Exercises and solutions: *Quadratic form and definiteness*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*. Checking your work on a computer is a recommended second step.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. For the following matrix, compute the quadratic form using the following vectors.

$$\begin{bmatrix} 2 & 5 \\ 5 & -2 \end{bmatrix}$$

a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **c)** $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ **e)** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **f)** $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

2. For the following matrix, compute the *normalized* quadratic form using the following vectors.

$$\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

a) $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$ **b)** $\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$ **c)** $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$ **e)** $\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$ **f)** $\begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

3. A certain 3x3 positive semidefinite square matrix has eigenvalues of 3 and 5. What is the third eigenvalue?

4. A certain 4x4 positive definite square matrix has eigenvalues of 1, -4, and 6. What is the fourth eigenvalue?

Answers

1. -

a) -2 **b)** 184 **c)** 18 **d)** 250 **e)** 2 **f)** -46

2. They're all zero! Are you surprised? Notice that the matrix is skew-symmetric, which means $\mathbf{A} = -\mathbf{A}^T$. Now consider the quadratic form:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T(-\mathbf{A}^T)\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x}$$

Considering the first and third terms, this means that the quadratic form is equal to its negative, for any skew-symmetric matrix and for all x. And the only number that equals its negative is 0.

3. It must be zero.

4. This is a trick question. If the matrix is positive definite, it cannot have a negative eigenvalue!