Assignment 3: CS 663, Fall 2021

Due: 7th October before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. You may discuss broad ideas with other students or ask me for any difficulties, but the code you implement and the answers you write must be your own. We will adopt a zero-tolerance policy against any violation.

Submission instructions: Follow the instructions for the submission format and the naming convention of your files from the submission guidelines file in the homework folder. Please see assignment3.zip in the homework folder. For all the questions, write your answers and scan them, or type them out in word/Latex. In eithe case, create a separate PDF file. The last two questions will also have code in addition to the PDF file. Once you have finished the solutions to all questions, prepare a single zip file and upload the file on moodle before 11:55 pm on 7th October. **Only one student per group should submit the assignment.** We will not penalize submission of the files till 10 am on 8th October. **No assignments will be accepted after this time.** Please preserve a copy of all your work until the end of the semester. **Your zip file should have the following naming convention:** RollNumber1_RollNumber2_RollNumber3.zip for three-member groups, RollNumber1_RollNumber2.zip for two-member groups and RollNumber1.zip for single-member groups.

1. Consider the two images in the homework folder 'barbara256.png' and 'kodak24.png'. Add zero-mean Gaussian noise with standard deviation $\sigma = 5$ to both of them. Implement a mean shift based filter and show the outputs of the mean shift filter on both images for the following parameter configurations: $(\sigma_s = 2, \sigma_r = 2); (\sigma_s = 0.1, \sigma_r = 0.1); (\sigma_s = 3, \sigma_r = 15)$. Comment on your results in your report. Repeat when the image is corrupted with zero-mean Gaussian noise of $\sigma = 10$ (with the same bilaterial filter parameters). Comment on your results in your report. Include all image ouputs as well as noisy images in the report. [20 points]
   Solutions and Marking Scheme: See code in homework folder. Basic mean shift implementation until convergence, 14 points. 5 points to be deducted if the student runs mean shift for a fixed number of iterations. 6 points for results for all parameter settings and noise values to be included in report. If the images are missing in the report, then deduct 3 points. Comments: larger σ_s, σ_r lead to more smoothing and smaller values lead to less smoothing.

2. Consider the barbara256.png image from the homework folder. Implement the following in MATLAB: (a) an ideal low pass filter with cutoff frequency $D \in \{40, 80\}$, (b) a Gaussian low pass filter with $\sigma \in \{40, 80\}$. Show the effect of these on the image, and display all filtered images in your report. Display the frequency response (in log absolute Fourier format) of all filters in your report as well. Comment on the differences in the outputs. Also display the log absolute Fourier transform of the original and filtered images. Comment on the differences in the outputs. Make sure you perform appropriate zero-padding while doing the filtering! [20 points]
   Solutions and Marking Scheme: 10 points for Gaussian filter and 10 points for ideal LPF. If zero-padding is not properly done and/or if the filter size is inaccurate, deduct 4 points for each part. Note: the size of the filter in the frequency domain must be at least $2H \times 2W$ where the image size is $H \times W$. The image also needs to be zero-padded to attain size at least $2H \times 2W$.

3. Prove the convolution theorem for 2D Discrete fourier transforms. [10 points]
   Solution: Let the two signals be $f(x, y)$ and $g(x, y)$ with size $N \times M$ and let their Discrete Fourier Transforms

be $F(u,v), G(u,v)$ respectively. Then we have:

$$DFT(f*g)(u,v) = \sum_{x=0}^{N-1}\sum_{y=0}^{M-1}(f*g)(x,y)\exp(-j2\pi(ux/N+vy/M)) \tag{1}$$

$$= \sum_{x=0}^{N-1}\sum_{y=0}^{M-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)g(x-n,y-m)\exp(-j2\pi(ux/N+vy/M)) \tag{2}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} g(x-n,y-m)\exp(-j2\pi(ux/N+vy/M)) \tag{3}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\exp(-j2\pi(nx/N+my/M))G(u,v) \text{ by Fourier shift theorem} \tag{4}$$

$$= G(u,v)\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\exp(-j2\pi(nx/N+my/M)) \tag{5}$$

$$= G(u,v)F(u,v). \text{ by defn. of the Fourier transform} \tag{6}$$

Note that the shifts in the above summations are all circular by definition.
Marking scheme: 1.5 points each for steps 1 and 2, 4 points for the use of the Fourier shift theorem and 3 points for completing the proof.

4. You can use the Fourier transform to compute the Laplacian of an image. But can you use the Fourier transform to compute the gradient magnitude at every pixel in an image? If yes, explain how you will do it. If not, explain why this is not possible. [10 points]
Solution: One cannot use the Fourier transform to compute the gradient magnitude of an image $f(x,y)$ which is given as $\sqrt{f_x^2(x,y)+f_y^2(x,y)}$ because the square-root operation is non-linear and cannot be implemented using the Fourier transform. The gradients themselves can be computed using the Fourier transform as they are convolution operations (via convolution theorem). The gradient squares can be computed using the dual of the convolution theorem since $f_x^2(x,y) = f_x(x,y)f_x(x,y)$ (pointwise multiplication in the spatial domain) can be implemented by frequency domain convolution.
Marking scheme: 10 points for correct answer with the justification that square-roots cannot be implemented by the Fourier transform. The explanation about the squared gradient magnitude is not required. If the student has mentioned that the squared magnitude can be computed using Fourier domain operations followed by a square-root operation, giving an overall affirmative answer, then full points are to be awarded.

5. If a function $f(x,y)$ is real, prove that its Discrete Fourier transform $F(u,v)$ satisfies $F^*(u,v) = F(-u,-v)$. If $f(x,y)$ is real and even, prove that $F(u,v)$ is also real and even. The function $f(x,y)$ is an even function if $f(x,y) = f(-x,-y)$. [15 points]
Solution: We have $F(u,v) = \sum_{x,y} f(x,y)\exp(-j2\pi(ux+vy)/N)$ and $F(-u,-v) = \sum_{x,y} f(x,y)\exp(j2\pi(ux+vy)/N) = \left(\sum_{x,y} f(x,y)\exp(-j2\pi(ux+vy)/N)\right)^* = F^*(u,v)$. Note that the conjugation step is valid only because $f(x,y)$ is real-valued.

Now consider that f is both real and even, i.e. $f(x,y) = f(-x,-y)$. Now we have:

$$F^*(u,v) = \Big(\sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(-j2\pi(ux+vy)/N)\Big)^* = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(ux+vy)/N) \text{ as } f \text{ is real} \quad (7)$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(-u(-x)-v(-y))/N) \quad (8)$$

$$= \sum_{x'=0}^{-N}\sum_{y=0}^{-N} f(x',y')\exp(j2\pi(-ux'-vy')/N) \quad (9)$$

replacing $x' \triangleq -x, y' \triangleq -y$ and also using $f(x',y') = f(-x',-y')$ as f is even

$$= \sum_{x'=0}^{N}\sum_{y=0}^{N} f(x',y')\exp(-j2\pi(ux'+vy')/N) \text{ using the periodic nature of the DFT} \quad (10)$$

$$= F(u,v). \quad (11)$$

This establishes that $F(u,v)$ is real-valued. We had already proved $F^*(u,v) = F(-u,-v)$ which establishes that $F(u,v) = F(-u,-v)$ and proves that F is even. Another way to prove that F is even, is to consider:

$$F(-u,-v) = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(-j2\pi(-ux-vy)/N) = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(ux+vy)/N) \quad (12)$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(-u(-x)-v(-y))/N) \quad (13)$$

$$= \sum_{x'=0}^{-N}\sum_{y=0}^{-N} f(-x',-y')\exp(j2\pi(-ux'-vy')/N) \text{ replacing } x' \triangleq -x, y' \triangleq -y \quad (14)$$

$$= \sum_{x'=0}^{N}\sum_{y=0}^{N} f(x',y')\exp(-j2\pi(ux'+vy')/N) \quad (15)$$

using the periodic nature of the DFT and also using $f(x',y') = f(-x',-y')$ as f is even

$$= F(u,v). \quad (16)$$

Marking scheme: 5 points for the proof that if $f(x,y)$ is real, then its Discrete Fourier transform $F(u,v)$ satisfies $F^*(u,v) = F(-u,-v)$. 5 points for the proof that the DFT of a real and even function is real, and 5 points for the proof that the DFT of a real and even function is even.

6. If $\mathcal{F}$ is the continuous Fourier operator, prove that $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$. Hint: Prove that $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$ and proceed further from there. [15 points]
Solution: We have $F(\mu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}dt$ and $f(t) = \int_{-\infty}^{+\infty} F(\mu)e^{j2\pi\mu t}d\mu$.

Hence $\mathcal{F}^2(f(t))(t) = \mathcal{F}(F(\mu))(t) = \int_{-\infty}^{+\infty} F(\mu)e^{-j2\pi\mu t}dt = f(-t)$ (comparing to the inverse Fourier transform expression for $f(t)$). Note that instead of t, we could have used another variable μ' (to distinguish it from μ), but this core argument will not change. This is also called Fourier duality. That is, if $F(\mu)$ is the Fourier transform of $f(t)$, then $f(-\mu)$ is the Fourier transform of $F(t)$.

Now, the Fourier transform of $f(-t)$ is equal to $F(-\mu)$. Hence $\mathcal{F}^3(f(t)) = F(-\mu)$. We now use an argument very similar to the one we used to obtain $\mathcal{F}^2(f(t))$. Hence $\mathcal{F}^4(f(t)) = \int_{-\infty}^{+\infty} F(-\mu)e^{-j2\pi\mu t}dt = x(t)$.
Marking scheme: For $\mathcal{F}^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$, there are 5 points each.

7. Provide an explanation for the presence of strong spikes in the center of the filters in the second sub-figure Of Fig. 1. Note that the fourier transform magnitudes of these filters are plotted in the first figure. [10 points]
Solution: We know that $f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\mu,\nu)e^{j2\pi(\mu x+\nu y)}d\mu d\nu$. The Fourier transform of all the three

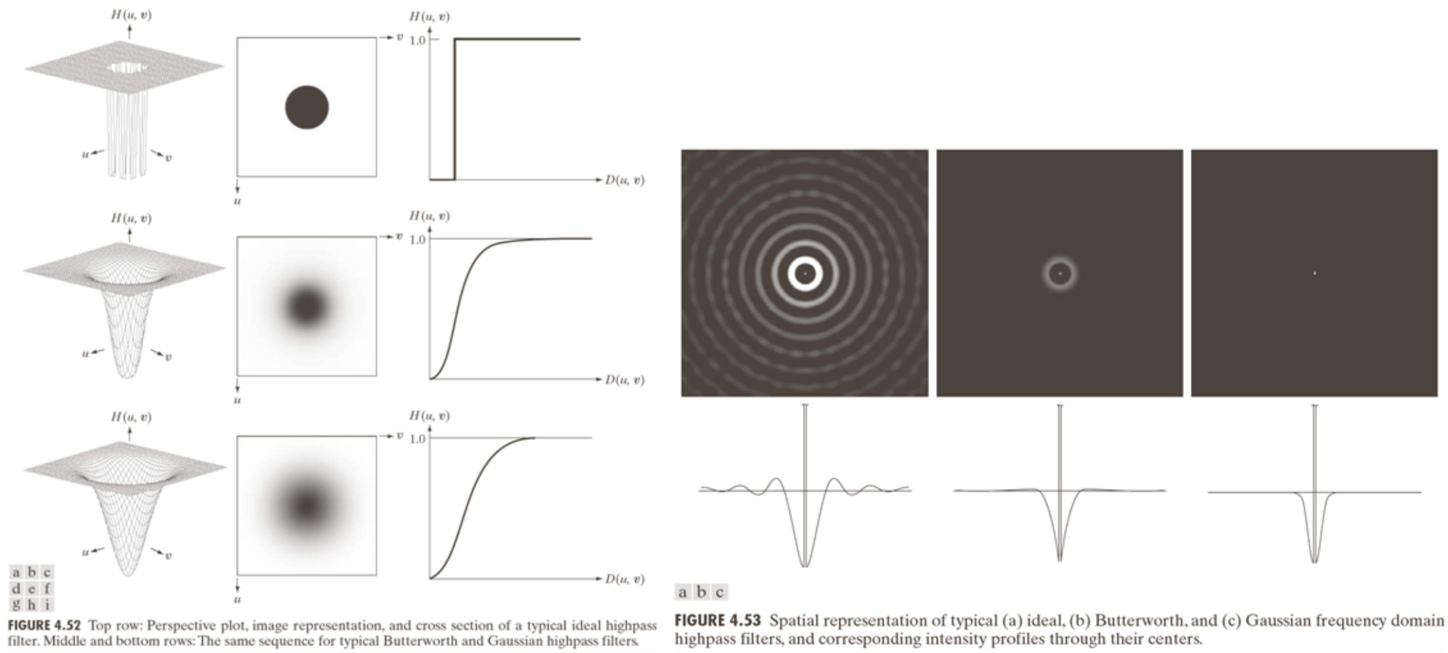

FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Figure 1: Figures required for the last question. Fourier domain (first figure) and spatial domain (second figure) representations of various filters.

high pass filters is real-valued and non-negative (by design, as can be seen by the left-hand-side figure). Now $f(0,0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\mu,\nu)d\mu d\nu$ which produces a large positive value leading to a spike in the center. At other locations, some of the values of $F(\mu,\nu)$ can get cancelled out as they are being multiplied with complex exponentials which have both positive and negative values depending on their location. Hence the values of $f(x,y)$ will never exceed $f(0,0)$ for $x \neq 0, y \neq 0$.

Marking scheme: Full points for any reasonable explanation about the central spike.