Exercises and solutions: *Singular value decomposition*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. For the following matrices, write out the sizes of the $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}^T$ matrices resulting from the SVD of that matrix.

a) $\mathbf{M} \in \mathbb{R}^{4\times5}$ **b)** $\mathbf{M} \in \mathbb{R}^{25\times25}$ **c)** $\mathbf{M} \in \mathbb{R}^{3\times17}$ **d)** $\mathbf{M} \in \mathbb{R}^{9\times4}$

2. For matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}^T$ of the SVD of a matrix, each component $\mathbf{u_i}\sigma_\mathbf{i}\mathbf{v_i}^T$ can be considered one "layer" of the matrix. What is the minimum number of layers that will perfectly reproduce the following matrices (note that this is not necessarily the same as the total number of layers from the full SVD).

a) $\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}$ **b)** $\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix}$ **c)** $\begin{bmatrix}1&2&3\\3&6&9\end{bmatrix}$ **d)** $\begin{bmatrix}1&0&6\\0&3&5\\1&5&0\end{bmatrix}$

3. Write MATLAB or Python code to implement the following experiment:

1. Generate a 2×3 matrix of random numbers.
2. Compute its SVD.
3. Compute two eigendecompositions using the matrix and its transpose.
4. Confirm that the eigenvalues and the singular values match for all three decompositions, as predicted by the math.
5. Plot the eigenvectors and singular vectors in 2D or 3D (as appropriate) to confirm that SVD and transpose+eigendecomposition produce the same eigenspaces.

Answers

1. -

a) $\mathbf{U}$: 4x4, $\mathbf{\Sigma}$: 4x5, $\mathbf{V}$: 5x5
b) $\mathbf{U}$: 25x25, $\mathbf{\Sigma}$: 25x25, $\mathbf{V}$: 25x25
c) $\mathbf{U}$: 3x3, $\mathbf{\Sigma}$: 3x17, $\mathbf{V}$: 17x17
d) $\mathbf{U}$: 9x9, $\mathbf{\Sigma}$: 9x4, $\mathbf{V}$: 4x4

2. -

a) 1 **b)** 1 **c)** 1 **d)** 3

3. Below is MATLAB code. Note that the eigenvectors and singular vectors may be in a different order (you can sort them if you want), but the subspaces match.

```
% the matrix
M = randn(2,3);

% SVD
[U,S,V] = svd(M);

% eigs
[Ue,De] = eig(M*M');
[Ve,Ds] = eig(M'*M);

% plot
figure(1), clf
subplot(121), hold on
plot([0 U(1,1)],[0 U(2,1)],'r','linew',3)
plot([0 U(1,2)],[0 U(2,2)],'k','linew',3)

plot([0 Ue(1,1)],[0 Ue(2,1)],'r--','linew',3)
plot([0 Ue(1,2)],[0 Ue(2,2)],'k--','linew',3)
axis([-1 1 -1 1]*1.5), grid on, axis square

subplot(122), hold on
plot3([0 V(1,1)],[0 V(2,1)],[0 V(3,1)],'r','linew',3)
plot3([0 V(1,2)],[0 V(2,2)],[0 V(3,2)],'k','linew',3)
plot3([0 V(1,3)],[0 V(2,3)],[0 V(3,3)],'b','linew',3)

plot3([0 Ve(1,1)],[0 Ve(2,1)],[0 Ve(3,1)],'r--','linew',3)
plot3([0 Ve(1,2)],[0 Ve(2,2)],[0 Ve(3,2)],'k--','linew',3)
plot3([0 Ve(1,3)],[0 Ve(2,3)],[0 Ve(3,3)],'b--','linew',3)

axis([-1 1 -1 1 -1 1]*1.5), grid on, axis square, rotate3d on
```