Q1. A function f(z) is said to be linear if $f(\alpha z_1 + z_2) = \alpha f(z_1) + f(z_2)$, where z_1, z_2 are two scalar values in the domain of f and α is a scalar constant. This definition extends to the case where $\boldsymbol{z}$ is a vector, i.e. $f(\boldsymbol{z})$ is linear if $f(\alpha \boldsymbol{z_1} + \boldsymbol{z_2}) = \alpha f(\boldsymbol{z_1}) + f(\boldsymbol{z_2})$. Now, we have seen the formula for bilinear interpolation which expresses image intensities as a bilinear function of x, y (spatial coordinates) in the form $v(x, y) = ax + by + cxy + d$ where a, b, c, d are scalar constants independent of x, y. Is $v(x, y)$ a linear function of x keeping y constant and vice-versa? Prove or disprove. Is it a linear function of $\boldsymbol{z} \triangleq (x, y)$?

Ans. $v(x, y)$ is given as $ax + by + cxy + d$. Applying the definition of linear function with variable as x keeping y constant:

$$\begin{aligned}
v(x) &= (a + cy)x + (by + d) \\
v(\alpha x_1 + x_2) &= (a + cy)(\alpha x_1 + x_2) + (by + d) \\
&= \alpha(a + cy)(x_1) + (a + cy)(x_2) + (by + d) \\
\alpha v(x_1) + v(x_2) &= \alpha((a + cy)x_1 + by + d) + (a + cy)x_2 + (by + d) \\
&= \alpha(a + cy)(x_1) + \alpha(by + d) + (a + cy)(x_2) + (by + d) \neq v(\alpha x_1 + x_2)
\end{aligned}$$

Hence it is not linear with respect to variable x keeping y as constant. Applying the definition of linear function with variable as y keeping x constant:

$$\begin{aligned}
v(y) &= (b + cx)y + (ax + d) \\
v(\alpha y_1 + y_2) &= (b + cx)(\alpha y_1 + y_2) + (ax + d) \\
&= \alpha(b + cx)(y_1) + (b + cx)(y_2) + (ax + d) \\
\alpha v(y_1) + v(y_2) &= \alpha((b + cx)y_1 + ax + d) + (b + cx)y_2 + (ax + d) \\
&= \alpha(b + cx)(y_1) + \alpha(ax + d) + (b + cx)(y_2) + (ax + d) \neq v(\alpha y_1 + y_2)
\end{aligned}$$

Hence it is also not linear with respect to variable y keeping a as constant. Applying the definition of linear function with variable $\boldsymbol{z} \triangleq (x, y)$:

$$\begin{aligned}
v(z) = v(x, y) &= ax + by + cxy + d \\
v(\alpha z_1 + z_2) &= v(\alpha(x_1, y_1) + (x_2, y_2)) = v(\alpha x_1 + x_2, \alpha y_1 + y_2) \\
&= a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha x_1 + x_2)(\alpha y_1 + y_2) + d \\
&= c\alpha^2(x_1 y_1) + c\alpha(x_1 y_2) + c\alpha(x_2 y_1) + c(y_1 y_2) + a\alpha(x_1) + a(x_2) + b\alpha(y_1) + b(y_2) + d \\
\alpha v(z_1) + v(z_2) &= \alpha v(x_1, y_1) + v(x_2, y_2) \\
&= \alpha(ax_1 + by_1 + cx_1 y_1 + d) + ax_2 + by_2 + cx_2 y_2 + d \\
&= c\alpha(x_1 y_1) + c(x_2 y_2) + a\alpha(x_1) + a(x_2) + b\alpha(y_1) + b(y_2) + d \neq v(\alpha z_1 + z_2)
\end{aligned}$$

Hence it not linear with respect to $\boldsymbol{z}$ as well.