Plan

- Basic Feasible Solution (BFS)
- Non degenerate BFS
- Degenerate BFS
- Simplex Algorithm

- Consider LPP(P)
  Max or Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{m\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $\qquad (*)$
  where $rank(A) = m$.

  $Fea(P) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.

- If (P) is of the form:

  Max or Min $\mathbf{c}^T\mathbf{x}$

  subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$,

  we add some variables and we can transform the set of constraints to get,

  $$A_{m\times n}\mathbf{x} + \mathbf{s}_{m\times 1} = [A : I]\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}_{m\times 1}, \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}.$$

  So it is now of the form of problem (*).
- If suppose we are given a problem of the type:
  Max or Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{k\times n}\mathbf{x} = \mathbf{b}_{k\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
  where $k > rank(A) = m$.

Basic Feasible Solution

- If $A_{k\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$ is consistent then throw away $k - m$ rows of A and the corresponding components of $\mathbf{b}$ to get an equivalent (having the same set of solutions) system of equations of the form (*).
- An $\mathbf{x} \in Fea(P)$ is called a **basic feasible solution** (**BFS**) of the LPP if the columns of the matrix A corresponding to the nonzero components of $\mathbf{x}$ are LI.
- An $\mathbf{x}$ satisfying the system $A\mathbf{x} = \mathbf{b}$, and the condition that the columns corresponding to the nonzero components are LI, is called a **basic solution** of the LPP.
- So a basic solution may not be a nonnegative vector, hence need not be a feasible solution of the LPP.
- A basic feasible solution of a LPP of the form (1), can have at most m strictly positive components (since $rank(A) = m$).

- **Example 1:** Consider the LPP (P)
  Max or Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{2\times 3}\mathbf{x} = \mathbf{b}_{2\times 1}$, $\mathbf{x} \geq \mathbf{0}$
  where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- Clearly $rank(A) = 2$.
- $\mathbf{x} = [1, 1, 0]^T$ is a **BFS** of (P) since it satisfies the conditions
  $A_{2\times 3}\mathbf{x} = \mathbf{b}_{2\times 1}$, $\mathbf{x} \geq \mathbf{0}$
  and the columns of A corresponding to the positive components of **x** are
  $[1, 1]^T$, $[2, 1]^T$, is **LI**.
- The basis matrix has columns $[1, 1]^T$,$[2, 1]^T$, basic variables are x_1, x_2.

- $\mathbf{x} = [2, 1, 1]^T$ is a **feasible** solution of (P) since it satisfies the conditions
  $A_{2\times3}\mathbf{x} = \mathbf{b}_{2\times1}$, $\mathbf{x} \geq \mathbf{0}$
  but the columns of A corresponding to the positive components of **x** are
  $[1, 1]^T$, $[2, 1]^T$, $[-1, -1]^T$ is **LD**,hence x is **not** a BFS.
- $\mathbf{x} = [0, 1, -1]^T$ is a **basic solution** of (P) since it satisfies the conditions
  $A_{2\times3}\mathbf{x} = \mathbf{b}_{2\times1}$
  and the columns of A corresponding to the nonzero components of **x** are
  $[2, 1]^T$, $[-1, -1]^T$, is **LI**.
  But **x** is **not** a feasible solution of (P).

Non degenerate BFS

- A basic feasible solution is called a **non degenerate BFS** if it has exactly m positive components, otherwise it is said to be a **degenerate** BFS.
- If $\mathbf{x}$ is a **non degenerate BFS**, then the columns of A corresponding the nonzero (positive) components of $\mathbf{x}$ form a basis of $\mathbb{R}^m$.
- If the positive components of $\mathbf{x}$ are $x_1, \ldots, x_m$, then $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$, forms a basis of $\mathbb{R}^m$, where $\tilde{\mathbf{a}}_i$ is the ith column of A.
- The $m \times m$ matrix, $B_{m\times m} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$ formed with columns $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$, of A is called the **basis matrix** corresponding to $\mathbf{x}$.
- The variables $x_1, \ldots, x_m$ are called the **basic variables**, and $x_{m+1} = x_{m+2} = \ldots = x_n = 0$, are called the **non basic variables** of the **BFS** $\mathbf{x}$.

Degenerate BFS

- If $\mathbf{x}$ is a **degenerate** BFS, then consider the columns of A corresponding to the nonzero components of $\mathbf{x}$. Let them be $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, $k < m$.
  Then consider $(m-k)$ LI columns of A, such that these $(m-k)$ columns of A together with $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, form a basis of $\mathbb{R}^m$.
- The matrix $B_{m\times m}$ formed with these m columns, (let $B_{m\times m} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$) is called **a** basis matrix corresponding to $\mathbf{x}$.
- The components $x_1, \ldots, x_m$ of $\mathbf{x}$ are called **basic variables** corresponding to $\mathbf{x}$ and the basis matrix B. The components $x_{m+1} = \ldots = x_n = 0$ are the **nonbasic variables**.

- **Example 2:** Consider the LPP (P)
  Max or Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{2\times 3}\mathbf{x} = \mathbf{b}_{2\times 1}$, $\mathbf{x} \geq \mathbf{0}$
  where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Clearly $rank(A) = 2$.
- $\mathbf{x} = [0, 1, 1]^T$ is a non degenerate **BFS** of (P) since it satisfies the conditions
  $A_{m\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$
  and the columns of A corresponding to the positive components of **x** are
  $[2, 1]^T, [-1, 0]^T$, is LI.
  Also the number of positive components of **x** is equal to $2 = rank(A)$.
- The basis matrix is unique upto a permutation of the columns.
  The basic variables are given by x_2, x_3.

Example 2 (continued)

- $\mathbf{x} = [1, 0, 0]^T$ is also a BFS of (P) since it satisfies the conditions
  $A_{2\times 3}\mathbf{x} = \mathbf{b}_{3\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
  the columns of A corresponding to the positive components of $\mathbf{x}$ is
  $[1, 1]^T$, is **LI**.
  But is a degenerate BFS since the number of positive components of $\mathbf{x}$ is strictly less 2.
- There are two different basis matrices B for $\mathbf{x}$.
  (i) B with columns $[1, 1]^T$,$[2, 1]^T$, **basic variables** x_1, x_2.
  (ii) B' with columns $[1, 1]^T$,$[-1, 0]^T$, **basic variables** x_1, x_3.

- Depending on the choice of the rest of the $(m-k)$ columns which are added, the same degenerate BFS $\mathbf{x}$ will correspond to different basis matrices B, and hence will have different **basic** and **nonbasic** variables.
- If $\mathbf{x}$ is a **BFS** then $\mathbf{x}_B = B^{-1}\mathbf{b}$, where $\mathbf{x}_B$ are the components of $\mathbf{x}$ corresponding to the basic variables.
- **BFS** $\mathbf{x}$ is of the form, $\mathbf{x} = \begin{bmatrix} B^{-1}\mathbf{b}_{m\times 1} \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix}$.
- Corresponding to **BFS** $\mathbf{x}$
  $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$,
  where, $\mathbf{c}_B^T$ are the components of $\mathbf{c}^T$ corresponding to the basic variables.
- If LPP (P) is the diet problem, (with $\geq$ inequalities changed to equalities in the original problem),
  then $\mathbf{x}$ gives the quantities of the food products F_j,
  $j = 1, \ldots, n$ in the diet.

- The food products $F_1, \ldots, F_m$, which correspond to the basic variables $x_1, x_2, \ldots, x_m$ of **x**, are the ones which are included in the diet in the quantities x_j.
- The other food products corresponding to the nonbasic variables of **x** (they have zero values) are **not** consumed.
- Let us assume that (P*) has atleast one **feasible solution** and let **x** be a BFS (degenerate or non degenerate) of the LPP (P*).
- Let $B = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$ be **a** (or **the**) basis matrix corresponding to **x**.
- For all $k = 1, 2, \ldots, n$, let

$$\tilde{\mathbf{a}}_k = \sum_{i=1}^{m} u_{ik}\tilde{\mathbf{a}}_i = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} \qquad \textbf{(*)}$$

  for some real u_{ik}'s, $i = 1, \ldots, m$, $k = 1, \ldots, n$,
- Then $\begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = B^{-1}\tilde{\mathbf{a}}_k$ for all $k = 1, 2, \ldots, n$.

- In order to obtain the same amount of nutrient as unit amount of F_k, $k = 1, \ldots, n$, one needs to consume (u_{1k} amount of F_1)+ (u_{2k} amount of F_2)+...+ (u_{mk} amount of F_m).
- Let z_k be the value of unit amount of F_k, if we include **only** $F_i, i = 1, \ldots, m$ in the diet.
- Then $z_k = u_{1k}c_1 + u_{2k}c_2 + \ldots + u_{mk}c_m = \sum_{i=1}^{m} u_{ik}c_i$
$$= [c_1, \ldots, c_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = \mathbf{c}_B^T B^{-1} \tilde{\mathbf{a}}_k.$$
- For all $k = 1, \ldots, m$, note that $z_k = c_k$.
- If $z_k > c_k$ then look for a better solution.

- the simplex table corresponding to BFS $\mathbf{x}$ is given by

	0	..	0	..	$c_s - z_s$	..	$c_k - z_k$	..	
	$B^{-1}\tilde{\mathbf{a}}_1$	..	$B^{-1}\tilde{\mathbf{a}}_m$	..	$B^{-1}\tilde{\mathbf{a}}_s$	..	$B^{-1}\tilde{\mathbf{a}}_k$	..	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	1	..	0	..	u_{1s}	..	u_{1k}	..	x_1
$\tilde{\mathbf{a}}_2$	0	..	0	..	u_{2s}	..	u_{2k}	..	x_2
$\vdots$	0	..	0	..	$\vdots$	..	$\vdots$	..	$\vdots$
$\tilde{\mathbf{a}}_r$	$\vdots$	..	$\vdots$	..	u_{rs}	..	u_{rk}	..	x_r
$\vdots$	$\vdots$	..	$\vdots$	..	$\vdots$	..	$\vdots$	..	$\vdots$
$\tilde{\mathbf{a}}_m$	0	..	1	..	u_{ms}	..	u_{mk}	..	x_m

- **Case 1:** $c_k - z_k < 0$ for atleast one k, $k = m+1, \ldots, n$.
  Let $c_s - z_s = min\{c_k - z_k : c_k - z_k < 0, \;\; k = m+1, \ldots, n\}$.
  Simplex algorithm says that, include x_s in the diet or make x_s a basic variable so the column $\tilde{\mathbf{a}}_s$ will be included in the basis matrix.
- If there exists, s, l, such that for both s, l,
  $c_s - z_s = c_l - z_l = min\{c_k - z_k : c_k - z_k < 0, \;\; k = m+1, \ldots, n\}$,
  then include any **one** of these food products in the diet(as a basic variable).
- Let $\mathbf{x}'$ be the new solution to be obtained.
  The components of $\mathbf{x}'$ are given by
  $x'_i = x_i - u_{is}x'_s$ for $i = 1, \ldots, m$,
  $x'_s \geq 0$ and
  $x'_i = 0$ for $i = m+1, \ldots, n, i \neq s$.

- If $\mathbf{x}'$ is feasible for the LPP, then $x'_s \geq 0$ is such that $x'_i \geq 0$ for all $i = 1, \ldots, m$.
- If $u_{is} \leq 0$, for some $i = 1, \ldots, m$, then for all $x'_s \geq 0$, $x'_i \geq 0$ for that i.
- Hence $x'_s \geq 0$, should be such that $x'_s \leq \frac{x_i}{u_{is}}$, $u_{is} > 0$ (if at all there is one such i).
- **Case 1a:** For some (atleast one) $i = 1, \ldots, m$, $u_{is} > 0$ (where s is as defined in **Case 1**).
  Let $\frac{x_r}{u_{rs}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$.
- $\frac{x_r}{u_{rs}}$ is called the **minimum ratio**.
- $\mathbf{x}' \geq \mathbf{0} \Rightarrow x'_s \leq \frac{x_r}{u_{rs}}$.
- Also $A\mathbf{x}' = \mathbf{b}$ which implies $\mathbf{x}' \in Fea(LPP)$.

- $\mathbf{c}^T\mathbf{x}' = \sum_{i=1}^{m} c_i x_i' + c_s x_s'$
  $= \mathbf{c}^T\mathbf{x} + x_s'(c_s - z_s)$,
  $\Rightarrow \mathbf{c}^T\mathbf{x}' \leq \mathbf{c}^T\mathbf{x}$.
- $\mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x}$ if $x_s' > 0$, which is when the **minimum ratio** $\frac{x_r}{u_{rs}} > 0$.
- Let $x_s' = \frac{x_r}{u_{rs}}$.
- Then $x_r' = x_r - u_{rs}\frac{x_r}{u_{rs}} = 0$
- x_s is called the **entering variable**, and x_r is called a **leaving variable**.
- If there exists $r, t \in \{m+1, \ldots, n\}$
  $\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$,
  then take any **one** of r, t as the **leaving variable**.

- $\mathbf{x}'$ is a **BFS** of the LPP.
- The basis matrix corresponding to $\mathbf{x}'$ is $B' = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_{r-1} \tilde{\mathbf{a}}_{r+1} \dots \tilde{\mathbf{a}}_m \tilde{\mathbf{a}}_s]$.
- $\tilde{\mathbf{a}}_k = \sum_{i=1, i\neq r}^{m} (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_s \frac{u_{rk}}{u_{rs}}$.
- For all $k = 1, \dots, n$ the new u_{ik}'s are given by, $u'_{ik} = u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}$ for $i = 1, \dots, r-1, r+1, m$ and $u'_{sk} = \frac{u_{rk}}{u_{rs}}$.
- $\mathbf{b} = \sum_{i=1, i\neq r}^{m} (x_i - \frac{u_{is}}{u_{rs}} x_r) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_s (\frac{x_r}{u_{rs}})$
- If z'_k denotes the new values of z_k then $z'_k = z_k + \frac{(c_s - z_s)}{u_{rs}} u_{rk}$ and $c_k - z'_k = (c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$.

- The simplex table corresponding to the new BFS $\mathbf{x}'$ is given by

	$c_s - z'_s$ $c_s - z_s - \frac{(c_s - z_s)}{u_{rs}} u_{rs} = 0$	$c_k - z'_k$ $(c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$	
	$B'^{-1}\tilde{\mathbf{a}}_s$	$B'^{-1}\tilde{\mathbf{a}}_k$	$B'^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	$u_{1s} - \frac{u_{1s}}{u_{rs}} u_{rs} = 0$	$u_{1k} - \frac{u_{1s}}{u_{rs}} u_{rk}$	$x_1 - \frac{u_{1s}}{u_{rs}} x_r$
$\tilde{\mathbf{a}}_2$	$u_{2s} - \frac{u_{2s}}{u_{rs}} u_{rs} = 0$	$u_{2k} - \frac{u_{2s}}{u_{rs}} u_{rk}$	$x_2 - \frac{u_{2s}}{u_{rs}} x_r$
$\vdots$	0	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_r$	$\frac{u_{rs}}{u_{rs}} = 1$	$\frac{u_{rk}}{u_{rs}}$	$\frac{x_r}{u_{rs}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_m$	$u_{ms} - \frac{u_{ms}}{u_{rs}} u_{rs} = 0$	$u_{mk} - \frac{u_{ms}}{u_{rs}} u_{rk}$	$x_m - \frac{u_{ms}}{u_{rs}} x_r$

- The entry u_{rs} of the previous table which is made 1 (by dividing) in this table is called the **pivot element**.

- **Case 1b:** For all $i = 1, \dots, m$, $u_{is} \leq 0$ (where s is as defined in **Case 1**).
- Then $\mathbf{x}' \geq \mathbf{0}$, for all $x'_s \geq 0$,
  hence $\mathbf{x}' \in Fea(LPP)$ for all $x'_s \geq 0$.
- Since $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + x'_s(c_s - z_s)$, $(c_s - z_s) < 0$.
  So given any $M \in \mathbb{R}$, by taking x'_s sufficiently large we can make $\mathbf{c}^T\mathbf{x}'$ smaller than M.
  So the LPP does **not** have an optimal solution.

- The set of all directions of $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, rank(A) = m\}$, is given by $D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq 0, A_{m\times n}\mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}\}$
- If for some basis matrix B and a column $\tilde{\mathbf{a}}_s$ of A, $B^{-1}\tilde{\mathbf{a}}_s \leq \mathbf{0}$ then

$$\mathbf{d}=\begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  is an extreme direction of S, where the entry **1** in the above vector is at the s th position.
- If $\mathbf{d}$ is as defined above then $\mathbf{c}^T\mathbf{d} < 0$.
- **Case 2:(Optimality Condition)** (sufficient condition) If $c_k - z_k \geq 0$ for all $k = 1, \ldots, n$. Then $\mathbf{x}$ is optimal for the LPP.

- If instead the LPP would have been a maximization problem as given below:
  Max $\mathbf{c}^T\mathbf{x}$
  subject to
  $A_{m\times n}\mathbf{x} = \mathbf{b}$,
  $\mathbf{x} \geq \mathbf{0}$, $rank(A) = m$
- **Case 1':** $c_k - z_k > 0$ for at least one k, $k = m+1, \ldots, n$.
  s th variable will be the entering variable if
  $c_s - z_s$=max$\{c_k - z_k : c_k - z_k > 0, \ \ k = m+1, \ldots, n\}$.
- **Case 1a:** remains unchanged.
- **Case 2':** (**Optimality condition** (sufficient condition)) If $c_k - z_k \leq 0$ for all $k = 1, \ldots, n$.
- The above optimality conditions (for max and min problems) are sufficient but not necessary for an optimal solution.