Exercises and solutions: *Matrix rank*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. Compute the rank of the following matrices.

a) $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 4 \\ 5 & -220 \\ 5 & 5/2 \end{bmatrix}$ **d)** $\begin{bmatrix} 2 & -4 & 10 \\ 2 & 3 & -4 \\ 4 & 2 & 0 \end{bmatrix}$

2. Compute the rank of the result of the following operations.

$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & -1 & 3 \\ 6 & -7 & 7 \end{bmatrix}$

a) $\mathbf{A}$ **b)** $\mathbf{B}$ **c)** $\mathbf{A}+\mathbf{B}$ **d)** $\mathbf{A}\mathbf{A}^{\mathbf{T}}$
e) $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ **f)** $\mathbf{A}\mathbf{B}^{\mathbf{T}}$ **g)** $(\mathbf{A}\mathbf{B}^{\mathbf{T}})^{\mathbf{T}}$ **h)** $\mathbf{A}\mathbf{A}^{\mathbf{T}}+\mathbf{A}\mathbf{A}^{\mathbf{T}}$

3. For the following $m \times m$ matrices, what value of λ would give each matrix rank $m-1$?

a) $\begin{bmatrix} 1 & 3 \\ 1 & \lambda \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 2 & \lambda \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & \lambda \\ 1 & 0 & 3 \end{bmatrix}$

4. Determine the maximum possible rank of the following operations.

$\mathbf{A} \in \mathbb{R}^{2\times3}, \quad \mathbf{B} \in \mathbb{R}^{3\times3}, \quad \mathbf{C} \in \mathbb{R}^{3\times4}, \quad \mathbf{D} \in \mathbb{R}^{3\times4}$

a) $\mathbf{A}$ **b)** $\mathbf{B}$ **c)** $\mathbf{C}$ **d)** $\mathbf{D}$
e) $\mathbf{C}^{\mathsf{T}}\mathbf{B}$ **f)** $\mathbf{C}^{\mathsf{T}}\mathbf{C}$ **g)** $\mathbf{A}\mathbf{D}$ **h)** $\mathbf{C}\mathbf{D}$
i) $\mathbf{B}+\mathbf{B}$ **j)** $\mathbf{C}+\mathbf{D}$ **k)** $\mathbf{B}\mathbf{A}^{\mathbf{T}}\mathbf{A}\mathbf{C}$ **l)** $\mathbf{B}\mathbf{A}^{\mathbf{T}}\mathbf{A}\mathbf{C}+\mathbf{D}$

Answers

1. -

a) $r = 1$ **b)** $r = 2$ **c)** $r = 2$ **d)** $r = 2$

2. -

a) 2 **b)** 2 **c)** 2 **d)** 2
e) 2 **f)** 2 **g)** 2 **h)** 2

3. -

a) $\lambda = 3$ **b)** $\lambda \neq 0$ **c)** $\lambda \neq 2$ **d)** $\lambda = 0$

4. -

a) 2 **b)** 3 **c)** 3 **d)** 3
e) 3 **f)** 3 **g)** 2 **h)** invalid multiplication
i) 3 **j)** 3 **k)** 2 **l)** 3