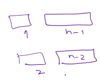
Agenda

- Quicksort has worst case quadratic runtime
- But how likely is this?
  - Average case analysis of quicksort: the technique of summation

Quicksort

$C_0 = 0$

$C_n = (n+1) + 2\left(\frac{\sum_{k=0}^{n-1} C_k}{n}\right)$



$n C_n = n(n+1) + 2 \sum_{0 \le k \le n-1} C_k = n^2 + n + 2 \sum C_k$

$(n-1) C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{0 \le k \le n-2} C_k = n^2 - 2n + \cancel{1} + n - \cancel{1} + 2\Sigma$

$n C_n - (n-1) C_{n-1} = 2n + 2 C_{n-1}$

$n C_n = (n-1) C_{n-1} + 2 C_{n-1} + 2n$

$= (n+1) C_{n-1} + 2n$

$a_n = n \quad b_n = n+1 \quad c_n = 2n$

$\sigma_n = \frac{(n-\cancel{1})(\cancel{n-2}) \cdots \cancel{1}}{(n+1)(n)(\cancel{n-1}) \cdots \cancel{3}\,\cancel{2}}\; 2$

$= \frac{2}{n(n+1)}$

$\frac{1}{\sigma_n} = \frac{n(n+1)}{2}$

$a_n T_n = b_n T_{n-1} + c_n$ — (1)

$T_n = 2 T_{n-1} + 1 \quad \leftrightarrow$

$a_n = 1 \quad b_n = 2 \quad c_n = 1$

$\sigma_n a_n T_n = \sigma_n b_n T_{n-1} + \sigma_n c_n$

$\sigma_n b_n = \underline{\sigma_{n-1} a_{n-1}}$ — (2) Design

$\sigma_n a_n T_n = \sigma_{n-1} a_{n-1} T_{n-1} + \sigma_n c_n$

<u>Def</u> $S_n = \sigma_n a_n T_n$ — (3)

$S_n = S_{n-1} + \sigma_n c_n$

$S_n = \sigma_0 a_0 T_0 + \sum_{1 \le k \le n} \sigma_k c_k$

$T_n = \frac{1}{\sigma_n a_n}\left[\sigma_1 b_1 T_0 + \sum_{1 \le k \le n} \sigma_k c_k\right]$ — (4)

$S_n = \sum_0^n c_k$

$S_n = S_{n-1} + c_n$

$S_0 = c_0$

$\sigma_n = \frac{\sigma_{n-1} a_{n-1}}{b_n}$

$= \frac{a_{n-1} a_{n-2} \cdots a_1}{b_n b_{n-1} \cdots b_2}$ — (5)

$$T_n = \frac{1(\cancel{n})(n+1)}{2\,\cancel{n}}\left[\sum_{1\le k\le n}\frac{(2k)\cdot 2}{k(k+1)}\right] \qquad H_n \leftrightarrow \int \frac{1}{x} \leftrightarrow \underline{\lg n}$$

$$= (2)(n+1)\sum_{1\le k\le n}\frac{1}{k+1}$$

$$= (2)(n+1)\left[\sum_{1\le k-1\le n}\frac{1}{k}\right] = (2)(n+1)\sum_{2\le k\le n+1} 1/k$$

$$= (2)(n+1)\left[\underbrace{\sum_{1\le k\le n} 1/k}_{H_n} - 1 + \left(\frac{1}{n+1}\right)\right]$$

$$\boxed{\text{Q.Sort is } O(n \log n)}$$

$$= 2(n+1)\left[H_n - \left(\frac{n}{n+1}\right)\right] = 2(n+1)H_n - 2n$$

$$T_n = 2T_{n-1} + 1$$

$$\frac{T_n}{2^n} = \underbrace{\frac{2}{2^n}T_{n-1}}_{\frac{T_{n-1}}{2^{n-1}}} + 2^{-n}$$

$$T_n = 2^n - 1$$

$$S_n = \frac{T_n}{2^n} \Rightarrow \boxed{S_n = S_{n-1} + 2^{-n}}$$

$$\Downarrow$$

$$\sum_{1\le k\le 1}^{n} 2^{-k}$$

$$\boxed{\sigma_n}$$