DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
July-Nov 2022 : MA 321 Optimization

Time : 6:00pm-7:15pm **25th August, 2022** **Maximum marks: 15**

Quiz-1

Notation: $Fea(P)$ is the feasible region of (P)
LI: Linearly Independent

1. Consider the following problem (P):

$$\text{Min } -2x_1 + 3x_2$$
$$\text{subject to } \quad x_1 + x_2 \geq 2$$
$$-2x_1 + x_2 \leq 2$$
$$x_1 - x_2 \leq 3$$
$$x_1 \geq 0, x_2 \geq 0.$$

(a) **Check** whether (P) has an optimal solution.

Solution: Since any **direction d** of $Fea(P)$ must satisfy $\mathbf{d} \geq 0$, and $d_1 - d_2 \leq 0$ or $d_1 \leq d_2$, we have $\mathbf{c}^T\mathbf{d} = -2d_1 + 3d_2 \geq 0$, hence (P) has an optimal solution.

(b) Give all the extreme directions of $Fea(P)$ and justify that they are indeed extreme directions of $Fea(P)$.

Solution: The two extreme directions are $[1, 2]^T$ and $[1, 1]^T$ (from the graph). (Ideally one has to check that any $\mathbf{d} \in R^2$ satisfying $d_1 = 0$ cannot be a direction, similarly any $\mathbf{d} \in R^2$ satisfying $d_2 = 0$ cannot be a direction. Also any $\mathbf{d} \in R^2$ satisfying $d_1 + d_2 = 0$ cannot be a direction, but it is okay if you have not done this much work)
Clearly $[1, 2]^T$ satisfies the conditions $d_1 + d_2 \geq 0, \quad -2d_1 + d_2 \leq 0, \quad d_1 - d_2 \leq 0,$ $d_1 \geq 0, d_2 \geq 0$ and it lies on the LI hyperplane $-2d_1 + d_2 = 0$.
Similarly $[1, 1]^T$ also satisfies the conditions $d_1+d_2 \geq 0, \quad -2d_1+d_2 \leq 0, \quad d_1-d_2 \leq 0,$ $d_1 \geq 0, d_2 \geq 0$ and it lies on the LI hyperplane $d_1 - d_2 = 0$.
Hence both $[1, 2]^T$ and $[1, 1]^T$ are extreme directions.

(c) If possible give an $\mathbf{u} \in \mathbb{R}^2$, an $\mathbf{x}_0 \in Fea(P)$ and an $a > 0$, such that both $\mathbf{x}_0 + a\mathbf{u}$ and $\mathbf{x}_0 - a\mathbf{u}$ are extreme points of $Fea(P)$. Is this $\mathbf{u}$ orthogonal to the normal of any defining hyperplane of $Fea(P)$? If yes, then give the hyperplane.

Solution: There will be various choices for this.
For example one choice could be $\mathbf{x}_0 = \frac{1}{2}[2, 0]^T + \frac{1}{2}[3, 0]^T = [\frac{5}{2}, 0]^T$, $a = \frac{1}{2}$, $\mathbf{u} = [1, 0]^T$ and it is orthogonal to the normal of the defining hyperplane $x_2 = 0$.

(d) Give the set of all points in $Fea(P)$ which can be expressed **uniquely** as a convex combination of the extreme points of $Fea(P)$. Justify.

Solution: The required set is clearly the set of all points in the triangle with corners at the points $[2, 0]^T$, $[3, 0]^T$ and $[0, 2]^T$.

(I do not expect you to write this proof in your exam but I am giving a proof of the above since a student after class asked me.
Let $\mathbf{x} = \lambda_1[2, 0]^T + \lambda_2[3, 0]^T + \lambda_3[0, 2]^T$, $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$.
If $\lambda_1 + \lambda_2 = 0$, then $\mathbf{x} = [0, 2]^T$ and is a point of the triangle.
If $\lambda_1 + \lambda_2 \neq 0$, then $\mathbf{x} = (\lambda_1 + \lambda_2)(\frac{\lambda_1}{\lambda_1+\lambda_2}[2, 0]^T + \frac{\lambda_2}{\lambda_1+\lambda_2}[3, 0]^T) + \lambda_3[0, 2]^T$,
which is clearly a convex convex combination of a corner point of the triangle and a point on the opposite side of the corner, hence is a point of the triangle with the corners as the extreme points.

Also given any point $\mathbf{x}$ of the triangle, if it lies on the boundary of the triangle then it can be expressed as a convex combination of atmost two corner points of the triangle, which are extreme points.
If $\mathbf{x}$ is an interior point of the triangle, then consider the line segment from a corner of the triangle to $\mathbf{x}$ and extend it further till it cuts the opposite side, call this point $\mathbf{y}$.
Then $\mathbf{x}$ can be written as a convex combination of that corner point and $\mathbf{y}$, but $\mathbf{y}$ lies on the boundary and can be expressed as a convex combination of two corner points of the triangle, hence $\mathbf{x}$ can be expressed as a convex combination of the three corner points of the triangle which are the extreme points.)

The fact that this representation is **unique** can be shown as follows:
Let $\mathbf{x} = \alpha_1[2, 0]^T + \alpha_2[3, 0]^T + \alpha_3[0, 2]^T$
and let $\mathbf{x} = \lambda_1[2, 0]^T + \lambda_2[3, 0]^T + \lambda_3[0, 2]^T$
where $0 \leq \alpha_i, \lambda_i \leq 1$, $\sum_i \alpha_i = 1$ and $\sum_i \lambda_i = 1$.
Then $\mathbf{0} = (\alpha_1 - \lambda_1)[2, 0]^T + (\alpha_2 - \lambda_2)[3, 0]^T + (\alpha_3 - \lambda_3)[0, 2]^T$,
which implies $\alpha_3 = \lambda_3$, which implies $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$.
Since $\alpha_1[2, 0]^T + \alpha_2[3, 0]^T = \lambda_1[2, 0]^T + \lambda_2[3, 0]^T$,
we can divide the equation throughout by $\alpha_1 + \alpha_2$ and we get new α_i, λ_i, $i = 1, 2$ such that in the above equation $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2 = 1$.
Then $\alpha_1[-1, 0]^T + [3, 0]^T = \lambda_1[-1, 0]^T + [3, 0]^T$, which gives $\alpha_1 = \lambda_1$, hence $\alpha_1 = \lambda_2$.

[2+2+2+2]

Answers given without any/proper justification will be awarded zero marks

2. Consider the following problem (P):
Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{2\times 5}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

where the **first row** of A is $[1, -1, 3, 2, 1]$, the **first column** of A is $[1, -2]^T$ and $\mathbf{b} = [0, 1]^T$.
Also let $\mathbf{d} = [1, 3, 0, 0, 0]^T$ be an extreme direction of $Fea(P)$.

(a) How many **distinct** directions does $Fea(P)$ have?

Solution: Since $\mathbf{d}$ is an extreme direction, $\mathbf{d}$ must lie on 4 LI hyperplanes defining D.
Three of them are $d_3 = 0, d_4 = 0, d_5 = 0$.
Since $d_1 > 0$ and $d_2 > 0$ and $\mathbf{d}$ does not lie on the hyperplane corresponding to the first row of A, hence $\mathbf{d}$ must lie on the hyperplane corresponding to the second row of A.
Hence $-2 + 3a_{22} = 0$, which gives $a_{22} = \frac{2}{3}$,
where a_{ij} is the (i, j) th element of A.
Check that $\mathbf{d} = [1, 1, 0, 0, 0]^T$ is an extreme direction of $Fea(P)$ which lies on the LI hyperplanes $d_3 = 0, d_4 = 0, d_5 = 0$ and the hyperplane corresponding to the first row of A.
So we have atleast two **distinct** extreme directions. Since the convex combination of directions is again a direction, the number of **distinct** directions is infinite.

(b) Determine if possible, whether the number of extreme points of $Fea(P)$ is < 7 or is ≥ 7.

Solution: Since an extreme point of $Fea(P)$ must lie on 5 LI defining hyperplanes and the number of rows of A is 2, hence at least 3 LI defining hyperplanes on which an extreme lies must come from the non negativity constraints.
So any extreme point must have atleast three components which are equal to 0.
1. $\mathbf{0}$ is an extreme point of $Fea(P)$, since $\mathbf{b} \geq \mathbf{0}$.
2. Consider extreme points of $Fea(P)$ with **only** one nonzero component.
Since the first component of $\mathbf{b}$ (say $\mathbf{b}_1$) is equal to 0, check the only type of feasible points with exactly 4 components equal to 0 is $x_1 = x_3 = x_4 = x_5 = 0$, and $x_2 > 0$. Note that $[0, \frac{3}{2}, 0, 0, 0]^T$ is an extreme point of $Fea(P)$.
3. For the extreme points which have exactly two components nonzero (because of the first row of A, and $\mathbf{b}_1 = 0$) the only possibilities are the following:
(i) $x_1 > 0, x_2 > 0$, other components equal to 0,
(ii) $x_3 > 0, x_2 > 0$, other components equal to 0,
(ii)$x_4 > 0, x_2 > 0$, other components equal to 0,
(iv) $x_5 > 0, x_2 > 0$, other components equal to 0.
Of which one can check that feasible points satisfying the condition $x_1 > 0, x_2 > 0$, other components equal to 0, are not extreme points, hence the total number of extreme points of $Fea(P)$ is ≤ 5.

(c) If (P') is obtained from (P) by changing the second column $\mathbf{a}_2$ of A to $\mathbf{a}_2 + [1, -1]^T$ (everything else in (P) and (P') are same) and (P') has an optimal solution then if possible give the optimal value of (P').

Solution: When the second column of A is changed as above, the second column of the changed matrix call it A' becomes $[0, -\frac{1}{3}]^T$ and the first row of A' is $[1, 0, 3, 2, 1]$.
Now consider the entries of the first row of A'. Since $\mathbf{b}_1 = 0$, any feasible $\mathbf{x}$ of (P'), must have $x_1 = 0, x_3 = 0, x_4 = 0, x_5 = 0$ or it should be of the form $[0, x_2, 0, 0, 0]^T$, $x_2 \geq 0$.
Hence $\mathbf{c}^T\mathbf{x} = c_2x_2$ for all feasible $\mathbf{x} \in Fea(P')$, where c_2 is the second component of $\mathbf{c}$.
Check that **all** $\mathbf{x}$ of the form $[0, x_2, 0, 0, 0]^T$, $x_2 \geq 0$ is feasible for (P') (or $[0, 1, 0, 0, 0]^T$ is a direction for $Fea(P')$).
Since given (P') has an optimal solution, the component c_2 of $\mathbf{c}$ must be non negative ($\mathbf{c}^T\mathbf{d} \geq 0$ implies $c_2 \geq 0$).
Hence $\mathbf{c}^T\mathbf{x} = c_2x_2 \geq 0$ for all feasible $\mathbf{x} \in Fea(P')$. Since $\mathbf{0} \in Fea(P')$ the optimal value is 0 and $\mathbf{0}$ is an optimal solution of (P').

[9]