Plan

- Examples of optimization problems
- Examples of Linear Programming Problems (LPP)
- Solution of an LPP by Graphical Method
- Extreme points and Corner points
- Exercises and Questions

Optimization Problems

- Nothing takes place in the world whose meaning is not that of some minimum or maximum: L. Euler
- Heron's problem: 'On Mirrors' (book related to laws of reflection of light, approx 1st century AD).
  A and B are two given points on the same side of a line L.
  Find a point D on L such that the sum of the distances from A to D and from B to D is a minimum.
  Nature breaks a ray of light at equal angles, if it does not unnecessarily want it to meander to no purpose.
- Extremal principles of nature: Laws of refraction of light.
  What characterizes the trajectory of light moving from one point to another in a non homogeneous medium is that it is traversed in a minimum time.

- Isoperimetric problem, Oldest version: ‘Aenid’ of Vergil. Escaping from her brother, the Phoenician princess Dido set off westward in search of a safe place to settle down. She liked a certain place now known as ‘Bay of Tunis’. Dido asked the local leader Yarb for as much land as could be ‘encircled with a bull’s hide’(9th century BC).
- Isoperimetric problem: Among all closed plane curves of a given length(perimeter), find the one that encloses the maximum area. Answer: Circle
- Isoepiphanic property of the Sphere: The Sphere encloses the largest volume among all closed surfaces with the same surface area.
- Solved by Archimedes, Schwartz.
- Easier problem: Among all rectangles of a given perimeter (say 20), find the one that encloses the maximum area.

  Maximize ac

  subject to, $a + c = 10,$

  $$a \geq 0, c \geq 0.$$

  Answer: Square

Linear Programming Problems

Diet Problem:

- Let there be m nutrients $N_1, N_2, ..., N_m$ and n food products, $F_1, F_2, ..., F_n$, available in the market which can supply these nutrients.

  For healthy survival a human being requires say atleast, b_i units of the i th nutrient, $i = 1, 2, \ldots, m$, respectively.

  Let a_{ij} be the amount of the i th nutrient (N_i) present in unit amount of the j th food product (F_j), and let $c_j, j = 1, 2, ..., n$ be the cost of unit amount of F_j.

  So the problem is to decide on a diet of minimum cost consisting of the n food products (in various quantities) so that one gets the required amount of each of the nutrients.

Formulation of the Diet Problem

Let x_j be the amount of the jth food product to be used in the diet (so $x_j \geq 0$ for $j = 1, 2, .., n$,) and the problem reduces to (under certain simplifying assumptions):

Min $\sum_{j=1}^{n} c_j x_j = \mathbf{c}^T \mathbf{x}$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, for $i = 1, 2, ..., m$,

$x_j \geq 0$ for all $j = 1, 2, \ldots, n$.

or as

$A\mathbf{x} \geq \mathbf{b}$, (or alternatively as $-A\mathbf{x} \leq -\mathbf{b}$) $\mathbf{x} \geq \mathbf{O}$,

where A is an $m \times n$ matrix (a matrix with m rows and n columns), the (i, j) th entry of A is given by a_{ij},
$\mathbf{b} = [b_1, b_2, \ldots, b_m]^T$ and $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, $\mathbf{O}$ is zero vector.

Transportation Problem: Let there be m supply stations, $S_1, S_2, .., S_m$ for a particular product (P) and n destination stations, $D_1, D_2, ..., D_n$ where the product is to be transported.

Let c_{ij} be the cost of transportation of unit amount of the product (P) from S_i to D_j.

Let s_i be the amount of the product available at S_i and let d_j be the corresponding demand at D_j.

The problem is to find x_{ij}, $i = 1, 2, .., m$, $j = 1, 2, ..., n$, where x_{ij} is the amount of the product to be transported from S_i to D_j such that the cost of transportation is minimum so that the demands are met.

Formulation of the Transportation Problem

The problem can be modelled as (under certain simplifying assumptions)

Min $\sum_{i,j} c_{ij}x_{ij}$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq s_i, \text{ for } i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \text{ for } j = 1, 2, ..., n,$$

$x_{ij} \geq 0$ for all $i = 1, 2, .., m, \quad j = 1, 2, .., n.$

Note that the constraints of the above LPP can again be written as:

Min $\mathbf{c}^T\mathbf{x}$

subject to $A\mathbf{x} \leq \mathbf{b}$,

$\mathbf{x} \geq \mathbf{0}$,

where A is a matrix with $(m + n)$ rows and $(m \times n)$ columns,

$\mathbf{x}$ is a vector with $m \times n$ components

and $\mathbf{b} = [s_1, \ldots, s_m, -d_1, \ldots, -d_n]^T$.

For example the 1st row of A (the row corresponding to the first supply constraint) is given by
$[1, 1, \ldots, 1, 0, \ldots, 0]^T$
that is 1 in the first n positions and 0's elsewhere.

The second row of A(the row corresponding to the second supply constraint) is given by
$[0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0]^T$
that is 1 in the $(n+1)$ th position to the $2n$ th position and 0's elsewhere.

The mth row of A (the row corresponding to the m th supply constraint) is given by
$[0, \ldots, 0, 1, 1, \ldots, 1]^T$
that is 1 in the $(m-1)n+1$ th position to the mn th position and 0's elsewhere.

The $(m+1)$ th row of A (the row corresponding to the first destination constraint) is given by
$[-1, 0, \ldots, 0, -1, 0, \ldots, 0, -1, 0, \ldots, 0]$,
that is -1 at the first position, -1 at the (n+1)th position, -1 at the (2n+1)th position, ..., -1 at the ((m-1)n +1) th position, etc and 0's elsewhere.

The $(m+n)$ th row of A (the row corresponding to the nth (last) destination constraint) is given by
$[0, \ldots, -1, 0, \ldots, -1, 0, \ldots, -1, 0, \ldots, -1, \ldots, 0, \ldots, -1]$,
that is -1 at the nth position, -1 at the 2n th position, -1 at the 3n th position, ..., -1 at the $(m \times n)$ th position, etc and 0's elsewhere.

Linear Programming Problem

Given $\mathbf{c} \in \mathbb{R}^n$, a column vector with n components, $\mathbf{b} \in \mathbb{R}^m$, a column vector with m components, and an $A \in \mathbb{R}^{m \times n}$, a matrix with m rows and n columns

A linear programming problem(LPP) is given by :
Max or Min $\mathbf{c}^T \mathbf{x}$

subject to $A\mathbf{x} \leq \mathbf{b}$ (or $A\mathbf{x} \geq \mathbf{b}$),

$\mathbf{x} \geq \mathbf{0}$.

The function $f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$ is called the objective function, the constraints $\mathbf{x} \geq \mathbf{0}$ are called the non negativity constraints.

Note that the above problem can also be written as:
Max or Min $\mathbf{c}^T\mathbf{x}$

$\mathbf{a}_i^T\mathbf{x} \leq b_i$ for $i = 1, 2, \ldots, m$,

$x_j \geq 0$ for $j = 1, 2, \ldots, n$, or $-\mathbf{e}_j{}^T\mathbf{x} \leq 0$ for all $j = 1, 2, \ldots, n$,

where $\mathbf{a}_i^T$ is the i th row of the matrix A, and $\mathbf{e}_j$ is the j th column of the identity matrix of order n, I_n.

Note that each of the functions

$\mathbf{a}_i{}^T\mathbf{x}$, for $i = 1, 2 \ldots, m$,

$-\mathbf{e}_j{}^T\mathbf{x}$, for $j = 1, 2, \ldots, n$,

and $\mathbf{c}^T\mathbf{x}$ are all linear functions from $\mathbb{R}^n \to \mathbb{R}$, hence the name **linear programming problem**.

Linear function, Feasible solution, Optimal Solution

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear map (linear function, linear transformation) if it satisfies the following:

  $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

  and $T(\alpha \mathbf{x}) = \alpha T(\mathbf{x})$ for all $\alpha \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$.
- An $\mathbf{x} \geq \mathbf{0}$ satisfying the constraints $A\mathbf{x} \leq \mathbf{b}$ (or $A\mathbf{x} \geq \mathbf{b}$) is called a feasible solution of the linear programming problem (LPP).
- The set of all feasible solutions of a LPP is called the feasible solution set or the feasible region of the LPP.

  Hence the feasible region of a LPP, denoted by Fea(LPP)is given by,
  Fea(LPP)$= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, A\mathbf{x} \leq \mathbf{b}\}$.

- Fea(LPP)$= \{\mathbf{x} \in \mathbb{R}^n : x_j \geq 0, j = 1, \ldots, n, \quad \mathbf{a}_i^T\mathbf{x} \leq b_i,\ i = 1, \ldots, m\}$
  $= \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{e}_j^T\mathbf{x} \leq 0,\ j = 1, \ldots, n, \quad \mathbf{a}_i^T\mathbf{x} \leq b_i,\ i = 1, \ldots, m\}$,
  where $\mathbf{a}_i^T$ is the i th row of the matrix A, and $\mathbf{e}_j$ is the jth standard unit vector, or the j th column of the identity matrix I_n.
- A feasible solution of an LPP is called an optimal solution if it minimizes or maximizes the objective function (depending on the nature of the problem). The set of all optimal solutions is called the optimal solution set of the LPP.
- If the LPP has an optimal solution, then the value of the objective function $\mathbf{c}^T\mathbf{x}$ where $\mathbf{x}$ is an optimal solution of the LPP is called the optimal value of the LPP.
- **Example:** Consider another problem,
  Min $-x + 2y$
  subject to
  $x + 2y \geq 1,\ -x + y \leq 1,\ 2x + 4y \leq 4, \quad x \geq 0, y \geq 0.$

Hyperplanes, Normals, Closed Half Spaces, Polyhedral set

- A subset H of $\mathbb{R}^n$ is called a hyperplane if it can be written as:
  H$= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = d\}$ for some $\mathbf{a} \in \mathbb{R}^n$ and $d \in \mathbb{R}$, or equivalently as

  H$= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) = 0\}$ for some $\mathbf{a} \in \mathbb{R}^n$, $d \in \mathbb{R}$, and $\mathbf{x}_0$ satisfying $\mathbf{a}^T\mathbf{x}_0 = d$. (**)
- So geometrically a hyperplane in $\mathbb{R}$ is just an element of $\mathbb{R}$ (a single point), in $\mathbb{R}^2$ it is just a straight line,

  in $\mathbb{R}^3$ it is just the usual plane we are familiar with.
- The vector $\mathbf{a}$ is called a normal to the hyperplane H, since it is orthogonal (or perpendicular) to each of the vectors on the hyperplane starting from (with tail at) $\mathbf{x}_0$.

- A collection of hyperplanes $H_1, \ldots, H_k$ in $\mathbb{R}^n$ is said to be **Linearly Independent (LI)** if the corresponding normal vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent as vectors in $\mathbb{R}^n$.
- Otherwise the collection of hyperplanes is said to be **Linearly Dependent (LD)**.
  Hence any set of $(n+1)$ hyperplanes in $\mathbb{R}^n$ is LD.
- Associated with the hyperplane H are two closed half spaces
  $H_1 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq d\}$ and
  $H_2 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \geq d\}$.

  Note that the hyperplane H, and the two half spaces H_1, H_2 are all closed subsets of $\mathbb{R}^n$ (since each of these sets contains all its boundary points, the boundary points being $\mathbf{x} \in \mathbb{R}^n$ satisfying the condition $\mathbf{a}^T\mathbf{x} = d$).

- A set which is the intersection of a finite number of closed half spaces is called a **polyhedral set**.
  Hence the feasible region of a LPP is a **polyhedral set**.
- Since the intersection of any collection of closed subsets of $\mathbb{R}^n$ is again a closed subset of $\mathbb{R}^n$, hence Fea(LPP) is always a **closed** subset of $\mathbb{R}^n$ (also geometrically you can see (or guess) that the feasible region of a LPP contains all its boundary points).
- The hyperplanes $\mathbf{a}_i^T\mathbf{x} = b_i$ for $i = 1, 2, \ldots, m$,
  $x_j = 0$ for $j = 1, 2, \ldots, n$, or $-\mathbf{e}_j{}^T\mathbf{x} = 0$ for all $j = 1, 2, \ldots, n$
  associated with an LPP are called its defining hyperplanes.
- The associated half spaces are
  $\mathbf{a}_i^T\mathbf{x} \leq b_i$, $i = 1, 2, \ldots, m$,
  $x_j \geq 0$, $j = 1, 2, \ldots, n$, or $-\mathbf{e}_j{}^T\mathbf{x} \leq 0$, $j = 1, 2, \ldots, n$.

Solution by graphical method of LPP's in two variables

- **Example 1:** Given the linear programming problem
  Max $5x + 2y$
  subject to
  $3x + 2y \leq 6$
  $x + 2y \leq 4$
  $x \geq 0, y \geq 0.$

  The optimal solution to be $x = 2$ and $y = 0$, which happened to be a corner point of the feasible region. Later we will again convince ourselves (by using more rigor) that this is indeed the optimal solution. The optimal value for this problem is 10.

- **Example 2:** Consider another problem,
  Min $-x + 2y$
  subject to

  $x + 2y \geq 1$
  $-x + y \leq 1,$
  $x \geq 0, y \geq 0.$

  The above linear programming problem does not have an optimal solution.
- **Example 3:** Note that in the above problem keeping the feasible region same, if we just change the objective function to Min $2x + y$, then the changed problem has a unique optimal solution.

- **Example 4:** Also in **Example 2** if we change objective function to Min $x + 2y$
  then the changed problem has infinitely many optimal solutions, although the set of optimal solutions is bounded.
- **Example 5:** Note that in **Example 2** keeping the feasible region same, if we just change the objective function to Min y, then the changed problem has infinitely many optimal solutions and the optimal solution set is unbounded.
- **Example 6:** Max $-x + 2y$
  subject to
  $x + 2y \leq 1$
  $-x + y \geq 1,$
  $x \geq 0,\ y \geq 0.$
  Clearly the feasible region of this problem is the empty set. So this problem is called **infeasible**, and since this problem does not have a feasible solution it obviously does not have an optimal solution.

- Question 1: Can there be exactly 2, 5, or say exactly 100 optimal solutions of a LPP?

  And do the set of optimal solutions have some nice geometric structure, that is if we have two optimal solutions then what about points in between and on the line segment joining these two solutions?
- Question 2: Is the set of optimal solutions of an LPP a convex set?

  That is, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two optimal solutions of a LPP, then are $\mathbf{y}$'s of the form $\mathbf{y} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$, $0 \leq \lambda \leq 1$, also optimal solutions of the LPP?
- A nonempty set $S \subseteq R^n$ is said to be a convex set if for all $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in S$, for all $0 \leq \lambda \leq 1$.
- $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$, $0 \leq \lambda \leq 1$ is called a convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.
- If in the above expression, $0 < \lambda < 1$, then the convex combination is said to be a strict convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

- Let us first try to answer **Question 2**.

  If the answer to this question is a YES then that would imply that if a LPP has more than one optimal solution then it should have infinitely many optimal solutions, so the answer to **Question 1** would be a NO.

  Answer to Question 2 is YES, that is the set of optimal solutions is indeed a convex set.

- **Question 3:** If the feasible region of a LPP is a nonempty, bounded set then does the LPP always have an optimal solution?

  The answer to this question is yes, due to a result by Weierstrass, called the Extreme Value Theorem given below:

- **Extreme Value Theorem:** If S is a nonempty, closed, bounded subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continuous, then f attains both its minimum and maximum value in S.

- **Question 4:** Whenever a LPP has an optimal solution does there always exist at least one corner point (points lying at the point of intersection of at least two distinct lines in case $n = 2$), at which the optimal value is attained?
- Given a LPP with a nonempty feasible region, $Fea(LPP) = S \subset \mathbb{R}^n$, an element $\mathbf{x} \in S$ is said to be an corner point of S, if $\mathbf{x}$ lies at the point of intersection of n linearly independent hyperplanes defining S.
- An $\mathbf{x} \in Fea(LPP)$ which does not lie on any of the defining hyperplanes of $Fea(LPP)$ is an interior point of $Fea(LPP)$
- An $\mathbf{x} \in Fea(LPP)$ which lies in atleast one defining hyperplane of the $Fea(LPP)$ is a boundary point of $Fea(LPP)$.

- Note that the way we have written our feasible region S, it has $(m+n)$ defining hyperplanes.
- Also, note that the **corner points** of the feasible region of the LPP cannot be written as a strict convex combination of two distinct points of the feasible region, or in other words those are all **extreme points** of the feasible region.
- Given a nonempty convex set, $S \subset \mathbb{R}^n$, an element $\mathbf{x} \in S$ is said to be an **extreme point** of S if $\mathbf{x}$ cannot be written as a strict convex combination of two distinct elements of S. That is if

  $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, for some $0 < \lambda < 1$ and $\mathbf{x}_1, \mathbf{x}_2 \in S$, then $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$.

- Theorem: If $S = Fea(LPP)$ is nonempty, where $Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{a}_i^T\mathbf{x} \leq b_i \text{ for all } i = 1, \ldots, m, \quad -\mathbf{e}_j^T\mathbf{x} \leq 0 \text{ for all } j = 1, \ldots, n\}$
  then $\mathbf{x} \in S$ is a corner point of S if and only if it is an extreme point of S.
  Proof: Key points
- Remark: From the above equivalent definition of extreme points of the feasible region of a LPP it is clear that the total number of extreme points of the feasible region is $\leq (m+n)_{C_n}$.

- Exercise: Think of a LPP such that the number of extreme points of the $Fea(LPP)$, is equal to that given by the upper bound.
- Exercise: If possible give an example of a LPP with $(m+n)$ constraints (including the non negativity constraints) such that the number of extreme points of the $Fea(LPP)$ is strictly greater than $(m+n)$ and is strictly less than $(m+n)_{C_n}$.
- Exercise: If possible give an example of a LPP with $(m+n)$ constraints (including the nonnegativity constraints) such that the number of extreme points of the $Fea(LPP)$ is strictly less than $(m+n)$.
- Question: Does the feasible region of a LPP (where the feasible region is of the form given before) always have an extreme point?

  Answer: **Yes**, we will see this later.
- Exercise: Think of a convex set defined by only one hyperplane. Will it have any extreme point?