Agenda

- Average case analysis of quicksort
  - Camouflage for the technique of summation
- The Fibonacci sequence (or Virahanka sequence) is the celebrated equation g(n) = g(n-1) + g(n-2)
  - What is the runtime of the "natural" algorithm
  - Camouflage for the method of generating functions

Generating function

$g_{-1} = g_{-2} = \cdots g_{-n} \cdots = 0$

$\langle g_0, g_1, \cdots g_n \rangle \leftrightsquigarrow G(z) = \sum_{k \geq 0}^{\infty} g_k z^k$

$g_n = g_{n-1} + g_{n-2}$

$\langle \binom{r}{0} \binom{r}{1} \cdots \binom{r}{2} \cdots \binom{r}{n} \cdots \rangle \quad \sum_{k \geq 0} \binom{r}{k} z^k \cdots (1+z)^r$

$G(z)$

$$F(z) = \sum g_k z^k = g_0 + g_1 z + g_2 z^2 + g_3 z^3 \quad (1)$$

$$z F(z) = g_0 z + g_1 z^2 + g_2 z^3 + g_3 z^4 \quad -(2)$$

$$z^2 F(z) = g_0 z^2 + g_1 z^3 + g_2 z^4 \quad -(3)$$

$$F(z)(1 - z - z^2) = g_0 + g_1 z = z$$

$g_0 = 0$
$g_1 = 1$
$g_2 = 1$

$$F(z) = \frac{z}{1 - z - z^2}$$

$$\text{LHS:- } \underline{1 + \alpha z + \alpha^2 z^2 + \cdots} = \frac{1}{1 - \alpha z} \leftrightsquigarrow \alpha^n$$

$$1 + \beta z + \beta^2 z^2 + \cdots = \frac{1}{1 - \beta z} \leftrightsquigarrow \beta^n$$

$$\therefore A\left(\frac{1}{1-\alpha z}\right) + B\left(\frac{1}{1-\beta z}\right) \leftrightsquigarrow A\alpha^n + B\beta^n$$

$$\frac{z}{1 - z - z^2} = \frac{A}{(1 - \alpha z)} + \frac{B}{(1 - \beta z)} \leftrightsquigarrow \underline{\frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n}$$

$$\Rightarrow A(1 - \beta z) + B(1 - \alpha z) = z \Rightarrow (A + B) + (-A\beta - B\alpha) z = z$$

$\therefore A + B = 0 \quad A = -B$

$\therefore -A\beta + A\alpha = 1$

$\Rightarrow A(\alpha - \beta) = 1$

$$\overset{\alpha}{\phi} = \frac{1}{2}(1 + \sqrt{5}) \qquad \overset{\beta}{\bar{\phi}} = \frac{1}{2}(1 - \sqrt{5})$$

$$1 - z - z^2 = (1 - \phi z)(1 - \bar{\phi} z)$$

$$\frac{1}{2}(1 + \sqrt{5} - 1 + \sqrt{5}) = \sqrt{5}$$

$$\therefore A = 1/\sqrt{5}$$

$$g_n = g_{n-1} + g_{n-2} \leftrightsquigarrow \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n)$$

$g_0 = 0$

$g_1 = 1$

$\approx O(\log n)$

<u>Method.</u>

$g_{-1} = g_{-2} = g_{-3} = \cdots 0$

1. Write a <u>single</u> equation for g_n in terms of all others

   $g(n) = g(n-1) + g(n-2) + \delta_1(n)$

2. Multiply LHS by z^n and do a summation over all n

   LHS $\quad G(z) = \sum_{n \geq 0} g_n z^n \leftrightsquigarrow g_n \qquad G(z) = z G(z) + z^2 G(z) + z$

   RHS $\quad$ expression involve $G(z)$

3. Solve for $G(z)$

   $G(z) = \dfrac{z}{1 - z - z^2}$

4. Expand $G(z)$ as a power series $\sum g_n z^n$

   $g_n = \dfrac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n)$

$g_0 = 0 \qquad g_{-1} = 0$

$g_1 = 1$

$\rightarrow g_n = g_{n-1} + g_{n-2} \;\times \quad \rightarrow g_n = g_{n-1} + g_{n-2} + \delta_1(n)$

$g_1 = g_0 + g_{-1}$
$= 0 + 0$
$g_1 = 0$

$$\underline{\delta_k(n)} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{o/wise} \end{cases} \qquad \underline{\langle 0, 0, 0, 0, 1, 0, 0 0 \rangle}$$

$\langle 0, 1, 0, 0, 0, 0 \rangle \quad 0z^0 + 1z^1 + 0 + 0 \cdots \quad \boxed{G(z) = z}$

$g_n \Leftrightarrow G(z) \qquad g_{n-1}$ [illegible] ?

$$g(n-1) \leftrightarrow \sum_{n \ge 0} g_{n-1} z^{n}$$

$$= \cancel{g_{-1}} z^0 + g_0 z^1 + g_1 z^2 + \cdots$$

$$= z\,(g_0 + g_1 z^2 + \cdots)$$

$$g(n-k) \qquad z^k$$