# Exercises and solutions: *Matrix multiplication*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

## Exercises

**1.** Determine whether each of the following operations is valid, and, if so, the size of the resulting matrix (note that $\odot$ indicates Hadamard multiplication).
$\mathbf{A} \in \mathbb{R}^{2\times3}, \quad \mathbf{B} \in \mathbb{R}^{3\times3}, \quad \mathbf{C} \in \mathbb{R}^{3\times4}$

**a)** $\mathbf{CB}$
**b)** $\mathbf{C^TB}$
**c)** $(\mathbf{CB})^\mathsf{T}$
**d)** $\mathbf{C^TBC}$
**e)** $\mathbf{ABCB}$
**f)** $\mathbf{ABC}$
**g)** $\mathbf{C^TBA^TAC}$
**h)** $\mathbf{B^TBCC^TA}$
**i)** $\mathbf{AA^T}$
**j)** $\mathbf{A^TA}$
**k)** $\mathbf{BBA^TABBCC}$
**l)** $(\mathbf{CBB^TCC^T})^\mathsf{T}$
**m)** $(\mathbf{A}+\mathbf{ACC^TB})^\mathsf{T}\mathbf{A}$
**n)** $\mathbf{C}+\mathbf{CA^TABC}$
**o)** $\mathbf{C}+\mathbf{BA^TABC}$
**p)** $\mathbf{B}+3\mathbf{B}+\mathbf{A^TA}-\mathbf{CC^T}$
**q)** $\mathbf{A}\odot(\mathbf{ABC})$
**r)** $\mathbf{A}\odot\mathbf{ABC}(\mathbf{BC})^\mathbf{T}$

**2.** Compute the following matrix multiplications. For each problem, apply two different methods of matrix multiplication and confirm that they give the same answer.

**a)** $\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}0&5\\2&2\end{bmatrix}$
**b)** $\begin{bmatrix}-3&2\\-2&3\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}$
**c)** $\begin{bmatrix}11&-5\\9&-13\end{bmatrix}\begin{bmatrix}3&1\\-8&.5\end{bmatrix}$
**d)** $\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$
**e)** $\begin{bmatrix}2&2\\1&3\end{bmatrix}\begin{bmatrix}10&1\\-5&4\end{bmatrix}$
**f)** $\begin{bmatrix}a&0\\b&0\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix}$
**g)** $\begin{bmatrix}a&b\\0&0\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix}$
**h)** $\begin{bmatrix}1&0&4\\0&1&1\\3&3&0\end{bmatrix}\begin{bmatrix}-2&-3&-1\\-1&-9&3\\0&1&5\end{bmatrix}$
**i)** $\begin{bmatrix}a&0&1\\0&b&0\\1&0&c\end{bmatrix}\begin{bmatrix}a&b&c\\1&2&3\\0&0&1\end{bmatrix}$

**3.** Compute the following matrix-vector products, if the operation is valid.

**a)** $\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$
**b)** $\begin{bmatrix}2\\3\end{bmatrix}^T\begin{bmatrix}2&0\\0&3\end{bmatrix}$
**c)** $\begin{bmatrix}2&1\\2&3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$
**d)** $\begin{bmatrix}2\\3\end{bmatrix}^T\begin{bmatrix}2&1\\2&3\end{bmatrix}$
**e)** $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$
**f)** $\begin{bmatrix}1&0&1\\0&-4&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\2\\1\end{bmatrix}$
**g)** $\begin{bmatrix}1\\2\\5\end{bmatrix}\begin{bmatrix}1&0&1\\0&-4&0\\1&0&1\end{bmatrix}$
**h)** $\begin{bmatrix}3\\3\\2\end{bmatrix}^T\begin{bmatrix}1&3&2\\6&1&5\\3&5&0\end{bmatrix}$

**4.** For the following pairs of matrices, vectorize and compute the vector dot product, then compute the Frobenius inner product as $tr(\mathbf{A}^\mathsf{T}\mathbf{B})$.

**a)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

**b)** $\begin{bmatrix} 0 & 5 \\ 7 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 13 & 14 \end{bmatrix}$

**c)** $\begin{bmatrix} 4 & -5 & 8 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}, \begin{bmatrix} 4 & -5 & 8 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

**d)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

**e)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

**f)** $\begin{bmatrix} 1 & 1 & 7 \\ 2 & 2 & 6 \\ 3 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

**5.** An $N \times N$ matrix $\mathbf{A}$ has $N^2$ elements. However, because the matrix product $\mathbf{A}^T\mathbf{A}$ is symmetric, not all elements are unique. Create two matrices (one $2 \times 2$ and one $3 \times 3$) that contain non-zero and non-repeating integers, and compute $\mathbf{A}^T\mathbf{A}$ for each matrix. Count the number of total elements and the number of unique elements. Then work out a formula for the number of unique elements in such a matrix.

# Answers

**1.** -

**a)** no
**b)** yes: $4 \times 3$
**c)** no
**d)** yes: $4 \times 4$
**e)** no
**f)** yes: $2 \times 4$
**g)** yes: $4 \times 4$
**h)** no
**i)** yes: $2 \times 2$
**j)** yes: $3 \times 3$
**k)** no
**l)** no
**m)** yes: $3 \times 3$
**n)** no
**o)** yes: $3 \times 4$
**p)** yes: $3 \times 3$
**q)** no
**r)** yes: $2 \times 3$

**2.** -

**a)** $\begin{bmatrix} 0 & 5 \\ 2 & 17 \end{bmatrix}$

**b)** $\begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix}$

**c)** $\begin{bmatrix} 73 & 8.5 \\ 131 & 2.5 \end{bmatrix}$

**d)** $\begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$

**e)** $\begin{bmatrix} 10 & 10 \\ -5 & 13 \end{bmatrix}$

**f)** $\begin{bmatrix} 2a & 3a \\ 2b & 3b \end{bmatrix}$

**g)** $\begin{bmatrix} 2a+4b & 3a+b \\ 0 & 0 \end{bmatrix}$

**h)** $\begin{bmatrix} -2 & 1 & 19 \\ -1 & -8 & 8 \\ -9 & -36 & 6 \end{bmatrix}$

**i)** $\begin{bmatrix} a^2 & ab & ac+1 \\ b & 2b & 3b \\ a & b & 2c \end{bmatrix}$

**3.** -

**a)** $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$

**b)** $\begin{bmatrix} 4 & 9 \end{bmatrix}$

**c)** $\begin{bmatrix} 7 \\ 13 \end{bmatrix}$

**d)** $\begin{bmatrix} 10 & 11 \end{bmatrix}$

**e)** $\begin{bmatrix} b \\ c \\ a \end{bmatrix}$

**f)** $\begin{bmatrix} 6 \\ -8 \\ 6 \end{bmatrix}$

**g)** invalid

**h)** $\begin{bmatrix} 27 & 22 & 21 \end{bmatrix}$

**4.** -

**a)** $a + 2b + 3c + 4d$
**b)** 63
**c)** 135
**d)** $a^2 + b^2 + ca + bd$
**e)** $a^2 + b^2 + c^2 + d^2$
**f)** undefined

**5.** A symmetric matrix can have up to $N(N+1)/2$ unique elements.