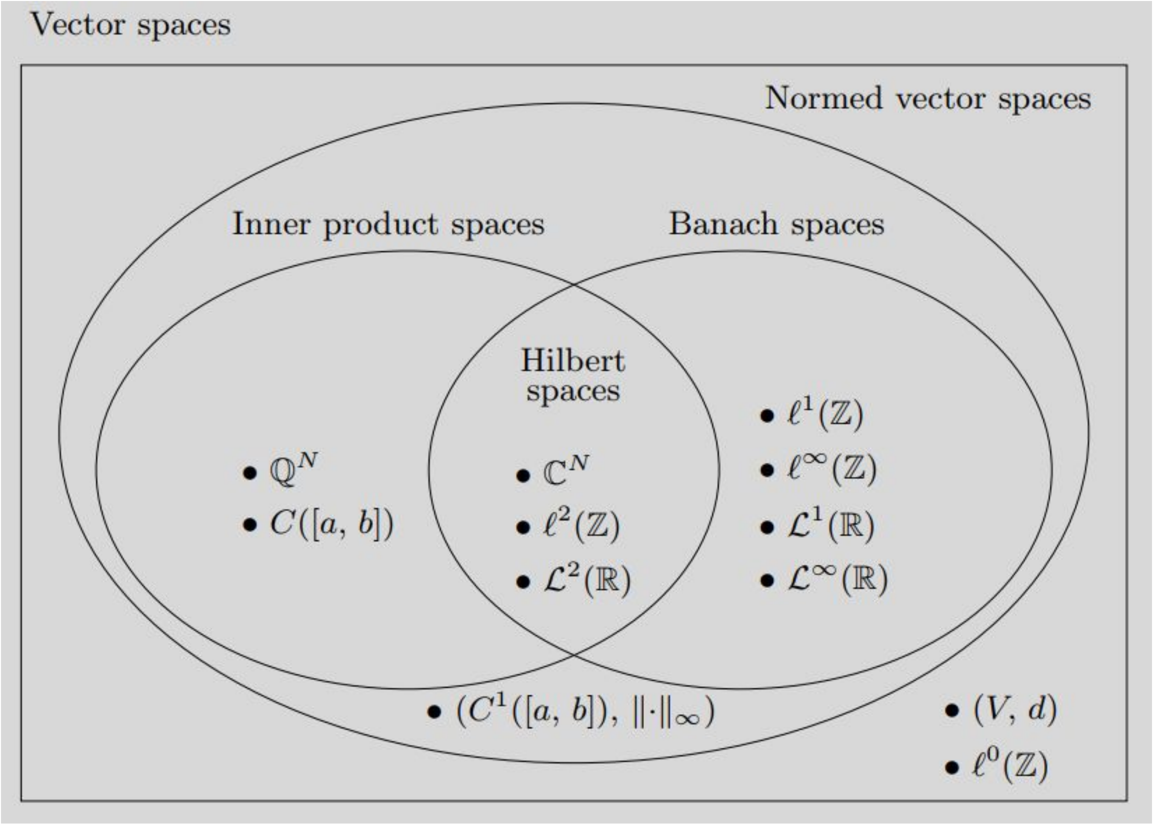
DEFINITION 2.15 (CAUCHY SEQUENCE OF VECTORS) A sequence of vectors x_0, x_1, ... in a normed vector space is called a *Cauchy sequence* when, given any $\varepsilon > 0$, there exists a K_ε such that

$$\|x_k - x_m\| < \varepsilon \qquad \text{for all } k,\, m > K_\varepsilon.$$

Definitions (notion of a computable vector space)

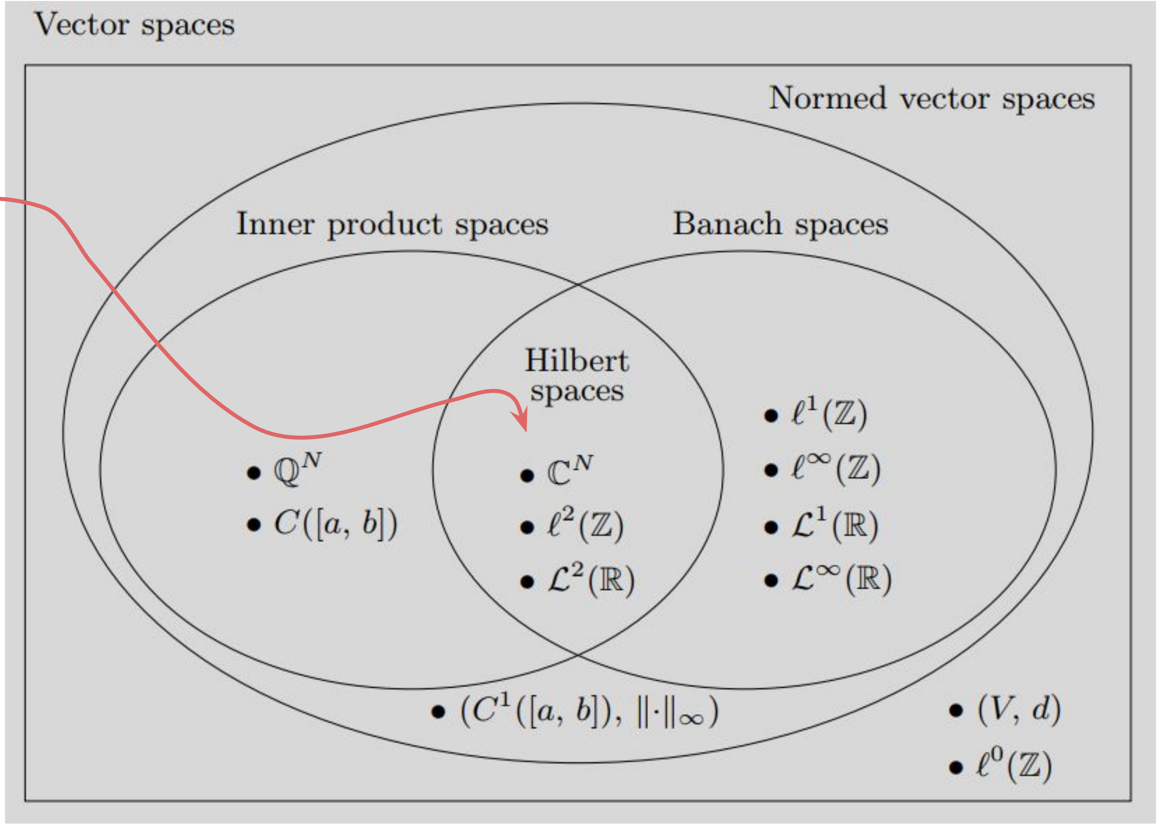
DEFINITION 2.16 (COMPLETENESS AND HILBERT SPACE) A normed vector space V is said to be *complete* when every Cauchy sequence in V converges to a vector in V. A complete inner product space is called a *Hilbert space*.

In a snapshot



Hilbert Space

To ease our analysis we will enforce that our signal/data resides (mostly)



Next lecture we will continue …