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Partial Differentiation, Euler's Theorem, Change of Variables

Partial Derivative of First and Second Order, Euler's Theorem for Homogenous Functions, Derivatives of Implicit Functions, Total Derivatives, Change of Variables.

FUNCTIONS OF SEVERAL VARIABLES

In Engineering problems one frequently comes across a variable quantity which depends for its values on two or more independent variables. For instance, the area of a rectangle depends upon its length and breadth and we say that this area is a function of two independent variables, the length and the breadth.

We use the notation $f(x, y)$ or $F(x, y)$ etc., to denote the value of the function at (x, y) and write $z \equiv f(x, y)$ or $z = F(x, y)$, etc., Also, we may write $z = z(x, y)$ where it should be clearly understood that in this notation z is used in two senses, namely, as a function and as a variable. The concept can be easily extended to the functions of three or more variables. Thus, $\omega = f(x, y, z)$ denotes the value of the function f at (x, y, z), a point in three dimensional space.

PARTIAL DERIVATIVES

Let us extend the concept of ordinary derivative of the function of one variable to the derivative of a function f (or z) of two independent variables x and y. Now, question arises, should we differentiate f with respect to x or y? The answer is simple : treat y as constant while differentiating f with respect to x and treat x as constant while differentiating f with respect to y. This way we define two different derivatives and call them partial derivatives to distinguish them from the ordinary derivative of a function of a single independent variable.

We denote the partial derivative of z (or f) with respect to x by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x \text{ or } z_x \text{ or } \left(\frac{\partial z}{\partial x}\right)_y.$$

and the partial derivative of z (or f) w.r.t. y by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y or z_y or $\left(\frac{\partial z}{\partial y}\right)_x$.

Thus, $$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} = \left(\frac{\partial f}{\partial x}\right)_y = \text{value of } \frac{\partial f}{\partial x} \text{ when } y \text{ is kept constant.}$$

and $$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \left(\frac{\partial f}{\partial y}\right)_x = \text{value of } \frac{\partial f}{\partial y} \text{ when } x \text{ is kept constant.}$$

Geometrically, $\dfrac{\partial z}{\partial x}\left(\text{or } \dfrac{\partial f}{\partial x}\right)$ **is the slope of the tangent drawn to the curve of intersection of the surface** $z = f(x, y)$ **with a plane parallel to the plane** $y = 0$, ***i.e.,*** ***ZOX*-** **plane.**

Similarly for, $\dfrac{\partial z}{\partial y}\left(\text{or } \dfrac{\partial f}{\partial y}\right)$. *[GGSIPU II Sem I Term 2011, I Term 2013]*

In general, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are functions of both x and y and therefore, we can also obtain the higher order partial derivatives of $f(x, y)$ as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy},$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}, \text{ etc.}$$

In $\dfrac{\partial^2 f}{\partial x\,\partial y}$ we first differentiate f partially with respect to y and then with respect to x, It should be noted that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$ in general, meaning thereby, that the order of partial differentiation is generally immaterial.

[redacted] **(a)** If $z = f(x + ay) + \phi(x - ay)$, then show that $z_{yy} = a^2 z_{xx}$.

(b) If $u = e^{xyz}$, find $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ *[GGSIPU II Sem End Term 2013]*

SOLUTION: (a) From $z = f(x + ay) + \phi(x - ay)$, we get

$$z_x = \frac{\partial z}{\partial x} = f'(x + ay)\frac{\partial}{\partial x}(x + ay) + \phi'(x - ay)\frac{\partial}{\partial x}(x - ay)$$

$$= f'(x + ay) + \phi'(x - ay)$$

where f' and ϕ' mean ordinary derivatives of f and ϕ with respect to $x + ay$ and $x - ay$ respectively.

Similarly, $z_y = \dfrac{\partial z}{\partial y} = f'(x + ay)\dfrac{\partial}{\partial y}(x + ay) + \phi'(x - ay)\dfrac{\partial}{\partial y}(x - ay)$

$$= a f'(x + ay) - a\phi'(x - ay).$$

Now, differentiating z_x and z_y again partially with respect to x and y respectively, gives

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x} f'(x + ay) + \frac{\partial}{\partial x}\phi'(x - ay)$$

$$= f''\frac{\partial}{\partial x}(x + ay) + \phi''\frac{\partial}{\partial x}(x - ay) = f''(x + ay) + \phi''(x - ay).$$

and $$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}[a f'(x + ay)] - \frac{\partial}{\partial y}[a\phi'(x - ay)]$$

$$= a f''\frac{\partial}{\partial y}(x + ay) - a\phi''\frac{\partial}{\partial y}(x - ay) = a^2 f''(x + ay) + (-a)^2\phi''(x - ay)$$

which clearly gives $z_{yy} = a^2 z_{xx}$.

Hence Proved.

(b) $u = e^{xyz}$ hence $\dfrac{\partial u}{\partial x} = e^{xyz}(yz) = yzu.$

Similarly $\dfrac{\partial u}{\partial y} = xz\,u$ and $\dfrac{\partial u}{\partial z} = xy\,u.$

Therefore,
$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}(xyu) = x\left(y\frac{\partial u}{\partial y}+u\right) = x[y(xzu) + u]$$
$$= ux[1 + xy\,z] = u(x + x^2yz)$$

Further
$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}[u(x+x^2yz)]$$
$$= (x+x^2yz)\frac{\partial u}{\partial x} + u(1+2xyz) = uyz\,(x + x^2yz) + u(1 + 2xyz)$$
$$= u[xy\,z(1 + xy\,z) + 1 + 2xy\,z] = u\,[1 + 3xy\,z + x^2y^2z^2]$$
$$= (1 + 3xyz + x^2y^2z^2)\,e^{xyz}.$$ **Ans.**

EXAMPLE 12.2 Show that, at a point on the surface $x^x\,y^y\,z^z = c$ where $x = y = z,$ we have
$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{x\,\log(ex)}.$$
[GGSIPU II Sem End Term 2010]

SOLUTION: From the given relation $x^x\,y^y\,z^z = c$ ...(1)

it is clear that we can take x as a function of y and z, or y as a function of z and x, or z as a function of x and y. Since we are to calculate $\dfrac{\partial^2 z}{\partial x \partial y}$, let us take here z as a function of x and y.

Taking logarithm in (1), gives $x \log x + y \log y + z \log z = \log c.$ ... (2)

Differentiating (2) partially with respect to y, gives

$$0 + y\cdot\frac{1}{y} + 1.\,\log y + z\cdot\frac{1}{z}\frac{\partial z}{\partial y} + \log z\cdot\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}. \quad \text{... (3)}$$

Similarly, differentiating (2) partially *w.r.t.* x, gives $\dfrac{\partial z}{\partial x} = \dfrac{-(1+\log x)}{1+\log z}.$ ... (4)

Next, differentiating (3) partially *w.r.t.* x, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(1+\log y)\frac{\partial}{\partial x}\frac{1}{(1+\log z)}$$
$$= -(1+\log y)\frac{(-1)}{(1+\log z)^2}\frac{\partial}{\partial x}(1+\log z) = \frac{1+\log y}{(1+\log z)^2}\frac{1}{z}\frac{\partial z}{\partial x}$$

Using (4) here, gives

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1+\log y}{(1+\log z)^2}\,\frac{(-1)(1+\log x)}{z(1+\log z)} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}.$$

At the point $x = y = z,$ we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{x(1+\log x)} = \frac{-1}{x(\log e+\log x)} = \frac{-1}{x\log(ex)}.$$ **Hence Proved.**

(a) If $z(x+y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

[GGSIPU II Sem End Term January 2011]

(b) If $z = e^{ax+by} f(ax-by)$ show that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$.

[GGSIPU II Sem End Term 2005]

SOLUTION: **(a)** From $z(x+y) = x^2+y^2$, we get

$$\frac{\partial z}{\partial x}(x+y) + z = 2x \quad \text{and} \quad \frac{\partial z}{\partial y}(x+y) + z = 2y$$

$$\Rightarrow \qquad (x+y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 2(x-y) \qquad \text{or} \qquad \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x-y)}{x+y}$$

and
$$(x+y)\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 2(x+y) - 2z.$$

$$\therefore \qquad \left[1 - \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\right] = 1 - \left[\frac{2(x+y-z)}{x+y}\right] = \frac{-(x+y-2z)}{x+y} = -1 + \frac{2z}{x+y}$$

$$= -1 + \frac{2(x^2+y^2)}{(x+y)^2} = \frac{x^2+y^2-2xy}{(x+y)^2} = \left(\frac{x-y}{x+y}\right)^2 = \frac{1}{4}\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2.$$

Hence Proved.

(b) Given that $z = e^{ax+by} f(ax-by)$, we have

$$\frac{\partial z}{\partial x} = ae^{ax+by} f(ax-by) + ae^{ax+by} f'(ax-by)$$

and
$$\frac{\partial z}{\partial y} = be^{ax+by} f(ax-by) - be^{ax+by} f'(ax-by)$$

Therefore
$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}[2f(ax-by)] = 2abz.$$

Hence the result.

If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$, show that

$$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z).$$

[GGSIPU II Sem End Term 2009]

SOLUTION: Presence of u_x, u_y and u_z indicates that we have to take u as a function of x, y and z.

Differentiating the given relation partially *w.r.t.* x, we get

$$\frac{2x}{a^2+u} - \frac{x^2 u_x}{(a^2+u)^2} - \frac{y^2 u_x}{(b^2+u)^2} - \frac{z^2 u_x}{(c^2+u)^2} = 0$$

or
$$u_x = \frac{2x}{K(a^2+u)} \quad \text{where} \quad K = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}.$$

Similarly, differentiating the given relation partially *w.r.t.* y and z separately, gives

$$u_y = \frac{2y}{K(b^2+u)} \quad \text{and} \quad u_z = \frac{2z}{K(c^2+u)}.$$

[redacted] If $V = (x^2 + y^2 + z^2)^{m/2}$ find the value of $m(\neq 0)$ which will make

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

[GGSIPU I Sem End Term 2003; I Sem End Term 2006]

SOLUTION: Given $V = (x^2 + y^2 + z^2)^{m/2}$ hence $\frac{\partial V}{\partial x} = \frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1} \cdot 2x$

and $$\frac{\partial^2 V}{\partial x^2} = mx\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2} \cdot 2x + m(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,x^2 + (x^2+y^2+z^2)].$$

Similarly for $\frac{\partial^2 V}{\partial y^2}$ and $\frac{\partial^2 V}{\partial z^2}$.

Thus, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,(x^2+y^2+z^2) + 3(x^2+y^2+z^2)]$$

$$= m(m+1)\,(x^2+y^2+z^2)^{\frac{m}{2}-1}.$$

$\therefore$ $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{when } m = 0 \text{ or } m = -1$$

Therefore the required non-zero value of m is -1. **Ans.**

[redacted] If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

[GGSIPU I Sem End Term 2004 Reappear; II Sem End Sem 2012; End Term 2013]

SOLUTION: Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, we have

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$\therefore$ $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \frac{3\,(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}.$$

Next $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\frac{3}{(x+y+z)}$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}.$$

Hence proved.

EXAMPLE 12.9. If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$ show that

$$\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = 1$$

SOLUTION: From $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, we can get

$$\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \quad \text{and} \quad \left(\frac{\partial y}{\partial u}\right)_\theta = -\frac{\sin\theta}{u^2}.$$

Next, to find $\left(\dfrac{\partial u}{\partial x}\right)_y$ and $\left(\dfrac{\partial u}{\partial y}\right)_x$ we have to express u as a function of x and y.

Eliminating θ in the given relations, we get $x^2 u^2 + y^2u^2 = 1$ or $u^2 = \dfrac{1}{x^2+y^2}$

Therefore, $2u\left(\dfrac{\partial u}{\partial x}\right)_y = \dfrac{-2x}{(x^2+y^2)^2} = -2xu^4$ or $\left(\dfrac{\partial u}{\partial x}\right)_y = -xu^3 = -u^2 \cos\theta.$

Similarly $\left(\dfrac{\partial u}{\partial y}\right)_x = -yu^3 = -u^2 \sin\theta.$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = \frac{-\cos\theta}{u^2}(-u^2\cos\theta) - \frac{\sin\theta}{u^2}(-u^2\sin\theta) = 1.$$

Hence Proved.

EXAMPLE 12.10. If $u = f(r)$ *where* $r^2 = x^2 + y^2 + z^2$, show that

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r).$$

SOLUTION: Since $u = f(r)$ we have $u_x = f'(r)\dfrac{\partial r}{\partial x}.$

From the relation $r^2 = x^2 + y^2 + z^2$, we get $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}.$

Similarly, we can have $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}.$

$$\therefore \quad u_x = \frac{\partial u}{\partial x} = f'(r)\frac{x}{r}.$$

Differentiating again *w.r.t* x partially, we get

$$u_{xx} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\left(\frac{1}{r} - \frac{x}{r^2}\cdot\frac{\partial r}{\partial x}\right) + \frac{x^2}{r^2}f''(r) = \frac{f'(r)}{r} - f'(r)\cdot\frac{x}{r^2}\cdot\frac{x}{r} + \frac{x^2}{r^2}f''(r)$$

$$= \frac{f'(r)}{r} + x^2\left(\frac{1}{r^2}f''(r) - \frac{f'(r)}{r^3}\right).$$

Similarly, $u_{yy} = \frac{f'(r)}{r} + y^2\left(\frac{1}{r^2}f''(r) - \frac{f'(r)}{r^3}\right)$ and $u_{zz} = \frac{f'(r)}{r} + z^2\left(\frac{1}{r^2}f''(r) - \frac{f'(r)}{r^3}\right)$

$$\therefore \quad u_{xx} + u_{yy} + u_{zz} = \frac{3f'(r)}{r} + (x^2 + y^2 + z^2)\left\{\frac{1}{r^2}f''(r) - \frac{f'(r)}{r^3}\right\}$$

$$= f''(r) + \frac{2}{r}f'(r) \text{ as } x^2 + y^2 + z^2 = r^2.$$

Hence Proved.

EXERCISE 12A

1. If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 2u.$

2. If V = log (tan x + tan y + tan z) find the value of
$$\sin 2x \frac{\partial V}{\partial x} + \sin 2y \frac{\partial V}{\partial y} + \sin 2z \frac{\partial V}{\partial z}.$$

3. Given that $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, find the value of $\dfrac{\partial^2 z}{\partial y \partial x}$

and verify that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}.$

4. If $u^2 = x^2 + y^2 + z^2$ find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}.$

5. If $u(x, t) = ae^{-gx} \sin(nt - gx)$ where a, g and n are constants, and $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$,

show that $g = \dfrac{1}{a}\sqrt{\dfrac{n}{2}}.$

6. If $u = (1 - 2xy + y^2)^{-1/2}$ prove that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2 u^3.$

And also evaluate $\dfrac{\partial}{\partial x}\left\{(1-x^2)\dfrac{\partial u}{\partial x}\right\} + \dfrac{\partial}{\partial y}\left\{y^2 \dfrac{\partial u}{\partial y}\right\}.$

7. If $u = e^{r\cos\theta}\cos(r\sin\theta)$ and $v = e^{r\cos\theta}\sin(r\sin\theta)$, show that
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

8. If $u = r^n$ and $r^2 = x^2 + y^2 + z^2$, prove that
$$u_{xx} + u_{yy} + u_{zz} = n(n+1)r^{n-2}.$$

9. Given that $u = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, evaluate $u_{xx} + u_{yy}$ in the simplified form.

10. Let $u = lx + my$ and $v = mx - ly$ where l and m are constants, show that
$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2} = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

11. If $u = \log(\tan x + \tan y)$, evaluate $\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y}.$

12. Let $xu + yv = 0$ and $\dfrac{u}{x} + \dfrac{v}{y} = 1$, then show that
$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{y^2+x^2}{y^2-x^2}.$$

13. If $z = (x + y)\left[1+\phi\left(\frac{y}{x}\right)\right]$ prove that $x\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x \partial y}\right) = y\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial y \partial x}\right)$.

14. If $u = e^{xyz} f\left(\frac{xz}{y}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\, xyzu = y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

and hence deduce that $x\frac{\partial^2 u}{\partial x \partial z} = y\frac{\partial^2 u}{\partial y \partial z}$.

15. Given $x = \cos\theta - r\sin\theta$ and $y = \sin\theta + r\cos\theta$, show that

(*i*) $\frac{\partial \theta}{\partial x} = \frac{-\cos\theta}{r}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$

(*ii*) $\frac{\partial^2 \theta}{\partial x^2} = \frac{\cos\theta}{r^3}(\cos\theta - 2r\sin\theta)$

16. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)+\frac{1}{r}f'(r)$. [GGSIPU II Sem End Term 2010]

HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_n y^n$ in which all the terms are of same degree, is called homogeneous function in x and y of degree n. The definition can be extended to functions of three or more variables. The above expression can also be written as

$$x^n [a_0 + a_1 (y/x) + a_2 (y/x)^2 + ... + a_n (y/x)^n] = x^n f(y/x)$$

where $f(y/x)$ is a n^{th} degree polynomial in (y/x). It is important to note here that we are now in a position to generalise $f(y/x)$ to include trigonometric, exponential and other functions.

Thus, $x^n f(y/x)$ defines a homogeneous function in x and y, of degree n, whatever may be the functional nature of f. For example, $x^4 \tan(y/x)$ is homogeneous in x and y of degree 4. Similarly, the expression $\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ is a homogeneous function of degree $-\dfrac{1}{6}$ because it can be written as

$$\frac{x^{1/3}\left\{1+(y/x)^{1/3}\right\}}{x^{1/2}\left\{1+(y/x)^{1/2}\right\}} = x^{\frac{1}{3}-\frac{1}{2}} g(y/x) = x^{-1/6} g(y/x).$$

In this connection a very useful theorem follows:

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function in x and y, of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$ [GGSIOPU II Sem End Term 2006]

PROOF : As explained above we can write $u = x^n f(y/x)$ or $u = x^n f(t)$ where $t = y/x$.

Then $$\frac{\partial u}{\partial x} = nx^{n-1}f(t) + x^n f'(t)\frac{\partial t}{\partial x} = nx^{n-1}f(t) + x^n f'(t)(-y/x^2)$$
$$= nx^{n-1}f(t) - x^{n-2}yf'(t)$$

and $$\frac{\partial u}{\partial y} = x^n f'(t)\frac{\partial t}{\partial y} = x^n f'(t)\frac{1}{x} = x^{n-1}f'(t)$$

Therefore, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(t) - x^{n-1}yf'(t) + x^{n-1}yf'(t) = nu.$$

Some Useful Deductions of Euler's Theorem

I. If u is a homogeneous function in x, y and z, of degree n, we can write $u = x^n f(y/x, z/x)$ and then it can be easily proved that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

II. If u is a homogeneous function in x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

PROOF : We already have $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \quad ...(1)$$

Differentiating (1) partially *w.r.t.* x and y separately, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad \text{... (2)}$$

and $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \quad \text{... (3)}$$

Multiplying (2) by x and (3) by y and adding these, gives

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

Using (1) here and the fact that $\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\,x\frac{\partial u}{\partial x}+(n-1)\,y\frac{\partial u}{\partial y}=n(n-1)u$$

III. If z is homogeneous in x and y of degree n, and z is a function of u as $z=f(u)$ then we have

(*i*) $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}$$

(*ii*) $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1] \quad \text{where} \quad g(u)=\frac{n\,f(u)}{f'(u)}.$$

Proof: (*i*) Since z is homogenous in x and y of degree n, we have $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$. ... (4)

Also, since $z=f(u)$, we have $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$.

Therefore (4) becomes $x\,f'(u)\dfrac{\partial u}{\partial x}+y\,f'(u)\dfrac{\partial u}{\partial y}=n\,f(u)$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}=g(u) \quad \text{where} \quad g(u)=\frac{n\,f(u)}{f'(u)}. \quad \text{... (5)}$$

(*ii*) Differentiating (5) partially *w.r.t.* x and y separately, gives

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x} \quad \text{... (6)}$$

and $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y} \quad \text{... (7)}$$

Multiplying (6) by x and (7) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g'(u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(g'(u)-1)=g(u)\,(g'(u)-1).$$

where $$g(u)=\frac{n\,f(u)}{f'(u)}.$$

[redacted] **(a)** If $z = x^2\tan^{-1}\left(\dfrac{y}{x}\right) - y^2\tan^{-1}\left(\dfrac{x}{y}\right)$, then prove that $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x}$ and evaluate $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy}$. *[GGSIPU II Sem I Term 2011]*

(b) If $z = x^4 y^2 \sin^{-1}\dfrac{x}{y} + \log x - \log y$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4 y^2 \sin^{-1}\frac{x}{y}.$$

SOLUTION: **(a)** Here $\tan^{-1}(y/x)$ and $\tan^{-1}(x/y)$ are both of degree zero and hence behave like constants. Therefore z is homogeneous function in x and y of degree two. Applying Euler's theorem, we get

$$x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = 2(2-1)\, z = 2z. \qquad \textbf{Ans.}$$

(b) Let us write $z = u + v$ where $u = x^4 y^2 \sin^{-1}\dfrac{x}{y}$ and $v = \log\left(\dfrac{x}{y}\right)$.

Observe here that $\dfrac{x}{y}$ is homogenous of degree 0, so is $\sin^{-1}\dfrac{x}{y}$ and, in turn, u is homogenous in x, y of degree $4 + 2$ (= 6). Also $\log\left(\dfrac{x}{y}\right)$ is homogenous in x and y of degree 0, therefore by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u \qquad \text{and} \qquad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.$$

Adding these, gives $x\dfrac{\partial}{\partial x}(u+v) + y\dfrac{\partial}{\partial y}(u+v) = 6u + 0$

or $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4 y^2 \sin^{-1}\frac{x}{y}. \qquad \textbf{Hence Proved.}$$

[redacted] **(*a*)** If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$

[GGSIPU II Sem End Term 2005]

(*b*) If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

[GGSIPU II Sem End Term 2007]

SOLUTION: **(*a*)** Given that $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ we have $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$.

Obviously $\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is homogeneous in x and y of degree 1/2 hence $\cos u$ is homogeneous of degree 1/2.

By Euler's theorem $x\dfrac{\partial}{\partial x}\cos u + y\dfrac{\partial}{\partial y}\cos u = \dfrac{1}{2}\cos u$

or $$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

(*b*) Given $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ hence $\sin u = \dfrac{x^2+y^2}{x+y}$.

Clearly $\sin u$ is a homogeneous function in x and y of degree one, hence by Euler's theorem

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = 1 \sin u$$

Hence the result.

or $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

EXAMPLE 12.13. If $u = \log\left(\dfrac{x^2+y^2}{x+y}\right)$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

Also find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y}\, y^2\dfrac{\partial^2 u}{\partial y^2}$.

[*GGSIPU IInd Sem End Term 2010*]

SOLUTION: $u = \log\left(\dfrac{x^2+y^2}{x+y}\right)$ $\therefore$ $\dfrac{x^2+y^2}{x+y} = e^u = f(u)$, say.

Here $f(u)$ is a homogeneous function in x and y of degree 1, hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = 1\cdot\frac{e^u}{e^u} = 1. \quad \text{Hence Proved.}$$

Also we know that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y}\, y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1]$

where $g(u) = \dfrac{nf(u)}{f'(u)}$ which is equal to 1 as shown above and $g'(u) = 0$.

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 1(0-1) = -1.$$ **Ans.**

EXAMPLE 12.14. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = \sin 2u(1 - 4\sin^2 u)$.

[*GGSIPU II Sem End Term 2006; Term I 2005; I Term 2011; End Term 2011*]

SOLUTION: Give that $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ hence $\tan u = \dfrac{x^3+y^3}{x-y} = f(u)$, say.

Clearly $\tan u$ is a homogeneous function in x and y of degree 2, hence by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = \sin 2u = g(u), \text{ say.}$$

Also, by Euler's theorem (Deduction III), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[.g'(u)-1]$$

$$= \sin 2u\,[2\cos 2u - 1] = \sin 4u - \sin 2u.$$ **Hence the result.**

EXAMPLE 12.15. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

[GGSIPU II Sem End Term 2009; I Term 2006; I Term 2012; End Term 2012]

SOLUTION: Since $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ we have $\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = f(u)$, say.

Clearly, $\sin u$ is homogeneous in x and y of degree 1/2, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u = g(u), \text{ say.}$$

Also by Euler's theorem (Deduction III), we know that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] = \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u-1\right] = \frac{1}{4}\tan u(\sec^2 u-2)$$

$$= \frac{1}{4}\frac{\sin u}{\cos u}\left[\frac{1}{\cos^2 u}-2\right] = -\frac{1}{4}\frac{\sin u}{\cos u}\cdot\frac{2\cos^2 u-1}{\cos^2 u}$$

$$= \frac{-1}{4}\frac{\sin u\cos 2u}{\cos^3 u}.$$

Hence the result.

EXAMPLE 12.16. If $u = \sin^{-1}(x^2+y^2)^{1/5}$ prove that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{2}{25}\tan u\,(2\tan^2 u-3).$$ *[GGSIPU I Sem II Term 2003]*

SOLUTION: Since $u = \sin^{-1}(x^2+y^2)^{1/5}$ we have $\sin u = (x^2+y^2)^{1/5} = f(u)$, Say.

Since $\sin u$ is homogeneous in x and y of degree 2/5, we have, by Euler's theorem

$$xu_x + yu_y = \frac{nf(u)}{f'(u)} = \frac{2}{5}\frac{\sin u}{\cos u} = \frac{2}{5}\tan u = g(u).$$

Again by Euler's theorem (Deduction III), we have

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = g(u)\,[g'(u)-1]$$

$$= \frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u-1\right] = \frac{2}{25}\tan u\,[2\tan^2 u-3].$$

Hence the result.

EXAMPLE 12.17. If $u = \dfrac{(x^2+y^2)^n}{2n(2n-1)} + x f\left(\dfrac{y}{x}\right) + \phi\left(\dfrac{x}{y}\right)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$.

SOLUTION: Let $u = u_1 + u_2 + u_3$ where $u_1 = \dfrac{(x^2+y^2)^n}{2n(2n-1)}$, $u_2 = x f\left(\dfrac{y}{x}\right)$, $u_3 = \phi\left(\dfrac{x}{y}\right)$.
Clearly u_1 is homogeneous in x and y of degree $2n$, u_2 is homogeneous in x and y of degree 1 and u_3 is homogenous in x and y of degree 0. Thus, applying Euler's theorem separately to u_1, u_2, u_3, we get

$$x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2} = 2n(2n-1)\,u_1$$

$$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 1(1-1)u_2 = 0$$

$$x^2 \frac{\partial^2 u_3}{\partial x^2} + 2xy \frac{\partial^2 u_3}{\partial x \partial y} + y^2 \frac{\partial^2 u_3}{\partial y^2} = 0\,(0-1)\,u_3 = 0$$

Adding these, gives $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2n\,(2n-1)\,u_1 + 0 + 0$

$$= 2n\,(2n-1)\,\frac{(x^2+y^2)^n}{2n\,(2n-1)} = (x^2+y^2)^n.$$ **Ans.**

EXAMPLE 12.18. (a) If $u = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right)$, show that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + xu_x + yu_y = n^2 u.$$

(b) If $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2 \tan u.$

SOLUTION: (a) Let $u = u_1 + u_2$ where

$$u_1 = x^n f\left(\frac{y}{x}\right) \quad \text{and} \quad u_2 = y^{-n} \phi\left(\frac{x}{y}\right).$$

Here u_1 is homogeneous in x and y of degree n while u_2 is homogenous in x and y of degree $-n$. Making use of Euler's theorem to u_1 and u_2 separately, we get

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = nu_1, \qquad \text{... (1)}$$

$$x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = -nu_2. \qquad \text{... (2)}$$

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = n\,(n-1)\,u_1 \qquad \text{... (3)}$$

and $$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = -n\,(-n-1)u_2. \qquad \text{... (4)}$$

Adding (1), (2), (3) and (4), gives

$$x^2 \frac{\partial^2}{\partial x^2}(u_1+u_2) + 2xy \frac{\partial^2}{\partial x \partial y}(u_1+u_2) + y^2 \frac{\partial^2}{\partial y^2}(u_1+u_2) + x\frac{\partial}{\partial x}(u_1+u_2) + y\frac{\partial}{\partial y}(u_1+u_2)$$
$$= n\,(n-1)\,u_1 + (-n)\,(-n-1)\,u_2 + nu_1 - nu_2$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u_1\,(n^2 - n + n) + u_2\,(n^2 + n - n) = n^2 u.$$

Hence Proved.

(b) Here u is not homogeneous function

but $$f(u) = \sin u = \frac{x^3+y^3+z^3}{ax+by+cz}$$ is homogeneous in x, y, z of degree 2.

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2f(u)}{f'(u)} = \frac{2\sin u}{\cos u} = 2\tan u.$$

Hence Proved.

EXERCISE 12B

1. If $u = \dfrac{x^3 + y^3}{y\sqrt{x}} + \dfrac{1}{x^7} \sin^{-1}\left(\dfrac{x^2 + y^2}{x^2 + 2xy}\right)$ find the value of

   $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + x\, u_x + y\, u_y$ at the point (1, 2).

2. If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$ show that $\dfrac{\partial u}{\partial x} = \dfrac{-y}{x} \dfrac{\partial u}{\partial y}$.

3. If $z = x^3 e^{-x/y}$ prove that $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = 6z$.

4. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, evaluate $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$.

5. If $y = x \sin u$, find the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

6. If $f(x, y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$, show that $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} + 2f = 0$.

7. Determine the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$ if $u = \sin^{-1} (x^3 + y^3)^{2/5}$.

8. (*a*) If $u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x - y}\right)$, then evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

   (*b*) If $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$ evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

9. If $u = \dfrac{x^2 y^2 z^2}{x^2 + y^2 + z^2} + \cos \dfrac{2xy + yz + zx}{x^2 + y^2 + z^2}$ evaluate $x\, u_x + y\, u_y + z\, u_z$.

10. If $u = \log \dfrac{x + y}{\sqrt{x^2 + y^2}} \sin^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that

    $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \tan\theta \left(\frac{1}{2} \sec^2\theta - 1\right) \log \frac{x + y}{\sqrt{x^2 + y^2}}$$

    where $\theta = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$.

11. If $u = \dfrac{1}{r} f(\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$, then evaluate $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y}$.

12. If $u = \log\sqrt{x^2 + y^2}$, evaluate $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$.

13. If $x = e^u \tan v$, $y = e^u \sec v$ find the value of

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

[GGSIPU II Sem End Sem 2012]

14. If $u = \log\dfrac{x^4 + y^4}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3.$

15. If $u(x, y) = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

16. If $u = \sin^{-1}\dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = -3\tan u.$

[GGSIPU II Sem I Ter 2014]

TOTAL DERIVATIVE

If $u = f(x, y)$ is a function of two independent variables x and y, and x and y are separately functions of a single independent variable t then u can be expressed as a function of t alone and then we can find the ordinary derivative $\frac{du}{dt}$ which is called the *total derivative* of u w.r.t. t, to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Let us be interested in finding $\frac{du}{dt}$ without actually substituting the values of x and y in terms of t, in $f(x, y)$. We derive below the relation between the total derivative $\frac{du}{dt}$ and the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. When t is given an increment δt, suppose x and y get increments δx and δy respectively and this, in turn, causes u to get an increment of δu. In other words, when t becomes $t + \delta t$, let x become $x + \delta x$ and y become $y + \delta y$ and consequently, u becomes $u + \delta u$. Thus, δx and δy both tend to 0 as δt tends to 0 and then, by definition,

$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \lim_{\delta t \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \left(\frac{dx}{dt}\right) + \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \left(\frac{dy}{dt}\right)$$

$$= \lim_{\delta x \to 0} \frac{\delta f(x, y+\delta y)}{\delta x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{since } u = f(x, y).$$

Thus, we have $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \text{... (1)}$$

and in terms of differentials this result can be better written as

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy \qquad \text{...(2)}$$

This du is called the total derivative of u.

This is a very important result and will be used very frequently in partial differentiation.

On extending this result to functions of three variables, if $u = u(x, y, z)$, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

or $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

Some Deductions

If $f(x, y) = 0$ is an implicit relation between x and y, *i.e.*, if y is an implicit function of x, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Therefore, $\frac{dy}{dx}$, in terms of partial derivatives of f w.r.t. x and y, is given by

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

In this case, we can also calculate $\frac{d^2y}{dx^2}$ in terms of partial derivatives. But, before that, let us introduce the following conventional notations:

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x} = r, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y} = t, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = s.$$

Thus, we can write $\frac{dy}{dx} = \frac{-p}{q}$.

Hence $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{p}{q}\right) = -\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}$

Now, using (1) we can write

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}.1 + \frac{\partial p}{\partial y}.\frac{dy}{dx} = r + s\left(\frac{-p}{q}\right) = \frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} = s + t\left(\frac{-p}{q}\right) = \frac{qs - pt}{q}$$

Putting the values of $\frac{dp}{dx}$ and $\frac{dq}{dx}$ in (3), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q.\frac{qr - ps}{q} - p.\frac{qs - pt}{q}\right] = -\frac{1}{q^3}\left[q^2r - 2\,pqs + p^2t\right]$$

Thus, we have $\frac{d^2y}{dx^2} = -\frac{(q^2r - 2pqs + p^2t)}{q^3}$.

CHANGE OF INDEPENDENT VARIABLES

Let $u = f(x, y)$ where $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$. ... (1)

It is frequently necessiated to change the expressions involving $u, x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ to the expressions involving $u, t_1, t_2, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}$ etc.

In this constext, the earlier result $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ can now be extended to the following

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1} \quad \text{and} \quad \frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}.$$

Next, if instead of (1), we are given that

$$u = u(t_1, t_2), \quad t_1 = \phi_1(x, y) \quad \text{and} \quad t_2 = \phi_2(x, y).$$

Then the equations of transformation become

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y}.$$

Conversion from cartesian to polar coordinates and *vice versa*, are frequently encountered examples. Thus, if $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

and other way round, we have the formulae

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

EXAMPLE 12.19. Find $\dfrac{dy}{dx}$ if (i) $(\cos x)^y - (\sin y)^x = 0$.

(ii) $y^x + x^y = (x + y)^{(x+y)}$

SOLUTION: (i) The given relation is $(\cos x)^y = (\sin y)^x$

Taking logarithm on both sides, gives $y \log(\cos x) = x \log(\sin y)$

Now, let us write $f(x, y) = y \log(\cos x) - x \log(\sin y) = 0$

So y can be taken as an imlicit function of x or *vice versa*.

Here $$\frac{\partial f}{\partial x} = \frac{y(-\sin x)}{\cos x} - 1.\log(\sin y) = -(y \tan x + \log \sin y)$$

and $$\frac{\partial f}{\partial y} = \log\cos x \,.\, 1 - \frac{x\cos y}{\sin y} = \log\cos x - x\cot y.$$

Therefore, $$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = \frac{(y\tan x + \log\sin y)}{\log\cos x - x\cot y}.$$ **Ans.**

(ii) Let $f(x, y) = y^x + x^y - (x + y)^{(x+y)} = 0$.

$$\therefore \quad \frac{\partial f}{\partial x} = y^x \log y + y \,.\, x^{y-1} - (x+y)^{x+y}[1 + \log(x+y)]$$

and $$\frac{\partial f}{\partial y} = x\,y^{x-1} + x^y \log x - (x+y)^{x+y}[1 + \log(x+y)]$$

Hence $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{y^x \log y + y\,x^{y-1} - (x+y)^{x+y}(1+\log(x+y))}{x\,y^{x-1} + x^y \log x - (x+y)^{x+y}(1+\log(x+y))}\right].$$ **Ans.**

EXAMPLE 12.20. If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

SOLUTION: Since $f(x, y) = 0$, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

And since $\phi(y, z) = 0$, we have

$$d\phi = \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \quad \therefore \quad \frac{dz}{dy} = -\frac{\partial \phi}{\partial y} \Big/ \frac{\partial \phi}{\partial z}$$

$$\therefore \quad \frac{dy}{dx} \cdot \frac{dz}{dy} = (-1)^2 \frac{\dfrac{\partial f}{\partial x} \dfrac{\partial \phi}{\partial y}}{\dfrac{\partial f}{\partial y} \dfrac{\partial \phi}{\partial z}} \quad \text{or} \quad \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y}.$$

Hence Proved.

EXAMPLE 12.21. If $x^n + y^n = a^n$, find $\dfrac{d^2y}{dx^2}$.

SOLUTION: Let $f(x, y) = x^n + y^n - a^n = 0$

then $\dfrac{\partial f}{\partial x} = p = n\, x^{n-1}$, $\dfrac{\partial f}{\partial y} = q = n\, y^{n-1}$,

$$\frac{\partial^2 f}{\partial x^2} = r = n\,(n-1)\, x^{n-2},\ s = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial p}{\partial y} = 0 \text{ and } \frac{\partial^2 f}{\partial y^2} = t = n\,(n-1)\, y^{n-2}.$$

Since $f(x, y) = 0$ we can take y as an implicit function of x.

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{q^3}\,[\,q^2 r - 2pqs + p^2 t\,]$$

$$= -\frac{1}{n^3 y^{3n-3}}\,[n^2 y^{2n-2} \cdot n(n-1) x^{n-2} - 2n^2 x^{n-1} y^{n-1} \cdot (0) + n^2 x^{2n-2} \cdot n(n-1) y^{n-2}]$$

$$= -\frac{n^3 (n-1)}{n^3 y^{3n-3}}\,[\,x^{n-2} y^{2n-2} + y^{n-2} x^{2n-2}]$$

$$= -\frac{(n-1)}{y^{3n-3}}\, x^{n-2} y^{n-2} (y^n + x^n) = -\frac{(n-1)\, x^{n-2} \cdot a^n}{y^{2n-1}}.$$

Ans.

EXAMPLE 12.22. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^3$ find the value of $\dfrac{dz}{dx}$ when $x = y = a$. *[GGSIPU IInd Sem End Term 2010]*

SOLUTION: Since $z = \sqrt{x^2 + y^2}$ we have, from total derivative concept,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{or} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \frac{dy}{dx}$$

$$\therefore \quad \frac{dz}{dx} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2x \cdot 1 + \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y \cdot \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + y^2}} \left[x + y \frac{dy}{dx} \right]$$

From the relation $x^3 + y^3 + 3axy = 5a^3$, we can have

$$\frac{dy}{dx} = \frac{-\dfrac{\partial}{\partial x}(x^3 + y^3 + 3axy)}{\dfrac{\partial}{\partial y}(x^3 + y^3 + 3axy)} = \frac{-(3x^2 + 3ay)}{3y^2 + 3ax} = -1 \text{ at } (a, a).$$

$$\therefore \quad \left(\frac{dz}{dx}\right)_{(a,a)} = \left[\frac{1}{\sqrt{x^2 + y^2}} \{x + y(-1)\} \right]_{(a,a)} = 0.$$

Ans.

EXAMPLE 12.23. If $z = xy\,f(y/x)$ and z is constant, show that

$$\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\dfrac{dy}{dx}\right)}{y\left(y - x\dfrac{dy}{dx}\right)}.$$ [GGSIPU II Sem I Term 2005]

SOLUTION: Since z is constant we write $xy\,f(y/x) = z =$ constant

$$\therefore \qquad \frac{dy}{dx} = \frac{-\dfrac{\partial}{\partial x}[xyf(y/x)]}{\dfrac{\partial}{\partial y}[xyf(y/x)]} = \frac{-[yf(y/x) + xyf'(y/x)(-y/x^2)]}{[xf(y/x) + xyf'(y/x)(1/x)]}$$

or
$$\frac{x\dfrac{dy}{dx}}{y} = \frac{\dfrac{y}{x}f'(y/x) - f(y/x)}{\dfrac{y}{x}f'(y/x) + f(y/x)}.$$

Applying componendo and dividendo here, we get

$$\frac{y + x\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = \frac{2\dfrac{y}{x}f'(y/x)}{2f(y/x)} = \frac{yf'(y/x)}{xf(y/x)}$$

$$\therefore \qquad \frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\dfrac{dy}{dx}\right)}{y\left(y - x\dfrac{dy}{dx}\right)}.$$ **Hence the result.**

EXAMPLE 12.24. **(a)** If $u = f(x - y, y - z, z - x)$ show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

(b) If $u = (e^{x-y}, e^{y-z}, e^{z-x})$ find the value of $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$.

[GGSIPU II Sem I Term Jan. 2011]

SOLUTION: **(a)** Let $x' = x - y$, $y' = y - z$, $z' = z - x$ then we have $u = f(x', y', z')$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial x} + \frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial x}$$

But $\dfrac{\partial x'}{\partial x} = 1$, $\dfrac{\partial y'}{\partial x} = 0$ and $\dfrac{\partial z'}{\partial x} = -1$, hence $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial x'} - \dfrac{\partial u}{\partial z'}$

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial y} = -\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'} + 0$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial z} + \frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial z} = 0 - \frac{\partial u}{\partial y'} + \frac{\partial u}{\partial z'}$$

Therefore, $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'} - \frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'} - \frac{\partial u}{\partial y'} + \frac{\partial u}{\partial z'} = 0.$$ **Hence Proved**

(b) Putting $X' = x - y,\ Y' = y - z,\ Z' = z - x$ we have $u = (X', Y', Z')$

then $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X'}\cdot\frac{\partial X'}{\partial x} + \frac{\partial u}{\partial Y'}\frac{\partial Y'}{\partial x} + \frac{\partial u}{\partial Z'}\frac{\partial Z'}{\partial x} = 1\cdot\frac{\partial u}{\partial X'} + 0\cdot\frac{\partial u}{\partial Y'} - 1\cdot\frac{\partial u}{\partial Z'}.$$

Similarly $$\frac{\partial u}{\partial y} = -1\cdot\frac{\partial u}{\partial X'} + 1\cdot\frac{\partial u}{\partial Y'} + 0\cdot\frac{\partial u}{\partial Z'} \quad \text{and} \quad \frac{\partial u}{\partial z} = 0\cdot\frac{\partial u}{\partial X'} - 1\cdot\frac{\partial u}{\partial Y'} + 1\cdot\frac{\partial u}{\partial Z'}$$

$\therefore$ $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ **Ans.**

EXAMPLE 12.25. (a) If $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$ and F is a function of x, y, z then show that $u\frac{\partial F}{\partial u} + u\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$.

[GGSIPU II Sem End Term 2014]

(b) If $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

[GGSIPU II Sem End Term 2009, End Term 2011]

SOLUTION: (a) By change of independent variables, we have

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u} = 1\cdot\frac{\partial F}{\partial x} + (v+w)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial v} = 1\cdot\frac{\partial F}{\partial x} + (w+u)\frac{\partial F}{\partial y} + wu\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial w} = 1\cdot\frac{\partial F}{\partial x} + (u+v)\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial z}$$

Therefore $$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = (u+v+w)\frac{\partial F}{\partial x} + 2(uv+vw+wu)\frac{\partial F}{\partial y} + 3uvw\frac{\partial F}{\partial z}$$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

Hence Proved.

(b) Here $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad \text{... (1)}$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta\frac{\partial z}{\partial x} + \cos\theta\frac{\partial z}{\partial y}. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta-2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

Hence Proved.

EXAMPLE 12.26. Let $z=f(x,y)$ where $x=e^u\cos v$, $y=e^u\sin v$, then show that

$$y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=e^{2u}\frac{\partial z}{\partial y}$$

and $$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right].$$

SOLUTION: We have $z=f(x,y)$ and x and y are functions of u and v,

Hence $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=e^u\cos v\frac{\partial z}{\partial x}+e^u\sin v\frac{\partial z}{\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} \quad ...(1)$$

and $$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=-e^u\sin v\frac{\partial z}{\partial x}+e^u\cos v\frac{\partial z}{\partial y}=-y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y} \quad ...(2)$$

$$\therefore\quad y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=xy\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}-xy\frac{\partial z}{\partial x}+x^2\frac{\partial z}{\partial y}$$

$$=(x^2+y^2)\frac{\partial z}{\partial y}=(e^{2u}\cos^2 v+e^{2u}\sin^2 v)\frac{\partial z}{\partial y}=e^{2u}\frac{\partial z}{\partial y}.$$

Next, squaring and adding (1) and (2), gives

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=x^2\left(\frac{\partial z}{\partial x}\right)^2+2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+y^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2-2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+x^2\left(\frac{\partial z}{\partial y}\right)^2$$

$$=(x^2+y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=(e^{2u}\cos^2 v+e^{2u}\sin^2 v)\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$=e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right].$$

or $$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$ **Hence the result.**

EXAMPLE 12.27. By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}$. Also, show that

$\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$ transforms into $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$. [GGSIPU II Sem I Term 2006]

SOLUTION: Since $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}.$$

$$\therefore \quad u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = (u\cos\alpha - v\sin\alpha)\frac{\partial z}{\partial x} + (u\sin\alpha + v\cos\alpha)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

Next
$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u}$$
$$= \left(\cos\alpha\frac{\partial^2 z}{\partial x^2} + \sin\alpha\frac{\partial^2 z}{\partial x\partial y}\right)\cos\alpha + \left(\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\frac{\partial^2 z}{\partial y^2}\right)\sin\alpha$$
$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\sin\alpha\cdot\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}.$$

and
$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial x}\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial v}$$
$$= \left(-\sin\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\frac{\partial^2 z}{\partial x\partial y}\right)(-\sin\alpha) + \left(-\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos\alpha\frac{\partial^2 z}{\partial y^2}\right)\cos\alpha$$
$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

Hence the result.

EXAMPLE 12.28. Transform the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates r and θ.

[GGSIPU II Sem End Term 2009; End Term 2006; End Term 2013]

SOLUTION: Since $x = r\cos\theta$, $y = r\sin\theta$ we have $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}.$$

From $r^2 = x^2 + y^2$, we have $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$.

Similarly, from $\theta = \tan^{-1}\dfrac{y}{x}$ we have $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin\theta}{r}$

and $\dfrac{\partial\theta}{\partial y} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$.

Therefore $\dfrac{\partial u}{\partial x} = \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}$ and $\dfrac{\partial u}{\partial y} = \sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial\theta}$.

Thus, in terms of operators we can write

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \quad\text{and}\quad \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Therefore, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \left(\cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}\right)\left(\cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}\right)$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

and $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \left(\sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}\right)\left(\sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial\theta}\right)$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$+\frac{\cos\theta}{r}\left[\sin\theta\frac{\partial^2 u}{\partial\theta\partial r} + \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}$$

Therefore, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = (\sin^2\theta + \cos^2\theta)\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\sin^2\theta + \cos^2\theta}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{\sin^2\theta + \cos^2\theta}{r}\dfrac{\partial u}{\partial r}$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Thus, the Laplace equation in cartesian form $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ gets transformed into polar form as $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$ **Ans.**

EXAMPLE 12.29. Let f be a composite function of u and v and u and v be functions of x and y, given by $u=x^2-y^2$, $v=2xy$, then show that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right).$$

SOLUTION: Given that $f=f(u,v)$ and $u=x^2-y^2$, $v=2xy$, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}(2y) \Rightarrow \frac{\partial}{\partial x}\equiv 2\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)$$

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}(-2y)+\frac{\partial f}{\partial v}(2x) \Rightarrow \frac{\partial}{\partial y}\equiv 2\left(-y\frac{\partial}{\partial u}+x\frac{\partial}{\partial v}\right)$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=4\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)$$

$$=4x\left(x\frac{\partial^2 f}{\partial u^2}+y\frac{\partial^2 f}{\partial u\,\partial v}\right)+4y\left(x\frac{\partial^2 f}{\partial u\,\partial v}+y\frac{\partial^2 f}{\partial v^2}\right)$$

$$=4x^2\frac{\partial^2 f}{\partial u^2}+8xy\frac{\partial^2 f}{\partial u\,\partial v}+4y^2\frac{\partial^2 f}{\partial v^2}.$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2}=4\left(-y\frac{\partial}{\partial u}+x\frac{\partial}{\partial v}\right)\left(-y\frac{\partial f}{\partial u}+x\frac{\partial f}{\partial v}\right)$$

$$=-4y\left(-y\frac{\partial^2 f}{\partial u^2}+x\frac{\partial^2 f}{\partial u\,\partial v}\right)+4x\left(-y\frac{\partial^2 f}{\partial u\,\partial v}+x\frac{\partial^2 f}{\partial v^2}\right)$$

$$=4y^2\frac{\partial^2 f}{\partial u^2}-8xy\frac{\partial^2 f}{\partial u\,\partial v}+4x^2\frac{\partial^2 f}{\partial v^2}$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\frac{\partial^2 f}{\partial u^2}+4(y^2+x^2)\frac{\partial^2 f}{\partial v^2}=4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right).$$

Hence Proved.

EXAMPLE 12.30. If $z = f(x, y)$, $x^2 = uv$, $y^2 = u/v$ then change the independent variables x and y to u and v in the equation $x^2 \dfrac{\partial^2 z}{\partial x^2} - 2xy \dfrac{\partial^2 z}{\partial xy} + y^2 \dfrac{\partial^2 z}{\partial y^2} + 2y \dfrac{\partial z}{\partial y} = 0.$

[*GGSIPU II Sem I Term 2010*]

SOLUTION: $x^2 = uv,\ y^2 = u/v \Rightarrow u^2 = x^2 y^2,\ v^2 = x^2/y^2,$ or $u = xy,\ v = x/y.$

Therefore $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = y \cdot \dfrac{\partial z}{\partial u} + \dfrac{1}{y}\dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = x\dfrac{\partial z}{\partial u} - \dfrac{x}{y^2}\dfrac{\partial z}{\partial v}$

$\Rightarrow$ $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2\dfrac{x}{y}\dfrac{\partial z}{\partial v} = 2v\dfrac{\partial z}{\partial v}.$ Squaring the operators on both sides we get

$\therefore$ $x^2 \dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 4v^2 \dfrac{\partial^2 z}{\partial v^2}$

Also, we have $2y\dfrac{\partial z}{\partial y} = 2u\dfrac{\partial z}{\partial u} - 2v\dfrac{\partial z}{\partial v}.$

Therefore the equation $x^2 \dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 z}{\partial v^2} + 2y\dfrac{\partial z}{\partial y} = 0$ becomes

$$4v^2 \frac{\partial^2 z}{\partial v^2} + 2u\frac{\partial z}{\partial u} - 2v\frac{\partial z}{\partial v} = 0. \qquad \textbf{Ans.}$$

EXERCISE 12C

1. If $u = x \log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\dfrac{du}{dx}$ at $(1, 1)$.

[GGSIPU II Sem I Term 2011]

2. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ prove that $\dfrac{d^2y}{dx^2} = \dfrac{\Delta}{(hx+by+f)^2}$

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & h & c \end{vmatrix}$.

3. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\dfrac{dy}{dx}$.

4. If $x^4 + y^4 + 4a^2xy = 0$, prove that $(y^3 + a^2x)^3 \dfrac{d^2y}{dx^2} = 2a^2xy(x^2y^2 + 3a^4)$.

5. The relations $f(x, y, z) =$ constant and $xyz =$ constant together define y as a function of x, then show that

$$\frac{dy}{dx} = \frac{-y\left(x\dfrac{\partial f}{\partial x} - z\dfrac{\partial f}{\partial z}\right)}{x\left(y\dfrac{\partial f}{\partial y} - z\dfrac{\partial f}{\partial z}\right)}.$$

6. If $u = \log(\tan x + \tan y + \tan z)$, find the value of $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z}$.

7. If $u = f\left(\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}\right)$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

8. If $u = x + ay$ and $v = x + by$ transforms the equation $2\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = 0$ into $\dfrac{\partial^2 z}{\partial u \partial v} = 0$, find the values of a and b.

9. If $z = f(x, y)$ where $x = \log u$, $y = \log v$ then show that

(i) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = u^2\dfrac{\partial^2 z}{\partial u^2} + v^2\dfrac{\partial^2 z}{\partial v^2} + u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}$.

(ii) $\dfrac{\partial^2 z}{\partial x \partial y} = uv\dfrac{\partial^2 z}{\partial u \partial v}$.

10. If $z = f(u, v)$ where $u = x + y$, $v = x - y$ show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 2\dfrac{\partial z}{\partial u}$.

11. If $z = f(x, y)$ where $x = u \cosh v$ and $y = u \sinh v$, then show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2.$$

12. If $z = f(x, y)$ where $x = uv$ and $y = \frac{u}{v}$, then show that

(*i*) $\frac{\partial z}{\partial x} = \frac{1}{2v}\frac{\partial z}{\partial u} + \frac{1}{2u}\frac{\partial z}{\partial v}$

(*ii*) $\frac{\partial z}{\partial y} = \frac{v}{2}\frac{\partial z}{\partial u} - \frac{v^2}{2u}\frac{\partial z}{\partial v}$.

13. If $u = f(ax^2 + 2hxy + by^2)$ and $v = \phi(ax^2 + 2hxy + by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

14. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

[*GGSIPU II Sem I Term 2013; I Sem I Term 2014*]

15. If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$ then show that $\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \phi^2} = 4xy\frac{\partial^2 f}{\partial x \partial y}$.

[*GGSIPU II Sem End Term 2014*]

16. If $f(x, y, z) = 0$ show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$.

[*GGSIPU II Sem I Term 2011; I Term 2012*]

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CHAPTER 13

Expansion of Functions of Two Variables, Approximations, Jacobians, Maxima and Minima

Applications of Partial Differentiation, Taylor's Theorem for Functions of Two Variables, Error and Approximations, Jacobians, Extreme Values of Functions of Several Variables (Maxima and Minima) Saddle Points, Lagrange's Method of Undetermined Multipliers.

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Let us first recall the Taylor's expansion of function of one variable as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots \qquad \dots (1)$$

where h is small. Now, consider the extension of above result to function of two independent variables, *i.e.*, to obtain the expansion of $f(x+h, y+k)$ in powers of h and k, both h and k being small.

In $f(x+h, \; y+k)$ first take $x+h$ as constant and expand $f(x+h, \; y+k)$ in powers of k using the relation (1), as

$$f(x+h, y+k) = f(x+h, y) + k\frac{\partial}{\partial y} f(x+h, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x+h, y) + \dots \qquad \dots (2)$$

Now expanding each term on the R.H.S. of (2) in powers of h by Taylor's theorem of one variable considering y as constant, we have

$$f(x+h, y+k) = \left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right] + k\frac{\partial}{\partial y}\left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right]$$

$$+ \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}\left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right] + \dots$$

or $$f(x+h, \; y+k) = f(x, \; y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots$$

This extension of Taylor's theorem is useful when we discuss the maxima and minima of function of two variables. If we put $x = 0, \; y = 0$ and write x for h and y for k we arrive at the extension of *Maclaurin's theorem for two independent variables* as

$$f(x,y) = f(0,0) + x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)}$$

$$+ \frac{1}{2!}\left\{x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)}\right\} + \dots$$

EXAMPLE 13.1. Expand $f(x, y) = e^{xy}$ about $(1, 1)$ upto second degree terms.

[GGSIPU IInd Sem. Ist Term 2005, 2010]

SOLUTION: $$f(x, y) = f(1, 1) + \left[(x-1)\frac{\partial f}{\partial x} + (y-1)\frac{\partial f}{\partial y}\right]_{(1,1)}$$
$$+\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x\partial y} + (y-1)^2\frac{\partial^2 f}{\partial y^2}\right]_{(1,1)} + \ldots$$
$$= e + (x-1)(ye^{xy})_{(1,1)} + (y-1)(xe^{xy})_{(1,1)} + \frac{1}{2}[(x-1)^2 e + 2(x-1)(y-1)e + (y-1)^2 e] + \ldots$$
$$= e(x+y-1) + \frac{e}{2}(x+y-2)^2 + \ldots \quad \textbf{Ans.}$$

EXAMPLE 13.2. Find the expansion of $\cos x \cos y$ is powers of x, y upto fourth degree terms.

[GGSIPU I Sem End Term 2004 Reappear]

SOLUTION: Expanding the function $f(x, y) = \cos x \cos y$ about the origin by Taylor's theorem and retaining terms upto fourth order, we have

$$f(x,y) = f(0,0) + (xf_x + yf_y) + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+\frac{1}{3!}[x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}] +$$
$$\frac{1}{4!}[x^4 f_{xxxx} + 4x^3 yf_{xxxy} + 6x^2y^2 f_{xxyy} + 4xy^3 f_{xyyy} + y^4 f_{yyyy}]$$

At the origin $(0, 0)$, we have

$$f(0,0) = \cos 0 \cos 0 = 1,\ f_x = -\sin x \cos y = 0 \text{ at } f_y = -\cos x \sin y = 0.$$
$$f_{xx} = -\cos x \cos y = -1,\ f_{xy} = \sin x \sin y = 0,\ f_{yy} = -\cos x \cos y = -1$$
$$f_{xxx} = \sin x \sin y = 0,\ f_{xxy} = +\cos x \sin y = 0,\ f_{xyy} = \sin x \cos y = 0$$
$$f_{yyy} = +\cos x \sin y = 0,\ f_{xxxx} = \cos x \cos y = 1,$$
$$f_{xxxy} = -\sin x \sin y = 0,\ f_{xxyy} = \cos x \cos y = 1,$$
$$f_{xyyy} = -\sin x \sin y = 0,\ f_{yyyy} = \cos x \cos y = 1.$$

$$\therefore \cos x \cos y = 1 + (0.x + 0.y) + \frac{1}{2!}(-1x^2 + 0.(2xy) - 1y^2) + \frac{1}{3!}[0x^3 + 0.(3x^2y) + 0.(3xy^2) + 0.y^3]$$
$$+\frac{1}{4!}[1.(x^4) + 0.(4x^3y) + 1.6(x^2y^2) + 0.(4xy^3) + 1.(y^4)]$$
$$= 1 - \frac{x^2}{2!} - \frac{y^2}{2!} + \frac{x^4}{4!} + \frac{6x^2y^2}{4!} + \frac{1}{4!}y^4 = 1 - \frac{1}{2}(x^2+y^2) + \frac{1}{24}(x^4 + 6x^2y^2 + y^4) \quad \textbf{Ans.}$$

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$$\cos x \cos y = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right]\left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\right]$$
$$= 1 - \frac{x^2}{2!} - \frac{y^2}{2!} + \frac{x^4}{4!} + \frac{y^4}{4!} + \frac{x^2y^2}{2!2!} \quad \text{keeping terms upto } 4^{th} \text{ orders}$$
$$= 1 - \frac{1}{2}(x^2+y^2) + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24}. \quad \textbf{Ans.}$$

EXAMPLE 13.3. Expand $e^x \log(1+y)$ in the neighbourhood of the origin retaining terms upto second degree in x and y.

[*GGSIPU II Sem End Term 2006 Reappear; End Term 2012*]

SOLUTION: We know that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

and $$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$$

Therefore $$e^x \log(1+y) = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]\left[y - \frac{y^2}{2} + \frac{y^3}{3} \ldots\right]$$

$$= y - \frac{y^2}{2} + xy \quad \text{retaining terms upto second degree.}$$ **Ans.**

EXAMPLE 13.4. Obtain the Taylor's linear approximation to the function $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point $(-1, 1)$. Also, find the maximum error in the region $x+1 < 0.1$, $y - 1 < 0.1$ [*GGSIPU II Sem End Term 2011*]

SOLUTION: $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1.$ $\quad (x_0, y_0) = (-1, 1).$

By Taylor's theorem, linear approximations to $f(x,y)$ about (x_0, y_0) is

$$f(x,y) = f(x_0, y_0) + (x - x_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,\,y_0)} + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0,\,y_0)} + \ldots$$

Here $f(-1,1) = 2 + 1 + 1 - 3 - 4 + 1 = -2.$

$$\frac{\partial f}{\partial x} = 4x - y + 3 = -2 \text{ at } (-1,1) \text{ and } \frac{\partial f}{\partial y} = -x + 2y - 4 = -1 \text{ at } (-1,1)$$

Hence $f(x,y) = -2 + (x+1)(-2) + (y-1)(-1).$

Here, the error term $= -2(x+1) - (y-1)$ $\therefore$ Maximum error $= |-2(0.1) - (0.1)| = 0.3$ **Ans.**

EXAMPLE 13.5. Expand $\sin(xy)$ in powers of $(x-1)$ and $(y - \pi/2)$ upto and including the second degree terms. [*GGSIPU I Sem End Term 2003*]

SOLUTION: $f(x,y) = \sin(xy)$, $f_x = y\cos(xy)$, $f_y = x\cos(xy)$,

$f_{xx} = -y^2\sin(xy)$, $f_{xy} = 1\cos(xy) - xy\sin(xy)$, $f_{yy} = -x^2\sin(xy)$.

The expansion of $f(x,y)$ about $(1, \pi/2)$ by Taylor's theorem, is

$$f(x,y) = f(1,\pi/2) + [(x-1)f_x + (y - \pi/2)f_y] + \frac{1}{2!}\left[(x-1)^2 f_{xx} + 2(x-1)\left(y - \frac{\pi}{2}\right)f_{xy} + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\right]$$

retaining terms upto second degree and taking derivatives at $(1, \pi/2)$.

At $(1, \pi/2)$, $f(x,y) = 1$, $f_x = \dfrac{\pi}{2}\cos\dfrac{\pi}{2} = 0$, $f_y = 1\cos\pi/2 = 0$,

$$f_{xx} = -\left(\frac{\pi}{2}\right)^2 .1, \quad f_{xy} = -\frac{\pi}{2}, \quad \text{and } f_{yy} = -1 \text{ at } (1, \pi/2).$$

$$\therefore \quad \sin(xy) = 1 + [0(x-1) + 0(y - \pi/2)] + \frac{1}{2!}\left[-\frac{\pi^2}{4}(x-1)^2 + 2\left(\frac{-\pi}{2}\right)(x-1)\left(y - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right)^2\right]$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y - \pi/2) - \frac{1}{2}(y - \pi/2)^2$$ **Ans.**

EXAMPLE 13.6. (a) Expand $f(x, y) = \tan^{-1}(xy)$, in powers of $(x-1)$ and $(y-1)$ and hence evaluate $f(0.9, 1.1)$. [GGSIPU II Sem I Term 2012]

(b) Obtain the Taylor's expansion of the function $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$ about $(1, 1)$ upto including second degree terms and compute $f(1.1, -0.9)$. [GGSIPU II Sem End Term 2009; I Term 2011]

SOLUTION: (a) Here $f(x, y) = \tan^{-1}(xy)$, $f(1, 1) = \pi/4$.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2xy^3}{(1+x^2y^2)^2},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x^3y}{(1+x^2y^2)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{(1+x^2y^2)1 - y\,2x^2y}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(x^2+y^2)^2}$$

At (1, 1), we have $\dfrac{\partial f}{\partial x} = \dfrac{1}{2}, \ \dfrac{\partial f}{\partial y} = \dfrac{1}{2}, \ \dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{2}, \ \dfrac{\partial^2 f}{\partial y^2} = \dfrac{1}{2}, \ \dfrac{\partial^2 f}{\partial x \partial y} = 0.$

$$\therefore \quad f(x,y) = f(1,1) + (x-1)\left(\frac{\partial f}{\partial x}\right)_{(1,1)} + (y-1)\left(\frac{\partial f}{\partial y}\right)_{(1,1)}$$

$$+ \frac{1}{2!}\left[(x-1)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} + 2(x-1)(y-1)\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)} + (y-1)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)}\right] + \dots$$

$$= \frac{\pi}{4} + \frac{1}{2}[(x-1)+(y-1)] + \frac{1}{4}\left[(x-1)^2 + (y-1)^2 + 0\right] + \dots \qquad \textbf{Ans.}$$

$$\therefore \quad f(0.9, 1.1) = \frac{\pi}{4} + \frac{1}{2}(-0.1+0.1) + \frac{1}{4}(0.01+0.01) = 0.785 \text{ approx.} \qquad \textbf{Ans.}$$

(b) Given $f(x, y) = \tan^{-1}(y/x)$. Taylor's expansion of $f(x, y)$ about $(1, 1)$ is

$$f(x,y) = f(1,1) + (x-1)\left(\frac{\partial f}{\partial x}\right)_{(1,1)} + (y-1)\left(\frac{\partial f}{\partial y}\right)_{(1,1)}$$

$$+ \frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x \partial y} + (y-1)^2\frac{\partial^2 f}{\partial y^2}\right]_{(1,1)} + \dots$$

Here $\dfrac{\partial f}{\partial x} = \dfrac{-y}{x^2+y^2} = -\dfrac{1}{2}$ at $(1, 1)$, $\dfrac{\partial f}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{1}{2}$ at $(1, 1)$

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2} = \dfrac{1}{2}$ at $(1, 1)$, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{-2xy}{(x^2+y^2)^2} \equiv -\dfrac{1}{2}$ at $(1, 1)$

and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2} \equiv 0$ at $(1, 1)$.

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

Here, $x - 1 = 1.1 - 1 = 0.1$, $\quad y - 1 = 0.9 - 1 = -0.1$

$$\therefore \quad f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 = 0.7862. \qquad \textbf{Ans.}$$

EXAMPLE 13.7. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ by using Taylor's series. *[GGSIPU IInd Sem End Term 2010]*

SOLUTION: By Taylor's theorem expansion of $f(x, y)$ about the point (a, b), is

$$f(x,y) = f(a,b)+\left[(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}\right]+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}+(y-b)^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-a)^3\frac{\partial^3 f}{\partial x^3}+3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2}+(y-b)^3\frac{\partial^3 f}{\partial y^3}\right]+\ldots$$

where derivatives are to be taken at (a, b).

Here $f(x, y) = x^2y + 3y - 2$ and $a = 1, b = -2$. $\therefore$ $f(1, -2) = -10$,

$$\frac{\partial f}{\partial x}=2xy,\quad \frac{\partial f}{\partial y}=x^2+3,\quad \frac{\partial^2 f}{\partial x^2}=2y,\quad \frac{\partial^2 f}{\partial x\partial y}=2x,\quad \frac{\partial^2 f}{\partial y^2}=0,\quad \frac{\partial^3 f}{\partial x^3}=0,\quad \frac{\partial^3 f}{\partial y^3}=0,$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=2,\quad \frac{\partial^3 f}{\partial x\partial y^2}=0.$$

$$\therefore\ x^2y+3y-2 = -10+\left[(x-1)(-4)+(y+2)(4)+\frac{1}{2!}\{(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)\}\right.$$

$$\left.+\frac{1}{6}\left[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)\right]+\ldots\right.$$

Thus, the expansion of $f(x, y)$ by Taylor series, about $(1, -2)$, is

$$x^2y + 3y - 2 = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2)$$

Ans.

ERRORS AND APPROXIMATIONS

Let u be a function of two variables x and y and let δx and δy be small changes made in x and y respectively and the resulting change in u be δu, then

$$\delta u = u(x+\delta x, y+\delta y) - u(x, y)$$

$$= [u(x+\delta x, y+\delta y) - u(x, y+\delta y)] + [u(x, y+\delta y) - u(x, y)]$$

$$= \frac{u(x+\delta x, y+\delta y)-u(x, y+\delta y)}{\delta x}\,\delta x+\frac{u(x, y+\delta y)-u(x,y)}{\delta y}\,\delta y.$$

Since $\displaystyle\lim_{\delta y\to 0}\frac{u(x, y+\delta y)-u(x,y)}{\delta y}=\frac{\partial u}{\partial y}$ and $\displaystyle\lim_{\delta x\to 0}\frac{u(x+\delta x, y+\delta y)-u(x, y+\delta y)}{\delta x}=\frac{\partial u}{\partial x}$

we can write $\displaystyle\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y$ approximately. ...(1)

If δx and δy are considered as small errors in the measurement of x and y, then δu, given by the above equation (1), represents the error in the calculated value of u.

EXAMPLE 13.8. Compute the approximate value of $(1.04)^{3.01}$.

[GGSIPU II Sem End Term 2006 Reappear; I Term 2011]

SOLUTION: Let $f(x, y) = x^y$. Here $x = 1$, $y = 3$ and $\delta x = 0.04$, $\delta y = 0.01$.

Therefore, $\delta f = \dfrac{\partial f}{\partial x}\delta x + \dfrac{\partial f}{\partial y}\delta y$ approximately.

$= yx^{y-1}\,\delta x + x^y \log x \,.\, \delta y = 3(1^{3-1})\,(0.04) = 0.12$

$\therefore$ $(1.04)^{3.01} = 1.12$ approx. **Ans.**

EXAMPLE 13.9. (a) Compute the value of $\left[(3.8)^2 + 2\,(2.1)^3\right]^{1/5}$ using the theory of approximation.

(b) If $f(x, y, z) = x^2y^3z^{1/10}$, using partial differentiation, find the approximate value of f when $x = 1.99$, $y = 3.01$, $z = 0.98$.

[GGSIPU II Sem I Term 2012; II Sem 2013]

SOLUTION: (a) Consider the function $u = (x^2 + 2y^3)^{\frac{1}{5}}$. Take $x = 4$ and $y = 2$ so that $x + \delta x = 3.8$ and $y + \delta y = 2.1$ then $\delta x = -0.2$ and $\delta y = 0{\cdot}1$.

The corresponding error δu in u, is given by

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y = \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}.2x\,\delta x + \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}.\,6y^2\,\delta y$$

$$= \frac{1}{5}\left(x^2 + 2y^3\right)^{\frac{-4}{5}}(2x\,\delta x + 6y^2\,\delta y)$$

$$= \frac{1}{5}\left(4^2 + 2(2)^3\right)^{-\frac{4}{5}}[\,2(4)\,(-0.2) + 6(2)^2\,(0.1)] \quad \text{at } x = 4, y = 2$$

$$= \frac{1}{5}(32)^{-\frac{4}{5}}[-1.6 + 2.4] = \frac{1}{5}\,2^{-4}\,(0.8) = 0.01.$$

Also, at $x = 4$, $y = 2$, we have $u = 2$.

$\therefore$ Required approximate value $= u + \delta u = 2.01$. **Ans.**

(b) Here $f(x, y, z) = x^2y^3z^{1/10} = 108$ at, $x = 2, y = 3, z = 1$.

and $\delta x = -0.01$, $\delta y = 0.01$ and $\delta z = -0.02$.

Using $f(x + \delta x, y + \delta y, z + \delta z) = f(x, y, z) + \left(\dfrac{\partial f}{\partial x}\delta x + \dfrac{\partial f}{\partial y}\delta y + \dfrac{\partial f}{\partial z}\delta z\right)$, we get

$$f(1.99, 3.01, 0.98) = 4 \times 27 \times 1^{1/10} + [-0.01\,(2xy^3z^{1/10}) + 0.01\,(3x^2y^2z^{1/10}) - \frac{0.02}{10}(x^2y^3z^{-9/10})]_{(2,\,3,\,1)} + \dots$$

$$= 108 + [-0.01(2 \times 2 \times 27 \times 1) + 0.01\,(3 \times 4 \times 9 \times 1) - 0.002\,(4 \times 27 \times 1)] + \dots$$

$= 108 - 0.01(108) + 0.01\,(108) - 0.002\,(108)$ approx.

$= 108\,(1 - 0.01 + 0.01 - 0.002)$ approx.

$= 107.784$ approx. **Ans.**

EXAMPLE 13.10. (a) Find the possible percentage error in computing the resistance r from the formula $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ if r_1 and r_2 are both in error by 2%.

[GGSIPU IInd Sem End Term 2010]

(b) In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m) the count of brick is 100 bricks per m^3. Find the error in the cost when the tape is streched 2% beyond its standard length. The cost of bricks is Rs. 2000 per thousand bricks.

SOLUTION : (a) From $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ or $r=\frac{r_1r_2}{r_1+r_2}$, we can have

$$-\frac{1}{r^2}\delta r \doteq -\frac{1}{r_1^2}\delta r_1-\frac{1}{r_2^2}\delta r_2 \quad \text{or} \quad \frac{\delta r}{r}=\frac{r}{r_1}\frac{\delta r_1}{r_1}+\frac{r}{r_2}\frac{\delta r_2}{r_2}=\frac{r_2}{r_1+r_2}\frac{\delta r_1}{r_1}+\frac{r_1}{r_1+r_2}\frac{\delta r_2}{r_2}$$

$$\therefore \quad \frac{\delta r}{r}\times 100 = \frac{r_2}{r_1+r_2}(2)+\frac{r_1}{r_1+r_2}(2)=2\% \qquad \textbf{Ans.}$$

(b) Volume $V=xyz$ or $\log V=\log x+\log y+\log z$.

$$\Rightarrow \quad \frac{\delta V}{V}=\frac{\delta x}{x}+\frac{\delta y}{y}+\frac{\delta z}{z} \quad \text{hence} \quad \frac{\delta V}{V}\times 100=2+2+2=6$$

or $$\delta V=\frac{6V}{100}=\frac{6(5\times 10\times 5)}{100}=15\text{m}^3.$$

The number of bricks in $\delta V = 15\times 100=1500$.

Therefore, the error in the cost $=\frac{1500\times 2000}{1000}=\text{Rs. }3000.$ **Ans.**

EXAMPLE 13.11. The acceleration of a piston is calculated from the formula $f=\omega^2 r\left(\cos\theta+\frac{r}{l}\cos 2\theta\right)$ where r and l are constants. If ω and θ suffer changes $\delta\omega$ and $\delta\theta$, then show that the resulting change δf in f, is given by

$$\frac{\delta f}{f}=2\frac{\delta\omega}{\omega}-\left(\frac{\sin\theta+\frac{2r}{l}\sin 2\theta}{\cos\theta+\frac{r}{l}\cos 2\theta}\right)\delta\theta$$

If $\theta=30^\circ$ and $\frac{r}{l}=\frac{1}{4}$, and θ and ω are each 1% less, find the corresponding percentage change in f.

SOLUTION: In the given relation, taking logarithm, we get

$$\log f=2\log\omega+\log r+\log\left(\cos\theta+\frac{r}{l}\cos 2\theta\right)$$

$$\therefore \quad \frac{\delta f}{f}=2\frac{\delta\omega}{\omega}+0+\frac{-\sin\theta-\frac{2r}{l}\sin 2\theta}{\cos\theta+\frac{r}{l}\cos 2\theta}\delta\theta \qquad \textbf{hence the first result.}$$

or $\dfrac{\delta f}{f}\times 100 = 2\dfrac{\delta\omega}{\omega}\times 100 - \dfrac{\sin\theta + \dfrac{2r}{l}\sin 2\theta}{\cos\theta + \dfrac{r}{l}\cos 2\theta}\left(\dfrac{\delta\theta\times 100}{\theta}\right).\theta.$

Taking $\theta = (\pi/6)$, $\dfrac{r}{l} = \dfrac{1}{4}$, $\dfrac{\delta\theta}{\theta}\times 100 = -1$ and $\dfrac{\delta\omega}{\omega}\times 100 = -1$, we get

$$\frac{\delta f}{f}\times 100 = 2(-1) - \frac{\sin 30^\circ + (2/4)\sin 60^\circ}{\cos 30^\circ + (1/4)\cos 60^\circ}.(-1)\left(\frac{\pi}{6}\right)$$

$$= -2 + \frac{\frac{1}{2}+\frac{1}{2}\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}+\frac{1}{8}}.\frac{\pi}{6} = -2 + \frac{4+2\sqrt{3}}{4\sqrt{3}+1}.\frac{\pi}{6}$$

$= -2 + 0.5 = 1{\cdot}5$ approximately.

$\therefore$ Error in f is 1.5 % approximately. **Ans.**

EXAMPLE 13.12. (a) If the sides and angles of a triangle ABC vary in such a way that its circum-radius remains constant, prove that $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=0$

[GGSIPU II Sem I Term 2006]

(b) In a plane triangle ABC if the sides a and b are kept constant, show that the variations of its angles are given by the relation

$$\frac{dA}{\sqrt{a^2-b^2\sin^2 A}} = \frac{dB}{\sqrt{b^2-a^2\sin^2 B}} = -\frac{dC}{c}.$$

SOLUTION: (a) By sine rule in triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad \text{(which is constant)}$$

$\Rightarrow$ $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$

hence $da = 2R\cos A\, dA$, $db = 2R\cos B\, dB$, $dc = 2R\cos C\, dC$.

$\therefore$ $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C} = 2R(dA+dB+dC)$

Since $A+B+C=\pi$ we have $dA+dB+dC=0$

Therefore $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=0$. **Hence Proved.**

(b) By the sine formula, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ or $a\sin B = b\sin A$. ...(1)

Taking differentials on both sides, we get $(a\cos B)\,dB = (b\cos A)\,dA$

$\Rightarrow$ $\dfrac{dA}{a\cos B} = \dfrac{dB}{b\cos A} = \dfrac{dA+dB}{a\cos B + b\cos A}$ (by componendo and dividendo) ...(2)

Next, $a \cos B = a\sqrt{1-\sin^2 B} = \sqrt{a^2 - a^2 \sin^2 B} = \sqrt{a^2 - b^2 \sin^2 A}$ using (1),

and $b \cos A = b\sqrt{1-\sin^2 A} = \sqrt{b^2 - b^2 \sin^2 A} = \sqrt{b^2 - a^2 \sin^2 B}$. using (1).

Also, by projection rule in Δ ABC we have $a \cos B + b \cos A = c$,

and since $A + B + C = \pi$, $dA + dB + dC = 0$ or $dA + dB = -dC$.

As such (2) becomes $\dfrac{dA}{\sqrt{a^2 - b^2 \sin A}} = \dfrac{dB}{\sqrt{b^2 - a^2 \sin^2 B}} = \dfrac{-dC}{c}$. **Hence Proved.**

EXAMPLE 13.13. The angles of a triangle are calculated from the sides a, b, c. If small changes $\delta a, \delta b, \delta c$ are made in measuring the sides a, b, c, show that the change in the angle A, is given by

$\delta A = \dfrac{a}{2\Delta}[\delta a - \delta b \cos C - \delta c \cos B]$ where Δ is the area of the triangle.

Also verify that $\delta A + \delta B + \delta C = 0$. [*GGSIPU II Sem End Term 2014*]

SOLUTION: We know that in triangle ABC, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$.

$$\therefore \quad -\sin A\, \delta A = \frac{2bc\,(2b\delta b + 2c\delta c - 2a\delta a) - (b^2+c^2-a^2)(2b\delta c + 2c\delta b)}{(2bc)^2}$$

or
$$-2\sin A\, \delta A = \frac{1}{(bc)^2}[2b^2 c\,\delta b + 2bc^2\delta c - 2abc\,\delta a - (b^2+c^2-a^2)(b\delta c + c\delta b)$$

$$= \frac{1}{(bc)^2}[\delta b\,(2b^2c - b^2c - c^3 + ca^2) + \delta c\,(2bc^2 - b^3 - bc^2 + ba^2) - 2abc\,\delta a]$$

$$= \frac{1}{(bc)^2}[c\delta b\,(b^2 + a^2 - c^2) + b\delta c\,(c^2 + a^2 - b^2) - 2abc\,\delta a]$$

$$= \frac{\delta b}{b}\left(\frac{b^2+a^2-c^2}{bc}\right) + \frac{\delta c}{c}\left(\frac{c^2+a^2-b^2}{bc}\right) - \frac{2a\delta a}{bc}.$$

$$\therefore \quad \delta A = -\frac{1.bc}{4\Delta}\left[\frac{\delta b}{b}\left(\frac{a^2+b^2-c^2}{bc}\right) + \frac{\delta c}{c}\left(\frac{c^2+a^2-b^2}{bc}\right) - \frac{2a\delta a}{bc}\right] \quad \text{where } \Delta = \frac{1}{2}bc\sin A.$$

$$= -\frac{1}{4\Delta}\left[\frac{\delta b}{b}\cos C.2ab + \frac{\delta c}{c}\cos B.2ac - 2a\delta a\right]$$

$$= -\frac{2a}{4\Delta}[\cos C\,\delta b + \cos B\,\delta c - \delta a] = \frac{a}{2\Delta}[\delta a - \delta b\cos C - \delta c\cos B].$$

Similarly, $\delta B = \dfrac{b}{2\Delta}[\delta b - \delta a\cos C - \delta c\cos A]$ and $\delta C = \dfrac{c}{2\Delta}[\delta c - \delta a\cos B - \delta b\cos A]$

$$\therefore \quad \delta A + \delta B + \delta C = \frac{1}{2\Delta}[a\delta a - a\delta b\cos C - a\delta c\cos B + b\delta b - b\delta a\cos C - b\delta c\cos A$$
$$+ c\delta c - c\delta a\cos B - c\delta b\cos A]$$

$$= \frac{1}{2\Delta}[\delta a\,(a - b\cos C - c\cos B) + \delta b\,(b - a\cos C - c\cos A)$$
$$+ \delta c\,(c - a\cos B - b\cos A)]$$

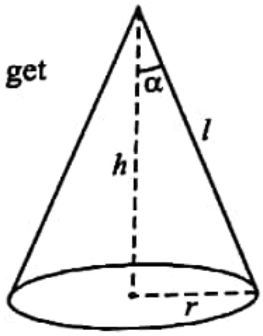
$$= \frac{1}{2\Delta}.(0) = 0$$ (using the projection rule for Δ ABC) **Hence Proved.**

EXAMPLE 13.14. The height h and the semi vertical angle α of a cone are measured and from these, A, the total area of the cone including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in A. Also show that if $\alpha = \pi/6$, an error of $+1\%$ in h will be compensated by an error of $-0.33°$ in α.

SOLUTION: Let r be the radius of the base and l the slant heihgt of the cone, then $r = h\tan\alpha$ and $l = h\sec\alpha$ $\therefore$ Total surface area $A = \pi rl + \pi r^2 = \pi h^2(\tan\alpha\sec\alpha + \tan^2\alpha)$

Hence $$\delta A = \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial \alpha}\delta\alpha$$
$$= 2\pi h(\tan\alpha\sec\alpha + \tan^2\alpha)\,\delta h + \pi h^2(\tan^2\alpha\sec\alpha + \sec^3\alpha + 2\tan\alpha\sec^2\alpha)\,\delta\alpha$$

or $$\delta A = 2\pi h\tan\alpha(\sec\alpha + \tan\alpha)\,\delta h + \pi h^2\sec\alpha(\sec\alpha + \tan\alpha)^2\,\delta\alpha$$



Putting $\alpha = \frac{\pi}{6}$ and $\frac{\delta h}{h}\times 100 = 1$, $\tan\alpha + \sec\alpha = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \sqrt{3}$, we get

$$\delta A = \frac{2\pi h^2}{100}\frac{1}{\sqrt{3}}(\sqrt{3}) + \pi h^2\frac{2}{\sqrt{3}}(3\delta\alpha) = 2\pi h^2(0.01) + \pi h^2\, 2\sqrt{3}\,\delta\alpha$$

However, as given $\delta A = 0$, therefore

$$2\pi h^2(0.01) + 2\sqrt{3}\,\pi h^2\,\delta\alpha = 0 \quad \text{or} \quad 0.01 + \sqrt{3}\,\delta\alpha = 0$$

$$\Rightarrow \quad \delta\alpha = -\frac{0.01}{\sqrt{3}}\text{ radians} = -\frac{0.01}{\sqrt{3}}\times\frac{180}{\pi}\text{ (in degrees)} = -0.33°.$$

Hence Proved.

EXAMPLE 13.15. A balloon in the form of a right cylinder of radius 1.5 m and height 4 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m how much change (%) will be in the volume of balloon?

SOLUTION : Volume of the balloon $= V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$.

Hence $$\delta V = \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h = \pi\, 2r\cdot\delta r\cdot h + \pi r^2\delta h + \frac{4}{3}\cdot\pi\, 3r^2\delta r$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\pi r[2h\,\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3} = \frac{2\delta r\cdot h + r\cdot\delta h + 4r\cdot\delta r}{rh + \frac{4}{3}r^2}$$

Here $r = 1.5$ m, $h = 4$ m. $\delta r = +0.01$ m and $\delta h = 0.05$ m

Therefore $$\frac{\delta V}{V} = \frac{2(0.01)(4) + 1.5(0.05) + 4(1.5)\,0.01}{(1.5)4 + \frac{4}{3}(1.5)^2} = \frac{0.08 + 0.075 + 0.06}{6 + 3} = 0.02389$$

and hence percentage error in volume $= \frac{\delta V}{V}\cdot 100 = 2.389\%$. **Ans.**

EXERCISE 13A

1. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ using Taylors theorem.
2. Find Taylor's expansion of the function $f(x, y) = e^{-x^2-y^2} \cos(xy)$ about the point $x_0 = 0,\ y_0 = 0$ upto three terms.
3. Expand $e^{a \sin^{-1} x}$ in ascending powers of x as far upto x^4 by Maclaurins series.
4. Expand x^y in powers of $(x - 1)$ and $(y - 1)$ upto third degree terms.
5. Find the percentage error in the calculated area of rectangle when an error of $+1\ \%$ is made in measuring its length and breadth.
6. The period of a simple pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum percentage error in T due to possible errors up to $1\ \%$ in l and $2.5\ \%$ in g.
7. In computing the volume of a right circular cone, errors of $2\ \%$ and $1\ \%$ are made in the height and the radius of the base respectively. Calculate the resulting percentage error in the volume.
8. A pile of bricks measures $2\ m \times 20\ m \times 6\ m$ and the count is 430 bricks to $1\ m^3$. Find the error in the cost when the tape is stretched by 1% beyond its standard length, the cost of bricks being Rs. 400 per thousand.
9. Find the approximate value of $[0.98^2 + 2.01^2 + 1.94^2]^{\frac{1}{2}}$.
10. The area of a triangle ABC is computed from the formula $\Delta = \frac{1}{2}\, bc \sin A$. Errors of 1%, 2% and 3% are made in measuring b, c and A. If $A = 45°$, find the percentage error in the calculated value of Δ.
11. The focal length of a mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v are both in error by 2%.
12. The resistance R of a circuit is found by the formula $I = \frac{E}{R}$. If there is an error of 0.1 ampear in reading I and 0.5 volts in E, find the corresponding possible percentage error in R when $I = 15$ amp and $E = 100$ volts.
13. The resonant frequency f in a circuit is given by $f = \frac{1}{2\pi\sqrt{LC}}$. If the measurements of inductance L and capacitance C are in error by $+2\%$ and -1% respectively, find the percentage error in the calculated value of f.
14. The inner dimensions of a rectangular box are measured to be 1, 1.5 and 3 metres subject to an error of ± 0.2 metres in each. Within what limits must the diagonal of the box lie?
15. At a distance of 120 ft. from the foot of a tower, the elevation of its top is 60°, if the possible error in measuring the distance and the elevation are 1 inch and 1 minute, find the approximate error in the calculated height.
16. The area of a triangle is calculated from the lengths a, b, c of its sides by using the formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. If a is diminished and b is increased by the same small amount k, show that the resulting change in the area is given by
$$\frac{\delta\Delta}{\Delta} = 2(a-b)\, k[c^2 - (a-b)^2]^{-1/2}$$
17. Find the approximate value of $\sqrt{24}\ (18)^{1/4}\ (30)^{1/3}$ using the concept of total derivative.

JACOBIANS

Let $u = u(x, y)$ and $v = v(x, y)$ be two continuous functions of the independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y.

Then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u and v with respect to x and y, and is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial (u, v)}{\partial (x, y)}$.

Similarly, if u, v, w are functions of x, y, z then Jacobian of u, v, w with respect to x, y, z is defined as $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial (u, v, w)}{\partial (x, y, z)}$.

CHAIN RULE FOR JACOBIANS

If u, v are functions of x, y and x, y are themselves functions of r, s then

$$\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \frac{\partial (u, v)}{\partial (r, s)}$$ [GGSIPU II Sem I Term 2011]

Proof:

$$\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \end{vmatrix} \quad \text{...(1)}$$

On the other hand, since $u = u(x, y)$, $v = v(x, y)$ and $x = x(r, s)$, $y = y(r, s)$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}.$$

Therefore (1) becomes $\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \frac{\partial (u, v)}{\partial (r, s)}$.

In general, $\frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (u_1, u_2, \ldots, u_n)} \cdot \frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (v_1, v_2, \ldots, v_n)} = \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (v_1, v_2, \ldots, v_n)}$.

Corollary of Chain Rule

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

In the above chain rule, if we replace r, s by u, v this corollary immediately follows.

If $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ then the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ is denoted by J′.

Thus, $J'=\dfrac{\partial(x,y)}{\partial(u,v)}$ and we have $JJ'=1$.

Also, in general, $\dfrac{\partial(x_1,x_2,\dots,x_n)}{\partial(u_1,u_2,\dots,u_n)}\cdot\dfrac{\partial(u_1,u_2,\dots,u_n)}{\partial(x_1,x_2,\dots,x_n)}=1.$

JACOBIAN OF IMPLICIT FUNCTIONS

If u and v are implicit functions of the variables x and y, connected by the relations

$$f_1(u,v,x,y)=0 \quad\text{and}\quad f_2(u,v,x,y)=0$$

Differentiating the above relations partially *w.r.t.* x and y separately, we get

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x}=0,\quad \frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}=0 \quad \text{...(1)}$$

and

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}=0,\quad \frac{\partial f_2}{\partial y}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}=0 \quad \text{...(2)}$$

Now,

$$\frac{\partial(f_1,f_2)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}\\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}-\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y}\\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y}\end{vmatrix}=(-1)^2\frac{\partial(f_1,f_2)}{\partial(x,y)} \quad \text{[using (1) and (2)]}$$

Thus, we have

$$\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\,\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

In the case when u, v, w are implicit functions of x, y, z, given by the relations

$$f_1(u,v,w,x,y,z)=0$$
$$f_2(u,v,w,x,y,z)=0$$
$$f_3(u,v,w,x,y,z)=0$$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\,\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}.$$

EXAMPLE 13.16. If $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, evaluate $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

SOLUTION: Since $r^2 = x^2 + y^2$ we have $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$

and as $\theta = \tan^{-1}(y/x)$, we have $\dfrac{\partial\theta}{\partial x} = \dfrac{-y/x^2}{1+y^2/x^2} = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin\theta}{r}$

and $\dfrac{\partial\theta}{\partial y} = \dfrac{1/x}{1+y^2/x^2} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$.

Therefore, $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial\theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}[\cos^2\theta + \sin^2\theta) = \frac{1}{r}$$ **Ans.**

EXAMPLE 13.17. Find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ where $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$.

[GGSIPU II Sem End Term 2009; End Term 2004 Reappear]

SOLUTION: We know that $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}. \quad ...(1)$$

Here $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

and $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Therefore, (1) gives $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$ **Ans.**

EXAMPLE 13.18. For the transformation $x = a(u+v)$, $y = b(u-v)$ and $u = r^2\cos 2\theta$, $v = r^2\sin 2\theta$, find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

SOLUTION: From the relations $x = a(u+v)$, $y = b(u-v)$, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$$

And from the relations $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix}$$

$$= 4r^3 \begin{vmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix} = 4r^3$$

By chain rule, $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(r,\theta)} = -2ab \cdot 4r^3 = -8\, abr^3.$ **Ans.**

EXAMPLE 13.19. If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$, find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.

[GGSIPU Ist Sem. End Term 2003]

SOLUTION: $$J = \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left(\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right) + \frac{x_3}{x_1}\left(\frac{x_1 x_2}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_2 x_3^2}\right) + \frac{x_2}{x_1}\left(\frac{x_3 x_1}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_3 x_2^2}\right)$$

$$= 0 + \frac{x_3}{x_1}\left(\frac{x_1}{x_3} + \frac{x_1}{x_3}\right) + \frac{x_2}{x_1}\left(\frac{x_1}{x_2} + \frac{x_1}{x_2}\right) = 2 + 2 = 4.$$ **Ans.**

EXAMPLE 13.20. If $x + y + z = u$, $y + z = uv$, $z = uvw$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

SOLUTION: The given relations can be written as

$$z = uvw, \quad y = uv - z = uv - uvw = uv\,(1 - w)$$

and $$x = u - (y + z) = u - uv = u\,(1 - v).$$

Therefore, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$ (Applying $R_2 \to R_2 + R_3$)

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = uv\,[u(1-v) + uv] = u^2 v.$$ **Ans.**

EXAMPLE 13.21. If $u = xyz, \quad v = x^2 + y^2 + z^2, \quad w = x + y + z$ find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

[GGSIPU II Sem I Term 2010; I Term 2014]

SOLUTION: Let us first calculate the value of

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} yz & z(x-y) & y(x-z) \\ 2x & 2(y-x) & 2(z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2z\,(x-y)\,(z-x) - 2y\,(y-x)\,(x-z) = -2\,(x-y)\,(z-x)\,(y-z)$$

Using the fact that $JJ' = 1$, we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = J' = \frac{1}{J} = \frac{-1}{2(x-y)\,(y-z)\,(z-x)}.$$ **Ans.**

EXAMPLE 13.22. If $u^3 + v + w = x + y^2 + z^2, \quad u + v^3 + w = x^2 + y + z^2$ and $u + v + w^3 = x^2 + y^2 + z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

[GGSIPU IInd Sem. Ist Term 2006; End Term 2006]

SOLUTION: The given relations can be written as implicit functions, as

$$f_1\,(u, v, w, x, y, z) = u^3 + v + w - x - y^2 - z^2 = 0$$
$$f_2\,(u, v, w, x, y, z) = u + v^3 + w - x^2 - y - z^2 = 0$$
$$f_3\,(u, v, w, x, y, z) = u + v + w^3 - x^2 - y^2 - z = 0$$

Now $$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix} = 3u^2\,(9v^2w^2 - 1) - 1\,(3w^2 - 1) + 1\,(1 - 3v^2)$$

$$= 27\,u^2v^2w^2 - 3u^2 - 3v^2 - 3\,w^2 + 2$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix} = -1\,(1 - 4yz) + 2y\,(2x - 4xz) - 2z\,(4xy - 2x)$$

$$= -1 + 4\,(xy + yz + zx) - 16\,xyz$$

Therefore $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\,\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} \Big/ \frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}$$

$$= \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$ **Ans.**

PARTIAL DERIVATIVES FROM IMPLICIT FUNCTIONS USING JACOBIANS

Suppose u, v are implicit functions of the independent variables x, y connected by the functional relations

$$f_1(u, v, x, y) = 0 \quad \text{... (1)}$$

and

$$f_2(u, v, x, y) = 0. \quad \text{... (2)}$$

To obtain partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ we partially differentiate (1) and (2) w.r.t. x and y separately, and get

$$\left.\begin{array}{l}\frac{\partial f_1}{\partial x}.1+\frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial x}=0\\ \frac{\partial f_2}{\partial x}.1+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}=0\end{array}\right\} \quad \text{... (3)}$$

and

$$\left.\begin{array}{l}\frac{\partial f_1}{\partial y}.1+\frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial y}=0\\ \frac{\partial f_2}{\partial y}.1+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}=0\end{array}\right\} \quad \text{... (4)}$$

Solving (3) for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, we get

$$\frac{\partial u}{\partial x} = -\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v}\end{vmatrix} \Bigg/ \begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix} = -\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$$

[*GGSIPU IInd Sem. Ist Term 2005*]

and

$$\frac{\partial v}{\partial x} = -\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial x}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial x}\end{vmatrix} \Bigg/ \begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix} = -\frac{\frac{\partial (f_1, f_2)}{\partial (u, x)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$$

Similarly, solving (4) for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial (f_1, f_2)}{\partial (y, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial (f_1, f_2)}{\partial (u, y)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}.$$

EXAMPLE 13.23. If $u^2 + xv^2 = x + y$, $v^2 + yu^2 = x - y$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, using Jacobians.

SOLUTION: The functional relations are

$$f_1(u, v, x, y) = u^2 + xv^2 - x - y = 0 \quad \text{and} \quad f_2(u, v, x, y) = v^2 + yu^2 - x + y = 0$$

Now

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}} = -\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v}\end{vmatrix} \Bigg/ \begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}$$

$$= -\begin{vmatrix} v^2-1 & 2xv \\ -1 & 2v \end{vmatrix} \Big/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix} = -\frac{1}{2}\frac{(v^3-v+xv)}{uv-uvxy} = \frac{-1(v^2-1+x)}{2u(1-xy)}.$$

and $$\frac{\partial v}{\partial y} = \frac{-\dfrac{\partial(f_1,f_2)}{\partial(u,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = -\begin{vmatrix} 2u & -1 \\ 2yu & 1 \end{vmatrix} \Big/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix} = \frac{-1}{2}\frac{(u+yu)}{(uv-xyuv)} = \frac{-(1+y)}{2v(1-xy)}.$$ **Ans.**

Example 13.24. Using Jacobians find $\dfrac{\partial u}{\partial x}$ if $u^2+xy^2-xy=0$ and $u^2+uvx+v^2=0.$

[GGSIPU II Sem I Term 2011]

Solution: Since $u^2=x(y-y^2)$ we can find $\dfrac{\partial u}{\partial x}$ without using Jacobians also, as follows:

$$2u\frac{\partial u}{\partial x} = y-y^2 \qquad \therefore \qquad \frac{\partial u}{\partial x} = \frac{y(1-y)}{2u} = \frac{u^2}{2ux} = \frac{u}{2x}.$$ **Ans.**

However, by the method of Jacobians, we have

$$f_1 \equiv u^2+xy^2-xy=0, \quad f_2 \equiv u^2+uvx+v^2=0$$

$$\therefore \qquad \frac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} y^2-y & 0 \\ uv & ux+2v \end{vmatrix} = -\frac{u^2}{x}(ux+2v)$$

and $$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 2u+vx & ux+2v \end{vmatrix} = 2u(ux+2v)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{\dfrac{u^2}{x}(ux+2v)}{2u(ux+2v)} = \frac{u}{2x}.$$ **Ans.**

Example 13.25. If $u=x+y^2, v=y+z^2, w=z+x^2,$ prove that $\dfrac{\partial x}{\partial u} = -(1+8xyz)^{-1}.$

[GGSIPU II Sem I Term 2005]

Solution: We are given three functional relations in $u, v, w, x, y, z,$ as

$$f_1 = u-x-y^2=0,$$
$$f_2 = v-y-z^2=0$$

and $$f_3 = w-z-x^2=0.$$

then $$\frac{\partial x}{\partial u} = \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = \frac{\begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}} = \frac{-1}{1+8xyz}.$$ **Hence Proved.**

JACOBIANS TO DETERMINE FUNCTIONAL DEPENDENCE

Jacobian is also used in determining whether or not two functions are functionally dependent. Two functions $f(x, y)$ and $\phi(x, y)$ are called functionally dependent if they are functions of each other.

Assume that $f(x,y)$ and $\phi(x,y)$ are functionally dependent then there exists a relation $F(f, \phi) = 0$.

Differentiating it partially *w.r.t.* x and y, we get

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial y} = 0$$

These are homogeneous equations. Their non-trivial solution would exist only when

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial \phi}{\partial y} \end{vmatrix} = 0, \quad i.e. \quad \frac{\partial(f, \phi)}{\partial(x, y)} = 0$$

Hence $f(x, y)$ and $\phi(x, y)$ **are functionally dependent if their Jacobian vanishes identically.**

The idea can be easily extended to three functions and, in general, to n functions as well.

EXAMPLE 13.26. Examine the functional dependence of $u = \dfrac{x-y}{1+xy}$ and $v = \tan^{-1}x - \tan^{-1}y$.

If dependent, find the relation. *[GGSIPU II Sem I Term 2011; I Term 2012]*

SOLUTION: $\dfrac{\partial u}{\partial x} = \dfrac{1+y^2}{(1+xy)^2}$, $\dfrac{\partial u}{\partial y} = \dfrac{-(1+x^2)}{(1+xy)^2}$, $\dfrac{\partial v}{\partial x} = \dfrac{1}{1+x^2}$ and $\dfrac{\partial v}{\partial y} = \dfrac{-1}{1+y^2}$.

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{1+y^2}{(1+xy)^2} & \dfrac{-(1+x^2)}{(1+xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{-1}{1+y^2} \end{vmatrix} = \frac{1}{(1+xy)^2}\begin{vmatrix} 1+y^2 & -(1+x^2) \\ \dfrac{1}{1+x^2} & \dfrac{-1}{1+y^2} \end{vmatrix} = 0$$

$\Rightarrow$ u and v are functionally dependent. Clearly the relation between them is $u = \tan v$. **Ans.**

EXAMPLE 13.27. Show that the functions $u = x + y + z$, $v = x^3 + y^3 + z^3 - 3xyz$, and $w = x^2 + y^2 + z^2 - xy - yz - zx$ are functionally dependent.

Find the relation between them. *[GGSIPU II Sem End Term 2005]*

SOLUTION: The functions u, v, w are functionally dependent if $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \\ 2x - y - z & 2y - z - x & 2z - x - y \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2 - yz & y^2 - x^2 + z(y-x) & z^2 - x^2 + y(z-x) \\ 2x - y - z & 3(y-x) & 3(z-x) \end{vmatrix}$$

$$= 6\begin{vmatrix} (y-x)(x+y+z) & (z-x)(x+y+z) \\ y-x & z-x \end{vmatrix} = 6(y-x)(z-x)(x+y+z)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Therefore u, v, w are functionally dependent.

Next, $u = x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) = vw$

Hence $u = vw$ is the desired relation. **Ans.**

EXERCISE 13B

1. Find the Jacobian of the following transformations
$u = \cos x,\ v = \sin x \cos y,\ w = \sin x \sin y \cos z$.
2. If (x, y, z) and (r, θ, ϕ) are respectively the cartesian and spherical polar coordinates of a point then show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.
3. For the transformation $x = e^u \cos v,\ y = e^u \sin v$ show that $\dfrac{\partial(u, v)}{\partial(x, y)} = e^{-2u}$.
4. Verify $JJ' = 1$ when $x = v^2 + w^2,\ y = w^2 + u^2,\ z = u^2 + v^2$. where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.
5. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.
6. If $u^3 + v^3 + w^3 = x + y + z,\ u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.
7. If $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$, find $\dfrac{\partial x}{\partial u}$.
8. If $x + y + z + u + v = a$ and $x^2 + y^2 + z^2 + u^2 + v^2 = b$ where a and b are constants, find $\left(\dfrac{\partial v}{\partial y}\right)_{(x,u)}$ and $\left(\dfrac{\partial y}{\partial v}\right)_{(x,z)}$.
9. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = xy + yz + zx$ investigate whether u, v, w are functionally dependent on x, y, z, or not. If yes, find the relation between them.
10. If $u = \sin^{-1} x + \sin^{-1} y,\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ find out if they are functionally dependent and if so, find the relation between them.
11. Investigate if $u = \dfrac{x - y}{x + z},\ v = \dfrac{x + z}{y + z}$ are functionally dependent and if so, find the functional relation between them.
12. If u and v are functions of x and y then show that $\dfrac{\partial(u, v)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(u, v)} = 1$
13. Given that $x = u + v + w,\ y = u^2 + v^2 + w^2,\ z = u^3 + v^3 + w^3$ find $\dfrac{\partial u}{\partial x}$.
14. Show that the functions $u = x + y + z,\ v = x^3 + y^3 + z^3 - 3xyz$,
$w = x^2 + y^2 + z^2 - xy - yz - zx$ are functionally dependent. Also find the relation between them. *[GGSIPU IInd Sem. End Term 2005]*
15. If $u = \dfrac{y^2}{2x},\ v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.
16. If $u^3 + v^3 = x + y,\ u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{y^2 - x^2}{2uv(u - v)}$. *[GGSIPU IInd Sem. I Term 2013]*
17. If u, v, w are the roots of the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$ find $\dfrac{\partial(u, v, w)}{\partial(a, b, c)}$.

MAXIMA AND MINIMA OF A FUNCTION OF TWO OR MORE VARIABLES

We shall simply extend the definition of maxima and minima of functions of one variable to functions of two variables. The function $f(x, y)$ has a maximum value for a certain pair of values of x and y, if this value, say $f_{\max}$, is greater than the values of $f(x, y)$ for all values of x and y in the small neighbourhood of the particular pair of values, *i.e.*, $f(x, y) > f(x+h, y+k)$ where h and k are small. Similary, a minimum value of $f(x, y)$ is defined.

Thus, the quantity $f(x, y) - f(x+h, y+k)$ must retain a constant sign for small variations in h and k. When this constant sign is positive, there exists a maximum value of $f(x, y)$ and when it is negative there exists a minimum value of $f(x, y)$. Clearly, if the sign does not remain constant, there will be *neither a maximum nor a minimum* and then the point is called *saddle point.*

To obtain the criterion for maxima and minima, recall the Taylor's expansion of $f(x + h, y + k)$ as

$$f(x+h, y+k) - f(x, y) = (hf_x + kf_y) + \frac{1}{2\,!}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}) + \ldots \qquad \ldots(1)$$

When h and k are small the first term on the R.H.S. of (1) governs the sign of the R.H.S. and hence, for constant sign on the R.H.S., we must have

$$f_x = p = 0 \quad \text{and} \quad f_y = q = 0 \qquad \ldots(2)$$

as preliminary condition for the existence of maxima and minima. A point satisfying the conditions in (2), is called a **stationary point or critical point**. Then, in that case, eqn. (1) becomes

$$f(x+h, y+k) - f(x, y) = \frac{1}{2\,!}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}) + \ldots \qquad \ldots(3)$$

Again since h and k are small, the sign of R.H.S. of (3) will be governed by the sign of its first term. Now, consider

$$h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy} = \frac{1}{f_{xx}}\left[h^2 f_{xx}^2 + 2hk\, f_{xy}\, f_{xx} + k^2\, f_{xx}\, f_{yy}\right]$$

$$= \frac{1}{f_{xx}}\left[(h\, f_{xx} + k\, f_{xy})^2 + k^2(f_{xx}\, f_{yy} - f_{xy}^2)\right] = \frac{1}{r}[(hr+ks)^2 + k^2(r+s^2)]$$

If $f_{xx}f_{yy} - f_{xy}^2 \geq 0$ or $rt - s^2 \geq 0$, the above expression in the square brackets is always positive and there will be maxima if $f_{xx} (= r)$ is negative and minima if $f_{xx} (= r)$ is positive. Whereas, if $f_{xx}f_{yy} - f_{xy}^2 < 0$ or $rt - s^2 < 0$, the expression in the square brackets can change in sign and therefore there will be neither maxima nor minima. Thus, provided that f_{xx}, f_{yy} and f_{xy} are not all zero, the criteria for the existence of maxima and minima runs as follows: .

(*i*) $f_x = f_y = 0$, or $p = q = 0$ the solution of which gives pairs of values of x and y, called *critical points or stationary points.*

(*ii*) Compute $f_{xx} (= r)$, $f_{yy} (= t)$ and $f_{xy} (= s)$ for each pair (x, y). If $f_{xx}\, f_{yy} - f_{xy}^2 > 0$, or $rt - s^2 > 0$, there will be a maxima if f_{xx} (or f_{yy}) < 0 and minimum if f_{xx} (or f_{yy}) > 0.

(*iii*) If $f_{xx}\, f_{yy} - f_{xy}^2 < 0$ there will be neither a maxima nor minima and points are called ***saddle points.***

EXAMPLE 13.28. Find the maximum and minimum values of the function
$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. [GGSIPU III Sem End Term 2009, 2013]

SOLUTION: $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\therefore \quad p = \dfrac{\partial f}{\partial x} = 4x^3 - 4x + 4y$ and $q = \dfrac{\partial f}{\partial y} = 4y^3 + 4x - 4y$.

For stationary points $p = 0,\ q = 0$, that is

$$x^3 - x + y = 0 \quad \text{and} \quad y^3 - y + x = 0 \qquad ...(2)$$

Adding (1) and (2), gives $x^3 + y^3 = 0 \quad \therefore \quad y = -x$

and substracting (2) from (1) gives $x^3 - y^3 - 2(x - y) = 0$ or $(x - y)(x^2 + y^2 + xy - 2) = 0$

$\Rightarrow \quad x = y$ or $x^2 + y^2 + xy = 2$.

Putting $y = -x$ we get $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$.

Thus, the stationary points are $(0, 0), (\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$.

Now, $r = 12x^2 - 4,\ t = 12y^2 - 4,\ s = 4$.

$\therefore \quad rt - s^2 = 16(3x^2 - 1)(3y^2 - 1) - 16$.

At $(0, 0)$, $r = -4, s = 4, t = -4$ so $rt - s^2 = 0$.

At $(\pm\sqrt{2}, \pm\sqrt{2})$, $r = 44 = t$ and $rt - s^2 > 0$.

Hence Max $f(x, y) = 8$ and Min $f(x, y) = -8$. **Ans.**

EXAMPLE 13.29. Find the dimensions of a rectangular box (without top) with a given volume, so that the material used is minimum. [GGSIPU IInd Sem. End Term 2007]

SOLUTION: Let the length, breadth and height of the box be x, y and z respectively and V be the given volume, then $V = xyz =$ constant.

The surface area $S = xy + 2yz + 2zx$. ... (1)

Substituting $z = \dfrac{V}{xy}$ in (1), we get $S = xy + 2(x + y)\dfrac{V}{xy} = xy + \dfrac{2V}{y} + \dfrac{2V}{x}$. ... (2)

Therefore, $\dfrac{\partial S}{\partial x} = y - \dfrac{2V}{x^2}$ and $\dfrac{\partial S}{\partial y} = x - \dfrac{2V}{y^2}$.

At stationary points $\dfrac{\partial S}{\partial x} = 0,\ \dfrac{\partial S}{\partial y} = 0$, *i.e.*, $x^2 y = 2V$ and $xy^2 = 2V$.

$\Rightarrow \quad xy(x - y) = 0$ which gives $x = 0,\ y = 0,\ x = y$ when $x = y,\ x^3 = y^3 = 2V$.

$\therefore$ Stationary points are $(0, 0)$ and $\left((2V)^{1/3}, (2V)^{1/3}\right)$.

Next, $\dfrac{\partial^2 S}{\partial x^2} = \dfrac{4V}{x^3},\quad \dfrac{\partial^2 S}{\partial x \partial y} = 1,\quad \dfrac{\partial^2 S}{\partial y^2} = \dfrac{4V}{y^3}$.

At $(0, 0)$ $\dfrac{\partial^2 S}{\partial x^2}$ and $\dfrac{\partial^2 S}{\partial y^2}$ are not defined, therefore, neither maxima nor minima at $(0, 0)$.

At $\left((2V)^{1/3}, (2V)^{1/3}\right)$, $\dfrac{\partial^2 S}{\partial x^2} = 2,\ \dfrac{\partial^2 S}{\partial y^2} = 2$ hence

$$\frac{\partial^2 S}{\partial x^2}\frac{\partial^2 S}{\partial y^2} - \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2 = 4 - 1 > 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial x^2} > 0$$

hence S has minimum value at $x = (2V)^{1/3},\ y = (2V)^{1/3}$.

At this point $z = \dfrac{V}{xy} = \dfrac{V}{(2V)^{1/3}.(2V)^{1/3}} = \dfrac{V^{1/3}}{2^{2/3}} = \dfrac{x}{2}.$

Therefore, the box should have a square base and the height should be half the length of the base for material being used to be least, in making the box. **Ans.**

EXAMPLE 13.30. Examine the function $f(x, y) = \sin x + \sin y + \sin(x + y)$ for maximum and minimum values. [GGSIPU II Sem End Term 2005]

SOLUTION: $f(x, y) = \sin x + \sin y + \sin(x + y)$

Hence $\dfrac{\partial f}{\partial x} = p = \cos x + \cos(x + y) = 2\cos\left(x+\dfrac{y}{2}\right)\cos\dfrac{y}{2}$

and $\dfrac{\partial f}{\partial y} = q = \cos y + \cos(x + y) = 2\cos\left(y+\dfrac{x}{2}\right)\cos\dfrac{x}{2}.$

For f to be maximum or minimum, $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$, that is,

$\cos\left(x+\dfrac{y}{2}\right)\cos\dfrac{y}{2} = 0$ and $\cos\left(y+\dfrac{x}{2}\right)\cos\dfrac{x}{2} = 0$

which gives $x = \pi$, $y = \pi$; $x = \pi/3$, $y = \pi/3$

∴ Critical points are $(\pi/3, \pi/3)$, and (π, π) if x and $y \in [0, \pi]$.

Next, $\dfrac{\partial^2 f}{\partial x^2} = r = -\sin x - \sin(x+y)$, $\dfrac{\partial^2 f}{\partial y^2} = t = -\sin y - \sin(x+y)$

and $\dfrac{\partial^2 f}{\partial x \partial y} = s = -\sin(x+y).$

At (π, π), $r = s = t = 0$, hence it is a saddle point.

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $r = -2\sin\pi/3 = -\sqrt{3}$, $t = -\sqrt{3}$, $s = -\sqrt{3}/2$. ∴ $rt - s^2 = \dfrac{9}{4} > 0$ and $r < 0$.

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and (π, π) is saddle point. **Ans.**

EXAMPLE 13.31. In a plane triangle ABC find the maximum value of $\cos A \cos B \cos C$. [GGSIPU IInd Sem End Term 2010; End Sem 2012]

SOLUTION: In triangle ABC, $A + B + C = \pi$, hence

$$\cos A \cos B \cos C = \cos A \cos B \cos(\pi - A - B) = -\frac{1}{2}[\cos(A+B) + \cos(A-B)]\cos(A+B)$$

$$= -\frac{1}{4}[1 + \cos(2A+2B) + \cos 2A + \cos 2B] = f(A, B), \text{ say.}$$

then $\dfrac{\partial f}{\partial A} = \dfrac{1}{2}\sin(2A+2B) + \dfrac{1}{2}\sin 2A$ and $\dfrac{\partial f}{\partial B} = \dfrac{1}{2}\sin(2A+2B) + \dfrac{1}{2}\sin 2B$.

For f to be maximum $\dfrac{\partial f}{\partial A} = 0$, $\dfrac{\partial f}{\partial B} = 0$ or $\sin(2A + B)\cos B = 0$ and $\sin(2B + A)\cos A = 0$.

Now if $\cos B = 0$ then $B = \dfrac{\pi}{2}$ which gives $\sin A \cos A = 0$ or $A = \dfrac{\pi}{2}$ which is not possible.

Therefore, $2A + B = \pi$ and $2B + A = \pi \Rightarrow A = \pi/3$, $B = \pi/3$.

Next, $\dfrac{\partial^2 f}{\partial A^2} = \cos(2A+2B)+\cos 2A$, $\dfrac{\partial^2 f}{\partial A \partial B} = \cos(2A+2B)$, $\dfrac{\partial^2 f}{\partial B^2} = \cos(2A+2B)+\cos 2B$.

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial^2 f}{\partial A^2} = -\cos\dfrac{\pi}{3} - \cos\dfrac{\pi}{3}$, $\dfrac{\partial^2 f}{\partial A \partial B} = -\cos\dfrac{\pi}{3}$, $\dfrac{\partial^2 f}{\partial B^2} = -\cos\dfrac{\pi}{3} - \cos\dfrac{\pi}{3}$.

Here, $\dfrac{\partial^2 f}{\partial A^2}\cdot\dfrac{\partial^2 f}{\partial B^2} - \left(\dfrac{\partial^2 f}{\partial A \partial B}\right)^2 = 4\cos^2\dfrac{\pi}{3} - \cos^2\dfrac{\pi}{3} = 3\cos^2\dfrac{\pi}{3} > 0$ and $\dfrac{\partial^2 f}{\partial A^2} < 0$, hence Maxima at $(\pi/3, \pi/3)$.

and Max. value of $(\cos A \cos B \cos C) = \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$. **Ans.**

EXAMPLE 13.32. Examine the function $f(x, y) = x^3 - 3x^2 - 4y^2 + 1$ for maximum and minimum.

[GGSIPU II Sem End Term 2006; End Term 2005]

SOLUTION: $f(x, y) = x^3 - 3x^2 - 4y^2 + 1$. Here $\dfrac{\partial f}{\partial x} = 3x^2 - 6x$, $\dfrac{\partial f}{\delta y} = -8y$.

For $f(x, y)$ to have maximum or minimum, $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\delta y}$.

Thus, the critical points are $(0, 0)$ and $(2, 0)$.

Next, $r = \dfrac{\partial^2 f}{\partial x^2} = 6x - 6$, $s = \dfrac{\partial^2 f}{\partial x \partial y} = 0$, $t = \dfrac{\partial^2 f}{\partial y^2} = -8$.

At $(0, 0)$, $r < 0, t < 0$ and $rt - s^2 = 48 > 0$ hence maxima.

At $(2, 0)$, $r = 6, s = 0, t = -8$ hence $rt - s^2 < 0$ $\therefore$ saddle point at $(2, 0)$. **Ans.**

EXAMPLE 13.33. Discuss the maxima and minima of $x^3y^2(1 - x - y)$.

[GGSIPU II Sem I Term 2006; End Term 2011]

SOLUTION: Given function is $f = x^3y^2(1 - x - y)$.

$\therefore$ $\dfrac{\partial f}{\partial x} = -3x^2y^2(x + y - 1) - x^3y^2 = -x^2y^2[4x + 3y - 3]$ and $\dfrac{\partial f}{\partial y} = -x^3y[2x + 3y - 2]$.

For f to be maximum or minimum $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$

$\Rightarrow$ $x = 0$, $y = 0$ and $x = \dfrac{1}{2}$, $y = \dfrac{1}{3}$. Thus, the critical points are $(0, 0)$ and $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$.

Next, $r = \dfrac{\partial^2 f}{\partial x^2} = -2xy^2(4x+3y-3) - 4x^2y^2$, $t = \dfrac{\partial^2 f}{\partial y^2} = -x^3(2x+3y-2) - 3x^3y$

and $s = \dfrac{\partial^2 f}{\partial x \partial y} = -2x^2y(4x+3y-3) - 3x^2y^2$

At $(0, 0)$ $r = s = t = 0$ hence neither nor maxima here

and at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$, $r = -\dfrac{1}{9}$, $s = \dfrac{-1}{12}$, $t = \dfrac{-1}{8}$ hence $rt - s^2 = \dfrac{1}{72} - \dfrac{1}{144}$.

Since $rt - s^2 > 0$ and r and t are negative, we have a maxima at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$. **Ans.**

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

[GGSIPU II Sem End Term 2006]

At occasions it would be necessary to find the maxima or minima of a function subject to one or two conditions (or constraints) being satisfied. Suppose we are to maximise or minimise the function $u = f(x, y)$ subject to the condition $g(x, y) = 0$. If we can solve the latter equation for y in terms of x and substitute it in the function $f(x, y)$, then the problem reduces to that of finding the maxima or minima of a function of a single variable x. However, usually it is not possible and we take recourse to Lagrange's method of undetermined multipliers as explained below:

Since $u = f(x, y)$ we have $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x}.1 + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$ ... (1)

and from the relation $g(x, y) = 0$ we have $\dfrac{\partial g}{\partial x}.1 + \dfrac{\partial g}{\partial y}\dfrac{dy}{dx} = 0$ ... (2)

From (2), we have $\dfrac{dy}{dx} = \dfrac{-\partial g/\partial x}{\partial g/\partial y} = -\dfrac{g_x}{g_y}$ on the curve $g(x, y) = 0$.

But on $g(x, y) = 0$ the function $u = f(x, y)$ becomes a function of x only, as mentioned above, hence the stationary points are given by $\dfrac{du}{dx} = 0$, that is,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\left(\frac{-\partial g/\partial x}{\partial g/\partial y}\right) = 0 \quad \text{or} \quad f_x\, g_y - f_y\, g_x = 0 \qquad ...(3)$$

which can be solved subject to the condition $g(x, y) = 0$. ...(4)

Algebraically, this is equivalent to finding the stationary points of a function F given by

$$F(x, y) = f(x, y) + \lambda g(x, y) \qquad ...(5)$$

where λ is a parameter to be determined.

The stationary points of $F(x, y)$ are given by the equations

$$f_x + \lambda\, g_x = 0 \quad \text{and} \quad f_y + \lambda\, g_y = 0$$

which will have a solution if the condition (3) is satisfied. The maxima or minima can then be obtained by examining the condition in the neighbourhood of the stationary points.

Further, suppose it is required to find the stationary values of a function of three variables, say

$$u = f(x, y, z) \qquad ...(1)$$

subject to conditions $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. ... (2)

In this case we construct the function F as

$$F = f + \lambda_1 \phi + \lambda_2 \psi \qquad ...(3)$$

where λ_1, λ_2 are called non-zero Lagrange's multipliers.

Next, form the equations $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial z} = 0$ ... (4)

Eliminating $x, y, z, \lambda_1, \lambda_2$ between (1), (2) and (4) we shall get an equation in u, the roots of which give the stationary values of $u = f(x, y, z)$.

The method would be more clear through the following examples.

EXAMPLE 13.34. Use Lagrange's method of multipliers to find the smallest and largest value of $x + 2y$ on the circle $x^2 + y^2 = 1$. *[GGSIPU II Sem End Term 2011]*

SOLUTION: Here $f(x,y) = x + 2y$, $\phi(x,y) = x^2 + y^2 - 1 = 0$.

Let $F = x + 2y + \lambda(x^2 + y^2 - 1)$ where λ is undetermined multiplier.

$$\therefore \quad \frac{\partial F}{\partial x} = 1 + 2\lambda x, \quad \frac{\partial F}{\partial y} = 2 + 2\lambda y. \quad \text{For } F \text{ to be Max. and Min.} \quad \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0.$$

$$\therefore \quad 1 + 2\lambda x = 0 \quad \text{and} \quad 2 + 2\lambda y = 0 \quad \text{or} \quad x = \frac{-1}{2\lambda}, \quad y = \frac{-1}{\lambda}$$

Since $x^2 + y^2 = 1$ we have $\dfrac{1}{4\lambda^2} + \dfrac{1}{\lambda^2} = 1$, $\quad \therefore \quad \lambda = \dfrac{+\sqrt{5}}{2}$

Thus, $\quad x + 2y = -\dfrac{1}{2\lambda} - \dfrac{2}{\lambda} = -\dfrac{5}{2\lambda}$

$\therefore$ Max $(x + 2y) = \sqrt{5}$ and Min $(x + 2y) = -\sqrt{5}$. **Ans.**

EXAMPLE 13.35. Determine the maxima of the function $u = (x + 1)(y + 1)(z + 1)$ subject to the condition $a^x b^y c^z = k$ where a, b, c, k are constants.

SOLUTION: From the given function and the given condition, we can write

$$\log u = \log(x + 1) + \log(y + 1) + \log(z + 1) \qquad \text{...(1)}$$

$$\text{and} \quad x \log a + y \log b + z \log c = \log k \qquad \text{...(2)}$$

Now, when u is maximum $\log u$ will also be maximum. For finding critical points, we differetiate (1) and (2) and get

$$\frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1} = 0 \qquad \text{...(3)}$$

$$\text{and} \quad \log a \, dx + \log b \, dy + \log c \, dz = 0. \qquad \text{...(4)}$$

Multiplying (4) by λ and then adding to (3) and equating to zero the co-efficients of dx, dy, dz, we get $\dfrac{1}{x+1} + \lambda \log a = 0$, $\quad \dfrac{1}{y+1} + \lambda \log b = 0$, $\quad \dfrac{1}{z+1} + \lambda \log c = 0$

$$\text{or} \quad \left.\begin{aligned} 1 + \lambda x \log a + \lambda \log a &= 0 \\ 1 + \lambda y \log b + \lambda \log b &= 0 \\ 1 + \lambda z \log c + \lambda \log c &= 0 \end{aligned}\right\} \qquad \text{...(5)}$$

Adding these, we get $3 + \lambda(x \log a + y \log b + z \log c) + \lambda(\log a + \log b + \log c) = 0$

$$\text{or} \quad 3 + \lambda \log k + \lambda \log(abc) = 0 \quad \therefore \quad \lambda = \frac{-3}{\log(k\,abc)}.$$

Substituting this value of λ in (5), we get

$$x + 1 = \frac{-1}{\lambda \log a} = \frac{\log(k\,abc)}{3 \log a}, \quad y + 1 = \frac{\log(k\,abc)}{3 \log b}, \quad z + 1 = \frac{\log(k\,abc)}{3 \log c}$$

Thus, the maxima is given by $x = \dfrac{\log\left(k\dfrac{bc}{a^2}\right)}{3 \log a}$, $\quad y = \dfrac{\log\left(k\dfrac{ca}{b^2}\right)}{3 \log b}$, $\quad z = \dfrac{\log\left(k\dfrac{ab}{c^2}\right)}{3 \log c}$. **Ans.**

EXAMPLE 13.36. The temperature T at any point (x, y, z) of space is given by $T = 400\,xyz^2$ find the highest temperature at the surface of the sphere $x^2 + y^2 + z^2 = 1$.

[GGSIPU II Sem I Term 2005]

SOLUTION: $T = 400\,xyz^2$ where $x^2 + y^2 + z^2 = 1$. Let $T' = xyz^2 = xy\,(1 - x^2 - y^2)$

hence $$\frac{\partial T'}{\partial x} = y\,(1 - x^2 - y^2) + xy\,(-2x) = -\,y\,(3x^2 + y^2 - 1)$$

and $$\frac{\partial T'}{\partial y} = x(1 - x^2 - y^2) + xy\,(-2y) = -\,x\,(3y^2 + x^2 - 1).$$

For T' to be maximum or minimum $\dfrac{\partial T'}{\partial x} = \dfrac{\partial T'}{\partial y} = 0$.

Thus, $y\,(3x^2 + y^2 - 1) = 0$ and $x\,(3y^2 + x^2 - 1) = 0$.

The above relations when solved for x and y, give $x = 0$, $y = 0$ and $x^2 = \dfrac{1}{4}$, $y^2 = \dfrac{1}{4}$.

Thus the critical points are $(0, 0), \left(\dfrac{1}{2}, \dfrac{1}{2}\right), \left(\dfrac{1}{2}, \dfrac{-1}{2}\right), \left(\dfrac{-1}{2}, \dfrac{1}{2}\right), \left(-\dfrac{1}{2}, \dfrac{-1}{2}\right)$.

Next, $r = \dfrac{\partial^2 T'}{\partial x^2} = -6xy$, $t = \dfrac{\partial^2 T'}{\partial y^2} = -6xy$ and $s = \dfrac{\partial^2 T'}{\partial x \partial y} = -3x^2 - 3y^2 + 1$.

At $(0, 0)$, $r = t = 0, s = 1$ so $rt < s^2$. Hence $(0, 0)$ is a saddle point.

At $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{-1}{2}\right)$, $r = -\dfrac{3}{4}, t = \dfrac{-3}{4}, s = \dfrac{-1}{2}$ $\therefore$ $rt - s^2 = \dfrac{5}{16} > 0$ and $r < 0$

and at $\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$, $r = \dfrac{3}{4} = t$ and $s = \dfrac{-1}{2}$ $\therefore$ $rt - s^2 = \dfrac{5}{16} > 0$ and $r > 0$.

$\therefore$ $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{-1}{2}\right)$ are maxima and $\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ are minima.

Thus maximum $T = 400.\,\dfrac{1}{2}.\dfrac{1}{2}.\left(1 - \dfrac{1}{4} - \dfrac{1}{4}\right) = 50$. **Ans.**

EXAMPLE 13.37. (*a*) Find the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the fulfilment of the condition $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$.

(*b*) If $\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} = 6$ find the values of x, y and z which makes $x + y + z$ maximum.

[GGSIPU II Sem End Term 2013]

SOLUTION: (*a*) Let $f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$ and $\phi(x, y, z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0$.

Now, consider the function $F = f + \lambda\,\phi$ where λ is an unknown non-zero quantity

or $$F = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right).$$

For stationary points, we have $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0.$

or $\quad 2a^3x - \dfrac{\lambda}{x^2} = 0, \quad 2b^3y - \dfrac{\lambda}{y^2} = 0, \quad 2c^3z - \dfrac{\lambda}{z^2} = 0$ or $2a^3x^3 = 2b^3y^3 = 2c^3y^3 = \lambda.$

or $\quad ax = by = cz = K,$ say. $\Rightarrow \quad x = \dfrac{K}{a}, \quad y = \dfrac{K}{b}, \quad z = \dfrac{K}{c}.$

Substituting these in $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1,$ we get $K = a + b + c.$

Hence the stationary point is given by $x = \dfrac{a+b+c}{a}, \; y = \dfrac{a+b+c}{b}, \; z = \dfrac{a+b+c}{c}$

$\therefore$ Stationary value of $f = \dfrac{a^3(a+b+c)^2}{a^2} + \dfrac{b^3(a+b+c)^2}{b^2} + \dfrac{c^3(a+b+c)^2}{c^2}$

$= (a + b + c)^3.$ **Ans.**

(*b*) $f(x, y, z) = x + y + z.$ Here $\phi(x, y, z) = \dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} - 6 = 0.$

Consider $F = f + \lambda\phi = x + y + z + \lambda\left(\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} - 6\right).$

$f(x, y, z)$ will be maximum when $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0.$

that is $\quad 1 - \dfrac{3\lambda}{x^2} = 0, \quad 1 - \dfrac{4\lambda}{y^2} = 0, \quad 1 - \dfrac{5\lambda}{z^2} = 0$ or $x = \sqrt{3\lambda}, \quad y = \sqrt{4\lambda}, \quad z = \sqrt{5\lambda}.$

Putting these in $\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{z}{z} = 6$ we get

$$\frac{3}{\sqrt{3\lambda}} + \frac{4}{\sqrt{4\lambda}} + \frac{5}{\sqrt{5\lambda}} = 6 \quad \text{or} \quad \sqrt{\lambda} = \frac{1}{6}\left[\sqrt{3} + \sqrt{4} + \sqrt{5}\right]$$

$$\therefore \quad x + y + z = \sqrt{\lambda}\left(\sqrt{3} + \sqrt{4} + \sqrt{5}\right) = \frac{1}{6}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2 \quad \textbf{Ans.}$$

EXAMPLE 13.38. Find the volume of the greatest rectangular parallelopiped that can be inscribed inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

[GGSIPU IInd Sem. End Term 2006; I Term 2012]

SOLUTION: Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ parallel to the co-ordinate axes. The volume V of the parallelopiped, is given by $V = 8\,xyz$ which is to be maximised

subject to the condition $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Consider the function $F = 8\,xyz + \lambda\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right)$ where λ is an unknown multiplier.

For stationary values $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0$

or $\quad 8yz + \dfrac{2\lambda x}{a^2} = 0, \quad 8zx + \dfrac{2\lambda y}{b^2} = 0, \quad 8xy + \dfrac{2\lambda z}{c^2} = 0.$

Eliminating λ between the above equations, taken in pairs, we get $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$.

Substituting these in the condition $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3} \quad \text{or} \quad x = +\frac{a}{\sqrt{3}},\ y = +\frac{b}{\sqrt{3}},\ z = +\frac{c}{\sqrt{3}}.$$

Note that as x increases, volume also increases. Hence V is maximum when

$x = \dfrac{a}{\sqrt{3}},\ y = \dfrac{b}{\sqrt{3}},\ z = \dfrac{c}{\sqrt{3}}.\quad \therefore$ Maximum volume $= 8xyz = \dfrac{8abc}{3\sqrt{3}}$. **Ans.**

EXAMPLE 13.39. Find the largest and the smallest distances from the origin to the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1 \quad \text{and the plane} \quad z = x + y.$$

SOLUTION: Let $u = x^2 + y^2 + z^2$ be the square of the distance of any point (x, y, z) on the ellipsoid from the origin. We are to maximise and minimise u subject to the constraints

$$\phi_1 = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0 \qquad ...(1)$$

and $\quad \phi_2 = x + y - z = 0. \qquad ...(2)$

Now consider the function $F = u + \lambda_1 \phi_1 + \lambda_2 \phi_2$

$$= x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2 (x + y - z).$$

where λ_1 and λ_2 are non-zero indetermined multipliers.

For critical points we have $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0.$

Hence $\quad 2x + \dfrac{\lambda_1 x}{2} + \lambda_2 = 0, \quad 2y + \dfrac{2\lambda_1 y}{5} + \lambda_2 = 0 \quad$ and $\quad 2z + \dfrac{2\lambda_1 z}{25} - \lambda_2 = 0.$

Above equations when solved for x, y, z, give

$$x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_1 + 50}. \qquad ...(3)$$

Substituting these in the constraint $x + y - z = 0$ and on dividing by $\lambda_2\ (\neq 0)$, we get

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0$$

or $\quad 2(2\lambda_1 + 10)(2\lambda_1 + 50) + 5(\lambda_1 + 4)(2\lambda_1 + 50) + 25(\lambda_1 + 4)(2\lambda_1 + 10) = 0$

or $\quad (17\lambda_1 + 75)(\lambda_1 + 10) = 0$ which gives $\lambda_1 = -10,\ -\dfrac{75}{17}$.

When $\quad \lambda_1 = -10 \quad$ we have, from (3) $\quad x = \dfrac{\lambda_2}{3}, \quad y = \dfrac{\lambda_2}{2}, \quad z = \dfrac{5}{6}\lambda_2.$

These values should satisfy (1), hence $\lambda_2^2\left(\frac{1}{36}+\frac{1}{20}+\frac{1}{36}\right)=1$ or $\lambda_2=\pm 6\sqrt{\frac{5}{19}}$.

This gives two critical points $\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right), \left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right)$.

$\therefore$ The stationary value of $x^2+y^2+z^2$ corresponding to these critical points, is

$$=\frac{(20+45+125)}{19}=10.$$

Next, when $\lambda_1=-\frac{75}{17}$ we have from (3). $x=\frac{34}{7}\lambda_2,\ \ y=-\frac{17}{4}\lambda_2,\ \ z=\frac{17}{28}\lambda_2.$

Substituting these in (1), gives $\lambda_2=\pm\frac{140}{17\sqrt{646}}$ which, in turn, gives the critical points as

$$\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right) \text{ and } \left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$$

The stationary value of $x^2+y^2+z^2$ corresponding to these critical points, is

$$=\frac{(1600+1225+25)}{646}=\frac{75}{17}.$$

Thus, the required maximum value is 10 and the minimum value is $\frac{75}{17}$. **Ans.**

EXAMPLE 13.40. (a) If $u=ax^2+by^2+cz^2$ where $x^2+y^2+z^2=1$ and $lx+my+nz=0$ prove that the stationary value of u satisfy the equation $\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{x^2}{c-u}=0.$

[GGSIPU II Sem End Term 2014]

(b) Show that the stationary values of $u=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$ where $lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, are roots of equation

$$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0.$$

SOLUTION: (a) We are to maximize or minimize $u=ax^2+by^2+cz^2$ subject to the conditions $\phi_1=x^2+y^2+z^2-1=0$ and $\phi_2=lx+my+nz=0$.

Let $F=u+\lambda\phi_1+\mu\phi_2$. For u to be maximum and minimum, we have

$$\frac{\partial F}{\partial x}=\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi_1}{\partial x}+\mu\frac{\partial \phi_2}{\partial x}=2ax+\lambda(2x)+\mu(l)=0 \quad ...(1)$$

$$\frac{\partial F}{\partial y}=\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi_1}{\partial y}+\mu\frac{\partial \phi_2}{\partial y}=2by+\lambda(2y)+\mu(m)=0 \quad ...(2)$$

$$\frac{\partial F}{\partial z}=\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi_1}{\partial z}+\mu\frac{\partial \phi_2}{\partial z}=2cz+\lambda(2z)+\mu(n)=0 \quad ...(3)$$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get

$2u+2\lambda(1)+\mu(0)=0$ hence $\lambda=-u$.

Putting $\lambda = -\mu$ in (1), (2), (3), we get

$$\therefore \quad x = \frac{-\mu l}{2a-2u}, \quad y = \frac{-\mu m}{2b-2u}, \quad z = \frac{-\mu n}{2c-2u}$$

Substituting these in $lx + my + nz = 0$, gives

$$-\frac{\mu l^2}{2a-2u} - \frac{\mu m^2}{2b-2u} - \frac{\mu n^2}{2c-2u} = 0$$

or $$\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$$ **Hence Proved.**

(b) Let us write $u = f(x, y, z) = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4},$ ... (1)

subject to $\phi(x, y, z) = lx + my + nz = 0$ ... (2)

and $\psi(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0.$ ... (3)

Now construct the function $F = f + \lambda_1 \phi + \lambda_2 \psi,$ ... (4)

where λ_1, λ_2 are non-zero Lagrangian multipliers.

For stationary values of u, we have

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \text{that is,}$$

$$\frac{2x}{a^4} + \lambda_1 l + \lambda_2 \frac{2x}{a^2} = 0 \qquad \text{...(5)}$$

$$\frac{2y}{b^4} + \lambda_1 m + \lambda_2 \frac{2y}{b^2} = 0 \qquad \text{...(6)}$$

$$\frac{2z}{c^4} + \lambda_1 n + \lambda_2 \frac{2z}{c^2} = 0. \qquad \text{...(7)}$$

Multiplying (5), (6), (7) by x, y, z respectively and adding, gives

$$2u + 0 \cdot \lambda_1 + 2\lambda_2 = 0 \quad \text{hence} \quad \lambda_2 = -u$$

Substituting this value of λ_2 in (5), (6) and (7), we get

$$x = \frac{-a^4 \lambda_1 l}{2(1-a^2 u)} \quad \text{and} \quad y = \frac{-b^4 \lambda_1 m}{2(1-b^2 u)}, \quad z = \frac{-c^4 \lambda_1 n}{2(1-c^2 u)}.$$

These values must satisfy (2), hence

$$-\frac{\lambda_1 a^4 l^2}{2(1-a^2 u)} - \frac{\lambda_1 b^4 m^2}{2(1-b^2 u)} - \frac{\lambda_1 c^4 n^2}{2(1-c^2 u)} = 0$$

Since $\lambda_1 \neq 0$ we have $\dfrac{a^4 l^2}{1-a^2 u} + \dfrac{b^4 m^2}{1-b^2 u} + \dfrac{c^4 n^2}{1-c^2 u} = 0.$

which gives the stationary values of u.

Hence Proved.

EXAMPLE 13.41. Find a point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find the minimum f.

[GGSIPU III Sem End Term 2004 (Reappear); End Term 2006 (Reappear); End Term 2012]

SOLUTION: We are to minimise $f = x^2 + y^2 + z^2$ such that $ax + by + cz = p$.

We can write $f = x^2 + y^2 + \dfrac{1}{c^2}(p - ax - by)^2$

$$= \left(1+\frac{a^2}{c^2}\right)x^2 + \left(1+\frac{b^2}{c^2}\right)y^2 - \frac{2ap}{c^2}x - \frac{2bp}{c^2}y + \frac{2abxy}{c^2} + \frac{p^2}{c^2}$$

Hence $$\frac{\partial f}{\partial x} = 2\left(1+\frac{a^2}{c^2}\right)x - \frac{2ap}{c^2} + \frac{2aby}{c^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = 2\left(1+\frac{b^2}{c^2}\right)y - \frac{2bp}{c^2} + \frac{2abx}{c^2}.$$

For maxima and minima of f, we have $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$.

Thus, $(a^2 + c^2)\,x + aby = ap$ and $(b^2 + c^2)\,y + abx = bp$

which, on solving for x and y, give $x = \dfrac{ap}{a^2+b^2+c^2}$ and $y = \dfrac{bp}{a^2+b^2+c^2}$.

which defines a critical point $\left(\dfrac{ap}{a^2+b^2+c^2}, \dfrac{bp}{a^2+b^2+c^2}\right)$.

Next, $$r = \frac{\partial^2 f}{\partial x^2} = 2\left(1+\frac{a^2}{c^2}\right), \quad s = \frac{\partial^2 f}{\partial x \partial y} = \frac{2ab}{c^2} \quad \text{and} \quad t = \frac{\partial^2 f}{\partial y^2} = 2\left(1+\frac{b^2}{c^2}\right).$$

$$\therefore \quad rt - s^2 = \frac{4}{c^4}(a^2+c^2)(b^2+c^2) - \frac{4a^2b^2}{c^4} = \frac{4}{c^4}[c^4 + a^2c^2 + b^2c^2] = \frac{4}{c^2}[a^2+b^2+c^2) > 0.$$

Here $r > 0$ hence at the critical point, function f has minima.

When $x = \dfrac{ap}{a^2+b^2+c^2}$, $y = \dfrac{bp}{a^2+b^2+c^2}$, we have

$$z = \frac{1}{c}[p - ax - by] = \frac{1}{c}\left[p - \frac{a^2p}{a^2+b^2+c^2} - \frac{b^2p}{a^2+b^2+c^2}\right] = \frac{pc}{a^2+b^2+c^2}$$

$$\therefore \quad \text{Min. } f = x^2 + y^2 + z^2 = \frac{p^2}{(a^2+b^2+c^2)^2}[a^2+b^2+c^2] = \frac{p^2}{a^2+b^2+c^2}.$$ **Ans.**

14

Partial Differential Equations—Formation, Solution of First Order Equations

Formation of Partial Differential Equations, Solution of First Order Equations of Lagrangian Form and Charpit's Method.

FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

As in the case of ordinary differential equations, partial differential equations can be formed by eliminating arbitrary constants (or parameters) from a given functional relation between the variables. Let us consider a functional relation $f(x, y, z, a, b) = 0$ ...(1)

in which f is some known function of x, y, z, and z is the dependent variable depending on x and y and a and b are arbitrary constants. To eliminate a and b from (1) we need two more equations which are formed through differentiation w.r.t. x and y partially.

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \qquad ...(2)$$

Similarly, differentiating (1) partially w.r.t. y, we get $\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0$ ...(3)

where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$.

Eliminating a and b from (1), (2) and (3), one can obtain a relation of the form

$$g(x, y, z, p, q) = 0. \qquad ...(4)$$

This is a partial differential equation of order one and (1) is its complete solution.

Next if, in place of arbitrary constants a and b, suppose there is **a functional relation**

$$\mathbf{F(u, v) = 0} \qquad ...(5)$$

where F is some arbitrary function and u and v are some known functions of x, y, z.

To form a partial differential equation here, let us assume that z is a dependent variable depending on x and y. Differentiating (5) partially w.r.t. x, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

or

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \qquad ...(6)$$

Similarly, differentiating (6) partially w.r.t. y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad ...(7)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (6) and (7), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

that is, $$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0$$

or $$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}=p\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)+q\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right).$$

or $$Pp + Qq = R \quad ...(8)$$

where $P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$, and $R=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$.

Here P, Q, R are known functions of x, y, z because u and v are known. The relation (8) is a partial differential equation of order one, and the relation (5), derived from it, is its solution. Thus, we observe that a first order partial differential equation in two independent and one dependent variable contains in its solution either two arbitrary constants or one arbitrary function.

For example, consider a situation where two arbitrary functions are to be eliminated to obtain the differential equation. Let $z = \phi_1(x + ay) + \phi_2(x - ay)$ ...(9)

where ϕ_1 and ϕ_2 are two arbitrary functions and z is the dependent variable with x and y as independent variables. Differentiating z partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x}=\phi'_1(x+ay)+\phi'_2(x-ay) \quad \text{and} \quad \frac{\partial z}{\partial y}=a\phi'_1(x+ay)-a\phi'_2(x-ay).$$

Also $$\frac{\partial^2 z}{\partial x^2}=\phi''_1(x+ay)+\phi''_2(x-ay) \quad ...(10)$$

and $$\frac{\partial^2 z}{\partial y^2}=a^2\phi''_1(x+ay)+a^2\phi''_2(x-ay). \quad ...(11)$$

Eliminating the arbitrary functions ϕ_1 and ϕ_2 in (10) and (11) gives the partial differential equation of second order as $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$ and (9) is its solution.

Thus, the complete solution of a second order partial differential equation will always involve two arbitrary functions and, in general, the solution of the partial differential equation of order n will contain n arbitrary functions.

EXAMPLE 14.1. (*a*) Form the partial differential equation by eliminating the arbitrary constants a and b from the relation $z = (x-a)^2 + (y-b)^2$.

(*b*) Construct a first order partial differential equation, if $z = ax + by + a^2 + b^2$ where a and b are arbitrary constants.

[*GGSIPU III Sem End Term 2012*]

SOLUTION: (*a*) Given that $z = (x-a)^2 + (y-b)^2$. ...(1)

Differentiating (1) partially w.r.t. x, we get $\dfrac{\partial z}{\partial x} = p = 2(x-a)$ ...(2)

Similarly differentiating (1) partially w.r.t. y, we get $\dfrac{\partial z}{\partial y} = q = 2(y-b)$ ...(3)

Substituting for $(x-a)$ and $(y-b)$ from (2) and (3) in (1), gives

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \quad \text{or} \quad 4z = p^2 + q^2. \qquad \textbf{Ans.}$$

(*b*) $$z = ax + by + a^2 + b^2$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b.$$

Hence the partial differential equation is $z = xp + yq + p^2 + q^2$. **Ans.**

EXAMPLE 14.2. Form the p.d.e. by eliminating the arbitrary function from

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

[*GGSIPU III Sem End Term 2010; III Term 2011*]

SOLUTION: The given relation can be written as

$$\frac{1}{2}(z - y^2) = f\left(\frac{1}{x} + \log y\right) \qquad ...(1)$$

Differentiating (1) partially w.r.t. x and w.r.t. y, gives

$$\frac{1}{2}(p-0) = f'\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right) \quad \text{and} \quad \frac{1}{2}(q - 2y) = f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

Eliminating f' in the above relations, we get

$$\frac{p}{q-2y} = \frac{-y}{x^2} \quad \text{or} \quad x^2p + yq = 2y^2. \qquad \textbf{Ans.}$$

EXAMPLE 14.3. (*a*) Form a partial differential equation from $(x-a)^2 + (y-b)^2 + z^2 = 1$

[*GGSIPU III Sem End Term 2004; II Term 2012*]

(*b*) Find the differential equation of all spheres whose centres lie on the z-axis.

[*GGSIPU III Sem End Term 2010*]

SOLUTION: (*a*) We have $(x-a)^2 + (y-b)^2 + z^2 = 1$ ...(1)

Differentiating (1) partially *w.r.t.* x and *w.r.t.* y, we get

$$2(x-a) + 2zp = 0 \quad \text{and} \quad 2(y-b) + 2zq = 0$$

or $$x - a = -pz \quad \text{and} \quad y - b = -qz$$

Substituting for $x-a$ and $y-b$ in (1), we get

$z^2(p^2 + q^2 + 1) = 1$ which is the required differential equation.

(*b*) Equation of the sphere having centre at $(0, 0, c)$ and radius r is $x^2+y^2+(z-c)^2=r^2$
Differentiating it partially w.r.t. x and w.r.t. y, we get

$$2x+2(z-c)p=0 \quad \text{and} \quad 2y+2(z-c)q=0$$

Eliminating $(z-c)$ in the above relations, gives $py-qx=0$. **Ans.**

EXAMPLE 14.4. (*i*) Form a partial differential equation from $x^2+y^2=(z-c)^2\tan^2\alpha$.

[*GGSIPU III Sem II Term 2006; III Sem II Term 2010*]

(*ii*) Form a partial differential equation from $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

[*GGSIPU III Sem End Term 2007; III Sem End Term 2013*]

SOLUTION: (*i*) Given equation is $x^2+y^2=(z-c)^2\tan^2\alpha$. ...(1)

Differentiating (1) partially *w.r.t.* x and *w.r.t.* y, we get

$$2x=2(z-c)\tan^2\alpha\,\frac{\partial z}{\partial x} \quad \text{and} \quad 2y=2(z-c)\tan^2\alpha\,\frac{\partial z}{\partial y}.$$

Eliminating c and α from the above relations, we get, $\dfrac{x}{y}=\dfrac{p}{q}$ or $py=qx$.

which is the required differential equation. **Ans.**

(*ii*) Differentiating the given equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ... (1)

partially *w.r.t.* x and *w.r.t.* y, we get $\dfrac{2x}{a^2}+\dfrac{2z}{c^2}p=0$ or $c^2x+a^2zp=0$ ...(2)

and $\dfrac{2y}{b^2}+\dfrac{2z}{c^2}q=0$ or $c^2y+b^2zq=0$ ... (3)

Differentiating (2) *w.r.t.* x partially, we get $c^2+a^2(zr+p^2)=0$ ... (4)

and then eliminating $\dfrac{c^2}{a^2}$ between (2) and (4), gives $-x(zr+p^2)+zp=0$

or $$xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2=pz$$ which is the required partial differential equation. **Ans.**

EXAMPLE 14.5. (*a*) Form a partial differential equation by eliminating the arbitrary function f from the relation $f(x^2+y^2, x^2-z^2)=0$

(*b*) Form the partial differential equation by eliminating the arbitrary function f and ϕ from the equation $z=f(y/x)+\phi(xy)$.

[*GGSIPU III Sem End Term 2009; III Sem End Term 2013*]

(*c*) Form the P.D.E. by eliminating the arbitrary functions from $z=yf(x)+xg(y)$.

[*GGSIPU III Sem End Term 2011*]

SOLUTION: (*a*) Let $u=x^2+y^2$ and $v=x^2-z^2$ then the given relation becomes
$f(u, v)=0$. ...(1)

Differentiating (1) partially w.r.t. x and y separately, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0 \quad ...(2)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0 \quad ...(3)$$

Here $\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=2y,\quad \frac{\partial u}{\partial z}=0,\quad \frac{\partial v}{\partial x}=2x,\quad \frac{\partial v}{\partial y}=0,\quad \frac{\partial v}{\partial z}=-2z.$

Writing p for $\frac{\partial z}{\partial x}$ and q for $\frac{\partial z}{\partial y}$, equations (2) and (3) become

$$2x\frac{\partial f}{\partial u}+(2x-2zp)\frac{\partial f}{\partial v}=0 \quad \text{and} \quad 2y\frac{\partial f}{\partial u}-2zq\frac{\partial f}{\partial v}=0$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ in the above equations, gives $\begin{vmatrix} x & x-pz \\ y & -qz \end{vmatrix}=0$

or $\qquad -xzq-xy+yzp=0 \qquad$ or $\qquad py-qx=xy/z \qquad$ **Ans.**

(*b*) Given $z=f\left(\frac{y}{x}\right)+\phi(xy)$ hence we have

$$p=\frac{\partial z}{\partial x}=f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)+\phi'(xy)\cdot y \quad \text{and} \quad q=\frac{\partial z}{\partial y}=f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)+\phi'(xy)\cdot x$$

Eliminating f' from in the above relations, we get

$$px+qy=2xy\phi'(xy) \quad ...(1)$$

$f'(\frac{y}{x}) = \frac{p-\phi'(xy)y}{(-y/x^2)}$

$q: \; p-\phi'(xy)y$

Differentiating (1) partially w.r.t x, we get

$$p\cdot 1+rx+ys=2y\phi'(xy)+2xy^2\phi''(xy). \quad ...(2)$$

Similarly, differentiating (1) partially w.r.t. y, we have

$$xs+q\cdot 1+yt=2x\phi'(xy)+2x^2y\phi''(xy). \quad ...(3)$$

Multiplying (2) by x and (3) by y and subtracting, we get

$$px+rx^2+xyr=xys+qy+yt$$

or $$px-qy=y^2t-x^2s. \qquad \textbf{Ans.}$$

(c) From $z=yf(x)+xg(y)$, we can get

$$\frac{\partial z}{\partial x}=p=yf'(x)+g(y) \quad \text{and} \quad \frac{\partial z}{\partial y}=q=f(x)+xg'(y)$$

Also $\frac{\partial^2 z}{\partial x^2}=r=\frac{\partial p}{\partial x}=yf''(x)$ and $\frac{\partial^2 z}{\partial y^2}=\frac{\partial q}{\partial y}=t=xg''(y)$ and $s=\frac{\partial^2 z}{\partial x\partial y}=f'(x)+g'(y).$

And $\frac{\partial s}{\partial x}=f''(x)$ and $\frac{\partial s}{\partial y}=g''(y)$

It follows that $\frac{\partial^2 z}{\partial x^2}=y\frac{\partial^3 z}{\partial x^2\partial y}.$ **Ans.**

EXAMPLE 14.6. Eliminate the arbitrary function 'f' from the equation $z = f\left(\frac{xy}{z}\right)$.

[GGSIPU III Sem End Term 2011]

SOLUTION: We have $z = f\left(\frac{xy}{z}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = p = f'\left(\frac{xy}{z}\right)y\left(\frac{z-px}{z^2}\right) \quad \text{and} \quad \frac{\partial z}{\partial y} = q = f'\left(\frac{xy}{z}\right)x\left(\frac{z-qy}{z^2}\right)$$

Eliminating f' in the above two relations, we get

$$\frac{p}{q} = \frac{z-px}{z-qy}\cdot\frac{y}{x} \quad \text{or} \quad px(z-qy) = qy(z-px)$$

or $\quad z(px - qy) = xy\,pq - xy\,pq = 0 \quad \Rightarrow \quad px = qy$ as $z \neq 0$. **Ans.**

EXAMPLE 14.7. (*i*) Form the partial differential equation by eliminating the arbitrary function f from $f(xy + z^2, x + y + z) = 0$.

[GGSIPU III Sem II Term 2006; II Term 2007]

(*ii*) Form a partial differential equation from $lx + my + nz = \phi(x^2 + y^2 + z^2)$

[GGSIPU III Sem End Term 2003; End Term 2012]

(*iii*) Form the p.d.e. by eliminating the arbitrary function from $z = (x + y)\,\phi(x^2 - y^2)$.

[GGSIPU III Sem End Term 2014]

SOLUTION: (*i*) The given relation is $f(u, v) = 0$ ... (1)

where $u = xy + z^2$ and $v = x + y + z$

Differentiating (1) partially *w.r.t.* x and *w.r.t.* y, we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$$

$\frac{\partial f}{\partial u}(y+2pz) = -\frac{\partial f}{\partial v}(1+p)$

$\frac{\partial f}{\partial u}$

or $\quad \frac{\partial f}{\partial u}(y + 2zp) + \frac{\partial f}{\partial v}(1 + p) = 0$ ...(2)

and $\quad \frac{\partial f}{\partial u}(x + 2zq) + \frac{\partial f}{\partial v}(1 + q) = 0$ ...(3)

Eliminating $\left(\dfrac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}}\right)$ in (2) and (3), we get $-\left(\frac{1+p}{y+2pz}\right) = -\left(\frac{1+q}{x+2zq}\right)$

or $\quad (x + 2zq)(1 + p) - (y + 2pz)(1 + q) = 0$

or $\quad p(x - 2z) + q(2z - y) = y - x$ is the required differential equation **Ans.**

(*ii*) The given relation is $lx + my + nz = \phi(u)$ ...(1)

where $u = x^2 + y^2 + z^2$.

Differentiating (1) partially *w.r.t.* x and *w.r.t.* y, we get

$$l + np = \phi'(u)\frac{\partial u}{\partial x} = \phi'(u)(2x + 2zp) \quad ...(2)$$

and $$m + nq = \phi'(u)\frac{\partial u}{\partial y} = \phi'(u)(2y + 2zq) \quad ...(3)$$

Eliminating $\phi'(u)$ in (2) and (3), gives

$$\frac{l+np}{2(x+pz)} = \frac{m+nq}{2(y+qz)} \quad \text{or} \quad ly + npy + lqz = mx + nqx + mpz$$

or $$z(mp - lq) = y(l + np) - x(m + nq). \quad \textbf{Ans.}$$

(*iii*) We have $$z = (x + y)\,\phi\,(x^2 - y^2)$$

$\therefore$ $$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,(2x) + 1 \cdot \phi\,(x^2 - y^2)$$

and $$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,(-2y) + 1 \cdot \phi\,(x^2 - y^2)$$

$\Rightarrow$ $$py + qx = 0 + (x + y)\,\phi\,(x^2 - y^2) = z$$

$\Rightarrow$ the p.d.e. is $py + qx = z$. **Ans.**

SOLVING LINEAR PARTIAL DIFFERENTIAL EQUATION OF ORDER ONE

A linear partial differential equation of first order with dependent variable z and independent variables x and y, of the form $\mathbf{Pp + Qq = R}$ is called **LAGRANGIAN FORM** ...(1)
where P, Q, R are functions of x, y, z, and is linear in p and q.

Suppose the solution of equation (1) is $\phi\,(u, v) = 0$, where ϕ is an arbitrary function and u and v are some definite functions of x, y, z such that $u = a$ and $v = b$ are two independent solutions of (1), where a and b are arbitrary constants.

As discussed and explained in the last article 'Formation of Partial Differential Equation' the functions P, Q, R are given by

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}, \quad Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} \quad \text{and} \quad R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}. \quad ...(2)$$

From $u = a$ and $v = b$ we have $du = 0$ and $dv = 0$

therefore $$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad ...(3)$$

and $$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad ...(4)$$

From (3) and (4), we can write

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

or $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad ...(5)$$

Because of relations (2), $u = a$ and $v = b$ are the solutions of equation (1). Once u and v are determined from the simultaneous equations (5) we can write the solution of (1) as

$$\phi(u, v) = 0 \qquad \text{or} \qquad u = F(v) \qquad \text{or} \qquad v = F(u).$$

Thus, to solve the linear partial differential equation $Pp + Qq = R$ ***we form a set of auxiliary simultaneous equations*** $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. ***If*** $u = a$ ***and*** $v = b$ ***are two independent solutions of these equations then the solution of the above partial differential equation is*** $\phi(u, v) = 0$ ***where*** ϕ ***is an arbitrary function.***

The procedure will be more clear through the following examples.

EXAMPLE 14.8. Obtain the general solution of the linear partial differential equation

$$y^2zp - x^2zq = x^2y.$$

SOLUTION: The given equation is of the Lagrangian form $Pp + Qq = R$ where $P = y^2z$, $Q = -x^2z$ and $R = x^2y$.

The auxiliary system of simultaneous equations is $\frac{dx}{y^2z} = \frac{dy}{-x^2z} = \frac{dz}{x^2y}$

From $\frac{dx}{y^2z} = \frac{dy}{-x^2z}$ we have $x^2dx + y^2dy = 0$, whose solution is $x^3 + y^3 = a.$

and from $\frac{dy}{-x^2z} = \frac{dz}{x^2y}$ we have $ydy + zdz = 0$, whose solution is $y^2 + z^2 = b$

Therefore the general solution is $f(x^3 + y^3, y^2 + z^2) = 0$, where f is an arbitrary function. **Ans.**

EXAMPLE 14.9. (a) Solve the partial differential equation

$$x(y - z)\,p + y\,(z - x)\,q = z\,(x - y).$$

(b) Find the general solution of the linear equation

$$x(y^2 - z^2)\,p + y\,(z^2 - x^2)\,q = z(x^2 - y^2).$$

SOLUTION: (a) The equation is of the Lagrangian form $Pp + Qq = R$ where $P = x(y - z)$, $Q = y(z - x)$ and $R = z(x - y)$.

The auxiliary system of simultaneous equations is $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$.

Since $x(y - z) + y(z - x) + z(x - y) = 0$ we take the multiplying factors 1, 1, 1 and add these to get

$dx + dy + dz = 0$ whose solution is $x + y + z = a$, a being an arbitrary constant.

Further, since $y - z + z - x + x - y = 0$, we take the multiplying factors $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ and add these to get $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$ whose solution is $xyz = b$, b being an arbitrary constant.

Thus, here $u = x + y + z$ and $v = xyz$ and therefore, the solution of the given equation is $\phi(u, v) = 0$, that is, $\phi(x + y + z, xyz) = 0$ **Ans.**

(b) The given linear equation is of the Lagrangian form $Pp + Qq = R$
where $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$.

The auxiliary equations are $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$

Since $y^2 - z^2 + z^2 - x^2 + x^2 - y^2 = 0$ we take multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ so as to get

$\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0$ which gives the solution as $xyz = a$ where a is an arbitrary constant.

Again, since $x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2) = 0$ we take multipliers x, y, z so as to get $xdx + ydy + zdz = 0$ which gives the solution as $x^2 + y^2 + z^2 = b$, where b is an arbitrary constant.

Therefore, the general solution is $f(xyz, x^2 + y^2 + z^2) = 0$. This can also be written either as
$xyz = \phi(x^2 + y^2 + z^2)$ or as $x^2 + y^2 + z^2 = \psi(xyz)$. **Ans.**

EXAMPLE 14.10. Find the general solution of the linear p.d.e.

$$(x^2 - y^2 - z^2)p + 2xyq = 2xz$$

[*GGSIPU II Sem I Term 2014*]

SOLUTION: The given equation is of the Lagrangian form $Pp + Qq = R$
where $P = x^2 - y^2 - z^2$, $Q = 2xy$ and $R = 2xz$.

The auxiliary system of simultaneous equations is $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

From $\dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ we have $\dfrac{dy}{y} = \dfrac{dz}{z}$, whose solution as

$$\log y = \log z + \log a \quad \text{or} \quad \frac{y}{z} = a.$$

Also, we can write $\dfrac{dz}{2xz} = \dfrac{xdx + ydy + zdz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2} = \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$

or $\dfrac{dz}{z} = \dfrac{2(xdx + ydy + zdz)}{x^2 + y^2 + z^2}$ whose solution is

$$\log(x^2 + y^2 + z^2) = \log z + b \quad \text{or} \quad \frac{x^2 + y^2 + z^2}{z} = b.$$

Thus, the general solution is $\phi(u, v) = 0$, where $u = \dfrac{y}{z}$, $v = \dfrac{x^2 + y^2 + z^2}{z}$

and ϕ is an arbitrary function. **Ans.**

EXAMPLE 14.11. (*i*) Solve $p - x^2 = q + y^2$. [*GGSIPU III Sem End Term 2006*]

(*ii*) Solve $\dfrac{y - z}{yz}p + \dfrac{z - x}{zx}q = \dfrac{x - y}{xy}$

[*GGSIPU III Sem End Term 2006; II Term 2010*]

SOLUTION: (*i*) Given differential equation is $p - x^2 = q + y^2$. or $p - q = y^2 + x^2$.

Its auxiliary equation is $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{y^2 + x^2}$

Here we have $dx + dy = 0$ hence $x + y = c_1$ where c_1 is an arbitrary constant.

Next $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{y^2 + x^2} = \frac{x^2dx - y^2dy - dz}{0}$

$\Rightarrow$ $x^2dx - y^2dy - dz = 0$ which on integration, gives

$$\frac{x^3}{3} - \frac{y^3}{3} - z = c_2 \quad \text{where } c_2 \text{ is an arbitrary constant.}$$

Thus the solution can be written as $x^3 - y^3 - 3z = f(x + y)$ where f is an arbitrary function. **Ans.**

(*ii*) Given partial differential equation is $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

Its auxiliary equation is $$\frac{dx}{\frac{y-z}{yz}} = \frac{dy}{\frac{z-x}{zx}} = \frac{dz}{\frac{x-y}{xy}}$$

Since $\frac{y-z}{yz} + \frac{z-x}{zx} + \frac{x-y}{xyz} = 0$, hence taking 1, 1, 1 as multipliers, gives

$dx + dy + dz = 0$ whose solution is $x + y + z = c_1$ where c_1 is an arbitrary constant.

Next, since $\frac{y-z}{xyz} + \frac{z-x}{xyz} + \frac{x-y}{xyz} = 0$, taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

which gives $\log x + \log y + \log z = c_2$ or $xyz = c_2'$ where c'_2 is an arbitrary constant.

Thus the required solution is $f(x + y + z, xyz) = 0$. **Ans.**

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE

The general form of the non-linear partial differential equation of order one, is

$$F(x, y, z, p, q) = 0 \quad \text{where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}. \qquad ...(1)$$

Before considering a general method for finding a complete solution of (1), we give below some special procedures for solving the following four types of equations:

TYPE I: $F(p, q) = 0$.

In this type the given equation does not contain any of the variables x, y and z.

Let its complete solution be $z = ax + by + c$.

then $\frac{\partial z}{\partial x} = p = a$ and $\frac{\partial z}{\partial y} = q = b$ hence $F(a, b) = 0$.

Thus, the complete solution is $\boldsymbol{z = ax + by + c}$ **where** $\boldsymbol{F(a, b) = 0}$ which has only two arbitrary constants.

EXAMPLE 14.12. Solve the equation $p^2 - q^2 = 1$.

SOLUTION: Here $F(p, q) = p^2 - q^2 - 1$.

The solution of the given equation is $z = ax + by + c$ where $F(a, b) = 0$, i.e. $a^2 - b^2 = 1$.

Hence b can be expressed in terms of a as $b = \sqrt{a^2 - 1}$. Thus, the complete solution of the given equation is $z = ax + \left(\sqrt{a^2 - 1}\right) y + c$ where a and c are arbitrary constants and c. **Ans.**

EXAMPLE 14.13. Solve the equation $pq + p + q = 0$

SOLUTION: Here $F(p, q) = pq + p + q$.

A complete solution of the given equation is $z = ax + by + c$
since $F(a, b) = 0$, that is, $ab + a + b = 0$

we can write $b = -\dfrac{a}{a+1}$ therefore the complete solution can be written as

$z = ax - \dfrac{a}{a+1}y + c$ which involves a and c as two arbitrary constants. **Ans.**

TYPE II: $z = px + qy + f(p, q)$

Assume the solution of the given equation as

$$z = ax + by + f(a, b) \quad \text{...(1)}$$

Since $\dfrac{\partial z}{\partial x} = p = a$ and $\dfrac{\partial z}{\partial y} = q = b$, replacing a and b by p and q respectively in (1), we get $z = px + qy + f(p, q)$ which is the given partial differential equation.

Therefore $z = ax + by + f(a, b)$ is definitely the complete solution of the equations of type II.

EXAMPLE 14.14. Solve $px + qy = z - 3p^{1/3}q^{1/3}$.

SOLUTION: The given equation can be written as $z = px + qy + 3p^{1/3}q^{1/3}$
which is of type II. where $f(p, q) = 3(pq)^{1/3}$.

Therefore, its complete solution is $z = ax + by + 3a^{1/3}b^{1/3}$
where a and b are arbitrary constants. **Ans.**

TYPE III: $F(p, q, z) = 0$.

In this type the independent variables are absent in the given equation. To solve it let us assume that z is a function of $(x + ay)$ where a is an arbitrary constant.

Setting $t = x + ay$, we have z as a function of t, hence

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dt}\frac{\partial t}{\partial x} = \frac{dz}{dt} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial y} = a\frac{dz}{dt}.$$

Substituting these values of p and q in the given equation, we get

$$F\left(\frac{dz}{dt}, a\frac{dz}{dt}, z\right) = 0$$

which is a first order ordinary differential equation with z as dependent variable and t as independent variable. If its solution is $z = \phi(t, b)$ where b is an arbitrary constant, then replacing t by $x + ay$ the complete solution of the given equation, becomes

$z = \phi(x + ay, b)$ involving two arbitrary constants a and b.

EXAMPLE 14.15. Solve the equation $z = p^2 + q^2$ where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$.

SOLUTION: The given equation belongs to Type III. Let us assume that z is a function of $x + ay$ where a is an arbitrary constant. Writing $t = x + ay$ so that

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial x} = \frac{dz}{dt} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{dt}\frac{\partial t}{\partial y} = a\frac{dz}{dt},$$

the given equation becomes $z = \left(\frac{dz}{dt}\right)^2 + a^2\left(\frac{dz}{dt}\right)^2$ or $\frac{dz}{\sqrt{z}} = \frac{dt}{\sqrt{1+a^2}}$

Its solution is $2\sqrt{z} = \frac{t}{\sqrt{1+a^2}} + \frac{b}{\sqrt{1+a^2}}$ where b is an arbitrary constant.

Thus, the complete solution is $4z(1 + a^2) = (t + b)^2$

or $4(1 + a^2)z = (x + ay + b)^2$, which involves two arbitrary constants a and b. **Ans.**

EXAMPLE 14.16. Solve $p(1 - q^2) = q(1 - z)$.

SOLUTION: Given equation is of type III. Setting $t = x + ay$ where a is an arbitrary constant, z being a function of t, we have $p = \frac{dz}{dt}$ and $q = a\frac{dz}{dt}$ and the given equation becomes

$$\frac{dz}{dt}\left|1 - a^2\left(\frac{dz}{dt}\right)^2\right| = a\frac{dz}{dt}(1 - z) \quad \text{or} \quad \frac{dz}{dt}\left|1 - a + az - a^2\left(\frac{dz}{dt}\right)^2\right| = 0$$

$\Rightarrow$ either $\frac{dz}{dt} = 0$ which gives $z = c$, or $a^2\left(\frac{dz}{dt}\right)^2 = 1 - a + az$

that is, $\frac{a\,dz}{\sqrt{1-a+az}} = dt$ whose solution is $2\sqrt{1-a+az} = t + b$

or $4(1 - a + az) = (x + ay + b)^2$ where b is an arbitrary constant.

Each of $z = c$ and $4(1 - a + az) = (x + ay + b)^2$ is a solution, latter being a complete solution. **Ans.**

TYPE IV: $F_1(x, p) = F_2(y, q)$.

In this type the dependent variable z is absent and we can separate terms of p and x on one side and q and y on the other side. Since x and y are independent variables, each of the expressions on the left and right hand side of the given equation must be a constant, say a. Thus, we get two separate differential equations as $F_1(x, p) = a$ and $F_2(y, q) = a$

where a is an arbitrary constant.

Solving the above first relation for p and second relation for q, suppose, we get

$$p = \phi_1(x, a) \quad \text{and} \quad q = \phi_2(y, a)$$

Further, since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$, substituting for p and q here, we get

$$dz = \phi_1(x, a)\,dx + \phi_2(y, a)\,dy.$$

Integration on both sides, yields $z = \int \phi_1(x, a)\,dx + \int \phi_2(y, a)\,dy + b$

where b is another arbitrary constant, thus we get the solution of the given equation.

EXAMPLE 14.17. Solve the partial differential equation $p - q = x^2 + y^2$

SOLUTION: Here the dependent variable z is absent and the given equation can be written as

$p - x^2 = q + y^2$. It is of type IV.

Setting $p - x^2 = a,\quad q + y^2 = a,$ we get

$p = x^2 + a$ and $q = a - y^2$ where a is an arbitrary constant.

Substituting these values in $dz = pdx + qdy$, we get

$$dz = (a + x^2)\,dx + (a - y^2)\,dy$$

which when integrated on both sides, yields

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b$$

Thus the required solution is $z = a(x + y) + \frac{1}{3}(x^3 - y^3) + b,$

where a and b are arbitrary constants. **Ans.**

Method of Transformation of Variables

If the given partial differential equation does not fall under any of the above types, sometimes a transformation of variables reduces the given equation to one of the above types.

The combination px, for instance, suggests the transformation

$x = e^X$ or $X = \log x$, because then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{1}{x}\frac{\partial z}{\partial X} \quad \text{or} \quad px = \frac{\partial z}{\partial X}.$$

Similarly, the combination qy suggests the transformation

$y = e^Y$ or $Y = \log y$, because then or $qy = \frac{\partial z}{\partial Y}$.

Also the appearance of $\frac{p}{z}$ and/or $\frac{q}{z}$ in the given equation, suggest the transformation $z = e^Z$ or $Z = \log z$, because then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ}\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \quad \text{or} \quad \frac{p}{z} = \frac{\partial Z}{\partial x}. \quad \text{And, similarly,} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}.$$

EXAMPLE 14.18. Solve $x^2p^2 + y^2q^2 = z^2$, where $p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$.

[GGSIPU IIIrd Sem. End Term 2005]

SOLUTION: The given equation can be written as $\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$

Applying the transformations X = log x, Y = log y, Z = log z, the above equation becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1. \qquad ...(1)$$

where $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$. Now it is of type I, i.e. $f(P, Q) = 0$.

The solution of (1) is $Z = aX + bY + \log c$. Here $f(P, Q) = P^2 + Q^2 - 1$
so that $f(a, b) = 0$ which gives $a^2 + b^2 = 1$

Therefore, the solution is $Z = aX + \sqrt{1-a^2}\ Y + \log c$ where a and c are arbitrary constants.

or $\log z = a \log x + \sqrt{1-a^2}\ \log y + \log c$ or $z = c\, x^a\, y^{\sqrt{1-a^2}}$. **Ans.**

EXAMPLE 14.19. Solve the equation $p^2 + q^2 = z^2 (x + y)$.

SOLUTION: The given equation can be written as $\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x + y$.

Now applying the transformation $Z = \log z$, we have

$p = z\frac{\partial Z}{\partial x}$, $q = z\frac{\partial Z}{\partial y}$ and the given equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y \quad \text{or} \quad \left(\frac{\partial Z}{\partial x}\right)^2 - x = y - \left(\frac{\partial Z}{\partial y}\right)^2$$ which is of Type IV.

Setting $\left(\frac{\partial Z}{\partial x}\right)^2 - x = a = y - \left(\frac{\partial Z}{\partial y}\right)^2$ where a is an arbitrary constant, we have

$$\frac{\partial Z}{\partial x} = \sqrt{x+a} \quad \text{and} \quad \frac{\partial Z}{\partial y} = \sqrt{y-a}$$

Putting these in $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$, we get

$$dZ = \sqrt{x+a}\ dx + \sqrt{y-a}\ dy$$

$$\Rightarrow \quad Z = \int\sqrt{x+a}\,dx + \int\sqrt{y-a}\ dy = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$$

or $\quad (3/2) \log z = (x + a)^{3/2} + (y - a)^{3/2} + b$

is the required complete solution where a and b are arbitrary constants. **Ans.**

EXAMPLE 14.20. Solve the equation $p^2x^2 = z (z - qy)$.

SOLUTION: We apply the transformations $X = \log x$ and $Y = \log y$ so that

$px = P = \frac{\partial z}{\partial X}$ and $qy = Q = \frac{\partial z}{\partial Y}$ and the given equation reduces to

$P^2 = z (z - Q)$ which is of type III. ...(1)

Hence, we can take z as a function of $X + aY$. where a is an arbitrary constants
Let us set here $t = X + aY$, then

$$\frac{\partial z}{\partial X} = \frac{dz}{dt}\frac{\partial t}{\partial X} = \frac{dz}{dt} \quad \text{and} \quad \frac{\partial z}{\partial Y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial Y} = a\frac{dz}{dt}$$

Therefore (1) becomes $\left(\frac{dz}{dt}\right)^2 = z\left(z - a\frac{dz}{dt}\right)$ or $\left(\frac{dz}{dt}\right)^2 + az\frac{dz}{dt} - z^2 = 0$

$$\Rightarrow \qquad \frac{dz}{dt} = \frac{-az \pm \sqrt{a^2z^2 + 4z^2}}{2} = \frac{z}{2}\left[-a \pm \sqrt{a^2+4}\right].$$

Let us consider $\dfrac{dz}{dt} = \dfrac{z}{2}\left[\sqrt{a^2+4} - a\right]$ which, on integration, gives

$$\int \frac{dz}{z} = \frac{1}{2}\int\left(\sqrt{a^2+4} - a\right) dt \qquad \text{or} \qquad \log z = \frac{1}{2}\left[\sqrt{a^2+4} - a\right](t+b)$$

or $$\log(z^2) = \left(\sqrt{a^2+4} - a\right)(X + aY + b)$$

or $$\log(z^2) = \left(\sqrt{a^2+4} - a\right)(\log x + a\log y + b)$$ which is the required solution

where a and b are arbitrary constants. **Ans.**

Complete Solution by Charpit's Method

Now comes a very general method to solve a first order non-linear partial differential equation

$$f(x, y, z, p, q) = 0 \qquad \text{...(1)}$$

Here z is a function of the independent variables x and y, hence, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \qquad \text{...(2)}$$

Let us now suppose that there exists another relation

$$F(x, y, z, p, q) = 0 \qquad \text{...(3)}$$

such that after solving (1) and (3) for p and q and substituting these values of p and q in (2), the equation (2) becomes integrable. Thus, z, p, q may be expressed as a function of x and y. Since these values identically satisfy (1) and (3), their derivatives w.r.t. x and y both must vanish.

Differentiating (1) and (3) partially w.r.t. x, gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(4)}$$

and $$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0. \qquad \text{...(5)}$$

Similarly, differentiating (1) and (3) partially w.r.t. y, gives

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(6)}$$

and $$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(7)}$$

Now eliminating $\dfrac{\partial p}{\partial x}$ from (4) and (5), yields

$$\frac{\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}p + \dfrac{\partial f}{\partial q}\dfrac{\partial q}{\partial x}}{\dfrac{\partial f}{\partial p}} = \frac{\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}p + \dfrac{\partial F}{\partial q}\dfrac{\partial q}{\partial x}}{\dfrac{\partial F}{\partial p}}$$

or $$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\cdot\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial p}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\cdot\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\cdot\frac{\partial f}{\partial p}\right)=0 \quad ...(8)$$

Similarly eliminating $\frac{\partial q}{\partial y}$ from (6) and (7), gives

$$\frac{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y}}{\frac{\partial f}{\partial q}}=\frac{\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}}{\frac{\partial F}{\partial q}}$$

or $$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\cdot\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial F}{\partial q}\cdot\frac{\partial f}{\partial z}-\frac{\partial f}{\partial q}\cdot\frac{\partial F}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\cdot\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\cdot\frac{\partial f}{\partial q}\right)=0 \quad ...(9)$$

But $\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}$ hence, adding (8) and (9) and rearranging, yields

$$\frac{\partial F}{\partial p}\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)+\frac{\partial F}{\partial q}\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)+\frac{\partial F}{\partial z}\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)+\frac{\partial F}{\partial x}\left(-\frac{\partial f}{\partial p}\right)+\frac{\partial F}{\partial y}\left(-\frac{\partial f}{\partial q}\right)=0 \quad ..(10)$$

This is a linear equation of order one with x, y, z, p, q as independent variables and F as dependent variable. Therefore, as in Lagrangian method, we can write an auxiliary system of simultaneous equations as

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{\frac{-\partial f}{\partial p}}=\frac{dy}{\frac{-\partial f}{\partial q}}=\frac{dF}{0} \quad ...(11)$$

Any integral (i.e., solution) of (11) will satisfy (10). The simplest relation involving atleast one of p and q may be taken as $F = 0$. Now, from $f = 0$ and $F = 0$ the values of p *and* q should be found in terms of x and y and should be substituted in (2) which, on integration, gives the desired solution.

EXAMPLE 14.21. Find the complete solution of the equation $px + qy = pq$.

SOLUTION: Here the differential equation is $f(x, y, z, p, q) = 0$ where $f \equiv px + qy - pq = 0$-

$$\therefore \quad \frac{\partial f}{\partial x}=p,\quad \frac{\partial f}{\partial y}=q,\quad \frac{\partial f}{\partial z}=0,\quad \frac{\partial f}{\partial p}=x-q,\quad \frac{\partial f}{\partial q}=y-p.$$

Therefore, the Charpit's auxiliary equations are

$$\frac{dp}{p+0}=\frac{dq}{q+0}=\frac{dz}{-p^2-q^2}=\frac{dx}{-(x-q)}=\frac{dy}{-(y-p)}=\frac{df}{0}$$

The first two of the above equations give $\log p = \log q + \log a$

or $p = aq$ where a is an arbitrary constant.

Now, writing aq for p in the given equation, we get

$$q\,(ax + y) = aq^2 \quad \text{or} \quad q=\frac{ax+y}{a} \quad \text{or} \quad p = ax + y.$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (ax + y)\, dx + \frac{ax + y}{a}\, dy$$

or $$adz = (ax + y)(adx + dy) = (ax + y)\, d(ax + y)$$

On integration, it yields: $az = \frac{1}{2}(ax + y)^2 + b$

which is a complete integral where a and b are arbitrary constants. **Ans.**

EXAMPLE 14.22. Solve $(p^2 + q^2)\, y = qz$. [*GGSIPU II Sem End Term 2014*]

SOLUTION: The given partial differential equation is $f \equiv (p^2 + q^2)\, y - qz = 0$

$$\therefore \quad \frac{\partial f}{\partial p} = 2py, \quad \frac{\partial f}{\partial q} = 2qy - z, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = p^2 + q^2.$$

The Charpit's auxiliary simultaneous equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

or $$\frac{dp}{0 - pq} = \frac{dq}{p^2 + q^2 - q^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z} = \frac{df}{0}$$

The first two of the above relations, give $\frac{dp}{-q} = \frac{dq}{p}$ or $pdp + qdq = 0$

which on integration gives $p^2 + q^2 = \text{constant} = a^2$, say.

Substituting $p^2 + q^2 = a^2$ in the give equation, we get $q = \frac{a^2 y}{z}$, and hence

$$p^2 = a^2 - \frac{a^4 y^2}{z^2} \quad \text{or} \quad p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}.$$

Now putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\, dx + \frac{a^2 y}{z}\, dy \quad \text{or} \quad \frac{zdz - a^2 ydy}{\sqrt{z^2 - a^2y^2}} = adx$$

which on integration gives $\sqrt{z^2 - a^2y^2} = ax + b$

or $z^2 - a^2y^2 = (ax + b)^2$ where a and b are arbitrary constants. **Ans.**

EXAMPLE 14.23. Apply Charpit's method to solve $2xz - px^2 - 2qxy + pq = 0$.

SOLUTION: The given equation can be written as $f \equiv 2xz - px^2 - 2qxy + pq = 0$.

The Charpit's auxiliary simultaneous equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{df}{0}$$

or $$\frac{dp}{2z-2px-2qy+2px} = \frac{dq}{-2qx+2qx} = \frac{dz}{px^2-pq+2xyq-pq} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{df}{0}$$

or $$\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{px^2+2xyq-2pq} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{df}{0}$$

The second expression above gives, $q = a$ where a is an arbitrary constant.

Putting $q = a$ in the given equation, we get

$$2xz - px^2 - 2axy + ap = 0 \quad \text{or} \quad p = \frac{2xz-2axy}{x^2-a}$$

$\therefore$ $dz = pdx + qdy$ becomes $dz = \dfrac{2x(z-ay)}{x^2-a}\,dx + ady$

or $$\frac{dz-ady}{z-ay} = \frac{2xdx}{x^2-a}$$

which on integration gives $\log(z-ay) = \log(x^2-a) + \log b$

or $z - ay = b(x^2-a)$ where b is the other arbitrary constant. Hence the complete solution.

Ans.

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Many functions satisfy these conditions, like all polynomials, circular functions like $\sin at$, $\cos at$, e^{at} and many others. The conditions are sufficient but not necessary for a function to have Laplace transform.

For example, $f(t) = t^{-1/2}$ for $t > 0$. This function is not piecewise continuous over $[0, k]$ since $\lim_{t \to 0} t^{-1/2} = \infty$. Nevertheless, $\int_0^k e^{-st} t^{-1/2} dt$ exists for all $k > 0, s > 0$.

Then $$L(f(t)) = \int_0^\infty e^{-st}\, t^{-1/2}\, dt \quad (\text{put } x = t^{1/2})$$

$$= \int_0^\infty e^{-sx^2}\, dx = \frac{2}{\sqrt{s}} \int_0^\infty e^{-z^2} dz \quad (\text{on putting } z = x\sqrt{s}\,) = \sqrt{\frac{\pi}{s}}$$

where we have used the result $\int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$.

Further, we can also define inverse Laplace transform.

If $f(t) = L^{-1}(\bar{f}(s); s \to t)$ or simply $f(t) = L^{-1}(\bar{f}(s))$.

then $f(t)$ is called the ***inverse Laplace transform*** of $\bar{f}(s)$.

LINEARITY THEOREM. Laplace transformation is linear, that is,

if $L(f_1(t)) = \bar{f}_1(s)$ and $L(f_2(t)) = \bar{f}_2(s)$

then for any constants c_1 and c_2

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

FIRST SHIFTING THEOREM

If $L(f(t)) = \bar{f}(s)$ **then** $L(e^{at} f(t)) = \bar{f}(s-a)$ $(s > a)$ *[GGSIPU III Sem End Term 2010]*

PROOF: $$L(e^{at} f(t)) = \int_0^\infty e^{-st} \{e^{at} f(t)\}\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-pt} f(t)\, dt = \bar{f}(p) = \bar{f}(s-a).$$

LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

(i) $f(t) = e^{at}$, $a > 0$.

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt \quad \text{where } s > a$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}[0-1] = \frac{1}{s-a}.$$

Thus $L(e^{at}) = \dfrac{1}{s-a}$ where $s > a$.

If we take here $a = 0$ we have $L(1) = \dfrac{1}{s}$.

(ii) $f(t) = \sin at$ or $\cos at$.

We know that $e^{ait} = \cos at + i \sin at$

Taking Laplace Transform on both sides, we get

$$L(e^{ait}) = L(\cos at) + iL(\sin at) \quad \text{by Linearity property.}$$

Using (i) we have $L(e^{ait}) = \dfrac{1}{s-ai} = \dfrac{s+ai}{s^2+a^2}$ Therefore, $L(\cos at) + iL(\sin at) = \dfrac{s+ai}{s^2+a^2}$.

$$\Rightarrow \quad L(\cos at) = \frac{s}{s^2+a^2} \quad \text{and} \quad L(\sin at) = \frac{a}{s^2+a^2}.$$

(iii) $f(t) = \sinh at$ or $\cosh at$. [GGSIPU III Sem End Term 2004]

$$L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2-a^2}$$

and $$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)} = \frac{s}{s^2-a^2}$$

Thus, $$L(\cosh at) = \frac{s}{s^2-a^2} \quad \text{and} \quad L(\sinh at) = \frac{a}{s^2-a^2}.$$

(iv) Let $f(t) = t^n$, then [GGSIPU III Sem End Term 2004]

$$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt, \quad \text{(now putting } st = x\text{)}$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\, dx = \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{provided } n > -1 \text{ and } s > 0.$$

If n is a positive integer $\Gamma(n+1) = n!$ then $L(t^n) = \dfrac{n!}{s^{n+1}}$.

Thus, $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$, **if n is a positive integer and $s > 0$.**

Notice here that, on setting $s = 1$, we get

$$n! = \int_0^\infty e^{-t} t^n\, dt \quad \text{for } n = 0, 1, 2, \ldots$$

This provides a way of expressing $n!$ in terms of Laplace transform.

For example, let $f(t) = t^3 - 4t + 5 + 3\sin 2t$, then

$$L\{f(t)\} = L(t^3) - 4L(t) + L(5) + 3L(\sin 2t)$$

$$= \frac{3!}{s^4} - 4\left(\frac{1}{s^2}\right) + \frac{5}{s} + 3\cdot\frac{2}{s^2+4} = \frac{5s^5+2s^4+20s^3-10s^2+24}{s^4(s^2+4)}.$$

(*v*) Making use of the first shifting theorem we can write a few more results as

$$L(e^{at} t^n) = \frac{\overline{|n+1}}{(s-a)^{n+1}}.$$

$$L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}, \quad L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2},$$

$$L(e^{at}\sinh bt) = \frac{b}{(s-a)^2-b^2}, \quad L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2-b^2}.$$

EXAMPLE 15.1. Find the Laplace transform of

(*i*) $\sin^2 t$ (*ii*) $\sin 2t \cos 3t$ (*iii*) $\cos^3 t$. (*iv*) $\sin t \cos t$

[GGSIPU End Term 2012]

SOLUTION: (*i*)
$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = L\left(\frac{1}{2}\right) - \frac{1}{2}L(\cos 2t)$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}. \quad \textbf{Ans.}$$

(*ii*)
$$L(\sin 2t\cos 3t) = L\left(\frac{1}{2}(\sin 5t - \sin t)\right) = \frac{1}{2}\cdot\frac{5}{s^2+25} - \frac{1}{2(s^2+1)}$$

$$= \frac{5(s^2+1)-(s^2+25)}{2(s^2+1)(s^2+25)} = \frac{2(s^2-5)}{(s^2+1)(s^2+25)}. \quad \textbf{Ans.}$$

(*iii*) Since $\cos 3t = 4\cos^3 t - 3\cos t$

$$\therefore \quad L(\cos^3 t) = L\left(\frac{3\cos t + \cos 3t}{4}\right) = \frac{3}{4}L(\cos t) + \frac{1}{4}L(\cos 3t)$$

$$= \frac{3}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s}{s^2+9} = \frac{s(3s^2+27+s^2+1)}{4(s^2+1)(s^2+9)} = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}. \quad \textbf{Ans.}$$

(*iv*)
$$L(\sin t\cos t) = L\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot\frac{2}{s^2+2^2} = \frac{1}{s^2+4} \quad \textbf{Ans.}$$

EXAMPLE 15.2. Find the Laplace transform of (*i*) $e^{-2t}(3\cos 4t - 2\sin 5t)$ (*ii*) $L(\sinh at \cos at)$.

[GGSIPU III Sem End Term 2010]

SOLUTION: (*i*)
$$L\{e^{-2t}(3\cos 4t - 2\sin 5t)\} = 3L(e^{-2t}\cos 4t) - 2L(e^{-2t}\sin 5t)$$

$$= 3\cdot\frac{s+2}{(s+2)^2+4^2} - 2\cdot\frac{5}{(s+2)^2+5^2}$$

$$= \frac{3(s+2)}{(s+2)^2+16} - \frac{10}{(s+2)^2+25}. \quad \textbf{Ans.}$$

(*ii*)
$$L(\sinh at\cos at) = L\left[\frac{1}{2}(e^{at}-e^{-at})\cos at\right] = \frac{1}{2}L(e^{at}\cos at) - \frac{1}{2}L(e^{-at}\cos at)$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2-2as+2a^2} - \frac{s+a}{s^2+2as+2a^2}\right]. \quad \textbf{Ans.}$$

EXAMPLE 15.3. (a) Let
$$f(t) = \frac{t}{a} \quad \text{for} \quad 0 < t < a$$
$$= 1 \quad \text{for} \quad t > a$$
Find the Laplace transform of $f(t)$.

(b) Obtain the Laplace transform of the function
$$f(t) = \sin t \quad \text{for} \quad 0 < t < \pi$$
$$= 0 \quad \text{for} \quad t > \pi$$
[GGSIPU III Sem End Term 2011]

SOLUTION: (a)
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^a e^{-st} \cdot \frac{t}{a}\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$$
$$= \left[\frac{t}{a}\left(\frac{e^{-st}}{-s}\right)\right]_0^a - \int_0^a \frac{1}{a}\left(\frac{e^{-st}}{-s}\right)dt + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$
$$= \frac{-e^{-as}}{s} - 0 + \frac{1}{as}\left[\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{s}(0 - e^{-as}) = \frac{1}{as^2}\left[1 - e^{-as}\right]. \quad \textbf{Ans.}$$

(b)
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} \sin t\,dt + 0 = \left[\frac{e^{-st}}{s^2+1}(-s\sin t - 1\cos t)\right]_0^\pi$$
$$= \frac{e^{-\pi s}}{s^2+1}(0+1) - \frac{1}{s^2+1}(0-1) = \frac{e^{-\pi s}+1}{s^2+1}. \quad \textbf{Ans.}$$

EXAMPLE 15.4. Find the Laplace transform of the function

(i) $$f(t) = \begin{cases} 2; & 0<t<\pi \\ 0; & \pi<t<2\pi \\ \sin t; & t>2\pi \end{cases}$$
[GGSIPU II Sem End Term 2006]

(ii) $$f(t) = \begin{cases} 2+t^2, & 0<t<2 \\ 6\quad , & 2<t<3 \\ 2t-5, & 3<t<\infty \end{cases}$$
[GGSIPU III Sem End Term 2009]

SOLUTION: (i)
$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt = \int_0^\pi 2e^{-st}dt + \int_\pi^{2\pi} 0e^{-st}dt + \int_{2\pi}^\infty \sin t\, e^{-st}dt$$
$$= \left.\frac{2e^{-st}}{-s}\right|_0^\pi + 0 + \left[\frac{e^{-st}}{1+s^2}(-s\sin t - 1\cos t)\right]_{2\pi}^\infty$$
$$= \frac{2}{s}(1-e^{-s\pi}) + \frac{e^{-2\pi s}}{1+s^2}(s\sin 2\pi + \cos 2\pi) = \frac{2}{s}(1-e^{-\pi s}) + \frac{e^{-2\pi s}}{1+s^2}$$
Ans.

(ii)
$$L(f(t)) = \int_0^\infty s^{-st} f(t)dt = \int_0^2 e^{-st}(2+t^2)\,dt + \int_2^3 6e^{-st}dt + \int_3^\infty (2t-5)e^{-st}dt$$

$$= \left[(2+t^2)\frac{e^{-st}}{-s}\right]_0^2 - \int_0^2 2t\frac{e^{-st}}{-s}dt + \left[\frac{6e^{-st}}{-s}\right]_2^3 + \left[(2t-5)\frac{e^{-st}}{-s}\right]_3^\infty - 2\int_3^\infty \frac{e^{-st}}{-s}dt$$

$$= \frac{-6}{s}e^{-2s} + \frac{2}{s} + \frac{2}{s}\left[\frac{te^{-st}}{-s}\right]_0^2 - \int_0^2 \frac{2e^{-st}}{s(-s)}\,dt - \frac{6}{s}(e^{-3s} - e^{-2s}) + 0 + \frac{e^{-3s}}{s} + \frac{2}{s}\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \frac{-6}{s}e^{-2s} + \frac{2}{s} - \frac{4}{s^2}e^{-2s} + \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^2 + \frac{6}{s}(e^{-2s} - e^{-3s}) + \frac{e^{-3s}}{s} + \frac{2}{s^2}e^{-3s}$$

$$= e^{-2s}\left(-\frac{6}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{6}{s}\right) + e^{-3s}\left(-\frac{6}{s} + \frac{1}{s} + \frac{2}{s^2}\right) + \frac{2}{s} + \frac{2}{s^3}$$

or $$L(f(t)) = \frac{2}{s} + \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right) + e^{-3s}\left(\frac{2}{s^2} - \frac{5}{s}\right)$$ **Ans.**

EXAMPLE 15.5. (*a*) Find the Laplace transform of $f(t)$ defined as

$$f(t) = |t-1| + |t+1|, \quad t \geq 0$$

[GGSIPU III Sem End Term 2005, End Term 2010]

(*b*) Find the Laplace transform of the function $f(t) = \begin{cases} 0, & 0 < t < 3 \\ (t-3)^2, & t > 3. \end{cases}$

[GGSIPU II Sem End Term 2011]

SOLUTION: (*a*) $f(t) = |t-1| + |t+1|, \quad t \geq 0$

$\therefore$ We can write
$$f(t) = 1 - t + 1 + t = 2 \quad \text{for} \quad 0 < t < 1$$
$$= t - 1 + t + 1 = 2t \quad \text{for} \quad t > 1.$$

$\therefore$
$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 2e^{-st}\,dt + \int_1^\infty 2t\,e^{-st}\,dt$$

$$= \left[2\frac{e^{-st}}{-s}\right]_0^1 + \left[2t\frac{e^{-st}}{-s}\right]_1^\infty - \int_1^\infty 2\frac{e^{-st}}{-s}dt$$

$$= \frac{2}{s}(1 - e^{-s}) + 0 + \frac{2}{s}e^{-s} + \frac{2}{s^2}e^{-s} = \frac{2}{s} + \frac{2}{s^2}e^{-s}$$

Therefore the Laplace transform of $f(t) = \frac{2}{s} + \frac{2}{s^2}e^{-s}$. **Ans.**

(*b*) $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 0\,e^{-st}dt + \int_3^\infty (t-3)^2 e^{-st}\,dt = 0 + \left[(t-3)^2\frac{e^{-st}}{-s}\right]_3^\infty - \int_3^\infty 2(t-3)\frac{e^{-st}}{-s}dt$$

$$= 0 + 0 + \frac{2}{s}\int_3^\infty (t-3)\,e^{-st}\,dt = \frac{2}{s}\left[(t-3)\frac{e^{-st}}{-s}\right]_3^\infty - \frac{2}{s}\int_3^\infty \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_3^\infty = \frac{2e^{-3s}}{s^3}.$$ **Ans.**

PROPERTIES OF LAPLACE TRANSFORM

I. **Change of Scale.** If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

[*GGSIPU II Sem End Term 2013*]

Proof: By definition $L\{f(at)\} = \int_0^{\infty} e^{-st} f(at)\, dt$ (now putting $at = x$)

$$= \int_0^{\infty} e^{-s\frac{x}{a}} f(x)\frac{dx}{a} = \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)x} f(x)\, dx = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

II. If $L\{f(t)\} = \bar{f}(s)$ then $L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$

Proof: We have $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt$

Differentiating both sides w.r.t. s (assuming differentiation valid), under the integral sign, we get

$$\frac{d}{ds}\bar{f}(s) = \int_0^{\infty}\frac{\partial}{\partial s}\left\{e^{-st} f(t)\right\} dt = \int_0^{\infty} -te^{-st} f(t)\, dt = -L\{tf(t)\}$$

Hence $L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$.

Replacing $f(t)$ by $t f(t)$ on both sides, we can get

$$L\{t^2 f(t)\} = (-1)^2\frac{d^2}{ds^2}\bar{f}(s)$$

In general, $L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\bar{f}(s)$, $n = 1, 2, 3, \ldots$

which can be easily verified using the method of Mathematical induction.

III. If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty}\bar{f}(s)\, ds$ [*GGSIPU III Sem End Term 2010*]

Proof: We have $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt$.

Integrating both sides w.r.t. s under the limits s to ∞, we get

$$\int_s^{\infty}\bar{f}(s)\, ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)\, dt\right] ds \quad \text{(changing the order of integration)}$$

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}\, ds\right] dt = \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_0^{\infty} f(t)\left[0 - \frac{e^{-st}}{(-t)}\right] dt = \int_0^{\infty} e^{-st}\frac{f(t)}{t}\, dt = L\left\{\frac{1}{t} f(t)\right\}$$

Therefore $L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty}\bar{f}(s)\, ds$.

Replacing $f(t)$ by $\frac{1}{t}f(t)$ in the above result, we can get

$$L\left\{\frac{1}{t^2}f(t)\right\} = \int_s^\infty\left[\int_s^\infty \bar{f}(s)\,ds\right]ds.$$

IV. Laplace Transform of Derivatives.

If $L\{f(t)\} = \bar{f}(s)$ and $f'(t)$ is continuous then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$ (integrating by parts)

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)\,dt$$

$$= \underset{t\to\infty}{\text{Lt}}\, e^{-st} f(t) - f(0) + s\bar{f}(s)$$

Since $\underset{t\to\infty}{\text{Lt}}\, e^{-st} f(t) = 0$ as $f(t)$ is of exponential order, we have

$$L\{f'(t)\} = s\bar{f}(s) - f(0).$$

Replacing $f(t)$ by $f'(t)$ in the above relation, we get

$$L\{f''(t)\} = s^2\bar{f}(s) - s f(0) - f'(0)$$

In general, $L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$ which can be easily verified using the principle of mathematical induction.

V. Laplace Transform of integrals.

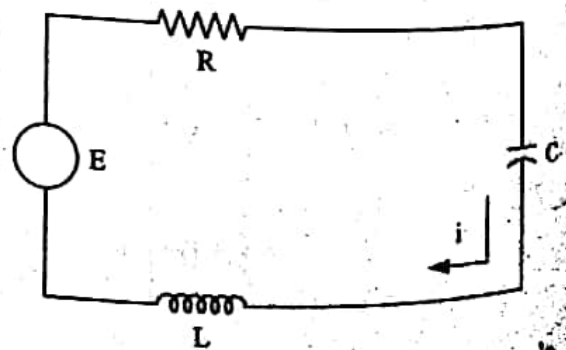
If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$. [GGSIPU III Sem End Term 2011]

Proof: $L\left\{\int_0^t f(u)\,du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,du\right]dt$

[integrating by parts taking e^{-st} as second function and keeping in mind $\frac{d}{dt}\int_0^t f(u)\,du = f(t)$]

$$\therefore \quad L\left(\int_0^t f(u)\,du\right) = \left[\frac{e^{-st}}{-s}\int_0^t f(u)\,du\right]_0^\infty - \int_0^\infty f(t)\cdot\frac{e^{-st}}{-s}\,dt = \frac{1}{s}\bar{f}(s).$$

In many problems of electrical engineering we come across integro-differential equations. Consider an electric circuit in series as shown in the adjoining figure. As per Kirchoff's second law in the L-C-R circuit we get the flow of current i satisfying the integero-differential equation



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E_0 \cos wt.$$

Here when we take Laplace transform on both sides of the above equation the above mentioned formula is used.

EXAMPLE 15.6. (a) Show that $L(t\sin at) = \dfrac{2as}{(s^2+a^2)^2}$ and $L(t\cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

(b) Find the Laplace transformation of $\sin^3 2t$. [GGSIPU II Sem End Term 2012]

SOLUTION: (a) Since $L(\sin at) = \dfrac{a}{s^2+a^2}$ and $L(\cos at) = \dfrac{s}{s^2+a^2}$, we have

$$L(t\sin at) = -\frac{d}{ds}\frac{a}{s^2+a^2} = -a\frac{d}{ds}(s^2+a^2)^{-1} = \frac{2as}{(s^2+a^2)^2}.$$

and $$L(t\cos at) = -\frac{d}{ds}\frac{s}{s^2+a^2} = -\frac{(s^2+a^2)1 - s\cdot 2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$ **Hence Proved.**

(b) Since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we have

$$\sin^3(2t) = \frac{1}{4}[3\sin 2t - \sin 6t]$$

$$\therefore\ L(\sin^3 2t) = \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t) = \frac{3}{4}\cdot\frac{2}{s^2+2^2} - \frac{1}{4}\frac{6}{s^2+6^2} = \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \frac{3}{2}\cdot\frac{32}{(s^2+4)(s^2+36)} = \frac{48}{(s^2+4)(s^2+36)}.$$ **Ans.**

EXAMPLE 15.7. Find the Laplace transform of

(i) $\sin at \sin bt$ [GGSIPU II Sem End Term 2007]

(ii) $t^2 e^{-2t}$ [GGSIPU II Sem End Term 2005]

(iii) $\dfrac{e^{-t}\sin t}{t}$ [GGSIPU II Sem End Term 2007]

(iv) $te^{-t}\sin 3t$. [GGSIPU II Sem End Term 2011; II Sem End 2012]

SOLUTION: (i) $\sin at \sin bt = \dfrac{1}{2}[\cos(a-b)t - \cos(a+b)t]$

$$\therefore\quad L(\sin at \sin bt) = \frac{1}{2}L\cos(a-b)t - \frac{1}{2}L\cos(a+b)t$$

$$= \frac{1}{2}\frac{s}{s^2+(a-b)^2} - \frac{1}{2}\frac{s}{s^2+(a+b)^2} = \frac{s}{2}\frac{[s^2+(a+b)^2 - s^2 - (a-b)^2]}{[s^2+(a-b)^2][s^2+(a+b)^2]}$$

or $$L(\sin at \sin bt) = \frac{2abs}{[s^2+(a-b)^2][s^2+(a+b)^2]}$$ **Ans.**

(ii) We know that if $L[f(t)] = \bar{f}(s)$ then $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)$

$$\therefore\quad L(t^2 e^{-2t}) = (-1)^2\frac{d^2}{ds^2}\frac{1}{s+2} = \frac{2}{(s+2)^3}$$ **Ans.**

(iii) We know that $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ $\therefore$ $L(e^{-t}\sin t) = \dfrac{1}{(s+1)^2+1}$.

Also since $L\left(\frac{1}{t}f(t)\right) = \int_s^\infty \bar{f}(s)\,ds$ where $L(f(t)) = \bar{f}(s)$, we have

$$L\left[\frac{1}{t}(e^{-t}\sin t)\right] = \int_s^\infty \frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

Therefore $L\left(\frac{1}{t}e^{-t}\sin t\right) = \cot^{-1}(s+1)$. **Ans.**

(*iv*) We know that $L[e^{-t}\sin 3t] = \frac{3}{(s+1)^2+9} = \frac{3}{s^2+2s+10}$

$\therefore$ $L[t(e^{-t}\sin 3t] = -\frac{d}{ds}\left(\frac{3}{s^2+2s+10}\right) = \frac{6s+6}{(s^2+2s+10)^2}$. **Ans.**

EXAMPLE 15.8. Find the Laplace transform of

(*i*) $2e^t \sin 4t \cos 2t$. (*ii*) $\frac{(e^{-at}-e^{-bt})}{t}$. (*iii*) $\frac{1-e^t}{t}$.

[*GGSIPU II Sem End Term 2012*]

SOLUTION: (*i*) By Linearity theorem

$$L(2\sin 4t\cos 2t) = L(\sin 6t + \sin 2t) = \frac{6}{s^2+6^2} + \frac{2}{s^2+2^2}$$

$\therefore$ By first shifting theorem

$$L(2e^t\sin 4t\cos 2t) = \frac{6}{(s-1)^2+36} + \frac{2}{(s-1)^2+4} = \frac{6}{s^2-2s+37} + \frac{2}{s^2-2s+5}.$$ **Ans.**

(*ii*) $L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$

Now, by the property of Laplace transform, we have

$$L\left(\frac{e^{-at}-e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds = \left[\log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \underset{s\to\infty}{\text{Lt}}\left[\log\frac{s+a}{s+b}\right] - \log\frac{s+a}{s+b} = \underset{s\to\infty}{\text{Lt}} \log\frac{1+\frac{a}{s}}{1+\frac{b}{s}} + \log\frac{s+b}{s+a}$$

$$= \log\left(\frac{1+0}{1+0}\right) + \log\frac{s+b}{s+a} = \log\frac{s+b}{s+a}$$ **Ans.**

(iii) $L(1-e^t) = \frac{1}{s} - \frac{1}{s-1}$

Since $L\left(\frac{1}{t}f(t)\right) = \int_s^\infty f(s)\,ds$ where $\bar{f}(s) = L(f(t))$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds = [\log s - \log(s-1)]_s^\infty$$

$$= \left[\log\frac{s}{s-1}\right]_s^\infty = \underset{s\to\infty}{\text{Lt}} \log\frac{s}{s-1} - \log\frac{s}{s-1} = \log 1 + \log\frac{s-1}{s} = \log\frac{s-1}{s}.$$ **Ans.**

EXAMPLE 15.9. (*i*) Evaluate $L\left(\dfrac{\sin at}{t}\right)$. Does $L\left(\dfrac{\cos at}{t}\right)$ exist? [*GGSIPU III Sem End Term 2007*]

(*ii*) Using Laplace transform, show that $L\displaystyle\int_0^t \frac{\cos at - \cos bt}{t}\,dt = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$

[*GGSIPU III Sem End Term 2007*]

(*iii*) Show that $L\displaystyle\int_0^t e^t\,\frac{\sin t}{t}\,dt = \frac{1}{s}\cot^{-1}(s-1)$.

[*GGSIPU III Sem End Term 2005; II Sem End Term 2009; II Sem I Term 2014*]

SOLUTION: (*i*) We know that $L(\sin at) = \dfrac{a}{s^2+a^2}$ and $L(\cos at) = \dfrac{s}{s^2+a^2}$.

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

and $\quad L\left(\dfrac{\cos at}{t}\right) = \displaystyle\int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$ which is non-existent. **Ans.**

(*ii*) $L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$\therefore \quad L\left[\frac{1}{t}(\cos at - \cos bt)\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) \qquad \left(\text{since } \lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2} = \log 1 = 0\right)$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

Now, using $\quad L\left(\displaystyle\int_0^t f(t)\,dt\right) = \dfrac{1}{s}\bar{f}(s)$, we have

$$L\left[\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right] = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$ **Ans.**

(*iii*) We know that $L(e^t \sin t) = \dfrac{1}{(s-1)^2+1}$

Hence $\quad L\left(\dfrac{e^t\sin t}{t}\right) = \displaystyle\int_s^\infty \frac{ds}{(s-1)^2+1} = \left[\tan^{-1}(s-1)\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$$

Therefore $\quad L\left[\displaystyle\int_0^t \frac{e^t\sin t}{t}\,dt\right] = \dfrac{1}{s}\cot^{-1}(s-1)$ **Ans.**

EXAMPLE 15.10. (*i*) Using Laplace transform, evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$

[GGSIPU III Sem End Term 2004]

(*ii*) Evaluate $L(\sin h^3 2t)$ [GGSIPU III Sem End Term 2003]

SOLUTION: $L(t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3}\frac{1}{(s^2+1)} = \frac{-d^2}{ds^2}\frac{(-1)\,2s}{(s^2+1)^2} = \frac{d^2}{ds^2}\frac{2s}{(s^2+1)^2}$

$$= \frac{d}{ds}\frac{(s^2+1)^2\, 2 - 2s \times 2(s^2+1)2s}{(s^2+1)^4} = \frac{d}{ds}\frac{2(s^2+1)-8s^2}{(s^2+1)^3}$$

$$= \frac{d}{ds}\frac{2-6s^2}{(s^2+1)^3} = \frac{(s^2+1)^3(-12s)-(2-6s^2)\,3\,(s^2+1)^2 \times 2s}{(s^2+1)^6}$$

$$= \frac{-12s\,(s^2+1)+12s\,(3s^2-1)}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

That is, $L(t^3 \sin t) = \int_0^\infty e^{-st} t^3 \sin t \, dt = \frac{24s\,(s^2-1)}{(s^2+1)^4}$

Putting here $s = 1$, we get $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0.$ **Ans.**

(*ii*) we know that $\sinh 3t = 3 \sinh t + 4 \sinh^3 t$ or $\sinh^3 t = \frac{1}{4}[\sinh 3t - 3 \sinh t]$

Therefore $\sinh^3 2t = \frac{1}{4}[\sinh 6t - 3 \sinh 2t]$

and $L(\sinh^3 2t) = \frac{1}{4}L(\sinh 6t) - \frac{3}{4}L(\sinh 2t) = \frac{1}{4}\cdot\frac{6}{s^2-6^2} - \frac{3}{4}\frac{2}{s^2-2^2}$

$$= \frac{3}{2}\frac{32}{(s^2-4)(s^2-36)} = \frac{48}{(s^2-4)(s^2-36)}.$$ **Ans.**

EXAMPLE 15.11. (a) Find the Laplace transform of the function $f(t)$ given by

$$f(t) = \begin{cases} \sin(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$$

(b) Given that $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}.$

SOLUTION: (a) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\alpha e^{-st}\cdot 0\, dt + \int_\alpha^\infty e^{-st} \sin(t-\alpha)\, dt$

$$= 0 + \int_0^\infty e^{-s(u+\alpha)} \sin u \, du \quad \text{on putting } u = t - \alpha$$

$$= e^{-\alpha s}\int_0^\infty e^{-su} \sin u \, du = e^{-\alpha s}\, L\{\sin u\} = \frac{e^{-\alpha s}}{s^2+1}.$$ **Ans.**

(b) Let $f(t) = 2\sqrt{\frac{t}{\pi}}$ and we are given $L\{f(t)\} = \frac{1}{s^{3/2}} = \bar{f}(s)$.

Then by the property on derivatives, we have

$$L\{f'(t)\} = s\bar{f}(s) - f(0) = s \cdot \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$$

But $f'(t) = \frac{2}{\sqrt{\pi}} \frac{d}{dt} \sqrt{t} = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}$

Therefore we have $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$. **Hence Proved.**

EXAMPLE 15.12. (a) Using Laplace transformation, show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$.

[GGSIPU II Sem End Term 2006]

(b) Evaluate $\int_0^\infty t\, e^{-2t} \cos t\, dt$ using L' transformation.

[GGSIPU II Sem, End Term 2006]

SOLUTION: (a) We know that $L(\sin a t) = \frac{a}{s^2 + a^2}$

Therefore $L\left(\frac{1}{t} \sin at\right) = \int_s^\infty \frac{a}{s^2 + a^2} ds = a \cdot \frac{1}{a}\left[\tan^{-1} \frac{s}{a}\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Thus $\int_0^\infty e^{-st} \frac{\sin at}{t} dt = \cot^{-1}\left(\frac{s}{a}\right)$

Taking $a = 1$ and $s = 0$ in the above relation, we get

$$\int_0^\infty \frac{\sin t}{t} dt = \cot^{-1}(0) = \frac{\pi}{2}.$$ **Hence Proved.**

(b) We know that $L(t \cos t) = -\frac{d}{ds} \frac{s}{s^2 + 1}$

or $\int_0^\infty e^{-st}\, t \cos t\, dt = -\frac{d}{ds} \frac{s}{s^2 + 1} = -\left[\frac{(s^2 + 1)1 - s \cdot 2s}{(s^2 + 1)^2}\right] = \frac{s^2 - 1}{(s^2 + 1)^2}$

Putting $s = 2$ on both sides, we get $\int_0^\infty e^{-2t}\, t \cos t\, dt = \frac{2^2 - 1}{(2^2 + 1)^2} = \frac{3}{25}$. **Ans.**

EXERCISE 15A

1. Find the Laplace transform of the function (*i*) $3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t$. (*ii*) te^{-t} (*iii*) $t^2 e^{-2t}$.
2. Find the Laplace transform of (*i*) $t \sin t$ (*ii*) $t^2 \cos t$ (*iii*) $\cos(2t+3)$.
3. Find the Laplace transform of (*i*) $t^3 e^{4t}$ (*ii*) $e^{-2t}\sqrt{t}$ (*iii*) $t^2 e^t \sin 4t$.

   [*GGSIPU II Sem End Term 2014*]
4. Find the Laplace transform of (*i*) $\int_0^t x^2 e^x \, dx$ (*ii*) $\int_0^t \cos^2 u \, du$.
5. Find the Laplace transform of the following functions

   (*i*) $f(t) = \begin{cases} 0, & 0 < t < 2 \\ 1, & t > 2 \end{cases}$ (*ii*) $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$
6. Find (*i*) $L\left(\dfrac{\cos 2t - \cos 3t}{t}\right)$ (*ii*) $L\left(\dfrac{1 - \cos t}{t^2}\right)$. (*iii*) $L(\sinh at \sin at)$

   [*GGSIPU II Sem End Term 2014*]
7. Evaluate (*i*) $\int_0^\infty e^{-2t} \sin^3 t \, dt$ [*GGSIPU III Sem End Term 2012*] (*ii*) $\int_0^\infty \dfrac{\cos 6t - \cos 4t}{t} dt$
8. If $L(\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$ find $L(\cos^2 at)$.
9. If $L(f(t)) = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & 0 < t < a \end{cases}$

   then prove that $L(g(t)) = e^{-as}\bar{f}(s)$.
10. (*i*) Find $L(\sin\sqrt{t})$.

    [*GGSIPU III Sem End Term 2010*]

    (*ii*) If $L(\sin\sqrt{t}) = \dfrac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$, show that $L\left(\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right) = \left(\dfrac{\pi}{p}\right)^{1/2} e^{-1/4p}$.

    [*GGSIPU II Sem I Term 2014*]
11. Find the Laplace transform of $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$ [*GGSIPU II Sem End Term 2009*]

INVERSE LAPLACE TRANSFORM

Given $\bar{f}(s)$ we are to find the object function $f(t)$ of which $\bar{f}(s)$ is the Laplace transform and we write $L^{-1}\{\bar{f}(s)\} = f(t)$.

We recall here the first shifting theorem as; if $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

This can also be stated as $L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$.

It will be quite useful in finding inverse Laplace transform of many functions, In addition to this the method of partial fractions is extremely useful in finding inverse Laplace transform of functions which are in the form of proper rational fractions.

Let us now list inverse Laplace transforms of some standard functions.

1. $L^{-1}\left(\frac{1}{s}\right) = 1.$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, n = 1, 2, 3, \ldots.$

3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$

4. $L^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) = \frac{e^{at}t^n}{n!}, n = 1, 2, 3, \ldots.$

5. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

6. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at.$

7. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at.$

8. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

9. $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt.$

10. $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt.$

11. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at.$ [GGSIPU III Sem End Term 2009]

Proof: Since $L(\sin at) = \frac{a}{s^2+a^2}$ we have $L(t\sin at) = -\frac{d}{ds}\frac{a}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2}$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at.$$

12. $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}[\sin at - at\cos at].$ [GGSIPU II Sem End Term 2013]

Proof:
$$L(t\cos at) = -\frac{d}{ds}\frac{s}{s^2+a^2} = \frac{s^2-a^2}{(s^2+a^2)^2} = \frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}$$
$$= \frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2} = L\left(\frac{1}{a}\sin at\right) - \frac{2a^2}{(s^2+a^2)^2}$$

Hence $L^{-1}\frac{2a^2}{(s^2+a^2)^2} = \frac{1}{a}\sin at - t\cos at$

or
$$L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}[\sin at - at\cos at].$$

EXAMPLE 15.13. Find the inverse Laplace transform of

$$(i)\ \frac{1}{s^2-5s+6} \quad (ii)\ \frac{s+7}{s^2+2s+5} \quad (iii)\ \frac{2s^2-6s+5}{s^3-6s^2+11s-6}.$$

[GGSIPU II Sem II Term 2005]

SOLUTION: (i) $L^{-1}\left(\frac{1}{s^2-5s+6}\right) = L^{-1}\left(\frac{1}{(s-3)(s-2)}\right) = L^{-1}\left(\frac{1}{s-3}-\frac{1}{s-2}\right) = e^{3t}-e^{2t}.$ **Ans.**

(ii) Let $\frac{s+7}{s^2+2s+5} = \bar{f}(s)$. We can write

$$\bar{f}(s) = \frac{s+7}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+2^2} + \frac{6}{(s+1)^2+2^2}$$

$$\therefore \quad f(t) = L^{-1}\left(\bar{f}(s)\right) = L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 6\cdot L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^{-t}\cos 2t + 6e^{-t}\frac{\sin 2t}{2} = e^{-t}[\cos 2t + 3\sin 2t].$$ **Ans.**

(iii) Resolving the given expression into partial fractions, we get

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$\Rightarrow \quad 2s^2-6s+5 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2).$$

Putting $s=1,2,3$ successively, we get $A=\frac{1}{2}$, $B=-1$, $C=\frac{5}{2}$.

$$\therefore\ L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right) = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

Ans.

EXAMPLE 15.14. Find the inverse Laplace transform of the following functions

$$(i)\ \frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3} \qquad (ii)\ \frac{5s+3}{(s-1)(s^2+2s+5)}$$

SOLUTION: (i) Let $\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3} \equiv \frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$

$$\Rightarrow \quad 6s^3-21s^2+20s-7 = A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1) \quad ...(1)$$

Putting $s=-1$ and 2 successively in (1), we get $A=2$ and $D=-1$.

Then, comparing the co-efficients of terms of various powers of s on both sides, gives

$B=4$ and $C=3$.

$$\therefore\ L^{-1}\left(\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}\right) = 2L^{-1}\left(\frac{1}{s+1}\right)+4L^{-1}\left(\frac{1}{s-2}\right)+3L^{-1}\left(\frac{1}{(s-2)^2}\right)-L^{-1}\frac{1}{(s-2)^3}$$

$$= 2e^{-t}+4e^{2t}+3t\,e^{2t}-\frac{t^2}{2}e^{2t} = 2e^{-t}+\left(4+3t-\frac{t^2}{2}\right)e^{2t}.$$ **Ans.**

(ii) Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$

$\Rightarrow \quad 5s+3 = A(s^2+2s+5)+(s-1)(Bs+C)$

Putting $s = 1$ on both sides, we get $A = 1$ and then comparing the coefficients of terms of various powers of s, gives $B = -1$, $C = 2$.

Therefore $L^{-1}\dfrac{5s+3}{(s-1)(s^2+2s+5)} = L^{-1}\dfrac{1}{s-1} + L^{-1}\left(\dfrac{-s+2}{s^2+2s+5}\right)$

$$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 3L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t. \qquad \textbf{Ans.}$$

EXAMPLE 15.15. Find the inverse Laplace transform of

(i) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$ (ii) $\dfrac{s}{s^4+4a^4}$.

[GGSIPU II Sem End Term 2006; End Term 2009; II Sem Term 2012]

SOLUTION: (i) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{p-4}{(p+5)(p+2)}$ where $p = s^2+2s$

$$= \frac{3}{p+5} - \frac{2}{p+2} = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2} = \frac{3}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+1^2}$$

$\therefore \quad L^{-1}\left\{\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\} = 3L^{-1}\left[\dfrac{1}{(s+1)^2+2^2}\right] - 2L^{-1}\left[\dfrac{1}{(s+1)^2+1^2}\right]$

$$= \frac{3}{2}e^{-t}\sin 2t - 2e^{-t}\sin t = \frac{e^{-t}}{2}[3\sin 2t - 4\sin t] \qquad \textbf{Ans.}$$

(ii) $\dfrac{s}{s^4+4a^4} = \dfrac{s}{(s^2+2a^2)^2-4a^2s^2} = \dfrac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$\therefore \quad L^{-1}\dfrac{s}{s^4+4a^4} = \dfrac{1}{4a}L^{-1}\dfrac{1}{(s-a)^2+a^2} - \dfrac{1}{4a}L^{-1}\dfrac{1}{(s+a)^2+a^2}$

$$= \frac{1}{4a}\cdot e^{at}\frac{\sin at}{a} - \frac{1}{4a}\cdot\frac{e^{-at}\sin at}{a} = \frac{\sin at}{4a^2}(e^{at}-e^{-at})$$

$$= \frac{1}{2a^2}\sin at \sinh at. \quad \textbf{Ans.}$$

Example 15.16. Find the inverse Laplace transform of

(i) $\dfrac{s+4}{s(s-1)(s^2+4)}$ [GGSIPU II Sem End Term 2005]

(ii) $\log\dfrac{s-1}{s}$ [GGSIPU III Sem End Term 2007]

(iii) $\dfrac{3s+5\sqrt{2}}{s^2+8}$ [GGSIPU III Sem End Term 2004]

Solution: (i) Resolving the given expression into partial fractions, we get

$$\frac{s+4}{s(s-1)(s^2+1)} = -\frac{4}{s}+\frac{5}{2(s-1)}+\frac{3s-5}{2(s^2+1)}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+1)}\right] = -4L^{-1}\left(\frac{1}{s}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-1}\right)+\frac{3}{2}L^{-1}\left(\frac{s}{s^2+1}\right)-\frac{5}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -4.(1)+\frac{5}{2}e^t+\frac{3}{2}\cos t-\frac{5}{2}\sin t \quad \textbf{Ans.}$$

(ii) Consider $f(t)=1-e^t$ then $L(f(t)) = \dfrac{1}{s}-\dfrac{1}{s-1} = \bar{f}(s)$

$$\Rightarrow \quad L\left(\frac{1}{t}f(t)\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds = [\log s-\log(s-1)]_s^\infty$$

$$= \left[\log\frac{s}{s-1}\right]_s^\infty = 0-\log\frac{s}{s-1} = \log\frac{s-1}{s}.$$

Therefore $L^{-1}\log\dfrac{s-1}{s} = \dfrac{1}{t}(1-e^t)$. **Ans.**

(iii) $$L^{-1}\left(\frac{3s+5\sqrt{2}}{s^2+8}\right) = 3L^{-1}\left(\frac{s}{s^2+8}\right)+5\sqrt{2}\,L^{-1}\left(\frac{1}{s^2+8}\right) = 3\cos 2\sqrt{2}\,t+\frac{5\sqrt{2}}{2\sqrt{2}}\sin 2\sqrt{2}\,t$$

Thus $$L^{-1}\left(\frac{3s+5\sqrt{2}}{s^2+8}\right) = 3\cos 2\sqrt{2}\,t+\frac{5}{2}\sin 2\sqrt{2}\,t. \quad \textbf{Ans.}$$

Example 15.17. Find the inverse Laplace transform of

(i) $\dfrac{s^2+2s-3}{s(s-3)(s+2)}$ [GGSIPU II Sem End Term 2006 Reappear]

(ii) $\dfrac{s^2}{s^4-a^4}$ [GGSIPU III Sem End Term 2003; II Sem End Term 2009]

Solution: (i) Resolving the given expression into partial fractions, we get

$$\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2s}+\frac{4}{5(s-3)}-\frac{3}{10(s+2)}$$

$$\therefore \quad L^{-1}\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{5}L^{-1}\left(\frac{1}{s-3}\right)-\frac{3}{10}L^{-1}\left(\frac{1}{s+2}\right) = \frac{1}{2}+\frac{4}{5}e^{3t}-\frac{3}{10}e^{-2t}$$

Thus, $L^{-1}\dfrac{s^2+2s-3}{s(s-3)(s+2)} = \dfrac{1}{2}+\dfrac{4}{5}e^{3t}-\dfrac{3}{10}e^{-2t}$ **Ans.**

(ii) $$\frac{s^2}{s^4-a^4} = \frac{s^2}{(s^2+a^2)(s^2-a^2)} = \frac{1}{2}\left(\frac{1}{s^2+a^2}+\frac{1}{s^2-a^2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{s^2}{s^4-a^4}\right) = \frac{1}{2}L^{-1}\frac{1}{s^2+a^2}+\frac{1}{2}L^{-1}\frac{1}{s^2-a^2} = \frac{1}{2a}\sin at+\frac{1}{2a}\sinh at$$

or $$L^{-1}\left(\frac{s^2}{s^4-a^4}\right) = \frac{1}{2a}(\sin at+\sinh at).$$ **Ans.**

EXAMPLE 15.18. **(a)** Find the inverse Laplace transform of $\log\dfrac{s+1}{s-1}$.

[GGSIPU II Sem End Term 2006 (Reappear); End Term 2013]

(b) Find the inverse laplace transform of $\dfrac{1}{s^3(s^2+1)}$.

[GGSIPU II Sem End Term 2010; End Term 2014]

SOLUTION: (a) Let $f(t) = \mathrm{L}^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\}$ hence $\bar{f}(s) = \log\left(\dfrac{s+1}{s-1}\right)$

then $$\mathrm{L}(t f(t)) = -\frac{d}{ds}\bar{f}(s) = -\frac{d}{ds}\log\frac{s+1}{s-1} = -\frac{d}{ds}\{\log(s+1)-\log(s-1)\}$$

$$= -\left\{\frac{1}{s+1}-\frac{1}{s-1}\right\} = \frac{2}{(s^2-1)}$$

$\therefore \quad t f(t) = 2\mathrm{L}^{-1}\dfrac{1}{s^2-1} = 2\sinh t$ Therefore $f(t) = \dfrac{2}{t}\sinh t.$ **Ans.**

(b) $$\frac{1}{s^3(s^2+1)} = \frac{1}{s}\left(\frac{1}{s^2(s^2+1)}\right) = \frac{1}{s}\left[\frac{1}{s^2}-\frac{1}{s^2+1}\right]$$

On using the property $L\displaystyle\int_0^t f(t)dt = \frac{1}{s}\bar{f}(s)$ and

$L^{-1}\left[\dfrac{1}{s^2}-\dfrac{1}{s^2+1}\right] = t-\sin t,$ we have

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t (t-\sin t)dt = \left[\frac{t^2}{2}+\cos t\right]_0^t = \frac{t^2}{2}+\cos t-1$$ **Ans.**

Example 15.19. Find the inverse Laplace transform of

$$(i)\ \frac{s}{(s^2-a^2)^2} \qquad (ii)\ \frac{1}{(s^2-a^2)^2} \qquad (iii)\ \frac{s}{s^4+s^2+1}.$$

[GGSIPU III Sem End Term 2010; II Sem End Term 2011]

Solution: (i) We know that $\quad L(\sinh at) = \dfrac{a}{s^2-a^2}$

hence $\quad L(t\sinh at) = -\dfrac{d}{ds}\dfrac{a}{s^2-a^2} = -a\dfrac{(-1)\cdot 2s}{(s^2-a^2)^2} = \dfrac{2as}{(s^2-a^2)^2}$

$\Rightarrow \quad L^{-1}\dfrac{s}{(s^2-a^2)^2} = \dfrac{t\sinh at}{2a}.$ **Ans.**

(ii) We know that $\ L(\cosh at) = \dfrac{s}{s^2-a^2}$

hence $\ L(t\cosh at) = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-a^2}\right) = -\dfrac{\{(s^2-a^2)\cdot 1 - s\cdot 2s\}}{(s^2-a^2)^2}$

$$= \frac{s^2-a^2+2a^2}{(s^2-a^2)^2} = \frac{1}{s^2-a^2} + \frac{2a^2}{(s^2-a^2)^2}$$

$\Rightarrow \quad t\cosh at = L^{-1}\dfrac{1}{s^2-a^2} + 2a^2 L^{-1}\dfrac{1}{(s^2-a^2)^2} = \dfrac{1}{a}\sinh at + 2a^2 L^{-1}\dfrac{1}{(s^2-a^2)^2}$

$\therefore \quad L^{-1}\dfrac{1}{(s^2-a^2)^2} = \dfrac{1}{2a^2}\left[\dfrac{-1}{a}\sinh at + t\cosh at\right] = \dfrac{at\cosh at - \sinh at}{2a^3}.$ **Ans.**

(iii) $\dfrac{s}{s^4+s^2+1} = \dfrac{s}{(s^2+1)^2-s^2} = \dfrac{s}{(s^2+s+1)(s^2-s+1)}$

$$= \frac{1}{2}\left\{\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right\} = \frac{1}{2\left\{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}} - \frac{1}{2\left\{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

Therefore $\ L^{-1}\left(\dfrac{s}{s^4+s^2+1}\right) = \dfrac{1}{2}\left[e^{t/2}\dfrac{1}{(\sqrt{3}/2)}\sin\left(\dfrac{\sqrt{3}t}{2}\right) - e^{-t/2}\dfrac{1}{(\sqrt{3}/2)}\sin\left(\dfrac{\sqrt{3}t}{2}\right)\right]$

$$= \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right) = \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\sinh\frac{t}{2}.$$ **Ans.**

Example 15.20. Find (i) $L^{-1}\log\left(1+\dfrac{1}{s^2}\right)$ (ii) $L^{-1}[\tan^{-1}(s-1)]$.

Solution: (i) Let $\bar{f}(s) = \log\left(1+\dfrac{1}{s^2}\right) = \log(s^2+1) - 2\log s$

$\therefore \quad \dfrac{d}{ds}\bar{f}(s) = \dfrac{2s}{s^2+1} - \dfrac{2}{s}.$

Hence $\ L^{-1}\left\{\dfrac{d}{ds}\bar{f}(s)\right\} = L^{-1}\dfrac{2s}{s^2+1} - L^{-1}\left(\dfrac{2}{s}\right) = 2\cos t - 2$

But $\quad L^{-1}\dfrac{d}{ds}\bar{f}(s) = -t f(t) \qquad$ therefore $\ -t f(t) = 2\cos t - 2$

$\Rightarrow \quad f(t) = \dfrac{2(1-\cos t)}{t}$ which is the required function. **Ans.**

(ii) Let $f(t) = L^{-1} \tan^{-1}(s-1)$ then $\bar{f}(s) = \tan^{-1}(s-1)$

$$\therefore \quad \frac{d}{ds}\bar{f}(s) = \frac{1}{(s-1)^2+1}$$

$$\Rightarrow \quad L^{-1}\frac{d}{ds}\bar{f}(s) = L^{-1}\frac{1}{(s-1)^2+1} = e^t \sin t$$

But $\quad L^{-1}\frac{d}{ds}\bar{f}(s) = -t f(t)$

hence $\quad -t f(t) = e^t \sin t \quad$ or $\quad f(t) = -\frac{1}{t}e^t \sin t$, which is the required function. **Ans.**

EXAMPLE 15.21. Find $L^{-1}\left[s \log \frac{s}{\sqrt{s^2+1}} + \cot^{-1} s\right]$.

SOLUTION: Let $L(f(t)) = F(s) = s \log \frac{s}{\sqrt{s^2+1}} + \cot^{-1} s, = s \log s - \frac{s}{2}\log(s^2+1) + \cot^{-1} s.$

$$F'(s) = 1 \cdot \log s + s \cdot \frac{1}{s} - \frac{s}{2} \cdot \frac{2s}{s^2+1} - \frac{1}{2}\log(s^2+1) - \frac{1}{s^2+1}$$

$$= \log s + 1 - \frac{s^2}{s^2+1} - \frac{1}{2}\log(s^2+1) - \frac{1}{s^2+1} = \log s - \frac{1}{2}\log(s^2+1)$$

Therefore $\quad F''(s) = \frac{1}{s} - \frac{s}{s^2+1} \quad \therefore \quad L^{-1}\{F''(s)\} = 1 - \cos t$

But $\quad L^{-1}\{F''(s)\} = (-1)^2 t^2 f(t)$

$$\Rightarrow \quad t^2 f(t) = 1 - \cos t \quad \text{or} \quad f(t) = \frac{1-\cos t}{t^2}.$$ **Ans.**

CONVOLUTION AND CONVOLUTION THEOREM

The function $\int_0^t f_1(u)\, f_2(t-u)\, du$ is called the *convolution* of the functions f_1 and f_2 and is denoted by $f_1 * f_2$. It is easy to verify that $f_1 * f_2 = f_2 * f_1$.

Let $f_1(t)$ and $f_2(t)$ be two functions of t and $L\{f_1(t)\} = \bar{f}_1(s)$ and $L\{f_2(t)\} = \bar{f}_2(s)$ then the theorem states that $L^{-1}\{\bar{f}_1(s)\,\bar{f}_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du = \int_0^t f_2(u)\, f_1(t-u)\, du$. In other words $L(f_1 * f_2) = \bar{f}_1(s)\,\bar{f}_2(s)$.

[GGSIPU III Sem End Term 2009; End Term 2013]

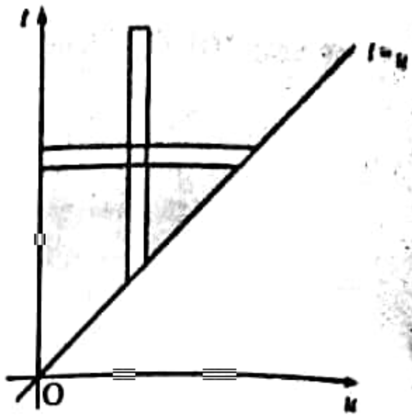
PROOF : By definition

$$L\left\{\int_0^t f_1(u)\, f_2(t-u)\, du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f_1(u)\, f_2(t-u)\, du\right] dt = \int_0^\infty \int_0^t e^{-st} f_1(u)\, f_2(t-u)\, du\, dt$$

[Now changing the order of integration, (see the figure)]

$$= \int_0^\infty \int_u^\infty e^{-st} f_1(u)\, f_2(t-u)\, dt\, du$$

$$= \int_0^\infty \left[\int_u^\infty e^{-st} f_2(t-u)\, dt\right] f_1(u)\, du$$



(Now putting $t - u = y$ in the inner integral)

$$= \int_0^\infty \left[\int_0^\infty e^{-s(u+y)} f_2(y)\, dy\right] f_1(u)\, du$$

$$= \int_0^\infty e^{-su} f_1(u)\, du \int_0^\infty e^{-sy} f_2(y)\, dy = \bar{f}_1(s)\, \bar{f}_2(s)$$

$$\Rightarrow \qquad L^{-1}\{\bar{f}_1(s)\, \bar{f}_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du.$$

Example 15.22. Use convolution to find (*i*) $L^{-1} \dfrac{1}{(s^2+a^2)^2}$. (*ii*) $L^{-1} \dfrac{s}{(s^2+a^2)^3}$

[GGSIPU III Sem End Term 2010]

Solution: (*i*) Since $L^{-1} \dfrac{1}{s^2+a^2} = \dfrac{1}{a} \sin at$, using convolution theorem here, we get

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1}{a} \sin au \cdot \frac{1}{a} \sin a(t-u)\, du$$

$$= \frac{1}{2a^2} \int_0^t [\cos a(2u-t) - \cos at]\, du = \frac{1}{2a^2} \left[\frac{1}{2a} \sin a(2u-t) - u \cos at\right]_0^t$$

$$= \frac{1}{2a^2} \left[\frac{1}{2a} \sin at - t \cos at + \frac{1}{2a} \sin at\right] = \frac{1}{2a^3} [\sin at - at \cos at]. \qquad \textbf{Ans.}$$

(*ii*) Since $L\left(\dfrac{t}{2a} \sin at\right) = \dfrac{s}{(s^2+a^2)^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$, applying convolution theorem we get

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2} \frac{1}{(s^2+a^2)}\right] = \int_0^t \frac{u}{2a} \sin au \cdot \frac{1}{a} \sin a(t-u)\, du = \frac{1}{2a^2} \int_0^t u \sin au \sin a(t-u)\, du$$

$$= \frac{1}{4a^2} \int_0^t u [\cos(2au - at) - \cos at]\, du$$

$$= \frac{1}{4a^2} \int_0^t u \cos(2au - at)\, du - \frac{1}{4a^2} \left[\frac{u^2}{2} \cos at\right]_0^t$$

$$= \frac{1}{4a^2} \left[\left\{u \frac{\sin(2au - at)}{2a}\right\}_0^t - \int_0^t \frac{1 \cdot \sin(2au - at)}{2a}\, du\right] - \frac{t^2}{8a^2} \cos at$$

$$= \frac{1}{4a^2} \left[\frac{t}{2a} \sin at + \frac{1}{4a^2} \{\cos(2au - at)\}_0^t\right] - \frac{t^2}{8a^2} \cos at$$

$$= \frac{t}{8a^3} \sin at + \frac{1}{16a^4} (\cos at - \cos at) - \frac{t^2}{8a^2} \cos at$$

$$= \frac{t}{8a^3} (\sin at - at \cos at). \qquad \textbf{Ans.}$$

Example 15.23. (*a*) Employ convolution theorem, to find

(*i*) $L^{-1}\left[\dfrac{1}{s\sqrt{s+4}}\right]$. (ii) $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]$

[GGSIPU III Sem End Term 2008] [GGSIPU II Sem End Term 2013]

(*b*) Applying convolution theorem show that

$$\int_0^t \sin u \cos(t-u)\, du = \frac{1}{2} t \sin t$$ [GGSIPU III Sem End Term 2010]

Solution: (*a*) (*i*) We know that $L^{-1}\left(\dfrac{1}{s}\right) = 1,\ L^{-1}\left(\dfrac{1}{\sqrt{s}}\right) = \dfrac{t^{-1/2}}{\Gamma 1/2} = \dfrac{1}{\sqrt{\pi t}}\ \therefore L^{-1}\left(\dfrac{1}{\sqrt{s+4}}\right) = \dfrac{e^{-4t}}{\sqrt{\pi t}}$,

therefore, by convolution theorem, we have

$$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^t \frac{e^{-4u}}{\sqrt{\pi u}}\cdot 1\, du = \frac{1}{\sqrt{\pi}}\int_0^t u^{-\frac{1}{2}} e^{-4u}\, du$$ **Ans.**

(*ii*) Since $L(t) = \dfrac{1}{s^2}$ and $L(te^{-t}) = \dfrac{1}{(s+1)^2}$, by convolution theorem, we have

$$L^{-1}\left(\frac{1}{s^2(s+1)^2}\right) = \int_0^t (t-u)\, u e^{-u} du = \int_0^t (ut-u^2)e^{-u} du$$

$$= -\left[(ut-u^2)e^{-u}\right]_0^t + \int_0^t (t-2u)\, e^{-u}\, du$$

$$= -(t^2-t^2)e^{-t} + 0 - \left[(t-2u)\, e^{-u}\right]_0^t + \int_0^t (-2)\, e^{-u} du$$

$$= 0 + t e^{-t} + t + 2\left[e^{-u}\right]_0^t = t e^{-t} + t + 2e^{-t} - 2$$

$$= (t+2)e^{-t} + t - 2.$$ **Ans.**

(*b*) By convolution theorem, we have $\displaystyle\int_0^t f_1(u)\, f_2(t-u)\, du = L^{-1}[\bar{f}_1(s)\bar{f}_2(s)]$

Here $f_1(u) = \sin u$ and $f_2(u) = \cos u$ and $\bar{f}_1(s) = \dfrac{1}{s^2+1}$ and $\bar{f}_2(s) = \dfrac{s}{s^2+1}$

$$\therefore \int_0^t (\sin u \cos(t-u)\, du = L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right) = L^{-1}\frac{s}{(s^2+1)^2}\Bigg) = \frac{1}{2} t \sin t.$$ **Hence Proved.**

EXAMPLE 15.24. Use convolution theorem to evaluate the Laplace transform of

(*i*) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ [GGSIPU II Sem End Term 2007; End Term 2010; End Term 2014]

(*ii*) $\dfrac{s^2}{(s^2+w^2)^2}$ [GGSIPU II Sem End Term 2006]

(*iii*) $\dfrac{1}{s(s+1)(s+2)}$ [GGSIPU II Sem End Term 2011]

SOLUTION: (*i*) By convolution theorem, we have

$$L^{-1}\left[\bar{f}_1(s)\,\bar{f}_2(s)\right] = \int_0^t f_1(u)\,f_2(t-u)\,du \quad \text{where} \quad \bar{f}_1(s) = L(f_1(t)) \quad \text{and} \quad \bar{f}_2(s) = L(f_2(t))$$

Since $L(\cos at) = \dfrac{s}{s^2+a^2}$ and $L(\cos bt) = \dfrac{s}{s^2+b^2}$, we have

$$L^{-1}\left[\frac{s}{s^2+a^2}\times\frac{s}{s^2+b^2}\right] = \int_0^t \cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}]\,du$$

$$= \left[\frac{1}{2(a-b)}\sin\{(a-b)u+bt\}+\frac{1}{2(a+b)}\sin\{(a+b)u-bt\}\right]_0^t$$

$$= \frac{1}{2(a-b)}(\sin at-\sin bt)+\frac{1}{2(a+b)}(\sin at+\sin bt)$$

$$= \frac{1}{2(a^2-b^2)}[2a\sin at-2b\sin bt]$$

or $$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at-b\sin bt}{a^2-b^2}.$$ **Ans.**

(*ii*) $$L^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+w^2}\times\frac{s}{s^2+w^2}\right] = \int_0^t \cos wu\cos w(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos wt+\cos w(t-2u)]\,du = \frac{1}{2}\left[u\cos wt-\frac{1}{2w}\sin w(t-2u)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos wt+\frac{1}{w}\sin wt\right]$$

Therefore $$L^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = \frac{t}{2}\cos wt+\frac{1}{2w}\sin wt.$$ **Ans.**

(*iii*) Applying the convolution theorem $L^{-1}[f(s)\, g(s) = \int_0^t f(u)\, g(t-u)\, du$ we get

$$L^{-1}\frac{1}{(s+1)(s+2)} = \int_0^t e^{-u}\, e^{-2(t-u)}du = \int_0^t e^{(u-2t)}du = \left[e^{u-2t}\right]_0^t = e^{-t} - e^{-2t}.$$

Next, $$L^{-1}\frac{1}{s}\left\{\frac{1}{(s+1)(s+2)}\right\} = \int_0^t (e^{-u} - e^{-2u})\, du$$

$$= \left[-e^{-u} + \frac{1}{2}e^{-2u}\right]_0^t = -e^{-t} + \frac{1}{2}e^{-2t} + 1 - \frac{1}{2}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$ **Ans.**

CHAPTER 16

Applications of Laplace Transform—Solving Differential Equations, Unit Step Function and Impulse Function

Laplace Transform of Unit Step Function, Periodic Function, Impulse Function, Solution of Ordinary Differential Equation by Laplace Transformation.

LAPLACE TRANSFORM OF A PERIODIC FUNCTION

[GGSIPU III Sem End Term 2010; End Term 2011; II Sem End Term 2013]

Let $f(t)$ be a periodic function with period T, that is, $f(t+T)=f(t)$, then by definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + \ \ldots$$

In I_2 putting $t = T + u$ so that $dt = du$, we get

$$I_2 = \int_T^{2T} e^{-st} f(t)\,dt = \int_0^T e^{-s(T+u)} f(T+u)\,du = e^{-sT}\int_0^T e^{-su} f(u)\,du \quad [\text{since } f(T+u)=f(u)]$$

$$= e^{-sT} I_1 \quad \text{where } I_1 = \int_0^T e^{-st} f(t)\,dt.$$

Similarly, in I_3 putting $t = 2T + u$ so that $dt = du$, we get

$$I_3 = \int_{2T}^{3T} e^{-st} f(t)\,dt = \int_0^T e^{-s(2T+u)} f(2T+u)\,du$$

$$= e^{-2sT}\int_0^T e^{-su} f(u)\,du \quad [\text{since } f(2T+u) = f(T+u) = f(u)]$$

$$= e^{-2sT} I_1.$$

Proceeding the same way, we get

$I_4 = e^{-3sT} I_1,\quad I_5 = e^{-4sT} I_1,$ and so on.

Therefore, $L\{f(t)\} = I_1 + e^{-sT} I_1 + e^{-2sT} I_1 + e^{-3sT} I_1 + \ldots .$

$= \dfrac{I_1}{1-e^{-sT}}$ as a sum of an infinite geometric series with common ratio less than 1.

Hence $$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

EXAMPLE 16.1. If $f(t) = t^2$ for $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find $L(f(t))$.
[GGSIPU III Sem End Term 2003]

SOLUTION: If $f(t)$ is a periodic function with period T, then

$$L(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$. Here $f(t) = t^2$ which is periodic with period 2, hence

$$L(f(t)) = \frac{\int_0^2 t^2 e^{-st}\,dt}{1-e^{-2s}} = \frac{1}{1-e^{-2s}}\left[\left\{t^2\frac{e^{-st}}{-s}\right\}_0^2 - \int_0^2 \frac{2te^{-st}}{-s}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} + \frac{2}{s}\left\{\frac{te^{-st}}{-s}\right\}_0^2 - \frac{2}{s}\int_0^2 \frac{1e^{-st}}{-s}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} - \frac{4}{s^2}e^{-2s} + \frac{2}{s^2}\left\{\frac{e^{-st}}{-s}\right\}_0^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} - \frac{4}{s^2}e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right]$$

$$= \frac{2e^{-2s}}{1-e^{2s}}\left[\frac{2}{s} + \frac{2}{s^2} + \frac{1}{s^3} - \frac{e^{2s}}{s^3}\right]$$

or
$$L(f(t)) = \frac{2e^{-2s}}{s^3(1-e^{2s})}\,[2s(1+s) + 1 - e^{2s}].$$ **Ans.**

EXAMPLE 16.2. (*i*) For the periodic function $f(t)$ of period 4, defined by

$$f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}, \text{ find } L(f(t)).$$

[GGSIPU III Sem End Term 2007]

(*ii*) Find the Laplace transform of $2c$ - periodic function

$$f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$$

[GGSIPU III Sem End Term 2006; End Term 2009, End Term 2005, End Term 2012]

SOLUTION: (*i*) Given $f(t+4) = f(t)$ and $f(t) = 3t, \quad 0 < t < 2$
$= 6, \quad 2 < t < 4.$

$\therefore$ For the periodic function $f(t)$ of period 4, we have

$$L(f(t)) = \frac{1}{1-e^{-4s}}\int_0^4 e^{-st} f(t)\,dt = \frac{1}{1-e^{-4s}}\left[\int_0^2 3te^{-st}\,dt + \int_2^4 6e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\left\{3\frac{t\,e^{-st}}{-s}\right\}_0^2 - \int_0^2 3\frac{e^{-st}dt}{-s} + 6\left\{\frac{e^{-st}}{-s}\right\}_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{-6}{s}e^{-2s} + \frac{3}{s}\left\{\frac{e^{-st}}{-s}\right\}_0^2 - \frac{6}{s}\{e^{-4s} - e^{-2s}\}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{-6}{s}e^{-2s} + \frac{3}{s^2}(1-e^{-2s}) - \frac{6}{s}e^{-4s} + \frac{6}{s}e^{-2s}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3}{s^2}(1-e^{-2s}) - \frac{6}{s}e^{-4s}\right] = \frac{3}{s^2(1+e^{-2s})} - \frac{6e^{-4s}}{s(1-e^{-4s})} \qquad \textbf{Ans.}$$

(*ii*) $f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$ which is $2c$ – periodic function

$$\therefore \quad Lf(t) = \frac{1}{1-e^{-2cs}}\int_0^{2c} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2cs}}\left[\int_0^c t\,e^{-st}dt + \int_c^{2c}(2c-t)\,e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[\left\{t\frac{e^{-st}}{-s}\right\}_0^c - \int_0^c 1\frac{e^{-st}dt}{-s} + \left\{(2c-t)\frac{e^{-st}}{-s}\right\}_c^{2c} + \int_c^{2c}\frac{e^{-st}}{-s}dt\right]$$

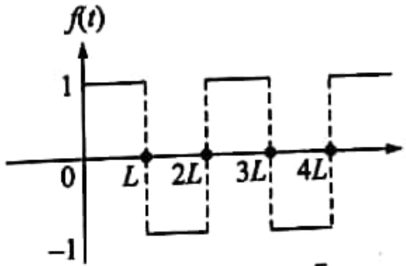
$$= \frac{1}{1-e^{-2cs}}\left[-\frac{c}{s}e^{-cs} - \frac{1}{s^2}(e^{-cs}-1) + \frac{c}{s}e^{-cs} + \frac{1}{s^2}(e^{-2cs} - e^{-cs})\right]$$

$$= \frac{1}{s^2(1-e^{-2cs})}\left[1 - 2e^{-cs} + e^{-2cs}\right] = \frac{1-e^{-cs}}{s^2(1+e^{-cs})} \qquad \textbf{Ans.}$$

EXAMPLE 16.3. Find the Laplace transform of a periodic function $f(t)$ given by

$$f(t) = \begin{cases} 1, & 0 < t < L \\ -1, & L < t < 2L \end{cases}.$$

SOLUTION: The function $f(t)$ is represents the square wave



$$\therefore \quad L\{f(t)\} = \frac{1}{1-e^{-2sL}}\int_0^{2L} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2sL}}\left[\int_0^L e^{-st}\cdot 1\,dt + \int_L^{2L} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2sL}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^L - \left[\frac{e^{-st}}{-s}\right]_L^{2L}\right\}$$

$$= \frac{1}{s(1-e^{-2sL})}\left[1 - 2e^{-sL} + e^{-2sL}\right] = \frac{(1-e^{-sL})^2}{s(1-e^{-sL})(1+e^{-sL})}$$

$$= \frac{1-e^{-sL}}{s(1+e^{-sL})} = \frac{1}{s}\frac{e^{sL/2} - e^{-sL/2}}{e^{sL/2} + e^{-sL/2}} = \frac{1}{s}\tanh\left(\frac{sL}{2}\right). \qquad \textbf{Ans.}$$

APPLICATIONS TO SOLVE DIFFERENTIAL EQUATIONS

In the methods discussed earlier for solving the differential equations we have been finding the complementary function, then particular integral and then evaluating the arbitrary constants with the help of given initial conditions. The Laplace transform method is much shorter specially for linear differential equations with constant coefficients. In this method, we first find the Laplace transform of both sides of differential equation. The result is a subsidiary algebraic equation in $\bar{y}(s)$ where $\bar{y}$ is the Laplace transform of $y(t)$. It is then solved for $\bar{y}$ and finally we apply the inverse transform which represents y in terms of t as solution of the differential equation.

We shall illustrate the use of Laplace transform technique to the following three types of differential equations :

(a) Linear differential equations with constant co-efficients.

(b) Linear differential equations with variable co-efficients.

(c) Simultaneous ordinary differential equations.

EXAMPLE 16.4. (a) Solve the intial value problem $y'' + ay' - 2a^2y = 0$, $y(0) = 6$, $y'(0) = 0$.

[GGSIPU II Sem End Term 2006]

(b) Solve $\dfrac{d^4x}{dt^4} - a^4x = 0$ where a is constant, using Laplace transform, given that $x = 1, x' = x'' = x''' = 0$ at $t = 0$.

SOLUTION: (a) Taking Laplace transform on both sides of $y'' + ay' - 2a^2y = 0$, we get

$$s^2\bar{y} - sy(0) - y'(0) + a[s\bar{y} - y(0)] - 2a^2\bar{y} = 0 \quad \text{where} \quad \bar{y} = L(y(x))$$

or $\quad (s^2 + as - 2a^2)\,\bar{y} = 6\,(s + a)$ since $y(0) = 6$ and $y'(0) = 0$, given.

or $$\bar{y} = \frac{6\,(s+a)}{(s-a)\,(s+2a)} = \frac{4}{s-a} + \frac{2}{s+2a}.$$

Taking inverse Laplace transform here, we get $\quad y(x) = 4e^{ax} + 2\,e^{-2ax}$ **Ans.**

(b) Taking Laplace transform on both sides of the given equation, we get

$$s^4\,\bar{x} - s^3\,x\,(0) - s^2\,x'\,(0) - sx''\,(0) - x'''\,(0) - a^4\,\bar{x} = 0 \quad \text{where} \quad \bar{x} = \text{L}\;\{x\,(t)\}$$

Using the given initial conditions, we get

$$s^4\,\bar{x} - s^3 - a^4\,\bar{x} = 0 \quad \text{or} \quad \bar{x} = \frac{s^3}{s^4 - a^4} = \frac{s^3}{(s^2 + a^2)\,(s-a)\,(s+a)}$$

Resolving into partial fractions, gives

$$\bar{x} = \frac{1}{4\,(s-a)} + \frac{1}{4\,(s+a)} + \frac{1}{2}\frac{s}{s^2 + a^2}$$

Taking inverse transform on both sides, gives

$$x\,(t) = \frac{1}{4}e^{at} + \frac{1}{4}e^{-at} + \frac{1}{2}\cos at \quad \text{or} \quad x\,(t) = \frac{1}{2}[\cosh at + \cos at]$$ **Ans.**

which is the required solution.

EXAMPLE 16.5. (*a*) Using Laplace transform, solve the equation

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0 \text{ under the conditions } y(0)=1,\ y'(0)=2,\ y''(0)=2.$$

[GGSIPU III Sem End Term 2004]

(*b*) Solve the differential equation using Laplace transform,

$$(D-1)(D-2)(D-3)x=5;\ x=0,\ \frac{dx}{dt}=1,\ \frac{d^2x}{dt^2}=0 \text{ at } t=0.$$

[GGSIPU II Sem End Term 2013]

SOLUTION: (*a*) Taking Laplace transform on both sides of the given equation, we get

$$s^3\bar{y}-s^2y(0)-sy'(0)-y''(0)+2[s^2\bar{y}-sy(0)-y'(0)]-[s\bar{y}-y(0)]-2\bar{y}=0$$

Since $y(0)=1,\ y'(0)=2, y''(0)=2,$ we get $\bar{y}(s^3+2s^2-s-2)-s^2-2s-2-2s-4+1=0$

or $$\bar{y}=\frac{s^2+4s+5}{s^3+2s^2-s-2}=\frac{s^2+4s+5}{(s-1)(s+1)(s+2)}=\frac{5}{3(s-1)}-\frac{1}{s+1}+\frac{1}{3(s+2)}$$

On taking inverse Laplace transform, we get $y=\frac{5}{3}e^{t}-e^{-t}+\frac{1}{3}e^{-2t}$

which is the required solution. **Ans.**

(*b*) The given equation is $(D^3-6D^2+11D-6)x=5$ where $D\equiv\frac{d}{dt}$.

Taking Laplace transform on both sides, gives

$$s^3\bar{x}-s^2x(0)-sx'(0)-x''(0)-6[s^2\bar{x}-sx(0)-x'(0)]+11[s\bar{x}-x(0)]-6\bar{x}=\frac{5}{s}$$

or $$(s^3-6s^2+11s-6)\bar{x}-s+6=\frac{5}{s}$$

$\therefore$ $$\bar{x}=\frac{\frac{5}{s}+s-6}{(s-1)(s-2)(s-3)}=\frac{s^2-6s+5}{s(s-1)(s-2)(s-3)}=\frac{s-5}{s(s-2)(s-3)}$$

$$=-\frac{5}{6s}+\frac{3}{2(s-2)}-\frac{2}{3(s-3)}$$

$\Rightarrow$ $$x(t)=-\frac{5}{6}+\frac{3}{2}e^{2t}-\frac{2}{3}e^{3t}.$$ **Ans.**

EXAMPLE 16.6. (*a*) Use Laplace transform to solve the equation

$$\frac{d^2y}{dx^2}+\frac{4dy}{dx}+8y=1 \text{ given that } y=0 \text{ and } \frac{dy}{dx}=1 \text{ at } x=0.$$

(*b*) Find the solution of the initial value problem $y''+4y'+4y=12t^2e^{-2t}$, $y(0)=2$ and $y'(0)=1$.

[GGSIPU II Sem End Term 2011]

SOLUTION: (*a*) Taking Laplace transform on both sides of the given equation, we get

$$s^2\bar{y}-sy(0)-y'(0)+4\{s\bar{y}-y(0)\}+8\bar{y}=\frac{1}{s} \quad \text{where } \bar{y}=\mathrm{L}(y(x);\ x\to s)$$

or $$\bar{y}(s^2+4s+8)=sy(0)+y'(0)+4y(0)+\frac{1}{s}=\frac{1}{s}+1 \quad \text{as } y(0)=0 \text{ and } y'(0)=1.$$

or $$\bar{y}=\frac{s+1}{s(s^2+4s+8)}=\frac{1}{8s}+\frac{-\frac{1}{8}s+\frac{1}{2}}{s^2+4s+8} \quad \text{(on resolving into partial fractions)}$$

$\therefore$ $$y=\frac{1}{8}\mathrm{L}^{-1}\left(\frac{1}{s}\right)-\frac{1}{8}\mathrm{L}^{-1}\frac{s-4}{s^2+4s+8}=\frac{1}{8}-\frac{1}{8}\mathrm{L}^{-1}\frac{s+2}{(s+2)^2+2^2}+\frac{3}{4}\mathrm{L}^{-1}\frac{1}{(s+2)^2+2^2}$$

$$=\frac{1}{8}-\frac{1}{8}e^{-2x}\cos 2x+\frac{3}{8}e^{-2x}\sin 2x\ \frac{1}{8}-\frac{e^{-2x}}{8}(\cos 2x-3\sin 2x).$$ **Ans.**

(*b*) Taking Laplace transform on both sides of the given eequation, we get

$$s^2\bar{y}-sy(0)-y'(0)+4[s\bar{y}-y(0)]+4\bar{y}=12L(t^2e^{-2t})=12\cdot\frac{2}{(s+2)^3}.$$

or $$(s^2+4s+4)\bar{y}-2s-1-8=\frac{24}{(s+2)^3} \quad \text{or} \quad (s+2)^2\bar{y}=2s+9+\frac{24}{(s+2)^3}$$

or $$\bar{y}=\frac{2s+4+5}{(s+2)^2}+\frac{24}{(s+2)^5}=\frac{2}{s+2}+\frac{5}{(s+2)^2}+\frac{24}{(s+2)^5}$$

$\Rightarrow$ $$y(t)=L^{-1}\left[\frac{2}{s+2}+\frac{5}{(s+2)^2}+\frac{24}{(s+2)^5}\right]=e^{-2t}[2+5t+24^2t^4]$$ **Ans.**

EXAMPLE 16.7. (*a*) Find the general solution of the equation $\frac{d^2x}{dt^2}+9x=\cos 2t$.

[GGSIPU III Sem End Term 2009]

(*b*) Solve the following differential equation, using Laplace transform

$$\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t,\ y(0)=1.$$

[GGSIPU II Sem II Term 2014]

SOLUTION: (*a*) Let the initial conditions be $x(0)=c_1$ and $x'(0)=c_2$.
Taking L′ transform on both sides of the given equation, we get

$$s^2\bar{x}-sx(0)-x'(0)+9\bar{x}=\frac{s}{s^2+2^2} \quad \text{where } \bar{x}=\mathrm{L}\{x(t)\}.$$

or $$(s^2+9)\bar{x}=c_1s+c_2+\frac{s}{s^2+4}$$

or $$\bar{x} = c_1 \frac{s}{s^2+9} + \frac{c_2}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$= c_1 \frac{s}{s^2+9} + \frac{c_2}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} \text{ (on resolving into partial fractions)}$$

Therefore $$x(t) = \left(c_1 - \frac{1}{5}\right) L^{-1}\left(\frac{s}{s^2+9}\right) + c_2 L^{-1}\left(\frac{1}{s^2+9}\right) + \frac{1}{5} L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \left(c_1 - \frac{1}{5}\right)\cos 3t + \frac{c_2}{3}\sin 3t + \frac{1}{5}\cos 2t = c'_1 \cos 3t + c'_2 \sin 3t + \frac{1}{5}\cos 2t$$

which is the required general solution where c'_1 and c'_2 are arbitrary constants. **Ans.**

(*b*) Using Laplace transform on the given differential equation, we get

$$s\bar{y} - y(0) + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1} \quad \text{where } \bar{y} = L(y(t);\, t \to s).$$

or $$\left(s + 2 + \frac{1}{s}\right)\bar{y} = 1 + \frac{1}{s^2+1} \quad \text{as } y(0) = 1$$

or $$\bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{1}{s+1} - \frac{3}{2(s+1)^2} + \frac{1}{2(s^2+1)} \text{ (on solving into partial fractions)}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{1}{s+1} - \frac{3}{2(s+1)^2} + \frac{1}{2(s^2+1)}\right] = e^{-t} - \frac{3t}{2}e^{-t} + \frac{1}{2}\sin t \quad \textbf{Ans.}$$

EXAMPLE 16.8. (a) A particle moves in a line so that its displacement x from a fixed point O at any time t, is given by $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t$.

If initially particle is at rest at $x = 0$, find its displacement at any time t.

(b) Use Laplace transform technique to solve the following initial value problem

$$y'' + \lambda^2 y = \cos \lambda t \quad \text{when } y(0) = 1,\; y\left(\frac{\pi}{2\lambda}\right) = 1.$$

[*GGSIPU II Sem End Term 2012*]

SOLUTION: (a) Taking Laplace transform on both sides of the given equation, we obtain

$$s^2\bar{x} - sx(0) - x'(0) + 4[s\bar{x} - x(0)] + 5\bar{x} = \frac{80(5)}{s^2+(5)^2} \quad \text{where } \bar{x} = L\{x(t)\}.$$

From the given initial conditions, we have $x(0) = x'(0) = 0$, hence

$$\bar{x}(s^2+4s+5) = \frac{400}{s^2+25} \quad \text{or} \quad \bar{x} = \frac{400}{(s^2+25)(s^2+4s+5)}$$

Resolving into partial fractions, we get $\bar{x} = \frac{-2s-10}{s^2+25} + \frac{2s+18}{s^2+4s+5}$

Now taking the inverse Laplace transform on both sides, we get

$$x(t) = -2\cos 5t - \frac{10}{5}\sin 5t + 2L^{-1}\left(\frac{s+2+7}{(s+2)^2+1}\right)$$

$$= -2\cos 5t - 2\sin 5t + 2e^{-2t}(\cos t + 7\sin t).$$

Therefore, the distance x, covered at any time t, is given by

$$x = -2(\cos 5t + \sin 5t) + 2e^{-2t}(\cos t + 7\sin t).$$ **Ans.**

(b) Taking L′ transform on both sides of $y'' + \lambda^2 y = \cos \lambda t$, we get

$$s^2\bar{y} - sy(0) - y'(0) + \lambda^2\bar{y} = \frac{s}{s^2+\lambda^2} \quad \text{where } \bar{y} = L(y(t), t \to s).$$

Using $y(0) = 1$ and letting $y'(0) = K$ here, we get

$$(s^2+\lambda^2)\bar{y} = \frac{s}{s^2+\lambda^2} + s + K \quad \text{or} \quad \bar{y} = \frac{s}{(s^2+\lambda^2)^2} + \frac{s+K}{s^2+\lambda^2}.$$

$$\therefore \quad y(t) = L^{-1}\frac{s}{(s^2+\lambda^2)^2} + L^{-1}\frac{s+K}{s^2+\lambda^2} = \frac{t}{2\lambda}\sin\lambda t + \cos\lambda t + \frac{K}{\lambda}\sin\lambda t$$

(since $L(\sin\lambda t) = \dfrac{\lambda}{s^2+\lambda^2}$, $L(\cos\lambda t) = \dfrac{s}{s^2+\lambda^2}$ and $L(t\sin\lambda t) = \dfrac{2\lambda s}{(s^2+\lambda^2)}$.)

Putting $t = \dfrac{\pi}{2\lambda}$, the above relation becomes

$$y\left(\frac{\pi}{2\lambda}\right) = \frac{\pi}{4\lambda^2}\sin\left(\frac{\lambda\pi}{2\lambda}\right) + \cos\left(\lambda\frac{\pi}{2\lambda}\right) + \frac{K}{\lambda}\sin\left(\frac{\lambda\pi}{2\lambda}\right) = 1,$$

or $$\frac{K}{\lambda} + \frac{\pi}{4\lambda^2} = 1 \quad \therefore \; K = \lambda\left(1 - \frac{\pi}{4\lambda^2}\right).$$

Thus, the solution is $y(t) = \dfrac{t}{2\lambda}\sin\lambda t + \cos\lambda t + \left(1 - \dfrac{\pi}{4\lambda^2}\right)\sin\lambda t.$ **Ans.**

EXAMPLE 16.9. Solve, using Laplace transform technique, the differential equation

(i) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x + e^{3x}$ where $y(0) = 1$, $y'(0) = -1$

[GGSIPU II Sem End Term 2007]

(ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$.

[GGSIPU II Sem End Term 2006 Reappear; End Term 2005]

SOLUTION: (i) Taking Laplace transform on both sides of the given equation, we get

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}, \quad \text{we get}$$

$$[s^2\bar{y} - sy(0) - y'(0)] - 3[s\bar{y} - y(0)] + 2\bar{y} = \frac{4}{s^2} + \frac{1}{s-3} \quad \text{where } \bar{y} = L(y(x)).$$

Since $y(0) = 1$ and $y'(0) = -1$, we have

$$(s^2 - 3s + 2)\bar{y} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

or $$(s^2 - 3s + 2)\bar{y} = \frac{4}{s^2} + \frac{1}{s-3} + s - 4 = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)}$$

or $$\bar{y} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \frac{2}{s^2} + \frac{3}{s} - \frac{1}{2(s-1)} - \frac{2}{s-2} + \frac{1}{2(s-3)}$$

(On resolving into partial fractions)

Taking inverse Laplace transform on both sides, we get

$$y = 2x + 3 - \frac{1}{2}e^x - 2e^{2x} + \frac{1}{2}e^{3x} \qquad \textbf{Ans.}$$

(*ii*) Taking Laplace transform on both sides of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, we get

$$[s^2\,\bar{y} - sy(0) - y'(0)] + 2[s\,\bar{y} - y(0)] + 5\,\bar{y} = \frac{1}{(s+1)^2+1} \quad \text{where } \bar{y} = L(y\,(t)).$$

or $$\bar{y}\,(s^2 + 2s + 5) = \frac{1}{s^2+2s+2} + 1, \quad \text{since} \quad y(0) = 0, \quad y'(0) = 1.$$

or $$\bar{y} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$$

$$= \frac{1}{3[(s+1)^2+1]} + \frac{2}{3[(s+1)^2+4]}$$

Taking inverse Laplace transform on both sides, we get the solution as

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\sin 2t = \frac{e^{-t}}{3}(\sin t + 2\sin 2t) \qquad \textbf{Ans.}$$

EXAMPLE 16.10. Solve the following differential equation with variable coefficients

$$t\frac{d^2x}{dt^2} - (t+2)\frac{dx}{dt} + 3x = t - 1, \text{ given that } x\,(0) = 0 \text{ and } x\,(2) = 9.$$

SOLUTION: Taking Laplace transform on both sides of the given equation, we get

$$L\,\{tx''\} - L\,(tx') - 2L\,(x') + 3L\,(x) = L\,(t-1) \qquad ...(1)$$

Here $L\,(x) = \bar{x}$, $\therefore$ $L(x') = s\bar{x} - x(0)$, and $L\,(t\,x') = -\frac{d}{ds}L\,(x') = -\frac{d}{ds}[s\bar{x} - x(0)]$

and $L\,(t\,x'') = -\frac{d}{ds}L\,(x'') = -\frac{d}{ds}[s^2\,\bar{x} - s\,x(0) - x'\,(0)]$

Using the initial condition $x(0) = 0$ and taking $x'\,(0) = k$, (1) becomes

$$-\frac{d}{ds}[s^2\,\bar{x}\,(s) - 0 - k] + \frac{d}{ds}[s\,\bar{x} - 0] - 2\,[s\,\bar{x} - 0] + 3\bar{x} = \frac{1}{s^2} - \frac{1}{s}$$

or $$-s^2\frac{d}{ds}\bar{x} - 2s\bar{x} + s\frac{d}{ds}\bar{x} + \bar{x} - 2s\bar{x} + 3\bar{x} = \frac{1}{s^2} - \frac{1}{s}$$

or $$(-s^2 + s)\frac{d}{ds}\bar{x} + 4\,(1-s)\,\bar{x} = \frac{1-s}{s^2} \qquad \text{or} \qquad \frac{d}{ds}\bar{x} + \frac{4}{s}\,\bar{x} = \frac{1}{s^3}$$

which is linear differential equation of first order in $\bar{x}$.

Integrating factor $= e^{\int \frac{4}{s} ds} = e^{4 \log s} = s^4$

$\therefore$ Solution is $\quad s^4 \bar{x} = \int \frac{1}{s^3} s^4 \, ds = \frac{s^2}{2} + c \quad$ or $\quad \bar{x} = \frac{1}{2s^2} + \frac{c}{s^4}$ where c is an arbitrary co...

Taking inverse transform on both sides, gives $x(t) = \frac{t}{2} + \frac{ct^3}{3!}$

Using the given condition $x(2) = 9$, we get $\quad c = 6$

$\therefore$ The required solution is $\qquad x(t) = \frac{t}{2} + t^3. \qquad$ **Ans.**

EXAMPLE 16.11. Solve the simultaneous equations

$$\frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t},$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0, \quad \text{given that} \quad x(0) = -1, \quad y(0) = 1.$$

SOLUTION: Taking Laplace transform on both sides of the given equations, we get

$$s\bar{x} - x(0) + s\bar{y} - y(0) + \bar{x} = -\frac{1}{s+1}$$

and $\qquad s\bar{x} - x(0) + 2[s\bar{y} - y(0)] + 2\bar{x} + 2\bar{y} = 0$

Using here $\quad x(0) = -1 \quad$ and $\quad y(0) = 1, \quad$ we get

$$(s+1)\,\bar{x} + s\bar{y} = -\frac{1}{s+1} \qquad \text{and} \qquad (s+2)\,\bar{x} + 2(s+1)\bar{y} = 1$$

Solving these simultaneous equations for $\bar{x}$ and $\bar{y}$, gives

$$\bar{x} = -\frac{s+2}{s^2+2s+2} \qquad \text{and} \qquad \bar{y} = \frac{s^2+3s+3}{(s^2+2s+2)(s+1)}.$$

or $\qquad \bar{x} = -\left[\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right] \qquad$ and $\qquad \bar{y} = \frac{1}{s+1} + \frac{1}{(s+1)^2+1}$

Taking inverse Laplace transform on the above relations, we get

$$x(t) = -e^{-t}(\cos t + \sin t) \qquad \text{and} \qquad y(t) = e^{-t}(1 + \sin t).$$ **Ans.**

EXAMPLE 16.12. Solve the following simultaneous equations

$$\frac{dx}{dt} + x + 3\int_0^t y\, dt = \cos t + 3\sin t \quad \text{and} \quad 2\frac{dx}{dt} + 3\frac{dy}{dt} + 6y = 0,$$

subject to the conditions $x = -3$, $y = 2$ at $t = 0$.

SOLUTION: Taking Laplace transform on both sides of the above equations, we get

$$s\bar{x} - x(0) + \bar{x} + 3 \cdot \frac{1}{s}\bar{y} = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

and $\qquad 2[s\bar{x} - x(0)] + 3[s\bar{y} - y(0)] + 6\bar{y} = 0 \quad$ where $\bar{x} = L(x(t)), \ \bar{y} = L(y(t))$.

or $\qquad (s+1)\,\bar{x} + \frac{3}{s}\,\bar{y} = \frac{s+3}{s^2+1} - 3 \qquad$ and $\qquad 2s\,\bar{x} + (3s+6)\bar{y} = 0$

Solving these simultaneous equations system for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = -\frac{(s+2)(3s-1)}{(s+3)(s^2+1)} \quad \text{and} \quad \bar{y} = -\frac{s(3s-1)}{(s+3)(s^2+1)}$$

Resolving into partial fractions, gives

$$\bar{x} = -\frac{1}{s+3} + \frac{1-2s}{s^2+1} \quad \text{and} \quad \bar{y} = \frac{2}{s+3} - \frac{2}{3}\cdot\frac{1}{s^2+1}$$

Now taking Laplace inverse transformation, we get

$$x(t) = -e^{-3t} + \sin t - 2\cos t, \quad y(t) = 2e^{-3t} - \frac{2}{3}\sin t.$$ **Ans.**

EXAMPLE 16.13. Solve the simultaneous equations

$$(D^2-3)x - 4y = 0, \quad x + (D^2+1)y = 0 \quad \text{where } D \equiv \frac{d}{dt} \quad \text{and} \quad t > 0,$$

$$\text{given that } x = y = \frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} = 2 \text{ at } t = 0.$$

[GGSIPU II Sem End Term 2006 Reappear]

SOLUTION: Taking Laplace transform on both sides of the given equations, we get

$$s^2\bar{x} - sx(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0 \quad \text{and} \quad \bar{x} + s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 0$$

where $\bar{x} = L(x(t))$, $\bar{y} = L(y(t))$.

Using $x(0) = y(0) = y'(0) = 0$ and $x'(0) = 2$, we get

$$(s^2-3)\,\bar{x} - 4\bar{y} = 2 \quad \text{and} \quad \bar{x} + (s^2+1)\bar{y} = 0$$

Eliminating $\bar{x}$ in in the above relations, we get

$$[(s^2+1)(s^2-3)+4]\,\bar{y} = -2$$

or
$$\bar{y} = \frac{-2}{s^4-2s^2+1} = \frac{-2}{(s-1)^2(s+1)^2}$$

$$= \frac{-1}{2(s-1)^2} - \frac{1}{2(s+1)^2} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

Taking inverse Laplace transform, gives

$$y(t) = \frac{-1}{2}e^{t}t - \frac{1}{2}e^{-t}t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \sinh t - t\cosh t$$

Putting this value of y in the given equation $x + (D^2+1)y = 0$, we get

$$\begin{aligned} x &= -y - D^2(\sinh t - t\cosh t) \\ &= -\sinh t + t\cosh t - \sinh t + D(t\sinh t + \cosh t) \\ &= -2\sinh t + t\cosh t + (t\cosh t + \sinh t + \sinh t) \\ &= 2t\cosh t \end{aligned}$$

∴ The soloution is $x = 2t\cosh t$, $y = \sinh t - t\cosh t$. **Ans.**

LAPLACE TRANSFORM OF SOME SPECIAL FUNCTIONS

Sometimes we are to find the solution of a differential equation of a physical system which is acted upon

(i) by a periodic force or a periodic voltage,

(ii) by an impulsive force or a voltage acting instantaneously at a certain time or a concentrated load acting at that time,

(iii) by a force acting on a part of the system or a voltage acting for a finite interval of time.

Let us study such functions and their transforms.

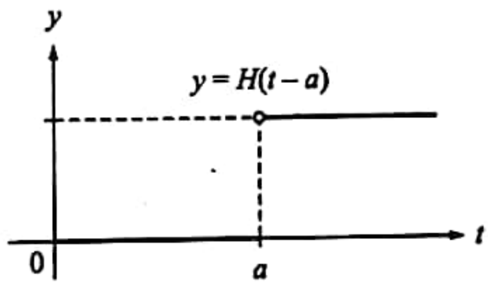
HEAVISIDE UNIT STEP FUNCTION

[GGSIPU III Sem End Term 2012; End Term 2013]

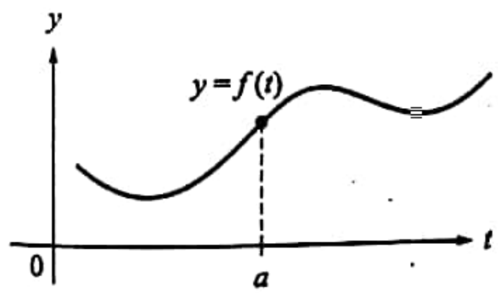
An important discontinuous function that finds very many applications in connection with Laplace transform and elsewhere is the **Unit Step Function** $H(t-a)$ or $U(t-a)$ with $a \geq 0$, also known as **Heaviside Step Function.**

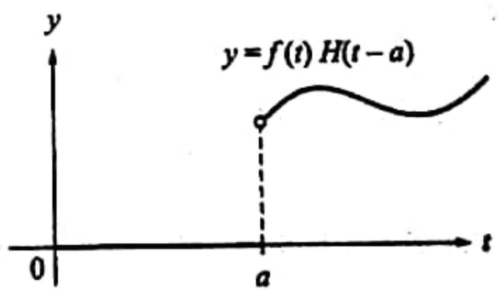
It is defined as $\quad H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq 0 \end{cases} \quad (a \geq 0)$ also written as $U(t-a)$

and is depicted in the adjoining figure.



The effect on $f(t)$ of multiplication by $H(t-a)$ is shown in the following figure.

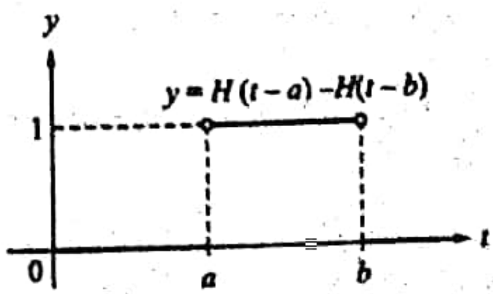




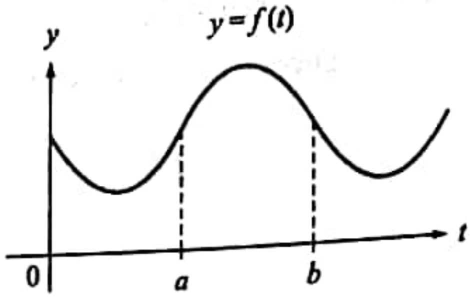
UNIT PULSE FUNCTION

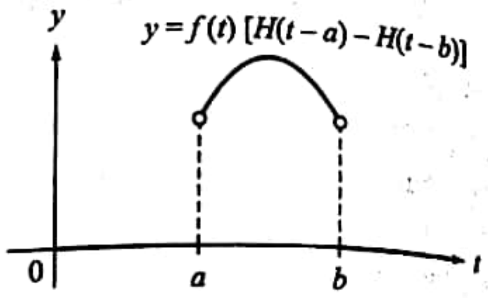
The unit pulse function $y = H(t-a) - H(t-b)$ is defined as

$$H(t-a) - H(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$$



The effect on $f(t)$ of multiplication by $H(t-a) - H(t-b)$ is shown in the following figure.





Laplace Transform of Unit Step Function and Unit Pulse Function

$$L[H(t-a)] = \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \quad \text{for } s > a \ge 0.$$

and $$L[H(t-a) - H(t-b)] = \int_a^b e^{-st}\,dt = \int_a^\infty e^{-st}\,dt - \int_b^\infty e^{-st}\,dt = \frac{1}{s}[e^{-as} - e^{-bs}].$$

SECOND SHIFTING THEOREM

Let $L(f(t)) = \bar{f}(s)$, then we have $L(f(t-a)\,H(t-a)) = e^{-as}\bar{f}(s)$.

Proof: By definition

$$L(f(t-a)\,H(t-a) = \int_0^\infty e^{-st} f(t-a)\,H(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

because $H(t-a) = 0$ for $t < a$ and $H(t-a) = 1$ for $t \ge a$.

Now, putting $x = t - a$ we get

$$L(f(t-a)\,H(t-a)) = \int_0^\infty e^{-s(a+x)} f(x)\,dx = e^{-as}\int_0^\infty e^{-sx} f(x)\,dx = e^{-as}\bar{f}(s)$$

If we take $f(t) = 1$ then $f(t-a) = 1$ and we get $L(H(t-a)) = e^{-as}\,L(1) = \frac{1}{s}e^{-as}$.

For example, let us find $L^{-1}\left(\frac{s\,e^{-3s}}{s^2+4}\right)$. We know that $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$.

Using second shifting theorem here, we get

$$L^{-1}\left(\frac{s\,e^{-3t}}{s^2+4}\right) = \cos 2\,(t-3)\,H(t-3).$$

UNIT IMPULSE FUNCTION (OR DIRAC-DELTA FUNCTION)

The concept of a very large force acting for a very short time frequently comes into being in mechanics. The unit impulse function is the limiting form of the function

$$\delta(t-a) = \frac{1}{\epsilon}, \quad a \le t \le a+\epsilon \qquad \text{[GGSIPU III Sem End Term 2012]}$$

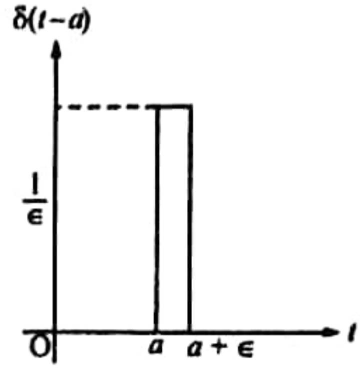
$$= 0 \quad \text{otherwise, where} \quad \epsilon \to 0.$$

From the adjoing figure it is clear that smaller the ϵ, more the height of strip because the area of the strip is always unity.

The above fact can be notionally stated as

$$\delta(t-a) = \infty \quad \text{for} \quad t = a$$
$$= 0 \quad \text{for} \quad t \ne a$$

such that $\int_0^\infty \delta(t-a)\,dt = 1$ for $a \ge 0$.



Filtering Property of Dirac-delta function:

Let $f(t)$ be continuous and integrable in $(0, \infty)$, then

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Proof: By the definition of Dirac-delta function, we have

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt$$

Using here the mean value theorem of integration, we get

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt = \frac{1}{\epsilon} f(t_0)\int_a^{a+\epsilon} dt = f(t_0), \text{ where } a < t_0 < a+\epsilon.$$

Therefore $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(t_0)$. Taking limit as $\epsilon \to 0$, we get

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a) \text{ as } f(t_0) \to f(a).$$

Laplace Transform of the Dirac-delta function is given by

$$L\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^{a+\epsilon} e^{-st}\,\frac{1}{\epsilon}\,dt + \int_\epsilon^\infty e^{-st}\,0\,dt$$

$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,dt = \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon} = \frac{1}{s\epsilon}\left[-e^{-s(a+\epsilon)} + e^{-as}\right] = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\epsilon}}{\epsilon}\right]$$

Taking limit as $\epsilon \to 0$, we have

$$L\{\delta(t-a)\} = \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\frac{1-e^{-s\epsilon}}{\epsilon} = \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[1-\left\{1-s\epsilon+\frac{s^2\epsilon^2}{2!}-\dots\right\}\right]$$

$$= \frac{e^{-as}}{s} \lim_{\epsilon \to 0} \left[s - \frac{s\epsilon^2}{2!} + \frac{s^2 \epsilon^3}{3!} - \dots \right]$$

$$= \frac{e^{-as}}{s} \cdot s = e^{-as}$$

Thus, $L\{\delta(t-a)\} = e^{-as}$. [GGSIPU II Sem End Term 2006]

In particular when $a = 0$ we have $L\{\delta(t)\} = 1$.

LAPLACE TRANSFORM OF $f(t)\,\delta(t-a)$

$$L\{f(t)\delta(t-a)\} = \int_0^\infty e^{-st} f(t)\,\delta(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t)\,0\,dt + \int_a^{a+\epsilon} e^{-st} f(t) \frac{1}{\epsilon}\,dt + \int_{a+\epsilon}^\infty e^{-st} f(t)\,0\,dt$$

$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st} f(t)\,dt = \frac{f(a)}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,dt \text{ \{assuming } f(a+\epsilon)=f(a)\}$$

$$= \frac{f(a)}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon} = f(a)\left(\frac{e^{-as} - e^{-as-s\epsilon}}{s\epsilon}\right) = f(a)\,e^{-as}\left[\frac{1-e^{-s\epsilon}}{s\epsilon}\right]$$

$$= f(a)\,e^{-as} \cdot 1. \quad \text{(On taking limit as } \epsilon \to 0)$$

Thus, $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$.

EXAMPLE 16.14. (a) Find the Laplace transform of the function

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

[GGSIPU II Sem End Term 2010]

(b) Express the function $f(t) = 2t$ for $0 < t < 5$
$= 10$ for $t > 5$
in terms of unit step function and find its Laplace transform.

[GGSIPU III Sem End Term 2011]

SOLUTION: (a) $f(t) = (t-1)[U(t-1) - U(t-2)] + (3-t)[U(t-2) - U(t-3)]$

$$= (t-1)\,U(t-1) - 2(t-2)\,U(t-2) + (t-3)\,U(t-3)$$

$$\therefore \quad L\{f(t)\} = L[(t-1)\,U(t-1)] - 2L[(t-2)\,U(t-2)] + L[(t-3)\,U(t-3)]$$

$$= e^{-s} L(t) - 2e^{-2s} L(t) + e^{-3s} L(t)$$

$$= (e^{-s} - 2e^{-2s} + e^{-3s})\frac{1}{s^2} = \frac{e^{-s}}{s^2}[1 - 2e^{-s} + e^{-2s}] = \frac{e^{-s}}{s^2}(1-e^{-s})^2. \quad \textbf{Ans.}$$

(b) $f(t) = 2t\{U(t) - U(t-5)\} + 10\,U(t-5)$

$$= 2t\,U(t) - 2(t-5)\,U(t-5)$$

Taking L′ transform and using second shifting theorem, we have

$$L(f(t)) = 2L\{t\,U(t)\} - 2L\{(t-5)\,U(t-5)\} = 2\frac{e^{-0s}}{s^2} - 2\frac{e^{-5s}}{s^2} = \frac{2}{s^2}(1-e^{-5s}). \quad \textbf{Ans.}$$

EXAMPLE 16.15. Find the Laplace transform of

(i) $t^2 \cup (t-3)$ [GGSIPU II Sem End Term 2014]

(ii) $e^{-3t} \cup (t-2)$.

[GGSIPU II Sem End Term 2005; III Sem End Term 2007; III Sem End Term 2003

SOLUTION: (i) We can write $t^2 \cup (t-3) = \{(t-3)^2 + 6(t-3) + 9\} \cup (t-3)$

We know that if $L(f(t)) = \bar{f}(s)$ then $L[f(t-a) \cup (t-a)] = e^{-as}\, \bar{f}(s)$.

$$\therefore \quad L[t^2 \cup (t-3) = L(t-3)^2 \cup (t-3) + 6\,L(t-3) \cup (t-3) + 9\,L[\,U\,(t-3)]$$

$$= e^{-3s}\frac{2!}{s^3} + 6\,e^{-3s}\frac{1}{s^2} + 9\frac{e^{-3s}}{s} = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Therefore the Laplace transform of $t^2 \cup (t-3) = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$ **Ans.**

(ii) We can write $e^{-3t} \cup (t-2) = e^{-3(t-2)} \times e^{-6} \cup (t-2)$

Since $L[f(t-a) \cup (t-a)] = e^{-as}\bar{f}(s)$

$$\therefore \quad L[e^{-3t} \cup (t-2)] = e^{-6}\,L[e^{-3(t-2)} \cup (t-2)] = e^{-6} \times e^{-2s}\frac{1}{s+3}$$

Hence $$L[e^{-3t} \cup (t-2)] = \frac{e^{-2(s+3)}}{s+3}.$$ **Ans.**

EXAMPLE 16.16. Find the inverse Laplace transform of (i) $\frac{e^{-\pi s}}{s^2+4}$. (ii) $\frac{(3s+1)}{s^2(s^2+4)}e^{-3s}$

[GGSIPU III Sem End Term 2009]

SOLUTION: (i) We know that $L^{-1}\frac{1}{s^2+4} = \frac{1}{2}\sin 2t$

therefore $$L^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] = \frac{1}{2}\sin 2(t-\pi)\,U(t-\pi)$$

$$= \begin{cases} 0, & \text{for } 0 < t < \pi \\ \frac{1}{2}\sin 2t, & \text{for } t > \pi \end{cases}$$ **Ans.**

(ii) Let $$\bar{f}(s) = \frac{3s+1}{s^2(s^2+4)} = \frac{(3s+1)}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right] = \frac{3}{4s} + \frac{1}{4s^2} - \frac{3s}{4(s^2+4)} - \frac{1}{4(s^2+4)}$$

then $$f(t) = L^{-1}(\bar{f}(s)) = \frac{3}{4} + \frac{t}{4} - \frac{3}{4}\cos 2t - \frac{1}{8}\sin 2t.$$

Now since $L\{f(t-a) \cup (t-a)\} = e^{-as}\bar{f}(s)$, we have

$$L^{-1}\left[\frac{(3s+1)e^{-3s}}{s^2(s^2+4)}\right] = \left[\frac{3+t-3}{4} - \frac{3}{4}\cos 2(t-3) - \frac{1}{8}\sin 2(t-3)\right]\cup(t-3)$$

$$= \frac{1}{8}[2t - \sin 2(t-3) - 6\cos 2(t-3)] \cup (t-3).$$ **Ans.**

EXAMPLE 16.17. Solve $\dfrac{d^2y}{dx^2}+4y = \cup(x-2)$ where $\cup$ is the unit step function and $y(0)=0$ and $y'(0)=1$.

[GGSIPU III Sem End Term 2006; End Term 2012]

SOLUTION: Taking Laplace transform on both sides of $y''+4y=\cup(x-2)$, we get

$$s^2\bar{y}-sy(0)-y'(0)+4\bar{y} = \frac{e^{-2s}}{s} \quad \text{or} \quad \bar{y}(s^2+4)=1+\frac{e^{-2s}}{s} \quad (\text{as } y(0)=0 \text{ and } y'(0)=1).$$

or
$$\bar{y} = \frac{1}{s^2+4}+\frac{e^{-2s}}{s(s^2+4)}$$

Therefore
$$y = L^{-1}\left(\frac{1}{s^2+4}\right)+L^{-1}\left[\frac{e^{-2s}}{s(s^2+4)}\right] = \frac{1}{2}\sin 2t + L^{-1}\left(\frac{e^{-2s}}{s(s^2+4)}\right).$$

and
$$L^{-1}\frac{1}{s(s^2+4)} = \int_0^t \frac{1}{2}\sin 2t\, dt = \left[\frac{-1}{4}\cos 2t\right]_0^t = \frac{1}{4}(1-\cos 2t) = \frac{1}{2}\sin^2 t.$$

Since $L[f(t-a)\cup(t-a)] = \bar{f}(s)e^{-as}$, we have

$$L^{-1}\left[\frac{e^{-2s}}{s(s^2+4)}\right] = \frac{1}{2}\sin^2(t-2)\cup(t-2)$$

Thus
$$y = \frac{1}{2}\sin 2t+\frac{1}{2}\sin^2(t-2)\cup(t-2)$$

which is the required solution. **Ans.**

EXAMPLE 16.18. Solve the initial value problem

$$y''+3y'+2y = H(t-\pi)\sin 2t, \quad y(0)=1, \quad y'(0)=0.$$

SOLUTION: Taking L'transform on both sides of the given equation, we get

$$s^2\bar{y}-sy(0)-y'(0)+3[s\bar{y}-y(0)]+2\bar{y} = L[H(t-\pi)\sin 2t]$$

or
$$s^2\bar{y}-s-0+3[s\bar{y}-1]+2\bar{y} = \frac{2e^{-\pi s}}{s^2+4} \quad \text{or} \quad (s^2+3s+2)\bar{y} = s+3+\frac{2e^{-\pi s}}{s^2+4}$$

or
$$\bar{y} = \frac{s+3}{(s+1)(s+2)}+\frac{2e^{-\pi s}}{(s+1)(s+2)(s^2+4)}$$

$$= \frac{2}{s+1}-\frac{1}{s+2}+e^{-\pi s}\left[\frac{2}{5}\left(\frac{1}{s+1}\right)-\frac{1}{4}\left(\frac{1}{s+2}\right)-\frac{1}{20}\left(\frac{2}{s^2+4}\right)-\frac{3}{20}\left(\frac{s}{s+4}\right)\right]$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = 2e^{-t}-e^{-2t}+H(t-\pi)\left[\frac{2}{5}e^{-(t-\pi)}-\frac{1}{4}e^{-2(t-\pi)}-\frac{1}{20}\sin 2(t-\pi)-\frac{3}{20}\cos 2(t-\pi)\right]$$ **Ans.**

EXAMPLE 16.19. Find $L\,[t\,U\,(t-4) - t^3\,\delta\,(t-2)]$

SOLUTION: $L\,[t \cup (t-4) = L[(t-4)\cup(t-4) + 4\cup(t-4)] = e^{-4s}L(t) + 4e^{-4s}L(1)$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

Next, for $t^3\,\delta\,(t-2)$, we consider $f(t) = t^3$ so $f(2) = 8$,

$\therefore$ $L\,\{t^3\,\delta\,(t-2)\} = f(2)\,e^{-2s} = 8e^{-2s}$

Therefore, $L\,[t\,U\,(t-4) - t^3\,\delta\,(t-2)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - 8e^{-2s}$. **Ans.**

CHAPTER

17 Complex Functions, Complex Number System, Differentiation and Analyticity, Conformal and Bilinear Mapping

Prerequisite for functions of complex variable, Complex Functions—Limit, Continuity, Differentiation and Analyticity, Cauchy-Riemann Equations, Conformal Mapping and Bilinear Mappings.

Prerequisite for Functions of Complex Variable

The theory of the functions of complex variables, particularly that of analytic functions, helps in solving many complex problems in heat conduction, fluid dynamics and electrostatics.

COMPLEX NUMBER SYSTEM — INTRODUCTION

$\sqrt{-1}$ is an imaginary quantity, denoted by i (pronounced as iota) so that $i^2 = -1$. Hence for any positive integer n, we have

$$i^{2n} = (i^2)^n = (-1)^n = \begin{cases} 1 & \text{when } n \text{ is even} \\ -1 & \text{when } n \text{ is odd} \end{cases} \quad \text{and} \quad i^{2n+1} = (i^2)^n\, i = \begin{cases} i & \text{when } n \text{ is even} \\ -i & \text{when } n \text{ is odd} \end{cases}$$

For a complex number $z = x + iy$ where x and y are real numbers, called real part and imaginary part of z respectively, we write $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

Argand Diagram

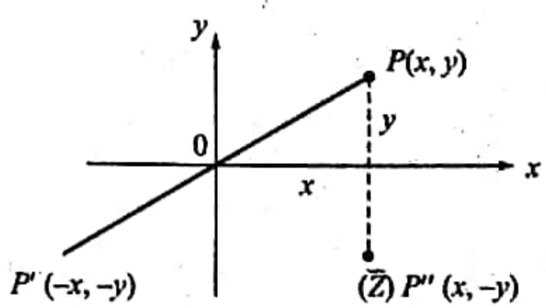
The complex number z can be represented by a point $P(x, y)$ with complex plane also called Argand plane. The complex number $-z$ is represented by a point $P'(-x, -y)$.

Modulus of z is denoted by $|z|$ and is defined as $|z| = \sqrt{x^2 + y^2}$.

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be **equal** if and only if $x_1 = x_2$ and $y_1 = y_2$.

Conjugate of the complex number z is denoted by $\bar{z}$ and is given by $\bar{z} = x - iy$.

Actually $\bar{z}$ is reflection of P in the X-axis.



Obviously, $\bar{\bar{z}} = z$, $|\bar{z}| = |z|$ and $z_1 = z_2 \leftrightarrow \bar{z}_1 = \bar{z}_2$

Algebra of Complex Numbers

Addition: $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

Multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Division: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\dfrac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

It clearly follows that $z\bar{z} = x^2 + y^2$, $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

$|z_1 z_2| = |z_1||z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

The distance between two complex numbers z_1 and z_2 is defined as

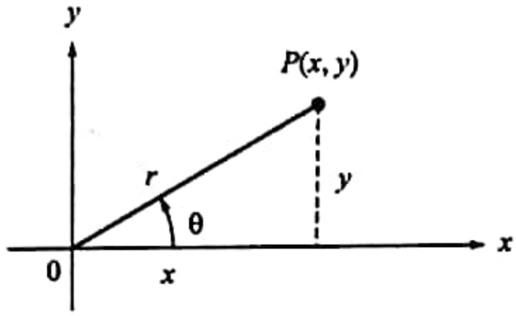
$$|z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Polar Form of a Complex Number

Let $z(= x + iy)$ be a complex number represented by a point $P(x, y)$ in the complex plane. Let $OP = r$ and $\angle POX = \theta$, then

$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore$ $z = r(\cos\theta + i\sin\theta)$ which is called **polar form** or **triangular form** of the complex number z.



Here $r = |z|$ = **Modulus or magnitude** of z, and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ = angle made by OP with the positive direction of X-axis and is called the **argument or amplitude** of z, denoted by arg (z).

Actually, arg (z) has infinitely many values since θ can be replaced by $\theta + 2n\pi$ where $n \in I$. Here, the value of θ lying in $-\pi < \theta \le \pi$ is called the **principal value** of θ and is denoted by Arg (z).

Thus, $\arg(z) = \text{Arg}(z) + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

Further, if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_1 + i\sin\theta_2) \\ &= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \end{aligned}$$

$\Rightarrow$ $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and $\arg(z_1 z_2) = \theta_1 + \theta_2 + 2n\pi$, $n \in I$.

Also, $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \quad (z_2 \neq 0)$$

$$= \frac{r_1[(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)]}{r_2[\cos^2\theta_2 - \sin^2\theta_2]}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$\Rightarrow$ $$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 + 2n\pi, \; n \in I$$

POWERS AND ROOTS OF A COMPLEX NUMBER.

DE MOIVRE'S THEOREM:

Let $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$

where n is any integer, positive or negative or a fraction.

Also, we have $|z^n| = r^n = |z|^n$ and $z^{-n} = r^{-n}(\cos n\theta - i\sin n\theta) = \dfrac{1}{z^n}$.

n^{th} root of z: n^{th} root of any number will be n in number, hence there will be n n^{th} root of z. To obtain them we proceed as follows.

$$z^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n} = r^{1/n}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)^{1/n},\ k \in I$$

$$= r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right],\ k = 0, 1, 2, \ldots, n-1.$$

This way we get n^{th} root of z (n in number).

Cube Roots of Unity:

Let $z^3 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi,\ k = 0, 1, 2, \ldots$

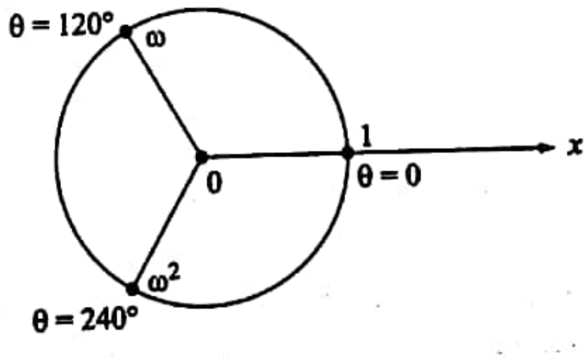
then $z = 1^{1/3} = (\cos 2k\pi + i\sin 2k\pi)^{1/3} = \cos\dfrac{2k\pi}{3} + i\sin\dfrac{2k\pi}{3},\ k = 0, 1, 2.$

Cube roots of unity are writte as $1, \omega, \omega^2$ where

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+i\sqrt{3}}{2} \quad \text{and} \quad \omega^2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \frac{-1-i\sqrt{3}}{2}.$$

Clearly $1 + \omega + \omega^2 = 0$.

As shown in the figure, the roots lie on the unit circle and divide the circumference of this circle into three equal parts.



n^{th} roots of unity:

If $z^n = 1$ then $z = 1^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$= \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n},\ k = 0, 1, 2, \ldots, n-1.$

These can be represented as $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$.

Here $\omega^n = 1$ and $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = \dfrac{1-\omega^n}{1-\omega} = 0.$

As in the case of cube roots of unity, all the n, n^{th} roots of unity lie on the unit circle and form the vertices of a regular polygon of n sides embedded in the unit circle. The roots divide the circumference of the unit circle into n equal parts.

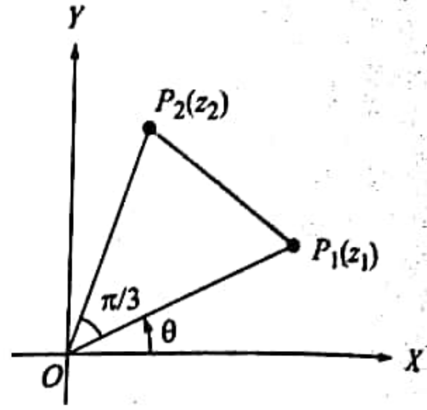
Triangular Identities:

For any two complex numbers z_1 and z_2 we have

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ (ii) $|z_1 - z_2| \ge |z_1| - |z_2|$.

EXAMPLE 17.1. Show that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 = z_1 z_2$.

SOLUTION: Let $OP_1 P_2$ be an equilateral triangle as shown in the figure.



Let $z_1 = r(\cos\theta + i\sin\theta)$, then

$$z_2 = r\left[\cos\left(\theta+\frac{\pi}{3}\right)+i\sin\left(\theta+\frac{\pi}{3}\right)\right]$$

Hence $$\frac{z_2}{z_1} = \frac{\cos\left(\theta+\frac{\pi}{3}\right)+i\sin\left(\theta+\frac{\pi}{3}\right)}{\cos\theta+i\sin\theta} = \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

and $$\frac{z_1}{z_2} = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{-1} = \cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$$

$$\Rightarrow \frac{z_1}{z_2}+\frac{z_2}{z_1} = 2\cos\frac{\pi}{3} = 1 \quad \text{or} \quad z_1^2+z_2^2 = z_1 z_2$$

Euler's Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) \quad \text{and} \quad \sin\theta = \frac{1}{2i}(\cos\theta - i\sin\theta).$$

Hyperbolic Functions: $\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$ and $\sin\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$.

Thus, we can write $\cos i\theta = \cosh\theta$ and $\sin i\theta = r\sin h\theta$.

Thus the complex circular fucntions can be split into real and imaginary parts as follows.

$\sin z = \sin(x+iy) = \sin x \cos hy + i\cos x \sin y$ and

$\cos z = \cos(x+iy) = \cos x \cos hy - i\sin x \sinh y$.

Similarly, for $\tan z$, $\cot z$, $\sec z$ and $\operatorname{cosec} z$.

Let $\omega = \sin^{-1} z$ when $z = \sin\omega$

The **complex inverse circular functions and complex inverse hyperbolic functions** can also be split into real and imaginary parts as follows.

then $z = \dfrac{e^{i\omega}-e^{-i\omega}}{2i}$ or $e^{2i\omega} - 2iz\,e^{i\omega} - 1 = 0$

Solving for $e^{i\omega}$, we get $e^{i\omega} = \frac{1}{2}\left[2iz \pm \sqrt{-4z^2+4}\right] = iz \pm \sqrt{1-z^2}$.

Taking $e^{i\omega} = iz+\sqrt{1-z^2}$ we have $i\omega = \log(iz+\sqrt{1-z^2})$

Thus, $\omega = \sin^{-1} z = -i\log(iz+\sqrt{1-z^2})$.

Similarly, other complex trigometric function can be defined as

$$\cos^{-1} z = -i\log(z+\sqrt{z^2-1})$$

$$\tan^{-1} z = -\frac{i}{2}\log\frac{1+iz}{1-iz} = \frac{i}{2}\log\frac{i+z}{i-z}, \quad z \neq \pm i$$

$$\cot^{-1} z = \tan^{-1}\frac{1}{z} = -\frac{i}{2}\log\left(\frac{z+i}{z-i}\right), \quad z \neq \pm i \text{ etc.}$$

Now we define complex inverse hyperbolic sine function as

$$\omega = \sinh^{-1} z \quad \text{when} \quad z = \sinh \omega$$

We can write $\sinh \omega = \dfrac{e^{\omega} - e^{-\omega}}{2} = z$ or $e^{2\omega} - 2z\, e^{\omega} - 1 = 0$

$$\Rightarrow \qquad e^{\omega} = \frac{1}{2}(2z \pm \sqrt{4z^2+4}) = z \pm \sqrt{z^2+1}$$

Let us take $e^{\omega} = z + \sqrt{z^2+1}$ or $\omega = \log(z+\sqrt{z^2+1})$

Therefore $\qquad \sinh^{-1} z = \log(z+\sqrt{z^2+1})$.

Similarly other complex inverse hyperbolic functions are defined as

$$\cosh^{-1} z = \log(z+\sqrt{z^2-1})$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \quad z \neq \pm 1.$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right), \quad z \neq \pm 1 \text{ etc.}$$

For example, let us split $\tan^{-1} z$ into real and imaginary parts.

Let $\qquad z = x + iy$ and $\tan^{-1} z = u + iv$ then

$$u + iv = \tan^{-1}(x+iy) \quad \text{and} \quad u - iv = \tan^{-1}(x-iy)$$

$$\Rightarrow \qquad 2u = \tan^{-1}(x+iy) + \tan^{-1}(x-iy) = \tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

and

$$2iv = \tan^{-1}(x+iy) - \tan^{-1}(x-iy) = \tan^{-1}\frac{2iy}{1+x^2+y^2} = i\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

Hence $\tan^{-1} z = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2x}{1-x^2-y^2}\right) + \dfrac{i}{2}\tanh^{-1}\left(\dfrac{2y}{1+x^2+y^2}\right)$.

As another illustration, let us find all values of $\sin^{-1} 2$ treating 2 as a complex number.

We know that $\sin^{-1} z = -i\log[iz+\sqrt{1-z^2}]$

$$\therefore \qquad \sin^{-1} 2 = -i\log[2i+\sqrt{-3}] = -i\log[i(2+\sqrt{3})] = -i[\log(2+\sqrt{3}) + \log e^{i(\pi/2+2n\pi)}]$$

$$= -i\left[\log(2+\sqrt{3}) + i(\pi/2+2n\pi\right], \; n = 0, \pm 1, \pm 2, \ldots$$

$$= \left(\frac{\pi}{2}+2n\pi\right) - i\log(2+\sqrt{3}) = \frac{\pi}{2}+2n\pi - i\cosh(2), \; n \in I.$$

The Concept of a Function of Complex Variable

If for each value of a complex variable $z\ (= x + iy)$ there corresponds one or more values of another complex variable $w\ (= u + iv)$ then w is said to be a function of z. In this section, the term function will mean a single valued function, *i.e.*, for one value of z we shall have unique value of w. This is written as $w = f(z)$, e.g., $w = z^2$, $w = \frac{1}{z}$, $w = e^z$, are all single valued while $w = z^{1/2}$ is not single valued, rather it is multiple valued.

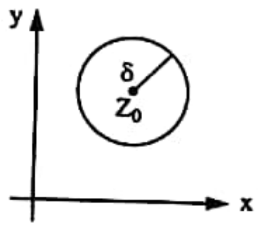
Limit of a Function $f(z)$

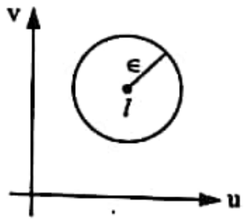
A function $w = f(z)$ is said to tend to a limit l as z tends to a point z_0 if, for a given small positive number $\in$, we can find a positive number δ such that

$$|f(z) - l| < \in \quad \text{for} \quad |z - z_0| < \delta.$$

Symbolically, we write $\underset{z \to z_0}{\text{Lt.}} f(z) = l.$

If no such number l exists, we say that $f(z)$ has no limit as $z \to z_0$. When the limit exists, then for every z in the δ-neighbourhood of z_0, the value of $f(z)$ lies in the $\in$-neighbourhood of l as shown in the adjoining figure.





Remember that if $\lim\limits_{z \to z_0} f(z)$ exists then it is unique. Since $z \to z_0$ from any direction along any straight line or along a curve the limit must be the same. If we obtain two different limits as $z \to z_0$ along two different paths, we conclude that the limit does not exist.

Remark 1. Let $f(z) = u(x, y) + i\, v(x, y)$, $z = x + iy$, $z_0 = x_0 + iy_0$, and $l = u_0 + i\, v_0$ then $\lim\limits_{z \to z_0} f(z) = l$ if and only if $\lim\limits_{x \to x_0,\, y \to y_0} u(x, y) = u_0$ and $\lim\limits_{x \to x_0,\, y \to y_0} v(x, y) = v_0$.

Remark 2. If $f(z)$ has a finite limit at $z = z_0$ then $f(z)$ is a bounded function in some neighbourhood of z_0.

Limit of function at $z = \infty$

The function $f(z)$ has a limit l as $z \to \infty$, if for any arbitrary small positive number $\in$, there exists a real number $\delta > 0$ such that $|f(z) - l| < \in$ whenever $|z| > \frac{1}{\delta}$.

In other words $\lim\limits_{z \to \infty} f(z) = \lim\limits_{z \to 0} f\left(\frac{1}{z}\right)$.

For example, $\lim\limits_{z \to \infty}\left(\frac{1}{z^2}\right) = 0$, $\lim\limits_{z \to -i}(z^2 - z) = i - 1$ and $\lim\limits_{z \to 2i}(3x + iy^2) = 4i$.

Also, $\lim_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right] = \lim_{z\to\infty}\dfrac{\left[\sqrt{z-2i}-\sqrt{z-i}\right]\left[\sqrt{z-2i}+\sqrt{z-i}\right]}{\left[\sqrt{z-2i}+\sqrt{z-i}\right]}$

$$= \lim_{z\to\infty}\frac{-i}{\sqrt{z-2i}+\sqrt{z-i}} = \lim_{z\to 0}\frac{-i}{\sqrt{\frac{1}{z}-2i}+\sqrt{\frac{1}{z}-i}} = \lim_{z\to 0}\frac{-i\sqrt{z}}{\sqrt{1-2iz}+\sqrt{1-iz}} = 0.$$

And $\lim_{z\to 0}\dfrac{z}{|z|}$ does not exist, since $\lim_{z\to 0}\dfrac{z}{|z|} = \lim_{x\to 0}\left[\lim_{y\to 0}\dfrac{x+iy}{\sqrt{x^2+y^2}}\right] = \lim_{x\to 0}\dfrac{x}{\sqrt{x^2}} = \pm 1$

and other way round $\cdot \lim_{z\to 0}\dfrac{z}{|z|} = \lim_{y\to 0}\left[\lim_{x\to 0}\dfrac{x+iy}{\sqrt{x^2+y^2}}\right] = \lim_{y\to 0}\dfrac{iy}{\sqrt{y^2}} = \pm i$.

Next, consider $\lim_{z\to 1} f(z) = \lim_{z\to 1}\dfrac{z^2-1}{z-1}$. We shall show that this limit has the value 2. Here $z_0 = 1$.

For this, we have to find a number $\delta > 0$, for a given small positive number $\in$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad \left|\frac{z^2-1}{z-1}-2\right| = |z-1|, \quad z \neq 1.$$

Thus, $|f(z) - 2| < \in$ whenever $0 < |z - 1| < \delta$. It implies that $\delta = \in$ in this case.
Also, it is quite easy to establish the following properties on limits.

If $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} f(z) = B$, then

(*i*) $\lim_{z\to z_0}[f(z) \pm g(z)] = A \pm B$

(*ii*) $\lim_{z\to z_0}[f(z)\, g(z)] = AB$

(*iii*) $\lim_{z\to z_0}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{A}{B}$, provided $B \neq 0$.

Continuity of Function

A complex function $f(z)$ is said to be continuous at a point $z = z_0$ if, for a given small positive number $\in$, we can find a positive number δ such that

$$|f(z) - f(z_0)| < \in \text{ whenever } |z - z_0| < \delta.$$

Thus, a function $f(z)$ is continuous at a point $z = z_0$, if the following three conditions are satisfied:

(*i*) $f(z_0)$ exists (*ii*) $\lim_{z\to z_0} f(z)$ exists (*iii*) $\lim_{z\to z_0} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a domain D if it is continuous at every point in D. Obviously, a function $f(z)$ which is not continuous at z_0 is said to be **discontinuous** at z_0.

However, if $f(z_0)$ exists and $\lim_{z\to z_0} f(z) = l$ but $f(z_0) \neq l$, then the point z_0 is called a point of **removable discontinuity**. In this case we **rediflme** the function $f(z)$ at $z = z_0$ such that $f(z_0) = l$ and the function can be made continuous at z_0.

Note the following:

1. If $f(z)$ and $g(z)$ are continuous at a point z_0, then $f(z) \pm g(z)$ and, $f(z)\,g(z)$ are continuous. Also, $\dfrac{f(z)}{g(z)}$ is continuous there, provided $g(z_0) \neq 0$.
2. If $f(z)$ is continuous in a closed domain D, then it is bounded in D, *i.e.*, $|f(z)| \le M$ for all z in D for some positive constant M.
3. The function $f(z)$ is continuous at $z = \infty$ if the function $f(1/z)$ is continuous at $z = 0$.
4. If the function $f(z)$ is continuous at $z = z_0$ then the function $\overline{f(z)}$ is also continuous at $z = z_0$.

It is easy to show that the functions, for instance, e^z, $\sin z$, $\cos z$ are all continuous for all z. Similarly the function z^n is continuous at all points in the finite complex plane for any positive integer n.

However the functions $f(z) = \begin{cases} \dfrac{I_m(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ and $f(z) = \begin{cases} \dfrac{\mathrm{Re}(z^2)}{|z|^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

are examples of discontinuous functions at $z = 0$. Also, the function $\log_e z$ is not continuous on the negative real axis including the point $z = 0$.

Further, if the function $w = f(z) = u + iv$ is continuous at $z = z_0$ then both $u(x, y)$ and $v(x, y)$ are continuous at $z_0 = (x_0, y_0)$ and conversely, if $u(x, y)$ and $v(x, y)$ both are continuous at (x_0, y_0) then $f(z)$ is continuous at $z = z_0$.

EXAMPLE 17.2. If $f(z) = \dfrac{(x+y)^2}{x^2+y^2}$ then show that $\lim\limits_{x \to 0}\left[\lim\limits_{y \to 0} f(z)\right] = 1$ and $\lim\limits_{y \to 0}\left[\lim\limits_{x \to 0} f(z)\right] = 1$ but $\lim\limits_{z \to 0} f(z)$ does not exist.

SOLUTION: Here $\lim\limits_{x \to 0}\left[\lim\limits_{y \to 0} f(z)\right] = \lim\limits_{x \to 0}\left[\lim\limits_{y \to 0} \dfrac{(x+y)^2}{x^2+y^2}\right] = \lim\limits_{x \to 0} \dfrac{x^2}{x^2} = 1$

and $\lim\limits_{y \to 0}\left[\lim\limits_{x \to 0} f(z)\right] = \lim\limits_{y \to 0}\left[\lim\limits_{x \to 0} \dfrac{(x+y)^2}{x^2+y^2}\right] = \lim\limits_{y \to 0} \dfrac{y^2}{y^2} = 1$

Thus, along two perpendicular paths, first one from some point $z = z_1$ vertically to the x-axis and then along the x-axis to $z_0 = 0$ and other one from $z = z_1$ horizontally to the y-axis and then along y-axis to $z_0 = 0$, the limits of $f(z)$ are the same. However, for $\lim\limits_{z \to 0} f(z)$ to exist, it is necessary that $f(z)$ approach the same limit along all paths leading to the origin and this is not the case here. In fact, along the paths

$y = mx$, we have $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} \dfrac{(x+y)^2}{x^2+y^2} = \lim\limits_{x \to 0} \dfrac{x^2(1+m)^2}{x^2(1+m^2)} = \dfrac{(1+m)^2}{1+m^2}$.

The limiting value here obviously dependents on m, that is, $f(z)$ approaches to different values along different lines and hence no limit exists.

EXAMPLE 17.3. Discuss the continuity of the function $f(z)$ at $z = 0$ where

$$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i\,\frac{x^3 + y^3}{x^2 + y^2}, \qquad z \neq 0$$
$$= 0, \qquad z = 0$$

SOLUTION: Here $f(z) = u + iv$ where $u(x, y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ and $v(x, y) = \dfrac{x^3 + y^3}{x^2 + y^2}$.

Now
$$\lim_{\substack{x \to 0 \\ y \to 0}} u(x, y) = \lim_{y \to 0} \frac{0 - y^3}{0 + y^2} = \lim_{y \to 0} (-y) = 0$$

and
$$\lim_{\substack{y \to 0 \\ x \to 0}} u(x, y) = \lim_{x \to 0} \frac{x^3 - 0}{x^2 + 0} = \lim_{x \to 0} x = 0$$

Similarly,
$$\lim_{\substack{x \to 0 \\ y \to 0}} v(x, y) = 0 = \lim_{\substack{y \to 0 \\ x \to 0}} v(x, y)$$

Hence $u(x, y)$ and $v(x, y)$ are continuous at $(0, 0)$. It follows that $f(z)$ is continuous at $z = 0$.

DIFFERENTIABILITY OF COMPLEX FUNCTION

The derivative of a function of a complex variable is defined precisely in the same way as in the real variable. But since the graph of a complex function cannot be drawn, the reader should note that the derivative is no longer the gradient (or slope) of a curve.

The derivative of $w = f(z)$ at z_0, is given by

$$w' = f'(z_0) = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Substituting $z - z_0 = \delta z$ in the above relation, we can write

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}.$$

From the definition of limit it follows that if $f'(z)$ is the derivative of $f(z)$ at z in a domain D, then for a given real small positive number ϵ, there exists a real positive number δ, such that

$$\left| \frac{f(z + \delta z) - f(z)}{\delta z} - f'(z) \right| < \epsilon \quad \text{whenever} \quad |\delta z| < \delta.$$

If we write $w = f(z)$ and $w + \delta w = f(z + \delta z)$, then $\dfrac{\delta w}{\delta z} = \dfrac{f(z + \delta z) - f(z)}{\delta z}$

and we can write $f'(z) = \lim\limits_{\delta z \to 0} \dfrac{\delta w}{\delta z} = \dfrac{dw}{dz}$.

Also, obviously, if $f(z)$ is differentiable at $z = z_0$ then it must be continuous at $z = z_0$.

Next, for illustration $f(z) = |z|^2$ is differentiable only at $z = 0$ and nowhere else, while $f(z) = z^n$, where n is a positive integer, is differentiable at every point in the finite complex plane. In general, any polynomial $P_n(z)$ in z is also differentiable at every point in the complex plane.

The formulae for differentiation of combinations of functions of a real variable have identical counterparts in the field of complex variables. Some familiar formulae are

$$\frac{d}{dz}(w_1 \pm w_2) = \frac{dw_1}{dz} \pm \frac{dw_2}{dz},$$

$$\frac{d}{dz}(w_1 w_2) = w_1 \frac{dw_2}{dz} + w_2 \frac{dw_1}{dz},$$

$$\frac{d}{dz}\left(\frac{w_1}{w_2}\right) = \frac{w_2\left(\frac{dw_1}{dz}\right) - w_1\left(\frac{dw_2}{dz}\right)}{w_2^2}, \qquad (w_2 \neq 0),$$

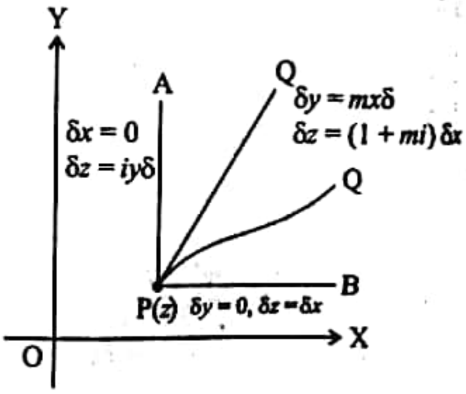
$$\frac{d}{dz}(w^n) = nw^{n-1}\frac{dw}{dz}$$

are valid when w_1, w_2 and w are differentiable functions of z.

Also, for composition of two functions f and g, we have

$$\frac{d}{dz}(f(g(z))) = \frac{df}{dg} \cdot \frac{dg}{dz}$$

The familiar formulae for differentiating exponential, trigonometric, and logarithmic functions of a real variable hold equally well for functions of complex variable. However, $\delta z = \delta x + i\delta y$ is itself a complex variable, and the difficulties involved in how δz approaches zero have no counter part in the differentiation of functions of real variable.



As is clear from the adjoining figure, δz can tend to zero in infinitely many ways, *i.e.*, a point Q $(z + \delta z)$ can tend to point P (z) along infinitely many paths. In particular, Q can approach P along the line AP on which δx is zero or along the line BP on which δy is zero. Clearly, for the derivative of $f(z)$ to exist, it is essential that $\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ must be the same no matter how δz approaches zero. The above fact will be illustrated in the following example.

EXAMPLE 17.4. (a) Show that $f(z) = \bar{z}$ has no derivative.

(b) Test the differentiability of the function

$$f(z) = \frac{(\bar{z})^2}{z} \text{ for } z \neq 0 \text{ and } f(0) = 0.$$

SOLUTION: (a) We have $f(z) = \bar{z} = x - iy$ hence .

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{x + \delta x - i(y + \delta y) - (x - iy)}{\delta x + i\delta y} = \frac{\delta x - i\delta y}{\delta x + i\delta y}$$

Now, if δz is real then $\delta y = 0$ and we have

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1.$$

And if δz is imaginary then $\delta x = 0$, and we have

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \lim_{\delta y \to 0} \frac{-i\delta y}{i\delta y} = -1.$$

More generally, if we let δz approach zero in such a way that $\delta y = m\delta x$, then

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \lim_{\delta x \to 0} \frac{\delta x - im\delta x}{\delta x + im\delta x} = \frac{1-im}{1+im} = \frac{1-m^2-2im}{1+m^2}$$

which implies that there are infinitely many values of the above limit depending upon the choice of m. Thus, it is clear that $\bar{z}$ $(= x - iy)$ has no derivative.

Actually there are many functions of z that do have derivative and in applications, only those functions are important which have derivatives. Our next task is to obtain conditions for the existence of the derivative of function of a complex variable.

(b) $$f(z) = u + iv = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}, \qquad z \neq 0$$

and $f(0) = 0$. Thus, $u = \dfrac{x^3-3xy^2}{x^2+y^2}$ and $v = \dfrac{y^3-3x^2y}{x^2+y^2}$

Hence $$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x \to 0} \frac{\dfrac{x^3-0}{x^2+0}}{x} = 1.$$

and $$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = 0, \quad \left(\frac{\partial v}{\partial x}\right)_{(0,0)} = 0 \quad \text{and} \quad \left(\frac{\partial v}{\partial y}\right)_{(0,0)} = 1.$$

For testing the differentiability of $f(z)$ at $z=0$, let us first take z approaching 0 along the x-axis, then

$$\lim_{(x,0)\to(0,0)} \frac{f(x+0i)-f(0)}{x+0i-0} = \lim_{x\to 0} \frac{x^2/x-0}{x-0} = 1$$

But if we let z approach 0 along the line $y = x$, then

$$\lim_{(x,x)\to(0,0)} \frac{f(x+ix)-f(0)}{x+ix-0} = \lim_{x\to 0} \frac{\dfrac{x^3-3x^3}{2x^2} + i\left(\dfrac{x^3-3x^3}{2x^2}\right)}{x(1+i)}$$

$$= \lim_{x\to 0} \frac{-x(1+i)}{x(1+i)} = -1$$

The two limits are not equal we conclude that $f(z)$ is not differentiable at the origin. **Ans.**

ANALYTIC FUNCTIONS

If $w = f(z)$ possesses a derivative at $z = z_0$ and at every point in some neighbourhood of z_0, then $f(z)$ is said to be *analytic at* z_0 and z_0 is called a *regular point* of $f(z)$.

If $f(z)$ is not analytic at z_0 and every neighbourhood of z_0 contains points at which $f(z)$ is analytic then z_0 is called a ***singular point*** of $f(z)$. A function which is analytic at every point of a region R is called ***analytic in R*** or ***holomorphic in R*** or ***regular in R***. Further, if a function is analytic at every point in the entire finite plane, it is said to be an ***entire function***. For example, a polynomial is an entire function since its derivative exists everywhere.

Following results follow immediately

(a) *If two functions are analytic in D, their sum and their product both are analytic in D.*

(b) *The quotient of two analytic functions is also analytic provided the function in the denominator does not vanish at any point in D.*

The reader should specifically note the difference between differentiability and analyticity.

CAUCHY-RIEMANN EQUATIONS

Let $f(z) = u + iv$ be a function of z defined in a domain D. The French mathematician Cauchy and the German mathematician Riemann discovered a pair of equations which u and v should satisfy at a point $z_0\ (x_0, y_0)$ in order that $f'(z)$ exists at $z = z_0$. These equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

PROOF: Since $f'(z_0)$ exists, we have $f'(z_0) = \lim\limits_{\delta z \to 0} \dfrac{f(z_0 + \delta z) - f(z_0)}{\delta z}$

Let $\quad f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$ and $\delta z = \delta x + i\delta y$, then

$$\frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \frac{u(x_0 + \delta x, y_0 + \delta y) - u(x_0, y_0) + i[v(x_0 + \delta x, y_0 + \delta y) - v(x_0, y_0)]}{\delta x + i\delta y}$$

$$\therefore \quad f'(z_0) = \lim_{(\delta x, \delta y) \to (0,0)} \frac{u(x_0 + \delta x, y_0 + \delta y) - u(x_0, y_0) + i[v(x_0 + \delta x, y_0 + \delta y) - v(x_0, y_0)]}{\delta x + i\delta y}$$

This limit must exist as $(\delta x, \delta y) \to (0, 0)$ in any manner we choose.

First of all we assume that δz is wholly real so $\delta y = 0$ and $\delta z = \delta x$ which gives

$$f'(z_0) = \lim_{\delta x \to 0} \frac{u(x_0 + \delta x, y_0) - u(x_0, y_0) + i[v(x_0 + \delta x, y_0) - v(x_0, y_0)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x_0 + \delta x, y_0) - u(x_0, y_0)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x_0 + \delta x, y_0) - v(x_0, y_0)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Next, we assume that δz is purely imaginary, so $\delta x = 0$ and $\delta z = i\delta y$ which gives

$$f'(z_0) = \lim_{\delta y \to 0} \frac{u(x_0, y_0 + \delta y) - u(x_0, y_0) + i[v(x_0, y_0 + \delta y) - v(x_0, y_0)]}{i\delta y}$$

$$= \frac{1}{i} \lim_{\delta y \to 0} \frac{u(x_0, y_0 + \delta y) - u(x_0, y_0)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x_0, y_0 + \delta y) - v(x_0, y_0)}{\delta y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Since $f'(z_0)$ is unique, the above two values of $f'(z_0)$ must be equal. Equating the real and imaginary parts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are known as Cauchy-Riemann equations which hold good for differentiable functions. Thus, **if the given function is differentiable (or analytic) it must satisfy the Cauchy-Riemann equations.**

But the converse is not necessarily true, i.e., the function may satisfy Cauchy-Riemann (C.R.) equations still may not be differentiable. Thus, ***the C.R. equations are only necessary conditions for the differentiability (or analyticity) of a complex function.***

One can notice an important fact that the Cauchy Riemann equations have arisen here from a consideration of only two of the infinitely many ways in which δz can approach zero. It is therefore, natural to expect additional conditions necessary to ensure that along the other paths also $\frac{\delta w}{\delta z}$ will approach the same limit $\frac{dw}{dz}$.

Sufficient conditions for a function to be analytic: Let $f(z) = u(x, y) + iv(x, y)$. If u and v are continuous and have continuous partial derivatives of first order in a domain D and also u and v satisfy the Cauchy-Riemann equations at all points in D then the function $f(z)$ is analytic in D and

$$f'(z) = u_x + i\, v_x = v_y - i\, u_y$$

EXAMPLE 17.5. A function $f(z)$ is defined as

$$f(z) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\,\dfrac{x^3 + y^3}{x^2 + y^2} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

Test the continuity and differentiability of $f(z)$ at the origin.

[GGSIPU II Sem End Term 2014]

SOLUTION: Since $f(z) = u + iv$, we have $u(x, y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ and $v(x, y) = \dfrac{x^3 + y^3}{x^2 + y^2}$.

To test the continuity of $f(z)$ at $z = 0$ we use the polar coordinates for convenience. Then

$$u = r(\cos^3\theta - \sin^3\theta) \quad \text{and} \quad v = r(\cos^3\theta + \sin^3\theta).$$

Clearly, u and v tend to zero as $r \to 0$ irrespective of the values of θ. Since $u(0, 0) = 0 = v(0, 0)$ it follows that $f(z)$ is continuous at $(0, 0)$. Thus $f(z)$ is continuous for all values of z.

Next, $$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1,$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y - 0} = \lim_{y \to 0} \frac{-y}{y} = -1,$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1,$$

and $$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y - 0} = \lim_{y \to 0} \frac{y}{y} = 1,$$

this shows that Cauchy-Riemann equations, are satisfied at the origin.

Finally, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ \text{along OY}}} \frac{0 - y^3 + i(0 + y^3)}{(0 + y^2)(0 + iy)} = \lim_{y \to 0} \frac{(i-1)y^3}{iy^3} = 1 + i$$

Now, let $z \to 0$ along with line $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{(x^2 + x^2)(x + ix)} = \lim_{x \to 0} \frac{2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{1+i}{2}$$

Since $f'(0)$ is not unique along two paths it follows that $f'(z)$ does not exist at $z = 0$. **Ans.**

EXAMPLE 17.6. (a) Let f be an analytic function in the domain D. If $|f(z)| = \lambda$ where λ is constant, then show that f is constant in D.

(b) Show that the function $\ln z$ is analytic for all z except when $\text{Re } z \le 0$.

[GGSIPU II Sem End Term 2013]

SOLUTION: (a) Suppose that $\lambda = 0$. Then $|f(z)|^2 = 0$, and hence $u^2 + v^2 = 0$ which implies that both $u = 0$ and $v = 0$ and therefore $f(z) = 0$ in D.

Next, suppose that $\lambda \ne 0$ then we can differentiate the equation $u^2 + v^2 = \lambda^2$ partially with respect to x and with respect to y to obtain the system of equations namely,

$$2u \cdot u_x + 2v \cdot v_x = 0 \quad \text{and} \quad 2u \cdot u_y + 2v \cdot v_y = 0. \qquad ...(1)$$

The Cauchy-Riemann equations can be used in the system of equations (1) to express the above system of equations as $u \cdot u_x - v \cdot u_y = 0$ and $v \cdot u_x + u \cdot u_y = 0$. ...(2)

Solving (2) for the unknowns u_x and u_y, we get

$$u_x = \frac{0}{u^2 + v^2} = 0 \quad \text{and} \quad u_y = \frac{0}{u^2 + v^2} = 0.$$

The conditions $u_x = 0$ and $u_y = 0$ together imply that $u(x, y) = c_1$ where c_1 is a constant and $v(x, y) = c_2$, where c_2 is another constant, and therefore $f(z) = c_1 + ic_2$ which is constant.

Hence Proved.

(b) It is easy to show that $\ln(x)$ is not continuous when $x \le 0$. Hence the $\ln(z)$ is not differentiable and hence not analytic when $\text{Re}(z) \le 0$. Further,

$$\ln(z) = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right) = u + iv \quad \therefore \quad u = \frac{1}{2}\ln(x^2 + y^2) \quad \text{and} \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Also, $$u_x = \frac{x}{x^2 + y^2} = v_y \quad \text{and} \quad u_y = \frac{y}{x^2 + y^2} = -v_x.$$

Since CR equations are satisfied for all (x, y) except at $(0, 0)$ and the first order partial derivatives of u and v are continuous everywhere except at $(0, 0)$, the given function is analytic everywhere except when $\text{Re } z \le 0$.

EXAMPLE 17.7. (a) Discuss the analyticity of the function $f(z) = z\,\bar{z}$.

[GGSIPU II Sem End Term 2014]

(b) Show that $f(z) = \dfrac{z}{z+1}$ is analytic at $z = \infty$.

SOLUTION: (a) $f(z) = z\bar{z} = |z|^2 = x^2 + y^2 = u + iv$. Hence $u = x^2 + y^2$, $v = 0$.

In this case the four first order partial derivatives are

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0 \text{ which are continuous everywhere.}$$

The Cauchy-Riemann equations in this case are $2x = 0$ and $2y = 0$

Therefore C.R. equations are satisfied only at the origin. Hence $z = 0$ is the only point at which $f'(z)$ exists, and therefore $f(z) = z\bar{z}$ is nowhere analytic except at the origin. **Ans.**

(b) The function $f(z)$ is analytic at $z = \infty$ if the function $f\left(\dfrac{1}{z}\right)$ is analytic at $z = 0$.

Since $f(z) = \dfrac{z}{z+1}$, $f\left(\dfrac{1}{z}\right) = \dfrac{1}{1+z}$.

Now $f\left(\dfrac{1}{z}\right)$ is differentiable at $z = 0$ and at all points in its neighbourhood. Hence the function $f\left(\dfrac{1}{z}\right)$ is analytic at $z = 0$ and, in turn, $f(z)$ is analytic at $z = \infty$.

EXAMPLE 17.8. (a) Use C.R. equations to show that $f(z) = z^3$ is analytic in the entire complex plane.

[GGSIPU II Sem End Term 2007]

(b) Show that $f(z) = \bar{z}$ is not differentiable at $z = 0$ and it is no where analytic.

[GGSIPU II Sem End Term 2006]

SOLUTION: (a) $f(z) = z^3 = (x + iy)^3 = x^3 - 3xy^2 + 3ix^2y - iy^3 = u + iv$

$\therefore \quad u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$.

Next, $\quad \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \dfrac{\partial u}{\partial y} = -6xy, \quad \dfrac{\partial v}{\partial x} = 6xy, \quad \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

we see that the Cauchy-Riemann equations are satisfied. Also since the partial derivatives of u and v are continuous hence the function is analytic for every z. **Hence Proved.**

(b) $f(z) = \bar{z} = x - iy = u + iv$ hence $u = x$ and $v = -y$, then

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1.$$

Here $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$ but $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ Hence $f(z)$ is analytic no where.

For derivative of $f(z)$ at $z = 0$ we have

$$f'(0) = \underset{\delta z \to 0}{Lt} \frac{f(0+\delta z) - f(0)}{\delta z} = \underset{\delta z \to 0}{Lt} \frac{\overline{\delta z} - 0}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{LT} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \begin{cases} 1 & \text{when } \delta z \to 0 \text{ along X-axis} \\ -1 & \text{when } \delta z \to 0 \text{ along Y-axis} \end{cases}$$

Therefore the limit does not exist hence $f(z)$ is not differentiable at the origin. **Hence Proved.**

EXAMPLE 17.9. Find the values of C_1 and C_2 such that the function

$f(z) = x^2 + C_1y^2 - 2xy + i(C_2x^2 - y^2 + 2xy)$ is analytic. Hence find $f'(z)$.
[GGSIPU II Sem I Term 2012]

SOLUTION: $f(z) = u + iv$ where $u = x^2 + C_1y^2 - 2xy$ and $v = C_2x^2 - y^2 + 2xy$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x - 2y, \quad \frac{\partial u}{\partial y} = 2C_1y - 2x, \quad \frac{\partial v}{\partial x} = 2C_2x + 2y \quad \text{and} \quad \frac{\partial v}{\partial y} = -2y + 2x.$$

Since $f(z)$ is analytic, C.R. equations are satisfied, that is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ is already true, and $2C_1y - 2x = -(2C_2x + 2y)$
which gives $C_1 = -1$ and $C_2 = 1$. **Ans.**

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i(2C_2x + 2y) = 2x - 2y + i(2x + 2y) \quad \text{as} \quad C_2 = 1$$

$$= 2z + 2iz = 2(1 + i)z. \qquad \textbf{Ans.}$$

EXAMPLE 17.10. (*a*) Determine 'p' such that the function

$f(z) = \frac{1}{2}\log x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is analytic. [GGSIPU II Sem I Term 2009]

(*b*) Determine where the Cauchy-Riemann equations are satisfied for the function $f(z) = 2x + i\,xy^2$. [GGSIPU II Sem End Term 2013]

SOLUTION: (*a*) $f(z) = u + iv = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$

Here $u = \frac{1}{2}\log(x^2 + y^2)$, $v = \tan^{-1}\left(\frac{px}{y}\right)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} \quad \text{and}$$

$$\frac{\partial v}{\partial x} = \frac{\frac{p}{y}}{1 + \frac{p^2x^2}{y^2}} = \frac{py}{y^2 + p^2x^2}, \quad \frac{\partial v}{\partial y} = \frac{-\frac{px}{y^2}}{1 + \frac{p^2x^2}{y^2}} = \frac{-px}{y^2 + p^2x^2}.$$

If $f(z)$ is analytic, C.R. equation are satisfied, hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\therefore \quad \frac{x}{x^2 + y^2} = \frac{-px}{y^2 + p^2x^2} \quad \text{and} \quad \frac{y}{x^2 + y^2} = \frac{-py}{y^2 + p^2x^2}$$

which gives $p = -1$ as the required value of p. **Ans.**

(b) $f(z) = u + iv$ hence $u = 2x$ and $v = xy^2$.

$\therefore \quad \dfrac{\partial u}{\partial x} = 2, \quad \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial v}{\partial x} = y^2$ and $\dfrac{\partial v}{\partial y} = 2xy$.

Cauchy-Riemann equations are $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$. $\therefore$ $2xy = 2$ and $y^2 = 0$

$xy = 1$ gives $x = \dfrac{1}{y}$ and $y = 0$ hence C.R. equations are no where satisfied.

CAUCHY-RIEMANN EQUATIONS IN POLAR FORM

Let $w = f(z)$ be analytic and we take z in polar form as $z = re^{i\theta}$, then

$$w = u + iv = f(re^{i\theta}) \qquad ...(1)$$

Differentiating it partially w.r.t. r, gives

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \qquad ...(2)$$

and differentiating (1) partially w.r.t. θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta} \qquad ...(3)$$

Eliminating $f'(re^{i\theta})$ in (2) and (3), we get

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad -r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$$

or $\quad \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ **and** $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ are the C.R. equations in polar form.

And from (2), we have the value of the derivative of $f(z)$ as

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\frac{\partial}{\partial r}(u + iv) = e^{-i\theta}\frac{\partial f}{\partial r}. \quad \textbf{Thus, } f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}.$$

EXAMPLE 17.11. Show that the function $f(z) = \begin{cases} e^{-z^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not analytic at the original though the Cauchy-Riemann equation are satisfied there.

[*GGSIPU II Sem. I Term 2010*]

SOLUTION: $\quad e^{-z^4} = e^{-r^4(\cos 4\theta + i\sin 4\theta)} = e^{-r^4\cos 4\theta} \cdot e^{-ir^4\sin 4\theta}$

$$= e^{-r^4\cos 4\theta}[\cos(r^4\sin 4\theta) - i\sin(r^4\sin 4\theta)] = u + iv$$

then $\quad u = e^{-r^4\cos 4\theta}\cos(r^4\sin 4\theta), \quad v = -e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)$.

$$\therefore \quad \frac{\partial u}{\partial r} = -e^{-r^4\cos 4\theta}\cdot 4r^3\cos 4\theta\cos(r^4\sin 4\theta) - e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)(4r^3\sin 4\theta)$$

$$= -4r^3 e^{-r^4\cos 4\theta}[\cos 4\theta\cos(r^4\sin 4\theta) + \sin 4\theta\sin(r^4\sin 4\theta)]$$

$$= -4r^3 e^{-r^4\cos\theta}\cos(4\theta - r^4\sin 4\theta)$$

and $$\frac{\partial u}{\partial \theta} = e^{-r^4\cos 4\theta}\cdot 4r^4 \sin 4\theta \cos(r^4 \sin 4\theta) - e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)\, 4r^4\cos 4\theta$$

$$= 4r^4 e^{-r^4\cos 4\theta}[\sin 4\theta\cos(r^4\sin 4\theta) - \cos 4\theta\cdot\sin(r^4\sin 4\theta)]$$

$$= 4r^4 e^{-r^4\cos 4\theta}\sin(4\theta - r^4\sin 4\theta).$$

Also $$\frac{\partial v}{\partial r} = 4r^3 e^{-r^4\cos 4\theta}\cos 4\theta\sin(r^4\sin 4\theta) - e^{-r^4\cos 4\theta}\cos(r^4\sin 4\theta)\, 4r^3\sin 4\theta$$

$$= -4r^3 e^{-r^4\cos 4\theta}\sin(4\theta - r^4\sin 4\theta).$$

and $$\frac{\partial v}{\partial \theta} = -e^{-r^4\cos 4\theta}\, 4r^4\sin 4\theta\sin(r^4\sin 4\theta) - e^{-r^4\cos 4\theta}\cos(r^4\sin 4\theta)\, 4r^4\cos 4\theta$$

$$= -4r^4 e^{-r^4\cos 4\theta}\cos(4\theta - r^4\sin 4\theta)$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Thus, C.R. equations are satified here. However, check the analyticity of the function at the origin.

$$f'(z) = \underset{z\to 0}{\text{Lt}}\frac{e^{-z^4} - 0}{z} = \underset{z\to 0}{\text{Lt}}\frac{1}{ze^{z^4}} = \underset{z\to 0}{\text{Lt}}\frac{1}{z\left(1 + z^4 + \frac{1}{21}z^8 + \dots\right)}$$

which is not existant at the origin hence not analytic at the origin.

EXAMPLE 17.12. If the function f is given by $f(z) = z^{1/2} = r^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ where the domain is $r > 0$ and $-\pi < \theta < \pi$, show that

$$f'(z) = \frac{1}{2z^{1/2}} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} - \frac{i}{2}r^{-1/2}\sin\frac{\theta}{2}.$$

SOLUTION: Since $f(z) = u + iv$ we have

$$u(r,\theta) = r^{1/2}\cos\frac{\theta}{2} \quad \text{and} \quad v(r,\theta) = r^{1/2}\sin\frac{\theta}{2}.$$

Hence $$\frac{\partial u}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}, \quad \frac{\partial u}{\partial \theta} = \frac{-1}{2}r^{1/2}\sin\frac{\theta}{2},$$

$$\frac{\partial v}{\partial r} = \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{1}{2}r^{1/2}\cos\frac{\theta}{2}.$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta} \quad \therefore \text{ C.R. equations are satisfied.}$$

Hence, $$\frac{\partial u}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}, \quad \frac{\partial u}{\partial \theta} = -\frac{1}{2}r^{1/2}\sin\frac{\theta}{2}, \quad \frac{\partial v}{\partial r} = \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{1}{2}r^{1/2}\cos\frac{\theta}{2}$$

Hence $f(z)$ is analytic. We know that in polar form

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\therefore \quad f'(z) = e^{-i\theta}\left[\frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} + i\frac{1}{2}r^{-1/2}\sin\frac{\theta}{2}\right]$$

$$= e^{-i\theta}\frac{1}{2}r^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2}r^{-1/2}e^{-i\theta}(e^{i\theta/2})$$

$$= \frac{1}{2}r^{-1/2}e^{-i\theta/2} = \frac{1}{2z^{1/2}}, \quad \textbf{hence the result.}$$

EXAMPLE 17.13. If $u(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0$ find an analytic function $f(z) = u + iv$.

[*GGSIPU II Sem. I Term 2010*]

SOLUTION: $u = \left(r - \frac{1}{r}\right)\sin\theta$ hence $\frac{\partial u}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta$ and $\frac{\partial u}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$

Since $f(z)$ is analytic, by Cauchy-Riemann equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$.

Therefore, we have $\frac{\partial v}{\partial r} = -\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta = \left(-1 + \frac{1}{r^2}\right)\cos\theta$ ...(1)

and $\frac{\partial v}{\partial \theta} = r\left(1 + \frac{1}{r^2}\right)\sin\theta = \left(r + \frac{1}{r}\right)\sin\theta$ ...(2)

$\therefore$ (2) gives $v = \int\limits_{r\,\text{constt.}} \left(r + \frac{1}{r}\right)\sin\theta\, d\theta = -\left(r + \frac{1}{r}\right)\cos\theta + c(r)$ ...(3)

$\Rightarrow \quad \frac{\partial v}{\partial r} = \left(-1 + \frac{1}{r^2}\right)\cos\theta + \frac{dc}{dr}.$ ...(4)

Comparing (1) and (4), we get $\frac{dc}{dr} = 0$ which means c is a pure constant.

From (4), we get $v = -\left(r + \frac{1}{r}\right)\cos\theta + c$

$\therefore \quad f(z) = u + iv = \left(r - \frac{1}{r}\right)\sin\theta - i\left(r + \frac{1}{r}\right)\cos\theta + c$ **Ans.**

EXAMPLE 17.14. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ in terms of z such that $v(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2$.

[*GGSIPU II Sem II Term 2011*]

SOLUTION: Here $v(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2$ and $f(z) = u(r, \theta) + iv(r, \theta) = f(re^{i\theta})$. ...(1)

Differentiating (1) partially w.r.t. r, and w.r.t. θ, we get

$\therefore \quad \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta$ and $\frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta$.

Since $f(z)$ is analytic, C.R. equations are satisfied, hence $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Differentiating $f(re^{i\theta}) = u + iv$ partially w.r.t. we get

$f'(re^{i\theta}) \cdot e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}. \quad \therefore \quad \frac{z}{r}f'(z) = \frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial r}$

$\therefore \quad f'(z) = \frac{1}{z}\left[-2r^2\sin 2\theta + r\sin\theta + i(2r^2\cos 2\theta - r\cos\theta)\right]$

$$= \frac{1}{z}\left[2r^2 i(\cos 2\theta + i\sin 2\theta) - ri(\cos\theta + i\sin\theta)\right]$$

$$= \frac{1}{z}[2ir^2 e^{2i\theta} - ire^{i\theta}] = \frac{1}{z}[2iz^2 - iz] = i(2z-1)$$

$\Rightarrow$ $f(z) = i(z^2 - z) + C + 2i$ (since v has a constant 2). **Ans.**

[Please note that V can be easily expressed in Cartesian form as $v = x^2 - y^2 - x - 2$].

HARMONIC FUNCTIONS

[*GGSIPU II Sem End Term 2012*]

*Any function $f(x, y)$ which possesses continuous partial derivatives of the first and second order and satisfies the Laplace equation $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is called the **harmonic function.***

It is of big practical importance that both the real part and the imaginary part of an analytic function satisfy the most important partial differential equation of physics–the Laplace equation which has frequent occurrence in gravitation, electrostatics, fluid flow, heat conduction, etc.

THEOREM: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then u and v satisfy the Laplace equation $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$

PROOF: Since $f(z) = u + iv$ is analytic in D, it necessarily satisfies the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad ...(1)$$

and

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \qquad ...(2)$$

Differentiating both sides of (1) w.r.t. x and (2) w.r.t. y, and adding we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 u = 0.$$

Similarly, differentiating both sides of (1) w.r.t. y and (2) w.r.t. x and subtracting, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 v = 0$$

which proves that u and v satisfy $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$

Thus, if $f(z) = u + iv$ is analytic function then both the component functions u and v are harmonic functions.

Here v **is said to be conjugate harmonic of** u **and** u **the conjugate harmonic of** v.

EXAMPLE 17.15. If $f(z)$ an analytic function of z show that

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|Rf(z)|^2 = 2|f'(z)|^2$$ [GGSIPU II Sem Term I 2006; I Term 2013]

SOLUTION: Let $f(z) = u + iv$ then $R f(z) = u$ and $\operatorname{Im} f(z) = v$.

$f(z)$ is given to be analytic in the z-plane hence C.R. equations are satisfied.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Next,
$$\frac{\partial^2}{\partial x^2}u^2 = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2$$

and
$$\frac{\partial^2 u^2}{\partial y^2} = \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right) = 2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2$$

$$\therefore \text{ L.H.S.} = \frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2} = 2u\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

Since $f(z)$ is analytic, Laplace equation holds for u and v hence $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0.$

And since $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we have

$$\text{L.H.S.} = 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Also since $f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ we have $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

Thus L.H.S $= 2|f'(z)|^2$ = RHS. **Hence the result.**

ORTHOGONALITY OF u AND v

[GGSIPU II Sem II Term 2011; I Term 2012; I Term 2013]

We now view the harmonic functions arising out of an analytic function, geometrically. Let $f(z) = u + iv$ be the analytic function, then consider a pair of families of curves

$$u(x, y) = C_1 \qquad \text{...(1)}$$

and
$$v(x, y) = C_2 \qquad \text{...(2)}$$

Differentiating (1) and (2) separately, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} = m_1, \text{ say, for curve (1) and } \frac{dy}{dx} = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y} = m_2, \text{ say, for curve (2).}$$

$$\text{Therefore} \quad m_1 m_2 = \frac{\dfrac{-\partial u}{\partial x}\Big/\dfrac{\partial u}{\partial y}}{\dfrac{-\partial v}{\partial x}\Big/\dfrac{\partial v}{\partial y}}.$$

Since Cauchy-Riemann equations are satisfied here, therefore $m_1 m_2 = -1$ which implies that (1) and (2) form an orthogonal system.

Thus we conclude that ***every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = C_1$ and $v(x, y) = C_2$ which form an orthogonal system.*** [GGSIPU II Sem I Term 2012]

FOR INSTANCE, suppose that we are to find the orthogonal trajectory of the family of curves $x^3 - 3xy^2 = C$. [GGSIPU II Sem I Term 2012]

Here, let $u = x^3 - 3xy^2 = C$ hence $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$ and $\dfrac{\partial u}{\partial y} = -6xy$.

Using C.R. equations, we have $\dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$ and $\dfrac{\partial v}{\partial x} = 6xy$

From $\dfrac{\partial v}{\partial x} = 6xy$ we can write $v = 3x^2y + K(y)$

$\therefore \quad \dfrac{\partial v}{\partial y} = 3x^2 + \dfrac{dK}{dy}$ which gives $\dfrac{dK}{dy} = -3y^2 \qquad \therefore \quad K = -y^3$

Therefore $v = 3x^2y - y^3$. Thus the trajectory orthogonal to $u = C$ is $3x^2y - y^3 = C'$. **Ans.**

MILNE-THOMPSON METHOD

The method provides a short cut to determine the analytic function $f(z)$ when u or v are given.

We have $z = x + iy$ and $\bar{z} = x - iy \quad \therefore \quad x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$. ...(1)

Now $w = f(z) = u(x, y) + i\,v(x, y) = u\left[\dfrac{z + \bar{z}}{2}, \dfrac{z - \bar{z}}{2i}\right] + i\,v\left[\dfrac{z + \bar{z}}{2}, \dfrac{z - \bar{z}}{2i}\right]$

This can be treated as an identity in two independent variables z and $\bar{z}$.

Now, if we take $\bar{z} = z$ in (1), it implies that $x = z, \quad y = 0$.

then $w = u(z, 0) + i\,v(z, 0)$.

Thus, to get $f'(z)$ terms of z we replace x by z and y by 0 in $u(x, y)$ and $v(x, y)$

occurring in $\dfrac{dw}{dz} = \dfrac{\partial u(x, y)}{\partial x} + i\dfrac{\partial v(x, y)}{\partial x}$.

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = \phi_1(x, y) = \phi_1(z, 0)$ and $\dfrac{\partial v}{\partial x} = \phi_2(x, y) = \phi_2(z, 0)$.

$\therefore \quad \dfrac{dw}{dz} = \phi_1(z, 0) + i\,\phi_2(z, 0)$. Integrating it w.r.t. z gives $w = f(z)$.

This useful method is provided by Milne & Thompson.

EXAMPLE 17.16. (a) Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$. Also prove that the given function satisfies Laplace equation. [*GGSIPU II Sem End Term 2007; End Term 2013*]

(b) Determine the analytic function $f(z) = u + iv$, if $v = \log(x^2 + y^2) + x - 2y$.

[*GGSIPU II Sem I Term 2006 Reappear*]

SOLUTION: (a) Given function is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$.

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x + 2, \quad \frac{\partial u}{\partial y} = -6xy - 6y.$$

$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -6x - 6.$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ which shows that u satisfies Laplace equation.

If $f(z) = u + iv$ is analytic, the C.R. equations are satisfied.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \text{Therefore} \quad \frac{\partial v}{\partial x} = 6xy + 6y$$

$$\text{Now} \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 - 3y^2 + 6x + 2 + 6ixy + 6iy.$$

Using the Milne-Thompson method here, by replacing x by z and y by 0, we get

$$f'(z) = 3z^2 + 6z + 2$$

Hence $f(z) = z^3 + 3z^2 + 2z + c$ where c is constant. **Ans.**

(b) Since $v = \log(x^2 + y^2) + x - 2y$, we have

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$$

$f(z)$ is given to be analytic hence C.R. equations are satisfied $\therefore \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$.

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{2y}{x^2 + y^2} - 2 + i\left[\frac{2x}{x^2 + y^2} + 1\right]$$

Applying here Milne Thomson method, that is, replacing x by z and y by 0, we get

$$f'(z) = 0 - 2 + i\left(\frac{2}{z} + 1\right) \quad \therefore \quad f(z) = (-2 + i)\,z + 2\,i\log z.$$ **Ans.**

EXAMPLE 17.17. (a) Find the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

(b) Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function.

[*GGSIPU II Sem End Term 2007; End Term 2012*]

SOLUTION: (a) Let $f(z) = u + iv$ where $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Hence $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (using C-R equations)

$$= \frac{(\cosh 2y - \cos 2x)\,2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x\,\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thompson's method we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

Therefore $f'(z) = \dfrac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i\,(0) = \dfrac{-2}{1 - \cos 2z} = -\operatorname{cosec}^2 z.$

Integrating both sides w.r.t. z, we get $f(z) = \cot z + i\,C$

where the constant of integration C is imaginary as u does not involve any constant. **Ans.**

(b) Given that $u = e^{-2xy}\sin(x^2 - y^2)$

hence $$\frac{\partial u}{\partial x} = e^{-2xy}\cos(x^2 - y^2)\cdot 2x - 2ye^{-2xy}\sin(x^2 - y^2)$$
$$= 2e^{-2xy}\,[x\cos(x^2 - y^2) - y\sin(x^2 - y^2)]$$

and $$\frac{\partial u}{\partial y} = e^{-2xy}\cos(x^2 - y^2)\,(-2y) - 2xe^{-2xy}\sin(x^2 - y^2)$$
$$= 2e^{-2xy}\,[-y\cos(x^2 - y^2) - x\sin(x^2 - y^2]$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} = -4ye^{-2xy}\,[x\cos(x^2 - y^2) - y\sin(x^2 - y^2)] + 2e^{-2xy}\,[1\cdot\cos(x^2 - y^2) - x\sin(x^2 - y^2)\cdot 2x - y\cos(x^2 - y^2)\cdot 2x]$$

and $$\frac{\partial^2 u}{\partial y^2} = +4xe^{-2xy}\,[y\cos(x^2 - y^2) + x\sin(x^2 - y^2)] + 2e^{-2xy}\,[y\sin(x^2 - y^2)(-2y) - \cos(x^2 - y^2) - x\cos(x^2 - y^2)(-2y]$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4e^{-2xy}\,[-xy\cos(x^2 - y^2) + y^2\sin(x^2 - y^2) + xy\cos(x^2 - y^2) + x^2\sin(x^2 - y^2)] + 2e^{-2xy}\,[\cos(x^2 - y^2) - 2x^2\sin(x^2 - y^2) - 2xy\cos(x^2 - y^2) - 2y^2\sin(x^2 - y^2) - \cos(x^2 - y^2) + 2xy\cos(x^2 - y^2)]$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Therefore u is harmonic. $u + iv$ is given to be analytic, hence using C.R. equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2e^{-2xy}\,[y\cos(x^2 - y^2) + x\sin(x^2 - y^2)].$$

Hence $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2e^{-2xy}\,[(x + iy)\cos(x^2 - y^2) - (y - ix)\sin(x^2 - y^2)]$$

Using Milne Thompson method we replace x by z and y by 0 and get

$$f'(z) = 2e^{\circ}\,[z\cos z^2 + iz\sin z^2] = 2z\,[\cos z^2 + i\sin z^2]$$

$\therefore$ $$f(z) = \int \cos z^2\cdot 2z\,dz + i\int \sin z^2\cdot 2z\,dz = \sin z^2 - i\cos z^2$$

Therefore the analytic function $f(z) = -ie^{iz^2}$. **Ans.**

EXAMPLE 17.18. If $f(z) = u + iv$ is an analytic function of $z\ (= x + iy)$ and $u - v = (x - y)(x^2 + 4xy + y^2)$, find $f(z)$.

SOLUTION: We have $u - v = (x - y)(x^2 + 4xy + y^2)$ ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$u_x - v_x = (x - y)(2x + 4y) + 1\,(x^2 + 4xy + y^2) = 3x^2 + 6xy - 3y^2 \quad ...(2)$$

and

$$u_y - v_y = (x - y)(4x + 2y) + (-1)(x^2 + 4xy + y^2) = 3x^2 - 6xy - 3y^2 \quad ...(3)$$

Using C-R equations $u_x = v_y$, $u_y = -v_x$ in (3), we get

$$-v_x - u_x = 3x^2 - 6xy - 3y^2 \quad ...(4)$$

Equations (2) and (4), give $u_x = 6xy$ and $v_x = -3x^2 + 3y^2$.

Since $f(z)$ is analytic, $f'(z) = u_x + i v_x = 6xy + 3i(y^2 - x^2)$ ...(5)

Using Mile-Thompson method we write replace x by z and y by 0 to get

$$f'(z) = 0 + 3i(0 - z^2) = -3iz^2$$

$\Rightarrow$ $f(z) = -iz^3 + C$ where. C is a complex arbitrary constant. **Ans.**

EXAMPLE 17.19. If $f(z)$ is a regular function, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

[GGSIPU II Sem. End Term 2012]

SOLUTION: Let $f(z) = u + iv$ hence $|f(z)|^2 = u^2 + v^2$.

$$\therefore \quad \frac{\partial}{\partial x}(u^2 + v^2) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

and

$$\frac{\partial^2}{\partial x^2}(u^2 + v^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2$$

Similarly

$$\frac{\partial^2}{\partial y^2}(u^2 + v^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2.$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)$$

$$= 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$+ 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2$$

Since $f(z)$ is a regular function, u and v are harmonic functions and hence satisfy the Laplace equation

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$$

As such we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Further, since $f(z)$ is regular, it satisfies C-R equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Therefore $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^2 = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 4\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = 4| f'(z)|^2$

Hence Proved.

EXAMPLE 17.20. (a) If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u + iv$ is analytic function of $z\ (= x + iy)$, find $f(z)$ in terms of z.

[*GGSIPU II Sem End Term 2014*]

(b) Determine the analytic function $f(z) = u + iv$, if

$u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$. [*GGSIPU II Sem I Term 2006*]

SOLUTION: Since $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} = \dfrac{\sin 2x}{\cos h2y - \cos 2x}$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{(\cos h\, 2y - \cos 2x)\, 2\cos 2x - 2\sin^2 2x}{(\cos h\, 2y - \cos 2x)^2} = \frac{2\cos 2x \cos h\, 2y - 2}{(\cos h\, 2y - \cos 2x)^2}$$

and $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = \dfrac{-\sin 2x \cdot 2\sin h\, 2y}{(\cos h\, 2y - \cos 2x)^2} = -\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}$ using C.R. equations as $f(z)$ is analytic function.

$$\Rightarrow \quad 2\frac{\partial u}{\partial x} = \frac{2(\cos 2x \cos h\, 2y - \sin 2x \sin h\, 2y) - 2}{(\cos h\, 2y - \cos 2x)^2}$$

$$\text{and} \quad 2\frac{\partial v}{\partial x} = \frac{2\cos 2x \cos h\, 2y - 2 + 2\sin 2x \sin h\, 2y}{(\cos h\, 2y - \cos 2x)^2}$$

Hence $\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{(\cos 2x \cos h\, 2y - \sin 2x \sin h\, 2y) + i(\cos 2x \cos h\, 2y + \sin 2x \sin h\, 2y) - (1+i)}{(\cos h\, 2y - \cos 2x)^2}$

Using Milne-Thompson method here, we have

$$f'(z) = \frac{\cos 2z\,(1+i) - (1+i)}{(1 - \cos 2z)^2} = \frac{-(1+i)}{1 - \cos 2z} = \frac{-(1+i)}{2\sin^2 z} = \frac{-(1+i)}{2}\operatorname{cosec}^2 z$$

Thereforem $f(z) = \left(\dfrac{1+i}{2}\right)\cos z$. **Ans.**

(b) Given $f(z) = u + iv \;\therefore\; i f(z) = iu - v$

Let $F(z) = (1 + i) f(z) = u - v + i(u + v) = U + iV$ where $U = u - v$ and $V = u + v$.

Given that $U = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial U}{\partial x} = \frac{2(\cos x - \cosh y)(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{4(\cos x - \cosh y)^2}$$

$$= \frac{-\sin x \cos x + \sin x \cosh y + \cos^2 x - \cos x \cosh y + \sin x \cos x + \sin^2 x - e^{-y} \sin x}{2(\cos x - \cosh y)^2}$$

$$= \frac{1 + \sin x \cosh y - \cos x \cosh y - e^{-y} \sin x}{2(\cos x - \cosh y)^2}$$

and
$$\frac{\partial U}{\partial y} = \frac{2(\cos x - \cosh y) e^{-y} - (\cos x + \sin x - e^{-y})(-2\sinh y)}{4(\cos x - \cosh y)^2}$$

$$= \frac{\sinh y (\cos x + \sin x) + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Since $f(z)$ is analytic, $(1 + i) f(z)$ is also analytic, that is, $U + iV$ is analytic hence C.R. equations are satisfied, for $F(Z)$, that is,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}.$$

$$\Rightarrow \quad \frac{\partial V}{\partial x} = \frac{e^{-y}(\cosh y + \sinh y - \cos x) - \sinh y(\cos x + \sin x)}{2(\cos x - \cosh y)^2}$$

Thus, we have $F'(z) = \dfrac{\partial U}{\partial x} + i \dfrac{\partial V}{\partial x}$ (using here Milne-Thompson method)

$$= \frac{1}{2(\cos z - 1)^2}[\{1 + \sin z \cdot 1 - \cos z \cdot 1 - 1 \cdot \sin z\} + i\{(1 - \cos z) - 0\}]$$

$$= \frac{1}{2(\cos z - 1)^2}[1 - \cos z + i(1 - \cos z)] = \frac{1+i}{2(1 - \cos z)}$$

Therefore
$$F(z) = \int F'(z) dz = \frac{1+i}{2} \int \frac{dz}{2\sin^2(z/2)} = -\left(\frac{1+i}{2}\right) \cot\frac{z}{2} + C$$

where C is constant of integration.

or
$$(1 + i) f(z) = -\left(\frac{1+i}{2}\right) \cot\frac{z}{2} + C$$ Now, since $f(\pi/2) = 0$ we have $C = \dfrac{1+i}{2}$

Hence
$$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$$ **Ans.**

EXAMPLE 17.21. Find the lines of force of the gravitational field whose equipotential lines are $e^x (x \cos y - y \sin y) = C$. Also find the complex potential.

SOLUTION: Let the complex potential be $w = u + i v$

where the equipotential lines are $u(x, y) = e^x (x \cos y - y \sin y) = c$.

Hence
$$\frac{\partial u}{\partial x} = e^x (x \cos y - y \sin y) + e^x(\cos y) = e^x [x \cos y - y \sin y + \cos y]$$

and
$$\frac{\partial u}{\partial y} = e^x (- x \sin y - y \cos y - \sin y).$$

Since Cauchy-Riemann equations are satisfied, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$$\therefore \quad \frac{\partial v}{\partial y} = e^x (x \cos y - y \sin y + \cos y)$$

which on integration w.r.t. y, gives $v(x, y) = e^x [x \sin y + y \cos y] + K(x)$

From this equation we can get

$$\frac{\partial v}{\partial x} = e^x [x \sin y + y \cos y] + e^x [\sin y] + K'(x)$$

$$= e^x (x \sin y + y \cos y + \sin y) + K'(x) = \frac{-\partial u}{\partial y}$$

$\Rightarrow \quad K'(x) = 0 \quad \therefore \; K(x) = \text{constant, say } C,$

$\therefore \quad v(x, y) = e^x (x \sin y + y \cos y) + C$ are the required lines of force.

Hence $\quad f(z) = u + iv = e^x (x \cos y - y \sin y) + i e^x (x \sin y + y \cos y) + C$

$$= e^x [x (\cos y + i \sin y) + i y (\cos y + i \sin y)] = e^{x+iy} (x + iy) + C$$

$$= z e^z + C$$ **Ans.**

EXERCISE 17A

1. If $\omega = f(z) = u + iv$ is analytic, show that

$$|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

2. Find $\lim\limits_{z \to 1+i} \dfrac{z^2 - 2i}{z^2 - 2z + 2}$.

3. A function $f(x, y)$ is given by

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^3 + y^3}, & x, y \neq 0 \\ 0, & x = y = 0 \end{cases}$$

Test the continuity of $f(x, y)$ at $(0, 0)$.

4. (a) The function $f(x, y) = \dfrac{xy}{x^2 + y^2}$. Find $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$.

(b) Show that the function $u = \dfrac{1}{2} \log (x^2 + y^2)$ is harmonic find its conjugate harmonic.

[GGSIPU II Sem. End Term 2010]

5. Show that the function $f(z) = e^{-y} \cos x + i e^{-y} \sin x$ is differentiable for all z and find its derivative.

6. (a) Show that $u(x, y) = xy^3 - x^3y$ is a harmonic function and find its conjugate harmonic function $v(x, y)$.

(b) Show that the function $u = \dfrac{1}{2} \log (x^2 + y^2)$ is harmonic and find its conjugate harmonic.

(GGSIPU II Sem. End Term 2010)

7. (i) Show that the function defined by $f(z) = |z|$ is not differentiable at the origin but is continuous everywhere.

(ii) Show that $\dfrac{d}{dz} (\bar{z})^2$ does not exist at any point except at $z = 0$.

8. Show that the functions defined by (i) $f(z) = z^3 + 1 - iz^2$ (ii) e^z are analytic everywhere. [GGSIPU IInd Sem. End Term 2005]

9. Obtain the singularities of the function

$$f(z) = \frac{\cos z}{z^2 + 1}.$$

10. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although the Cauchy-Riemann equations are satisfied there.

11. (a) If $u = \dfrac{x}{x^2 + y^2}$, find its conjugate harmonic v such that $u + iv$ is analytic.

(b) Let $u = e^x \cos y$. Find its conjugate harmonic and construct the analytic function $f(z) = u + iv$. [GGSIPU II Sem End Term 2005]

12. (a) Find the values of k such that $u = e^{kx} \cos y$ is the real part of the analytic function $u + iv$. Also find this function.

(b) Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a conjugate harmonic v of u. [GGSIPU II Sem End Term 2006]

13. Let $f(z) = u + iv$ be an analytic function. If $u = v^2$ show that $f(z)$ is constant.

14. Determine the value(s) of 'p' such that the function

$$f(z) = r^2 \cos 2\theta + i r^2 \sin p\theta \quad \text{is analytic.}$$

15. (a) Prove that $u = y^3 - 3x^2 y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function.

(b) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. [GGSIPU II Sem End Term 2011]

16. Given
$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \qquad z \neq 0$$
$$= 0 \qquad \text{when } z = 0$$

Test if $f(z)$ satisfies Cauchy-Riemann equations at the origin. Is $f(z)$ analytic at the origin ?

17. If $\omega = \phi + i\psi$ represents the complex potential function in an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, find the function ϕ and ω. [GGSIPU II Sem I Term 2001]

18. If $f(z) = u + iv$ is analytic, show that u and v both satisfy the relation

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

19. (a) If $f(z) = u + iv$ is analytic function of $z(= x + iy)$ and $u + v = (x + y)\ (2 - 4xy + x^2 + y^2)$, then find u and v and the function $f(z)$.

(b) If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z. [GGSIPU II Sem II Term 2014]

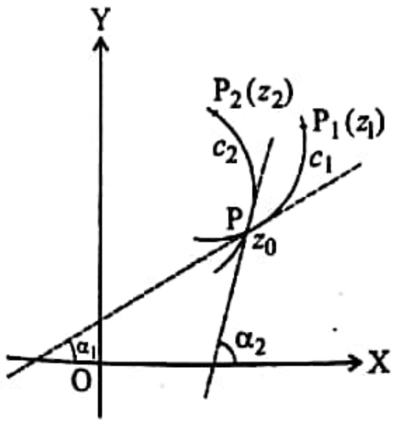
MAPPINGS OR TRANSFORMATIONS

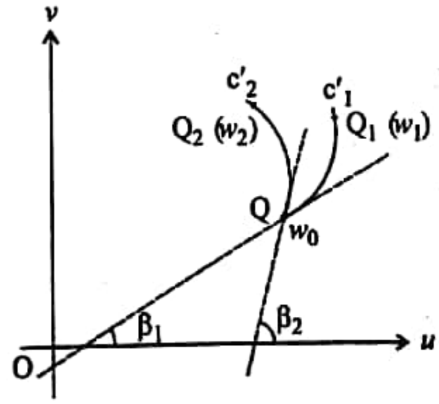
In the theory of real variables, $y = f(x)$ represents a correspondence between the points on the X-axis and on the Y-axis. By plotting these points and joining them by a smooth curve, we get the graph of the function. In the theory of complex variables, however, it is not possible to have a graph of the complex function, since we now have four variables x, y, u and v, for that a four dimensional graph paper is required which is not possible. But if we consider two planes, one known as z-plane for representing the variable $z\ (= x + iy)$ and the other known as w-plane for representing the variable $w = u + iv$, we can then establish a correspondence between the points on the z-plane and those on the w-plane with the help of the equation $u + iv = f(x + iy)$. Thus, corresponding to each point (x, y) on the z-plane, we shall have a point (u, v) on the w-plane and the corresponding points are called *images* of each other. The relation $w = f(z)$ is called a *mapping* or *transformation* and we say that a curve or region in z-plane is transformed into or mapped upon a corresponding curve or region in the w-plane.

CONFORMAL MAPPING : Suppose two curves C_1 and C_2 intersect in the z-plane at the point $z = z_0$ at an angle α. If their corresponding image curves C'_1 and C'_2 in w-plane intersect at the point $w = w_0$ at the same angle α and if the sense of rotation is also preserved then the mapping is said to be *conformal.* Conformal mappings have lot many applications in the solution of the problems in heat conduction, fluid flow and electrostatic potential.

THEOREM : *The mapping defined by an analytic function $f(z)$ is conformal, except at the critical points, that is, the points at which the derivative $f'(z)$ is zero.*

PROOF : Let the analytic function $w = f(z)$ map the curves C_1, C_2 in the z-plane into the curves C'_1, C'_2 in the w-plane. Also let z_0 be the point of intersection of the curves C_1, C_2 and z_1, z_2 be points on the curves C_1, C_2 respectively, adjoining the point z_0, and w_0, w_1, w_2 be the images of z_0, z_1, z_2 respectively in the w-plane.





Suppose the distances of the points P_1, P_2 from $P(z_0)$ are the same, then $|z_1 - z_0| = |z_2 - z_0| = r$

$$\therefore \qquad z_1 - z_0 = r\,e^{i\theta_1} \quad \text{and} \quad z_2 - z_0 = r\,e^{i\theta_2}$$

Now as z_1 and z_2 tend to z_0, the chords joining z_0 with z_1 and with z_2 become tangents to the curves C_1 and C_2 respectively. Therefore, as $r \to 0$, we get

$$\lim_{z_1 \to z_0} \theta_1 = \alpha_1 \quad \text{and} \quad \lim_{z_2 \to z_0} \theta_2 = \alpha_2$$

where α_1 and α_2 are the angles made by the tangents to the curves C_1 and C_2 with x-axis respectively at $z = z_0$.

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Next, let $w_1 - w_0 = \rho_1 e^{i\phi_1}$ and $w_2 - w_0 = \rho_2 \, e^{i\phi_2}$.

Since $f(z)$ is analytic and $f'(z_0) \neq 0$, suppose $f'(z_0) = \text{R}\, e^{i\psi}$ which remains the same in both the cases when $z_1 \to z_0$ or $z_2 \to z_0$.

Now for the curve C_1 we have

$$f'(z_0) = \text{R}\, e^{i\psi} = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = \lim_{P_1 \to P_0} \frac{\rho_1 \, e^{i\phi_1}}{r\, e^{i\theta_1}}$$

$$= \left[\lim_{z_1 \to z_0} \left(\frac{\rho_1}{r}\right)\right] e^{i(\phi_1 - \theta_1)} = \text{R}\; e^{i(\phi_1 - \theta_1)}.$$

If β_1 and β_2 are the angles made by the tangents to C_1' and C_2' at their point of intersection w_0 with the U-axis, then

$$\lim_{z_1 \to z_0} (\phi_1 - \theta_1) = \psi \quad \text{or} \quad \beta_1 - \alpha_1 = \psi.$$

Similarly, for the curve C_2 in the z-plane when $z_2 \to z_0$, we get

$$\lim (\phi_2 - \theta_2) = \psi \quad \text{or} \quad \beta_2 - \alpha_2 = \psi.$$

Therefore, we have $\quad \beta_2 - \beta_1 = \alpha_2 - \alpha_1$.

Thus the curves C'_1 and C'_2 intersect at the same angle as C_1 and C_2 do, that is, under **the conformal mapping $w = f(z)$. The angle between the curves is preserved.**

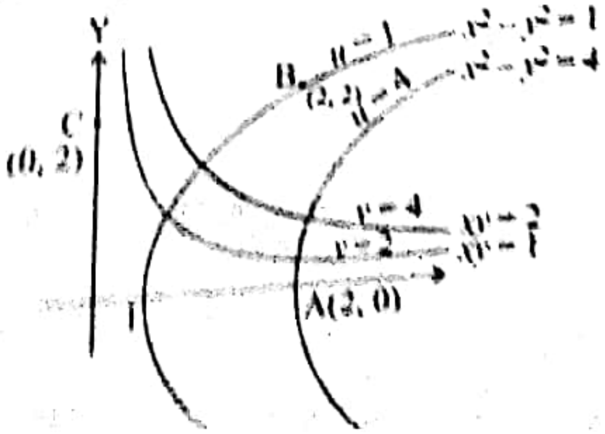
In particular, if two curves intersect orthogonally in the z-plane, the corresponding curves will also intersect orthogonally in the w-plane if the mapping (or transformation) is conformal.

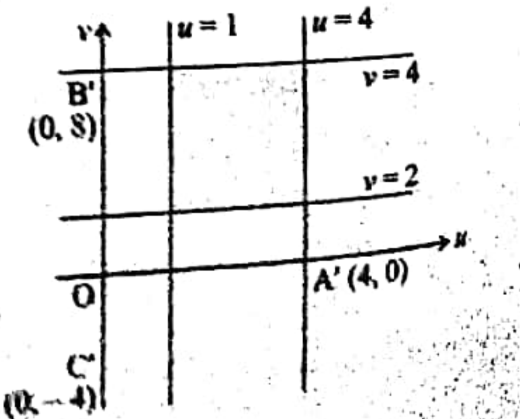
It is important to note that while such a transformation may rotate or change the dimension of any small figure, it definitely preserves the angle and the shape. This, however, is not true at a point where $f'(z) = 0$. At such a point $w_2 - w_0$ and $w_1 - w_0$ are of the order smaller than that of $z_2 - z_0$ and $z_1 - z_0$ and in that case $\lim \dfrac{\rho_1}{r}$ may not be equal to $\lim \dfrac{\rho_2}{r}$ and also $\lim (\phi_2 - \theta_2)$ may not be the same as $\lim (\phi_1 - \theta_1)$ at the point where $f'(z) = 0$.

EXAMPLE 17.22. Discuss the conformal mapping $w = z^2$.

SOLUTION: We have $u + i\,v = (x + i\,y)^2$ which gives $u = x^2 - y^2$ and $v = 2\,xy$.

Here the origin in the z-plane gets mapped into the origin in w-plane. The points A (2, 0), B (2, 2), C (0, 2) in the z-plane have images A′(4, 0), B′(0, 8) and C′(− 4, 0) in the w-plane. The curve $x^2 - y^2 = C_1$ is mapped into $u = C_1$ a straight line in w-plane parallel to v-axis. The curve $2\,xy = C_2$ in z-plane is mapped into $v = C_2$ another straight line in w-plane parallel to u-axis as depicted in the adjoining figure. The shaded regions in the two figures map into each other. Thus $w = f(z)$ gives a





transformation of curves and regions in the z-plane to other curves and regions in the w-plane. Therefore, by a suitable choice of $f(z)$, a problem with complicated boundaries may be converted into one with simple boundaries. For example, a flow in a channel bounded by the hyperbolas $xy = 1$ and $xy = 2$ transforms to a flow between the straight lines $u = 2$ and $v = 4$ by the transformation $w = f(z)$. Also note that $w = z^2$ is a conformal mapping.

EXAMPLE 17.23. For a conformal mapping $\omega = z^2$, show that

(i) coefficient of magnification at $z = 2 + i$ is $\sqrt{5}$

(ii) angle of rotation at $z = 2 + i$ is $\tan^{-1} 0.5$

(iii) the circle $|z - 1| = 1$ maps to a cardiode $\rho = 2(1 + \cos\phi)$ where $\omega = \rho e^{i\phi}$ in the ω plane.

[GGSIPU II Sem II Term 2005]

SOLUTION: In the polar form let $z = re^{i\theta}$ and $\omega = \rho e^{i\phi}$.

Here coefficient of magnification $= \dfrac{\rho}{r}$ **and angle of rotation** $= \phi - \theta$.

Consider the point $z = 2 + i = re^{i\theta}$ then $r = \sqrt{2^2 + 1} = \sqrt{5}$ and $\theta = \tan^{-1}(1/2)$.

The mapping is $\omega = z^2$ hence $\omega = \rho e^{i\phi} = r^2 e^{2i\theta}$

$\Rightarrow \rho = r^2 = (\sqrt{5})^2 = 5$ and $\phi = 2\theta = 2\tan^{-1}(1/2)$.

Therefore the coefficient of magnification $= \dfrac{5}{\sqrt{5}} = \sqrt{5}$

and the angle of rotation $= \phi - \theta = 2\tan^{-1} 1/2 - \tan^{-1}(1/2) = \tan^{-1}(0.5)$.

Next, consider the mapping $\omega = z^2$ on the circle $|z - 1| = 1 \Rightarrow z - 1 = e^{i\theta}$.

or $z = 1 + \cos\theta + i\sin\theta = 2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2 = 2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)$

Now $\omega = z^2 = 4\cos^2\theta/2(\cos\theta/2 + i\sin\theta/2)^2 = 2(1 + \cos\theta)(\cos\theta + i\sin\theta)$

But $\omega = \rho e^{i\phi} \Rightarrow \rho = 2(1 + \cos\theta)$ and $\phi = \theta$

Hence mapping of the circle $|z - 1| = 1$ into the ω plane, is

$\rho = 2(1 + \cos\phi)$ which is the cardiode. **Ans.**

EXAMPLE 17.24. Determine the region of the w-plane into which the region $1/2 \le x \le 1$ and $1/2 \le y \le 1$ is mapped by the transformation $w = z^2$.

[GGSIPU II Sem II Term 2003]

SOLUTION: The mapping $w = u + iv = z^2$ means $u = x^2 - y^2$, $v = 2xy$ ... (1)

Eliminating y in the above the relations we get $v^2 = -4x^2(u - x^2)$ ... (2)

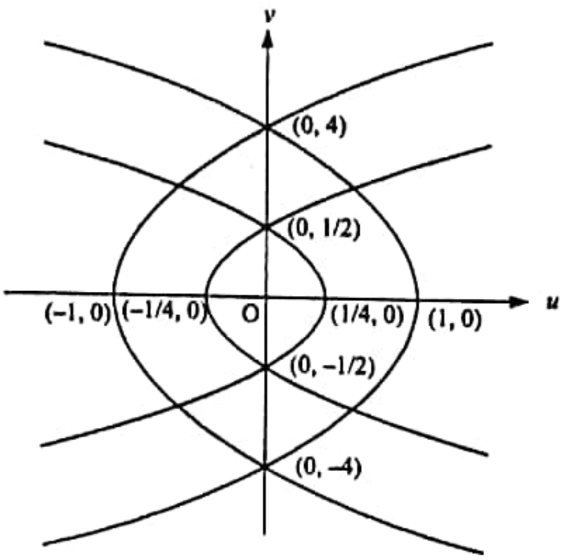
$\therefore$ line $x = 1/2$ is mapped to the parabola $v^2 = -\left(u - \dfrac{1}{4}\right)$ ... (3)

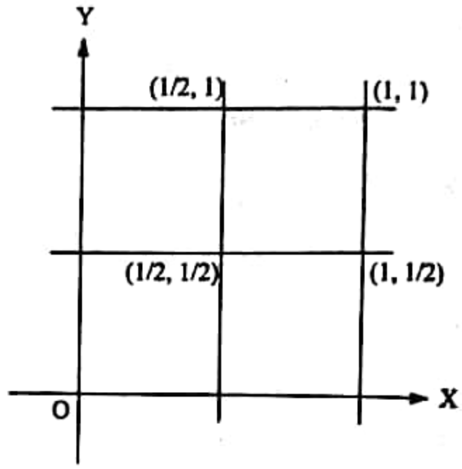
and the line $x = 1$ is mapped to the parabola $v^2 = -4(u - 1)$. .... (4)

Thus the area $\dfrac{1}{2} \le x \le 1$ is mapped to the region between the parabolas (3) and (4), as shown in the figure. Similarly eliminating x in the relations (1), we get $v^2 = 4y^2(u + y^2)$

Now the line $y = 1/2$ is mapped to the parabola $v^2 = \left(u + \dfrac{1}{4}\right)$ ... (5)

and the line $y = 1$ is mapped to the parabola $v^2 = 4(u + 1)$. ... (6)





Thus the area $1/2 \le y \le 1$ is mapped to the region between the parabolas (5) and (6) as shown in the figure. Therefore the rectangular area $1/2 \le (x, y) \le 1$ is mapped to the shaded area in w-plane as shown in the above figure.

Ans.

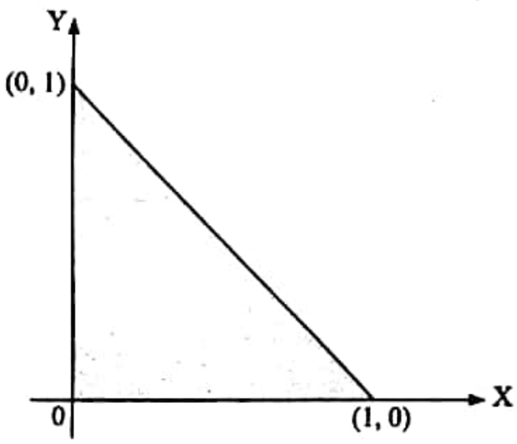
EXAMPLE 17.25. Under the mapping $w = ze^{ip/4}$ find the region in w-plane corresponding to the triangular region on the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

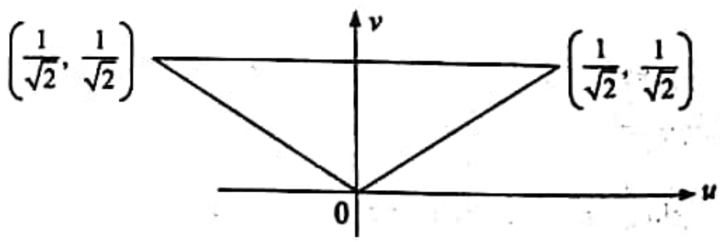
SOLUTION: $w = ze^{i\pi/4} = (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = (x+iy)\left(\frac{1+i}{\sqrt{2}}\right) = u + iv$

Hence $u = (x-y)/\sqrt{2}$ $v = (x+y)/\sqrt{2}$.

For $x = 0$ we get $u = -y/\sqrt{2}$ and $v = y/\sqrt{2}$ $\Rightarrow$ $v = -u$ is the line mapping.

For $y = 0$ we get $u = x/\sqrt{2}$ and $v = x/\sqrt{2}$ $\Rightarrow$ $v = u$ is the line mapping.





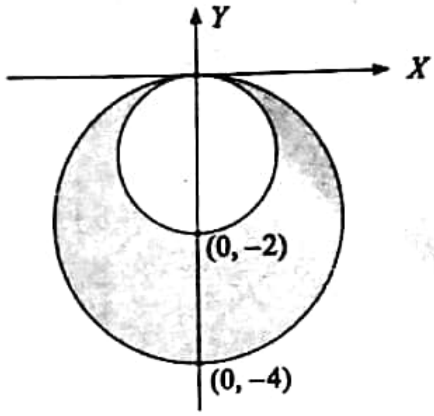
Further, putting $x + y = 1$ we get $v = 1/\sqrt{2}$ a line parallel to u-axis.

Thus, the triangular region in the z-plane is mapped on a triangular region in w-plane bounded by the lines $v = u$, $v = -u$ and $v = 1/\sqrt{2}$. as shown in the adjacent figure. **Ans.**

EXAMPLE 17.26. Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the region graphically. [GGSIPU II Sem End Term 2012]

SOLUTION: $w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u + iv$, Hence $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$

$\Rightarrow \frac{u}{v} = \frac{-x}{y}$ or $x = \frac{-yu}{v}$.



Now let us find the image of $y = a$ $\therefore$ $x = \frac{-au}{v}$

and $v = \frac{-a}{x^2+a^2} = \frac{-a}{\frac{a^2u^2}{v^2} + a^2}$ or $u^2 + v^2 + \frac{v}{a} = 0$.

Therefore, for $y = \frac{1}{2}$ we have $u^2 + v^2 + 2v = 0$

and when $y = \frac{1}{4}$ we get $u^2 + v^2 + 4v = 0$.

Thus, $\frac{1}{4} \le y \le \frac{1}{2}$ is mapped into area between the above mentioned two circles shown as shaded.

Ans.

THEOREM: A harmonic function $\phi(x, y)$ remains harmonic under a one to one conformal mapping $w = f(z)$.

PROOF : Since $\phi(x, y)$ is harmonic, we have $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

Since $f(z) = u + iv$ is analytic, u and v both are harmonic so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Now $$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial v}\frac{\partial u}{\partial y} \quad \text{(using C.R. equations)} \quad ...(1)$$

and $$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial u}{\partial x} \quad \text{(using C.R. equations).} \quad ...(2)$$

Hence $$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial v}\cdot\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial \phi}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left\{\frac{\partial^2 \phi}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial v}{\partial x}\right\} - \frac{\partial \phi}{\partial v}\frac{\partial^2 u}{\partial x \partial y}$$

$$- \frac{\partial u}{\partial y}\left\{\frac{\partial^2 \phi}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v^2}\frac{\partial v}{\partial x}\right\}$$

$$= \frac{\partial \phi}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial \phi}{\partial v} \cdot \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial v^2}\left(\frac{\partial u}{\partial y}\right)^2$$

(using C.R. equations)

Similarly, $$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial u} \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)^2.$$

Adding the above two, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)\left|\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right|^2 = \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2$$

$$= \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)|f'(z)|^2$$

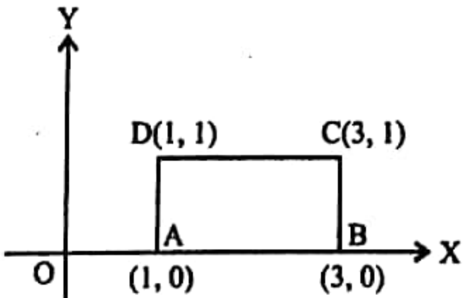
As $f'(z) \neq 0$ the equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ gets transformed to $\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0.$

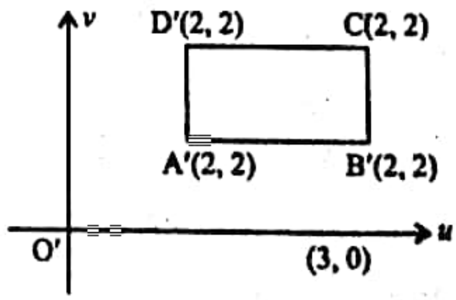
This property of conformal mapping of transforming harmonic function into harmonic function, is used in solving problems of potential fields in a number of engineering problems.

Now we intend to introduce the **Bilinear transformation** which is one of the most important conformal mapping. Bilinear transformation is composed of (*i*) translation (*ii*) rotation and magnification and (*iii*) inversion. In addition to these we shall also introduce the mappings $w = \sin z$ and $w = e^z$.

I. TRANSLATION : $w = z + C$ where $C = a + ib$ (a, b real constants)

In this transformation the figure in the z-plane is simply displaced (or shifted) in the direction of C. If $z = x + iy$ and $w = u + iv$ the transformation equations are $u = x + a$, $v = y + b$, which shows that the image of the region in z-plane is simply translated. The two regions have **same shape, size and orientation.** Consider the following example.





As depicted in the above figures the transformation $w = z + (1 + 2i)$ maps the rectangle ABCD in the z-plane into the rectangle A'B'C'D' in the w-plane by simple translation.

II. MAGNIFICATION AND ROTATION: $w = Az$ where A is complex constant.

Taking in polar form, if $A = a e^{i\alpha}$, $z = r e^{i\theta}$, $w = \rho e^{i\phi}$, then we have

$$\rho e^{i\phi} = a r e^{i(\theta + \alpha)} \Rightarrow \rho = ar \text{ and } \phi = \theta + \alpha.$$

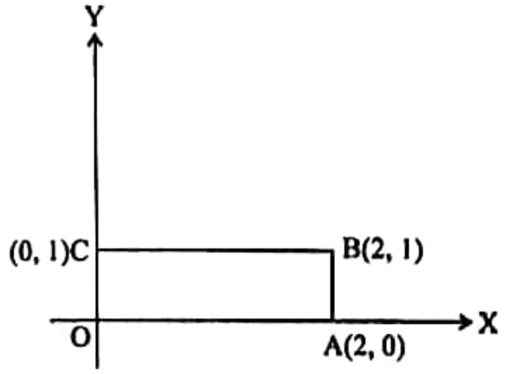
Thus, a point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(ar, \theta + \alpha)$ in the w-plane. This transformation consists of magnification (or contraction) of the radius vector of P by a

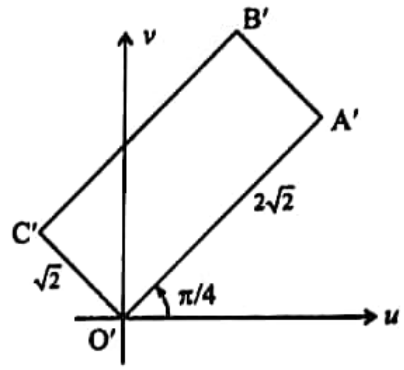
and its rotation through an angle α. Under this transformation any figure in the z-plane gets transformed into a geometrically similar figure in the w-plane.

If $z = re^{i\theta}$ is transformed to $w = \rho e^{i\phi}$ then we define **coefficient of magnification as** ρ/r **and angle of rotation as** $\phi - \theta$.

For example, consider a rectangle OABC in z-plane which gets mapped to another rectangle O′A′B′C′ under the transformation $w = (1 + i)z$ as shown in the adjoining figures.

Since $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \pi/4$, the rectangle OABC gets rotated through an angle $\pi/4$ and each side gets magnified $\sqrt{2}$ times.





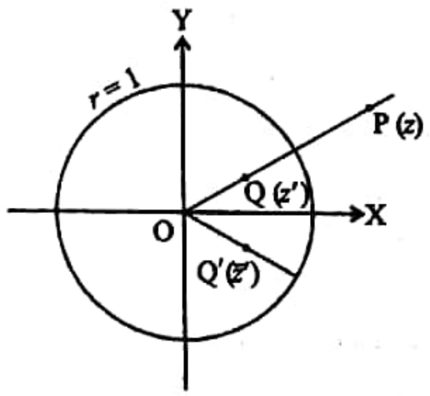
III. INVERSION AND REFLECTION $w = 1/z$.

In polar coordinates $w = \dfrac{1}{z}$ becomes $\rho e^{i\phi} = \dfrac{1}{r} e^{-i\theta}$ which gives $\rho = \dfrac{1}{r}$ and $\phi = -\theta$.

Here $\rho = \dfrac{1}{r}$ **is the inversion and** $\phi = -\theta$ **is the reflection.**

Consider a unit circle in the z-plane with centre at the origin. Here the points outside this unit circle map into points inside the unit circle and vice versa.

The unit circle clearly maps into a unit circle. This is a one to one mapping, *i.e.*, one point of z-plane corresponds with one point of w-plane, the exception being the origin which is a singular point. A small region near the origin inside the circle $r = 1/R$ maps into the exterior of the circle $\rho = R$. If we take R sufficiently large we can map any small neighbourhood of the origin into the exterior of the circle $\rho = R$. We generally say this as "$z = 0$ maps into the point at infinity".



This transformation consists of two consecutive mappings $w = \bar{z}'$ and $z' = \dfrac{1}{r} e^{i\theta}$. If P and Q represent z and z' respectively and $r = 1$ is the unit circle, then by the mapping $z' = \dfrac{1}{r} e^{i\theta}$, the point P is mapped to Q so that $OQ = \dfrac{1}{OP} = \dfrac{1}{r}$. Point Q (z) is called *inverse of P (z)*, and the mapping is called ***inversion of z w.r.t. the unit circle*** $r = 1$.

Next, the transformation $w = \bar{z}'$ maps Q (z') to Q′ ($\bar{z}'$) which is the *reflection* of Q about X-axis.

BILINEAR TRANSFORMATION (OR FRACTIONAL MAPPING)

The transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, is called *bilinear transformation.* ...(1)

It is also known as ***Mobius*** **or** ***linear fractional transformation.***

Inverse of (1) is $z = \dfrac{b - dw}{cw - a}$ which is also a bilinear transformation. ...(2)

Equation (2) on simplification, gives $cwz + dw - az - b = 0$ ...(3)

which is linear in both z and w, that is why the term *BILINEAR*. Here, if $c = 0$ equation (3) reduces to a linear transformation. On the other hand, if $c \neq 0$ then (3) can be written as the combination of the following three consecutive mappings,

$$z_1 = cz + d, \quad w_1 = \frac{1}{z_1}, \quad w = \frac{a}{c} + \frac{bc - ad}{c} w_1 \qquad ...(4)$$

On differentiating (1) we obtain $\dfrac{dw}{dz} = \dfrac{ad - bc}{(cz+d)^2}$.

If $ad - bc \neq 0$ then $\dfrac{dw}{dz} \neq 0$ for any z and hence bilinear transformation is **conformal** for all z.

However, if $ad - bc = 0$ then $\dfrac{dw}{dz} = 0$ for all z and the bilinear transformation (1) is not conformal.

A very important and equally useful property of bilinear mapping is, that it maps circles into circles and straight lines into straight lines. Let us witness it.

Consider the equation $\left|\dfrac{z-p}{z-q}\right| = \text{K}$ ...(5)

It is the locus of a point whose distances from the points 'p' and 'q' are in the ratio K : 1.

In particular, let 'p' be chosen as the origin (0, 0) and 'q' as (1, 0), then (5) becomes

$$\left|\frac{x+iy}{(x-1)+i(y-0)}\right| = \text{K}$$

or $\sqrt{x^2+y^2} = \text{K}\sqrt{(x-1)^2+y^2}$ or $x^2+y^2 = \text{K}^2(x^2+y^2-2x+1)$. ...(6)

For $\text{K} \neq 1$ (6) represents a circle, but for $\text{K} = 1$ it represents a straight line. Therefore (5) represents a circle for $\text{K} \neq 1$ and a straight line when $\text{K} = 1$.

Let us find the transformation of (5) in z-plane into w-plane. Using (2), equation (5) becomes

$$\left|\frac{\dfrac{b-dw}{cw-a} - p}{\dfrac{b-dw}{cw-a} - q}\right| = \text{K} \quad \text{or} \quad \left|\frac{b - dw - pcw + ap}{b - dw - qcw + aq}\right| = \text{K}$$

or $$\left|\frac{b + ap - w(d+pc)}{b + aq - w(d+qc)}\right| = \text{K} \quad \text{or} \quad \left|\frac{d+pc}{d+qc}\right| \left|\frac{w - \dfrac{b+ap}{d+pc}}{w - \dfrac{b+aq}{d+qc}}\right| = \text{K}$$

or $$\left|\frac{w - \dfrac{b+ap}{d+pc}}{w - \dfrac{b+aq}{d+aq}}\right| = \left|\frac{d+qc}{d+pc}\right| \text{K} = \text{K}' \qquad ...(7)$$

Since (7) is of the form (5) it also represents circles (or straight lines) in the w-plane according as $K' \neq 1$ (or $K' = 1$). Therefore the bilinear transformation (1) maps circles into circles and straight lines into straight lines.

Another aspect of it is that to every point in the z-plane (except $z = -d/c$) there corresponds a unique point in the w-plane. Similarly, to every point in the w-plane (except the point $w = a/c$) there corresponds a unique point in the z-plane. Thus, the bilinear transformation defines a one-one correspondence (except the point at ∞ in both the planes) between the z-plane and the w-plane.

Fixed Point or Invarient Point:

A point $z = z_0$ is called a fixed point or invariant point of the bilinear mapping if $w(z_0) = z_0$.

$\therefore$ By definition $w = \dfrac{az+b}{cz+d} = z \quad \Rightarrow \quad cz^2 - (a-d)z - b = 0.$

It is quadratic in z hence must have two roots if $c \neq 0$. Thus, such transformation will have at the most two fixed points.

In the transformation $w = \dfrac{az+b}{cz+d}$, it appears that there are four unknowns while the fact is that it has *three* unknowns, since we can write it as $w = (a'z + 1)/(c'z + d')$.

DETERMINING THE BILINEAR MAPPING

If we are given that the points z_1, z_2, z_3 in z plane, map into the points w_1, w_2, w_3 in the w-plane respectively, then the bilinear mapping $w = \dfrac{az+b}{cz+d}$ can be determined as follows.

First let us define the **cross ratio of four variables** t_1, t_2, t_3 **and** t_4 **as** $\dfrac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$. **It is denoted by** (t_1, t_2, t_3, t_4).

We have $w_i = \dfrac{az_i+b}{cz_i+d} \quad (i = 1, 2, 3).$

$$\therefore \quad w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)} \quad \text{and} \quad w - w_3 = \frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}.$$

$$\text{Hence} \quad w_2 - w_1 = \frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)} \quad \text{and} \quad w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$\Rightarrow \quad \frac{w-w_1}{w-w_3} = \frac{(z-z_1)(cz_3+d)}{(z-z_3)(cz_1+d)} \quad \text{and} \quad \frac{w_2-w_1}{w_2-w_3} = \frac{(z_2-z_1)(cz_3+d)}{(z_2-z_3)(cz_1+d)}.$$

It follows that
$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

which gives the required bilinear transformation. Observe here that the L.H.S. is the cross ratio of w, w_1, w_2, w_3 and RHS is the cross ratio of z, z_1, z_2, z_3. The above equation can be used even when one of the points in the z-plane or w-plane is at infinity.

The following examples will illustrate the determination of the bilinear transformation.

EXAMPLE 17.27. Show that the mapping $w = e^z$ is conformal in the whole of the z-plane.

SOLUTION: Let $z = x + iy$ and $w = \rho e^{i\phi}$ then $w = e^z$, gives

$$\rho e^{i\phi} = e^{x+iy} = e^x e^{iy} \Rightarrow \rho = e^x \text{ and } \phi = y.$$

Consider the mapping of the straight line $x = a$ in z-plane onto the w-plane which gives $\rho = e^a$ = constant, which is a circle in the w-plane in the anticlockwise direction. Similarly, the straight line $y = b$ is mapped into $\phi = b$ which is a radius vector in the w-plane.

The angle between the lines $x = a$ and $y = b$ in the z-plane is a right angle. The corresponding angle in the w-plane between the circle ρ = constant and the radius vector $\phi = b$ is also a right angle which establishes that the mapping $w = e^z$ is conformal. **Hence Proved.**

EXAMPLE 17.28. Show that the relation $w = \dfrac{5-4z}{4z-2}$ transforms the circle $|z| = 1$ into the circle of radius unity in w-plane and find the centre of the circle.

[GGSIPU II Sem End Term 2011]

SOLUTION: Given: $w = \dfrac{5-4z}{4z-2} \Rightarrow z = \dfrac{2w+5}{4w+4}$.

Now $|z| = 1$ gives $|2w+5| = |4w+4|$. If $w = u+iv$, then we have

$$|2u+5+2iv| = |4u+4+4iv| \text{ or } (2u+5)^2+4v^2 = 16(u+1)^2+16v^2$$

or $$12u^2+12v^2+12u = 9 \text{ or } \left(u+\frac{1}{2}\right)^2+v^2 = 1.$$

which is a circle with radius unity and centre $\left(-\dfrac{1}{2}, 0\right)$. **Ans.**

EXAMPLE 17.29. Show that the mapping $w = \dfrac{i-z}{i+z}$ maps the real axis of the z-plane onto the circle $|w| = 1$, and the half-plane $y > 0$ onto the interior of the unit circle $|w| < 1$ in the w-plane. [GGSIPU II Sem End Term 2006]

SOLUTION: $$w = \frac{i-z}{i+z} \quad \therefore \quad |w| = \frac{|i-z|}{|i+z|} = \frac{|-x-i(y-1)|}{|x+i(y+1)|} = \sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}}$$

$\therefore$ $|w| = 1$ gives $\dfrac{x^2+(y-1)^2}{x^2+(y+1)^2} = 1$ or $2y = 0$ which gives the real axis in z-plane.

Next, $|w| < 1$ gives $\dfrac{x^2+(1-y)^2}{x^2+(1+y)^2} < 1$ which gives $y > 0$.

Thus, the half-plane $y > 0$ corresponds to the interior of the circle $|\omega| = 1$.

Hence Proved.

EXAMPLE 17.30. Obtain the condition under which the mapping $w = \dfrac{az+b}{cz+d}$ maps a straight line of z-plane into a unit circle of w-plane.

SOLUTION: We know that equation of a straight line in z-plane is of the form $\left|\dfrac{z-p}{z-q}\right| = 1$ and of unit circle in w-plane is $|w| = 1$.

Now $\quad w = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\left(\dfrac{z+b/a}{z+d/c}\right)$ hence $|w| = \left|\dfrac{a}{c}\right|\left|\dfrac{z+b/a}{z+d/c}\right|$.

Here $\left|\dfrac{z+b/a}{z+d/c}\right| = 1$ is the straight line is z-plane.

Thus $|w| = 1$ only when $\left|\dfrac{a}{c}\right| = 1$ or $|a| = |c|$

which is the required condition. **Ans.**

EXAMPLE 17.31. Discuss the transformation $w = z + \dfrac{1}{z}$ and show that it maps the circle $|z| = a$ into an ellipse. Discuss the case when $a = 1$. Also, show that the radius vector $\arg(z) = \alpha$ $(\alpha < \pi/4)$ is mapped to a branch of a hyperbola whose eccentricity is $\sec\alpha$.

SOLUTION: This special transformation $w = z + \dfrac{1}{z}$ was introduced by **Joukowski.**

Here $$\frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2}$$

therefore, it is conformal everywhere except at the points $z = 1$ and $z = -1$, which corresponds to the points $w = 2$ and $w = -2$ of the w-plane. Let us change to polar coordinates, then

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow \quad u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta. \quad \text{...(1)}$$

Eliminating θ in the above two relations of (1), we get $$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \quad \text{...(2)}$$

and, eliminating r in (1), we get $$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \quad \text{...(3)}$$

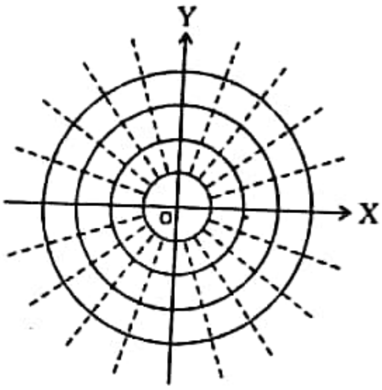
From (2) it follows that the circles r = constant of z-plane are mapped into a family of ellipses in the w-plane. These ellipses are confocal since

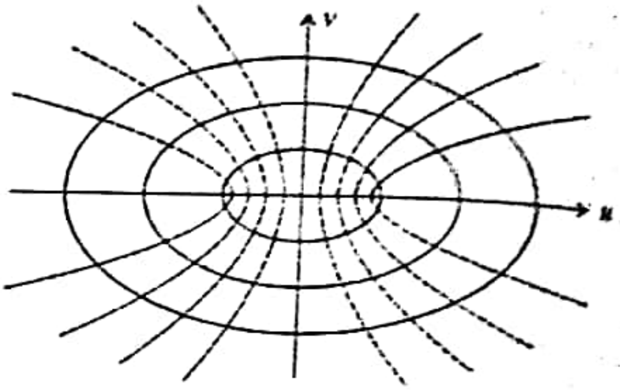
$$\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4, \quad \text{a constant.}$$

In particular, the unit circle $(r = 1)$ in z-plane gives from (1)

$$u = 2\cos\theta, \; v = 0 \quad \Rightarrow |u| \le 2, \; v = 0$$

that is, the unit circle flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w-plane.





Next, from (3) it follows that the radial lines θ = constant of the z-plane, map into a family of hyperbolas which are all confocal as shown in the above figure.

EXAMPLE 17.32. Find the bilinear transformation which maps the points $i, -i, 1$ of z-plane into $0, 1, \infty$ of the w-plane respectively.

[GGSIPU II Sem End Term 2007 Reapear; II Term 2013]

SOLUTION: We know that the bilinear transformation under which points z_1, z_2, z_3 in z-plane are mapped to the points w_1, w_2, w_3, in the w-plane, is give by

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \qquad \text{... (1)}$$

In the given problem $z_1 = i$, $z_2 = -i$, $z_3 = 1$ and $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$.
On the L.H.S. of (1) dividing the numerator and denominator by w_3, we get

$$\frac{(w-w_1)}{\left(\frac{w}{w_3}-1\right)}\cdot\frac{\left(\frac{w_2}{w_3}-1\right)}{(w_2-w_1)} = \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}$$

or
$$\frac{w-0}{0-1}\cdot\frac{0-1}{1-0} = \frac{z-i}{z-1}\cdot\frac{-i-1}{-i-i} \qquad \left(\text{since } \frac{1}{w_3}=0\right)$$

or
$$w = \frac{1+i}{2i}\left(\frac{z-i}{z-1}\right) = \frac{1-i}{2}\left(\frac{z-i}{z-1}\right)$$

which is the required bilinear transformation.

Ans.

EXAMPLE 17.33. (a) Determine the fractional transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = 1, w_2 = -i, w_3 = -1$ respectively. [GGSIPU II Sem End Term 2006]

(b) Determine the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ into $w_1 = i,\ w_2 = -1,\ w_3 = -i$.

SOLUTION: (a) The fractional transformation which maps the points z_1, z_2, z_3 in z-plane to w_1, w_2, w_3, in w-plane, is

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} = \frac{z-z_1}{z/z_3-1}\cdot\frac{z_2/z_3-1}{z_2-z_1}$$

Feeding $z_1 = 0$, $z_2 = 1$, $z_3 - \infty$ and $w_1 = 1$, $w_2 = -i$, $w_3 = -1$, we get

$$\frac{w-1}{w+1}\cdot\frac{-i+1}{-i-1} = \frac{z-0}{0-1}\cdot\frac{0-1}{1-0}$$

or $$\frac{w-1}{w+1}\,i = z \quad \text{or} \quad w = \frac{i+z}{i-z}.$$

Thus $w = \dfrac{i+z}{i-z}$ is the required fractional transformation. **Ans.**

(b) We cannot directly apply the formula

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)},$$ since in this case $z_3 = \infty$.

We can rewrite it as $$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{(z-z_1)(z_2/z_3-1)}{(z/z_3-1)(z_2-z_1)}$$

Now, substituting the given points, we get

$$\left(\frac{w-i}{w+i}\right)\cdot\left(\frac{-1+i}{-1-i}\right) = \frac{(z-0)(0-1)}{(0-1)(1-0)} \quad \text{or} \quad \frac{w-i}{w+i}(-i) = z$$

Solving for w we get the required transformation as

$$w = -i\,\frac{z-i}{z+i} = -\left(\frac{iz+1}{z+i}\right). \qquad \textbf{Ans.}$$

EXAMPLE 17.34. (a) Find the bilinear transformation which maps $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence find the image of $|z| < 1$. Also, find the invariant points of this transformation. [*GGSIPU II Sem II Term 2011*]

(b) Find the bilinear transformation which maps the points $z = 0, -1$ and i onto the point $\omega = i, 0$ and ∞ in ω-plane. Also, find the image of the unit circle $|z| = 1$. [*GGSIPU II Sem II Term 2014*]

SOLUTION: (a) The points z_1, z_2, z_3 in z-plane make images w_1, w_2, w_3 in the uv-plane, Bilinear mapping is $$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}.$$

$$\Rightarrow \quad \frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \quad \text{or} \quad \frac{w-i}{w+i} = \frac{i(z-1)}{z+1} = \frac{iz-i}{z+1}.$$

Applying componendo and dividendo here, we get

$$\frac{2w}{2i} = \frac{z(1+i)+1-i}{z(1-i)+1+i} \quad \text{or} \quad w = \frac{1+\left(\frac{1+i}{1-i}\right)z}{1-\left(\frac{1+i}{1-i}\right)z} = \frac{1+iz}{1-iz} \qquad \text{...(1)} \quad \textbf{Ans.}$$

To find the image of $|z|<1$, we write z in terms of w from (1) as $z = \dfrac{w-1}{i(w+1)}$

$\therefore \quad |z| = \dfrac{|w-1|}{|w+1|}$. Then $|z|<1$ gives $|w-1|<|w+1|$

Thus, the distance of w from 1 is less than distance of w from -1, that is, $\text{Re}(w) > 0$.

For invariant transformation, $w = z$ hence $z = \dfrac{1+iz}{1-iz}$

or $-iz^2+z-iz-1=0$ or $z^2+(1+i)z-i=0.$

$\therefore \quad z=\dfrac{-(1+i)\pm\sqrt{(1+i)^2+4i}}{2}=-\dfrac{1}{2}(1+i)\pm\dfrac{1}{2}\sqrt{3(2i)}=-\dfrac{1}{2}(1+i)\pm\dfrac{\sqrt{3}}{2}(1+i)=\dfrac{1+i}{2}(-1\pm\sqrt{3})$. **Ans.**

(b) We know that the bilinear mapping

$$\frac{\omega-\omega_1}{\omega-\omega_3}\cdot\frac{\omega_2-\omega_3}{\omega_2-\omega_1}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

maps the points z_1, z_2, z_3 in the z-plane onto the points w_1, w_2, w_3 in ω-plane.

Here $z_1=0, \quad z_2=-1, \quad z_3=i$ and $\omega_1=i, \omega_2=0, \omega_3=\infty$.

Since $\omega_3=\infty$ the above mapping can be written as

$$\frac{\omega-\omega_1}{\omega/\omega_3-1}\cdot\frac{\omega_2/\omega_3-1}{\omega_2-\omega_1}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{or} \quad \frac{\omega-i}{0-1}\cdot\frac{0-1}{0-i}=\frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

or $$(\omega-i)\,i=\frac{z(1+i)}{z-i} \quad \text{or} \quad \frac{\omega}{1}=\frac{z+1}{z-i} \quad \textbf{Ans.}$$

Applying componendo and dividendo here, we get

$$\frac{\omega+1}{\omega-1}=\frac{2z+1-i}{1+i} \quad \text{or} \quad \frac{2z}{1+i}=\frac{\omega+1}{\omega-1}+i=\frac{(1+i)\,\omega+1-i}{\omega-1}$$

or $$2z=\frac{(1+i)^2\omega+2}{\omega-1} \quad \text{or} \quad z=\frac{i\omega+1}{\omega-1}.$$

To find the image of $|z|=1$ under the above mapping, we have

$$|i\omega+1|=|\omega-1| \quad \text{or} \quad |i(u+iv)+1|=|u+iv-1| \quad \text{or} \quad |-v+1+iu|=|u-1+iv|$$

$$\Rightarrow \quad u^2+(v-1)^2=(u-1)^2+v^2 \quad \text{or} \quad u+v=0$$

which is a straight line through the origin in the ω-plane. **Ans.**

EXAMPLE 17.35. Find the transformation which maps the points $z=1, -i, -1$ to the points $w=i, 0, -i$ respectively. Show that this transformation maps the region outside the circle $|z|=1$ into the half space $R(w)\geq 0$.

[GGSIPU II Sem II Term 2006; II Term 2003]

SOLUTION: The bilinear transformation from z-plane to w-plane maps the points $z_1=1, \quad z_2=-i, \quad z_3=-1$ in z-plane to the points $w_1=i, \quad w_2=0, \quad w_3=-i$ in the w-plane.

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \quad \text{or} \quad \frac{w-i}{w+i}\cdot\frac{0+i}{0-i}=\frac{z-1}{z+1}\cdot\frac{-i+1}{-i-1}$$

or $$\frac{w-i}{w+i}=\frac{z-1}{z+1}\,\frac{1-i}{i+1}=-i\,\frac{z-1}{z+1}=\frac{i-iz}{z+1}$$

Applying componendo and dividendo here, we get

$$\frac{2w}{2i}=\frac{z(1-i)+(1+i)}{z(1+i)+(1-i)}$$

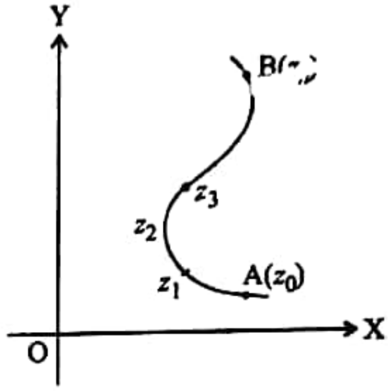
$$\therefore \quad \frac{w}{i}=\frac{1-i}{1+i}\cdot\frac{z+\dfrac{1+i}{1-i}}{z+\dfrac{1-i}{1+i}}=(-i)\,\frac{z+i}{z-i}.$$ Thus, $w=\dfrac{z+i}{z-i}$ is the required bilinear mapping. **Ans.**

CHAPTER

18

Complex Integration—Taylor and Laurent's Series, Residue Theorem, Evaluation of Real Integrals

Complex Integration, Line Integrals, Cauchy's Integration Theorem, Cauchy's Integration Formula Zeros and Singularity, Taylor's and Laurent's Series, Residue and Residue Theorem, Evaluation of Real Integrals: Integration around Unit Circle, around Semicircle and around Rectangles.



Let $f(z)$ be a continuous function of two independent variables x and y, its integral can be defined only as a line integral. Let C be any continuous curve from A (z_0) to B (z_n) in the z-plane. We divide C into n parts by the points $z_1, z_2, \ldots, z_{n-1}$ as shown in the adjoining figure and let us write $\delta z_i = z_i - z_{i-1}$, $i = 1, 2, \ldots, n$ then

$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \delta z_i \, f(z_i)$ where each δz_i approaches to 0, if it exists, is called the line integral of $f(z)$, taken along the curve C and is denoted by $\int_C f(z)\,dz$.

If $f(z) = u(x, y) + i\,v(x, y)$ where $z = x + iy$, we have

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \qquad \ldots(1)$$

which means that the evaluation of a line integral of complex function can be reduced to the evaluation of two line integrals of real functions.

Further, if the curve C has parametric equation $x = \phi(t)$, $y = \psi(t)$, then $u(x, y) + i\,v(x, y)$ and $x + iy$ both can be expressed in terms of t, and (1) becomes

$$\int_C f(z)\,dz = \int_a^b f\{z(t)\}\frac{dz}{dt}\,dt \quad \text{where} \quad t = a \text{ and } t = b \text{ are the extremities of the curve C.}$$

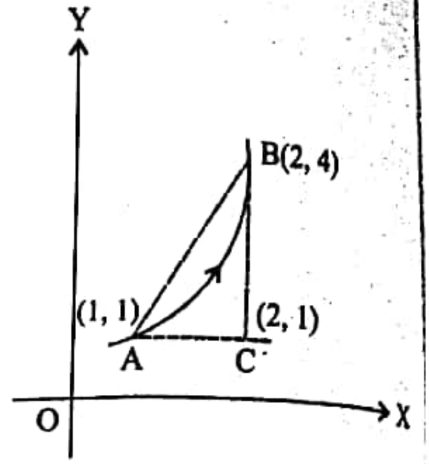
EXAMPLE 18.1. Evaluate $\int_A^B z^2\,dz$ where $A = (1, 1)$, $B = (2, 4)$, along

(*i*) the line segment AC parallel to X-axis and then along CB parallel to Y-axis.

(*ii*) the straight line AB joining the two points A and B.

(*iii*) The curve $C : y = x^2$.

SOLUTION: (*i*) For any point on AC we have $y = 1$, $z = x + i$, $dy = 0$ and for any point on CB we have $x = 2$, $z = 2 + iy$, $dx = 0$ (see the figure).



$$\int_A^B z^2\, dz = \int_{AC} z^2\, dz + \int_{CB} z^2\, dz$$

$$= \int_1^2 (x+i)^2\, dx + \int_1^4 (2+iy)^2\, i\,dy$$

$$= \frac{1}{3}\Big[(x+i)^3\Big]_1^2 + \frac{i}{3i}\Big[(2+iy)^3\Big]_1^4$$

$$= \frac{1}{3}\,[(2+i)^3-(1+i)^3] + \frac{1}{3}\,[(2+4i)^3-(2+i)^3]$$

$$= \frac{1}{3}\,[(2+4i)^3-(1+i)^3] = \frac{1}{3}(1+3i)\,[(2+4i)^2+(1+i)^2+(2+4i)\cdot(1+i)]$$

$$= \frac{1}{3}(1+3i)(-14+24i) = \frac{-86}{3} - 6i. \qquad \textbf{Ans.}$$

(*ii*) The equation of the line AB is $y = 3x - 2$.

For any point on the line AB, we have $z = x + i\,(3x - 2)$ and $dz = (1 + 3i)\, dx$,

hence $$\int_{AB} z^2\, dz = \int_1^2 [(1+3i)\,x-2i]^2\,(1+3i)\,dx = \frac{1}{3}\Big[\{(1+3i)\,x-2i\}^3\Big]_1^2$$

$$= \frac{1}{3}\,[(2+4i)^3-(1+i)^3] = \frac{-86}{3} - 6i. \qquad \textbf{Ans.}$$

(*iii*) $\int_C z^2\, dz = \int_C (x+iy)^2\,(dx+i\,dy)$. The parametric equation of the curve C is $x = t$, $y = t^2$ hence from A to B, t varies from 1 to 2.

$$\therefore \quad \int_C z^2\, dz = \int_1^2 (t+it^2)\,(dt+2it\,dt) = \int_1^2 (t^2-t^4+2it^3)(1+2it)\,dt$$

$$= \int_1^2 [t^2-5t^4+i\,(4t^3-2t^5)]\,dt = \left[\frac{t^3}{3} - t^5 + i\left(t^4 - \frac{t^6}{3}\right)\right]_1^2 = \frac{-86}{3} - 6i. \qquad \textbf{Ans.}$$

Note here that the values of the integral in (*i*), (*ii*) and (*iii*) are all equal to the difference of $\frac{1}{3}(x+iy)^3$ evaluated at two ends and does not depend on the path traversed. This is so because $f(z)$ is analytic (as will be clear in the next article). In such cases the integral can be directly evaluated by the formulae for integration of the function of a single variable. However, if $f(z)$ is not analytic (as in the case of next example) the integral will depend on the path traversed.

EXAMPLE 18.2. If $f(z) = x^2 + i\,xy$ evaluate $\int_A^B f(z)\, dz$ where A = (1, 1) and B (2, 4), along

(*i*) the straight line AB

(*ii*) the curve $C: x = t, y = t^2$.

SOLUTION: (*i*) For any point on the line AB, $z = x + i\,(3x-2)$ and $f(z) = x^2 + i\,x\,(3x-2)$ since the equation of line *AB* is $y = 3x - 2$. (see figure of the last example).

$$\therefore \quad \int_{AB} f(z)\, dz = \int_{x=1}^{x=2} [x^2 + i\,x\,(3x-2)]\; d\,\{x+i\,(3x-2)\} = \int_1^2 [(1+3i)\,x^2 - 2\,i\,x](1+3i)\,dx.$$

$$= (1+3i)\left[(1+3i)\frac{x^3}{3} - ix^2\right]_1^2 = (1+3i)\left[\frac{7}{3}(1+3i) - 3i\right]$$

$$= (1+3i)\left[\frac{7}{3} + 4i\right] = -\frac{29}{3} + 11i \qquad \textbf{Ans.}$$

(ii) $$\int_C f(z)\,dz = \int_C (x^2 + ixy)\,d(x+iy) = \int_{t=1}^{2} (t^2 + it^3)\,d(t + it^2)$$

$$= \int_1^2 (t^2 + it^3)(1 + 2it)\,dt = \int_1^2 (t^2 - 2t^4 + 3it^3)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{2t^5}{5} + \frac{3it^4}{4}\right]_1^2 = -\frac{151}{15} + \frac{45}{4}i.$$

The values of $\int_{AB} f(z)\,dz$ in (i) and (ii) for two different contours are not equal since $f(z)$ in this example is not analytic. **Ans.**

EXAMPLE 18.3. Evaluate $\int_C (z-a)^n\,dz$ where C is the circle with centre 'a' and radius r.

Also discuss the case when $n = -1$, that is, evaluate $\int_C \frac{dz}{z-a}$.

[GGSIPU II Sem End Term 2009]

SOLUTION: The equation of the circle C is $|z - a| = r$ or $z - a = re^{i\theta}$, $0 \le \theta \le 2\pi$. Here $dz = ire^{i\theta}\,d\theta$, hence for $n \ne -1$, we have

$$\int_C (z-a)^n\,dz = \int_0^{2\pi} r^n e^{ni\theta}\, ire^{i\theta}\,d\theta = r^{n+1}\int_0^{2\pi} e^{(n+1)i\theta} i\,d\theta = i\frac{r^{n+1}}{(n+1)}\left[e^{(n+1)i\theta}\right]_0^{2\pi} = 0. \qquad \textbf{Ans.}$$

However, when $n = -1$

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} = 2\pi i. \qquad \textbf{Ans.}$$

EXAMPLE 18.4. Evaluate the integral $I = \int_C \operatorname{Re}(z^2)\,dz$ from $z = 0$ to $2 + 4i$ along the parabola $y = x^2$.

[GGSIPU II Sem End Term 2011]

SOLUTION: $I = \int_C \operatorname{Re}(z^2)\,dz = \int_C (x^2 - y^2)(dx + idy)$ where $C : y = x^2$ from $(0, 0)$ to $(2, 4)$.

$$\therefore\quad I = \int_0^2 (x^2 - x^4)(dx + 2ix\,dx) = \int_0^2 (x^2 - x^4)(1 + 2ix)\,dx = \int_0^2 (x^2 + 2ix^3 - x^4 - 2ix^5)\,dx$$

$$= \left[\left(\frac{x^3}{3} - \frac{x^5}{5}\right) + 2i\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\right]_0^2 = \frac{8}{3} - \frac{32}{5} + i\left(\frac{16}{2} - \frac{64}{3}\right) = \frac{-56}{15} - \frac{40i}{3}. \qquad \textbf{Ans.}$$

EXAMPLE 18.5. Evaluate $\int_0^{1+i}(x^2-iy)dz$ along the path (i) $y=x$ (ii) $y=x^2$.

[GGSIPU II Sem End Term 2007; End Term 2006 Reappear]

SOLUTION: Given integral $I=\int_0^{1+i}(x^2-iy)dz = \int_0^{1+i}(x^2-iy)(dx+i\,dy)$.

(i) Along the path $y=x$, $I=\int_0^1(x^2-ix)(dx+i\,dx) = \int_0^1(x^2-ix)(1+i)\,dx$

$$=(1+i)\left(\frac{1}{3}-\frac{i}{2}\right) = \frac{5}{6}-\frac{i}{6} \quad \textbf{Ans.}$$

(ii) Along the path $y=x^2$, $I=\int_0^1(x^2-ix^2)(dx+2ix\,dx) = \int_0^1(1-i)x^2(1+2i\,x)\,dx$

$$=(1-i)\left[\frac{x^3}{3}+2i\frac{x^4}{4}\right]_0^1 = (1-i)\left(\frac{1}{3}+\frac{i}{2}\right) = \frac{1}{6}(5+i). \quad \textbf{Ans.}$$

EXAMPLE 18.6. Evaluate $\int_0^{3+i} z^2\,dz$ along the parabola $y^2=x/3$.

[GGSIPU II Sem End Term 2010]

SOLUTION: $I=\int_0^{3+i} z^2\,dz = \int_{(0,0)}^{(3,1)}(x^2-y^2+2ixy)(dx+idy)$. The curve is $x=3y^2$, hence

$$I=\int_0^1(9y^4-y^2+6iy^3)(6y+i)\,dy = \int_0^1(54y^5-6y^3+36iy^4+9iy^4-iy^2-6y^3)\,dy$$

$$=9-3+9i-\frac{i}{3} = 6+\frac{26i}{3} \quad \textbf{Ans.}$$

EXAMPLE 18.7. Evaluate the integral $\oint_C \bar{z}\,dz$ along the right half of the circle $C:|z|=2$.

[GGSIPU II Sem End Term 2012]

SOLUTION: For the right half of the circle $C:|z|=2$ we have $z=2e^{i\theta}$, $-\pi/2\le\theta\le\pi/2$

$\therefore$ $\bar{z}=2e^{-i\theta}$ and $dz=2i\,e^{i\theta}\,d\theta$.

Therefore $$\int_C \bar{z}\,dz = \int_{-\pi/2}^{\pi/2} 2e^{-i\theta}\cdot 2ie^{i\theta}\,d\theta = 4i\int_{-\pi/2}^{\pi/2} d\theta = 4\pi i. \quad \textbf{Ans.}$$

THE FUNDAMENTAL THEOREM OF CONTOUR INTEGRATION

"Let a continuous function $f(z)$ be the derivative of an analytic function $F(z)$. Let $f(z)$ and $F(z)$ both have domain D and C be a contour in D from $z=z_1$ to $z=z_2$, then $\int_C f(z)\,dz = F(z_2)-F(z_1)$."

PROOF: Let $f(z) = u(x, y) + iv(x, y)$ and $F(z) = U(x, y) + iV(x, y)$ and let C be given by $z(t) = x(t) + iy(t)$, $a \le t \le b$ and $z(a) = z_1$, $z(b) = z_2$.

Since $f(z) = F'(z)$, we can write $f = u + iv = F' = (U + iV)' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}$.

using Cauchy-Riemann equations, for F (z) is analytic.

$$\Rightarrow \quad u = \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad v = \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

Therefore
$$\int_C f(z)\,dz = \int_a^b (u + iv)\,d(x + iy) = \int_a^b (u + iv)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt$$
$$= \int_a^b \left[\left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}\right) + i\left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}\right)\right]dt \qquad \text{...(2)}$$

But $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}$ and $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$

$$\therefore \quad \int_C f(z)\,dz = \int_a^b \left(\frac{dU}{dt} + i\frac{\partial V}{\partial t}\right)dt = \int_a^b F'\,dt = F\{z(b)\} - F\{z(a)\} = F(z_2) - F(z_1).$$

A consequence of the above theorem is that if f is the continuous derivative of an analytic function F then for any closed contour C in the domain D, it follows that

$\int_C f(z)\,dz = 0$, since $z_2 = z_1$ as C is a closed contour.

Properties of Contour Integrals

(*i*) If the contour C is divided into two parts C_1 and C_2, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

(*ii*) If the sense of integration is reversed, the sign of the integral also changes, *i.e.*,

$$\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$$

(*iii*) $\int_C \{k_1 f_1(z) + k_2 f_2(z)\}\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$ where k_1, k_2 are constants.

(*iv*) If L is the length of a contour C and M the upper bound of $f(z)$, i.e., $|f(z)| \le M$ on the track of C, then $\left|\int_C f(z)\,dz\right| \le ML$

To show this property we write

$$|f(z_1)\,\delta z_1 + f(z_2)\,\delta z_2 + \dots + f(z_n)\,\delta z_n| \le |f(z_1)\,\delta z_1| + |f(z_2)\,\delta z_2| + \dots + |f(z_n)\,\delta z_n|$$
$$\le |f(z_1)|\,|\delta z_1| + |f(z_2)|\,|\delta z_2| + \dots + |f(z_n)|\,|\delta z_n|$$
$$\le M(|\delta z_1| + |\delta z_2| + \dots + |\delta z_n|) \le ML.$$

Taking the limit as $n \to \infty$ and $\Delta Z_r \to 0$ for all r we get $\left|\int_C f(z)\,dz\right| \le ML$.

EXERCISE 18A

1. Evaluate $\int_0^{1+i} (x - y + ix^2)\, dz$

   (*a*) along the straight line from (0, 0) to (1, 1). [*GGSIPU II Sem End Term 2005, 2010*]

   (*b*) over the path along the lines $y = 0$ and $x = 1$. [*GGSIPU II Sem End Term 2007*]

   (*c*) over the path along the lines $x = 0$ and $y = 1$.

2. Evaluate $\int_0^{2+i} (\bar{z})^2\, dz$ along

   (*i*) the line $y = x/2$

   (*ii*) the real axis to 2 and then vertically to $2 + i$.

3. Evaluate $\int_{1-i}^{2+i} (2z + 3)\, dz$ along

   (*i*) the path $x = t + 1$, $y = 2t^2 - 1$

   (*ii*) the straight line from $1 - i$ to $2 + i$.

4. Evaluate $\int_C |z|\, dz$ where C is the contour

   (*a*) the straight line from $z = -i$ to $z = i$

   (*b*) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.

   [*GGSIPU IInd Sem. End Term 2006; Sem, End Term 2006 Reappear*]

5. Evaluate $\oint_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

6. Evaluate $\int_{1-2i}^{3+i} (2z + 3)\, dz$

   (*a*) along the path $x = 2t + 1$, $y = 4t^2 - t - 2$, $0 \le t \le 1$.

   (*b*) along the straight line joining $1 - 2i$ and $3 + i$.

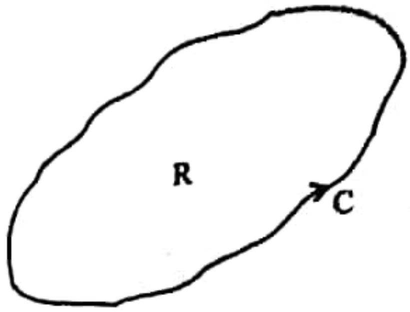
   (*c*) along the straight lines from $1 - 2i$ to $1 + i$ and then to $3 + i$.

7. Evaluate $\oint_C (z - z^2)\, dz$ where C is the upper half of the circle $|z| = 1$.

   [*GGSIPU II Sem End Term 2009*]

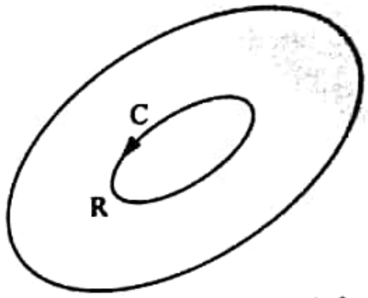
CONTOUR: A simple closed path is sometimes called **contour** and an integral over such a path is called **contour integral.**

SOME DEFINITIONS



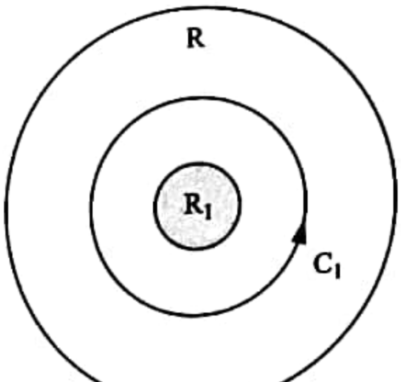
1. Simple Curve. A simple curve is one which does not cross itself. A simple curve encloses *a* **simply connected region** such that every *closed* curve lying in it can be contracted indefinitely without passing out of it as shown in the adjoining figure.

2. Multiple Curve. A multiple curve is one which crosses itself. As such, if a curve crosses itself once, it contains two regions R_1 and R_2 as shown in the adjoining figure. Same way, if multiple curve crosses itself twice it will contain more than two regions.

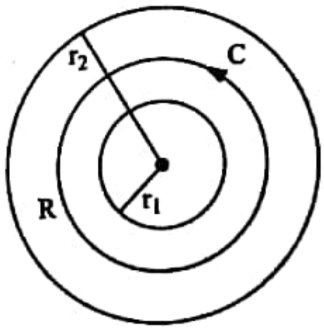


3. Simply Connected Region (or Domain): A region R is called simply connected if any simple closed curve lying in R can be shrunk to a point inside R without crossing it, as shown in adjoing figure.

4. Multiply Connected Region (or Domain): A domain which is not simply connected is called multiply connected. Thus a region R in which every closed curve lying in it, cannot be shrunk indefinitely without passing out of it, is multiple connected. A plain sheet of paper from which some part(s) are removed is an example of multiply connected region. In the adjoining figure, a region R_1 is removed from the region R. Now, if a closed curve C_1 is shrunk then it cannot be shrunk to a point in R since R_1 is not there. Another example of multiply connected domain is the annular region $r_1 < |z| < r_2$ as shown in the adjoining figure, because every closed curve C between the circles $|z| = r_1$ and $|z| = r_2$ cannot be shrunk to a point.

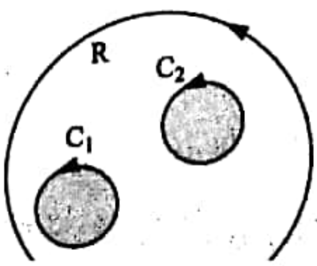


double connected region

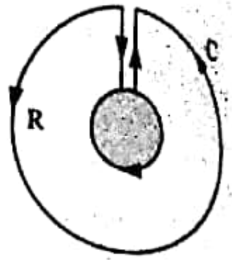


annular region double connected

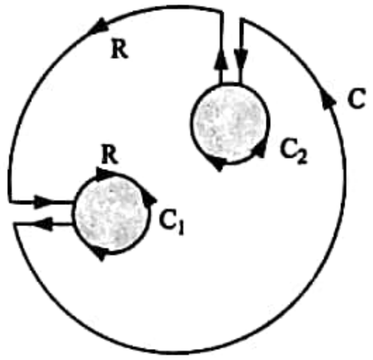
In otherwords, one can say that a multiply connected region has holes (or holes) in it. If it has one hole it is called **doubly connected** region whereas if it has two holes it is called triply connected region and so on.



Next, any multiply connected region can be converted into a simply connected region by introducing sufficient number of small cuts in the region. In case of doubly connected region we insert a small cut as shown in the adjoining figure and result is a simply connected region.



If the region is triply connected we introduce two cuts to convert it into a simply connected region as shown in the adjacent figure.



Idea can be extended to multiply connected region by introducing three or more cuts as required so as to convert it into a simply connected region. The concept is quite useful in evaluating integrals along a curve C which is bounding a multiply connected region.

CAUCHY'S INTEGRAL THEOREM

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a closed contour C, then $\int_C f(z)\,dz = 0.$

[*GGSIPU II Sem End Term 2007 Reappear; End Term 2006*]

PROOF: Let R be the region inside and on the closed contour C where C is described in the anti-clockwise direction and let $f(z) = u + iv$ be analytic throughout the region R in the xy-plane. Then, $dz = dx + i\,dy$ and

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \qquad ...(1)$$

Now let us recall Green's theorem discussed in the Chapter 'Vector Integration', According to it if P (x, y), Q (x, y) and the partial derivatives $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ are continuous over the region R bounded by a closed curve C, then $\displaystyle\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$ ...(2)

Since $f(z)$ is analytic, $f'(z)$ exists in R hence u, v and their partial derivatives are continuous in R. Using (2) in (1), yields

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \qquad ...(3)$$

Since $f(z)$ is analytic, C-R equations are satisfied, *i.e.*, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Therefore, from (3) we obtain $\int_C f(z)\,dz = 0.$

This proof was provided by Cauchy, in which $f'(z)$ is assumed to be continuous. Later on another proof of the theorem was given by the French Mathematician Goursat. His proof does not require the continuity of $f'(z)$ but that proof is beyond the scope of this book.

For instance, $\int_C e^z dz = 0$ and $\int_C \sin z dz = 0$ for any closed path C since e^z and $\sin z$ are analytic.

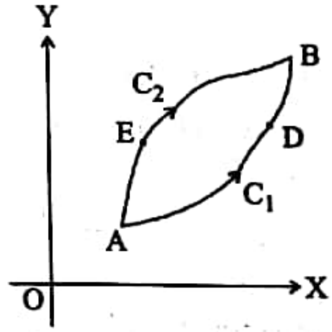
Also consider $\int_C \sec z dz$ if $C : |z| = 1$, since $\sec z$ is not analytic at $z = \pm \pi/2, \pm 3\pi/2$, but all these lie outside C, $\therefore \int_C \sec z\, dz = 0$.

Similarly, $\int_C \frac{dz}{z^2+4} = 0$, $C : |z| = 1$, since $z = \pm 2i$ both lie outside C.

COROLLARY. (Independence of Path) *If $f(z)$ is analytic in R and if two points A and B in R are joined by two different curves C_1 and C_2 lying wholly in R, then*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

Proof: Let C_1 be the curve ADB and C_2 the curve AEB, see the figure. Applying Cauchy's theorem to the closed contour ADBEA, we have



$$\int_{ADBEA} f(z)\, dz = 0$$

or $\int_{ADB} f(z)\, dz + \int_{BEA} f(z)\, dz = 0$ or $\int_{ADB} f(z)\, dz - \int_{AEB} f(z)\, dz = 0$

$\therefore \int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$, hence the corollary.

Thus the line integral of an analytic function is independent of the path and is only the function of the limits of integration only.

EXAMPLE 18.8. Evaluate $\int_C z^2\, dz$ using Cauchy's integral theorem, where $C : |z| = 1$.

SOLUTION: $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u + iv$. $\therefore u = x^2 - y^2$, $v = 2xy$.

Hence $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$ and $\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Since z^2 is analytic, C.R. equations hold good. Therefore by Cauchy's integral theorem, the value of the given integral is zero. **Ans.**

EXAMPLE 18.9. Evaluate $\int_c \frac{dz}{z^2}$ where $c : |z| = 1$.

SOLUTION: $c : |z| = 1 \quad \therefore \quad z = e^{i\theta}$, θ varying from 0 to 2π.

$$\therefore \quad \int_c \frac{dz}{z^2} = \int_0^{2\pi} \frac{i\, e^{i\theta}\, d\theta}{e^{2i\theta}} = i\int_0^{2\pi} e^{-i\theta}\, d\theta = -\left[e^{-i\theta}\right]_0^{2\pi} = 1 - e^{-2\pi i} = 0. \qquad \textbf{Ans.}$$

This is not due to cauchy's itegral theorem as $\frac{1}{z^2}$ is not analytic at $z = 0$.

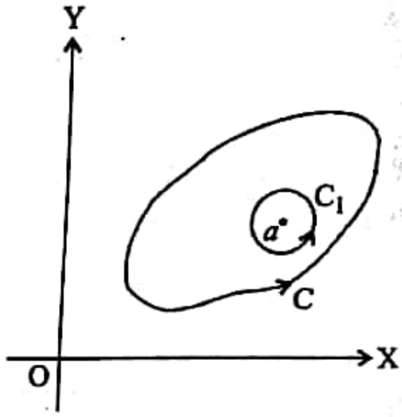
Actually, analyticity of $f(z)$ is sufficient but not necessary for $\int_c f(z)\, dz$ to be zero.

CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz.$$

[*GGSIPU II Sem II Term 2003; II Term 2006; II Term 2013*]



Since $f(z)$ is analytic inside and on the closed contour C, hence $\frac{f(z)}{z-a}$ also is analytic at every point inside C except at the point $z = a$. Draw a very small circle C_1 with centre at $z = a$ and radius $\in$ such that C_1 lies wholly inside C. Now $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1, Hence by the corollary of Cauchy's integral theorem, we have $\int_C \frac{f(z)}{z-a}\, dz = \int_{C_1} \frac{f(z)}{z-a}\, dz$. ...(1)

For any point on C_1, we have $|z - a| = \in \quad \therefore \quad z - a = \in e^{i\theta}$ so that $dz = i \in e^{i\theta}\, d\theta$,

hence $$\int_{C_1} \frac{f(z)}{z-a}\, dz = \int_{C_1} \frac{f(a + \in e^{i\theta})}{\in e^{i\theta}}\, i \in e^{i\theta}\, d\theta = i \int_0^{2\pi} f(a + \in e^{i\theta})\, d\theta. \qquad ...(2)$$

Now we take the limit when the circle C_1 shrinks to the point a so that $\in \to 0$, then the R.H.S. of (2) will tend to $i\int_0^{2\pi} f(a)\, d\theta = i f(a) \int_0^{2\pi} d\theta = 2\pi i f(a)$

Therefore, from (1), we have $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz \qquad ...(3)$$

which is known as Cauchy's integral formula.

COROLLARY. (DERIVATIVES OF ANALYTIC FUNCTION)

The utility of the Cauchy's integral formula can be extended by differentiating the above formula partially with respect to the parameter a. Therefore, differentiating (3) partially w.r.t. 'a' we get

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{(z-a)^2}.$$

Again differentiating (4) repeatedly, partially w.r.t. 'a', gives

$$f''(a) = \frac{1\cdot 2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^3}, \quad f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^4}, \text{ and so on.}$$

In general, $f^{(n)}(a) = \dfrac{n!}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$ $n = 0, 1, 2, 3, \ldots$

EXAMPLE 18.10. (*a*) Prove that $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i$ where $C : |z-a| = r$

[*GGSIPU II Sem End Term 2007*]

(*b*) Use Cauchy's integral formula to calculate $\displaystyle\int_C \frac{z^2+1}{z(2z+1)}\,dz$ where $C : |z| = 1$.

[*GGSIPU II Sem End Term 2011*]

SOLUTION: (*a*) Given integral is $\displaystyle\int_C \frac{dz}{z-a}$. The integrand has a singularity at $z = a$ which is inside the circle C with centre '*a*' and radius r.

Applying here the Cauchy's integral formula $\displaystyle\int_C \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a)$ as $f(z)\,(=1)$ is analytic inside and on C and '*a*' lies inside C, hence $\displaystyle\int_C \frac{1}{z-a}\,dz = 2\pi i\,[1]_{z=a} = 2\pi i.$ **Hence Proved.**

(*b*) Let $I = \displaystyle\int_C \frac{z^2+1}{z(2z+1)}\,dz = \int_C (z^2+1)\left[\frac{1}{z} - \frac{2}{2z+1}\right]dz = \int_C \frac{z^2+1}{z}\,dz - \int_C \frac{z^2+1}{z+1/2}\,dz$ where $C : |z| = 1$

By Cauchy's integral formula, since $z = 0$ and $z = -1/2$ both lie inside C, we have

$$I = 2\pi i\,(z^2+1)_{z=0} - 2\pi i\,(z^2+1)_{z=-1/2} = 2\pi i - 2\pi i\left(\frac{1}{4}+1\right) = \frac{-\pi i}{2}.$$ **Ans.**

EXAMPLE 18.11. Evaluate (*i*) $\displaystyle\oint_C \frac{e^{-z}}{z+1}\,dz$ where $C : |z| = 2$. (*ii*) $\displaystyle\int_C \frac{e^z}{z-2}\,dz$ where $C : |z| = 3$,

(*iii*) $\displaystyle\int_C \frac{e^z}{z-2}\,dz$ where $C : |z| = 1$.

[*GGSIPU II Sem End Term 2006 Reappear; End Term 2007 Reappear*]

SOLUTION: (*i*) In the given integral e^{-z} is analytic inside and on the circle $C : |z| = 2$, and $z = -1$ is inside C. Therefore by Cauchy's integral formula

$$\int_C \frac{e^{-z}}{z+1}\,dz = 2\pi i\,[e^{-z}]_{z=-1} = 2\pi i\, e.$$

(*ii*) $z = 2$ is the singularity lying in $C : |z| = 3$ and e^z is analytic inside and on C, therefore by Cauchy's integral formula

$$\int_C \frac{e^z}{z-2}\,dz = 2\pi i\left[e^z\right]_{z=2} = 2\pi i e^2.$$ **Ans.**

(*iii*) $z = 2$ lies outside the circle $C : |z| = 1$ and e^z is analytic inside and on C, hence $\displaystyle\oint_C \frac{e^z}{z-2}\,dz = 0$ by Cauchy's integral theorem. **Ans.**

EXAMPLE 18.12. (*a*) Evaluate $\int_C \frac{5z-2}{z^2-z}\,dz$ where $C : |z| = 2.$

(*b*) Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$ where (*i*) $C : |z-1| = 1.$ (*ii*) $C : |z|\ 2$

[*GGSIPU II Sem II Term 2013; End Term 2013*]

SOLUTION: (*a*) Here $\int_C \frac{5z-2}{z^2-z}\,dz = \int_C (5z-2)\left[\frac{1}{z-1}-\frac{1}{z}\right]dz = \int_C \frac{(5z-2)}{z-1}\,dz - \int_C \frac{5z-2}{z}\,dz.$

Since both the points $z = 1$ and $z = 0$ lie within the circle $C : |z| = 2,$ we have by Cauchy's integral formula

$$\int_C \frac{(5z-2)}{z-1}\,dz = 2\pi i\,[5z-2]_{z=1} = 6\pi i$$

and
$$\int_C \frac{5z-2}{z}\,dz = 2\pi i\,[5z-2]_{z=0} = -4\pi i$$

Therefore $$\int_C \frac{5z-2}{z^2-z}\,dz = 6\pi i - (-4\pi i) = 10\pi i$$ **Ans.**

(*b*) (*i*) The function $\frac{3z^2+z}{z^2-1}$ has singularities at $z^2-1=0,$ *i.e.*, at $z=1$ and $z=-1.$ The circle $C : |z-1| = 1,$ has centre at $z = 1$ and radius 1.

Point $z=-1$ lies outside C whereas $z = 1$ lies inside C, therefore, by Cauchy's integral formula, we have

$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{\left(\frac{3z^2+z}{z+1}\right)}{z-1}\,dz = 2\pi i\left[\frac{3z^2+z}{z+1}\right]_{z=1} = 4\pi i.$$ **Ans.**

(*ii*) $I = \int_C \frac{3z^2+z}{z^2-1}\,dz$ where $C: |z| = 2.$

The integrand has singularities at $z = \pm 1$ both lying inside $C.$

Since $\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right),$ we have

$$I = \int_C \frac{3z^2+z}{2(z-1)}\,dz - \int_C \frac{3z^2+z}{2(z+1)}\,dz$$

Using Cauchy's integral formula here, we get

$$I = \left[\frac{3z^2+z}{2}\right]_{z=1}(2\pi i) - \left[\frac{3z^2+z}{2}\right]_{z=-1}(2\pi i) = \pi i\,[4-2] = 2\pi i.$$ **Ans.**

EXAMPLE 18.13. (a) Evaluate $\int_C \left[\frac{2z}{(z-2)^2} + \frac{3(z-1)}{(z-2)^3}\right] dz$ where C is the circle $|z| = 3$.

(b) Evaluate $\int_c \frac{e^z\, dz}{z(1-z)^3}$ where C is (*i*) $|z| = \frac{1}{2}$, (*ii*) $|z-1| = \frac{1}{2}$.

SOLUTION: (*a*) The given integral can be written as $\int_C \frac{2z^2 - z - 3}{(z-2)^3} dz$ where $C : |z| = 3$.

Here we take $f(z) = 2z^2 - z - 3$ and 'a' $= 2$ which lies inside C.
Using the Cauchy's derived integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\, dz}{(z-a)^{n+1}} \quad \text{for } n = 2, \quad \text{we get}$$

$$f''(2) = \frac{2!}{2\pi i}\int_C \frac{2z^2 - z - 3}{(z-2)^3} dz \quad \text{Here} \quad f'(z) = 4z - 1 \quad \text{and} \quad f''(z) = 4 = f''(2).$$

Therefore, $\int_C \frac{2z^2 - z - 3}{(z-2)^3} dz = \pi i f''(2) = 4\pi i$ **Ans.**

(*b*) (*i*) Here $z = 0$ and $z = 1$ are the singularities of the integrand, of which only $z = 0$ lies inside $C : |z| = 1/2$.
Hence by Cauchys intergal formula, given integral equals

$$\int_C \frac{e^z/(1-z)^3}{z} dz = 2\pi i\left[\frac{e^z}{(1-z)^3}\right]_{z=0} = 2\pi i. \qquad \textbf{Ans.}$$

(*ii*) Out of the singularities $z = 0$ and $z = 1$ of the integrand, only $z = 1$ lies inside $C : |z-1| = \frac{1}{2}$. Hence by Cauchy's integral formula

$$\int_C \frac{e^z\, dz}{z(1-z)^3} = -\int_C \frac{e^z/z\, dz}{(z-1)^3} = \frac{-2\pi i}{2!} \operatorname*{Lt}_{z\to 1}\left[\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)\right]$$

$$= -\pi i \lim_{z\to 1}\frac{d}{dz}\left[\frac{ze^z - e^z}{z^2}\right] = -\pi i \operatorname*{Lt}_{z\to 1}\frac{z^2 e^z (z-2) + 2ze^z}{z^4}$$

$$= -\pi i \operatorname*{Lt}_{z\to 1}\frac{e^z}{z^3}(z^2 - 2z + 2) = -\pi i e \qquad \textbf{Ans.}$$

EXAMPLE 18.14. (*i*) Evaluate $\int_C \frac{e^{2z}\, dz}{z^2 - 3z + 2}$ where C is the circle $|z| = 3$.

[GGSIPU II Sem End Term 2006]

(*ii*) Evaluate $\int_C \frac{e^z\, dz}{(z-1)(z-4)}$ where $C : |z| = 2$.

[GGSIPU II Sem End Term 2007]

SOLUTION: (*i*) Since $\frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ we have

$$I = \int_C \frac{e^{2z}\, dz}{z^2 - 3z + 2} = \int_C \frac{e^{2z}\, dz}{z-2} - \int_C \frac{e^{2z}\, dz}{z-1} \quad \text{where } C : |z| = 3.$$

EXAMPLE 18.13. (a) Evaluate $\int_C \left[\frac{2z}{(z-2)^2}+\frac{3(z-1)}{(z-2)^3}\right]dz$ where C is the circle $|z|=3$.

(b) Evaluate $\int_c \frac{e^z\,dz}{z(1-z)^3}$ where C is (*i*) $|z|=\frac{1}{2}$, (*ii*) $|z-1|=\frac{1}{2}$.

SOLUTION: (*a*) The given integral can be written as $\int_C \frac{2z^2-z-3}{(z-2)^3}dz$ where $C:|z|=3$.

Here we take $f(z)=2z^2-z-3$ and 'a' = 2 which lies inside C.

Using the Cauchy's derived integral formula

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} \quad \text{for } n=2, \text{ we get}$$

$$f''(2)=\frac{2!}{2\pi i}\int_C \frac{2z^2-z-3}{(z-2)^3}dz \quad \text{Here } f'(z)=4z-1 \text{ and } f''(z)=4=f''(2).$$

Therefore, $\int_C \frac{2z^2-z-3}{(z-2)^3}dz = \pi i f''(2) = 4\pi i$ **Ans.**

(*b*) (*i*) Here $z=0$ and $z=1$ are the singularities of the integrand, of which only $z=0$ lies inside $C:|z|=1/2$.

Hence by Cauchys intergal formula, given integral equals

$$\int_C \frac{e^z/(1-z)^3}{z}dz = 2\pi i\left[\frac{e^z}{(1-z)^3}\right]_{z=0} = 2\pi i.$$ **Ans.**

(*ii*) Out of the singularities $z=0$ and $z=1$ of the integrand, only $z=1$ lies inside $C:|z-1|=\frac{1}{2}$. Hence by Cauchy's integral formula

$$\int_C \frac{e^z\,dz}{z(1-z)^3} = -\oint_C \frac{e^z/z\,dz}{(z-1)^3}=\frac{-2\pi i}{2!}\underset{z\to1}{\text{Lt}}\left[\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)\right]$$

$$= -\pi i\lim_{z\to1}\frac{d}{dz}\left[\frac{ze^z-e^z}{z^2}\right]=-\pi i\underset{z\to1}{\text{Lt}}\frac{z^2e^z(z-2)+2ze^z}{z^4}$$

$$= -\pi i\underset{z\to1}{\text{Lt}}\frac{e^z}{z^3}(z^2-2z+2)=-\pi ie$$ **Ans.**

EXAMPLE 18.14. (*i*) Evaluate $\int_C \frac{e^{2z}\,dz}{z^2-3z+2}$ where C is the circle $|z|=3$.

[*GGSIPU II Sem End Term 2006*]

(*ii*) Evaluate $\int_C \frac{e^z\,dz}{(z-1)(z-4)}$ where $C:|z|=2$.

[*GGSIPU II Sem End Term 2007*]

SOLUTION: (*i*) Since $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$ we have

$$I=\int_C \frac{e^{2z}\,dz}{z^2-3z+2}=\int_C \frac{e^{2z}\,dz}{z-2}-\int_C \frac{e^{2z}\,dz}{z-1} \quad \text{where } C:|z|=3.$$

Both the singularities $z = 1$ and $z = 2$ lie inside the circle $|z| = 3$.

Therefore, by Cauchy's integral formula

$$I = 2\pi i\, [e^{2z}]_{z=2} - 2\pi i\, [e^{2z}]_{z=1} = 2\pi i\, (e^4 - e^2).$$ **Ans.**

(*ii*) $\oint_C \frac{e^2\, dz}{(z-1)(z-4)}$. Here the integrand has singularities at $z = 1$ and $z = 4$ of which only $z = 1$ lies inside C the circle $|z| = 2$.

Hence by Cauchy's integral formula

$$\oint_C \frac{e^z/(z-4)}{(z-1)} dz = 2\pi i \left[\frac{e^z}{z-4}\right]_{z=1} = \frac{2\pi i\, e}{1-4} = -\frac{2\pi i\, e}{3}.$$ **Ans.**

EXAMPLE 18.15. (*a*) Using Cauchy's integral formula, evaluate

$$\int_C \frac{z^4\, dz}{(z+1)(z-i)^2} \quad \text{where } C \text{ is the ellipse } 9x^2 + 4y^2 = 36.$$

[*GGSIPU II Sem II Term 2003*]

(*b*) Evaluate $\oint_C \frac{(z-3)\, dz}{z^2+2z+5}$ where (*i*) $C : |z| = 1$, (*ii*) $C : |z + 1 + i| = 2$

[*GGSIPU II Sem End Term 2010*]

SOLUTION: (*a*) $\int_C f(z)dz = \int_C \frac{z^4\, dz}{(z+1)(z-i)^2}$ where $C : \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$.

$f(z)$ has singularities at $z = -1$ and $z = i$ and both lie inside C.

Now $$\frac{1}{(z+1)(z-i)^2} = \frac{-i}{2(z+1)} + \frac{1-i}{2(z-i)^2} + \frac{i}{2(z-i)}$$ (on resolving into partial fractions).

$$\therefore \quad \int_C f(z)dz = \frac{-i}{2}\int_C \frac{z^4}{z+1}dz + \frac{1-i}{2}\int_C \frac{z^4}{(z-i)^2}dz + \frac{i}{2}\int_C \frac{z^4}{z-i}dz$$

Employing Cauchy's integral formula in the above relation, we get

$$\int_C f(z)\, dz = \frac{-i}{2}\cdot 2\pi i\,[z^4]_{z=-1} + \frac{i}{2}\cdot 2\pi i\,[z^4]_{z=i} + \frac{1-i}{2}\cdot 2\pi i\,[4z^3]_{z=i}$$
$$= \pi(-1)^4 - \pi(i)^4 + \pi(i+1)\, 4i^3 = 4\pi(1-i).$$ **Ans.**

(*b*) (*i*) Let $I = \oint_C \frac{(z-3)\, dz}{z^2+2z+5}$ where $C : |z| = 1$.

Singularities of the integrand are given by $z^2 + 2z + 5 = 0$ i.e., $z = -1 \pm 2i$

Now, since $z = -1 \pm 2i$ both lie outside $C : |z| = 1$ hence $\oint_C \frac{z-3}{z^2+2z-5} = 0$.

(*ii*) $C : |z + 1 + i| = 2$. Here the singularity $z = -1 + 2i$ lies outside the curve C whereas $z = -1 - 2i$ lies inside C. $\therefore$ By Cauchy's integral formula

$$\oint_C \frac{(z-3)\, dz}{z^2+2z+5} = \oint_C \frac{(z-3)/(z+1-2i)}{(z+1+2i)} dz = 2\pi i\left[\frac{z-3}{z+1-2i}\right]_{z=-1-2i}$$
$$= 2\pi i\left[\frac{-1-2i-3}{-1-2i+1-2i}\right] = \pi(i+2)$$ **Ans.**

EXAMPLE 18.16. (a) Use Cauchy's integral formula to evaluate $\oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$.

[GGSIPU II Sem II Term 2005; End Term 2012]

(b) State and prove Cauchy's integral formula and evaluate $\oint_C \frac{z-1}{(z+1)^2(z-2)} dz$ where $C : |Z - i| = 2$.

[GGSIPU II Sem II Term 2012]

SOLUTION: (a) $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ has poles at $z = 1$ and $z = 2$ both lying inside $C : |z| = 3$. Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, using Cauchy's integral formula, we have

$$\begin{aligned}\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz &= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz \\ &= 2\pi i\,[\sin \pi z^2 + \cos \pi z^2]_{z=2} - 2\pi i\,[\sin \pi z^2 + \cos \pi z^2]_{z=1} \\ &= 2\pi i\,[\sin 4\pi + \cos 4\pi] - 2\pi i\,[\sin \pi + \cos \pi) = 4\pi i\end{aligned}$$

$\therefore$ The value of the given integral is $4\pi i$. **Ans.**

(b) Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$. Resolving into partial fractions $f(z) = \frac{1}{9(z-2)} + \frac{2}{3(z+1)^2} - \frac{1}{9(z+1)}$.

Clearly $f(z)$ has signularities at $z = 2$ and at $z = -1$, both lying inside $C : |z| = 3$.

$$\therefore \quad \oint_C f(z)\, dz = \frac{1}{9}\oint_C \frac{dz}{z-2} - \frac{1}{9}\oint \frac{dz}{(z+1)} + \frac{2}{3}\oint \frac{dz}{(z+1)^2}$$

Using Cauchy's integral formula, we get

$$\oint_C f(z)\, dz = 2\pi i\left(\frac{1}{9}\right) - 2\pi i\left(\frac{1}{9}\right) + \frac{2}{3}\cdot 2\pi i\left[\frac{d}{dz}(1)\right]_{z=-1} = 0. \qquad \textbf{Ans.}$$

EXAMPLE 18.17. (a) State and prove Cauchy's integral formula and hence find the value of (i) $F(3.5)$, (ii) $F(i)$, (iii) $F'(-1)$ and $F''(-i)$,

if $F(a) = \int_C \frac{4z^2 + z + 5}{z-a} dz$ where C is the ellipse $(x/2)^2 + (y/3)^2 = 1$.

[GGSIPU II Sem II Term 2006]

(b) Using Cauchy's integral formula, evaluate $\oint_C \frac{e^z\, dz}{(z^2 + \pi^2)^2}$ where C is $|z| = 4$.

[GGSIPU II Sem II Term 2014]

SOLUTION: (a) Cauchy's integral formula has already been stated and proved.

If $f(z)$ is analytic inside and on a closed curve C and $z = a$ lies inside or on C

then by Cauchy's integral formula $\oint_C \frac{f(z)\, dz}{z-a} = 2\pi i\, f(a)$.

(*i*) Since $C : (x/2)^2 + (y/3)^2 = 1$ and $z = 3.5$ does not lie in the ellipse C, hence $\dfrac{f(z)}{z-a}$ is analytic inside and on C hence value of the integral $\oint_C \dfrac{f(z)\,dz}{z-a} = 0$ where $f(z) = 4z^2 + z + 5$.

Therefore, $F(3.5) = 0$ **Ans.**

(*ii*) $z = i$, lies inside C.

$$\therefore \quad \oint_C \frac{f(z)\,dz}{z-i} = \oint_C \frac{4z^2+z+5}{(z-i)}dz = 2\pi i\, f(i) = F(i)$$

$$= 2\pi i\,[4z^2 + z + 5] \quad \text{at} \quad z = i$$

$$= 2\pi\,(i-1). \text{ Hence } F(i) = 2\pi\,(i-1).$$ **Ans.**

(*iii*) $z = -1$ also lies in C. Here $F(z) = 2\pi i\,[4z^2 + z + 5]$

Therefore $F'(z) = 2\pi i\,[8z + 1]$ and hence $F'(-1) = 2\pi i\,(-8 + 1) = -14\pi i$ **Ans.**

(*iv*) $z = -i$ also lies in C and $F(z) = 2\pi i\,[4z^2 + z + 5]$

$$F''(z) = 16\pi i \quad \text{hence } F''(-i) = 16\pi i$$ **Ans.**

(b) $$I = \oint_C \frac{e^z\,dz}{(z^2+\pi^2)^2} = \oint_C \frac{e^z\,dz}{(z+\pi i)^2\,(z-\pi i)^2}$$

The integrand has double poles at $z = \pm\pi i$ and both these lie inside the circle $|z| = 4$.

Let $$\frac{1}{(z+\pi i)^2\,(z-\pi i)^2} = \frac{1}{(2\pi i)^2\,(z-\pi i)^2} + \frac{1}{(-2\pi i)^2\,(z+\pi i)^2} + \frac{A}{z-\pi i} + \frac{B}{z+\pi i}$$

$$\Rightarrow \qquad 1 = -\frac{1}{4\pi^2}[(z-\pi i)^2 + (z+\pi i)^2] + A(z-\pi i)\,(z+\pi i)^2 + B(z+\pi i)\,(z-\pi i)^2$$

Comparing the coefficients of z^3 on both sides here, we get $O = A + B$ and comparing the constants on both sides we get

$$1 = -\frac{1}{4\pi^2}(-\pi^2 - \pi^2) + A(-\pi^2)\,(-\pi i) + B(-\pi^2)\,(\pi i) \quad \text{or} \quad A - B = \frac{-i}{2\pi^3}.$$

Hence $A = \dfrac{-i}{4\pi^3}$ and $B = \dfrac{i}{4\pi^3}$

Thus, $$I = \int_C \frac{e^z\,dz}{-4\pi^2(z-\pi i)^2} + \int_C \frac{e^z\,dz}{-4\pi^2(z+\pi i)^2} - \frac{i}{4\pi^3}\int_C \frac{e^z\,dz}{z-\pi i} + \frac{i}{4\pi^3}\int_C \frac{e^z\,dz}{z+\pi i}$$

Using Cauchy's integral formula, we get

$$I = \frac{-2\pi i}{4\pi^2}\left[\frac{d}{dz}e^z\right]_{z=\pi i} - \frac{2\pi i}{4\pi^2}\left[\frac{d}{dz}e^z\right]_{z=-\pi i} - \frac{i}{4\pi^3}(2\pi i)\,e^{\pi i} + \frac{i}{4\pi^3}(2\pi i)\,e^{-\pi i}$$

$$= \frac{-i}{2\pi}e^{\pi i} - \frac{i}{2\pi}e^{-\pi i} + \frac{1}{2\pi^2}e^{\pi i} - \frac{1}{2\pi^2}e^{-\pi i} = -\frac{i}{2\pi}(-1-1) + \frac{1}{2\pi^2}(-1+1) = \frac{i}{\pi}$$ **Ans.**

MORERA'S THEOREM *(as converse of Cauchy's theorem) : If $f(z)$ is continuous in a simply connected domain D and if $\int_C f(z)\,dz = 0$ around every closed curve C lying in D then $f(z)$ is analytic in D.*

Since $\int_C f(z)\,dz = 0$ where C is any closed curve lying in D, the line integral of $f(z)$ from a fixed point $z_0 = x_0 + iy_0$ to a variable point $z\,(= x + iy)$ along C must be a function of z, that is,

$$\int_{z_0}^{z} f(z)\,dz = \int_{(x_0,y_0)}^{(x,y)} (u+iv)(dx+i\,dy) = h(z) \qquad \text{...(1)}$$

Let $h(z) = s + i\,t$, then (1) gives

$$s + i\,t = \int_{(x_0,y_0)}^{(x,y)} (u+iv)(dx+i\,dy) = \int_{(x_0,y_0)}^{(x,y)} (u\,dx - v\,dy) + i\int_{(x_0,y_0)}^{(x,y)} (u\,dy + v\,dx).$$

$$\Rightarrow \qquad s = \int_{(x_0,y_0)}^{(x,y)} (u\,dx - v\,dy) \quad \text{and} \quad t = \int_{(x_0,y_0)}^{(x,y)} (u\,dy + v\,dx). \qquad \text{...(2)}$$

Partially differentiating (2) under the integral sign, w.r.t. x and y, we get

$$\frac{\partial s}{\partial x} = u, \quad \frac{\partial s}{\partial y} = -v, \quad \frac{\partial t}{\partial x} = v, \quad \frac{\partial t}{\partial y} = u.$$

Thus, $\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$ and $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$. Further, as $f(z)$ is given to be continuous so are u and v. Thus, s and t have continuous partial derivatives and satisfy the Cauchy-Riemann equations, hence $h(z)$ is analytic. Then

$$h'(z) = \frac{\partial s}{\partial x} + i\frac{\partial t}{\partial x} = u + i\,v = f(z).$$

Thus, $f(z)$ is the derivative of the analytic function $h(z)$ hence $f(z)$ is analytic by the corollary of the Cauchy's integral formula.

CAUCHY'S INEQUALITY: *If $f(z)$ is analytic within and on the circle $C : |z-a| = r$, then*

$$|f^{(n)}(a)| \quad \le \quad \frac{Mn!}{r^n}$$

where M is the maximum value of $|f(z)|$ on C.

In the corollary of Cauchy's integral formula, we had

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} \Rightarrow |f^{(n)}(a)| = \frac{n!}{2\pi}\left|\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\int_C \frac{|f(z)||dz|}{|z-a|^{n+1}} \le \frac{n!\,M}{2\pi r^{n+1}}\int_C |dz| \qquad \text{(since } |f(z)| \le M\text{)}$$

$$\le \frac{n!\,M}{2\pi r^{n+1}}\int_C dr = \frac{n!\,M\,2\pi r}{2\pi r^{n+1}} = \frac{Mn!}{r^n}.$$

LIOUVILLE'S THEOREM: If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.

Since $f(z)$ is bounded, there exists a constant M such that $|f(z)| \le M$, for all z. Therefore, from the above mentioned Cauchy's inequality taking $n = 1$, we have

$$|f'(a)| \le \frac{M}{r}$$

which can be made as small as we please by taking r sufficiently large. Therefore $f'(a) = 0$. Since a is any point, it follows that $f'(z) = 0$ for all z. Hence $f(z)$ is a constant.

EXERCISE 18B

1. Verify Cauchy's integral theorem for the function $f(z) = z^2$ taking C as the circle $|z| = 3$.

2. (a) Evaluate $\int_C \frac{z^2+1}{z^2-1} dz$ where C is the circle with centre at $z = 1$ and unit radius.

   (b) Evaluate $\int_C \frac{z^2-z+1}{z-1} dz$ where C is the contour (a) $|z| = 1$, (b) $|z| = \frac{1}{2}$.

3. Evaluate the integral $\int_C \left(\frac{2}{z} + \frac{3}{z^2}\right) dz$ where C is a path enclosing the origin.

4. Use Cauchy's integral formula to evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 2$.

   [GGSIPU II Sem End Term 2009; II Term 2011]

5. Evaluate $\int_C \frac{z^2+z+1}{z^2-7z+2} dz$ where C is the ellipse $4x^2 + 9y^2 = 1$.

6. Evaluate $\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz$ where C is the circle $|z| = 1$.

7. Verify Cauchy's integral theorem for the function

   $f(z) = \frac{1}{z^2+2z+2}$ when the contour C is the circle $|z| = 1$.

8. Evaluate $\int_C \frac{z\,dz}{9+9z-z^2-z^3}$ where $C : |z| = 2$.

9. Evaluate $\int_C \frac{\cos z}{z^2} dz$ where C is any closed contour around the origin.

10. Show that $\oint_C \frac{dz}{z^2(z-3)} = \frac{-2\pi i}{9}$ where C is the circle $|z| = 2$.

11. Evaluate $f(2)$ and $f(3)$ if $f(a) = \oint \frac{2z^2-z-2}{z-a} dz$ where C: $|z| = 2.5$.

    [GGSIPU II Sem II Term 2010]

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SERIES OF COMPLEX TERMS AND IT'S CONVERGENCE

Let $(u_1 + i\,v_1) + (u_2 + i\,v_2) + ... + (u_n + i\,v_n) + ...$ ...(1)

be an infinite series of complex terms where u's and v's are real numbers. The series (1) is said to be convergent if $\Sigma\, u_n$ and $\Sigma\, v_n$ both are convergent. If $\Sigma\, u_n$ converges to the sum A and $\Sigma\, v_n$ converges to the sum B then series (1) converges to the sum $A + i\,B$.

Also, if (1) is convergent, then $\lim_{n\to\infty} (u_n + iv_n) = 0$.

The series (1) is said to be ***absolutely convergent*** if the series

$|u_1 + i\ v_n| + |u_2 + i\,v_2| + ... + |u_n + i\,v_n| + ...$ is convergent.

Next, let the series $f_1(z) + f_2(z) + f_3(z) + ... + f_n(z) + ...$ ...(2)

be convergent and have the sum $S(z)$ and let $S_n(z)$ denote the nth partial sum (*i.e.*, the sum of first n terms) then the series (2) is said to be **uniformly convergent** in a region R if, given any small positive number $\in$ there exists a positive number N (depending upon $\in$ but not on z) such that for every z in R, we have $|S(z) - S_n(z)| < \in$ for all $n > N$.

As in the case of real number series, *the* **WEIRSTRASS'S M TEST** holds for a series of complex terms also. Thus, *the series (2) is uniformly convergent in a region R if there exists a convergent series ΣM_n of real positive constants, such that $|f_n(z)| \le M_n$ for all z in R.*

An important observation follows that **"a uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term".**

POWER SERIES AND ITS CONVERGENCE

If a power series $a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$ converges for $z = z_1$, then it converges absolutely for every value of z satisfying $|z| < |z_1|$.

Proof: Since the series $a_0 + a_1 z_1 + a_2 z_1^2 + a_3 z_1^3 + ... + a_n z_1^n + ...$ ...(1)

is given to be convergent, we have $\lim_{n\to\infty} a_n z_1^n = 0$. This means that we can find a number M such that $|a_n z_1^n| < M$ for every value of n.

Therefore $|a_0| + |a_1 z| + |a_2 z^2| + |a_3 z^3| + ...$

$$= |a_0| + |a_1 z_1|\,|z/z_1| + |a_2 z_1^2|\,|z/z_1|^2 + |a_3 z_1^3|\,|z/z_1|^3 + ...$$

$$< M + MK + MK^2 + ... \text{ where } K = |z/z_1|.$$

The geometric series $\Sigma\, M K^n$ is convergent for $K < 1$, *i.e.*, for $|z| < |z_1|$, hence the series (1) is convergent for $|z| < |z_1|$.

This implies that the series (1) is absolutely convergent at every point inside a circle with centre at the origin and radius $|z_1|$. The largest of such circles inside which the series (1) converges, is called the circle of convergence. Suppose $|z| = R$ is the circle of convergence and $|z| = R_1$ is a circle within the circle $|z| = R$ then from the above discussion

$$|a_n z^n| < M\left(\frac{R_1}{R}\right)^n \text{ for every point inside the circle } |z| = R_1.$$

Therefore, by Wierstrass's M test, the series is also uniformly convergent inside this circle.

As in the case of real series, it can be shown that a uniformly convergent series of continuous complex functions is itself continuous and is term by term integrable : It can also be shown here that this integration can be extended to all points within the circle of convergence.

Subtracting (3) from (2) we get (1), as $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} \dfrac{f(t)\,dt}{(t-a)^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2} \dfrac{f(t)\,dt}{(t-a)^{-n+1}}$.

The above result can also be written as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \dots + \frac{a_{-n}}{(z-a)^n} + \dots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{where } a_n = \frac{1}{2\pi i}\oint_C \frac{f(t)\,dt}{(t-a)^n}.$$

Here $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the REGULAR PART of the expansion of $f(z)$ and $\sum_{n=1}^{\infty} \dfrac{a_{-n}}{(z-a)^n}$ is called the PRINCIPAL PART of the expansion of $f(z)$.

EXAMPLE 18.18. (*a*) Find the Laurent's expansion of $f(z) = \dfrac{1}{z(z-1)^2}$ at the point $z = 1$.

(*b*) Expand $\dfrac{1}{z(z^2-3z+2)}$ for the region $1 \le |z| < 2$.

[GGSIPU II Sem End Term 2011]

SOLUTION: (*a*) To obtain the expansion of $f(z)$ about the point $z=1$, we put $z_1 = z-1$ or $z = 1 + z_1$, then we have

$$f(z) = \frac{1}{(1+z_1)z_1^2} = \frac{1}{z_1^2}(1+z_1)^{-1} = \frac{1}{z_1^2}(1 - z_1 + z_1^2 - z_1^3 + \dots)$$

$$= \frac{1}{z_1^2} - \frac{1}{z_1} + 1 - z_1 + z_1^2 - z_1^3 + \dots$$

$$= \frac{1}{(z-1)^2} - \frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2} \quad \text{which is valid for } 0 < |z-1| < 1. \qquad \textbf{Ans.}$$

(*b*) Let $f(z) = \dfrac{1}{z(z^2-3z+2)} = \dfrac{1}{z(z-1)(z-2)} = \dfrac{1}{2z} - \dfrac{1}{z-1} + \dfrac{1}{2(z-2)}$ (since $1 < z < 2$)

$$= \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1} = \frac{1}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right) - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\dots\right)$$

$$= -\frac{1}{2z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \dots - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\right). \qquad \textbf{Ans.}$$

EXAMPLE 18.19. Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in Laurent's series, valid for

(a) $1<|z|<3$ (b) $|z|>3$

(c) $0<|z+1|<2$ (d) $|z|<1.$

SOLUTION: (a) Resolving the given function of z into partial fractions, we get

$$f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z+3}\right)$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}\quad(\text{since } |z|>1 \text{ and } |z|<3)$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\ldots\right]\qquad \textbf{Ans.}$$

(b) For $|z|>3$

$$f(z)=\frac{1}{2}\left[\frac{1}{z+1}-\frac{1}{z+3}\right]=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\right)-\frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\ldots\right)$$

$$=\frac{1}{2}\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\ldots\right)-\frac{1}{2}\left(\frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\frac{27}{z^4}+\ldots\right)$$

$$=\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\ldots\qquad \textbf{Ans.}$$

(c) For $|z+1|<2$

$$f(z)=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z+1+2}\right)=\frac{1}{2(z+1)}-\frac{1}{4}\left(1+\frac{z+1}{2}\right)^{-1}$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}\left[1-\frac{z+1}{2}+\frac{(z+1)^2}{4}-\frac{(z+1)^3}{8}+\ldots\right]$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+\frac{(z+1)^3}{32}-\ldots\qquad \textbf{Ans.}$$

(d) For $|z|<1$

$$f(z)=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z+3}\right)=\frac{1}{2}(1+z)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}(1-z+z^2-z^3+\ldots)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\ldots\right)$$

$$=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2+\ldots+(-1)^n\frac{1}{2}(1-3^{-n-1})z^n+\ldots\qquad \textbf{Ans.}$$

Since $|z|<1$ the series becomes a Taylor's series.

EXAMPLE 18.20. Find the Taylor's series and Laurent's series of $f(z) = \dfrac{-2z+3}{z^2-3z+2}$ with centre at the origin.

[GGSIPU II Sem End Term 2006]

SOLUTION: $f(z) = \dfrac{-2z+3}{z^2-3z+2} = \dfrac{-2z+3}{(z-1)(z-2)} = \dfrac{-1}{z-1} - \dfrac{1}{z-2}$

For Taylor's expansion of $f(z)$ we take $|z| < 1$, then

$$f(z) = (1-z)^{-1} + \frac{1}{2}(1-\frac{z}{2})^{-1}$$

$$= (1 + z + z^2 + z^3 + ...) + \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+...\right)$$

$$= \frac{3}{2}+\frac{5}{4}z+\frac{9}{8}z^2+\frac{17}{16}z^3 + ...$$

which is the Taylor's expansion of $f(z)$ about the origin. **Ans.**

Next, for Laurent's expansion of $f(z)$ consider the annular region $1 \le |z| \le 2$, then

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2} = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right]+\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+...\right]$$

$$= \left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+...\right)-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right)$$

which is the Laurent's expansion of $f(z)$ about the origin. **Ans.**

EXAMPLE 18.21. (*a*) Expand the function $\dfrac{1-\cos z}{z^3}$ in Laurent's series about the point $z = 0$.

[GGSIPU II Sem End Term 2007 Reappear]

(*b*) Expand $\dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$ in Laurent's series.

[GGSIPU II Sem II Term 2013]

SOLUTION: (*a*) The function is $f(z) = \dfrac{1-\cos z}{z^3}$. To obtain its Laurent's expansion about the origin we use the expansion $\cos z = 1 - \dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dfrac{z^6}{6!} + ...$

$$\therefore \quad f(z) = \frac{1}{z^3}\left[1-(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+...)\right] = \frac{1}{z^3}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-...\right]$$

or $\quad f(z) = \dfrac{1}{2z}-\dfrac{z}{4!}+\dfrac{z^3}{6!} - ...$ is the required expansion. **Ans.**

(*b*) Given function $f(z) = \dfrac{e^{2z}}{(z-1)^3} = \dfrac{e^{2(z-1)+2}}{(z-1)^3}$

or $$f(z) = e^2 \cdot \frac{e^{2(z-1)}}{(z-1)^3} = \frac{e^2}{(z-1)^3}\left[1+2(z-1)+\frac{4(z-1)^2}{2!}+\frac{8(z-1)^3}{3!}+...\right]$$

$$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{2!(z-1)}+\frac{8}{3!}+\frac{16}{4!}(z-1)+\frac{32(z-1)^2}{5!}+...\right]$$

$$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{z-1}+\frac{4}{3}+\frac{4}{3}(z-1)+...\right]$$

which is Laurent's series of $f(z)$. **Ans.**

EXAMPLE 18.22. (*a*) Find the expansion of $f(z) = \dfrac{1}{z-z^3}$ in the region $1 < |z-1| < 2$.

[*GGSIPU II Sem End Term 2010*]

(*b*) Find Laurent's series for $f(z) = \dfrac{7z-2}{z^3-z^2-2z}$ in the region

(*i*) $0<|z+1|<1$ [*GGSIPU II Sem II Term 2003*]

(*ii*) $1<|z+1|<3$. [*GGSIPU II Sem II Term 2011, End Term 2014*]

SOLUTION: (*a*) $f(z) = \dfrac{1}{z(1-z^2)}$ is to be expanded in powers of $(z-1)$ in $1 < |z-1| < 2$.

$$f(z) = \frac{-1}{(z-1)\,z(z+1)} = \frac{-1}{z-1}\left[\frac{1}{z}-\frac{1}{z+1}\right] = \frac{-1}{z-1}\left[\frac{1}{z-1+1}-\frac{1}{z-1+2}\right]$$

$$= \frac{-1}{z-1}\left[\frac{1}{z-1}\left(1+\frac{1}{z-1}\right)^{-1}-\frac{1}{2}\left(1+\frac{z-1}{2}\right)^{-1}\right] = \frac{1}{2(z-1)}\left[1+\frac{z-1}{2}\right]^{-1}-\frac{1}{(z-1)^2}\left[1+\frac{1}{z-1}\right]^{-1}$$

$$= \frac{1}{2(z-1)}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+...\right]-\frac{1}{(z-1)^2}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+...\right]$$

$$= \left[\frac{1}{2(z-1)}-\frac{1}{4}+\frac{1}{8}(z-1)-\frac{1}{16}(z-1)^2+...\right]+\left[\frac{-1}{(z-1)^2}+\frac{1}{(z-1)^3}-\frac{1}{(z-1)^4}+\frac{1}{(z-1)^5}...\right]$$

or $$f(z) = \left[-\frac{1}{4}+\frac{1}{8}(z-1)-\frac{1}{16}(z-1)^2+...\right]+\left[\frac{1}{2(z-1)}-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}-\frac{1}{(z-1)^4}+...\right]$$

which is the required Laurent's expansion of $f(z)$. **Ans.**

(*b*) $$f(z) = \frac{7z-2}{z^3-z^2-2z} = \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z}+\frac{2}{z-2}-\frac{3}{z+1} = \frac{1}{z+1-1}+\frac{2}{z+1-3}-\frac{3}{z+1}$$

(*i*) For $0 < |z+1| < 1$

$$f(z) = -[1-(z+1)]^{-1} - \frac{2}{3}\left[1-\frac{z+1}{3}\right]^{-1} - \frac{3}{z+1}$$ (for Laurent's expansion)

$$= -\left[1+(z+1)+(z+1)^2+\ldots\right] - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\ldots\right] - \frac{3}{z+1}$$

$$= -\left(1+\frac{2}{3}\right) - \left(1+\frac{2}{9}\right)(z+1) - \left(1+\frac{2}{27}\right)(z+1)^2 + \ldots - \left(\frac{3}{z+1}\right)$$

or $$f(z) = -\frac{5}{3} - \frac{11}{9}(z+1) - \frac{29}{27}(z+1)^2 + \ldots + \left(\frac{-3}{z+1}\right).$$ **Ans.**

(*ii*) For $1 < |z+1| < 3$

$$f(z) = \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1} - \frac{3}{z+1}$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\ldots\right] - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\ldots\right].$$

or $$f(z) = \frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \ldots - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\ldots\right]$$ **Ans.**

EXAMPLE 18.23. Expand $\dfrac{z}{(z^2-1)(z^2+4)}$ in $1 < |z| < 2$.

[*GGSIPU II Sem II Term 2005; II Sem II Term 2010*]

SOLUTION: $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$ where $1 < |z| < 2$ or $1 < |z|^2 < 4$

or $$f(z) = \frac{z}{5}\left[\frac{1}{z^2-1} - \frac{1}{z^2+4}\right] = \frac{z}{5}\left[\frac{1}{z^2}\left(1-\frac{1}{z^2}\right)^{-1} - \frac{1}{4}\left(1+\frac{z^2}{4}\right)^{-1}\right]$$

$$= \frac{1}{5z}\left(1-\frac{1}{z^2}\right)^{-1} - \frac{z}{20}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{5z}\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\ldots\right] - \frac{z}{20}\left[1-\frac{z^2}{4}+\frac{z^4}{4^2}-\frac{z^6}{4^3}+\ldots\right]$$

or $$f(z) = \frac{1}{5}\left[\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^5}+\ldots\right] - \frac{1}{20}\left[z-\frac{z^3}{4}+\frac{z^5}{4^2}-\frac{z^7}{4^3}+\ldots\right]$$

which is the Laurent's expansion of $f(z)$ in $1 < |z| < 2$.

ZEROS OF AN ANALYTIC FUNCTION

Let $f(z)$ be an analytic function. If $f(z_0) = 0$ then z_0 is called the zero *of* $f(z)$.

If $f(z_0) = 0$ and $f'(z_0) \neq 0$ then z_0 is called **a simple zero or a zero of first order.** If $f(z_0) = f'(z_0) = 0$ and $f''(z_0) \neq 0$ then z_0 is called **a zero of second order.** Extending it, we say that z_0 is **a zero of order** n, if

$$f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(n)}(z_0) = 0 \quad \text{but} \quad f^{(n+1)}(z_0) \neq 0.$$

For example, $z^2 \sin z$ has a zero of third order at $z = 0$ and simple zeros at $z = \pm \pi, \pm 2\pi, \dots$ and $(1 - \cos z)$ has zeros of order two at $z = 0, \pm 2\pi, \pm 4\pi, \dots$.

Next, suppose that the analytic function $f(z)$ has a zero of order n at $z = z_0$, then its Taylor's series is of the form

$$\begin{aligned} f(z) &= b_n (z - z_0)^n + b_{n+1} (z - z_0)^{n+1} + b_{n+2} (z - z_0)^{n+2} + \dots \\ &= (z - z_0)^n [b_n + (z - z_0) b_{n+1} + (z - z_0)^2 b_{n+2} + \dots] \\ &= (z - z_0)^n [b_n + S], \text{ say.} \end{aligned}$$

By taking $|z - z_0|$ sufficiently small, we can make $|b_n| > |z - z_0| |S|$.

Thus, neighbourhood of z_0 can be found where $b_n + (z - z_0) S$ is not zero.

It implies that $f(z)$ can not have another zero in this neighbourhood. Thus we have the fact that **zeros of an analytic function are *isolated*.**

For example, $\dfrac{1}{1 - z}$ has a simple zero at unity and $\cos z$ has simple zeros at $z = \pm \pi/2, \pm 3\pi/2, \dots$ and $\left(\dfrac{z+1}{z^2+1}\right)^3$ has a zero at $z = -1$ of order three.

TYPES OF SINGULARITY OF A FUNCTION

A point z_0 at which a function $f(z)$ is not analytic is known as a singular point or singularity of $f(z)$. There are different types of singularities as follows.

(*a*) **Isolated Singularity.** If the function $f(z)$ is analytic at every point in the neighbourhood of a point z_0 except at z_0 itself, then z_0 is called an *isolated singular point* or *isolated singularity*. For example, $f(z) = 1/z$ then we have $f'(z) = -\dfrac{1}{z^2}$ it follows that $f(z)$ is analytic at every point except at $z = 0$, hence $z = 0$ is an isolated singularity. Also, $f(z) = \dfrac{1}{z^3 (z^2 + 1)}$ has three isolated singularities at $z = 0, z = \pm i$.

(*b*) **Removable Singularity:** If $f(z)$ is not defined at $z = z_0$ but $\lim_{z \to z_0} f(z)$ exists then $z = z_0$ is called a *removable singularity* of $f(z)$. Note here that in this case the principal part of the Laurent series is zero and $\lim_{z \to z_0} f(z) = a_0$. Thus, if we define $f(z_0) = a_0$ the function $f(z)$ becomes analytic at $z = z_0$.

For example $f(z) = \dfrac{\sin z}{z}$ has a removable singularity at $z = 0$, since $\lim_{z \to 0} \dfrac{\sin z}{z} = 1$ exists and we can redefine $f(z)$ as $f(z) = \dfrac{\sin z}{z}$ when $z \neq 0$ and $f(z) = 1$ when $z = 0$.

Also, $f(z) = \dfrac{\sin h\,(z-a)}{z-a}$ has a removable singularity at $z = a$, since we can write

$$f(z) = \frac{1}{z-a}\left[z-a+\frac{(z-a)^3}{3!}+\frac{(z-a)^5}{5!}+\ldots\right]$$

$$= 1+\frac{(z-a)^2}{3!}+\frac{(z-a)^4}{5!}+\ldots.$$

Here $f(z)$ does not contain any negative power of $(z-a)$. It is easy to see that the singularity in this case can be removed by redefining the function as follows :

$$f(z) = \frac{\sin h\,(z-a)}{z-a} \quad \text{when} \quad z \neq a$$

$$= 1 \quad \text{when} \quad z = a.$$

(*c*) **Essential Singularity.** *In the Laurent's expansion of $f(z)$ about $z = z_0$, if the principal part contains infinite number of terms then $z = z_0$ is called an essential singularity of $f(z)$.*

For example, $f(z) = \sin\dfrac{1}{z-1}$ has an essential singularity at $z = 1$, since

$$\sin\frac{1}{1-z} = \frac{1}{1-z}-\frac{1}{3!}\frac{1}{(1-z)^3}+\frac{1}{5!}\frac{1}{(1-z)^5}-\ldots.$$

And e^{-1/z^2} has essential singularity at $z = 0$. [*GGSIPU II Sem II Term 2013*]

(*d*) **POLES:** If in the principal part all the coefficients $b_{n+1}, b_{n+2}, \ldots$ are zero after a particular term b_n, then the Laurent's series of $f(z)$ reduces to

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots+\frac{b_n}{(z-z_0)^n}$$

and the singularity at $z = z_0$ is called a ***pole of order n.***

A pole of order one is called a *simple pole.*

For example, $\dfrac{z^2-2z+5}{z-2}$ has a simple pole at $z = 2$, since $\dfrac{z^2-2z+5}{z-2} = z+\dfrac{5}{z-2}$.

Also, $f(z) = \dfrac{4}{(z-2)^3}+\dfrac{3}{(z+1)^5}$ has a pole of order 5 at $z = -1$ and pole of order 3 at $z = 2$.

EXAMPLE 18.24. (*a*) Find the type of singularity of the function $f(z) = \dfrac{1-e^{2z}}{z^3}$ at $z = 0$

[*GGSIPU II Sem End Term 2006*]

(*b*) Find the nature and location of signularities of $f(z) = (z+1)\sin\left(\dfrac{1}{z-2}\right)$.

[*GGSIPU II Sem II Term 2011*]

SOLUTION: (*a*) $f(z) = \dfrac{1-e^{2z}}{z^3} = \dfrac{1}{z^3}\left[1-\left(1+2z+\dfrac{4z^2}{2!}+\dfrac{8z^3}{3!}+\dfrac{16z^4}{4!}+\ldots\right)\right]$

$$= \frac{1}{z^3}\left[-2z-2z^2-\frac{4}{3}z^3-\frac{2}{3}z^4-\ldots\right] = -\left[\frac{2}{z^2}+\frac{2}{z}+\frac{4}{3}+\frac{2}{3}z+\ldots\right]$$

Therefore $f(z)$ has a pole of order two at $z = 0$. **Ans.**

(*b*) The given fucntion $f(z)$ has singularity at $z=2$.

Here $$f(z) = (z+1)\sin\left(\frac{1}{z-2}\right) = (z+1)\left[\frac{1}{z-2} - \frac{1}{3!}\frac{1}{(z-2)^3} + \frac{1}{5!}\frac{1}{(z-2)^5} - \ldots\right]$$

Clearly, $z=2$ is an essential singularity of $f(z)$. **Ans.**

RESIDUE OF A COMPLEX FUNCTION

The coefficient b_1 of $\frac{1}{z-z_0}$ in the Laurent's expansion of $f(z)$ about an isolated singularity z_0, is called the **residue** of $f(z)$ at $z = z_0$. As defined earlier $b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$.

If z_0 is a simple pole, we have $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0}$.

Hence $$(z-z_0) f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + b_1$$

Therefore $$b_1 = \lim_{z\to z_0} [(z-z_0) f(z)]$$

which provides a simple method of calculating the residue of $f(z)$ at its simple pole $z = z_0$.

If $f(z)$ has a pole of order m at $z = z_0$ then the Laurent's expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m}$$

hence $$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \ldots + b_m.$$

Differentiating this equation w.r.t. z, $(m-1)$ times and taking limit as $z \to z_0$, we get

$$b_1 = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}.$$

This gives a simple method of calculating the residue of $f(z)$ at its pole $z = z_0$ of order m.

EXAMPLE 18.25. Determine the poles and residue there for the function.

(*a*) $\frac{z}{\cos z}$ (*b*) $z \cos\left(\frac{1}{z}\right)$ (*c*) $\frac{1-e^{2z}}{z^4}$

[GGSIPU II Sem II Term 2013]

SOLUTION: (*a*) Let $f(z) = \frac{z}{\cos z}$. The singularities of $f(z)$ are given by $z = \left(n+\frac{1}{2}\right)\pi$, $n \in I$.

Denoting this value as z_0, we have

$$\lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} \frac{z(z-z_0)}{\cos z} = \lim_{z\to z_0} \frac{z\left(z - n\pi - \frac{\pi}{2}\right)}{\cos z}$$ which is of the form $\frac{0}{0}$.

Applying L'Hospital's rule, we get

$$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{2z-n\pi-\frac{\pi}{2}}{-\sin z} = \frac{2z_0-n\pi-\frac{\pi}{2}}{-\sin z_0} = \frac{n\pi+\frac{\pi}{2}}{-\sin\left(n\pi+\frac{\pi}{2}\right)} = (-1)^{n+1}\left(n+\frac{1}{2}\right)\pi,$$

so $z=\left(n+\frac{1}{2}\right)\pi$ gives the simple poles of $\frac{z}{\cos z}$, and residues equal to $(-1)^{n+1}\left(n+\frac{1}{2}\right)\pi$. **Ans.**

(*b*) $f(z) = z\cos\left(\frac{1}{z}\right) = z\left[1-\frac{1}{2!z^2}+\frac{1}{4!}\frac{1}{z^4}-\frac{1}{6!}\frac{1}{z^6}+\ldots\right] = z-\frac{1}{2!z}+\frac{1}{4!}\frac{1}{z^3}-\frac{1}{6!}\frac{1}{z^5}+\ldots$

The poles of $f(z)$ are at $z=0$ only and the residue of $f(z)$ at $z=0$ is $-\frac{1}{2!}$. **Ans.**

(*c*) $f(z) = \frac{1}{z^4}\left[1-\left(1+2z+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\frac{16z^4}{4!}+\frac{32z^5}{5!}+\ldots\right)\right] = -\left[\frac{2}{z^3}+\frac{2}{z^2}+\frac{4}{3z}+\frac{2}{3}+\ldots\right]$

$\therefore$ At $z=0$, $f(z)$ has pole at $z=0$ of order 3 and Residue of $f(z)$ at $z=0$ is equal to $-\frac{4}{3}$. **Ans.**

EXAMPLE 18.26. (*a*) Determine the poles of the function $f(z) = \frac{z^2}{(z-1)(z-2)^2}$ and the residue at each pole. [*GGSIPU II Sem End Term 2007 Reappear; End Term 2005 Reappear*]

(*b*) Find the nature and location of the singularity of the function $f(z) = z^2 e^{z^{-1}}$. Hence find the residue of $f(z)$ at its pole inside the circle $|z| = 2$.

[*GGSIPU II Sem II Term 2014*]

SOLUTION: (*a*) $f(z) = \frac{z^2}{(z-1)(z-2)^2}$. It's poles are at $z=1$ and $z=2$.

At $z=1$, $f(z)$ has a simple pole, whereas at $z=2$ it has a pole of order 2.

$$\operatorname*{Res}_{z=1} f(z) = \operatorname*{Lt}_{z\to1}(z-1)f(z) = \operatorname*{Lt}_{z\to1}\frac{z^2}{(z-2)^2} = 1,$$

and $$\operatorname*{Res}_{z=2} f(z) = \operatorname*{Lt}_{z\to2}\frac{1}{(2-1)!}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \operatorname*{Lt}_{z\to2}\frac{d}{dz}\left(\frac{z^2}{z-1}\right) = \operatorname*{Lt}_{z\to2}\frac{(z-1)\,2z-z^2\cdot 1}{(z-1)^2} = 0.$$

Therefore, the residue of $f(z)$ at $z=1$ is 1 and at $z=2$ it is 0. **Ans.**

(*b*) $$f(z) = z^2e^{1/z} = z^2\left[1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\ldots\right]$$

or $$f(z) = z^2+z+\frac{1}{2!}+\frac{1}{3!z}+\frac{1}{4!z^2}+\ldots \quad ...(1)$$

$\therefore$ $f(z)$ has singularity at $z=0$ and it is an essential singularity. It lies inside $|z|=2$ and the residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{z}$ in (1).

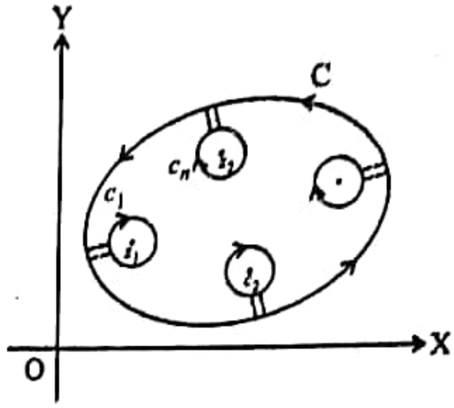
$\therefore$ Res $(f(z)$ at $z=0$ is equal to $\frac{1}{6}$. **Ans.**

CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic at all points in a simple closed curve C except for a finite number of isolated singularities $z_1, z_2, ..., z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \quad \textit{(sum of the residues of } f(z) \textit{ at } z = z_1, z_2, ..., z_n).$$

Let us enclose each of the singular points $z_1, z_2, ..., z_n$ by small (non-intersecting) circles $C_1, C_2, ..., C_n$ lying wholly inside C such that it encloses no other singular point. Then $f(z)$ is analytic in the multiply connected region lying between the curves C and $C_1, C_2, ..., C_n$. This region can be converted into a simply connected region by giving suitable thin cuts as shown by dotted lines in the adjoining figure. The boundary Γ of the simply connected region consists of the curve C (counter clockwise) the circles $C_1, C_2, ..., C_n$ (traversed all clockwise) and the cuts from C to these circles (traversed clockwise and anticlockwise both).



Now, by Cauchy's integral theorem $\int_\Gamma f(z)\,dz = 0$.

Let us here omit the integrals over the cuts since these are clockwise as well as anticlockwise, then we have

$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz - ... - \int_{C_n} f(z)\,dz = 0$$

or
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + ... + \int_{C_n} f(z)\,dz$$

where all the integrations are carried out in counter clockwise direction.

But, as discussed earlier, $\int_{C_i} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_i} f(z)$

$$\therefore \quad \int_C f(z)\,dz = 2\pi i \sum \operatorname*{Res}_{z=z_i} f(z) = 2\pi i \text{ (sum of the residues at } z_1, z_2, ..., z_n).$$

EXAMPLE 18.27. If C is the circle $|z| = 3$ prove that $\int_C \frac{(z+3)\,dz}{(z+1)^2(z-2)} = 0$.

[GGSIPU Sem End Sem 2005, End Term 2007 reppear]

SOLUTION: The function $f(z) = \frac{z+3}{(z+1)^2(z-2)}$ is analytic on C and at all points inside C except at the poles $z = -1$ and $z = 2$.

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z+3}{(z+1)^2} = \frac{5}{9}$$

and
$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)] = \lim_{z\to -1}\frac{d}{dz}\left(\frac{z+3}{z-2}\right) = \lim_{z\to -1}\frac{-5}{(z-2)^2} = \frac{-5}{9}$$

Therefore using the Cauchy's residue theorem

$$\int_C \frac{z+3}{(z+1)^2(z-2)}\,dz = 2\pi i\left(\frac{-5}{9}+\frac{5}{9}\right) = 0. \quad \textbf{Hence Proved.}$$

EXAMPLE 18.28. (*a*) Evaluate the complex integral $\int_C \frac{z^2\, dz}{(z-1)^2\,(z+2)}$ where $C: |z| = 3$.

[GGSIPU II Sem End Term 2005]

(*b*) Evaluate $\int_C \frac{e^z\, dz}{(z+1)^2\,(z-2)}$, $C: |z-1| = 3$. [GGSIPU II Sem End Term 2010]

SOLUTION: (*a*) $\int_C f(z)\, dz = \int_C \frac{z^2\, dz}{(z-1)^2\,(z+2)}$ where $C: |z| = 3$

$f(z)$ has poles of order two at $z = 1$ and at $z = -2$ there is a simple pole.

$$\operatorname*{Res}_{z=-2} f(z) = \operatorname*{Lt}_{z\to-2} (z+2)\, f(z) = \operatorname*{Lt}_{z\to-2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$$\operatorname*{Res}_{z=1} f(z) = \operatorname*{Lt}_{z\to1} \frac{d}{dz}[(z-1)^2\, f(z)] = \operatorname*{Lt}_{z\to1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \operatorname*{Lt}_{z\to1} \frac{z^2+4z}{(z+2)^2} = \frac{5}{9}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of the residues at the singularities lying inside } C)$$

$$= 2\pi i \left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i$$

(*b*) The poles of $f(z) = \frac{e^z}{(z+1)^2\,(z-2)}$ are at $z = -1$ and $z = 2$ and both of them lie inside $C: |z-1| = 3$.

$$\operatorname*{Res}_{z\to-1} f(z) + \operatorname*{Res}_{z=2} f(z) = \operatorname*{Lt}_{z\to-1} \frac{d}{dz}\left(\frac{e^z}{z-2}\right) + \operatorname*{Lt}_{z\to2} \frac{e^z}{(z+1)^2}$$

$$= \operatorname*{Lt}_{z\to-1} \frac{(z-2)\, e^z - e^z}{(z-2)^2} + \frac{e^2}{9} = \frac{-4e^{-1}}{9} + \frac{e^2}{9}$$

By Cauchys residue theorem $\int_C f(z)\, dz = 2\pi i$ (sum of the residues at the poles).

$\therefore$ Value of the given integral $= 2\pi i\left[\frac{e^2}{9} - \frac{4}{9e}\right] = \frac{2\pi i}{9}\left(e^2 - \frac{4}{e}\right)$ **Ans.**

EXAMPLE 18.29. State residue theorem and use it to evaluate

$\int_C \frac{dz}{z^8(z+4)}$ where C is the circle (i) $|z| = 2$ (ii) $|z+2| = 3$.

[GGSIPU II Sem II Term 2005]

SOLUTION: $\int_c f(z)dz = \oint_C \frac{dz}{z^8(z+4)}$. Here $f(z)$ has singularities at $z = 0$ and $z = -4$

(I) $C: |z| = 2$. Here, only the pole at $z = 0$ lies inside C.

$\therefore$ $\int_C \frac{dz}{z^8(z+4)} = 2\pi i$. (Residue of $f(z)$ at $z = 0$ which is pole of [illegible]

$$= 2\pi i \underset{z\to 0}{Lt} \frac{1}{7!}\frac{d^7}{dz^7}\left[\frac{z^8}{z^8(z+4)}\right] = \frac{2\pi i}{7!} \underset{z\to 0}{Lt} \frac{d^7}{dz^7}\left(\frac{1}{z+4}\right) = \frac{2\pi i}{7!} \underset{z\to 0}{Lt} \frac{(-1)^7\, 7!}{(z+4)^8} = \frac{-2\pi i}{4^8}.$$

(ii) $C: |z+2| = 3$. Here both the poles $z = 0$ and $z = 4$ lie inside C.

$$\underset{z=-4}{\text{Res}} f(z) = \underset{z\to -4}{Lt} \frac{(z+4)}{z^8(z+4)} = \underset{z\to -4}{Lt} \frac{1}{z^8} = \frac{1}{(-4)^8} = \frac{1}{(4)^8},$$

and $\underset{z=0}{\text{Res}} f(z) = \dfrac{-1}{4^8}$ (as obtained in case (i)).

$$\therefore \quad \int_C \frac{dz}{z^8(z+4)} = 2\pi i \times (\text{Sum of the residues at } z = 0 \text{ and } z = -4) = 2\pi i\left(-\frac{1}{(4)^8} + \frac{1}{4^8}\right) = 0.$$

Thus the value of the given integral is $\dfrac{-2\pi i}{(4)^8}$ in (i) and 0 in (ii). **Ans.**

EXAMPLE 18.30. Using residue theorem, evaluate $\oint_C \dfrac{dz}{z^4+1}$ where C is the circle $x^2 + y^2 = 2x$.

[GGSIPU II Sem II Term 2005, II Term 2006]

SOLUTION: The function $f(z) = \dfrac{1}{z^4+1}$ has poles given by $z^4 + 1 = 0$

or $\quad z = (-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = \cos\dfrac{2n\pi+\pi}{4} + i\sin\dfrac{2n\pi+\pi}{4}$, $n = 0, 1, 2, 3.$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4},\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4},\ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \left(\pm\frac{1\pm i}{\sqrt{2}}\right)$$

The circle C has centre at $(1, 0)$ and radius 1. The poles of $f(z)$, which lie within C, are $\dfrac{1+i}{\sqrt{2}}$ and $\dfrac{1-i}{\sqrt{2}}$.

$$\text{Residue of } f(z) \left(\text{at } z = \frac{1+i}{\sqrt{2}}\right) = \underset{z\to\frac{1+i}{\sqrt{2}}}{Lt} \frac{z - \frac{1+i}{\sqrt{2}}}{z^4+1} = \underset{z\to\frac{1+i}{\sqrt{2}}}{Lt}\left(\frac{1}{4z^3}\right)$$

$$= \frac{1}{4\cdot\left(\frac{1+i}{\sqrt{2}}\right)^3} = \frac{2\sqrt{2}}{4\cdot(2i)(1+i)} = \frac{-i(1-i)}{4\sqrt{2}} = \frac{-1-i}{4\sqrt{2}},$$

$$\text{and Residure of } f(z) \left(\text{at } z = \frac{1-i}{\sqrt{2}}\right) = \underset{z\to\frac{1-i}{\sqrt{2}}}{Lt} \frac{z - \frac{1-i}{\sqrt{2}}}{z^4+1} = \underset{z\to\frac{1-i}{\sqrt{2}}}{Lt} \cdot\frac{1}{4z^3}$$

$$= \frac{1}{4\left(\frac{1-i}{\sqrt{2}}\right)^3} = \frac{2\sqrt{2}}{4(-2i)(1-i)} = \frac{i-1}{4\sqrt{2}}.$$

By residue theorem

$$\int_C \frac{dz}{z^4+1} = 2\pi i \text{ (sum of the residues)}$$

$$= 2\pi i\left[\frac{-1-i}{4\sqrt{2}} + \frac{i-1}{4\sqrt{2}}\right] = -\frac{\pi i}{\sqrt{2}}.$$

Ans.

EXERCISE 18C

1. (a) Expand $\frac{1}{z}$ by Taylor's series about the point $z = 1$.
   (b) Find the Taylor's expansion of the function
   $$f(z) = \frac{2z^3+1}{z^2+z} \quad \text{about the point } z = i.$$ [GGSIPU II Sem II Term 2006]
2. (a) Expand $\tan z$ as a Maclaurin's series in powers of z.
   (b) Expand $f(z) = \frac{1}{(z+1)^2}$ about the point $z = -i$. [GGSIPU II Sem II Term 2014]
3. Determine the Laurent's expansion for $f(z) = \frac{1}{(1-z)(-z+2)}$ valid for the domain $1 < |z| < 2$.
4. (a) Determine the order of the pole of the function $f(z) = \frac{\sinh z}{z^5}$.
   (b) What type of singularity have the following functions
   (i) $\frac{\tan z}{z}$ (ii) $e^{z^{-3}}$ [GGSIPU II Term End Term 2009]
5. Show that when $|z+1| < 1$, $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$. (GGSIPU II Sem End Term 2009)
6. Find the residue of $f(z) = \frac{3z-4}{z(z-1)(z-2)}$ at each of its poles.
7. Obtain the residue of $f(z) = \left(\frac{z+1}{z-1}\right)^3$ at its pole.
8. Evaluate the integral $\int_C \frac{e^z - 1}{z(z+1)}\, dz$ where $C : |z| = 2$.
9. Find the value of the integral $\int_C \frac{dz}{z^3(z+4)}$ where $C : |z| = 2$.
10. Determine the residues of $f(z) = \frac{z^2}{(z-2)(z^2+1)}$ at its poles.
11. Find the residues of $\frac{1}{z(z+2)^3}$ at $z = 0$ and $z = -2$. [GGSIPU II Sem End Term 2009]
12. Evaluate $\oint_C \frac{e^z\, dz}{(z-1)(z+3)^2}$ where (i) $C : |z| = 3/2$ (ii) $C : |z| = 10$.
13. (a) Evaluate $\int_C \tan z\, dz$ where $C : |z| = 2$.
    (b) Evaluate the integral $\int_C e^{1/z^2}\, dz$ where C is the circle $|z| = 2$.
14. (a) Evaluate $\int_C \frac{\sin z\, dz}{z^6}$ where C is the circle $|z| = 2$ taken in the anticlockwise direction.
    (b) Evaluate $\int_C \frac{dz}{z \sin z}$ where C is the unit circle $|z| = 1$ described in the positive

EVALUATION OF REAL DEFINITE INTEGRALS

The Cauchy's Residue theorem provides a simple and effective method for evaluating a certain type of real integrals. Some of the commonly faced types are illustrated below :

I. Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

In this type we consider θ as the argument of z on the unit circle $z = e^{i\theta}$, then the above integral becomes a contour integral of a rational function of z around the unit circle. Such a process is known as contour integration.

And $$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Here $$dz = i e^{i\theta} d\theta \quad \text{or} \quad d\theta = \frac{dz}{iz}.$$

Also, as θ varies from 0 to 2π, as z moves one round along the unit circle C.

Hence $$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\frac{dz}{iz}. \quad (C : |z| = 1).$$

Then, we use Cauchys Residue theorem to evaluate the integral on the R.H.S. of the above relation.

EXAMPLE 18.31. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$, $a^2 < 1$. *[GGSIPU II Sem End Term 2006]*

SOLUTION: Putting $z = e^{i\theta}$, the given integral for $C : |z| = 1$, becomes

$$\int_C \frac{dz/(iz)}{1 - a\left(z + \frac{1}{z}\right) + a^2} = \int_C \frac{dz}{i\{(1+a^2)z - az^2 - a\}} = \int_C \frac{dz}{i(1-az)(z-a)}.$$

The integrand has poles at $z = a$, $z = 1/a$ but only the first one lies inside the unit circle C. The residue at this pole, is equal to

$$\lim_{z \to a} \frac{1}{i(1-az)} = \frac{1}{i(1-a^2)}$$

Therefore, by the Residue theorem, the given integral has the value $\dfrac{2\pi i}{i(1-a^2)} = \dfrac{2\pi}{1-a^2}$. **Ans.**

EXAMPLE 18.32. Evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2}$.

SOLUTION: Applying the property of definite integration,

$$\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

we get $$I = \int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

$$= \frac{1}{2}\int_C \frac{dz/(iz)}{\left[2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2}$$ (using the substitution $z = e^{i\theta}$)

$$= \frac{1}{2}\int_C \frac{dz \cdot 4z^2}{iz\,(4z+z^2+1)^2} = \frac{2}{i}\int_C \frac{z\,dz}{(z^2+4z+1)^2}$$ where $C : |z| = 1$.

The poles of $f(z) = \dfrac{2z}{i(z^2+4z+1)^2}$ are at $z = -2 \pm \sqrt{3}$ both of order two.

Let $z_1 = -2+\sqrt{3}$ and $z_2 = -2-\sqrt{3}$, out of these only $-2+\sqrt{3}$ lies inside C.

$$\therefore \quad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{d}{dz}[(z-z_1)^2 f(z)] = \lim_{z\to z_1} \frac{d}{dz}\left[\frac{2}{i}\frac{z}{(z-z_1)^2}\frac{(z-z_1)^2}{(z-z_2)^2}\right]$$

$$= \lim_{z\to z_1} \frac{d}{dz}\left[\frac{2z}{i(z-z_2)^2}\right] = \lim_{z\to z_1} \frac{-2(z+z_2)}{i(z-z_2)^3}$$

$$= \frac{-2}{i}\frac{z_1+z_2}{(z_1-z_2)^3} = \frac{-2}{i}\frac{(-4)}{(2\sqrt{3})^3} = \frac{1}{3i\sqrt{3}}$$

Therefore $$I = 2\pi i \operatorname*{Res}_{z=z_1} f(z) = \frac{2\pi i}{3i\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}.$$ **Ans.**

EXAMPLE 18.33. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > |b|$ and hence, show that

$$\int_0^{\pi} \frac{d\theta}{17-8\cos\theta} = \frac{\pi}{15}.$$ *[GGSIPU II Sem End Term 2007; II Term 2010]*

SOLUTION: Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$. Let us take $C : |z| = 1 \;\therefore\; z = e^{i\theta}$ so $dz = ie^{i\theta}\,d\theta$

then $$I = \int_C \frac{dz}{iz\left[a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right]} = \frac{2}{i}\int \frac{dz}{2az+bz^2+b}$$

The singularities of the integrand are the roots of $bz^2 + 2az + b = 0$

or $$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$ out of the which $z = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ lies inside C.

$\therefore$ By Cauchy's integral formula $$I = \frac{2}{i}\int_C \frac{dz}{b\left[z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right]\left[z-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$

$$2\pi i\left(\frac{2}{i}\right)\frac{1}{b\cdot\frac{2}{b}\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Next, taking $a = 17$ and $b = -8$ we get $\displaystyle\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta} = \frac{2\pi}{15} \;\therefore\; \int_0^{\pi} \frac{d\theta}{17-8\cos\theta} = \frac{\pi}{15}.$ **Ans.**

EXAMPLE 18.34. Apply the calculus of residues to evaluate

$$\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta} \quad \text{where} \quad |a|<1 \qquad \text{[GGSIPU II Sem End Term 2003]}$$

SOLUTION: Given integral can be written as $I = \frac{1}{2}\int_0^{2\pi} \frac{a\, d\theta}{a^2+\sin^2\theta}$

Here consider C as the unit circle $|z|=1$, $z=e^{i\theta}$ and $dz = ie^{i\theta}\, d\theta$ (by property of definite integral)

$$\therefore \quad I = a\int_0^{2\pi} \frac{d\theta}{2a^2+(1-\cos 2\theta)} = a\int_C \frac{dz}{iz\left[2a^2+1-\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\right]}$$

$$= \frac{a}{i}\int \frac{zdz}{(4a^2+2)\,z^2-z^4-1} = ai\int \frac{zdz}{z^4+1-4z^2a^2-2z^2}$$

$$= ai\int \frac{zdz}{(z^2-2az-1)\,(z^2+2az-1)} = ai\int \frac{zdz}{[(z-a)^2-(a^2+1)]\,[z+a)^2-(a^2+1)]}$$

$$= ai\int \frac{zdz}{\left(z-a+\sqrt{a^2+1}\right)\left(z-a-\sqrt{a^2+1}\right)\left(z+a+\sqrt{a^2+1}\right)\left(z+a-\sqrt{a^2+1}\right)}$$

The poles of the integrand are at $(\pm a \pm \sqrt{a^2+1})$. Out the these $z_1 = a+\sqrt{a^2+1}$ and $z_2 = -a-\sqrt{a^2+1}$ are lying outside C while $z_3 = -a+\sqrt{a^2+1}$ and $z_4 = a-\sqrt{a^2+1}$ lie inside C.

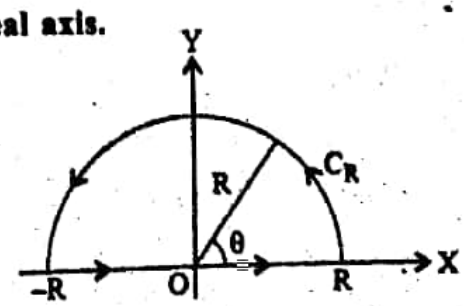
Now residue of the integrand at $z_3 = \underset{z\to z_3}{Lt} \dfrac{z}{\left(z-a+\sqrt{a^2+1}\right)\left(z-a-\sqrt{a^2+1}\right)\left(z+a+\sqrt{a^2+1}\right)}$

$$= \frac{-a+\sqrt{a^2+1}}{\left(-2a+2\sqrt{a^2+1}\right)\left(-2a\ 2\sqrt{a^2+1}\right)} = \frac{1}{-8a\sqrt{a^2+1}}$$

Similarly, the residue of the integrand at $z = z_4$, is equal to $\dfrac{1}{-8a\sqrt{a^2+1}}$

$$\therefore \quad \text{Value of given integral} = ai\cdot\frac{2\pi i\cdot 2}{-8a\sqrt{a^2+1}} = \frac{\pi}{2\sqrt{a^2+1}} \quad \textbf{Ans.}$$

II. Integrals of the type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$ **where** $f(x)$ **and** $F(x)$ **are polynomials in** x **such that** $\frac{x f(x)}{F(x)} \to 0$ **as** $x\to\infty$**, and** $F(x)$ **has no zeros on the real axis.**



To evaluate $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$ we consider $\int_C \frac{f(z)}{F(z)}\,dz$ where C is the closed contour consisting of the real axis from $-R$ to R and the semi-circle C_R of radius R in the upper half-plane. For sufficiently large R, the value of the integral is

$$= 2\pi i \text{ [sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane.]}$$

$$\text{Now} \int_C \frac{f(z)\,dz}{F(z)} = \int_{-R}^{R} \frac{f(z)\,dz}{F(z)} + \int_{C_R} \frac{f(z)\,dz}{F(z)} = \int_{-R}^{R} \frac{f(x)\,dx}{F(x)} + \int_{C_R} \frac{f(z)\,dz}{F(z)} \text{ (as } z = Re^{i\theta} \text{ on } C_R)$$

$$= \int_{-R}^{R} \frac{f(x)\,dx}{F(x)} + \int_0^{\pi} \frac{f(Re^{i\theta})\,Re^{i\theta}}{F(Re^{i\theta})}\,i\,d\theta$$

For large R, $\left|\int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta} \cdot i\,d\theta\right|$ is of the order of $\frac{R\,f(R)}{F(R)}$ which is given to tend to zero as $R \to \infty$. Therefore, the second integral on the R.H.S. vanishes when $R \to \infty$ and we are left with $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$. Following examples will make the concept more clear.

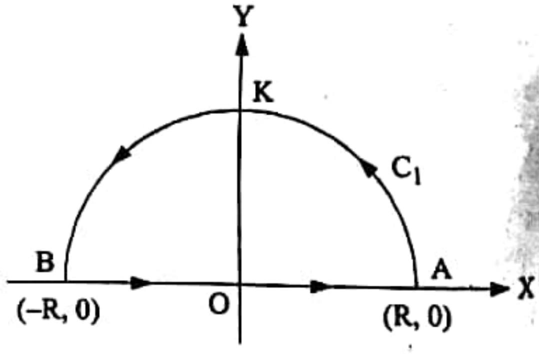
EXAMPLE 18.35. Apply Residue theorem to evaluate $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx$

[*GGSIPU II Sem II Term 2006, End Term 2011*]

SOLUTION: Since $\cos ax$ is the real part of e^{aix} we consider the function $f(z) = \frac{e^{aiz}}{z^2+1}$ which has poles at $z = i$ and $z = -i$ of which only $z = i$ is in the upper half plane.

$$\therefore \quad \text{Residue of } f(z) \text{ (at } z = i) = \underset{z \to i}{Lt} \frac{(z-i)\,e^{aiz}}{z^2+1}$$

$$= \underset{z \to i}{Lt} \frac{e^{aiz}}{z+i} = \frac{e^{-a}}{2i}.$$



Let C be the contour consisting of C_1 the semi-circle AKB and the line segment from B to A along x-axis.

$$\text{By residue theorem} \int_C f(z)\,dz = \int_{C_1} \frac{e^{aiz}\,dz}{(z^2+a^2)} + \int_{-R}^{R} \frac{e^{aix}\,dx}{x^2+a^2}$$

$$\text{But} \int_C \frac{a^{aiz}}{z^2+1}\,dz = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a},$$

The semicircle C_1 is $|z| = R$ where R is large, $z = Re^{i\theta}$, $0 \le \theta \le \pi$.

$$\therefore \quad \int_{C_1} \frac{e^{aiz}\,dz}{z^2+1} = \int_{C_1} \frac{e^{aiRe^{i\theta}} \cdot Rie^{i\theta}\,d\theta}{R^2 e^{2i\theta}+1} \to 0 \text{ as } R \to \infty.$$

$$\text{and} \quad \int_{BA} \frac{e^{aix}\,dx}{x^2+1} = \int_{-R}^{R} \frac{e^{aix}\,dx}{x^2+1} \to \int_{-\infty}^{\infty} \frac{e^{aix}\,dx}{x^2+1} \text{ as } R \to \infty.$$

$$\text{Thus,} \quad \pi e^{-a} = 0 + \int_{-\infty}^{\infty} \frac{e^{aix}\,dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{\cos ax + i \sin ax}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx + 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\cos ax\,dx}{x^2+1} = \pi e^{-a} \qquad \therefore \quad \int_0^{\infty} \frac{\cos ax\,dx}{x^2+1} = \frac{\pi e^{-a}}{2} \qquad \textbf{Ans.}$$

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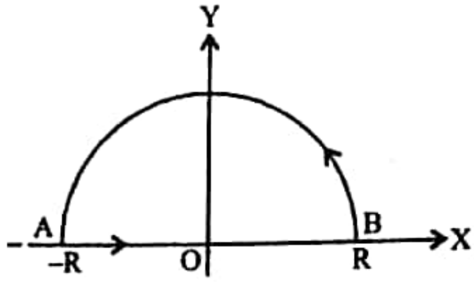
EXAMPLE 18.36. (a) Evaluate $\int_0^\infty \frac{dx}{x^4+1}$. [GGSIPU II Sem II Term 2011; II Term 2013]

(b) Using residue theorem, prove that $\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$. [GGSIPU II Sem II Term 2014]

SOLUTION: (a) Since the integrand is even function of x, we have



$$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^4+1}.$$

Further, since $\frac{x}{x^4+1} \to 0$ as $x \to \infty$, we have

$$\int_C \frac{1}{z^4+1}\,dz = \int_{-R}^{R} \frac{dx}{x^4+1} + \int_0^\pi \frac{R e^{i\theta}\, i\,d\theta}{R^4 e^{4i\theta}+1} \quad \text{(taking } z = R e^{i\theta})$$

where C consists of the semi-circle $|z| = R$ above the X-axis and the line segment from A to B along real axis.

Taking the limit as $R \to \infty$, we see that $\frac{R e^{i\theta}}{R^4 e^{4i\theta}+1} = \frac{R^{-3} e^{i\theta}}{e^{4i\theta}+R^{-4}} \to 0,$

hence the second integral on the R.H.S. of the above equation, vanishes.

The poles of the function $\frac{1}{z^4+1}$ are given by $z^4+1=0$ or $z^4 = -1 = e^{i(2n+1)\pi}$,

$n = 0, 1, 2, 3$, hence $z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$. Out of these four singularities, only the first two lie in the upper half plane.

The residue at the simple pole $z = z_0$ of $f(z)$ is equal to

$$\lim_{z\to z_0} \frac{(z-z_0)}{z^4+1} = \lim_{z\to z_0} \frac{1}{4z^3} \text{ (by L'Hospital's rule)} = \frac{1}{4z_0^3}.$$

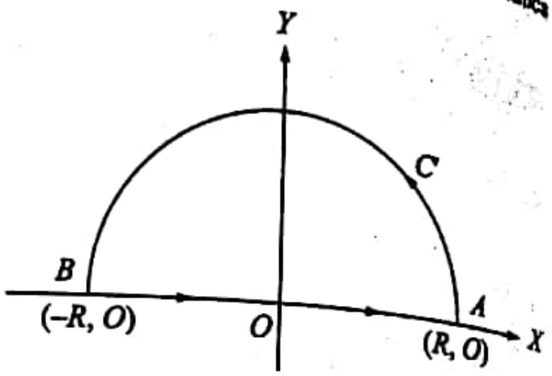
$\therefore$ Sum of the residues at $z = e^{i\pi/4}$ and $e^{3i\pi/4}$ is equal to

$$\frac{1}{4}[e^{-3\pi i/4} + e^{-9\pi i/4}] = \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = -\frac{i}{2\sqrt{2}}$$

$$\therefore \quad \int_C \frac{dz}{z^4+1} = 2\pi i\left(\frac{-i}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

Therefore $\int_{-\infty}^\infty \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$ and, in turn $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$. **Ans.**



(b) Consider $I = \int_C \frac{dz}{z^6+1} = \int_C f(z)\,dz$ where C is closed contour consisting of C′, the semi-circle, $|z| = R$ (R being large) from A(R, 0) to B(–R, 0) above the X-axis in anti-clockwise direction, and the portion of X-axis from B to A, (see figure).

The singularities of $f(z)$ are given by

$$z^6 + 1 = 0 \quad \text{or} \quad z = (-1)^{1/6} = (\cos \pi + i \sin \pi)^{1/6}$$

or
$$z = \cos\left(\frac{\pi + 2r\pi}{6}\right) + i \sin\left(\frac{\pi+2r\pi}{6}\right) = e^{i(2r+1)\pi/6}, \quad r = 0, 1, 2, 3, 4, 5.$$

Out of these six poles, $e^{i\pi/6}$, $e^{3\pi i/6}$ and $e^{5\pi i/6}$ lie inside C.

By residue theorem $\int_C \frac{dz}{z^6+1} = 2\pi i$ (sum of the residues of $f(z)$ at $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5\pi i/6}$)

$$\operatorname*{Res}_{z=e^{i\pi/6}} f(z) = \lim_{z=e^{i\pi/6}} f(z)\left(\frac{z-e^{i\pi/6}}{z^6+1}\right) = \lim_{z\to e^{i\pi/6}}\left(\frac{1}{6z^5}\right) = \frac{1}{6}e^{-5\pi i/6},$$

$$\operatorname*{Res}_{z=e^{i\pi/2}} f(z) = \lim_{z\to e^{i\pi/2}}\left(\frac{z-e^{i\pi/2}}{z^6+1}\right) = \lim_{z\to e^{i\pi/2}} \frac{1}{6z^5} = \frac{1}{6\,e^{5i\pi/2}} = \frac{1}{6}e^{-i\pi/2}$$

and
$$\operatorname*{Res}_{z=e^{5\pi i/6}} f(z) = \lim_{z\to e^{5\pi i/6}}\left(\frac{z-e^{5\pi i/6}}{z^6+1}\right) = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6\,e^{25\pi i/6}} = \frac{1}{6}e^{-\pi i/6}$$

$\therefore$
$$\int_C \frac{dz}{z^6+1} = \frac{2\pi i}{6}\left[e^{-5\pi i/6} + e^{-i\pi/2} + e^{-\pi i/6}\right]$$

$$= \frac{\pi i}{3}\left[\cos\left(\pi - \frac{\pi}{6}\right) - i \sin\left(\pi - \frac{\pi}{6}\right) + \cos\frac{\pi}{2} - i \sin\frac{\pi}{2} + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$$

$$= \frac{\pi i}{3}\left[-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} + 0 - i + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right] = \frac{2\pi}{3}.$$

Also
$$\int_C \frac{dz}{z^6+1} = \int_{C'}\frac{dz}{z^6+1} + \int_{-R}^{R}\frac{dx}{x^6+1}.$$ On C′, $|z| = R$, $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$.

$$= \int_0^{\pi} \frac{Rie^{i\theta}d\theta}{R^6 e^{6i\theta}} + 2\int_0^R \frac{dx}{x^6+1}$$

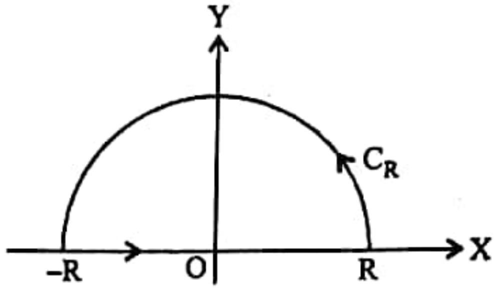
as $R \to \infty$ $\int_0^{\pi}\frac{iRe^{i\theta}d\theta}{R^6e^{6i\theta}+1} \to 0$ and $\int_0^R \frac{dx}{x^6+1} \to \int_0^{\infty}\frac{dx}{x^6+1}$.

Therefore, $\frac{2\pi}{3} = 0 + 2\int_0^{\infty}\frac{dx}{x^6+1}$ or $\int_0^{\infty}\frac{dx}{x^6+1} = \frac{\pi}{3}$. **Ans.**

EXAMPLE 18.37. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+1)^2}$

SOLUTION: Since $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+1)^2} = \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+1)^2}$ let us consider the integral $\int_C \frac{e^{iz}}{(z^2+1)^2}\, dz$

where C is the closed curve consisting of the semi-circle C_R of radius R with centre at the origin and real axis segment from $x = -R$ to $x = R$.



Here $f(z) = \frac{e^{iz}}{(z^2+1)^2}$ has second order poles at $z = \pm i$, out of which only $z = i$ lies in the upper half plane.

Also, since $\frac{x\, e^{ix}}{(x^2+1)^2}$ tends to zero as x tends to infinity, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\, dx + \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\, dz$$

As discussed earlier $\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\, dz \to 0$ as $R \to \infty$ $\therefore$ $\int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x^2+1)^2} = \int_C f(z)\, dz$

and hence $\int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z)$

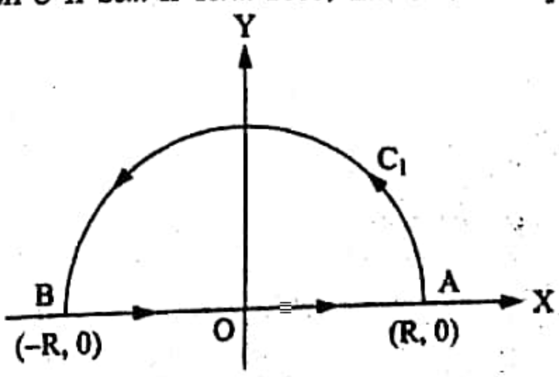
$$= 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2\, e^{iz}}{(z-i)^2\,(z+i)^2} = 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{e^{iz}}{(z+i)^2}$$

$$= 2\pi i \lim_{z\to i} \frac{i\, e^{iz}(z+i) - 2\, e^{iz}}{(z+i)^3} = 2\pi i\, \frac{e^{-1}[i(i+i)-2]}{(2i)^3} = \frac{\pi}{e}.$$ **Ans.**

EXAMPLE 18.38. Using contour integration show that $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

[GGSIPU II Sem II Term 2005; End Term 2013]

SOLUTION: The given integral $= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$.



Now consider $\int_C \frac{dz}{(z^2+a^2)^2}$ where C consists of C_1, the semicircle $|z| = R$ from A to B in anticlockwise direction and the line from B to A along x-axis.

$$\therefore \quad \int_C \frac{dz}{(z^2+a^2)^2} = \int_{C_1} \frac{dz}{(z^2+a^2)^2} + \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2}$$

The integrand $\frac{1}{(z^2+a^2)^2}$ has poles at $z = ai$ and $z = -ai$ both being of order two, out of which only $z = ai$ lies inside C. Therefore, by Cauchy's residue theorem

$$\int_C \frac{dz}{(z^2+a^2)^2} = 2\pi i \operatorname*{Res}_{z=ai} \frac{1}{(z^2+a^2)^2} = 2\pi i \operatorname*{Lt}_{z\to ai} \cdot \frac{d}{dz} \frac{(z-ai)^2}{(z+ai)^2 (z-ai)^2}$$

$$= 2\pi i \operatorname*{Lt}_{z\to ai} \frac{d}{dz} \frac{1}{(z+ai)^2} = 2\pi i \operatorname*{Lt}_{z\to ai} \frac{-2}{(z+ai)^3} = \frac{-4\pi i}{(2ai)^3} = \frac{\pi}{2a^3}.$$

Further $$\int_{C_1} \frac{dz}{(z^2+a^2)^2} = \int_0^{\pi} \frac{Ri e^{i\theta}\, d\theta}{(R^2 e^{2i\theta}+a^2)^2} \to 0 \quad (\text{as } R \to \infty)$$

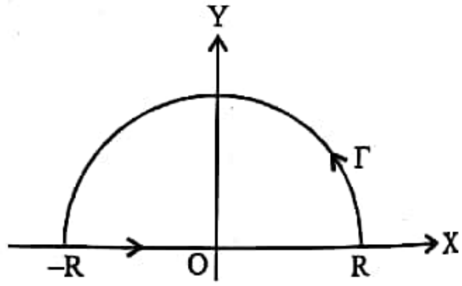
and $$\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} \to \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} \ (\text{as } R \to \infty)$$

$\therefore$ $$\int_C \frac{dz}{(z^2+a^2)^2} = 0 + \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

Hence $$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.$$ **Ans.**

EXAMPLE 18.39. Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \cdot \frac{dx}{(x^2+2x+2)} = \frac{7\pi}{50}.$

SOLUTION: The contour C consists of the semi circle Γ and the real axis segment from $-$ R to R, taken in counter clockwise direction.



The poles of $f(z) = \frac{z^2}{(z^2+1)^2 (z^2+2z+2)}$ are $z = i$ and $z = -1+i$ of order 2 and 1 respectively which lie inside C.

$$\text{Residue of } f(z)_{z=1} = \lim_{z\to i} \frac{d}{dz}\left[\frac{(z-i)^2 z^2}{(z+i)^2 (z-i)^2 (z^2+2z+2)}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2 (z^2+2z+2)}\right] = \frac{9i-12}{100} \text{ (on simplification).}$$

and $$\text{Residue of } f(z)_{z=-1+i} = \lim_{z\to -1+i} \frac{(z+1-i)\, z^2}{(z^2+1)^2 (z+1+i)(z+1-i)}$$

$$= \lim_{z\to -1+i} \frac{z^2}{(z^2+1)^2 (z+1+i)} = \frac{3-4i}{25} \text{ (on simplification).}$$

Therefore, using Cauchy's residue theorem, we get

Then $$\int_C \frac{z^2\, dz}{(z^2+1)^2 (z^2+2z+2)} = 2\pi i\left[\frac{9i-12}{100} + \frac{3-4i}{25}\right] = \frac{7\pi}{50}$$

or $$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2\,(x^2+2x+2)} + \int_{\Gamma} \frac{z^2\,dz}{(z^2+1)^2\,(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral tends to 0 on the same lines as discussed in the last problems, we get the result.

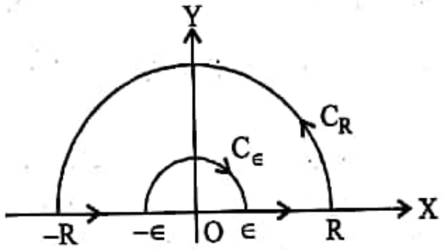
Hence Proved.

III. Integrals of the type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$ when $F(x)$ has zeros on the real axis.

In such cases we proceed in the same way as in type II except that the contour C is indented at the singularities on the real axis by small semi-circles so that the singularities are not included in C, as illustrated in the following example.

EXAMPLE 18.40. Evaluate $\int_{0}^{\infty} \frac{\sin ax}{x}\,dx,\ a > 0.$ [GGSIPU II Sem II Term 2012]

SOLUTION: Consider the function $f(z) = \frac{e^{aiz}}{z}$. It has a simple pole at $z = 0$. Consider the integral $\int_C f(z)\,dz$ where C consists of the parts of real axis from $-R$ to $-\in$ and from $\in$ to R, the small semi-circle $C_\in$ with centre at the origin and radius $\in$ which is small, and the large semi-circle C_R as shown in the adjoining figure.



Now $f(z)$ is analytic inside C (the only singularity $z = 0$ has been deleted by indenting the origin by drawing $C_\in$ therefore, by Cauchy's integral theorem, we have

$$\int_{\in}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz + \int_{-R}^{-\in} f(x)\,dx + \int_{C_\in} f(z)\,dz = 0 \qquad ...(1)$$

For C_R we have $z = Re^{i\theta},\ 0 \le \theta \le \pi$ and for $C_\in$ we have $z = \in e^{i\theta},\ 0 \le \theta \le \pi.$

Now $$\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{e^{aiRe^{i\theta}}}{Re^{i\theta}} \cdot Rie^{i\theta}\,d\theta = i\int_0^{\pi} e^{aiR(\cos\theta + i\sin\theta)}\,d\theta = i\int_0^{\pi} e^{-aR\sin\theta}\,e^{aiR\cos\theta}\,d\theta$$

$$\therefore \quad \left|\int_{C_R} f(z)\,dz\right| \le \int_0^{\pi} e^{-aR\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta \to 0 \quad \text{as} \quad R \to \infty.$$

Further $$\int_{C_\in} f(z)\,dz = \int_{C_\in} \frac{e^{ai\in e^{i\theta}}\,i\in e^{i\theta}}{\in e^{i\theta}}\,d\theta = i\int_{C_\in} e^{ai\in(\cos\theta + i\sin\theta)}\,d\theta$$

$$= i\int_{\pi}^{0} e^{ai\in(\cos\theta + i\sin\theta)}\,d\theta \longrightarrow i\int_{\pi}^{0} e^{0}\,d\theta = -i\pi, \quad \text{as } \in \to 0.$$

Next, $$\int_{-R}^{-\in} f(x)\,dx = \int_{-R}^{-\in} \frac{e^{aix}}{x}\,dx \quad (\text{putting } x = -t)$$

$$= \int_{R}^{\in} \frac{e^{-ait}}{-t}(-dt) = -\int_{\in}^{R} \frac{e^{-ait}}{t}\,dt = -\int_{\in}^{R} \frac{e^{-aix}}{x}\,dx$$

$$\therefore \quad \int_{\epsilon}^{R} f(x)\,dx + \int_{-R}^{-\epsilon} f(x)\,dx = \int_{\epsilon}^{R} \frac{1}{x}(e^{aix} - e^{-aix})\,dx = \int_{\epsilon}^{R} \frac{2i}{x} \sin ax\,dx.$$

This tends to $\int_0^{\infty} \frac{2i}{x} \sin ax\,dx$ as $\epsilon \to 0$ and $R \to \infty$. Therefore, (1) becomes

$$2i \int_0^{\infty} \frac{\sin ax}{x}\,dx - i\pi = 0 \quad \therefore \quad \int_0^{\infty} \frac{\sin ax}{x}\,dx = \frac{\pi}{2}.$$ **Ans.**

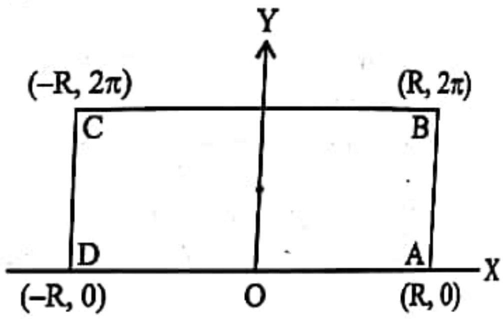
IV. Integrals around rectangular contours.

The evaluation of this type of integrals is illustrated in the following example.

EXAMPLE 18.41. Show that $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1}\,dx = \frac{\pi}{\sin a\pi}$ where $0 < a < 1$

SOLUTION: Let $f(z) = \frac{e^{az}}{e^z + 1}$ and consider $\int_C f(z)\,dz$ where C is the rectangle ABCD with vertices A (R, 0), B (R, 2π), C (–R, 2π) and D (–R, 0), R being positive and large.

The poles of $f(z)$ are given by $e^z + 1 = 0$ or $e^z = -1 = e^{(2n+1)\pi i}$ or $z = (2n+1)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ The only pole lying inside the rectangle ABCD is $z = \pi i$. Therefore, using Cauchy's residue theorem, we get



$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z)$$

$$= 2\pi i \lim_{z \to \pi i} \frac{(z - \pi i)\,e^{az}}{(e^z + 1)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= 2\pi i \lim_{z \to \pi i} \frac{a e^{az}(z - \pi i) + 1 \cdot e^{az}}{e^z} \quad \text{(on using L'Hospital's rule)}$$

$$= 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i\, e^{a\pi i} \quad \text{since} \quad e^{\pi i} = -1. \qquad \ldots(1)$$

Next, $\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$

$$= \int_0^{2\pi} f(R + iy)\,i\,dy + \int_{+R}^{-R} f(x + 2\pi i)\,dx + \int_{2\pi}^{0} f(-R + iy)\,i\,dy + \int_{-R}^{R} f(x)\,dx$$

since $z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.

$$\therefore \int_C f(z)\,dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy} + 1}\,dy - \int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{e^{(x+2\pi i)} + 1}\,dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy} + 1}\,dy + \int_{-R}^{R} \frac{e^{ax}}{e^x + 1}\,dx \ldots(2)$$

In the first integral on the R.H.S. of (2), we have

$$\left|\frac{e^{a(R+iy)}}{e^{R+iy}+1}\right| \leq \frac{e^{aR}}{e^{R}-1} \longrightarrow 0 \text{ as } R \to \infty \text{ since } 0<a<1.$$

Similarly, in the third integral on the R.H.S. of (2), we have

$$\left|\frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\right| \leq \frac{e^{-aR}}{1-e^{-R}} \longrightarrow 0 \text{ as } R \to \infty \text{ as } a>0 \text{ here.}$$

$$\therefore \quad \int_C f(z)\, dz = -e^{2a\pi i} \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\, dx + \int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{e^x+1} = (1-e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{e^x+1}$$

Thus from (1) and (3), we get $(1-e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{e^x+1} = -2\pi i e^{a\pi i}$

or $$\int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{e^x+1} = \frac{-2\pi i}{a^{-\pi/2}-e^{\pi i}} = \frac{\pi}{\sin a\pi}.$$

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CHAPTER

19

Multiple Integrals—Double and Triple Integrals

Double Integrals, Change of Order of Integration, Triple Integrals and Their Application in finding Area and Volume.

DOUBLE INTEGRAL—DEFINITION AND EVALUATION

Let $f(x, y)$ be a continuous function of x and y, defined within and on a region R bounded by a closed curve C in the xy-plane. Let the region R be subdivided into n subregions of areas $\delta A_1, \delta A_2, ..., \delta A_n$. Let (x_r, y_r) be any point within the r^{th} subregion δA_r.

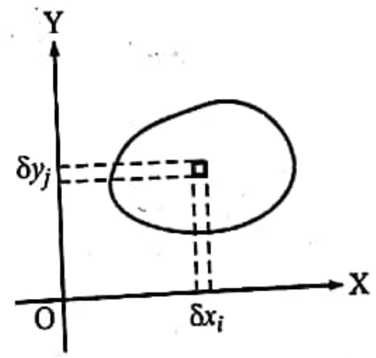
Now consider the sum $\sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r$

The limit of this sum as $n \to \infty$ and $\delta A_r \to 0$ $(r = 1, 2, ..., n)$, is defined as the double integral of $f(x, y)$ over the region R and is written as

$$\iint_R f(x, y)\, dA. \qquad ...(1)$$

To evaluate the double integral we subdivide R by lines parallel to the coordinate axes creating a rectangular grid. Since the area of the typical rectangle is $\delta x_i\, \delta y_j$, we have

$$\iint_R f(x, y)\, dx\, dy = \lim_{n \to \infty} \sum_{i, j=1}^{n} f(x_i, y_j)\, \delta x_i\, \delta y_j.$$

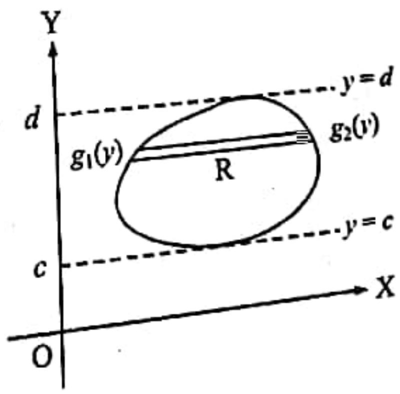


The above definition can obviously be extended to three dimensions giving rise to triple integrals.

The double integral over a region R is evaluated by two successive single integrals. Suppose the region R is described by the inequalities

$$c \le y \le d \text{ and } g_1(y) \le x \le g_2(y) \qquad ...(2)$$

as shown in the adjoining figure,

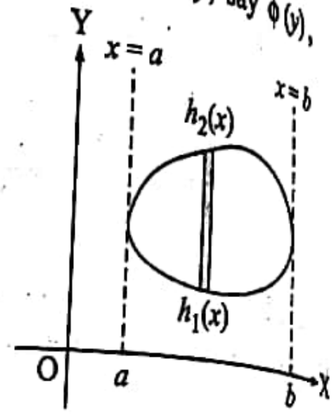


then $$\iint_R f(x,y)\, dx\, dy = \int_{y=c}^{y=d} \left[\int_{x=g_1(y)}^{x=g_2(y)} f(x, y)\, dx \right] dy \qquad ...(3)$$

First the inner integral $\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx$ is evaluated with respect to x treating y as constant. Since here the integrand and the limits are functions of y, the value of this integral would be a function of y, say $\phi(y)$,

hence $$\iint_R f(x,y)\,dx\,dy = \int_c^d \phi(y)\,dy \qquad \text{...(4)}$$

which can be easily evaluated.



Alternatively, if the region R is described by the inequalities

$a \le x \le b$ and $h_1(x) \le y \le h_2(x)$ .... (5)

as shown in the adjoining figure, then we write

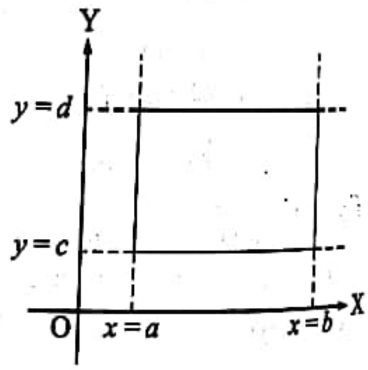
$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx.$$

$$= \int_{x=a}^{x=b}\left[\int_{y=h_1(x)}^{y=h_2(x)} f(x,y)\,dy\right]dx. \qquad \text{...(6)}$$

The inner integral $\int_{y=h_1(x)}^{y=h_2(x)} f(x,y)\,dy$ is a function of x, equal to F (x) say, then

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx = \int_a^b F(x)\,dx \qquad \text{...(7)}$$

which can be easily evaluated.



If the region R is a rectangle bounded by the lines $x = a, x = b;\ \ y = c, y = d$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d\int_a^b f(x,y)\,dx\,dy \qquad \text{...(8)}$$

from which it is quite clear that the *order of integration is immaterial when both limits of x and y are constants.*

EXAMPLE 19.1. Evaluate (i) $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx.$ (ii) $\displaystyle\int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}\,x\,dx\,dy.$

SOLUTION: (i) Let $$I = \int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx = \int_0^1\left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\right]dx.$$

We first evalaute $\displaystyle\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy$ where x has been treated as constant, hence

$$\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy = \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}(\tan^{-1}1-\tan^{-1}0) = \frac{\pi}{4\sqrt{1+x^2}}$$

$$\therefore \quad I = \int_0^1 \frac{\pi\,dx}{4\sqrt{1+x^2}} = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1 = \frac{\pi}{4}\log\left(1+\sqrt{2}\right). \qquad \textbf{Ans.}$$

(ii) Both the limits of integration for x and y are constants hence we can integrate with respect to any variable first. Let us integrate, for convinience, first w.r.t. x, then

$$I = \int_0^\infty \left[\int_0^\infty e^{-x^2(1+y^2)}\,x\,dx\right]dy = \int_0^\infty dy \int_0^\infty e^{-x^2(1+y^2)}\,x\,dx$$

Putting $x^2(1+y^2) = t$ in the inner integral and treating y as constant, we get

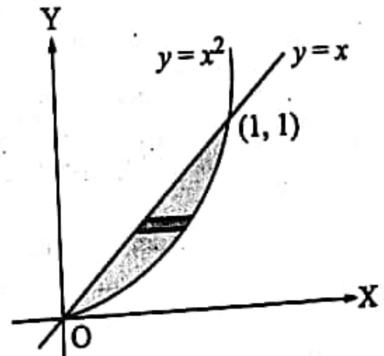
$$\int_0^\infty e^{-x^2(1+y^2)}\,x\,dx = \int_0^\infty e^{-t}\frac{dt}{2(1+y^2)} = \frac{1}{2(1+y^2)}\int_0^\infty e^{-t}\,dt$$

$$= \frac{1}{2(1+y^2)}[-e^{-t}]_0^\infty = \frac{1}{2(1+y^2)}$$

$$\therefore \quad I = \int_0^\infty \frac{1}{2(1+y^2)}\,dy = \frac{1}{2}[\tan^{-1}y]_0^\infty = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}. \qquad \textbf{Ans.}$$

EXAMPLE 19.2. Evaluate $\iint xy\,(x+y)\,dx\,dy$ over the region bounded by the line $y = x$ and the curve $y = x^2$. *[GGSIPU I Sem End Term 2004 Reapperar]*

SOLUTION: The region of integration is as shown in the adjoining figure. Let $I = \iint xy\,(x+y)\,dx\,dy$.



Let us first integrate with respect to x and then with respect to y. As shown in the adjoing figure we take an elementary strip parallel to x-axis and the limits for x are clearly from the line $x = y$ to the curve $x = \sqrt{y}$ and then the limits for y will be from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1\int_y^{\sqrt{y}} xy\,(x+y)\,dx\,dy = \int_0^1\left[\frac{x^3}{3}y+\frac{x^2}{2}y^2\right]_{x=y}^{x=\sqrt{y}}dy = \int_0^1\left[\frac{y^2\sqrt{y}}{3}+\frac{y^3}{2}-\left(\frac{y^4}{3}+\frac{y^4}{2}\right)\right]dy$$

$$= \int_0^1\left(\frac{y^{\frac{5}{2}}}{3}+\frac{y^3}{2}-\frac{5}{6}y^4\right)dy = \left[\frac{y^{\frac{7}{2}}}{3}\cdot\frac{2}{7}+\frac{y^4}{8}-\frac{5}{6}\cdot\frac{y^5}{5}\right]_0^1 = \frac{2}{21}+\frac{1}{8}-\frac{1}{6} = \frac{3}{56}. \qquad \textbf{Ans.}$$

CHANGE OF ORDER OF INTEGRATION

We have seen that, to evaluate a double integral we integrate first w.r.t. one variable (y or x depending upon the limits) considering the other variable as constant and then integrate with respect to the variable. However, if the limits are constant, the order of integration is immaterial, as stated earlier, and in that case we have

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy \quad i.e. \quad \int_a^b dx\int_c^d f(x,y)\,dy = \int_c^d dy\int_a^b f(x,y)\,dx.$$

But if the limits are variable and the integrand $f(x, y)$ in the double integral is either difficult or even impossible to integrate in the given order, then we reverse the order of integration and corresponding change is made in the limits of integration. The new limits are obtained by geometrical considerations, therefore a clear sketch of the curve should be drawn. Sometimes, in changing the order of integration we have to split up the region of integration and the new integral is expressed as the sum of a number of double integrals. The concept will be more clear through the following illustration.

EXAMPLE 19.3. Change the order of integration and evaluate $\displaystyle\int_0^a\int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy.$

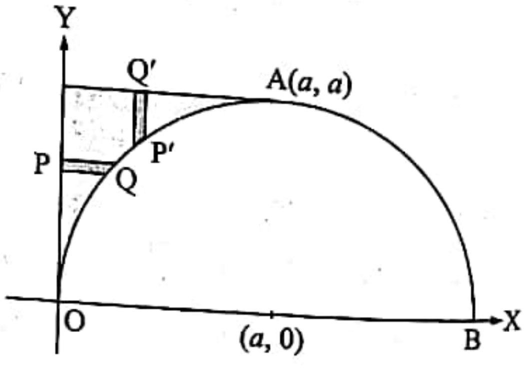
[GGSIPU IInd Sem, Ist Term 2005]

SOLUTION: In the given integral, as it stands, it is to be integrated first w.r.t. x and then w.r.t. y which is complicated, therefore it requires the change of order of integration for which the region over which the integration is to be done, is to be found first.

The region is bounded by $x = 0$, $x = a-\sqrt{a^2-y^2}$

or $(x-a)^2 = a^2 - y^2$ or $x^2 + y^2 - 2ax = 0$ (circle)

and $y = 0$, $y = a$, as shown in the adjoing figure.



Actually from the equation of the circle, we get $x = a \pm \sqrt{a^2-y^2}$; *i.e.* for one value of y we have two values of x. Here $x = a-\sqrt{a^2-y^2}$ represents the arc OA and $x = a+\sqrt{a^2-y^2}$ the arc AB of the circle. Thus, as it is, we have elementary strip PQ parallel to X-axis as shown in the figure.

When the order of integration is changed, we take the strip P′Q′ parallel to Y-axis. Here y varies from the circle $y = \sqrt{2ax-x^2}$ to the line $y = a$ then the strip P′Q′ moves parallel to itself from $x = 0$ to $x = a$.

Therefore $$I = \int_0^a\int_{\sqrt{2ax-x^2}}^a \frac{xy\log(x+a)}{(x-a)^2}\,dy\,dx = \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left[\int_{\sqrt{2ax-x^2}}^a y\,dy\right]dx$$

$$= \int_0^a \frac{x\log(x+a)}{2(x-a)^2}[a^2-(2ax-x^2)]\,dx$$

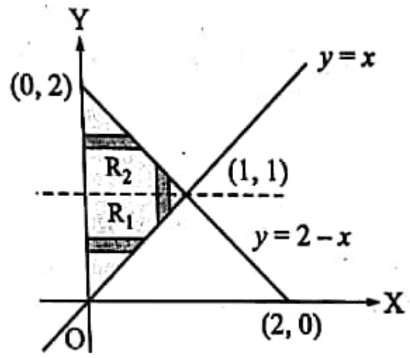
$$= \int_0^a \frac{x}{2} \log(x+a)\, dx \quad \text{(now integrating it by parts)}$$

$$= \left[\log(x+a) . \frac{x^2}{4}\right]_0^a - \int_0^a \frac{x^2\, dx}{4(x+a)} = \frac{a^2}{4}\log 2a - \frac{1}{4}\int_0^a \left(x - a + \frac{a^2}{x+a}\right) dx$$

$$= \frac{a^2}{4}\log 2a - \frac{1}{4}\left[\frac{x^2}{2} - ax + a^2 \log(x+a)\right]_0^a$$

$$= \frac{a^2}{4}\log 2a - \frac{1}{4}\left[\frac{a^2}{2} - a^2 + a^2\log 2a - a^2 \log a\right] = \frac{a^2}{8}[1 + 2\log a].$$ **Ans.**

EXAMPLE 19.4. Change the order of integration and hence evaluate $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$.



SOLUTION: In the given integral, as it is, the elementary strip has to be taken parallel to y-axis. Region of integration is the shaded portion as shown in the adjoining figure bounded by the lines $y = x$, $y = 2 - x$ and $x = 0, x = 1$. On changing the order of integration the elementary strip has to be taken parallel to x-axis, for which the region of integration, has to be divided into two regions R_1 and R_2.

$$\text{Therefore } I = \int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx = \iint_{R_1} \frac{x}{y}\, dx\, dy + \iint_{R_2} \frac{x}{y}\, dx\, dy$$

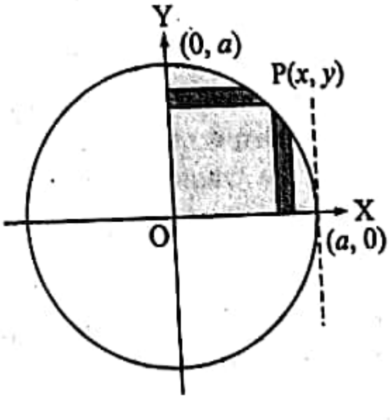
Region R_1 is bounded by $0 \le x \le y$, $0 \le y \le 1$ and R_2 is bounded by $0 \le x \le 2 - y$, $1 \le y \le 2$.

$$\therefore \quad I = \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\, dx\, dy = \int_0^1 \frac{1}{y} dy \int_0^y x\, dx + \int_1^2 \frac{1}{y} dy \int_0^{2-y} x\, dx$$

$$= \int_0^1 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y dy + \int_1^2 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^{2-y} dy = \int_0^1 \frac{1}{y} . \frac{y^2}{2} dy + \int_1^2 \frac{1}{2y}(2-y)^2\, dy$$

$$= \left[\frac{y^2}{4}\right]_0^1 + \int_1^2 \left(\frac{2}{y} - 2 + \frac{y}{2}\right) dy = \frac{1}{4} + \left[2\log y - 2y + \frac{y^2}{4}\right]_1^2 = 2\log 2 - 1.$$ **Ans.**

EXAMPLE 19.5. Evaluate $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dy\, dx}{(1+e^y)\sqrt{a^2 - x^2 - y^2}}$.



SOLUTION: The given integral is to be integrated first w.r.t. y which is not feasible, therefore we change the order of integration. As is clear from the given limits, the region of integration is the positive quadrant of the circle $x^2 + y^2 = a^2$. With the change of order of integration we have to take the elementary strip now parallel to x-axis and limits would change accordingly. (See figure.)

$$I = \int_0^a \int_0^{\sqrt{a^2 - y^2}} \frac{dx\, dy}{(1+e^y)\sqrt{a^2 - x^2 - y^2}}$$

$$= \int_0^a \frac{1}{1+e^y}\left[\int_0^{\sqrt{a^2-y^2}} \frac{dx}{\sqrt{a^2-y^2-x^2}}\right] dy = \int_0^a \frac{1}{1+e^y}\left[\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_{x=0}^{x=\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{1}{1+e^y}(\sin^{-1}1 - \sin^{-1}0)\, dy = \frac{\pi}{2}\int_0^a \frac{dy}{1+e^y}$$

Putting $e^y = t \quad \therefore e^y\, dy = dt$

$$I = \frac{\pi}{2}\int_1^{e^a} \frac{dt}{t(1+t)} = \frac{\pi}{2}\int_1^{e^a}\left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

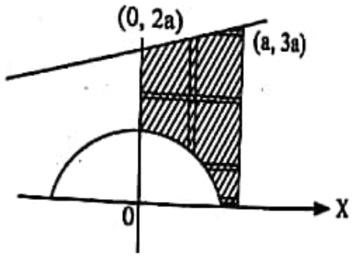
$$= \frac{\pi}{2}\left[\log\frac{t}{t+1}\right]_1^{e^a} = \frac{\pi}{2}\left[\log\frac{e^a}{e^a+1} - \log\frac{1}{2}\right] = \frac{\pi}{2}\log\frac{2e^a}{1+e^a}.$$ **Ans.**

EXAMPLE 19.6. (a) Change the order of integration in $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} f(x, y)\, dx\, dy$.

[GGSIPU II Sem Sem I Term 2013; II Term 2014]

(b) Change the order of integration in the double integral $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\, dy\, dx$.

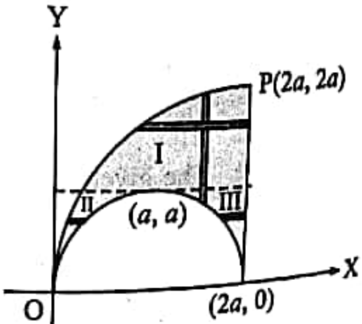
SOLUTION: The area of integration is given by $y = \sqrt{a^2-x^2}$ to $y = x + 2a$ and $x = 0$ to $x = a$, as shown in the adjoining figure as shaded area. On changing the order of integration the integral becomes



$$\int_{y=2a}^{3a} \int_{x=y-2a}^{a} f(x, y)\, dy\, dx + \int_{y=a}^{2a} \int_{x=0}^{a} f(x, y)\, dy\, dx + \int_{y=0}^{a} \int_{\sqrt{a^2-y^2}}^{a} f(x, y)\, dy\, dx$$ **Ans.**

(b) The order of integration in the given integral is first w.r.t. y and then w.r.t x. To evaluate the given integral, as it stands, the elementary strip has to be taken parallel to Y-axis and its lower end is on $y = \sqrt{2ax - x^2}$, *i.e.*, the circle $x^2 + y^2 - 2ax = 0$ and the upper end on $y = \sqrt{2ax}$, *i.e.* the parabola $y^2 = 2ax$. Then the strip is moved parallel to itself from $x = 0$ to $x = 2a$. Therefore the shaded portion between the parabola and the circle is the region of integration.

On changing the order of integration we are to integrate first w.r.t. x and then w.r.t. y. The elementary strip is taken parallel to X-axis. To cover the whole of the shaded area the region has to be divided into following three parts, as shown in the adjoining figure:



Region I. The strip extends from the parabola $y^2 = 2ax$, *i.e.* $x = \frac{y^2}{2a}$, to the straight line $x = 2a$. The strip is then moved parallel to itself from $y = a$ to $y = 2a$ to cover the region I. Therefore, the part of double integral in this region is I_1, given by

$$I_1 = \int_a^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dx\, dy.$$

Region II. The strip extends from the parabola $y^2 = 2ax$, *i.e.*, $x = \frac{y^2}{2a}$ to the circle $x^2 + y^2 - 2ax = 0$, *i.e.* $x = a - \sqrt{a^2 - y^2}$ as is clear from the figure. Then the strip is taken from $y = 0$ to $y = a$ to cover this region. Therefore the part of the integral in this region is I_2, given by

$$I_2 = \int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x, y)\, dx\, dy.$$

Region III. The strip extends from the circle $x^2 + y^2 - 2ax = 0$, *i.e.* $x = a + \sqrt{a^2 - y^2}$ to the line $x = 2a$. The strip is then taken from $y = 0$ to $y = a$ which covers this region. Denoting this part of integral by I_3, we have

$$I_3 = \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\, dx\, dy.$$

As such, the given integral is equal to $I_1 + I_2 + I_3$.

Thus, $$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\, dy\, dx = \int_a^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dx\, dy + \int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x, y)\, dx\, dy + \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\, dx\, dy.$$

Ans.

EXAMPLE 19.7. Express $\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\, dy\, dx + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} x\, dy\, dx$ as a single integral and evaluate it.

[GGSIPU I Sem End Term 2004 Reapperar; I Term 2008]

SOLUTION: Let $I_1 = \int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\, dy\, dx$ and $I_2 = \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} x\, dy\, dx$.

Let R_1 and R_2 be the regions over which I_1 and I_2 are being integrated respectively and are depicted by the shaded portion in Fig. I and Fig. II.

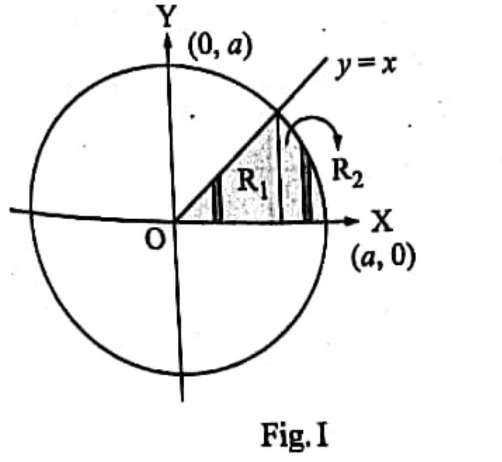


Fig. I

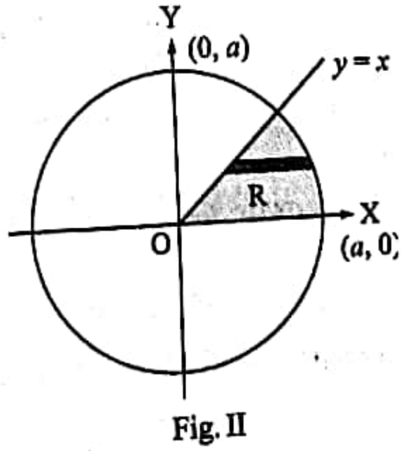


Fig. II

As is clear from Fig. II, $R = R_1 + R_2$

$$\therefore \quad I = I_1 + I_2 = \iint_R x\,dx\,dy. \qquad \textbf{Ans.}$$

For evaluating I we change the order of integration hence the elementary strip has to be taken parallel to x-axis from $x = y$ to $x = \sqrt{a^2 - y^2}$, *i.e.*, the circle $x^2 + y^2 = a^2$.

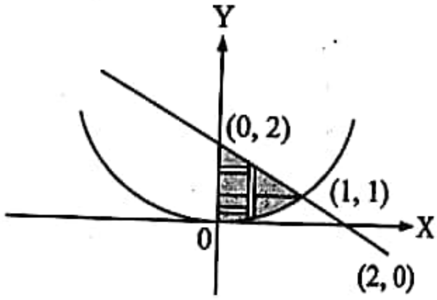
$$\therefore \quad I = \int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_0^{\frac{a}{\sqrt{2}}} \left[\int_y^{\sqrt{a^2-y^2}} x\,dx\right] dy = \int_0^{\frac{a}{\sqrt{2}}} \left[\frac{x^2}{2}\right]_y^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^{\frac{a}{\sqrt{2}}} (a^2 - y^2 - y^2)\,dy = \frac{1}{2}\left[a^2 y - \frac{2y^3}{3}\right]_0^{\frac{a}{\sqrt{2}}} = \frac{a^3}{3\sqrt{2}}. \qquad \textbf{Ans.}$$

EXAMPLE 19.8. Express $\int_0^1 \int_{x^2}^{2-x} f(x, y)\,dydx$ as a sum of two double integrals.

[GGSIPU II Sem I Term 2011]

SOLUTION: For the given integral, area of integration is shown as shaded area in the adjoining figure. Initially, there is vertical strip from $y = x^2$ to $y = 2 - x$ then x varies from $x = 0$ to $x = 1$. Changing the order of integration we divide the area into two parts having two horizontal strips, one moving from $x = 0$ to $x = \sqrt{y}$ and y from $y = 0$ to $y = 1$ and second from $x = 0$ to $x = 2 - y$ and and y from $y = 1$ to $y = 2$.



$$\therefore \quad \int_0^1 \int_{x^2}^{2-x} f(x, y)\,dydx = \int_0^1 \int_0^{\sqrt{y}} f(x, y)\,dxdy + \int_1^2 \int_0^{2-y} f(x, y)\,dxdy. \qquad \textbf{Ans.}$$

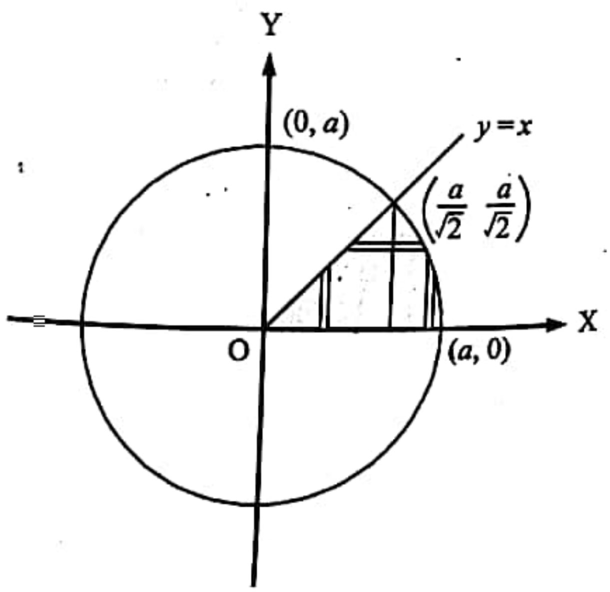
EXAMPLE 19.9. Change the order of integration in the integral $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} xdx\,dy$ and then evaluate it.

[GGSIPU I Sem End Term 2004 (Reappear); II Sem I term 2008]

SOLUTION: The area of integration in the given integral

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x\,dx\,dy$$ the is shaded portion in the figure.

Changing the order of integration, the area has to be divided into two parts by the line $x = a/\sqrt{2}$, then we have



$$I = \int_0^{a/\sqrt{2}}\int_0^x x\,dy\,dx + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^{a/\sqrt{2}} x\,[y]_0^x\,dx + \int_{a/\sqrt{2}}^{a} x\,[y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \int_0^{a/\sqrt{2}} x^2dx + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx = \left[\frac{x^3}{3}\right]_0^{a/\sqrt{2}} + \int_{a/\sqrt{2}}^{0} -t^2dt \text{ (On putting } a^2 - x^2 = t^2)$$

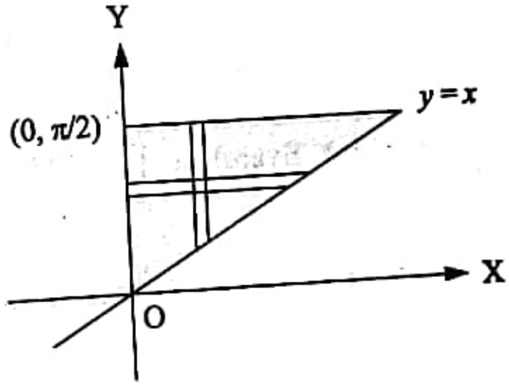
$$= \frac{a^3}{3.2\sqrt{2}} + \left[\frac{t^3}{3}\right]_0^{a/\sqrt{2}} = \frac{a^3}{6\sqrt{2}} + \frac{a^3}{6\sqrt{2}} = \frac{a^3}{3\sqrt{2}} = \frac{a^3\sqrt{2}}{6}$$ **Ans.**

EXAMPLE 19.10. Sketch the region of integration and evaluate

$$\int_0^{\pi/2} dy\int_0^y \cos 2y\sqrt{1-a^2\sin^2 x}\,dx$$ [*GGSIPU I Sem End Term 2003*]

SOLUTION: The given integral can be written as $I = \int_0^{\pi/2}\int_0^y \cos 2y\sqrt{1-a^2\sin^2 x}\,dy\,dx$

Now, let us change the order of integration, then it equals



$$I = \int_0^{\pi/2}\int_{y=x}^{\pi/2} \cos 2y\sqrt{1-a^2\sin^2 x}\,dy\,dx$$

$$= \int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\left[\frac{1}{2}\sin 2y\right]_x^{\pi/2} dx$$

$$= \int_0^{\pi/2}\left(0-\frac{1}{2}\sin 2x\right)\sqrt{1-a^2\sin^2 x}\,dx \text{ (On putting } 1 - a^2\sin^2 x = t^2),$$

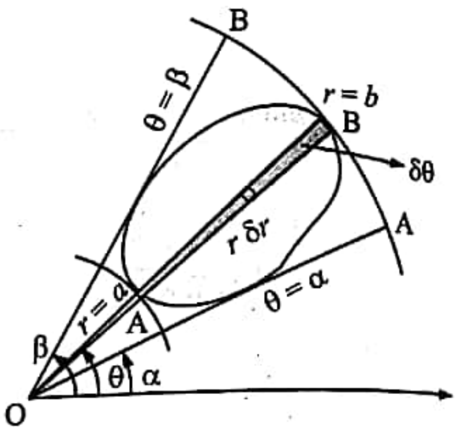
$$I = \frac{1}{a^2}\int_1^{\sqrt{1-a^2}} t.t\,dt = \left[\frac{t^3}{3a^2}\right]_1^{\sqrt{1-a^2}} = \frac{1}{3a^2}[(1-a^2)^{3/2}-1]$$ **Ans.**

DOUBLE INTEGRALS IN POLAR CO-ORDINATES

In the polar co-ordinates we divide the region of integration into a mesh formed by circular arcs and straight lines, given by r = constant, which are circles,

and θ = constant, which are straight lines

as shown in the adjoining figure. Here the element in the mesh is $\delta r . r\,\delta\theta$.



Now, if, at the point $P(r, \theta)$, the function is $f(r, \theta)$ then, over the wedge AB, the sum $\underset{\delta r \to 0}{\text{Lt}}\ \delta\theta \sum_{A}^{B} f(r,\theta) . r\,\delta r$

can be written as $\delta\theta \displaystyle\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr$

where $r_1(\theta)$ and $r_2(\theta)$ arc the radii vectors of two curves when θ is kept constant, while integrating w.r.t. r. Finally, summing for all wedges between $\theta = \alpha$ and $\theta = \beta$, we get

$$\underset{\delta\theta \to 0}{\text{Lt}} \sum_{\alpha}^{\beta} \delta\theta \sum_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\,\delta r = \int_{\theta=\alpha}^{\theta=\beta} d\theta \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr.$$

Thus, the double integral in the cartesian form $\iint_R f(x,y)\,dx\,dy$ becomes $\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$ when changed to polar co-ordinates. In the latter integral the region R should be expressed in the polar co-ordinates. Here also, we can go in for changing the order of integration, if needed.

EXAMPLE 19.11. **(a)** Evaluate $\displaystyle\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dx\, dy.$

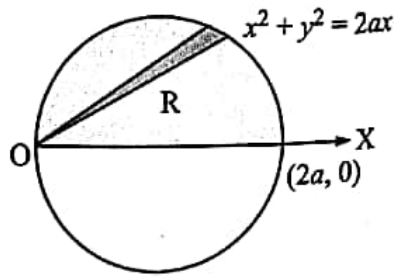
(b) Change into polar coordinates and hence evaluate

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

[*GGSIPU I Sem End Term 2003*]

SOLUTION: **(a)** The region of integration is the upper half of the circle $x^2 + y^2 = 2ax$. If we transform to polar co-ordinates, the region becomes $r = 2a\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$, (see figure). Therefore the given integral when converted to polar coordinates, becomes

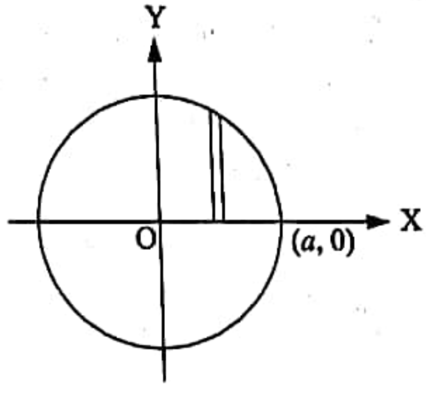
$$\int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 . r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$



$$= \frac{1}{4}(2a)^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} 4a^4 = \frac{3}{4}\pi a^4.$$ **Ans.**

(b) The given integral is $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$



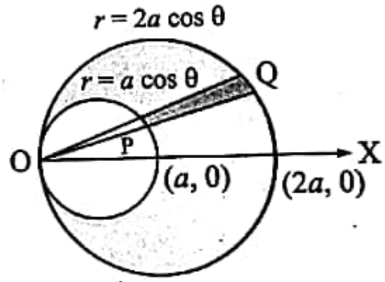
The area over which the integration is to be carried out, is the shaded portion in the adjoining figure. Converting to polar coordinates by $x = r\cos\theta$, $y = r\sin\theta$ so that $r^2 = x^2 + y^2$ and $dxdy = r\, d\theta\, dr$.

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r \cdot r d\theta\, dr = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\theta = \frac{a^3}{3} \cdot \frac{\pi}{2} = \pi a^3/6 \qquad \textbf{Ans.}$$

EXAMPLE 19.12. Evaluate $\iint y^2\, dx\, dy$ over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$.

SOLUTION: Let us convert to polar coordinates. The region of integration is bounded by the circle $x^2 + y^2 - ax = 0$ or $r = a\cos\theta$ and the circle $x^2 + y^2 - 2ax = 0$ or $r = 2a\cos\theta$. The element $dx\, dy = r\, dr\, d\theta$. The elementary strip, as shown in the adjoining figure, extends from $r = a\cos\theta$ to $r = 2a\cos\theta$. To cover the region of integration, θ varies from $\theta = -\pi/2$ to $\theta = \pi/2$, the integrand being $y^2 = (r\sin\theta)^2$.

$$\therefore \quad \text{Given integral} = \int_{-(\pi/2)}^{\pi/2} \int_{a\cos\theta}^{2a\cos\theta} r^2 \sin^2\theta\, r\, dr\, d\theta = \int_{-(\pi/2)}^{\pi/2} \sin^2\theta\, d\theta \int_{a\cos\theta}^{2a\cos\theta} r^3\, dr$$



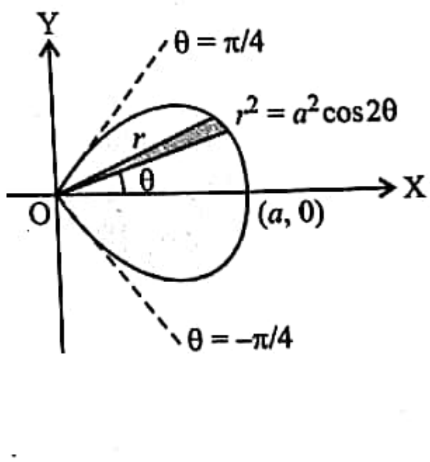
$$= \int_{-(\pi/2)}^{\pi/2} \sin^2\theta \left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{15}{4} a^4 \int_{-(\pi/2)}^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta$$

$$= \frac{15}{4} a^4\, 2 \int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{15a^4}{2} \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{15\pi a^4}{64}. \qquad \textbf{Ans.}$$

EXAMPLE 19.13. Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{r^2 + a^2}}$ over one loop of the leminiscate $r^2 = a^2\cos 2\theta$.

SOLUTION: The loop of the curve $r^2 = a^2\cos 2\theta$ is lying between $\theta = \pm\frac{\pi}{4}$. We integrate first w.r.t. r under the limits $r = 0$ to $r = a\sqrt{\cos 2\theta}$ and then, to cover the entire loop, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.



$\therefore$ The given integral becomes

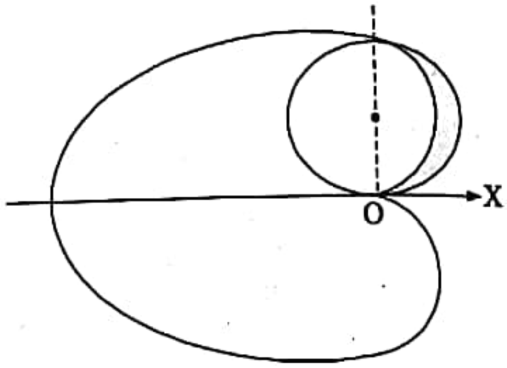
$$= \int_{-(\pi/4)}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\, dr\, d\theta}{\sqrt{r^2 + a^2}} = \int_{-(\pi/4)}^{\pi/4} d\theta \int_0^{a\sqrt{\cos 2\theta}} \frac{r\, dr}{\sqrt{r^2 + a^2}}$$

$$= \int_{-(\pi/4)}^{\pi/4} d\theta \left[\sqrt{r^2+a^2}\right]_0^{a\sqrt{\cos 2\theta}} = 2a \int_0^{\pi/4} \left(\sqrt{1+\cos 2\theta} - 1\right) d\theta$$

$$= 2a \int_0^{\pi/4} \left(\sqrt{2}\cos\theta - 1\right) d\theta = 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4} = 2a\left(1 - \frac{\pi}{4}\right).$$ **Ans.**

EXAMPLE 19.14. Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardiode $r = a(1 - \cos\theta)$. [*GGSIPU II Sem I Term 2006*]

SOLUTION: The required area is the shaded portion in the adjacent figure and is equal to



$$\int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{1}{2}r^2\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}a^2 \int_0^{\pi/2} [\sin^2\theta - (1-\cos\theta)^2]\,d\theta$$

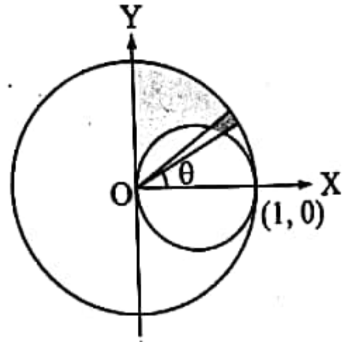
$$= \frac{a^2}{2} \int_0^{\pi/2} (-1 + 2\cos\theta - \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = a^2(1 - \pi/4)$$ **Ans.**

EXAMPLE 19.15. Evaluate the double integral

$$\int_0^1 x e^{-x^2} dx \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{y e^{-y^2}}{x^2+y^2} dy.$$

SOLUTION: The region of integration is the shaded portion bounded by the upper halves of circles $x^2+y^2=1$ and $x^2+y^2=x$ and by $x=0$ and $x=1$ as shown in the figure. Presence of (x^2+y^2) prompts to change to polar co-ordinates. The circle $x^2+y^2=x$ becomes $r=\cos\theta$ and $x^2+y^2=1$ becomes $r=1$, hence the given integral is equal to



$$\int_0^{\pi/2} \int_{r=\cos\theta}^{1} \frac{r\cos\theta . r\sin\theta\, e^{-r^2}}{r^2} . r\,d\theta\,dr$$

$$= \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_{r=\cos\theta}^{1} r e^{-r^2} dr$$

$$= \int_0^{\pi/2} \sin\theta\cos\theta\left[-\frac{1}{2}e^{-r^2}\right]_{r=\cos\theta}^{1} d\theta = \frac{1}{2}\int_0^{\pi/2}\left(e^{-\cos^2\theta} - e^{-1}\right)\sin\theta\cos\theta\,d\theta$$

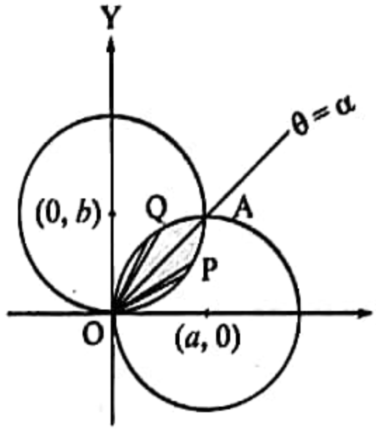
$$= \frac{1}{2}\int_0^{\pi/2} e^{-\cos^2\theta}\sin\theta\cos\theta\,d\theta - \frac{e^{-1}}{2}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{4}\left[e^{-\cos^2\theta}\right]_0^{\pi/2} - \frac{1}{4e}[\sin^2\theta]_0^{\pi/2} = \frac{1}{4}(1-e^{-1}) - \frac{1}{4e} = \frac{1}{4}(1-2e^{-1}).$$ **Ans.**

EXAMPLE 19.16. Evaluate $\iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$ over the area common to

$$x^2+y^2-2ax=0 \text{ and } x^2+y^2-2by=0,\quad a>b>0.$$

SOLUTION: The region of integration is the common portion of the circle $x^2+y^2-2ax=0$ (centre $(a, 0)$, radius a) and the circle $x^2+y^2-2by=0$ (centre $(0, b)$, radius b) which on changing to polar co-ordinates become $r = 2a\cos\theta$, $r = 2b\sin\theta$.



Their points of intersection are O the pole and A $\left(\sqrt{a^2+b^2}, \alpha\right)$ where $\tan\alpha = \frac{a}{b}$. The region of integration, shown in the adjoining figure as shaded area, can be divided into two subregions OPAO and OAQO as shown in the figure. Then

$$I = \iint\limits_{OPAQO} \frac{(x^2+y^2)^2\,dx\,dy}{x^2y^2}$$

$$= \iint\limits_{OPAO} \frac{r^4.r\,dr\,d\theta}{r^4\sin^2\theta\cos^2\theta} + \iint\limits_{OAQO} \frac{r^4.r\,dr\,d\theta}{r^4\sin^2\theta\cos^2\theta}$$

$$= \int_0^{\alpha}\int_0^{2b\sin\theta} \frac{r\,dr\,d\theta}{\sin^2\theta\cos^2\theta} + \int_{\alpha}^{\pi/2}\int_0^{2a\cos\theta} \frac{r\,dr\,d\theta}{\sin^2\theta\cos^2\theta}$$

$$= \int_0^{\alpha} \frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{2b\sin\theta} d\theta + \int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{2a\cos\theta} d\theta$$

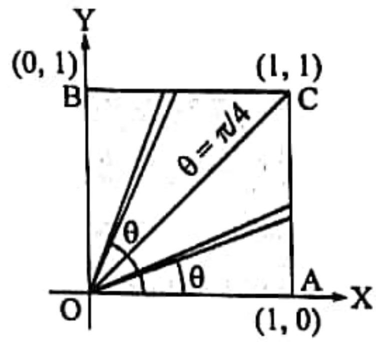
$$= 2b^2\int_0^{\alpha}\sec^2\theta\,d\theta + 2a^2\int_{\alpha}^{\pi/2}\operatorname{cosec}^2\theta\,d\theta$$

$$= 2b^2[\tan\theta]_0^{\alpha} + 2a^2[-\cot\theta]_{\alpha}^{\pi/2} = 2b^2\tan\alpha + 2a^2\cot\alpha$$

$$= 2b^2\cdot\frac{a}{b} + 2a^2\cdot\frac{b}{a} = 4ab. \qquad \textbf{Ans.}$$

EXAMPLE 19.17. Evaluate $\iint \frac{dx\,dy}{\left(1+x^2+y^2\right)^{\frac{3}{2}}}$ over the area of the square bounded by

$$x=0,\ y=0,\ x=1,\ y=1.$$



SOLUTION: For conveniences we change over to polar coordinates. Line $(x = 1)$ in polar co-ordinates is $r\cos\theta = 1$ or $r = \sec\theta$ and the line $y = 1$ is $r\sin\theta = 1$ or $r = \operatorname{cosec}\theta$.

The region of integration is the square which is divided into two regions OACO and OCBO as shown in the adjoining figure.

$$I = \iint \frac{dx\,dy}{\left(1+x^2+y^2\right)^{3/2}} = \iint\limits_{OACO} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}} + \iint\limits_{OCBO} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}}$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\operatorname{cosec}\theta} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}}$$

$$= \int_0^{\pi/4} \left[\frac{-1}{\sqrt{1+r^2}}\right]_0^{\sec\theta} d\theta + \int_{\pi/4}^{\pi/2} \left[\frac{-1}{\sqrt{1+r^2}}\right]_0^{\operatorname{cosec}\theta} d\theta$$

$$= \int_0^{\pi/4} \left[1 - \frac{1}{\sqrt{1+\sec^2\theta}}\right] d\theta + \int_{\pi/4}^{\pi/2} \left[1 - \frac{1}{\sqrt{1+\operatorname{cosec}^2\theta}}\right] d\theta$$

$$= \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \frac{\cos\theta\,d\theta}{\sqrt{1+\cos^2\theta}} + \int_{\pi/4}^{\pi/2} d\theta - \int_{\pi/4}^{\pi/2} \frac{\sin\theta\,d\theta}{\sqrt{1+\sin^2\theta}} = \frac{\pi}{2} - \mathrm{I}_1 - \mathrm{I}_2$$

where $\mathrm{I}_1 = \int_0^{\pi/4} \frac{\cos\theta\,d\theta}{\sqrt{2-\sin^2\theta}}$ and $\mathrm{I}_2 = \int_{\pi/4}^{\pi/2} \frac{\sin\theta\,d\theta}{\sqrt{2-\cos^2\theta}}$.

In I_1 put $\sin\theta = u$ and in I_2 put $\cos\theta = v$, then

$$\mathrm{I}_1 = \int_0^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{2-u^2}} = \left[\sin^{-1}\frac{u}{\sqrt{2}}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6}$$

and
$$\mathrm{I}_2 = \int_{\frac{1}{\sqrt{2}}}^{0} \frac{-dv}{\sqrt{2-v^2}} = \left[\sin^{-1}\frac{v}{\sqrt{2}}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6}$$

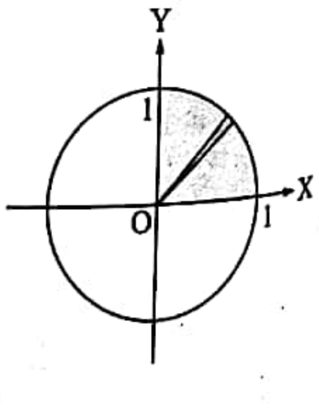
$\therefore$
$$\mathrm{I} = \frac{\pi}{2} - \frac{\pi}{6} - \frac{\pi}{6} = \frac{\pi}{6}.$$
Ans.

EXAMPLE 19.18. (a) Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{m/2} dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$

(b) Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

[GGSIPU II Sem End Term 2014]

SOLUTION: (a) The region of integration in the given double integral is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. Transforming these x, y to elliptical polar co-ordinates $x = ar\cos\theta$, $y = br\sin\theta$ so that $dx\,dy = abr\,d\theta\,dr$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ get transformed to the circle $r^2 = 1$, *i.e.*, $r = 1$, the polar circle with centre (0, 0) and radius 1 in the positive quadrant.



$$\therefore \quad \mathrm{I} = \int_0^{\pi/2}\int_0^1 ar\cos\theta . br\sin\theta\,(r^2\cos^2\theta + r^2\sin^2\theta)^{m/2}\,ab\,r\,d\theta\,dr$$

$$= \frac{a^2 b^2}{2} \int_0^{\pi/2} \int_0^1 \sin 2\theta \, r^{m+3} \, d\theta \, dr = \frac{a^2 b^2}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2} \left[\frac{r^{m+4}}{m+4} \right]_0^1$$

$$= \frac{a^2 b^2}{2(m+4)}. \qquad \textbf{Ans.}$$

(b) $I = \iint_A (x+y)^2 \, dx \, dy$ where A is the area under $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

On A, $x = \pm a$, $y = \pm \frac{b}{a}\sqrt{a^2 - b^2}$ $\therefore$ $I = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + 2xy + y^2) \, dy \, dx$

Using properties of definite integrals, we get

or $$I = 2\int_{-a}^{a} \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2 + 0) \, dy \, dx = 2\int_{-a}^{a} \left[\frac{y^3}{3} + x^2 y \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[\frac{1}{3}\frac{b^3}{a^3}(a^2 - x^2)^{3/2} + x^2 \frac{b}{a}\sqrt{a^2 - x^2} \right] dx = 4\int_0^a \left[\frac{b^3}{3a^3}(a^2 - x^3)^{3/2} + \frac{b}{a}x^2 \cdot (a^2 - x^2)^{1/2} \right] dx$$

$$= 4\int_0^{\pi} \left[\frac{b^3}{3a^3} a^3 \cos^3\theta + \frac{b}{a} a^2 \sin^2\theta \cdot a\cos\theta \right] a\cos\theta \, d\theta \qquad \text{(on putting } x = a\sin\theta)$$

$$= \frac{4}{3} ab^3 \frac{\left|\!\underline{\tfrac{1}{2}}\right. \left|\!\underline{\tfrac{5}{2}}\right.}{2\left|\!\underline{3}\right.} + 4a^3 b \frac{\left|\!\underline{3/2}\right. \left|\!\underline{3/2}\right.}{2\left|\!\underline{3}\right.} = \frac{1}{3} ab^3 \cdot \frac{3}{2} \cdot \frac{1}{2} \left(\left|\!\underline{\tfrac{1}{2}}\right. \right)^2 + a^3 b \frac{1}{2} \cdot \frac{1}{2} \left(\left|\!\underline{\tfrac{1}{2}}\right. \right)^2$$

$$= a^3 b \cdot \frac{\pi}{4} + ab^3 \frac{\pi}{4} = \frac{\pi}{4} ab(a^2 + b^2) \qquad \textbf{Ans.}$$

TRIPLE INTEGRATION

CARTESIAN COORDINATES

The triple integration of a function $f(x, y, z)$ defined over a closed parallelopiped, is expressed in the same manner as in the case of a double integral. Thus, we first obtain the sum

$$S = \sum_{i, j, k=1}^{n} f(x_i, y_j, z_k)\, \delta x_i\, \delta y_j\, \delta z_k$$

then the triple integral of $f(x, y, z)$ over a region V is the limiting value to which the sum S will tend to as $n \to \infty$. Thus, $\iiint_V f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k)\, \delta x_i\, \delta y_j\, \delta z_k$

The integral $\int_{x=a}^{b} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x, y, z)\, dx\, dy\, dz$ is generally evaluated as

$$\int_{x=a}^{b} \left[\int_{y=\phi_1(x)}^{\phi_2(x)} \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} f(x, y, z)\, dz \right\} dy \right] dx.$$

EXAMPLE 19.19. Evaluate (i) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ (ii) $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$

[GGSIPU II Sem I Term 2012]

SOLUTION: (i) We can write the given integral as

$$I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y}.e^z\, dz\, dy\, dx = \int_0^a \int_0^x e^{x+y} [e^z]_0^{x+y}\, dy\, dx$$

$$= \int_0^a \int_0^x e^{x+y} (e^{x+y} - 1)\, dy\, dx = \int_0^a \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^x . e^y \right]_{y=0}^{x} dx$$

$$= \int_0^a \left[\frac{1}{2} e^{2x} (e^{2x} - 1) - e^x (e^x - 1) \right] dx = \left[\frac{1}{8} e^{4x} - \frac{3}{4} e^{2x} + e^x \right]_0^a \frac{1}{8} e^{4a} - \frac{3}{4} e^{2a} + e^a - \frac{3}{8}.$$ **Ans.**

(ii) $$I = \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx = \int_0^{\log 2} \int_0^x e^{x+y} \int_0^{x+\log y} e^z\, dz\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^x e^{x+y} (e^{x+\log y} - e^0)\, dy\, dx = \int_0^{\log 2} \int_0^x (y e^{2x+y} - e^{x+y})\, dy\, dx$$

$$= \int_0^{\log 2} \left[e^{2x} \int_0^x y e^y dy - \left\{ e^{x+y} \right\}_0^x \right] dx = \int_0^{\log 2} e^{2x} \left[\left\{ y e^y \right\}_0^x - \int_0^x e^y\, dy \right] - \int_0^{\log 2} \left[\left\{ e^{x+y} \right\}_0^x \right] dx$$

$$= \int_0^{\log 2} e^{2x} (x e^x - e^x + 1)\, dx - \left[\frac{1}{2} e^{2x} - e^x \right]_0^{\log 2}$$

$$= \int_0^{\log 2} x e^{3x}\, dx - \left[\frac{1}{3}e^{3x}\right]_0^{\log 2} + \left[\frac{1}{2}e^{2x}\right]_0^{\log 2} - \left[\frac{1}{2}e^{2\log 2} - e^{\log 2} - \frac{1}{2} + 1\right]$$

$$= \left[x\frac{e^{3x}}{3}\right]_0^{\log 2} - \int_0^{\log 2} \frac{e^{3x}}{3}\, dx - \frac{1}{3}e^{3\log 2} + \frac{1}{3} + \frac{1}{2}\left(e^{2\log 2} - 1\right) - \frac{1}{2}e^{2\log 2} + e^{\log 2} - \frac{1}{2}$$

$$= \frac{1}{3}\log 2 \cdot e^{3\log 2} - \frac{1}{9}\left[e^{3x}\right]_0^{\log 2} - \frac{1}{3}e^{3\log 2} + \frac{1}{3} - 1 + 2$$

$$= \frac{8}{3}\log 2 - \frac{1}{9}(8-1) - \frac{1}{3}\cdot 8 + \frac{4}{3} = \frac{8}{3}\log 2 - \frac{19}{9}$$ **Ans.**

EXAMPLE 19.20. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz dy dx.$

[*GGSIPU I Sem End Term 2004 Reppear; II Term 2014*]

SOLUTION: Given integral $= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2-z^2)}}\, dz dy dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (\sin^{-1}1 - \sin^{-1}0) dy dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi^2}{8}$$ **Ans.**

EXAMPLE 19.21. Evaluate $\iiint_R \frac{dx\, dy\, dz}{(x+y+z+1)^3}$ where R is the region bounded by the planes

$x = 0,\quad y = 0,\ z = 0$ and $x + y + z = 1.$

[*GGSIPU II Sem I Term 2010*]

SOLUTION: Given integal $= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\, dz\, dy\, dx$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\, dx = -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x} dx = -\frac{1}{2}\int_0^1 \left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x}\right) dx$$

$$= -\frac{1}{2}\left[\frac{x}{4} - \frac{x^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1 = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right].$$ **Ans.**

EXAMPLE. 19.22. Obtain the value of the triple integral $\iiint\limits_{V} z^2\, dx\, dy\, dz$ over the volume V common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

SOLUTION: From the equation of the sphere, we have $z = \pm\sqrt{a^2 - x^2 - y^2}$.

$$I = \iint\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} z^2\, dz\, dy\, dx = 2\iint\int_{0}^{\sqrt{a^2-x^2-y^2}} z^2\, dz\, dy\, dx$$

$$= 2\iint\left[\frac{z^3}{3}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx \quad = \frac{2}{3}\iint (a^2 - x^2 - y^2)^{3/2}\, dy\, dx.$$

Now x and y vary as per cylinder $x^2 + y^2 = ax$ or $r = a\cos\theta$, converting to polar co-ordinates.

$$I = \frac{2}{3}\int_{-(\pi/2)}^{\pi/2}\int_{0}^{a\cos\theta} (a^2 - r^2)^{3/2}\, r\, dr\, d\theta.$$

$$= -\frac{2}{15}\int_{-\pi/2}^{\pi/2}\left[(a^2 - r^2)^{5/2}\right]_0^{a\cos\theta} = -\frac{2}{15}\int_{-(\pi/2)}^{\pi/2}[a^5\sin^5\theta - a^5]\, d\theta$$

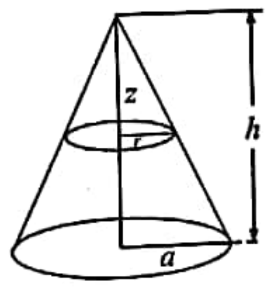
$$= 0 + \frac{2}{15}a^5\,[\theta]_{-(\pi/2)}^{\pi/2} = \frac{2\pi a^5}{15}.$$ **Ans.**

CYLINDRICAL POLAR CO-ORDINATES

The use of cylindrical coordinates r, θ, z is particularly suited for problems in which there is an axis of symmetry of the solid. In place of the volume element $dx\, dy\, dz$ we use $r\, d\theta\, dr\, dz$ as element with cross sectional area $r\, dr\, d\theta$ and height dz.

EXAMPLE 19.23. If a denotes the radius of the base and h the altitude of a right circular cone, express its volume as a triple integral and evaluate it using cylindrical coordinates.

SOLUTION: With cylindrical coordinates r, θ, z we have from the adjoing figure



$$\frac{z}{r} = \frac{h}{a} \quad \text{or} \quad z = \frac{rh}{a}$$

Hence the volume of the given cone is

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=\frac{rh}{a}}^{h} r\, dr\, d\theta\, dz = \int_0^{2\pi}\int_0^{a} [z]_{\frac{rh}{a}}^{h}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a}\left(h - \frac{rh}{a}\right) r\, dr\, d\theta = h\int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^3}{3a}\right) d\theta = \pi\, a^2 h/3.$$ **Ans.**

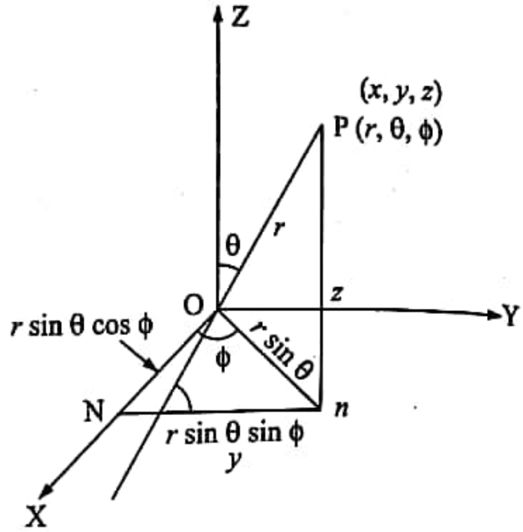
SPHERICAL POLAR CO-ORDIANTES

If there is symmetry about a point in any problem it would be quite convenient to use spherical coordinates

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$



The volume element in spherical polar co-ordinates is

$$dV = dr \cdot r\, d\theta \cdot r \sin\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

EXAMPLE 19.24. Evaluate $\iiint xyz\, dx\, dy\, dz$ over the volume in the positive octant of the sphere

$$x^2 + y^2 + z^2 = a^2.$$

SOLUTION: $I = \iiint xyz\, dx\, dy\, dz = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta\cos\phi \cdot r\sin\theta\sin\phi \cdot r\cos\theta \,.\, r^2\sin\theta\; dr\, d\theta\, d\phi$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^5 \cos\phi \sin\phi \sin^3\theta \cos\theta\; dr\, d\theta\, d\phi$$

$$= \left[\frac{r^6}{6}\right]_0^a \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi \int_0^{\pi/2} \sin^3\theta \cos\theta\, d\theta$$

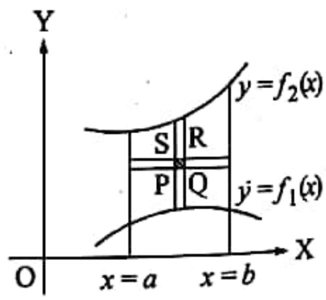
$$= \frac{a^6}{6} \cdot \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} = \frac{a^6}{48}.$$

Ans.

APPLICATIONS OF DOUBLE AND TRIPLE INTEGRALS

I. Area By Double Integration

(a) Cartesian Co-ordinates. Let us find the area enclosed between the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a, x = b$.



See adjoing figure. Let the area be divided into rectangular elements of the type PQRS of area $\delta x\, \delta y$ where $P \equiv (x, y)$ and $R \equiv (x + \delta x, y + \delta y)$. Now, when we move this element parallel to y-axis from $y = f_1(x)$ to $y = f_2(x)$ we get a vertical strip of area

$$\delta x \sum_{y=f_1(x)}^{y=f_2(x)} \delta y = \delta x \int_{y=f_1(x)}^{f_2(x)} dy$$

Adding up all such strips between the extreme ordinates at $x = a$ and $x = b$, we get the required area equal to

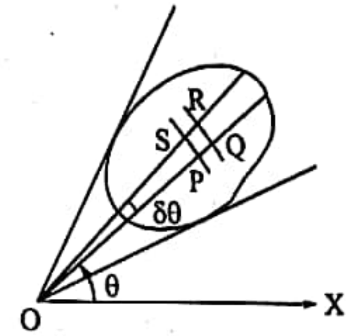
$$\lim_{\delta x \to 0} \sum_a^b \delta x \int_{f_1(x)}^{f_2(x)} dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} dy = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$$

Similarly, the area enclosed between the curves $x = f_1(y)$ and $x = f_2(y)$ and the extreme abscissas at $y = c$ and $y = d$ can be written as $\int_c^d \int_{f_1(y)}^{f_2(y)} dx\, dy$.

(b) Polar Co-ordinates. As shown in the adjoining figure, the area of the element PQRS $= r\,\delta\theta\,\delta r$ (shaded). If the total area is divided into such curvelinear rectangles then the sum $\Sigma\Sigma\, r\,\delta\theta\,\delta r$ taken over all these rectangles gives the limiting value of area A, as

$$A = \Sigma\Sigma r\,\delta\theta\,\delta r = \iint r\,d\theta\,dr$$

where limits are to be chosen so as to cover the entire area.

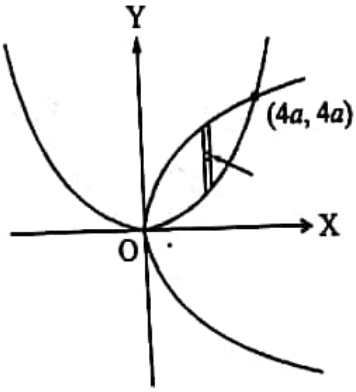


EXAMPLE 19.25. (a) Prove, by the method of double integration, that the area lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

[*GGSIPU IInd Sem. Ist Term 2005; I Term 2011*]

(b) Find the area bounded by the curve $(2a - x)\,y^2 = x^3$ and its asymptote using double integration concept.

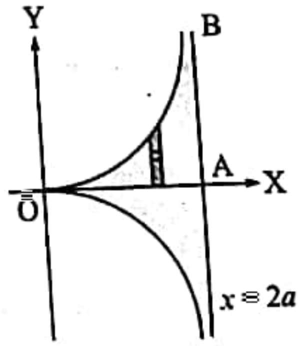
SOLUTION: (a) The required area is



$$\int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dy\,dx = \int_0^{4a} [y]_{x^2/4a}^{\sqrt{4ax}}\,dx = \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx$$

$$= \left[\sqrt{4a}\cdot\frac{2}{3}x^{3/2} - \frac{x^3}{3.4a}\right]_0^{4a} = \frac{16}{3}a^2. \qquad \textbf{Ans.}$$

(b) The shape of the curve is as shown in the adjoining figure. Curve $y^2(2a - x) = x^3$ has an asymptote parallel to y-axis as $x = 2a$.



Required area, because of symmetry about the axis of x, is equal to

$$2\iint_{OABO} dx\,dy = 2\int_0^{2a}\int_0^{\frac{x^{3/2}}{\sqrt{2a-x}}} dy\,dx = 2\int_0^{2a}[y]_0^{\frac{x^{3/2}}{\sqrt{2a-x}}}dx = 2\int_0^{2a}\frac{x^{3/2}\,dx}{\sqrt{2a-x}}$$

$$= 2\int_0^{\pi/2}\frac{(2a\sin^2\theta)^{3/2}}{\sqrt{2a - 2a\sin^2\theta}}\,4a\sin\theta\cos\theta\,d\theta \quad (\text{on putting } x = 2a\sin^2\theta)$$

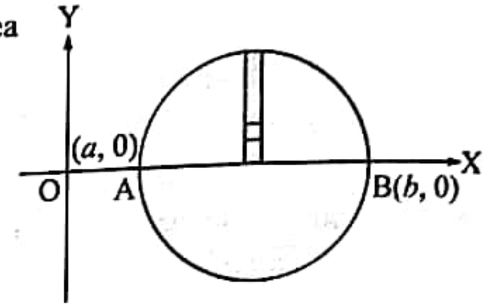
$$= 16a^2\int_0^{\pi/2}\frac{\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 16a^2\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad (\text{applying reduction formula}) = 3\pi a^2. \qquad \textbf{Ans.}$$

EXAMPLE 19.26. (a) Find the area of the loop of the curve $y^2 = (x-a)(b-x)$, $0 < a < b$.

(b) Find the area of the loop of the curve $x(x^2+y^2) = a(x^2-y^2)$ using double integration.

SOLUTION: (a) The given curve is a circle symmetrical about x-axis and extends from $x = a$ to $x = b$. The shaded portion, as shown in the figure, is the required area



$$= 2\int_a^b \int_0^{\sqrt{(x-a)(b-x)}} dx\, dy = 2\int_a^b \sqrt{(x-a)(b-x)}\, dx$$

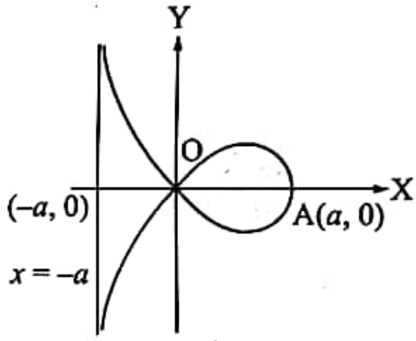
$$= 2\int_a^b \sqrt{(b-a)\sin^2 t\,(b-a)\cos^2 t} \cdot 2(b-a)\sin t \cos t\, dt$$

(on putting $x = a\cos^2 t + b\sin^2 t$, so that $dx = 2(b-a)\sin t \cos t\, dt$)

$\therefore$ The required area $= 4(b-a)^2 \int_0^{\pi/2} \sin^2 t \cos^2 t\, dt = 4(b-a)^2 \dfrac{\Gamma\frac{3}{2} \cdot \Gamma\frac{3}{2}}{2\Gamma 3} = \dfrac{\pi}{4}(b-a)^2$. **Ans.**

(b) The equation of the curve can be written as $y^2 = x^2 \dfrac{(a-x)}{a+x}$

The curve is symmetrical about x-axis. The shape of the curve is as shown in the adjoining figure. The required area A of the loop is equal to



$$A = 2\int_0^a \int_0^{x\sqrt{\frac{a-x}{a+x}}} dy\, dx = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\, dx$$

$$= 2\int_0^a \frac{x(a-x)}{\sqrt{a^2-x^2}}\, dx$$

(Now, putting $x = a\sin\theta$ $\therefore dx = a\cos\theta\, d\theta$)

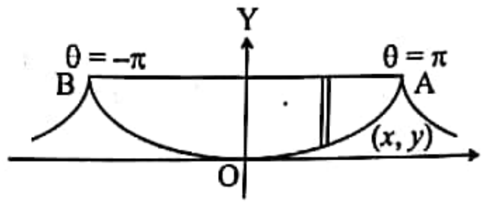
$$A = 2\int_0^{\pi/2} \frac{a\sin\theta\,(a - a\sin\theta)\, a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= 2a^2 \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta = a^2\left(2 - \frac{\pi}{2}\right).$$ **Ans.**

EXAMPLE 19.27. Show that the area between one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ and its base is $3\pi a^2$.

SOLUTION: Required area A is the shaded portion as shown in the figure. By symmetry,

$$\text{A} = (\text{area AOBA}) = 2\int_0^{a\pi} \int_{a(1-\cos\theta)}^{2a} dy\, dx$$



$$= 2\int_0^{a\pi} [y]_{a(1-\cos\theta)}^{2a}\, dx$$

$$= 2\int_0^{a\pi}[2a - a(1-\cos\theta)]\,dx \quad \text{(here } dx = a(1+\cos\theta)\,d\theta\text{)}$$

$$= 2a\int_0^{\pi}(1+\cos\theta)\,a(1+\cos\theta)d\theta = 2a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta \quad \left(\text{Putting now } \frac{\theta}{2} = t\right)$$

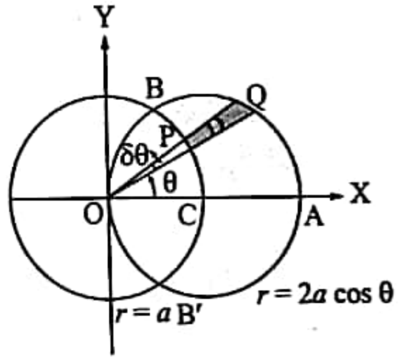
$$= 2a^2\int_0^{\pi/2}4\cos^4 t\,.2dt = 16\cdot a^2\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 3\pi\,a^2. \qquad \textbf{Ans.}$$

EXAMPLE 19.28. (a) Find the area lying inside the circle $x^2 + y^2 - 2ax = 0$ and outside the circle $x^2 + y^2 = a^2$ using double integrtion.

(b) Find the area enclosed by the lamicon $r = a + b\cos\theta$, $a > b$.

SOLUTION: (a) The centres of the two given circles are $(a, 0)$ and $(0, 0)$ and radii of both of them is a. The required area is the shaded portion as shown in the adjacent figure. The two circles are $r = a$ and $r = 2a\cos\theta$ in polar coordinates, and intersect in points B and B′, given by

$$2a\cos\theta = a \quad \text{or} \quad \cos\theta = \frac{1}{2} \quad \therefore\ \theta = \pm\frac{\pi}{3}.$$



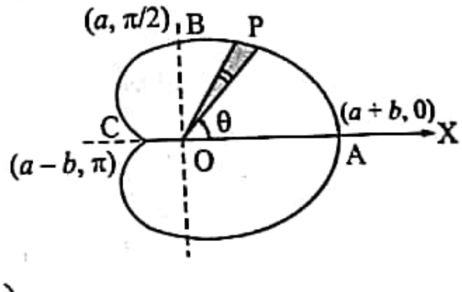
By symmetry, the required area is equal to

$$= 2\iint_{CABC} r\,dr\,d\theta = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_a^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/3}(4a^2\cos^2\theta - a^2)\,d\theta = a^2\int_0^{\pi/3}[2(1+\cos 2\theta) - 1]\,d\theta$$

$$= a^2\left[\theta + \frac{2\sin 2\theta}{2}\right]_0^{\pi/3} = a^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right). \qquad \textbf{Ans.}$$

(b) The curve is symmetrical about the x-axis as shown in the figure, hence the required area



$$= 2\int_0^{\pi}\int_0^{a+b\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a+b\cos\theta} d\theta$$

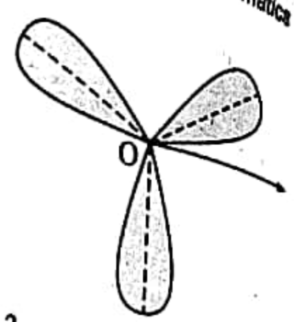
$$= \int_0^{\pi}(a+b\cos\theta)^2\,d\theta = \int_0^{\pi}(a^2 + 2ab\cos\theta + b^2\cos^2\theta)\,d\theta$$

$$= [a^2\theta + 2ab\sin\theta]_0^{\pi} + \frac{b^2}{2}\int_0^{\pi}(1+\cos 2\theta)\,d\theta = \pi\left(a^2 + \frac{b^2}{2}\right). \qquad \textbf{Ans.}$$

EXAMPLE 19.29. Find area enclosed by the leaves of the following curves

(i) $r = a\sin 3\theta$. (ii) $r = a\cos 4\theta$.

SOLUTION: (i) We know that $r = a\sin n\theta$ has n leaves if n is odd, and $2n$ leaves when n is even. Therefore $r = a\sin 3\theta$ has three symmetrical leaves and one leaf extends from $\theta = 0$ to $\theta = \pi/3$, as shown in the adjoing figure.



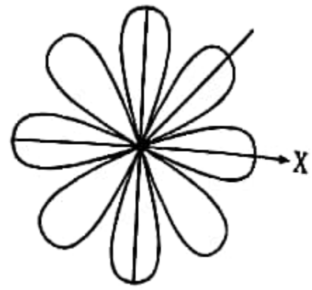
$$\therefore \quad \text{Required area} = 3\int_0^{\pi/3}\int_0^{a\sin 3\theta} r\, d\theta\, dr = 3\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^{a\sin 3\theta} d\theta$$

$$= \frac{3}{2}\int_0^{\pi/3} a^2 \sin^2 3\theta\, d\theta$$

$$= \frac{3a^2}{4}\int_0^{\pi/3}(1-\cos 6\theta)\, d\theta = \frac{3a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi a^2}{4}.$$

Ans.

(*ii*) We know that $r = a\cos n\theta$ has n leaves if n is odd and $2n$ leaves when n is even. Thus, $r = a\cos 4\theta$ will have eight symmetrical leaves. One leaf extends from $\theta = -\pi/8$ to $\theta = \pi/8$.



$$\text{Therefore, required area} = 8\int_{-(\pi/8)}^{\pi/8}\int_0^{a\cos 4\theta} r\, dr\, d\theta$$

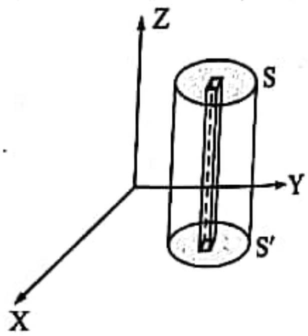
$$= 8\int_{-(\pi/8)}^{\pi/8}\left[\frac{r^2}{2}\right]_0^{a\cos 4\theta} d\theta = \frac{8}{2}\int_{-(\pi/8)}^{\pi/8} a^2\cos^2 4\theta\, d\theta$$

$$= 4a^2\int_0^{\pi/8} 2\cos^2 4\theta\, d\theta = 4a^2\int_0^{\pi/8}(1+\cos 8\theta)\, d\theta = \frac{\pi a^2}{2}.$$

Ans.

II. Volumes of solids by Double Integration

Consider a surface $z = f(x, y)$. Let S be a portion of this surface and S$'$ be its projection on the XY plane. We wish to show that the volume of the solid bounded by the surface $z = f(x, y)$ and the surrounding surface can be expressed as a double integral. Divide S$'$ into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to OX and OY. On each of these rectangles as base erect a prism with its length parallel to z-axis. Volume of this typical elementary prism between S$'$ and the given surface $z = f(x, y)$, is



$$\delta V = z\,\delta x\,\delta y = f(x, y)\,\delta x\,\delta y$$

Therefore, the total volume is given by

$$V = \iint_{S'} f(x, y)\, dx\, dy$$

where S$'$ is the region of integration.

EXAMPLE 19.30. Obtain the volume bounded by the surface $z = c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)$ and the quadrant of the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

SOLUTION: In solid geometry $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ represents a cylinder whose axis is along z-axis and guiding curve is the ellipse. Required volume V is given by

$$V = \iint z\, dx\, dy = \iint_C c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right) dx\, dy \quad \text{where} \quad C: \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Let us use elliptic polar co-ordinates $x = a\, r \cos\theta,\ y = br \sin\theta$ so that $dx\, dy = ab\, r\, d\theta\, dr$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$, hence

$$V = abc \int_0^{\pi/2} \int_0^1 (1 - r\cos\theta)(1 - r\sin\theta)\, r\, dr\, d\theta$$

$$= abc \int_0^{\pi/2} \int_0^1 [r - r^2(\cos\theta + \sin\theta) + r^3 \cos\theta \sin\theta]\, dr\, d\theta$$

$$= abc \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{3}(\cos\theta + \sin\theta) + \frac{r^4}{4}\sin\theta\cos\theta\right]_0^1 d\theta$$

$$= abc \int_0^{\pi/2} \left[\frac{1}{2} - \frac{1}{3}(\cos\theta + \sin\theta) + \frac{1}{4}\sin\theta\cos\theta\right] d\theta$$

$$= abc \left[\frac{\theta}{2} - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta + \frac{1}{8}\sin^2\theta\right]_0^{\pi/2} = abc\left(\frac{\pi}{4} - \frac{13}{24}\right).$$ **Ans.**

III. Volumes of Solids as Triple Integral

To express the volume of a solid as triple integral we note that the volume of an elementary cuboid with its faces parallel to the co-ordinate planes is $dx\, dy\, dz$ and therefore, the volume of the solid is given by

$$V = \iiint dx\, dy\, dz$$

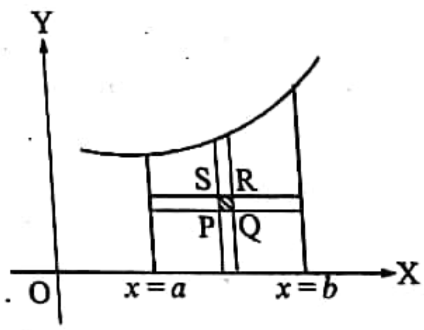
where the limits of integration w.r.t. z (if we integrate first w.r.t. z) are z_1 and z_2 obtained from the equation of the top and the bottom of the given surface, and then the double integration is w.r.t. x and y carried out over the area of projection of the given solid on the XY-plane.

IV. Volume of Solid of Revolution

Consider the area A bounded by $y = f(x)$, the ordinates $x = a$ and $x = b$ and the x-axis. Let this area revolve about x-axis then a solid is generated whose volume is to be determined. Consider an elementary area. PQRS where $P \equiv (x, y)$ and R $(x + \delta x, y + \delta y)$. The area of the rectangle is $\delta x\, \delta y$. When revolved about x-axis the elementary area generates an elementary solid of volume $\pi(y + \delta y)^2\, \delta x - \pi y^2\, \delta x$

$$= \pi\,[(y + \delta y)^2 - y^2]\, \delta x = \pi\,[2y\, \delta y\, \delta x + \delta y^2 \,.\, \delta x]$$

$$= 2\pi y\, \delta x\, \delta y \quad \text{neglecting the higher order infinitesimals.}$$



Therefore, the total volume of the solid of revolution of area A about X-axis, is equal to

$$V = 2\pi \int_a^b \int_0^{f(x)} y\, dy\, dx$$

Similarly, if the area is revolved about Y-axis, then the volume of the solid so generated, is

$$2\pi \iint_A x\, dx\, dy \quad \text{with appropriate limits.}$$

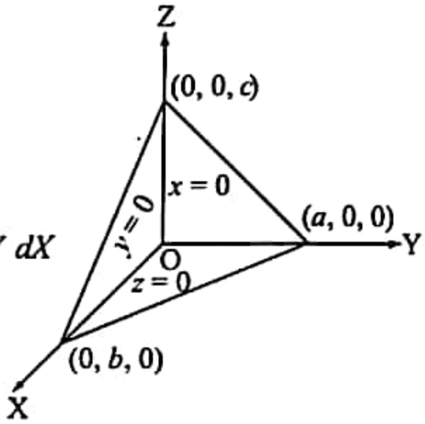
In **polar co-ordinates** the corresponding formula for **volume of solid of revolution of area A** about X-axis, becomes

$$\iint_A 2\pi r \sin\theta \, . \, r\, d\theta\, dr = 2\pi \iint_A r^2 \sin\theta \, d\theta\, dr$$

Similarly, the volume of solid of revolution of the area A about Y-axis $= 2\pi \iint_A r^2 \cos\theta \, d\theta\, dr$.

EXAMPLE 19.31. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. [GGSIPU II Sem. End Term 2003]

SOLUTION: The region of integration R is enclosed by the coordinate planes $x=0, y=0, z=0$ and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Putting $x=a\mathrm{X},\ y=b\mathrm{Y},\ z=c\mathrm{Z}$, the equation of the plane becomes $\mathrm{X}+\mathrm{Y}+\mathrm{Z}=1$. Then Z varies from $\mathrm{Z}=0$ to $\mathrm{Z}=(1-\mathrm{X}-\mathrm{Y})$,
Y varies from $\mathrm{Y}=0$ to $\mathrm{Y}=1-\mathrm{X}$
and X varies from $\mathrm{X}=0$ to $\mathrm{X}=1$.



Here $dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(x,y,z)}\, d\mathrm{X}\, d\mathrm{Y}\, d\mathrm{Z} = abc\, d\mathrm{X}\, d\mathrm{Y}\, d\mathrm{Z}$

$$\text{Volume V of the tetrahedron} = \iiint dx\,dy\,dz = abc\int_0^1\int_0^{1-\mathrm{X}}\int_0^{1-\mathrm{X}-\mathrm{Y}} d\mathrm{Z}\, d\mathrm{Y}\, d\mathrm{X}$$

or $$\mathrm{V} = abc\int_0^1\int_0^{1-\mathrm{X}} (1-\mathrm{X}-\mathrm{Y})\, d\mathrm{Y}\, d\mathrm{X}$$

$$= abc\int_0^1\left[\frac{(1-\mathrm{X}-\mathrm{Y})^2}{-2}\right]_0^{1-\mathrm{X}} d\mathrm{X} = \frac{abc}{2}\int_0^1 (1-\mathrm{X})^2\, d\mathrm{X}$$

$$= \frac{abc}{2}\left[\frac{(1-\mathrm{X})^3}{-3}\right]_0^1 = \frac{abc}{-6}(0-1) = \frac{abc}{6}.$$ **Ans.**

CHANGE OF VARIABLES USING JOCOBIANS

Sometimes it becomes difficult to evaluate double and triple integrals by usual methods, in such situations change of variables facilitates in transforming the given complicated integral into a simpler integral involving new variables.

For instance, let the variables x and y be replaced by new variables u and v by the transformation $x=f_1(u,v),\ \ y=f_2(u,v)$, then

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f\{f_1(u,v), f_2(u,v)\}\,|\mathrm{J}|\,du\,dv \quad \text{where} \quad \mathrm{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Here the region of integration R in xy-plane gets transformed to the region R′ in the uv-plane and J is known as Jacobian of transformation.

The idea can be extended to triple integral, as given below

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint f[f_1(u, v, w), f_2(u, v, w), f_3(u, v, w)]\, |J|\, du\, dv\, dw$$

where $x = f_1(u, v, w), \quad y = f_2(u, v, w), \quad z = f_3(u, v, w),$

and $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

EXAMPLE 19.32. Evaluate $\iint_R \cos\frac{x-y}{x+y}\, dx\, dy$ where 'R' is bounded by $x = 0, \; y = 0, \; x + y = 1.$

[GGSIPU II Sem I Term 2013]

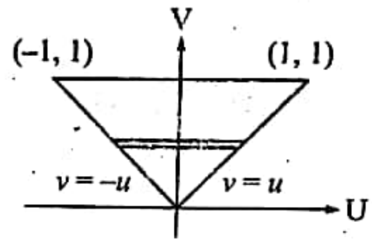
SOLUTION: Let $u = x - y$ and $v = x + y$ which gives $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$.

Thus, the region $x = 0$ means $u + v = 0$ or $v = -u$

and the region $y = 0$ means $v - u = 0$ or $v = u$

and the region $x + y = 1$ means $v = 1$.

Region R is in the form of shaded area, as shown in the figure.



Next, $$dx\, dy = \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} du\, dv = \frac{1}{2}\, du\, dv$$

$$\therefore \quad \iint_R \cos\frac{x-y}{x+y}\, dx\, dy = \frac{1}{2}\int_0^1 \int_{-v}^{v} \cos\frac{u}{v}\, du\, dv = \frac{1}{2}\int_0^1 \left[v \sin\frac{u}{v}\right]_{u=-v}^{v} dv.$$

$$= \frac{1}{2}\int_0^1 v\,(\sin 1 - \sin(-1)\, dv = \sin 1\left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2}\sin 1. \quad \textbf{Ans.}$$

EXAMPLE 19.33. Evalvate $\iint_R (x+y)^2\, dxdy$ where R is the parallelogram in the x-y-plane with vertices $(1, 0), (3, 1), (2, 2), (0, 1)$ using the transformation

$u = x + y, \; v = x - 2y.$ [GGSIPU II Sem I Term 2006]

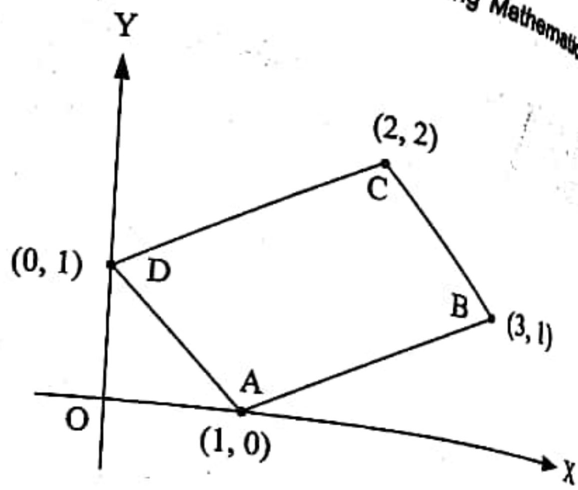
SOLUTION: R is parallelogram $ABCD$ as shown in the adjacent figure. Equation of line AB is $x - 2y = 1$ and of line DC is $x - 2y = -2$. Equation of line DA is $x + y = 1$ and of line CB is $x + y = 4$. Take $x + y = u$ and $x - 2y = v$, the lines can also be represented as $u = 1, \; u = 4$ and $v = -2, v = 1$.

We also know by change of variables that

$$du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dxdy$$

and $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

$\therefore$ $$dxdy = \frac{\partial(x,y)}{\partial(u,v)}\,dudv = \frac{-1}{3}\,dudv$$



Therefore $$\iint_R (x+y)^2\,dxdy = \iint_R u^2\left(\frac{-1}{3}\right)dudv = \frac{-1}{3}\int_{-2}^{1}\int_{1}^{4} u^2\,dudv = \frac{-1}{3}\int_{-2}^{1}\left[\frac{u^3}{3}\right]_1^4 dv$$

$$= -7\,(1+2) = -21. \quad \textbf{Ans.}$$

CHAPTER

20

Vector Differentiation, Gradient, Divergence and Curl

Differentiation of Vectors, Gradient, Divergence and Curl, Directional Derivative, Vector Identities.

Elementary ideas of vector analysis in the form of vector algebra, the reader is supposed to have. The differentiation and integration of vector fields has immense applications in a wide range of scientific and engineering disciplines. To mention a few we have rigid dynamics, fluid dynamics, heat transfer, continuum mechanics, electro magnetism, theory of relativity.

VECTOR DIFFERENTIATION

Let $\vec{V}$ be a vector, continuous and single valued function of a scalar variable t. The ideas of limit, continuity and differentiability can easily be extended to the vector function $\vec{V}(t)$. Thus, $\vec{V}(t)$ is continuous at $t = t_0$, if $\lim\limits_{t \to t_0} \vec{V}(t) = \vec{V}(t_0)$.

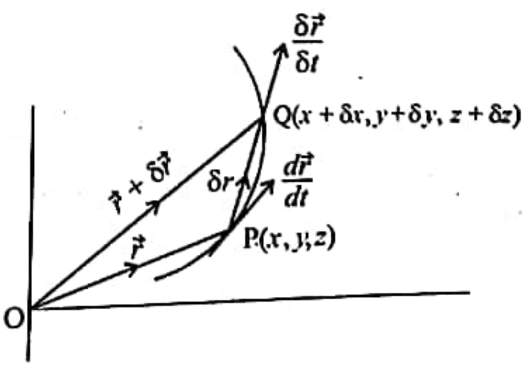
Similarly, the derivative of $\vec{V}(t)$ with respect to t is quite similar to that used for scalar functions. Thus, corresponding to a small change δt in t, let there be a change $\delta \vec{V}$ in $\vec{V}$, that is,

$$\delta \vec{V} = \vec{V}(t + \delta t) - \vec{V}(t)$$

then
$$\frac{d}{dt} \vec{V}(t) = \lim_{\delta t \to 0} \frac{\vec{V}(t+\delta t) - \vec{V}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\delta \vec{V}}{\delta t}.$$

Geometrically, if a vector function $\vec{V}(t)$ represents a set of position vectors of points on some curve C, then $\vec{V}(t)$ defines a curve C with equation $\vec{r} = \vec{V}(t)$.

Now the increment $\delta \vec{r}$ in $\vec{r}$ corresponding to the increment δt in t, is represented by the vector $\vec{PQ}$ in the adjoining figure. As $\delta t \to 0$, the point Q approaches to the point P. The chord $\vec{PQ}$ tends to coincide with the tangent to the curve at P. Then the limiting value of $\frac{\delta \vec{r}}{\delta t}$ as $\delta t \to 0$, will be a vector whose direction will be limiting direction of $d\vec{r}$, *i.e.*, tangent at P. This limiting value is called the derivative of $\vec{r}$ w.r.t. t, denoted by $\frac{d\vec{r}}{dt}$.



In particular, if the scalar variable t is replaced by the arc length s of the curve C measured from some fixed point on C, then we have

$$\lim_{\delta s\to 0}\left|\frac{\delta \vec{r}}{\delta s}\right| = \lim_{Q\to P}\frac{\text{chord PQ}}{\text{arc PQ}} = 1$$

Here $\dfrac{\delta\vec{r}}{\delta s}$ is a unit vector along the tangent to the curve C at P. The concepts of vector differentiation can easily be extended to the sum and product of vectors as follows, provided the original order of vectors is preserved in dealing with the vector product.

General Rules of Vector Differentiation: For vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, we have

$$\frac{d}{dt}(\vec{A}\pm\vec{B}) = \frac{d\vec{A}}{dt}\pm\frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(u\vec{A}) = \frac{du}{dt}\vec{A} + u\frac{d\vec{A}}{dt} \quad \text{where } u \text{ is a scalar function of } t.$$

Also, $$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt} \quad\text{and}\quad \frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Next, $$\frac{d}{dt}(\vec{A}\cdot\vec{B}\times\vec{C}) = \frac{d\vec{A}}{dt}\cdot\vec{B}\times\vec{C} + \vec{A}\cdot\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\cdot\left(\vec{B}\times\frac{d\vec{C}}{dt}\right).$$

and $$\frac{d}{dt}\{\vec{A}\times(\vec{B}\times\vec{C})\} = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C}) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

If the vector $\vec{A}(t)$ is expressed in terms of its components as

$$\vec{A} = A_1(t)\,i + A_2(t)\,j + A_3(t)\,k$$

then $$\frac{d\vec{A}}{dt} = \frac{dA_1}{dt}i + \frac{dA_2}{dt}j + \frac{dA_3}{dt}k.$$

Higher order derivatives of a vector function can be similarly defined.

Thus, if $\vec{r} = xi + yj + zk$ represents the position vector of a particle at any time t, then its velocity $\vec{v}$ and acceleration $\vec{a}$ are given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k \quad\text{and}\quad \vec{a} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$$

where $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$ are the components of velocity v, and $\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$ are the components of acceleration $\vec{a}$ of the particle along the three coordinate axes.

If $\vec{V}$ is a vector function of three scalar variables x, y, z then the partial derivatives $\dfrac{\partial\vec{V}}{\partial x}, \dfrac{\partial\vec{V}}{\partial y}, \dfrac{\partial\vec{V}}{\partial z}$ are defined in the same way as in the case of scalar functions.

For example, $\vec{V}_x = \dfrac{\partial\vec{V}}{\partial x} = \lim\limits_{\delta x\to 0}\dfrac{\vec{V}(x+\delta x, y, z) - \vec{V}(x,y,z)}{\delta x}$ provided the limit exists.

Higher order partial derivatives also are defined analogously.

TWO IMPORTANT RESULTS:

(1) The necessary and sufficient condition for $\vec{V}(t)$ to have a constant magnitude, is

$$\vec{V} \cdot \frac{d\vec{V}}{dt} = 0.$$

To show that the condition is necessary, let $\vec{V}$ be a vector of constant magnitude, then we have

$$\vec{V} \cdot \vec{V} = |\vec{V}|^2 = \text{constant}$$

$$\therefore \quad \frac{d}{dt}(\vec{V} \cdot \vec{V}) = 0 \quad \text{or} \quad \frac{d\vec{V}}{dt} \cdot \vec{V} + \vec{V} \cdot \frac{d\vec{V}}{dt} = 0$$

or $$2\vec{V} \cdot \frac{d\vec{V}}{dt} = 0 \quad \text{or} \quad \vec{V} \cdot \frac{d\vec{V}}{dt} = 0$$ which establishes the necessary part.

For sufficiency part, suppose $\vec{V} \cdot \frac{d\vec{V}}{dt} = 0 \Rightarrow \frac{d\vec{V}}{dt} \cdot \vec{V} + \vec{V} \cdot \frac{d\vec{V}}{dt} = 0$ or $\frac{d}{dt}(\vec{V} \cdot \vec{V}) = 0$

$\Rightarrow \vec{V} \cdot \vec{V} = \text{constant}$, hence $|\vec{V}| = \text{constant}$.

Thus, the condition is necessary as well as sufficient.

(2) The necessary and sufficient condition for any vector $\vec{V}$ to have a constant direction, is

$$\vec{V} \times \frac{d\vec{V}}{dt} = 0.$$

[*GGSIPU II Sem II Term 2011*]

First we show the sufficiency part. Let $|\vec{V}|$ be denoted by V, then

$$\vec{V} = |\vec{V}|\,\hat{V} = V\,\hat{V} \quad \text{and} \quad \frac{d\vec{V}}{dt} = \frac{dV}{dt}\hat{V} + V\frac{d\hat{V}}{dt}.$$

Therefore $$\vec{V} \times \frac{d\vec{V}}{dt} = V\hat{V} \times \left(\frac{dV}{dt}\hat{V} + V\frac{d\hat{V}}{dt}\right) = V\frac{dV}{dt}(\hat{V} \times \hat{V}) + V^2\,\hat{V} \times \frac{d\hat{V}}{dt}$$

$$= V^2\,\hat{V} \times \frac{d\hat{V}}{dt} \quad (\text{since } \hat{V} \times \hat{V} = 0).$$

As $\vec{V} \times \frac{d\vec{V}}{dt} = 0$ we have $\hat{V} \times \frac{d\hat{V}}{dt} = 0 \Rightarrow \frac{d\hat{V}}{dt}$ is parallel to $\hat{V}$.

And, since $\hat{V} \cdot \hat{V} = 1$ we have $\frac{d\hat{V}}{dt} \cdot \hat{V} + \hat{V} \cdot \frac{d\hat{V}}{dt} = 0 \Rightarrow \frac{d\hat{V}}{dt} \cdot \hat{V} = 0$

$\Rightarrow \quad \frac{d\hat{V}}{dt}$ is perpendicular to $\hat{V}$.

Thus, $\frac{d\hat{V}}{dt}$ is parallel as well as perpendicular to $\hat{V}$ which is possible only when $\hat{V}$ is a constant vector meaning thereby that V has constant direction and hence the condition is sufficient.

Next, to establish the necessary part, suppose $\vec{V}$ has a constant direction then $\hat{V}$ also has a constant direction and hence is a constant vector.

$\Rightarrow \quad \frac{d\hat{V}}{dt} = 0$ and, in turn, $\hat{V} \times \frac{d\hat{V}}{dt} = 0$ which establishes the necessary part.

EXAMPLE 20.1. A particle moves along a curve whose parametric equation in terms of time t, is $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$. Find its velocity and acceleration at time $t = 2$.

SOLUTION: If $\vec{r}$ is the position vector of the particle at any time t, then

$$\vec{r} = xi + yj + zk = 3t^2 i + (t^2 - 2t)\, j + t^3 k.$$

Then, the velocity $\vec{V}$ of the particle at a time t, is given by

$$\vec{V} = \frac{d\vec{r}}{dt} = 6ti + (2t - 2)\, j + 3t^2 k$$

and the acceleration $\vec{a}$ of the particle at time t, is given by

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6i + 2j + 6tk$$

At time $t = 2$, we have

$$\left[\frac{d\vec{r}}{dt}\right]_{t=2} = 12i + 2j + 12k \quad \text{and} \quad \left[\frac{d^2\vec{r}}{dt^2}\right]_{t=2} = 6i + 2j + 12k.$$ **Ans.**

EXAMPLE 20.2. Find the angle between the tangents to the curve $\vec{r} = t^2 i - 2t j + t^3 k$ at the points $t = 1$ and $t = 2$.

SOLUTION: The tangent at any point 't' is given by $\dfrac{d\vec{r}}{dt} = 2ti - 2j + 3t^2 k$

Therefore the tangents $\vec{T}_1$ and $\vec{T}_2$ at $t = 1$ and $t = 2$ respectively, are given by

$$\vec{T}_1 = 2i - 2j + 3k \quad \text{and} \quad \vec{T}_2 = 4i - 2j + 12k.$$

If θ is the angle between $\vec{T}_1$ and $\vec{T}_2$, then

$$\cos\theta = \frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1|\,|\vec{T}_2|} = \frac{2(4)+(-2)(-2)+3(12)}{\sqrt{2^2+(-2)^2+3^2}\,\sqrt{4^2+(-2)^2+12^2}} = \frac{48}{\sqrt{17}\sqrt{164}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{24}{\sqrt{17}\sqrt{41}}\right)$$ is the required angle. **Ans.**

EXAMPLE 20.3. If $\vec{r}$ is the position vector of a particle of mass m at time t and $\vec{F}$ is the external force applied on the particle, then show that the moment M of the force $\vec{F}$ about its origin O, is given by

$$\vec{M} = \frac{d\vec{H}}{dt}$$ where $\vec{H} = \vec{r} \times m\vec{V}$ and $\vec{V}$ is the velocity of the particle at time t.

SOLUTION: Moment M of the force $\vec{F}$ about the origin O, is given by

$$\vec{M} = \vec{r} \times \vec{F} = \vec{r} \times \left(m\frac{d\vec{V}}{dt}\right)$$

$$\frac{d\vec{H}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{V}) = \frac{d\vec{r}}{dt} \times m\vec{V} + \vec{r} \times \frac{d}{dt}(m\vec{V})$$

$$= \vec{V} \times (m\vec{V}) + \vec{r} \times \frac{d}{dt}(m\vec{V}) = 0 + \vec{r} \times \left(m\frac{d\vec{V}}{dt}\right)$$

$$\therefore \quad \vec{M} = \vec{r} \times m\frac{d\vec{V}}{dt} = \frac{d\vec{H}}{dt}.$$ **Ans.**

SCALAR AND VECTOR POINT FUNCTIONS AND LEVEL SURFACES

A variable quantity which depends for its value on its position only, that is, upon the coordinates of the points, say (x, y, z) in space, is called a *point function*. There are two types of point functions.

(a) Scalar Point Function: If a scalar quantity ϕ is a function of a point, that is, depends on its position (x, y, z) in space, then $\phi(x, y, z)$ is called a scalar point function. The scalar point function does not depend upon the choice of coordinate system. For example, density of a body, the temperature of a body at any instant, potential of a body are all examples of scalar point function. Pressure in a fluid at a point usually depends on its depth hence it is also a scalar point function.

In dealing with the differentiation of a scalar point function $\phi(x, y, z)$, following formulae will be frequently used $\quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad$ and $\quad \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial s}.$

(b) Vector Point Function: If a vector quantity $\vec{V}$ depends for its value on its position (x, y, z) in space, then $\vec{V}(x, y, z)$ is called a vector point function. Velocity of a moving fluid, force, gravitational force or electrical intensity are all examples of vector point function.

While differentiating a vector point function $\vec{V}(x, y, z)$, following formulae will be frequently used.

$$d\vec{F} = \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial y}dy + \frac{\partial \vec{F}}{\partial z}dz \quad \text{and} \quad \frac{\partial \vec{F}}{\partial s} = \frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial s}.$$

(c) LEVEL SURFACE: Let $\phi(x, y, z)$ be a scalar point function, continuous in a region. The surface drawn in space containing all those points where $\phi(x, y, z)$ has same value, is called a *level surface*. Equipotential surfaces and isothermal surfaces are some examples of level surfaces.

GRADIENT OF A SCALAR POINT FUNCTION

The vector differential operator $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ is denoted by the symbol ∇ and is called **Del** (or Nabla). It operates on any scalar point function $\phi(x, y, z)$ say, and we get a vector quantity $\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$ which is called the *gradient* of the scalar point function $\phi(x, y, z)$. This is also written as **gradient ϕ or simply grad ϕ.** Actually, the operator ∇ is a differential operator but vector in nature.

Thus $\quad \text{grad } \phi = \nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$

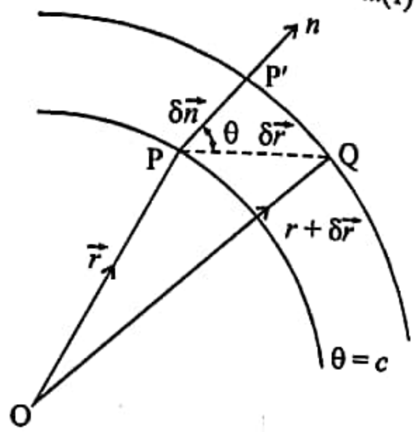
PHYSICAL AND GEOMETRICAL SIGNIFICANCE OF GRADIENT ϕ

Let P (x, y, z) be a point with position vector $\vec{r}$, then $\vec{r} = xi + yj + zk$

$\therefore \quad d\vec{r} = idx + jdy + kdz$

Next, $\vec{\nabla}\phi \cdot d\vec{r} = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right) \cdot (idx + jdy + kdz)$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \qquad ...(1)$$



Since $\vec{\nabla}\phi$ is vector, we want to know its direction as well as magnitude. Let Q $(x + \delta x,\ y + \delta y,\ z + \delta z)$ be a neighbouring point with position vector $\vec{r} + \delta\vec{r}$. Consider the level surface through P at every point of which the value of the scalar point function is ϕ while the level surface through Q has value of the function as $\phi + \delta\phi$ throughout, (see figure).

Thus, $\phi + \delta\phi = \phi(x + \delta x,\ y + \delta y,\ z + \delta z)$

$$= \phi(x, y, z) + \delta x\frac{\partial\phi}{\partial x} + \delta y\frac{\partial\phi}{\partial y} + \delta z\frac{\partial\phi}{\partial z} + \ldots$$

(by Taylor's series)

$\Rightarrow \quad \delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$ neglecting higher order terms.

$$= \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right) \cdot (i\,\delta x + j\,\delta y + k\,\delta z)$$

$$= \vec{\nabla}\phi \cdot \delta\vec{r} \quad \text{where} \quad \delta\vec{r} = \vec{PQ}. \qquad ...(2)$$

As the point Q approaches the point P, $\delta\phi \to 0$ and PQ becomes tangent to the level surface at P which indicates that **the direction of $\nabla\phi$ at P is along the normal to the level surface of P.**

Further, let the normal at P meet the level surface of Q at P′ then $\vec{PP'} = \delta\vec{n}$ = projection of PQ along the normal. Clearly $\frac{\partial\phi}{\partial r}$ is the rate of change of ϕ along $\vec{PQ}$ and $\frac{\partial\phi}{\partial n}$ is the rate of change of ϕ along PP′, *i.e.*, along the normal at P.

$\therefore \quad \frac{\delta\phi}{\delta r} = \frac{\delta\phi}{\delta n}\frac{\delta n}{\delta r} = \frac{\delta\phi}{\delta n}\cos\theta$ (see the above figure).

As $Q \to P$, $\lim\frac{\delta\phi}{\delta r} = \frac{\partial\phi}{\partial r}$ and $\lim\frac{\delta\phi}{\delta n} = \frac{\partial\phi}{\partial n}$

$$\Rightarrow \quad \frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial n}\cos\theta \quad \text{hence} \quad \frac{\partial\phi}{\partial r} \le \frac{\delta\phi}{\delta n}. \qquad ...(3)$$

Therefore, the **rate of change of ϕ is maximum in the direction of the normal.**

Since δn is the projection of δr along the normal, we have $dn = \hat{n} \cdot d\vec{r}$ ...(4)

In relation (1) we had $d\phi = \vec{\nabla}\phi \cdot d\vec{r}$ and let us write $d\phi = \frac{\partial\phi}{\partial n}dn$ ...(5)

Using (4) and (5) here we get $\nabla\phi \cdot d\vec{r} = \frac{\partial \phi}{\partial n}\hat{n} \cdot d\vec{r} \Rightarrow \nabla\phi = \frac{\partial \phi}{\partial n}\hat{n}$.

This shows that grad ϕ at the point P represents the maximum rate of change of ϕ which is along the outward drawn normal to the level surface at P. In other words, the vector $\nabla\phi$ is in the direction in which ϕ increases most rapidly and $-\nabla\phi$ is in the direction in which ϕ decreases most rapidly.

It is easy to derive the following corollaries on gradient:

(1) $\nabla (u \pm v) = \nabla (u) \pm \nabla (v)$.

(2) $\nabla (au) = a\nabla (u)$.

(3) $\nabla (uv) = \nabla (u)\, v + u\nabla (v)$.

(4) $\nabla [f(u)] = f'(u)\, \nabla u$.

(5) $\nabla (\vec{a} \cdot \vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

Let us prove (4) and (5) as (1), (2) and (3) are obvious.

(4) Let $v = f(u)$ where u is a scalar function of x, y, z.

then
$$\nabla (v) = i\frac{\partial v}{\partial x}+j\frac{\partial v}{\partial y}+k\frac{\partial v}{\partial z} = i\frac{\partial f(u)}{\partial x}+j\frac{\partial f(u)}{\partial y}+k\frac{\partial f(u)}{\partial z}$$
$$= i f'(u)\frac{\partial u}{\partial x} + j f'(u)\frac{\partial u}{\partial y} + k f'(u)\frac{\partial u}{\partial z} = f'(u)\left\{i\frac{\partial u}{\partial x}+j\frac{\partial u}{\partial y}+k\frac{\partial u}{\partial z}\right\}$$
$$= f'(u)\, \nabla u.$$

(5) Let $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{r} = xi + yj + zk$

hence $\vec{a} \cdot \vec{r} = a_1 x + a_2 y + a_3 z$

then
$$\nabla (\vec{a} \cdot \vec{r}) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(a_1 x + a_2 y + a_3 z) = ia_1 + ja_2 + ka_3 = \vec{a}.$$

EXAMPLE 20.4. If $r^2 = x^2 + y^2 + z^2$, show that $\nabla f(r) = \frac{f'(r)}{r}\vec{r}$ and hence show that

(i) $\nabla (r) = \hat{r}$

(ii) $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}$ [GGSIPU II Sem II Term 2013]

(iii) $\nabla (r^n) = n\, r^{n-2}\, \vec{r}$

(iv) $\nabla \int r^n dr = r^{n-1}\vec{r}$

[GGSIPU II Sem End Term 2006 Reappear]

SOLUTION: By definition

$$\nabla f(r) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right) f(r) = i f'(r)\frac{\partial r}{\partial x} + j f'(r)\frac{\partial r}{\partial y} + k f'(r)\frac{\partial r}{\partial z}$$
$$= f'(r)\left(i\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+k\frac{\partial r}{\partial z}\right)$$

From $r^2 = x^2 + y^2 + z^2$ we have $2r\frac{\partial r}{\partial x} = 2x$, hence $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\therefore \quad \nabla f(r) = f'(r)\left[i\frac{x}{r}+j\frac{y}{r}+k\frac{z}{r}\right] \quad \text{or} \quad \nabla f(r) = \frac{1}{r} f'(r)\, \vec{r}. \qquad ...(1)$$

(*i*) Take $f(r) = r$, then from (1), $\nabla(r) = \frac{1}{r}(1)\vec{r} = \hat{r}$.

(*ii*) Take $f(r) = \frac{1}{r}$, then from (1), $\nabla\left(\frac{1}{r}\right) = \frac{1}{r}\left(-\frac{1}{r^2}\right)\vec{r} = -\frac{1}{r^3}\vec{r}$.

(*iii*) Let us take $f(r) = r^n$, hence from (1), we get

$$\nabla(r^n) = \frac{1}{r}(n\, r^{n-1})\,\vec{r} = n\, r^{n-2}\,\vec{r}.$$

(*iv*) We take $f(r) = \int r^n dr$ so that $f'(r) = r^n$.

$\therefore$ Using (1) here, we get $\nabla \int r^n dr = \frac{1}{r}\cdot r^n\,\vec{r} = r^{n-1}\,\vec{r}$. **Hence Proved.**

EXAMPLE 20.5. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, show that ∇u, ∇v, ∇w are coplanar vectors.

SOLUTION:

$$\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z} = i\,(1) + j\,(1) + k\,(1) = i + j + k,$$

$$\nabla v = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z} = i\,(2x) + j\,(2y) + k\,(2z) = 2\,(xi + yj + zk)$$

and

$$\nabla w = i\frac{\partial w}{\partial x} + j\frac{\partial w}{\partial y} + k\frac{\partial w}{\partial z} = i\,(y + z) + j\,(z + x) + k\,(x + y).$$

We know that, $\nabla u, \nabla v, \nabla w$ will be coplanar when $\nabla u \cdot (\nabla v \times \nabla w) = 0$.

Here

$$\nabla u \cdot (\nabla v \times \nabla w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} \quad \left(\text{applying } R_3 \to R_3 + \frac{1}{2}R_2\right)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = (x + y + z)\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Therefore, $\nabla u, \nabla v, \nabla w$ are coplanar. **Hence Proved.**

EXAMPLE 20.6. (*a*) Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$ [*GGSIPU II Sem End Term 2007 Reappear*]

(*b*) Find a unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

[*GGSIPU II Sem End Term 2013*]

SOLUTION: (*a*) Equation of the given surface is $\phi \equiv x^3 + y^3 + 3xyz - 3 = 0$.

Since $\nabla\phi$ is a vector normal to the surface,

$$\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} = i(3x^2 + 3yz) + j(3y^2 + 3xz) + k(3z^2 + 3xy)$$

$$\therefore \quad [\nabla\phi]_{(1,\,2,\,-1)} = i(3 - 6) + j(12 - 3) + k(3 + 6) = 3(-i + 3j + 3k)$$

Therefore, unit vector along the normal to the given surface, is

$$\frac{(-i + 3j + 3k)}{\sqrt{1 + 9 + 9}} = \frac{1}{\sqrt{19}}(-i + 3j + 3k).$$ **Ans.**

(b) $\phi(x, y, z) = x^2y + 2xz - 4$. We know that $\nabla\phi$ is vector normal to the given surface $\phi = 0$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = i(2xy + 2z) + j(x^2) + k(2x) = -2i + 4j + 4k \quad \text{at} \quad (2, -2, 3).$$

$\therefore$ Unit vector normal to the surface at $(2, -2, 3)$

$$= \frac{-2i+4j+4k}{\sqrt{4+16+16}} = \frac{1}{3}(-i+2j+2k). \qquad \textbf{Ans.}$$

EXAMPLE 20.7. Determine the constant 'a' such that at any point of intersection of the two spheres $(x - a)^2 + y^2 + z^2 = 3$ and $x^2 + (y - 1)^2 + z^2 = 1$, their tangent planes are perpendicular to each other.

SOLUTION: Let $\phi_1 = (x - a)^2 + y^2 + z^2$ and $\phi_2 = x^2 + (y - 1)^2 + z^2$.

Then $$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z} = 2(x - a)\,i + 2yj + 2zk$$

and $$\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z} = 2xi + 2(y - 1)\,j + 2zk.$$

These vectors $\nabla\phi_1$ and $\nabla\phi_2$ are along the normals to the two spheres at a point (x, y, z) of their intersection. The tangent planes to the two spheres will be perpendicular to each other when the normals are perpendicular to each other for which

$$[2(x - a)\,i + 2yj + 2zk] \cdot [2xi + 2(y - 1) + 2zk] = 0 \quad \text{or} \quad x(x - 1) + y(y - 1) + z^2 = 0$$

or $$x^2 + y^2 + z^2 - ax - y = 0. \qquad ...(1)$$

At the point $P(x, y, z)$ of intersection of given spheres, we have

$$x^2 + y^2 + z^2 - 2ax + a^2 - 3 = 0 \quad \text{and} \quad x^2 + y^2 + z^2 - 2y = 0.$$

Adding these two and dividing by 2, we get or $x^2 + y^2 + z^2 - ax - y + \frac{1}{2}a^2 - \frac{3}{2} = 0$. ...(2)

Comparing (1) and (2), we get $\frac{a^2}{2} - \frac{3}{2} = 0$ or $a = \pm\sqrt{3}$. **Ans.**

EXAMPLE 20.8. If $\vec{A} = 2x^2i - 3yzj + xz^2k$ and $\phi = 2z - x^3y$, find $A \cdot \nabla\phi$ and $A \times \nabla\phi$ at the point $(1, -1, 1)$.

[GGIPU II Sem End Term 2007]

SOLUTION: Given $\phi = 2z - x^3y$ as a scalar function and $\vec{A} = 2x^2i - 3yzj + xz^2k$ as a vector function,

$$\therefore \quad \nabla\phi = i\frac{\partial}{\partial x}(2z - x^3y) + j\frac{\partial}{\partial y}(2z - x^3y) + k\frac{\partial}{\partial z}(2z - x^3y) = -3x^2yi - x^3j + 2k$$

At $(1, -1, 1)$, $\nabla\phi = 3i - j + 2k$ and $\vec{A} = 2i + 3j + k$ $\therefore$ $\vec{A} \cdot \nabla\phi = 3(2) - 1(3) + 2(1) = 5$.

And $$A \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix} = i(6 + 1) + j(3 - 4) + k(-2 - 9) = 7i - j - 11k. \qquad \textbf{Ans.}$$

DIRECTIONAL DERIVATIVE

As discussed earlier, $\frac{\partial \phi}{\partial s}$ represents the rate of change of ϕ in any direction and is called *directional derivative*. **The directional derivative in the direction of vector $\vec{a}$ is $\nabla\phi \cdot \hat{a}$. The directional derivatives of ϕ is maximum along the normal to the level surface of ϕ and is equal to $|\nabla\phi| = \frac{\partial \phi}{\partial n}$.**

EXAMPLE 20.9. Compute the directional derivative of $x^2 + y^2 + 4xz$ at $(1, -2, 2)$ in the direction of the vector $2i - 2j - k$. [*GGSIPU II Sem End Term 2005*]

SOLUTION: The scalar function $f = x^2 + y^2 + 4xz$.

Now $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = (2x + 4z)\, i + 2yj + 4xk = 10i - 4j + 4k$ at $(1, -2, 2)$.

The directional derivative of f at $(1, -2, 2)$ along the vector $2i - 2j - k$

$$= (10i - 4j + 4k) \cdot \frac{(2i - 2j - k)}{\sqrt{4+4+1}} = \frac{1}{3}(20 + 8 - 4) = 8. \qquad \textbf{Ans.}$$

EXAMPLE 20.10. Find the directional derivative of the scalar point function $\phi = 3e^{2x - y + z}$ at the point A $(1, 1, -1)$ in the direction towards the point B $(-3, 5, 6)$.

[*GGSIPU II Sem II Term 2010*]

SOLUTION: $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(3e^{2x-y+z}) = 3e^{2x - y + z}(2i - j + k)$

and $[\nabla\phi]_{(1,\,1,\,-1)} = 3\,(2i - j + k)\, e^{2-1-1} = 6i - 3j + 3k.$

Here, $\vec{AB} = (-3i + 5j + 6k) - (i + j - k) = -4i + 4j + 7k.$

Therefore, the required directional derivative is the component of $6i - 3j + 3k$ in the direction of the vector $-4i + 4j + 7k$,

$$= (6i - 3j + 3k) \cdot \frac{(-4i + 4j + 7k)}{\sqrt{(-4)^2 + 4^2 + 7^2}} = \frac{-24 - 12 + 21}{\sqrt{16+16+49}} = \frac{-5}{3}. \qquad \textbf{Ans.}$$

EXAMPLE 20.11. (a) If $f(x, y) = x^2 - xy - y + y^2$ show that all the points where the directional derivative in the direction of $\hat{b} = \frac{i + \sqrt{3}j}{2}$ is zero, lie on a line. Find the equation of the line.

[*GGSIPU II Sem End Term 2011*]

(b) If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a, b, c.

[*GGSIPU IInd Sem. End Term 2006*]

SOLUTION: (a) Give that $f = x^2 - xy - y + y^2$ then $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} = i\,(2x - y) + j\,(-x - 1 + 2y)$.

Here $\hat{b} = (i + \sqrt{3}j)/2$,

$\therefore$ Directional derivative of f along $\hat{b} = \nabla f \cdot \left(\dfrac{i+\sqrt{3}j}{2}\right) = \dfrac{1}{2}\left[(2x-y)+\sqrt{3}(-x-1+2y)\right]$.

Since this directional derivative is given to be zero, we have

$$2x-y=0 \quad \text{and} \quad -x-1+2y=0 \quad \Rightarrow \quad x=\frac{1}{3},\ y=\frac{2}{3}.$$

Since the line is along the vector $\vec{b}$ its slope is $\sqrt{3}$.

$\therefore$ Equation of the line is $y-\dfrac{2}{3}=\sqrt{3}\left(x-\dfrac{1}{3}\right)$. **Ans.**

(b) Since $\phi = ax^2y + by^2z + cz^2x$, we have

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z} = (2axy + cz^2)\,i + (2byz + ax^2)\,j + (2czx + by^2)\,k$$

$\therefore$ $[\nabla\phi]_{(1,\,1,\,1)} = (2a + c)\,i + (2b + a)\,j + (2c + b)\,k$

This is along the normal to the surface $\phi\,(x, y, z) = 0$ hence $[\nabla\phi]_{(1,\,1,\,1)}$ is the maximum directional derivative.

But this is given to be parallel to the line $\dfrac{x-1}{2}=\dfrac{y-3}{-2}=\dfrac{z}{1}$, therefore

$$\frac{2a+c}{2}=\frac{2b+a}{-2}=\frac{2c+b}{1}$$

From the first two, we have $3a + 2b + c = 0$ and from the last two, we have $a + 4b + 4c = 0$.

Solving the above equations for $a, b, c,$ we get $\dfrac{a}{4}=\dfrac{b}{-11}=\dfrac{c}{10}=\lambda$, say

Hence $a = 4\lambda$, $b = -11\lambda$, $c = 10\lambda$.

Since $|\nabla\phi|_{(1,\,1,\,1)} = 15$, we have $(2a + c)^2 + (2b + a)^2 + (2c + b)^2 = 15^2$

or $(8\lambda + 10\lambda)^2 + (-22\lambda + 4\lambda)^2 + (20\lambda - 11\lambda)^2 = 15^2$.

$$\therefore \quad \lambda^2 = \frac{15^2}{18^2+18^2+9^2} \quad \text{or} \quad \lambda = \pm\frac{5}{9}.$$

Therefore $a = \pm\dfrac{20}{9}$, $b = \pm\dfrac{55}{9}$, $c = \pm\dfrac{50}{9}$. **Ans.**

EXAMPLE 20.12. What is the directional derivative of $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ at $(-1, 2, 1)$?

[*GGSIPU II Sem II Term 2011*]

SOLUTION: $\nabla\phi = i\dfrac{\partial\phi}{\partial x}+j\dfrac{\partial\phi}{\partial y}+k\dfrac{\partial\phi}{\partial z} = y^2i+(2xy+z^3)j+3yz^2k$

$\therefore$ $[\nabla\phi]_{(2,-1,\,1)} = i-3j-3k$.

Let the other surface be $\phi_1 = x\log z - y^2 = -4$, then $\nabla\phi_1 = \sum i\dfrac{\partial\phi_1}{\partial x} = (\log z)\,i-2yj+\dfrac{x}{z}k$.

$\therefore$ $[\nabla\phi_1]_{(-1,\,2,\,1)} = 0i-4j-k = \vec{N}$, say.

Hence the directional derivative of ϕ along $\vec{N}$, is

$$(\nabla\phi)_A \cdot \frac{\vec{N}}{|\vec{N}|} = (i-3j-3k)\cdot\frac{(-4j-k)}{\sqrt{16+1}} = \frac{15}{\sqrt{17}}.$$ **Ans.**

EXAMPLE 20.13. (a) Find the directional derivative of the scalar function $f(x, y, z) = x^2 + xy + z^2$ at the point $A(1, -1, -1)$ in the direction of the line AB where B has the coordinates $(3, 2, 1)$ [GGSIPU II Sem End Term 2007]

(b) Find the directional derivative of the scalar function $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at a point $(3, 1, 2)$ in the direction of the vector (yz, zx, xy). [GGSIPU II Sem II Term 2005]

SOLUTION: (a) $f = x^2 + xy + z^2$, $\therefore\ \nabla f = i\dfrac{\partial f}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z} = (2x + y)i + xj + 2zk.$

$\therefore\quad (\nabla f)_A = [\nabla f]_{(1,-1,-1)} = (2-1)i + 1j - 2k = i + j - 2k.$

Also, $\overrightarrow{AB} = (3-1)i + (2+1)j + (1+1)k = 2i + 3j + 2k$

$\therefore$ The directional derivative of f at A along $\overrightarrow{AB}$

$$= [\nabla f]_{(1,-1,1)} \cdot \hat{AB} = (i + j - 2k).\frac{(2i+3j+2k)}{\sqrt{2^2+3^2+2^2}}$$

$$= \frac{1}{\sqrt{17}}(2+3-4) = \frac{1}{\sqrt{17}} \qquad \textbf{Ans.}$$

(b) $\nabla\phi = \sum i\dfrac{\partial\phi}{\partial x} = \sum i\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) = -\sum\dfrac{i}{r^2}\dfrac{\partial r}{\partial x} = -\sum\dfrac{ix}{r^3}$ where $r^2 = x^2 + y^2 + z^2$

Let $\vec{a}$ = vector $(yz, zx, xy) = iyz + jzx + kxy$, then the directional derivative of ϕ along $\hat{a} = \nabla\phi \cdot \hat{a}$

$$= -\frac{1}{r^3}(xi + yj + zk) \cdot \frac{iyz + jzx + kxy}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}} = \frac{-3xyz}{(x^2+y^2+z^2)^{3/2}\sqrt{y^2z^2+z^2x^2+x^2y^2}}$$

$$= \frac{-3.3.1.2}{(3^2+1^2+2^2)^{3/2}\sqrt{3^2.1^2+3^2.2^2+1^2.2^2}} = \frac{-9}{49\sqrt{14}} \text{ at } (3, 1, 2). \qquad \textbf{Ans.}$$

DIVERGENCE OF A VECTOR POINT FUNCTION

When the vector differential operator ∇ operates scalarly on a vector point function, the result is a scalar quantity, called the **divergence** of the vector point function. Thus, if $\vec{F}(x, y, z)$ is a vector point function then $\nabla \cdot \vec{F}$ denotes the divergence of $\vec{F}$ and

$$\nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (F_1 i + F_2 j + F_3 k) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \Sigma\frac{\partial F_1}{\partial x}$$

where F_1, F_2, F_3 are the components of $\vec{F}$.

The divergence of $\vec{F}$ can also be written as

$$\nabla \cdot F = i\cdot\frac{\partial \vec{F}}{\partial x} + j\cdot\frac{\partial \vec{F}}{\partial y} + k\cdot\frac{\partial \vec{F}}{\partial z} = \Sigma i\cdot\frac{\partial \vec{F}}{\partial x}$$

As an illustration $\quad \nabla \cdot \vec{r} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right) \cdot (xi + yj + zk)$

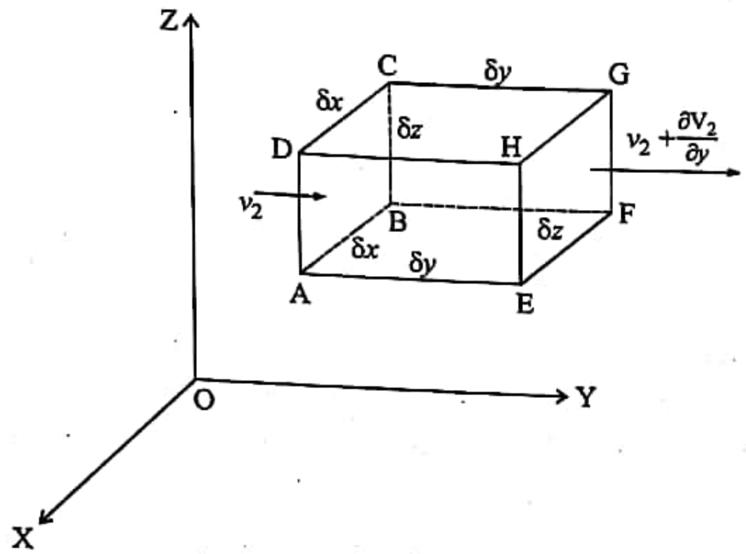
$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$$

In particular, if $\nabla \cdot \vec{F} = 0$, the vector field $\vec{F}$ is called SOLENOIDAL.

Physical Interpretation of Divergence of a Vector Field

To have a physical concept of the divergence of a vector point function, consider the flow of a fluid with velocity $V = V_1 i + V_2 j + V_3 k$ at a point $A(x, y, z)$. Consider a small rectangular parallelopiped with edges $\delta x, \delta y, \delta z$ parallel to the coordinate axes OX, OY, OZ respectively in the fluid with one vertex at $A(x, y, z)$.

Let us consider the fluid flow parallel to Y-axis, entering the face ABCD and coming out of the face EFGH. (See adjoining figure.) Amount of the fluid entering the face ABCD in unit time $= V_2(x, y, z)\,\delta x\,\delta z$. Amount of the fluid coming out of the parallel face EFGH in unit time $= V_2(x, y+\delta y, z)\,\delta x\,\delta z$.



Thus, the rate of fluid flow across the elementary volume along Y-axis

$$= V_2(x, y+\delta y, z)\,\delta x\,\delta z - V_2(x, y, z)\,\delta x\,\delta z$$
$$= \delta x\,\delta z\,[V_2(x, y+\delta y, z) - V_2(x, y, z)]$$
$$= \delta x\,\delta z\left[\left\{V_2(x, y, z) + \delta y\frac{\partial V_2}{\partial y} + \frac{(\delta y)^2}{2!}\cdot\frac{\partial^2 V_2}{\partial y^2} + \ldots\right\} - V_2(x, y, z)\right]$$
$$= \delta x\,\delta y\,\delta z\frac{\partial V_2}{\partial y} \quad \text{neglecting the higher order terms.}$$

Similarly, the rate of fluid flow along X-axis is $\frac{\partial V_1}{\partial x}\,\delta x\,\delta y\,\delta z$ and along z-axis is $\frac{\partial V_3}{\partial z}\,\delta x\,\delta y\,\delta z$.

Thus, the total amount of the fluid crossing the elementary volume, in unit time, is

$$\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)\delta x\,\delta y\,\delta z$$

Therefore, the rate of fluid flow through unit volume $= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\ (= \nabla \cdot \vec{V})$

Thus, **the divergence of the fluid velocity $\vec{V}$ represents the rate of fluid flow through unit volume.** It is also known as **FLUID FLUX**. Similarly, if $\vec{V}$ represents the electric flow then divergence $\vec{V}$ is the amount of electric flux which crosses a unit volume in unit time.

In case of fluid flow if $\text{div}(\vec{V}) = 0$, the fluid is said to be **incompressible**.

EXAMPLE 20.14. (a) Find the value of λ for which the vector $\vec{u} = (x+3y)i + (y-2z)j + (x+\lambda z)k$ is a solenoidal vector. [GGSIPU II Sem End Term 2005]

(b) If $\vec{r} = xi + yj + zk$, show that $\nabla \cdot (r^n \vec{r}) = (n+3)\, r^n$.

SOLUTION: (a) For $\vec{u}$ to be solenoidal $\nabla \cdot \vec{u} = 0$

or $\dfrac{\partial}{\partial x}(x+3y) + \dfrac{\partial}{\partial y}(y-2z) + \dfrac{\partial}{\partial z}(x+\lambda z) = 0$ or $1 + 1 + \lambda = 0$

Hence $\lambda = -2$ for $\vec{u}$ to be solenoidal. **Ans.**

(b) $$\nabla \cdot (r^n \vec{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (r^n xi + r^n yj + r^n zk)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n \frac{\partial x}{\partial x} + x\, nr^{n-1}\frac{\partial r}{\partial x} + r^n \frac{\partial y}{\partial y} + y\, nr^{n-1}\frac{\partial r}{\partial y} + r^n\frac{\partial z}{\partial z} + z\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right) = (n+3)\, r^n.$$ **Hence Proved.**

EXAMPLE 20.15. If $u = x^2 + y^2 + z^2$ and $\vec{V} = xi + yj + zk$ show that $div\,(u\vec{V}) = 5u$.

[GGSPU II Sem II Term 2006]

SOLUTION: Given that $u = x^2 + y^2 + z^2$ and $\vec{V} = xi + yj + zk$.

$\therefore$ $\nabla u = 2xi + 2yj + 2zk$ and $\nabla . V = 1 + 1 + 1 = 3$

$$\text{div}\,(u\,\vec{V}) = \nabla.(u\,\vec{V}) = (\nabla u).V + u(\nabla . \vec{V})$$
$$= (2xi + 2yj + 2zk).(xi + yj + zk) + 3u$$
$$= 2x^2 + 2y^2 + 2z^2 + 3u = 2u + 3u = 5u.$$ **Hence the result.**

EXAMPLE 20.16. If $\vec{a}$ is a constant vector and $\vec{r} = xi + yj + zk$, show that

(i) div $\vec{a} \times \vec{r} = 0$ (ii) div $[(\vec{a}\cdot\vec{r})\,\vec{a}] = a^2$ (iii) div $[a \times (r \times a)] = 2a^2$.

SOLUTION: (i) $$\text{div}\,(\vec{a}\times\vec{r}) = \Sigma i\frac{\partial}{\partial x}\cdot(\vec{a}\times\vec{r}) = \Sigma i\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{r})$$

$$= \Sigma i\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{r} + \vec{a}\times\frac{\partial\vec{r}}{\partial x}\right) = 0 + \Sigma i\cdot\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)$$ since $\vec{a}$ is constant.

$$= \Sigma i\cdot(\vec{a}\times i) = 0.$$

(ii) $$\text{div}\,\{(\vec{a}\cdot\vec{r})\,\vec{a}\} = \Sigma\, i\frac{\partial}{\partial x}\cdot\{(\vec{a}\cdot\vec{r})\vec{a}\} = \Sigma\, i\cdot\frac{\partial}{\partial x}\{(\vec{a}\cdot\vec{r})\vec{a}\}$$

$$= \Sigma\, i\cdot\left\{\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r})\vec{a} + (\vec{a}\cdot\vec{r})\frac{\partial\vec{a}}{\partial x}\right\} = \Sigma\, i\cdot\left\{\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r})\right\}\vec{a} + 0$$

$$= \Sigma\, i\cdot\left(\frac{\partial \vec{a}}{\partial x}\cdot\vec{r}+\vec{a}\cdot\frac{\partial \vec{r}}{\partial x}\right)\vec{a} = 0+\Sigma\, i\cdot(\vec{a}\cdot i)\vec{a} = \Sigma\,(\vec{a}\cdot i)\,(i\cdot\vec{a}) = \Sigma\, a_1^{\,2}$$

$$= a_1^2 + a_2^2 + a_3^2 = a^2 \quad (\text{where } \vec{a} = a_1 i + a_2 j + a_3 k)$$

(iii) $$\text{div}\,\{\vec{a}\times(\vec{r}\times\vec{a})\} = \sum i\cdot\frac{\partial}{\partial x}\left\{(\vec{a}\cdot\vec{a})\vec{r}-(\vec{a}\cdot\vec{r})\vec{a}\right\} = \sum i\cdot\frac{\partial}{\partial x}\left\{(a^2)\vec{r}\right\}-\sum i\cdot\frac{\partial}{\partial x}\left\{(\vec{a}\cdot\vec{r})\vec{a}\right\}$$

$$= \Sigma\, i\cdot\left\{\left(\frac{\partial}{\partial x}a^2\right)\vec{r}+a^2\cdot\frac{\partial \vec{r}}{\partial x}\right\}-\Sigma\, i\cdot\left\{\left(\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r})\right)\vec{a}+(\vec{a}\cdot\vec{r})\frac{\partial \vec{a}}{\partial x}\right\}$$

$$= \Sigma\, i\cdot(0+a^2 i)-\Sigma\, i\cdot\left\{\left(\frac{\partial \vec{a}}{\partial x}\cdot\vec{r}+\vec{a}\cdot\frac{\partial \vec{r}}{\partial x}\right)\right\}\vec{a}+0$$

$$= \Sigma\, a^2 - \Sigma\, i\cdot\{0+\vec{a}\cdot i\}\,\vec{a} = 3a^2 - \Sigma\,(\vec{a}\cdot i)\,(i\cdot\vec{a}) = 3a^2 - \Sigma\, a_1^2$$

$$= 3a^2 - a^2 = 2a^2.$$

Hence Proved.

THE CURL OF A VECTOR FIELD

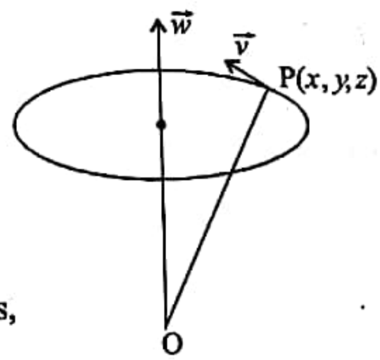
When a vector point function is operated vectorially by the operator ∇, the result is a vector quantity called *curl* of the vector. Thus, if $\vec{F}\ (x, y, z)$ is the vector point function, then

$$\text{curl}\ \vec{F} = \nabla\times\vec{F} = \Sigma\, i\times\frac{\partial \vec{F}}{\partial x} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+j\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+k\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

where F_1, F_2, F_3 are components of $\vec{F}$.

A vector $\vec{F}$ is called *IRROTATIONAL* (or CONSERVATIVE) if its curl is zero, that is, if $\nabla\times\vec{F} = 0$.

PHYSICAL INTERPRETATION OF CURL



Consider a rigid body rotating uniformly about a given axis, *i.e.*, rotating with a constant angular velocity. If $\vec{V}$ is the linear velocity of a point $P\,(x, y, z)$ of the body which is rotating with a constant angular velocity ω radians per second about a given axis (see the adjoining figure), then

$$\vec{V} = \vec{\omega}\times\vec{r} \quad \text{where } \vec{r} = xi + yj + zk.$$

Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ where $\omega_1, \omega_2, \omega_3$ are constants,

then $$\vec{V} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\, i + (\omega_3 x - \omega_1 z)\, j + (\omega_1 y - \omega_2 x)\, k.$$

$$\therefore \quad \text{curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= i(\omega_1 + \omega_1) + j(\omega_2 + \omega_2) + k(\omega_3 + \omega_3) = 2\vec{\omega}$$

Hence $\vec{\omega} = \frac{1}{2}\,\text{curl}\,\vec{V} = \frac{1}{2}(\nabla \times \vec{V})$ which shows that the angular velocity of a uniformly rotating body is one half of the curl of the linear velocity of any point of the body.

An important point is note worthy here. If $\vec{F}$ **is irrotational (or conservative) there exists a scalar point function ϕ such that $\vec{F} = \nabla\phi$ and ϕ is called the SCALAR POTENTIAL of $\vec{F}$.**

The formula for finding the scalar potential ϕ for an irrotational field $\vec{F}$ is given by

$d\phi = F_1 dx + F_2 dy + F_3 dz$ which is directly integrable.

EXAMPLE 20.17. If $\vec{r} = xi + yj + zk$ and $\vec{a}$ is a constant vector, show that

(i) $\vec{r}$ is irrotional, (ii) $\text{curl}\,(\vec{a} \times \vec{r}) = 2\vec{a}$.

[*GGSIPU II Sem End Term 2013*]

SOLUTION: (i) $\text{Curl}\,\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + j\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0.$

hence $\vec{r}$ is irrotational.

(ii) Let $\vec{a} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are constants.

hence $\vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\, i + (a_3 x - a_1 z) j + (a_1 y - a_2 x)\, k$

and $\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix} = i(a_1 + a_1) + j(a_2 + a_2) + k(a_3 + a_3) = 2\vec{a}.$

Hence Proved.

EXAMPLE 20.18. If $\vec{r} = xi + yj + zk$ prove that $\text{curl}\,(r^n \vec{r}) = 0$

[*GGSIPU I Sem II Term 2003*]

SOLUTION: $r^n \vec{r} = r^n xi + r^n yj + r^n zk$ where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \quad \text{curl}\,(r^n \vec{r}) = \nabla \times (r^n \vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] + j\left[\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right] + k\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$= \sum i\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] = \sum i\left[z.\left(nr^{n-1}\frac{\partial r}{\partial y}\right) - y\left(nr^{n-1}\frac{\partial r}{\partial z}\right)\right]$$

$$= \sum i\left[z.\left(nr^{n-1}\frac{y}{r}\right) - y\left(nr^{n-1}\frac{z}{r}\right)\right] = nr^{n-2}\sum i(yz - yz) = 0$$

Hence Proved.

EXAMPLE 20.19. If $\vec{V} = (x^2 - y^2 + 2xz)\, i + (xz - xy - yz)\, j + (z^2 + x^2)\, k$ is a vector field, find curl $\vec{V}$. Show that the vectors given by curl $\vec{V}$ at $P_0(1, 2, -3)$ and $P_1(2, 3, -2)$, are orthogonal. *[GGSIPU II Sem End Term 2005]*

SOLUTION: $\vec{V} = (x^2 - y^2 + 2xz)\, i + (xz - xy - yz)\, j + (z^2 + x^2)\, k$

$$\therefore \quad \text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy - yz & z^2 + x^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(xz - xy - yz)\right] + j\left[\frac{\partial}{\partial z}(x^2 - y^2 + 2xz) - \frac{\partial}{\partial x}(z^2 + x^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(xz - xy - yz) - \frac{\partial}{\partial y}(x^2 - y^2 + 2xz)\right]$$

$$= i\,[0 - (x - y)] + j\,[2x - 2x] + k\,[z - y) - (-2y + 0)]$$

$$= (y - x)\, i + 0j + (z + y)\, k$$

$\therefore$ Curl $\vec{V}$ at the point P_0 (1, 2, –3) $= i - k$ and Curl $\vec{V}$ at the point P_1 (2, 3, –2) $= i + k$

Since $(i - k).\,(i + k) = 0$, curl $\vec{V}$ at P_0 and P_1 are orthogonal. **Hence the result.**

VECTOR IDENTITIES

Given a scalar point function ϕ and vector point functions $\vec{F}$ and $\vec{G}$, following properties in the form of vector identities are important and very useful :

(1) **div $(\phi\vec{F}) = \vec{F} \cdot \nabla\phi + \phi$ div $\vec{F}$** or $\nabla \cdot (\phi\vec{F}) = (\nabla\phi) \cdot \vec{F} + \phi\,(\nabla \cdot \vec{F})$

$$\nabla \cdot (\phi\vec{F}) = \Sigma\, i\,\frac{\partial}{\partial x}\cdot(\phi\vec{F}) = \Sigma\, i \cdot \frac{\partial}{\partial x}(\phi\vec{F}) = \Sigma\, i \cdot \left(\frac{\partial\phi}{\partial x}\vec{F} + \phi\frac{\partial\vec{F}}{\partial x}\right) = \Sigma\, i \cdot \left(\frac{\partial\phi}{\partial x}\vec{F}\right) + \phi\,\Sigma\, i \cdot \frac{\partial\vec{F}}{\partial x}$$

$$= \vec{F} \cdot \Sigma\, i\frac{\partial\phi}{\partial x} + \phi\Sigma\left(i\frac{\partial}{\partial x}\cdot\vec{F}\right) = \vec{F} \cdot \nabla\phi + \phi\,(\nabla \cdot \vec{F}).$$

(2) **Curl $(\phi\vec{F})$ = (grad ϕ) $\times \vec{F} + \phi$ curl $\vec{F}$** or $\nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi\,(\nabla \times \vec{F})$

$$\nabla \times (\phi\vec{F}) = \Sigma\, i\frac{\partial}{\partial x}\times(\phi\vec{F}) = \Sigma\, i \times \frac{\partial}{\partial x}(\phi\vec{F}) = \Sigma\, i \times \left(\frac{\partial\phi}{\partial x}\vec{F} + \phi\frac{\partial\vec{F}}{\partial x}\right)$$

$$= \sum i\times\left(\frac{\partial\phi}{\partial x}\vec{F}\right)+\sum i\times\left(\phi\frac{\partial\vec{F}}{\partial x}\right) = \sum\left(i\frac{\partial\phi}{\partial x}\right)\times\vec{F}+\phi\sum i\times\frac{\partial\vec{F}}{\partial x} = \nabla\phi\times\vec{F}+\phi(\nabla\times\vec{F})$$

(3) $\operatorname{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot \text{Curl}\,\vec{F} - \vec{F}\cdot \text{curl}\,\vec{G}$ or $\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$

$$\nabla\cdot(\vec{F}\times\vec{G}) = \sum i\frac{\partial}{\partial x}\cdot(\vec{F}\times\vec{G}) = \sum i\cdot\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum i\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) = \sum i\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)-\sum i\cdot\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right)$$

$$= \sum\vec{G}\cdot\left(i\times\frac{\partial\vec{F}}{\partial x}\right)-\sum\vec{F}\cdot\left(i\times\frac{\partial\vec{G}}{\partial x}\right) = \vec{G}\cdot\sum\left(i\frac{\partial}{\partial x}\times\vec{F}\right)-\vec{F}\cdot\sum\left(i\frac{\partial}{\partial x}\times\vec{G}\right)$$

$$= \vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G}).$$

(4) $\text{Curl}\,(\vec{F}\times\vec{G}) = (\operatorname{div}\vec{G})\,\vec{F} - (\operatorname{div}\vec{F})\,\vec{G} + (\vec{G}\cdot\nabla)\,\vec{F} - (\vec{F}\cdot\nabla)\,\vec{G}$

or $\nabla\times(\vec{F}\times\vec{G}) = (\nabla\cdot\vec{G})\,\vec{F} - (\nabla\cdot\vec{F})\,\vec{G} + (\vec{G}\cdot\nabla)\,\vec{F} - (\vec{F}\cdot\nabla)\,\vec{G}$

$$\text{Curl}\,(\vec{F}\times\vec{G}) = \nabla\times(\vec{F}\times\vec{G}) = \sum i\frac{\partial}{\partial x}\times(\vec{F}\times\vec{G}) = \sum i\times\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum i\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) = \sum i\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)-\sum i\times\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right)$$

Consider $$\sum i\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) = \sum\left[(i\cdot\vec{G})\frac{\partial\vec{F}}{\partial x}-\left(i\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}\right] = \sum(\vec{G}\cdot i)\frac{\partial\vec{F}}{\partial x}-\sum\left(i\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}$$

$$= \sum\left(\vec{G}\cdot i\frac{\partial}{\partial x}\right)\vec{F}-\sum\left(i\frac{\partial}{\partial x}\cdot\vec{F}\right)\vec{G} = (\vec{G}\cdot\nabla)\,\vec{F}-(\nabla\cdot\vec{F})\vec{G}$$

Interchanging $\vec{F}$ and $\vec{G}$ in the above relation, we get

$$\sum i\times\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right) = (\vec{F}\cdot\nabla)\,\vec{G}-(\nabla\cdot\vec{G})\,\vec{F}$$

$\therefore$ $$\nabla\times(\vec{F}\times\vec{G}) = \{(\vec{G}\cdot\nabla)\,\vec{F}-(\nabla\cdot\vec{F})\,\vec{G}\}-\{(\vec{F}\cdot\nabla)\,\vec{G}-(\nabla\cdot\vec{G})\,\vec{F}\}$$

or $$\nabla\times(\vec{F}\times\vec{G}) = (\nabla\cdot\vec{G})\,\vec{F}-(\nabla\cdot\vec{F})\,\vec{G}+(\vec{G}\cdot\nabla)\,\vec{F}-(\vec{F}\cdot\nabla)\,\vec{G}.$$

(5) $\text{Grad}\,(\vec{F}\cdot\vec{G}) = \vec{F}\times\text{Curl}\,\vec{G}+\vec{G}\times\text{Curl}\,\vec{F}+(\vec{F}\cdot\nabla)\,\vec{G}+(\vec{G}\cdot\nabla)\,\vec{F}$

or $\nabla(\vec{F}\cdot\vec{G}) = \vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}\cdot\nabla)\,\vec{G}+(\vec{G}\cdot\nabla)\,\vec{F}$

$$\nabla(\vec{F}\cdot\vec{G}) = \sum i\frac{\partial}{\partial x}(\vec{F}\cdot\vec{G}) = \sum i\left(\frac{\partial\vec{F}}{\partial x}\cdot\vec{G}+\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right) = \sum i\left(\frac{\partial\vec{F}}{\partial x}\cdot\vec{G}\right)+\sum i\left(\frac{\partial\vec{G}}{\partial x}\cdot\vec{F}\right) \quad ...(1)$$

Now consider

$$\vec{G}\times\left(i\times\frac{\partial \vec{F}}{\partial x}\right)=\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)i-(\vec{G}\cdot i)\frac{\partial \vec{F}}{\partial x}\Rightarrow\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)i=\vec{G}\times\left(i\times\frac{\partial \vec{F}}{\partial x}\right)+(\vec{G}\cdot i)\frac{\partial \vec{F}}{\partial x}$$

$$\therefore\ \Sigma\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)i=\Sigma\,\vec{G}\times\left(i\times\frac{\partial \vec{F}}{\partial x}\right)+\Sigma\,(\vec{G}\cdot i)\frac{\partial \vec{F}}{\partial x}=\Sigma\,\vec{G}\times\left(i\frac{\partial}{\partial x}\times\vec{F}\right)+\Sigma\left(\vec{G}\cdot i\frac{\partial}{\partial x}\right)\vec{F}$$

$$\Sigma\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)i=\vec{G}\times(\nabla\times\vec{F})+(\vec{G}\cdot\nabla)\,\vec{F} \qquad \text{...(2)}$$

Interchanging $\vec{F}$ and $\vec{G}$ in the last relation, we get

$$\Sigma\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)i=\vec{F}\times(\nabla\times\vec{G})+(\vec{F}\cdot\nabla)\,\vec{G} \qquad \text{...(3)}$$

Using (2) and (3) in (1), we get

$$\nabla(\vec{F}\cdot\vec{G})=\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}\cdot\nabla)\,\vec{G}+(\vec{G}\cdot\nabla)\,\vec{F}.$$

Following are the properties on vector differentiation dealing with the double operation of ∇ on a single function.

(6) **Div (grad ϕ) = $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}$ or $\nabla\cdot(\nabla\phi)=\nabla^2\phi=\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}$**

$$\nabla^2\phi=(\nabla\cdot\nabla)\,\phi=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\phi$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.$$

[*GGSIPU II Sem End Term 2006*]

(7) **Curl (grad ϕ) = 0 or $\nabla\times(\nabla\phi)=0$.**

$$\nabla\times(\nabla\phi)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right)=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z\end{vmatrix}$$

$$=i\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)+j\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)+k\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)=0.$$

[*GGSIPU II Sem II Term 2011*]

(8) **Div (Curl $\vec{F}$) = 0 or $\nabla\cdot(\nabla\times\vec{F})=0$.**

Let $\vec{F}=F_1 i+F_2 j+F_3 k$ then

$$\nabla\times\vec{F}=\begin{vmatrix} i & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=i\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+j\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+k\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$\therefore\ \nabla\cdot(\nabla\times\vec{F})=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left\{i\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+j\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+k\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.$$

(9) Curl (Curl $\vec{F}$) = grad (div $\vec{F}$) − $\nabla^2 \vec{F}$ or $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$

Let $\vec{F} = F_1 i + F_2 j + F_3 k$ then

$$\nabla \times (\nabla \times \vec{F}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left\{i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\}$$

$$= k\left(\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2}\right) - j\left(\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y}\right) - k\left(\frac{\partial^2 F_3}{\partial y^2} - \frac{\partial^2 F_2}{\partial y \partial z}\right) + i\left(\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2}\right)$$
$$+ j\left(\frac{\partial^2 F_3}{\partial z \partial y} - \frac{\partial^2 F_2}{\partial z^2}\right) - i\left(\frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_3}{\partial z \partial x}\right)$$

$$= \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right) i + \left(\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial z \partial y} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}\right) j$$
$$+ \left(\frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2}\right) k$$

$$= \left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right) i$$
$$+ \left(\frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial y^2} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial y^2} - \frac{\partial^2 F_2}{\partial z^2}\right) j$$
$$+ \left(\frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^2 F_3}{\partial z^2} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} - \frac{\partial^2 F_3}{\partial z^2}\right) k$$

$$= i\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_1 i$$
$$+ j\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_2 j$$
$$+ k\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_3 k$$

$$= \nabla\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1 i + F_2 j + F_3 k)$$

or $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$.

We could arrive at the same result more conveniently by applying the symbolic method as follows:

$$\text{Curl (Curl F)} = \nabla \times (\nabla \times F)$$

Treating ∇'s as algebraic vectors ∇_1 and ∇_2 and then using the formula of vector triple product, we get

$$\nabla \times (\nabla \times F) = \nabla_1 \times (\nabla_2 \times F) = \nabla_2(\nabla_1 \cdot F) - (\nabla_1 \times \nabla_2)F$$
$$= \nabla(\nabla \cdot F) - \nabla^2 F \quad \text{(dropping the suffixes of dels).}$$

EXAMPLE 20.20. If $\vec{a}$ and $\vec{b}$ are constant vectors, show that

$$(i)\ \nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{\vec{a}}{r^n} - n\frac{(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r} \qquad (ii)\ \nabla\times\{\vec{a}\times(\vec{b}\times\vec{r})\} = \vec{a}\times\vec{b}.$$

SOLUTION: (i) $\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}\vec{a}\cdot\vec{r}\right) = \frac{1}{r^n}\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\nabla\left(\frac{1}{r^n}\right)$

$$= \frac{1}{r^n}\vec{a} + (\vec{a}\cdot\vec{r})\nabla\left(\frac{1}{r^n}\right) = \frac{\vec{a}}{r^n} + (\vec{a}\cdot\vec{r})(-n)r^{-n-2}\vec{r}$$

(since $\nabla(a\cdot r) = \vec{a}$, and $\nabla(r) = \frac{1}{r}\vec{r}$)

$$= \frac{1}{r^n}\vec{a} - \frac{n(a\cdot r)}{r^{n+2}}\vec{r}.$$

(ii) $\vec{a}\times(\vec{b}\times\vec{r}) = (a\cdot r)\vec{b} - (a\cdot b)\vec{r}$

$\therefore\ \nabla\times[a\times(b\times r)] = \nabla\times(\vec{a}\cdot\vec{r})\vec{b} - \nabla\times\{(a\cdot b)\vec{r}\}$

$= \nabla(\vec{a}\cdot\vec{r})\times\vec{b} + (\vec{a}\cdot\vec{r})\nabla\times\vec{b} - \{\nabla(\vec{a}\cdot\vec{b})\times\vec{r} + (\vec{a}\cdot\vec{b})\nabla\times\vec{r}\}$

$= \vec{a}\times\vec{b} + 0 + 0 + 0$, (since $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ and $\nabla\times\vec{r} = 0$). **Hence Proved.**

EXAMPLE 20.21. If $\vec{a}$ is a constant vector point function, show that

$$\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{-\vec{a}}{r^3} + \frac{3(\vec{a}\cdot\vec{r})}{r^5}\vec{r}.$$

SOLUTION: $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \left(\nabla\cdot\frac{\vec{r}}{r^3}\right)\vec{a} - (\vec{a}\cdot\nabla)\frac{1}{r^3}\vec{r}$ (note the steps)

Now, $\nabla\cdot\frac{\vec{r}}{r^3} = \nabla(r^{-3})\cdot\vec{r} + \frac{1}{r^3}\nabla\cdot\vec{r}$

$= (-3r^{-5}\vec{r})\cdot\vec{r} + \frac{1}{r^3}(3) = -3r^{-5}r^2 + \frac{3}{r^3} = 0$ (as $\vec{r}\cdot\vec{r} = r^2$)

and $(\vec{a}\cdot\nabla)\vec{r}\,r^{-3} = r^{-3}(\vec{a}\cdot\nabla)\vec{r} + \vec{r}(\vec{a}\cdot\nabla)r^{-3} = r^{-3}\vec{a} + [\vec{a}\cdot\nabla r^{-3}]\vec{r}$

$\therefore\ \nabla\left(\vec{a}\times\frac{\vec{r}}{r^3}\right) = 0 - \frac{\vec{a}}{r^3} - (-3)\frac{\vec{a}\cdot\vec{r}}{r^5}\vec{r} = -\frac{\vec{a}}{r^3} + \frac{3(\vec{a}\cdot\vec{r})}{r^5}\vec{r}$. (Since $\nabla r^{-3} = -3\cdot r^{-5}\vec{r}$)

Hence Proved.

EXAMPLE 20.22. Show that div (grad r^n) = $\nabla^2(r^n) = n(n+1)\, r^{n-2}$ where $r = |\vec{r}|$.

[GGSIPU II Sem II Term 2011; End Term 2012]

SOLUTION: div (grad r^n) = $\nabla \cdot \nabla (r^n)$

Now $\nabla(r^n) = \sum i \dfrac{\partial}{\partial x}(r^n) = \sum i n r^{n-1} \dfrac{\partial r}{\partial x} = n \sum i r^{n-1} \dfrac{x}{r} = n r^{n-2} \sum (ix)$

$$\therefore \quad \nabla^2(r^n) = \nabla \cdot (\nabla r^n) = n \sum \frac{\partial}{\partial x}(r^{n-2} x) = n \sum \left[r^{n-2}(1) + x \frac{\partial}{\partial x}(r^{n-2}) \right]$$

$$= n r^{n-2}(3) + n \sum x (n-2)\, r^{n-3} \left(\frac{\partial r}{\partial x} \right)$$

$$= 3n r^{n-2} + (n-2)\, n \sum x r^{n-3} \frac{x}{r} = 3n r^{n-2} + n(n-2)\, r^{n-4} (r^2)$$

$$= n(n+1)\, r^{n-2}.$$ **Hence Proved.**

EXAMPLE 20.23. If $r^2 = x^2 + y^2 + z^2$, show that $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$ and hence deduce that $\nabla^4 f(r) = f^{iv}(r) + \dfrac{4}{r} f'''(r)$.

[GGSIPU II Sem II Term 2006]

SOLUTION: We know that $\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$

Let $\phi = f(r)$ then $\dfrac{\partial \phi}{\partial x} = \dfrac{d\phi}{dr} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} f'(r)$

hence $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{x}{r} f'(r) \right] = \dfrac{x}{r} \dfrac{\partial}{\partial x} f'(r) + f'(r) \dfrac{\partial}{\partial x}\left(\dfrac{x}{r} \right)$

$$= \frac{x}{r}\left(\frac{d}{dr} f'(r) \right) \cdot \frac{\partial r}{\partial x} + f'(r) \frac{r(1) - x \dfrac{\partial r}{\partial x}}{r^2} = \frac{x^2}{r^2} f''(r) + f'(r)\left(\frac{1}{r} - \frac{x}{r^2} \cdot \frac{x}{r} \right)$$

$$= \frac{x^2}{r^2} f''(r) + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r).$$

Similarly $\dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{y^2}{r^2} f''(r) + \dfrac{1}{r} f'(r) - \dfrac{y^2}{r^3} f'(r)$ and $\dfrac{\partial^2 \phi}{\partial z^2} = \dfrac{z^2}{r^2} f''(r) + \dfrac{1}{r} f'(r) - \dfrac{z^2}{r^3} f'(r)$

$$\therefore \quad \nabla^2 f(r) = \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3}{r} f'(r) - \frac{f'(r)}{r^3}(x^2 + y^2 + z^2)$$

or

$$\nabla^2 f(r) = f''(r) + \frac{3}{r} f'(r) - \frac{1}{r} f'(r) = f''(r) + \frac{2}{r} f'(r). \quad \text{...(1)}$$

Further, to derive the expression for $\nabla^4 f(r)$ let us write

$F(r) = f''(r) + \dfrac{2}{r} f'(r)$ then $\nabla^4 f(r) = \nabla^2 F(r) = F''(r) + \dfrac{2}{r} F'(r)$ ...(2)

Now $F'(r) = \dfrac{d}{dr}\left[f''(r) + \dfrac{2}{r} f'(r) \right] = f'''(r) + \dfrac{2}{r} f''(r) - \dfrac{2}{r^2} f'(r)$ ...(3)

and

$$F''(r) = f^{(iv)}(r) + \frac{d}{dr}\left[\frac{2}{r} f''(r) \right] - \frac{d}{dr}\left[\frac{2}{r^2} f'(r) \right]$$

$$= f^{(iv)}(r) + \frac{2}{r} f'''(r) - \frac{2}{r^2} f''(r) - \frac{2}{r^2} f''(r) + \frac{4}{r^3} f'(r)$$

$$= f^{(iv)}(r) + \frac{2}{r} f'''(r) - \frac{4}{r^2} f''(r) + \frac{4}{r^3} f'(r) \quad ...(4)$$

Using (3) and (4) in (2), we get

$$\nabla^4 f(r) = f^{(iv)}(r) + \frac{2}{r} f'''(r) - \frac{4}{r^2} f''(r) + \frac{4}{r^3} f'(r) + \frac{2}{r}\left[f'''(r) + \frac{2}{r} f''(r) - \frac{2}{r^2} f'(r)\right]$$

$$= f^{(iv)}(r) + \frac{4}{r} f'''(r).$$

Hence the result.

EXAMPLE 20.24. A vector field $\vec{F}$ is given by $\vec{F} = (x^2 - y^2 + x)i - (2xy + y)j$

Show that the field is irrotational and find its scalar potential.

[GGSIPU II Sem II Term 2006; End Term 2009]

SOLUTION: $\vec{F} = (x^2 - y^2 + x)i - (2xy + y)j$

For $\vec{F}$ to be irrotational $\nabla \times \vec{F} = 0$ where $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$.

Here $\nabla \times \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) \times [(x^2 - y^2 + x)i - (2xy + y)j] = -k2y - k(-2y) = 0$

$\therefore$ $\vec{F}$ is irrotational. To obtain its scalar potential ϕ, say, we have $\vec{F} = \nabla \phi$

$$\frac{\partial \phi}{\partial x} = x^2 - y^2 + x, \quad \frac{\partial \phi}{\partial y} = -2xy - y$$

On integration *w.r.t.* x we get

$$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + C(y) \quad \therefore \quad \frac{\partial \phi}{\partial y} = -2xy + \frac{dC}{dy}$$

But $\frac{\partial \phi}{\partial y} = -2xy - y$. Hence $\frac{dC}{dy} = -y \Rightarrow C = \frac{-y^2}{2}$.

$\therefore$ $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$ is the required scalar potential. **Ans.**

EXAMPLE 20.25. (a) Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Also, find the scalar potential.

[GGSIPU II Sem End Term 2009; End Term 2006]

(b) Prove $\vec{F} = r^2\vec{r}$ is irrotational. Find the scalar function.

[GGSIPU II Sem End Term 2011]

SOLUTION: (a) $\vec{F} = \frac{\vec{r}}{r^3} = \frac{xi + yj + zk}{r^3}$, hence

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \Sigma \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \Sigma\left(\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right) = \frac{3}{r^3} - \frac{3}{r^5}(x^2 + y^2 + z^2) = 0$$

Therefore, $\vec{F}$ is a solenoidal vector.

Next, Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r^3 & y/r^3 & z/r^3 \end{vmatrix} = \sum i\left[\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right]$

$$= \sum i\left[z\frac{\partial}{\partial y}r^{-3} - y\frac{\partial}{\partial z}r^{-3}\right] = \sum i\left[z(-3)r^{-4}\frac{y}{r} - y(-3)r^{-4}\frac{z}{r}\right] = 0.$$

Therefore, $\vec{F}$ is irrotational also. Thus, there exists a scalar potential ϕ such that

$$\vec{F} = \nabla\phi \quad \text{and} \quad \vec{F}\cdot\overline{dr} = d\phi \quad \text{where } d\vec{r} = i\,dx + j\,dy + k\,dz.$$

$\therefore$ $$d\phi = \frac{x\,dx + y\,dy + z\,dz}{r^3} = \frac{r\,dr}{r^3} = \frac{dr}{r^2} \Rightarrow \phi = \int\frac{dr}{r^2} = -\frac{1}{r} + C.$$ **Ans.**

(b) $\nabla\times\vec{F} = \nabla\times(r^2\vec{r}) = \nabla(r^2)\times\vec{r} + r^2(\nabla\times\vec{r}) = \nabla(r^2)\times\vec{r}$ since $\nabla\times\vec{r} = 0$

$$= \left[\sum i\frac{\partial}{\partial x}(r^2)\right]\times\vec{r} = \left[\sum i2r\frac{\partial r}{\partial x}\right]\times\vec{r}$$

$$= 2r\left[\sum\frac{ix}{r}\right]\times\vec{r} = 2\vec{r}\times\vec{r} = 0 \quad \text{since } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \vec{r} = \Sigma xi$$

Hence $\vec{F}$ is irrotational.

Next, let $\phi(r)$ be the scalar function so that $\vec{F} = \nabla\phi$.

or $$r^2\vec{r} = \sum i\frac{\partial\phi(r)}{\partial x} = \sum i\frac{d\phi}{dr}\frac{\partial r}{\partial x} \quad \text{or} \quad r^2\sum xi = \sum i\frac{x}{r}\frac{d\phi}{dr} \Rightarrow \frac{d\phi}{dr} = r^3$$

Therefore, $\phi(r) = \frac{1}{4}r^4 + C.$ **Ans.**

EXAMPLE 20.26. Show that $\vec{F} = (6xy + z^3)\,i + (3x^2 - z)\,j + (3xz^2 - y)\,k$ is irrotational. Find scalar ϕ such that $\vec{F} = \sigma$. [GGSIPU II Sem II Term 2012]

SOLUTION: $$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = i(-1+1) + j(3z^2 - 3z^2) + k(6x - 6x) = 0$$

Hence $\vec{F}$ is irrotational. Let $\nabla\phi = \vec{F}$, then

$$\frac{\partial\phi}{\partial x} = 6xy + z^3, \quad \frac{\partial\phi}{\partial y} = 3x^2 - z \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 3xz^2 - y.$$

$\therefore$ $$\phi = 3x^2y + xz^3 + A(y, z) \quad \text{then} \quad \frac{\partial\phi}{\partial y} = 3x^2 + \frac{\partial A}{\partial y}$$

$\Rightarrow$ $$\frac{\partial A}{\partial y} = -z \quad \therefore\ A = -yz + B(z), \text{ then } \phi = 3x^2y + xz^3 - yz + B(z)$$

$\therefore$ $$\frac{\partial\phi}{\partial z} = 3xz^2 - y + \frac{dB}{dz} \Rightarrow \frac{dB}{dz} = 0 \text{ hence } B \text{ is a pure constant.}$$

Thus $\phi = 3x^2y + xz^3 - yz + K$ where K is an arbitrary pure constant. **Ans.**

EXAMPLE 20.27. Show that the field of force given by $\vec{F} = (y^2 \cos x + z^3)\, i + (2y \sin x - 4)\, j + (3xz^2 + 2)\, k$ is conservative and find its scalar potential. Also find the work done in moving the particle in the field from a point A (0, 1, – 1) to point B (π/2, – 1, 2). [GGSIPU II Sem End Term 2006]

SOLUTION:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= i\,(0 - 0) + j\,(0 + 3z^2 - 3z^2) + k\,(2y \cos x - 2y \cos x - 0) = 0.$$

$\therefore\ \vec{F}$ represents a conservative (or irrotational) field. The corresponding scalar potential ϕ is given by $\nabla\phi = \vec{F} = (y^2 \cos x + z^3)\, i + (2y \sin x - 4)\, j + (3xz^2 + 2)\, k$

Also we have $d\phi = \dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial y}dy + \dfrac{\partial \phi}{\partial z}dz = \nabla\phi \cdot \overline{dr}$ where $d\vec{r} + i\, dx + j\, dy + k\, dz$.

This,
$$d\phi = (y^2 \cos x + z^3)\, dx + (2y \sin x - 4)\, dy + (3xz^2 + 2)\, dz$$
$$= (y^2 \cos x\, dx + 2y \sin x\, dy) + (z^3 dx + 3xz^2 dz) + (-4dy + 2dz)$$
$$= d\,(y^2 \sin x + xz^3 - 4y + 2z).$$

Hence $\phi = y^2 \sin x + xz^3 - 4y + 2z + c$

Next, the required work done $= \displaystyle\int F \cdot dr = \int_A^B \nabla\phi \cdot dr = \int_A^B \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \right) = \int_A^B d\phi$

$$= [\phi]_{(0,1,-1)}^{(\pi/2,-1,2)} = \left[y^2 \sin x + xz^3 - 4y + 2z\right]_{(0,1,-1)}^{(\pi/2,-1,2)} = 4\pi + 15. \qquad \textbf{Ans.}$$

EXAMPLE 20.28. For what values of b and c, $\overline{F} = (y^2 + 2czx)\, i + y(bx + cz)\, j + (y^2 + cx^2)\, k$ is irrotational. Find the scalar ϕ such that $\overline{F} = \nabla\phi$.

[GGSIPU II Sem. End Term 2010 II Term 2011]

SOLUTION: $\overline{F} = (y^2 + 2czx)\, i + (bxy + cyz)\, j + (y^2 + cx^2)k$ is the given vector field.

$$\therefore\quad \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + 2czx & bxy + cyz & y^2 + cx^2 \end{vmatrix} = i\,(2y - cy) + j\,(2cx - 2cx) + k\,(by - 2y)$$

For $\overline{F}$ to be irrotational $\nabla \times F = 0$ hence $b = 2$ and $c = 2$.

Thus, $\overline{F} = (y^2 + 4zx)\, i + 2y\,(x + z)\, j + (y^2 + 2x^2)\, k.$

Now, let us find scalar ϕ so that $\overline{F} = \nabla\phi$ which gives

$$\frac{\partial \phi}{\partial x} = y^2 + 4zx, \quad \frac{\partial \phi}{\partial y} = 2y\,(x + z), \quad \frac{\partial \phi}{\partial z} = y^2 + 2x^2$$

On integrating $\dfrac{\partial \phi}{\partial x} = y^2 + 4zx$, we get $\phi = xy^2 + 2x^2z + k\,(y, z).$

$$\therefore \quad \frac{\partial \phi}{\partial y} = 2xy + \frac{\partial k}{\partial y}. \quad \text{But } \frac{\partial \phi}{\partial y} = 2y(x+z), \text{ hence } \frac{\partial k}{\partial y} = 2yz,$$

which on integrations gives $k = y^2 z + c(z)$.

$$\therefore \quad \phi = xy^2 + 2x^2 z + y^2 z + c(z).$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial z} = 2x^2 + y^2 + \frac{dc}{dz}, \quad \text{but } \frac{\partial \phi}{\partial z} = y^2 + 2x^2 \quad \therefore \quad \frac{dc}{dz} = 0 \Rightarrow c \text{ is a pure constant.}$$

Therefore, $\phi = xy^2 + 2x^2 z + y^2 z$ without any loss of generality. **Ans.**

EXAMPLE 20.29. (a) A fluid motion is given by $\vec{V} = (y+z)i + (z+x)j + (x+y)k$. Is this motion irrotational? If so, find the velocity potential. Is the motion possible for incompressible fluid? [GGSIPU II Sem End Term 2007 Reappear]

(b) Show that $\vec{F} = (2xy + z^3)i + x^2 j + 3z^2 xk$ is conservative field. Find its scalar potential and also the work done in moving a particle form (1, –2, 1), to (3, 1, 4). [GGSIPU II Sem II Term 2013]

SOLUTION: (a) Velocity vector $\vec{V} = (y+z)i + (z+x)j + (x+y)k$.

$$\therefore \quad \text{Curl } \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = i(1-1) + j(1-1) + k(1-1) = 0.$$

Hence motion is irrotational. Let the velocity potential be ϕ so that

$$\vec{V} = \nabla \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = y + z, \quad \frac{\partial \phi}{\partial y} = z + x, \quad \frac{\partial \phi}{\partial z} = x + y$$

Integrating $\frac{\partial \phi}{\partial x} = y + z$ *w.r.t.* x, we get $\phi = xy + xz + c(y, z)$.

Differentiating partially *w.r.t.* y, we get

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial c}{\partial y} \quad \text{but } \frac{\partial \phi}{\partial y} = z + x. \quad \text{Hence } \frac{\partial c}{\partial y} = z \quad \Rightarrow \quad c = yz + k(y).$$

Thus $\phi = xy + xz + yz + k(y)$. This should satisfy $\frac{\partial \phi}{\partial z} = x + y$.

Therefore the velocity potential $\phi = xy + yz + zx$.

Also, we have $\nabla . \vec{V} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y) = 0$

Hence, the motion is possible for incompressible fluid since $\vec{V}$ is incompressible. **Ans.**

(b) $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3z^2x) - \frac{\partial}{\partial z}(x^2)\right] + j\left[\frac{\partial}{\partial z}(2xy+z^3) - \frac{\partial}{\partial x}(3z^2x)\right] + k\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right]$$

$= i(0-0) + j\,(3z^2 - 3z^2) + k\,(2x-2x)] = 0$. Thus $\vec{F}$ is conservative field.

Hence there exists a scalar potential ϕ such that $\vec{F} = \nabla\phi$.

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3, \quad \frac{\partial\phi}{\partial y} = x^2 \text{ and } \frac{\partial\phi}{\partial z} = 3z^2x.$$

From the first relation, on integration w.r.t. x we get

$$\phi = x^2y + xz^3 + H(y, z), \text{ then } \frac{\partial\phi}{\partial y} = x^2 + \frac{\partial H}{\partial y} = x^2 \text{ hence } \frac{\partial H}{\partial y} = 0$$

In turn $H = G(z)$. Thus $\phi = x^2y + xz^3 + G(z)$.

Next, then $\frac{\partial\phi}{\partial z} = 3xz^2 + \frac{dG}{dz}$ But $\frac{\partial\phi}{\partial z} = 3xz^2$. $\Rightarrow \frac{dG}{dz} = 0$

Therefore G is a pure constant, say c. Thus $\phi = x^2y + xz^3 + c$

Further, the work done by $\vec{F}$, in moving the particle from $(1, -2, 1)$ to $(3, 1, 4)$ is equal to

$$\phi(3, 1, 4) - \phi(1, -2, 1) = \left[x^2y + xz^3 + c\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= (9 + 192) - (-2 + 1) = 202 \qquad \textbf{Ans.}$$

EXAMPLE 20.30. If $\vec{A} = x^2yzi + xyz^2j + y^2zk$ determine cur curl $\vec{A}$.

[GGSIPU II Sem End Term 2011]

SOLUTION: We know that $\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

Now $$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(y^2z) = 2xyz + xz^2 + y^2.$$

$$\therefore \quad \nabla(\nabla\cdot\vec{A}) = i\frac{\partial}{\partial x}(2xyz + xz^2 + y^2) + j\frac{\partial}{\partial y}(2xyz + xz^2 + y^2) + k\frac{\partial}{\partial z}(2xyz + xz^2 + y^2)$$

$$= i\,(2yz + z^2) + j\,(2xz + 2y) + k\,(2xy + 2xz)$$

and $$\nabla^2 \vec{A} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(x^2yz\,i + xyz^2\,j + y^2z\,k\right)$$

$$= (2yz\,i + 0j + 0k) + (0i + 0j + 2z\,k) + (0i + 2xy\,j + 0k) = 2yz\,i + 2xy\,j + 2z\,k,$$

$$\therefore \quad \nabla \times (\nabla \times A) = i(2yz + z^2) + j(2xz + 2y) + k\,(2xy + 2xz) - (2yz\,i + 2xy\,j + 2z\,k)$$

$$= iz^2 + j\,(2xz + 2y - 2xy) + k\,(2xy + 2xz - 2z).$$ **Ans.**

Aliter

$$\Delta \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & z^2xy & y^2z \end{vmatrix} = i\,(2yz - 2zxy) + j\,(x^2y - 0) + k\,(yz^2 - x^2z)$$

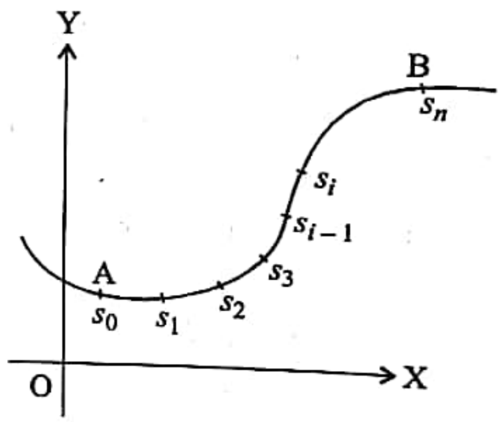
$$\Delta \times \nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - 2zxy & x^2y & yz^2 - x^2z \end{vmatrix} = i\,(z^2 - 0) + j\,(2y - 2xy - 2xz) + k\,(2xy - 2z + 2xz)$$

21 Vector Integration, Line Integral, Surface Integral, Stoke's Theorem and Gauss's Theorem

Vector Integration, Line Integral, Surface Integral, Volume Integrals, Green's Theorem, Stoke's Theorem and Gauss's Divergence Theorem.

Here we shall be introducing the line integrals, surface integrals and volume integrals which have applications in physical problems. Thereafter we shall see transformation of line integral to surface integral through Stoke's theorem and the transformation of volume integral into surface integral through Gauss Divergence Theorem. These theorems are important because of their applications in various problems.

LINE INTEGRAL



We are familiar with the definite integral $I = \int_a^b f(x)\,dx$ carried out along the x-axis. This can be extended to integration along a curve. For this let us consider a curve C defined in terms of s, the arc length, taken as parameter, and let $\vec{F}(s)$ be a vector point function of s. Let the interval $[a, b]$ be divided into n parts $s_{i-1} \le s \le s_i$, $i = 1, 2, ..., n$ where $s = s_0$ at A (when $x = a$) and $s = s_n$ at B (when $x = b$) (see the adjoining figure).

Let $\delta\vec{r}_i = \vec{r}(s_i) - \vec{r}(s_{i-1})$ and let $F_1, F_2, ..., F_n$ be the vector components of $\vec{F}$ along the tangent to the curve at the points of division. Next, we form the sum of the scalar products $\vec{F}_1 \cdot \delta\vec{r}_1, \vec{F}_2 \cdot \delta\vec{r}_2, ..., \vec{F}_n \cdot \delta\vec{r}_n$, denoted by S_n, that is, $S_n = \sum_{i=1}^{n} \vec{F}_i \cdot \delta\vec{r}_i$

Taking the limit of the sum S_n as $n \to \infty$, is defined as the line integral of $\vec{F}$, taken along the curve C and is written as

$$\int_C \vec{F} \cdot \vec{dr} \quad \text{where} \quad \vec{dr} = idx + jdy + kdz \quad \text{or} \quad \text{as } \int_C F \cdot \hat{T}\, ds$$

where $\hat{T}$ is unit vector along the tangent to the curve and ds is an element of arc of the curve C.

If the path of integration is a closed curve, we use the symbol $\oint_C \vec{F} \cdot \vec{dr}$.

Let $\vec{F} = F_1 i + F_2 j + F_3 k$ and $\vec{dr} = i dx + j dy + k dz$, then we have

$$\int_C \vec{F}\cdot\vec{dr} = \int_C (F_1 dx + F_2 dy + F_3 dz).$$

In practice, if $\vec{F} = \vec{v}$ where $\vec{v}$ is the velocity of the fluid particle, then the line integral $\oint_C \vec{v}\cdot\vec{dr}$ is called the **CIRCULATION** of the velocity around a closed curve C. Clearly, the circulation is zero in a conservative field.

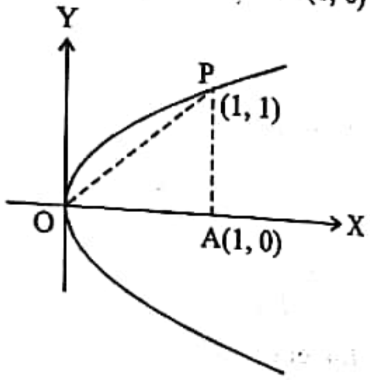
The line integral $\int_C F\cdot dr$ may or may not depend on the path of integration. Let us illustrate this fact by the following problems in two dimensions.

EXAMPLE 21.1. Let the vector field be $\vec{F} = x^2 i - xyj$. Find its line integral from the origin O to the point P (1, 1)

(*i*) along the straight line OP

(*ii*) along the parabola $y^2 = x$

(*iii*) along the x-axis from O to A (1, 0) and then parallel to Y-axis from A (1, 0) to P (1, 1).



SOLUTION: (*i*) Equation of the line OP is $y = x$ so that $dy = dx$

$$\int_{\text{line OP}} \vec{F}\cdot dr = \int_{\text{line OP}} (x^2 i - xyj)\cdot(i dx + j dy)$$

$$= \int_{\text{line OP}} x^2 dx - xy\,dy = \int_{\text{line OP}} (x^2 dx - x^2\,dx) = 0.$$

(*ii*) Equation of the parabola is $y^2 = x$ so that $2y dy = dx$

$$\therefore \int_{\text{curve OP}} \vec{F}\cdot dr = \int_{\text{curve OP}} \left(x^2 dx - xy\,dy\right) = \int_0^1 \left(x^2 - \frac{x}{2}\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{4}\right]_0^1 = \frac{1}{12}.$$

(*iii*) Here, we integrate $\vec{F}$ from O to A along X-axis and from A to P parallel to Y-axis.

$$\therefore \int_{\text{OAP}} \vec{F}\cdot dr = \int_{\text{OAP}} (x^2 dx - xy\,dy) = \int_{\text{OA}} x^2 dx - 0 + \int_{\text{AP}} (0 - 1y\,dy)$$

$$= \int_0^1 x^2 dx - \int_0^1 y\,dy = \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

This shows from (*i*), (*ii*) and (*iii*) that the line integral of $\vec{F}$ depends upon the path along which it is integrated hence it is different in (*i*), (*ii*) and (*iii*).

Ans.

EXAMPLE 21.2. Consider the vector field $\vec{F} = y^2 i + 2xyj$ and integrate it from the origin O to the point P (1, 1) along the paths (*i*), (*ii*) and (*iii*) of the Example 21.1.

SOLUTION: (*i*) $\int_{\text{line OP}} \vec{F} \cdot dr = \int_{\text{line OP}} (y^2 i + 2xy\, j) \cdot (i\, dx + j\, dy) = \int_{\text{line OP } (y=x)} y^2 dx + 2xy\, dy = \int_0^1 x^2 dx + 2x^2 dx$

$$= \int_0^1 3x^2 dx = \left[x^3\right]_0^1 = 1.$$

(*ii*) $\int_{\text{curve OP}} \vec{F} \cdot dr = \int_{y^2=x} y^2 dx + 2xy\, dy = \int_0^1 x\, dx + x\, dx = \left[x^2\right]_0^1 = 1.$

(*iii*) $\int_{\text{OAP}} \vec{F} \cdot dr = \int_{\text{OAP}} (y^2 dx + 2xy\, dy) = \int_{\text{OA}} 0 + 0 + \int_{\text{AP}} 0 + 2 \cdot 1 \cdot y\, dy = 0 + \int_0^1 (0 + 2y)\, dy = \left[y^2\right]_0^1 = 1.$

Here, we note that the line integral of $\vec{F} = y^2 i + 2xyj$ along the paths (*i*), (*ii*), (*iii*) is the same. **Ans.**

Actually here the line integral is independent of the path, since $\vec{F}$ is conservative.

We can easily show that if $\vec{F}$ **is the gradient of some scalar point function** ϕ, **the line integral of** $\vec{F}$ **is independent of the path** as follows.

Suppose $\vec{F} = \nabla\phi$ then $\int \vec{F} \cdot dr = \int_A^B \nabla\phi \cdot dr = [d\phi]_a^b = [\phi]_a^b = \phi(b) - \phi(a).$

Thus, in this case the line integral of $\vec{F}$ depends only on the end points A and B and not on the paths joining A and B.

EXAMPLE 21.3. Find the work done in moving a particle in a force field $\vec{F} = 3xy\, i - 5z\, j + 10\, xk$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$, from $t = 0$ to $t = 2$.

[*GGSIPU II Sem. End Term 2010*]

SOLUTION: Work done $= \int_C \vec{F} \cdot \vec{dr} = \int_C (3xyi - 5zj + 10xk) \cdot (idx + jdy + kdz) = \int_c (3xy\, dx - 5z\, dy + 10x\, dz)$

The curve C is $x = t^2 + 1$, $y = 2t^2$, $z = t^3$, from $t = 0$ to 2.

$$\therefore \quad \text{Work done} = \int_0^2 3(t^2+1) \cdot (2t^2) \cdot (2t)\, dt - 5t^3 \cdot (4t)\, dt + 10(t^2+1)\, 3t^2\, dt$$

$$= \int_0^2 [12(t^5 + t^3) - 20t^4 + 30\,(t^4 + t^2)]\, dt = \int_0^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\, dt$$

$$= \left[\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right]_0^2 = 16\,(8 + 4 + 3 + 5) = 320 \text{ units}$$ **Ans.**

EXAMPLE 21.4. Evaluate $\int_C (x^2+y^2)\,dx + (y+2x)\,dy$ where C is the boundary of the region in the first quadrant, that is bounded by the curves $y^2 = x$ and $x^2 = y$.

[*GGSIPU II Sem End Term 2011*]

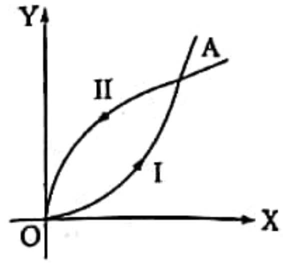
SOLUTION: Curve C consists of the curve I, from O to A, along $y=x^2$ and the curve II, from A to O, along $y^2=x$.

$$\therefore \quad \int_C (x^2+y^2)\,dx + (y+2x)\,dy = \int_0^1 (x^2+x^4)\,dx + (x^2+2x)\,2x\,dx + \int_1^0 (y^4+y^2)\,2y\,dy + (y+2y^2)\,dy$$

$$= \int_0^1 (5x^2+x^4+2x^3+4x^2)\,dx - \int_0^1 (2y^5+2y^3+y+2y^2)\,dy$$

$$= \left[\frac{5x^3}{3}+\frac{x^5}{5}+\frac{2x^4}{4}+\frac{4x^3}{3}\right]_0^1 - \left[\frac{2y^6}{6}+\frac{2y^4}{4}+\frac{y^2}{2}+\frac{2y^3}{3}\right]_0^1$$

$$= \frac{5}{3}+\frac{1}{5}+\frac{1}{2}+\frac{4}{3}-\frac{1}{3}-\frac{1}{2}-\frac{1}{2}-\frac{2}{3} = \frac{17}{10}. \text{ Ans.}$$



EXAMPLE 21.5. (a) If a force $\vec{f} = 2x^2y\,i + 3xy\,j$ displaces a particle in the XY-plane along the curve $y = 4x^2$ from $(0, 0)$ to $(1, 4)$, find the work done.

[*GGSIPU II Sem. End Term 2010*]

(b) Show that $\int (yz-1)\,dx + (z+zx+z^2)\,dy + (y+xy+2yz)\,dz$ is independent of the path of integration from $(1, 2, 2)$ to $(2, 3, 4)$. Evaluate the integral.

[*GGSIPU II Sem II Term 2012*]

SOLUTION: (a) Work done $\int_c \vec{f}\cdot\vec{dr} = \int_c (2x^2yi+3xyj)\cdot(idx+jdy) = \int_c 2x^2y\,dx + 3xy\,dy$

where $c : y = 4x^2$ hence $dy = 8x\,dx$ and x varies from 0 to 1.

$$\therefore \quad \text{Work done} = \int_0^1 2x^2(4x^2)\,dx + 3x(4x^2)\,8x\,dx = \int_0^1 (8x^4+96x^4)\,dx$$

$$= 104\left[\frac{x^5}{5}\right]_0^1 = \frac{104}{5} \text{ units}$$

Ans.

(b) Given integral $= \int \vec{F}\cdot\vec{dr}$ where $\vec{F} = (yz-1)\,i + (z+zx+z^2)\,j + (y+xy+2yz)\,k$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz-1 & z+zx+z^2 & y+xy+2yz \end{vmatrix}$$

$$= i\,(1+x+2z-1-x-2z) + j\,(y-y) + k\,(z-z) = 0$$

Hence $\overline{F}$ is irrotational vector field. Thus, there must exist a scalar ϕ such that $\overline{F} = \nabla\phi$.

$\therefore$ Given integral $= \int \overline{F}\cdot\overline{dr} = \int \nabla\phi\cdot(i\,dx + j\,dy + k\,dz) = \int \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int d\phi$

$= \phi\,(2, 3, 4) - \phi\,(1, 2, 2).$

Now let us find ϕ. From $\overline{F} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$, we get

$$\frac{\partial\phi}{\partial x} = yz - 1, \quad \text{...(1)}$$

$$\frac{\partial\phi}{\partial y} = z + zx + z^2 \quad \text{...(2)}$$

$$\frac{\partial\phi}{\partial z} = y + xy + 2yz. \quad \text{...(3)}$$

From (1), we get $\phi = xyz - x + A\,(y, z)$

$\therefore \quad \frac{\partial\phi}{\partial y} = xz + \frac{\partial A}{\partial y}$. Comparing it with (2) we get $\frac{\partial A}{\partial y} = z + z^2$

$\Rightarrow \quad A = yz + yz^2 + B\,(z) \quad \therefore \quad \phi = xyz - x + yz + yz^2 + B\,(z).$

$\Rightarrow \quad \frac{\partial\phi}{\partial z} = xy + y + 2yz + \frac{dB}{dz}$

Comparing it with (3) we get $\frac{dB}{dz} = 0 \quad \therefore \quad B$ is a pure constant.

Hence $\phi = xyz - x + yz + yz^2 + B.$

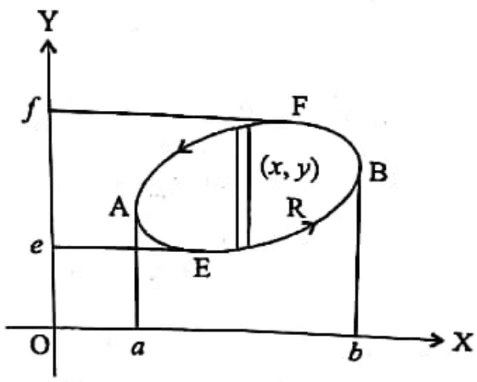
Thus, given integral $= \phi\,(2, 3, 4) - \phi\,(1, 2, 2) = 82 - 15 = 67.$ **Ans.**

GREEN'S LEMMA IN THE PLANE

If two functions U (x, y) and V (x, y) and their partial derivatives are single valued and continuous over a plane region R bounded by a closed curve C, then

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

[GGSIPU II Sem II Term 2005; Second Term 2007; End Term 2007]



PROOF: Let the closed curve C be such that any straight line parallel to the coordinate axes, cuts C in at the most two points. The curve C is divided into parts AEB and BFA where A and B correspond to the minimum and maximum values of x. Let the equations of the curves AEB and AFB (see the adjoining figure) be $y = y_1\,(x)$ and $y = y_2\,(x)$ respectively. If R is the region bounded by the curve C, we have

$$\iint_R \frac{\partial u}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} \frac{\partial u}{\partial y}dy\right]dx$$

$$= \int_{x=a}^{b}\left[u(x,y)\right]_{y=y_1(x)}^{y_2(x)}\,dx = \int_a^b \left[u(x,y_2)-u(x,y_1)\right]dx$$

$$= -\int_a^b u(x,y_1)\,dx - \int_b^a u(x,y_2)\,dx = -\oint_C u(x,y)$$

$$\Rightarrow \quad \int_C u\,dx = -\iint_R \frac{\partial u}{\partial y}\,dx\,dy \qquad \text{...(1)}$$

Similarly, let the equations of the curve EAF and EBF be $x = x_1(y)$ and $x = x_2(y)$ respectively, then

$$\iint_R \frac{\partial v}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=x_1(y)}^{x_2(y)} \frac{\partial v}{\partial x}dx\right]dy = \int_e^f \left[v(x_2,y)-v(x_1,y)\right]dy$$

$$= \int_f^e v(x_1,y)\,dy + \int_e^f v(x_2,y)\,dy = \oint_C v\,dy$$

$$\Rightarrow \quad \oint_C v\,dy = \iint_R \frac{\partial v}{\partial x}\,dx\,dy. \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy.$$

Now, let us Green's lemma in the vector notation, as follows.

We have $u\,dx + v\,dy = (ui + vj)\cdot(i\,dx + j\,dy) = \vec{F}\cdot\vec{dr}$

where $\vec{F} = ui + vj$ and $\vec{r} = xi + yj$ so that $\vec{dr} = idx + jdy$.

$$\text{Then} \quad \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = -\frac{\partial v}{\partial z}\,i + \frac{\partial u}{\partial z}\,j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k.$$

$$\text{Therefore,} \quad (\nabla\times F)\cdot k = -\frac{\partial v}{\partial z}i\cdot k + \frac{\partial u}{\partial z}j\cdot k + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k\cdot k = 0 + 0 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Thus, the Green's Lamma in the plane, can be written as

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_R (\vec{\nabla}\times\vec{F})\cdot k\;dx\,dy$$

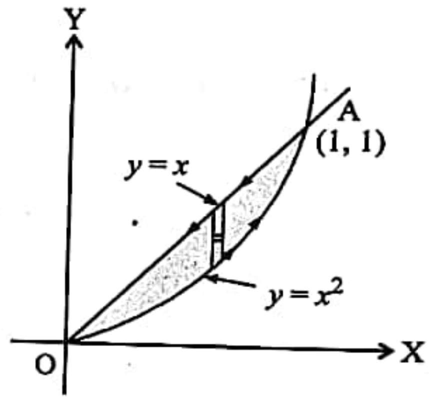
A generalization of this to a surface S in space, having a curve C as boundary leads quite naturally to Stoke's theorem to be discussed latter.

EXAMPLE 21.6. Verify Green's Lemma in the xy-plane for $\oint_C (xy+y^2)\,dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

[GGSIPU II Sem End Term 2010]

SOLUTION: Straight line $y = x$ and the parabola $y = x^2$ intersect at the points O (0, 0) and A (1, 1). The positive direction in traversing C is, as shown in the adjoining figure.

Along $y = x^2$ from O to A, the given line integral $\int (u\,dx + v\,dy)$ where $u = xy + y^2$, $v = x^2$, equals



$$\int_0^1 (x(x^2)+x^4)\,dx + x^2(2x)\,dx = \int_0^1 (3x^3+x^4)\,dx = \left[\frac{3x^4}{4}+\frac{x^5}{5}\right]_0^1 = \frac{19}{20}.$$

Along $y = x$ from A to O, the given integral equals

$$\int_1^0 (x(x)+x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx = -1.$$

Therefore, the given line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

Next $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy = \iint_R (x-2y)\,dx\,dy$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[x(x) - x^2 - (x^3 - x^4)\right] dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = -\frac{1}{20}.$$

Hence the Green's lemma is verified.

EXAMPLE 21.7. (a) A vector field $\vec{F}$ is given by $\vec{F} = (\sin y)\,i + x\,(1 + \cos y)\,j$.

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z = 0$.

(b) Using Green's theorem in the plane, evaluate $\int [\cos y\,i + x(1-\sin y)\,j].d\vec{r}$ for a closed curve given by $x^2 + y^2 = 1$, $z = 0$

[GGSIPU I Sem II Term 2006]

SOLUTION: (a) $\int_C \vec{F}\cdot d\vec{r} = \int_C [\sin y\,dx + x(1+\cos y)\,dy]$ where $C : \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, z = 0$.

It is not easy to evaluate the integral directly whereas using Green's lemma which is applicable here, will facilitate the job. By Green's lemma, the given line integral is equal to

$$\iint_S \left[\frac{\partial}{\partial x} x(1+\cos y) - \frac{\partial}{\partial y}\sin y\right] dx\,dy \quad \text{where S is the region inside the closed curve } C.$$

$$= \iint_S (1+\cos y - \cos y)\,dx\,dy = \iint_S dx\,dy = \text{area of } S = \pi ab$$

which is the area under the ellipse $C : \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z = 0$. **Ans.**

(b) By Green's theorem $\oint_c (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

where C is closed curve $x^2 + y^2 = 1,\ z = 0$ enclosing area R

In vector terms, we have $\int_c \vec{F}.\vec{dr} = \iint_R \text{curl } \vec{F}\cdot\vec{ds}$

Here $\vec{F} = (i \cos y) + x(1 - \sin y)j$ and $\vec{r} = xi + yj \quad \therefore \quad d\vec{r} = i dx + j\,dy.$

$$\therefore \qquad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & x(1-\sin y) & 0 \end{vmatrix}$$

$$= k\left[\frac{\partial}{\partial x}\{x(1-\sin y)\} - \frac{\partial}{\partial y}\cos y\right] = k(1 - \sin y + \sin y) = k$$

$$\Rightarrow \qquad \int_c \vec{F}.\vec{dr} = \iint_R k.\vec{ds} = \iint_R k\cdot k\,dx\,dy = \iint_R dx\,dy = \pi$$

(since area of the circle $x^2 + y^2 = 1,\ z = 0$ is π). **Ans.**

EXAMPLE 21.8. Use Green's theorem to find the value of the line integral

$$\int_C \vec{F}\cdot\vec{dr} \quad \text{where } \vec{F} = y^3 i - x^3 j \text{ and } C : x^2 + y^2 = a^2,\ z = 0.$$

SOLUTION : Here $\vec{F} = y^3 i - x^3 j = ui + vj \quad \therefore \quad u = y^3,\ v = -x^3.$

By Green's theorem $\int_C u\,dx + v\,dy = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

where S is the plane area bounded by the closed curve C: $x^2 + y^2 = a^2,\ z = 0.$

Since $u = y^3$ and $v = -x^3$ we have $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -3x^2 - 3y^2$

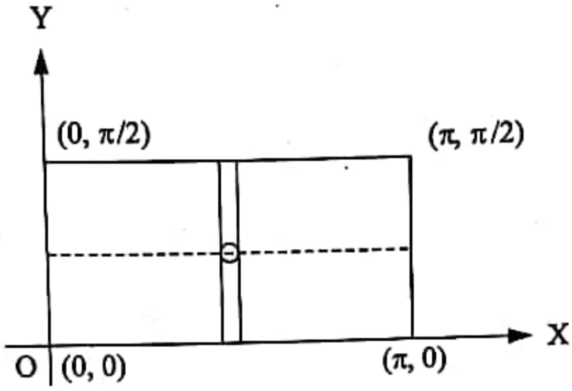
$\therefore \quad \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_S -3(x^2 + y^2)\,dx\,dy$ where S is the region inside the circle C

Changing to polar coordinates the above integral becomes

$$-3\int_0^{2\pi}\int_0^a r^2 r\,dr\,d\theta = -3\int_0^{2\pi} d\theta \int_0^a r^3 dr = -\frac{3}{2}\pi a^4.$$ **Ans.**

EXAMPLE 21.9. Evaluate by Green's theorem, $\int_c e^{-x} (\sin y\, dx + \cos y\, dy)$ where C is the rectangle with vertices $(0, 0), (\pi, 0), (\pi, \pi/2), (0, \pi/2)$.

[*GGSIPU II Sem End Term 2006 Reappear; End Term 2012*]



SOLUTION: By Green's theorem $\oint_c f\, dx + g\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx dy$

where C is a closed curve enclosing the region R. In the given case, $f = e^{-x} \sin y$, $g = e^{-x} \cos y$.

$$\therefore \int_c e^{-x} \sin y\, dx + e^{-x} \cos y\, dy = \iint_R \left[\frac{\partial}{\partial x}(e^{-x} \cos y) - \frac{\partial}{\partial y}(e^{-x} \sin y)\right] dx dy$$

where R is the region enclosed in the rectangle with vertices $(0, 0), (\pi, 0), (\pi, \pi/2), (0, \pi/2)$.

$$= \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dy dx = -2 \iint_R e^{-x} \cos y\, dy dx$$

$$= -2 \int_0^{\pi} e^{-x} dx \int_0^{\pi/2} \cos y\, dy = 2\,(e^{-\pi} - 1).$$

Ans.

SURFACE INTEGRAL

Let $\vec{F}$ be a vector point function and S a given two sided surface. Subdivide the area S into p elements of area δS_i, $i = 1, 2, ..., p$. Let $P_i(x_i, y_i, z_i)$ be any point in δS_i and let $\hat{n}_i$ be unit normal vector drawn outwards to δS_i at the point P_i. Denote $\vec{F}(x_i, y_i, z_i)$ by $\vec{F}_i$.

Now consider $\sum_{i=1}^{p} (\vec{F}_i \cdot \hat{n}_i)\, \delta S_i$ where $\vec{F}_i \cdot \hat{n}_i$ is the normal component of $\vec{F}_i$ at P_i. Then the limit of this sum, as $p \to \infty$, in such a way that the largest dimension of δS_i approaches to zero, if it exists, is called the surface integral of the normal component of $\vec{F}$ over S and is denoted by $\iint_S \vec{F} \cdot \hat{n}\, ds$.

Thus, we have $$\iint_S \vec{F} \cdot \hat{n}\, ds = \lim_{p \to \infty} \sum_{i=1}^{p} (\vec{F}_i \cdot \hat{n}_i)\, \delta S_i = \iint_S \vec{F} \cdot \vec{ds}$$

where $\vec{ds}$ is the vector $(dy\,dz\, i + dz\,dx\, j + dx\,dy\, k)$.

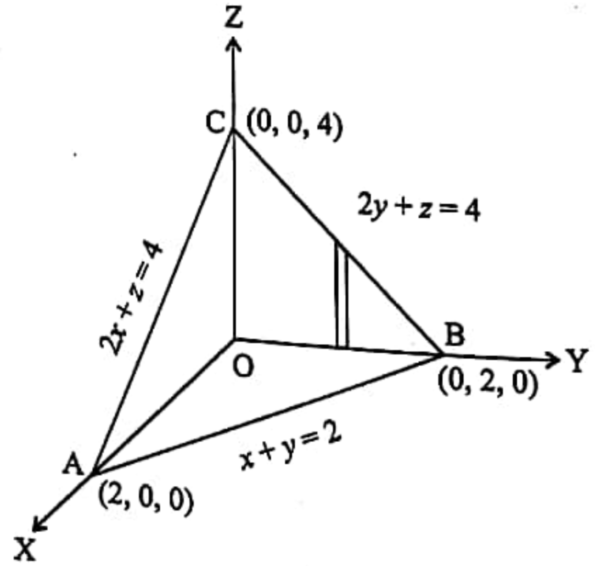
If $\vec{F} = F_1 i + F_2 j + F_3 k$ then $\iint_S \vec{F} \cdot \vec{ds} = \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy$.

EXAMPLE 21.10. Evaluate $\iint_S \vec{A} \cdot \vec{ds}$ where $\vec{A} = xi + (z^2 - zx)\, j - xy\, k$ and S is the surface of the triangle with vertices (2, 0, 0), (0, 2, 0), (0, 0, 4).

SOLUTION: Here S is the plane surface $\frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1$ or $2x + 2y + z = 4$ bounded by the lines $x + y = 2,\ z = 0$; $2y + z = 4,\ x = 0$ and $2x + z = 4,\ y = 0$. (See figure)

Since $\vec{ds} = i\, dy\, dz + j\, dz\, dx + k\, dx\, dy$, we have

$$\vec{A} \cdot \vec{ds} = [xi + (z^2 - zx)\, j - xyk] \cdot (idydz + jdzdx + kdxdy)$$
$$= xdydz + (z^2 - zx)\, dzdx - xy\, dx\, dy$$



$$\therefore \quad \iint_S \vec{A}\cdot\vec{ds} = \int\int_{\Delta OBC} x\, dy\, dz + \int\int_{\Delta OAC} (z^2 - zx)\, dz\, dx - \int\int_{\Delta OAB} xy\, dx\, dy$$

For points on the given surface, we have $x = 2 - y - \frac{1}{2}z$

$$\text{Hence } \iint_S \overline{A}\cdot\overline{ds} = \int_0^2\int_0^{4-2y}\left(2-y-\frac{1}{2}z\right)dz\,dy + \int_0^2\int_0^{4-2x}(z^2-zx)\,dz\,dx - \int_0^2\int_0^{2-x} xy\,dy\,dx$$

$$= \int_0^2\left[2z - yz - \frac{z^2}{4}\right]_0^{4-2y} dy + \int_0^2\left[\frac{z^3}{3} - \frac{xz^2}{2}\right]_0^{4-2x} dx - \int_0^2\left[x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^2 (2-y)^2\,dy + \frac{8}{3}\int_0^2 (2-x)^3\,dx - \frac{5}{2}\int_0^2 x(2-x)^2\,dx$$

$$= \frac{8}{3} + \frac{32}{3} - \frac{10}{3} = 10.$$ **Ans.**

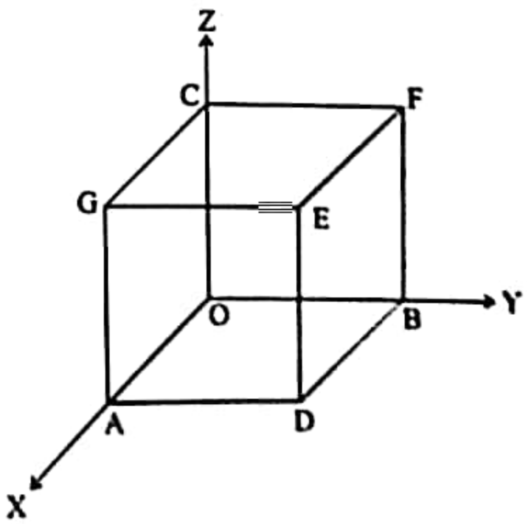
EXAMPLE 21.11. If $\vec{F} = 4xzi - y^2j + yzk$ evaluate $\iint_S \vec{F}\cdot\overline{ds}$ where S is the surface of the cube bounded by $x = 0,\ x = 1;\ y = 0,\ y = 1;\ z = 0,\ z = 1.$

SOLUTION: The given integral has to be calculated over the six faces of the given unit cube (see figure). For the face ADEG ($x = 1$), we have $\hat{n} = i,\ ds = dydz$, hence

$$\int\int_{ADEG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4zi - y^2j + yzk)\cdot i\,dy\,dz = \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 4z\,dz\int_0^1 dy = 2.$$

For the face OBFC ($x = 0$), we have $\hat{n} = -i,\ ds = dydz$, hence

$$\int\int_{OBFC} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4(0)zi - y^2j + yzk)\cdot(-i)\,dy\,dz = 0$$



For the face DBFE ($y = 1$), we have $\hat{n} = j,\ ds = dxdz$, hence

$$\int\int_{DBFE} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xzi - 1^2j + (1)zk)\cdot j\,dx\,dz$$

$$= \int_0^1\int_0^1 -dx\,dz = -1$$

For the face OAGC ($y = 0$), we have $\hat{n} = -j,\ ds = dx\,dz$, hence

$$\int\int_{OAGC} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xzi - 0j + 0k)\cdot(-j)\,dx\,dz = 0.$$

For the face GEFC ($z = 1$), we have $\hat{n} = k,\ ds = dxdy$, hence

$$\int\int_{EEFC} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4x)(1)\,i - y^2 j + y(1)k)\cdot k\, dx\, dy = \int_0^1\int_0^1 y\, dx\, dy = \frac{1}{2}.$$

and for the face OBDA ($z = 0$), we have $\hat{n} = -k$, $ds = dx\, dy$, hence

$$\int\int_{OBDA} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 \left[(4x(0)i - y^2 j + y(0)k\right]\cdot(-k)\, dx\, dy = 0$$

Adding these, yields

$$\iint_S \vec{F}\cdot\hat{n}\, ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}. \qquad \textbf{Ans.}$$

EXAMPLE 21.12. Find the value of the surface integral $\iint_S 2x^2 y\, dy\, dz - y^2 dz\, dx + 4xz^2\, dx\, dy$ where S is the curved surface of the cylinder $y^2 + z^2 = 9$, bounded by the planes $x = 0$, $x = 2$.

SOLUTION: We know that $\hat{n}\, ds = i\, dy\, dz + j\, dz\, dx + k\, dx\, dy$ in terms of the projection of $\vec{ds}$ on the coordinate planes. Taking $\vec{F} = 2x^2 yi - y^2 j + 4xz^2 k$, the given integral can be written as $\iint_S \vec{F}\cdot\hat{n}\, ds$.

To find $\hat{n}$ for the surface S, let $\phi = y^2 + z^2 - 9$ then $\nabla\phi = 0i + 2yj + 2zk$

hence $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2yj + 2zk}{2\sqrt{y^2+z^2}} = \dfrac{yj + zk}{3}$ $\quad\therefore\quad \vec{F}\cdot\hat{n} = 0 - \dfrac{y^3}{3} + \dfrac{4}{3}xz^3.$

Thus, the given integral $= \dfrac{1}{3}\iint_S (-y^3 + 4xz^3)\, ds.$

To evaluate it, let us employ the cylindrical polar coordinates $y = r\sin\theta$, $z = r\cos\theta$, $x = x$ so that $ds = r\, d\theta\, dx$.

$\therefore$ The above Integral $= \dfrac{1}{3}\iint_S (-r^3\sin^3\theta + 4xr^3\cos^3\theta)\, r\, d\theta\, dx$ (here $r = 3$)

$$= \frac{1}{3}\int_0^{2\pi}\int_0^2 (-81\sin^3\theta + 4(81)x\cos^3\theta)\, dx\, d\theta = 27\int_0^{2\pi}\left[-x\sin^3\theta + 2x^2\cos^3\theta\right]_0^2 d\theta$$

$$= -54\int_0^{2\pi}\sin^3\theta\, d\theta + 216\int_0^{2\pi}\cos^3\theta\, d\theta = 0 + 432\int_0^{\pi}\cos^3\theta\, d\theta = 0. \qquad \textbf{Ans.}$$

(since $\int_0^{2\pi}\sin^3\theta\, d\theta = 0$ and $\int_0^{2\pi}\cos^3\theta\, d\theta = 2\int_0^{\pi}\cos^3\theta\, d\theta = 0$ by property of definite integrals.)

STOKE'S CIRCULATION THEOREM

If S is an open surface (two sided) bounded by a closed non-intersecting curve C and if a vector field $\vec{F}$ has continuous partial derivatives, then

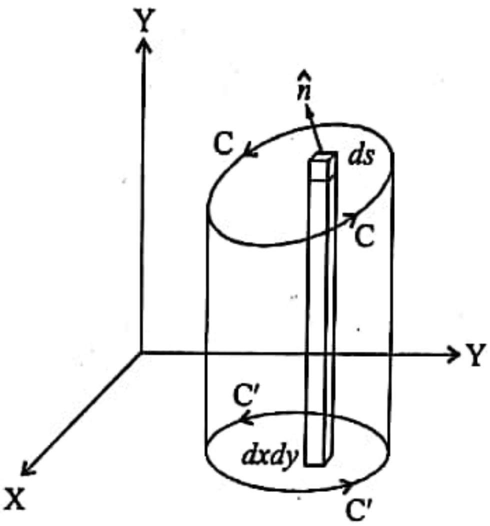
$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint_S (\nabla\times\vec{F})\cdot\vec{ds}$$

[GGSIPU II Sem II Term 2010[

where C is traversed in positive direction.

PROOF: Consider the case when the equation of the given surface S can be written as $z = f(x, y)$, $y = g(x, z)$ and $x = h(y, z)$ where f, g, h are continuous and have continuous partial derivatives.

Let the surface S be oriented upwards and take its equation as $z = f(x, y)$. Let us write $\phi(x, y, z) = z - f(x, y) = 0$, then the outward drawn normal is given by



$$\hat{n} = \frac{-\left(\frac{\partial f}{\partial x}\right)i - \left(\frac{\partial f}{\partial y}\right)j + k}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}. \quad \text{...(1)}$$

Also, $\hat{n} = i\cos\alpha + j\cos\beta + k\cos\gamma$, where α, β, γ are the angles which the unit normal makes with the positive directions of X, Y, Z-axis respectively. Comparing the above two relations, we have

$$\frac{\cos\alpha}{-\frac{\partial f}{\partial x}} = \frac{\cos\beta}{-\frac{\partial f}{\partial y}} = \frac{\cos\gamma}{1}. \quad \text{...(2)}$$

Therefore, we need to show that

$$\oint_C F_1\, dx + F_2 dy + F_3\, dz = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds \quad \text{...(3)}$$

Using the equation of the surface as $Z = f(x, y)$ we need to prove that

$$\oint_C F_1(x, y, z)\, dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial r}\cos\gamma\right) ds. \quad \text{...(4)}$$

Let R be the projection of the surface S on XY-plane and C_1 be the projection of the bounding curve C on the XY-plane. Then

(using Green's theorem)

$$\oint_C F_1(x, y, z)\, dx = \oint_{C_1} F_1(x, y, f(x, y)\, dx$$

$$= \iint_R -\frac{\partial F_1}{\partial y}[x, y, f(x, y)]\, dx\, dy = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right) dx\, dy$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z} \frac{\cos\beta}{\cos\gamma} \right) \cos\gamma \, ds \quad \text{since } dx\,dy = ds \cos\gamma.$$

Hence $$\oint_C F_1(x, y, z)\, dx = -\iint_S \left(\frac{\partial F_1}{\partial y} \cos\gamma - \frac{\partial F_1}{\partial z} \cos\beta \right) ds = \iint_S \left(\frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma \right) ds$$

Hence the result in (4) is proved.

Similarly, assuming the equation of the surface as $y = g(x, z)$ and $z = h(x, y)$ we can prove the equality of the terms corresponding to the components $F_2(x, y, z)$ and $F_3(x, y, z)$.

IMPORTANT COROLLARY: If two surfaces S_1 and S_2 have same bounding curve C, then

$$\iint_{S_1} (\nabla \times F) \cdot \vec{ds} = \iint_{S_2} (\nabla \times F) \cdot \vec{ds} \qquad \textbf{since each is equal to } \int_C \vec{F} \cdot \vec{dr}.$$

EXAMPLE 21.13. (a) Verify Stoke's theorem for $\vec{F} = (2x - y)\, i - yz^2 j - y^2 z k$ where S is the upper half of the surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

[*GGSIPU II Sem End Term 2005; End Term 2010*]

(b) Verify Stoke's theorem for the function $\vec{F} = x^2 i + xy\, j$ integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a$ and $y = a$.

[*GGSIPU II Sem End Term 2013*]

SOLUTION: (a) The boundary C of the given open surface S is the circle in the *xy*-plane with centre at the origin and radius one. Let $x = \cos t$, $y = \sin t$, $z = 0, 0 \le t \le 2\pi$ be the parametric equation of C, then

$$\oint_C F \cdot dr = \oint_C (2x - y)\, dx - yz^2 dy - y^2 z\, dz = \int_C (2x - y)\, dx + 0 + 0 \qquad (\text{as } z = 0)$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt = \int_0^{2\pi} \left[-\sin 2t + \frac{1 - \cos 2t}{2} \right] dt$$

$$= \left[\frac{1}{2}\cos 2t + \frac{t}{2} - \frac{1}{4}\sin 2t \right]_0^{2\pi} = 0 + \pi + 0.$$

Next $$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = i(-2yz + 2yz) + j(0 - 0) + k(0 + 1) = k.$$

$\therefore$ $$\iint_S (\nabla \times F) \cdot \hat{n}\, ds = \iint_S k \cdot \hat{n}\, ds = \iint_{S'} dx\, dy \qquad (\text{as } \hat{n}\; k\; ds = dx\, dy)$$

where S′ is the projection of S on the *xy*-plane. Therefore, the above integral equals

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx = 2\,(2) \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \int_0^1 \sqrt{1 - x^2}\, dx = \pi$$

hence **the Stoke's theorem is verified.**

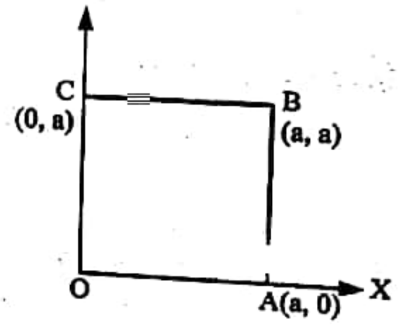
(b) According to Stoke's theorem $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$.

Here $\vec{F} = x^2 i + xy\,j \quad \therefore \quad \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = k\,(y-0) = ky$

Thus $$\iint \nabla\times F\cdot\hat{n}\,ds = \iint ky\cdot k\,dx\,dy = \int_0^a\int_0^a y\,dy\,dx = \frac{1}{2}a^3$$

and $$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2 i + xy\,j)\cdot(i\,dx + j\,dy) = \int_C x^2 dx + xy\,dy$$

where C is the closed curve $OABC$.
Along OA, $y = 0$, x varies from 0 to a,
along AB, $x = a$ and y varies from 0 to a,
along BC', $y = a$ and x varies from a to 0.
and along CO, $x = 0$ and y varies from a to 0.



$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^a x^2 dx + \int_0^a ay\,dy + \int_a^0 x^2 dx + \int_a^0 0\,dy = \frac{a^3}{2}.$$

Hence Stoke's theorem is verified.

EXAMPLE 21.14. Verify Stoke's theorem for the function $\vec{F} = xyi + yzj + z^2k$ over the cube whose vertices are (0, 0, 0), (a, 0, 0), (0, a, 0), (0, 0, a), (a, a, 0), (0, a, a), (a, 0, a), (a, a, a) if the face of the cube in XOY plane is missing.

SOLUTION: As mentioned earlier, if S_1 and S_2 are two surfaces having the same boundary curve C, then we have

$$\int\int_{S_1}(\nabla\times\vec{F})\cdot ds = \int\int_{S_2}(\nabla\times\vec{F})\cdot ds$$ since both the above integrals are equal to $\int_C \vec{F}\cdot d\vec{r}$.

In the present case, in place of considering five faces of the open cube, we consider the single face in the XY-plane.

Here $$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & z^2 \end{vmatrix} = i\,(0-y) + j\,(0-0) + k\,(0-x) = -iy - kx.$$

Now, for the bounding curve, we have

$$\int\int_S(\nabla\times\vec{F})\cdot\hat{n}\,ds = \int\int_S(-iy-kx)\cdot k\,ds = \int\int_S -x\,dx\,dy = \int_0^a\int_0^a -x\,dx\,dy = -\int_0^a\left[\frac{x^2}{2}\right]_0^a dy = -\frac{a^3}{2}$$

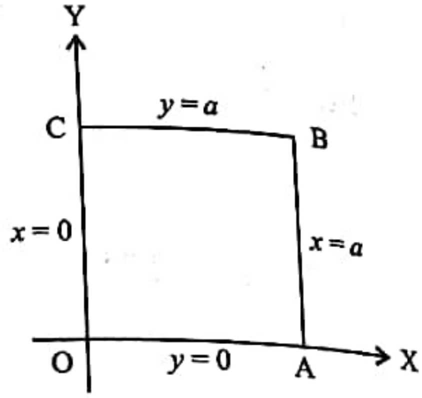
Here $\vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + z^2\,dz = xy\,dx$ since $z = 0$ for C.

$\therefore \quad \int_{OA} \vec{F}\cdot\vec{dr} = \int_{OA} xy\, dx = 0 \quad$ (as $y=0$ on OA),

$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{AB} xy\, dx = 0$ (as $x=a$ along AB so that $dx=0$),

$\int_{BC} \vec{F}\cdot dr = \int_{BC} xy\, dx = \int_a^0 ax\, dx = -\frac{a^3}{2} \quad$ (as $y=a$ on BC),

and $\quad \int_{CO} \vec{F}\cdot dr = \int_{CO} xy\, dx = 0 \quad$ (as $x=0$, along CO)



Adding the above four integrals, we get $\int_C \vec{F}\cdot dr = -\frac{a^3}{2}$.

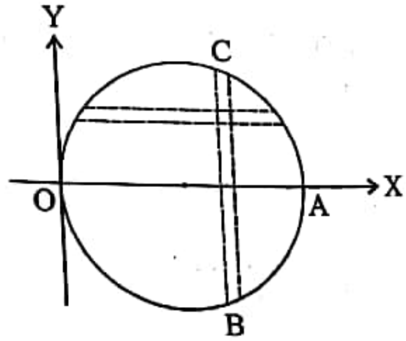
Thus $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot\vec{dr} = \frac{-a^3}{2}$ and **hence Stoke's theorem is verified.**

EXAMPLE 21.15. Find the value of the surface integral $\iint_S (\nabla\times\vec{F})\cdot\vec{ds}$ taken over the portion of the surface $x^2+y^2-2ax+az=0$ and the bounding curve in the plane $z=0$ where $\vec{F}=(y^2+z^2-x^2)\, i+(z^2+x^2-y^2)\, j+(x^2+y^2-z^2)\, k$.

SOLUTION: The given surface meets the plane $z=0$ in the circle $x^2+y^2-2ax=0,\ z=0$.

The surface integral of $\nabla\times\vec{F}$ over the given surface is the same as the surface integral of $\nabla\times F$ over the area of the circle $x^2+y^2-2ax=0,\ z=0$.

We have $\vec{F}=(y^2+z^2-x^2)\, i+(z^2+x^2-y^2)\, j+(x^2+y^2-z^2)\, k$,



$$\therefore\ \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$$= i\,(2y-2z)+j\,(2z-2x)+k\,(2x-2y).$$

Next, for the area OBAC (where $z=0$), we have

$$\nabla\times\vec{F} = 2yi-2xj+(2x-2y)\,k \quad \text{and} \quad \vec{ds}=\hat{n}\,ds = k\,dx\,dy$$

$$\therefore\ \iint_S \nabla\times\vec{F}\cdot\vec{ds} = \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} (2x-2y)\,dy\,dx = \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} 2x\,dy\,dx - \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} 2y\,dy\,dx$$

$$= 2\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} 2x\,dy\,dx - 0 \quad \text{(by property of definite integrals)}$$

$$= 4\int_0^{2a} x[y]_0^{\sqrt{2ax-x^2}}\,dx = 4\int_0^{2a} x\sqrt{2ax-x^2}\,dx \quad \text{(putting } x = 2a\sin^2\theta)$$

$$= 64\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta = \frac{64\,\overline{|5/2}\,\overline{|3/2}}{2\,\overline{|4}} = \frac{64}{12}\cdot\frac{3}{2}\cdot\left(\frac{1}{2}\overline{|\frac{1}{2}}\right)^2$$

$$= 2\pi a^2$$

Ans.

EXAMPLE 21.16. (a) Evaluate $\oint \vec{F}.d\vec{r}$ by Stoke's theorem where $F = y^2i + x^2j - (x+z)y^2k$ and C is the bofindary of the triangle with vertices (0, 0, 0), (1, 0, 0) and (1, 1, 0).

[GGSIPU II End Sem End Term 2007]

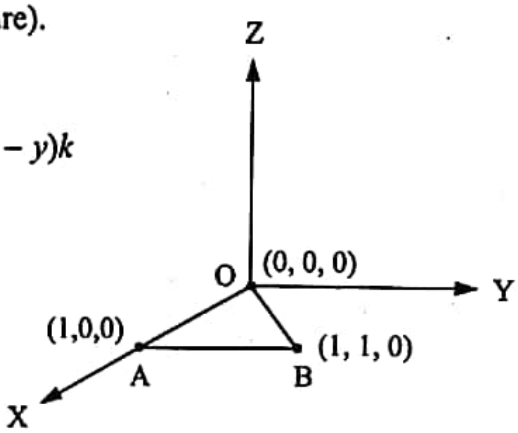
(b) Use Stoke's theorem to evaluate $\int_C \vec{V}\cdot\vec{dr}$ where $\vec{V} = y^2i + xyj + xzk$ and C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ oriented in the positive direction.

[GGSIPU II Sem End Term 2014]

SOLUTION: (a) By Stoke's theorem $\int_c \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}.d\vec{s}$ where $\vec{F} = y^2i + x^2i - (x+z)y^2k$,

C is the triangle OAB and S is the region inside C (see figure).

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z)y^2 \end{vmatrix} = -2y(x+z)i + y^2j + 2(x-y)k$$



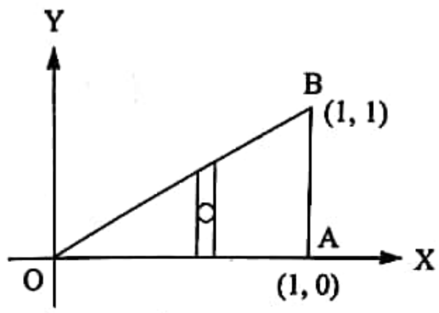
$$\therefore \iint_s \nabla\times\vec{F}\cdot\vec{ds} = \iint_S \nabla\times\vec{F}.k\,dxdy$$

$$= \iint_s [-2y(x+z)i.k + y^2j.k + 2(x-y)k\cdot k]dxdy$$

$$= \iint_{\Delta OAB} 2(x-y)dxdy = 2\int_0^1\int_0^x (x-y)dydx$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right)dx$$

$$= \int_0^1 x^2dx = \frac{1}{3}$$



Ans.

(b) C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ and hence C : $x^2 + y^2 = 9$, $z = 0$.

$\vec{V} = y^2i + xyj + xzk$. Using Stoke's theorem, we have

$\int_C \vec{V}\cdot\vec{dr} = \iint_s \nabla\times\vec{V}\cdot\vec{ds}$ where S is the area under the curve C, that is, the area inside $x^2+y^2=9$.

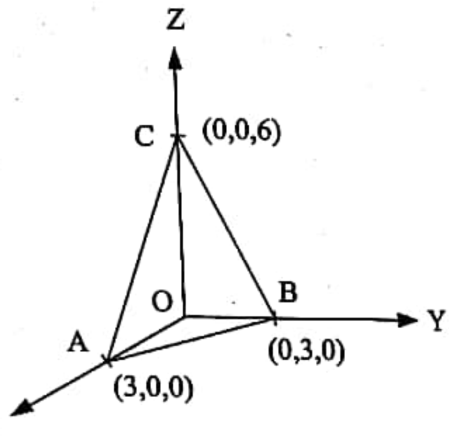
$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = i\,(0) + j\,(0 - z) + k\,(y - 2y) = -zj - yk.$$

$$\therefore \quad \int_C \vec{V} \cdot \vec{dr} = \iint_S (-zj - yk) \cdot (-k)\, dxdy = \iint_S y\, dydx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y\, dydx = 0 \quad \textbf{Ans.}$$

EXAMPLE 21.17. (a) Evaluate $\oint_C \vec{F}.d\vec{s}$ where $\vec{F} = xyi - x^2j + (x + z)k$ and S in the region of the plane $2x + 2y + z = 6$ in the first octant. [*GGSIPU II Sem End Term 2006*]

(b) Evaluate $\oint_C \vec{V} \cdot \vec{dr}$ using Stoke's theorem if $\vec{V} = (3x + 2z)i + (x + 3y)j + (2y - 3z)k$ and C is the curve of intersection of the plane $6x + 3y + 4z = 12$ with the coordinate planes. [*GGSIPU II Sem End Term 2011*]

SOLUTION: (a) $\vec{F} = xyi - x^2j + (x + z)k$ and the surface S is the plane as shown in the adjoining figure. Equation of the plane is $2x + 2y + z = 6$.



$\therefore$ Unit vector along the normal to the plane

$$= \hat{n} = \frac{2i + 2j + k}{\sqrt{4 + 4 + 1}} = \frac{1}{3}\,(2i + 2j + k)$$

$\iint_s \vec{F}.d\vec{s} = \iint_s \vec{F}.\hat{n}\,ds$. Also we have $dxdy = \hat{n}.k\; ds$

$$\therefore \quad ds = \frac{dxdy}{\frac{1}{3}(2i + 2j + k).k} = 3\; dxdy$$

$$\therefore \quad \iint_s \vec{F}.d\vec{s} = \iint_s [xyi - x^2 j + (x + z)k].\frac{1}{3}(2i + 2j + k).3dxdy$$

$$= \int\int_{\Delta OAB} (2xy - 2x^2 + x + z)dxdy = \int\int_{\Delta OAB} (2xy - 2x^2 + x + 6 - 2x - 2y)dxdy$$

$$= \int_0^3 \int_0^{3-x} (2xy - 2x^2 - x - 2y + 6)dydx = \int_0^3 [xy^2 - 2x^2 y - xy - y^2 + 6y]_0^{3-x} dx$$

$$= \int_0^3 [x(3 - x)^2 - 2x^2(3 - x) - x(3 - x) - (3 - x)^2 + 6(3 - x)]dx$$

$$= \int_0^3 (3 - x)[x(3 - x) - 2x^2 - x - (3 - x) + 6]dx = \int_0^3 (x - 3)(3x^2 - 3x - 3)dx$$

$$= 3\int_0^3 (x^3 - 4x^2 + 2x + 3)dx = 3\left[\frac{x^4}{4} - \frac{4x^3}{3} + x^2 + 3x\right]_0^3 = \frac{27}{4} \quad \textbf{Ans.}$$

(b) $\vec{V} = (3x+2z)i + (x+3y)j + (2y-3z)k.$

$$\therefore \quad \nabla\times\vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x+2z & x+3y & 2y-3z \end{vmatrix} = i(2-0)+j(2-0)+k(1-0) = 2i+2j+k.$$

If $\hat{n}$ is unit vector along the normal to the plane $6x+3y+4z=12$, then

$$\hat{n} = \frac{6i+3j+4k}{\sqrt{36+9+16}} = \frac{6i+3j+4k}{\sqrt{61}}.$$

By Stoke's theorem $\oint_C \vec{V}\cdot\overline{dr} = \iint_S (2i+2j+k)\cdot\hat{n}\,\overline{dS}$

$$\therefore \quad \oint_C \vec{V}\cdot\overline{dr} = \iint_S \frac{(2i+2j+k)\cdot(6i+3j+4k)}{\sqrt{61}}\,dS = \frac{1}{\sqrt{61}}\iint_S (12+6+4)\,dS = \frac{22}{\sqrt{61}}\iint_S dS$$

The given plane is $6x+3y+4z=12$ or $\frac{x}{2}+\frac{y}{4}+\frac{z}{3}=1$. It meets the coordinate axes in $(2,0,0),(0,4,0)$ and $(0,0,3)$ whose area $=\sqrt{61}$. $\therefore$ $\oint_C \vec{V}\cdot\overline{dr} = 22.$

Ans.

GAUSS DIVERGENCE THEOREM

The surface integral of the normal component of a vector point function $\vec{F}$ taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of $\vec{F}$ taken throughout the volume V bounded by S, i.e.,

$$\iint_S \vec{F}\cdot\vec{ds} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$$

[GGSIPU II Sem End Term 2006 Reappear]

Let $\vec{F} = F_1 i + F_2 j + F_3 k$ then $\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}.$

In the rectangular coordinate system $dv = dx\,dy\,dz$. Let the unit vector $\hat{n}$ along the normal to the surface S at the point (x, y, z) be given by $\hat{n} = n_1 i + n_2 j + n_3 k$ making angles α, β, γ with x-axis, y-axis, z-axis respectively, so that

$$n_1 = \hat{n}\cdot i = \cos\alpha, \quad n_2 = \hat{n}\cdot j = \cos\beta \quad \text{and} \quad n_3 = \hat{n}\cdot k = \cos\gamma$$

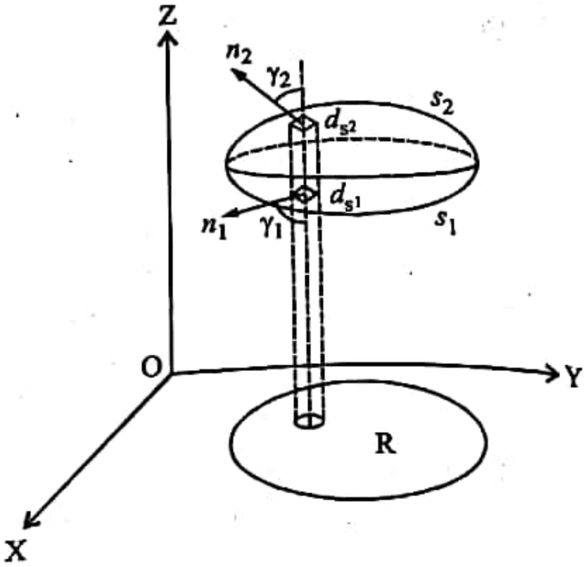
Then $\vec{F}\cdot\hat{n} = (F_1 i + F_2 j + F_3 k)\cdot(n_1 i + n_2 j + n_3 k) = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$

therefore the above statement of divergence theorem can be writtten as

$$\iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,ds = \iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

Proof: Let us prove $\iint_S F_3\cos\gamma\,ds = \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz.$

Here S is the closed surface such that any line parallel to the coordinate axes, cuts S in at most two points. Let the equations of the lower and upper portions S_1 and S_2 of the surface S be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. If R denotes the projection of S on the xy plane, then



$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dy\, dx$$

$$= \iint_R \left[F_3(x, y, z)\right]_{z=f_1(x,y)}^{f_2(x,y)} dx\, dy$$

$$= \iint_R \left[F_3(x, y, f_2) - F_3(x, y, f_1)\right] dy\, dx$$

For the upper portion S_2, $dx\, dy = \cos\gamma_2\, ds_2 = k \cdot n_2\, ds_2$ since the normal n_2 to S_2 makes an acute angle γ_2 with k.

For the lower portion S_1, $dy\, dx = -\cos\gamma_1\, ds_1 = -k \cdot n_1\, ds_1$ since the normal n_1 to S_1 makes an obtuse angle γ_1 with k.

$$\therefore \quad \iint_R F_3(x, y, f_2)\, dy\, dx = \iint_{S_2} F_3\, k \cdot \hat{n}_2\, ds_2 \quad \text{and} \quad \iint_R F_3(x, y, f_1)\, dy\, dx = -\iint_{S_1} F_3\, k \cdot \hat{n}_1\, ds_1$$

$$\therefore \quad \iint_R F_3(x, y, f_2)\, dy\, dx - \iint_R F_3(x, y, f_1)\, dy\, dx = \iint_{S_2} F_3\, k \cdot \hat{n}_2\, ds_2 + \iint_{S_1} F_3\, k \cdot \hat{n}_1\, ds_1$$

$$= \iint_S F_3\, k \cdot \hat{n}\, ds.$$

Thus $\displaystyle\iiint_V \frac{\partial F_3}{\partial z}\, dv = \iint_S F_3\, k \cdot \hat{n}\, ds$.

Similarly, by projecting S on the other coordinates planes, we get

$$\iiint_V \frac{\partial F_1}{\partial x}\, dv = \iint_S F_1\, i \cdot n\, ds \quad \text{and} \quad \iiint_V \frac{\partial F_2}{\partial y}\, dv = \iint_S F_2\, j \cdot n\, ds.$$

Adding these gives

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv = \iint_S (F_1 i + F_2 j + F_3 k) \cdot \hat{n}\, ds \quad \text{or} \quad \iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$$

EXAMPLE 21.18. Verify the divergence theorem for the vector field $\vec{F} = 4xi - 2y^2j + z^2k$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

[GGSIPU II Sem End Term 2009; End Term 2005; End Term 2006; End Term 2010; End Term 2014]

SOLUTION: This being a case of closed surface we have, by divergence theorem

$$\iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds.$$

Here $\vec{F} = 4xi - 2y^2j + z^2k$.

The volume integral $= \iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V \left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(z^2)\right] dx\,dy\,dz$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z-4yz+z^2\right]_{z=0}^{3} dx\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\,dy\,dx$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 21\,dy\,dx - 0 \quad \text{(since area of the circle} = \pi(2^2))$$

$= 84\,\pi$ (on simplification)

The surface S consists of three surfaces, one being the base S_1 $(z = 0)$, second being the top S_2 $(z = 3)$ and third the curved surface S_3 of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$.

Thus the surface integral $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1}\vec{F}\cdot\hat{n}\,ds_1 + \iint_{S_2}\vec{F}\cdot\hat{n}\,ds_2 + \iint_{S_3}\vec{F}\cdot\hat{n}\,ds_3$

On S_1 $(z = 0)$, $\hat{n} = -k$ and $\vec{F} = 4xi - 2y^2j$ $\therefore$ $\vec{F}\cdot\hat{n} = 0$ therefore $\iint_{S_1}\vec{F}\cdot\hat{n}\,ds_1 = 0$.

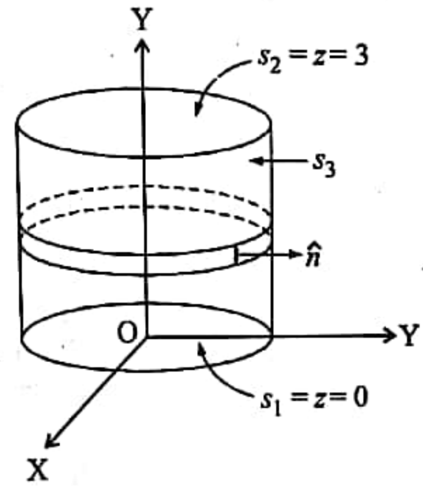
On S_2 $(z = 3)$, $\hat{n} = k$ and $\vec{F} = 4xi - 2y^2j + 9k$ $\therefore$ $\vec{F}\cdot\hat{n} = 9$

therefore $\iint_{S_2}\vec{F}\cdot\hat{n}\,ds_2 = 9\iint_{S_2} ds_2 = 9\,(\pi\,2^2) = 36\,\pi$.

On S_3, the curved surface of the cylinder $(x^2 + y^2 = 4)$, the normal to $x^2 + y^2 = 4$ is in the direction of $\nabla(x^2 + y^2) = 2xi + 2yj$

$\therefore$ The unit vector $\hat{n}$ along the normal to the curved surface, is given by

$$\hat{n} = \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{xi+yj}{2} \quad (\text{as } x^2 + y^2 = 4)$$



hence $\vec{F}\cdot\hat{n} = (4xi - 2y^2j + z^2k)\cdot\frac{xi+yj}{2} = 2x^2 - y^3$

$$\therefore \iint_{S_3}\vec{F}\cdot\hat{n}\,ds_3 = \iint_{S_3}(2x^2-y^3)\,ds_3$$

Writing $x = 2\cos\theta$, $y = 2\sin\theta$ then $ds_3 = 2\,d\theta\,dz$,

and we get $\iint_{S_3}\vec{F}\cdot\hat{n}\,ds_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[2(2\cos\theta)^2-(2\sin\theta)^3\right]2\,dz\,d\theta$

$$= 48\int_0^{2\pi}(\cos^2\theta-\sin^3\theta)\,d\theta = 48\,\pi + 0.$$

Thus, the given surface integral $= 0 + 36\pi + 48\pi = 84\pi$, which agrees with the value of the volume integral.

Hence Proved.

EXAMPLE 21.19. (a) Use divergence theorem to evaluate $\iint \bar{F}.d\bar{s}$ where $\bar{F} = x^3 i + y^3 j + z^3 k$ and S in the surface of the sphere $x^2 + y^2 + z^2 = a^2$

[GGSIPU II Sem End Term 2007 Reappear]

(b) Use the divergence theorem to evaluate $\iint_S \bar{V} \cdot \hat{n}\, ds$ where $\bar{V} = (2x + z)\, i + yzj + z^2 k$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

[GGSIPU II Sem End Term 2011]

SOLUTION: (a) Here S in the closed surface hence by divergence theorem

$$\iint_S \bar{F}.d\bar{s} = \iiint_V \nabla.\bar{F}\, dv = \iiint_V \nabla.(x^3 i + y^3 j + z^3 k)\, dxdydz$$

$$= \iiint (3x^2 + 3y^2 + 3z^2) dx\, dy\, dz$$

Transforming it into spherical polar coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ and $z = r\cos\theta$, and using $dxdydz = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} drd\theta d\phi = r^2 \sin\theta\, drd\theta d\phi$

$$\text{given Integral} = 3\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{a} r^2.r^2 \sin\theta\, drd\phi d\theta$$

$$= 3\int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi \int_{r=0}^{a} r^4 dr = 3.2.2\pi.\ \frac{a^5}{5} = 12\pi a^5/5. \qquad \textbf{Ans.}$$

(b) Using Gauss divergence theorem $\iint_S \bar{V} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \bar{V})\, dv.$

$$\nabla \cdot \bar{V} = \frac{\partial}{\partial x}(2x + z) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = 2 + 3z.$$

$\therefore \iint_S \bar{V} \cdot \hat{n}\, ds = \iiint_V (2 + 3z)\, dxdydz$ Converting to spherial polar coordinates

$$= \iiint_V (2 + 3r\cos\theta)\, r^2 \sin\theta\, drd\theta d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} (2r^2 + 3r^3 \cos\theta) \sin\theta\, drd\theta d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \left(\frac{2r^3}{3} + \frac{3r^4}{4}\cos\theta\right)_{r=0}^{a} \sin\theta\, d\theta d\phi = \int_{\theta=0}^{\pi/2} \left(\frac{2a^3}{3} + \frac{3a^4}{4}\cos\theta\right) \sin\theta\, d\theta\, [\phi]_0^{2\pi}$$

$$= 2\pi\left[\frac{2a^3}{3}(-\cos\theta) + \frac{3a^4}{8}\sin^2\theta\right]_0^{\pi/2} = \frac{4}{3}\pi a^3 + \frac{3\pi a^4}{4}. \qquad \textbf{Ans.}$$

EXAMPLE 21.20. The vector field $\vec{F} = x^2 i + zj + yzk$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface S, evaluate the surface integral $\iint_S \vec{F} \cdot \vec{ds}$.

SOLUTION: The surface integral $\iint_S \vec{F} \cdot \vec{ds}$ has to be evaluated as the sum of six integrals corresponding to the six faces of the cuboid. Since S is a closed surface, the Gauss divergence theorem is applicable, hence $\iint_S \vec{F} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{F}\, dv$

Therefore, in place of evaluating the surface integral $\iint_S \vec{F} \cdot \vec{ds}$ as the sum of six integrals, we shall evaluate the volume integral $\iiint_V \nabla \cdot \vec{F}\, dv$.

Now $\quad \text{div } \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (x^2 i + zj + yzk) = 2x + 0 + y$

$$\therefore \iiint_V \nabla \cdot \vec{F}\, dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (2x+y)\, dz\, dy\, dx = \int_0^a\int_0^b (2x+y)[z]_0^c\, dy\, dx = c\int_0^a\int_0^b (2x+y)\, dy\, dx$$

$$= c\int_0^a \left[2xy + \frac{y^2}{2}\right]_0^b dx = c\int_0^a \left(2xb + \frac{b^2}{2}\right) dx$$

$$= cb\left[x^2 + \frac{bx}{2}\right]_0^a = bc\left(a^2 + \frac{ab}{2}\right) = abc\left(a + \frac{b}{2}\right)$$ **Ans.**

EXAMPLE 21.21. (a) Verify Gauss divergence theorem for the vector $\bar{F} = yi + xj + z^3 x$ taken over the cylindrical region $x^2 + y^2 = 9;\ z = 0,\ z = 6$. [GGSIPU II Sem II Term 2006]

(b) Given $\bar{U} = (x, -y, z)$ and $S: x^2 + y^2 = a^2, z = 0, z = b$, verify divergence theorem. [GGSIPU II Sem II Term 2005]

SOLUTION: (a) Here $\bar{F} = yi + xj + z^3 k$ and S is the total surface of the circular cylinder which consists of (i) the curved surface for which $\hat{n} = \dfrac{2xi + 2yj}{\sqrt{4x^2 + 4y^2}} = \dfrac{xi + yj}{3}$, (ii), circular plate at $z = 0$ for which $\hat{n} = -k$ and (iii) the circular plate at $z = 6$ for which $\hat{n} = k$.

$$\therefore \iint_s \bar{F}.\hat{n} ds = \iint_{(i)} (yi + xj + z^3 k).\frac{xi + yj}{3} ds + \iint_{(ii)} (yi + xj + 0^3 k).(-k) ds + \iint_{(iii)} (yi + xj + 6^3 k).k ds$$

converting to cylindrical polar coordinates $x = 3\cos\theta$, $y = 3\sin\theta$, $z = z$, $ds = rd\theta dz = 3d\theta dz$, we get

$$\frac{1}{3}\iint_{(i)} 2xy ds = \frac{2}{3}\int_0^6\int_0^{2\pi} 3\cos\theta \cdot 3\sin\theta \cdot 3\, d\theta dz$$

$$= \frac{2}{3}(27)\int_0^6 dz \int_0^{2\pi} \sin\theta\cos\theta\, d\theta = 108\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = 0$$

$$\iint_{(ii)} (yi + xj)\cdot(-k)\, ds = 0 \quad \text{and} \quad \iint_{(iii)} (yi + xj + 6^3 k)\cdot k ds = 6^3 \text{ (area of circle} = 6^3(9\pi).$$

Next, $$\int\int_V\int \nabla.\vec{F}\, dv = \iiint_V \left(\frac{\partial}{\partial x}y + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}z^3\right)dxdydz = \iiint_V 3z^2 dxdydz$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^6 3z^2\, dzdydx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [z^3]_0^6 dydx$$

$$= 6^3 \text{ (area of the circle)} = 6^3\,(9\pi).$$

Hence the divergence theorem is verified.

(b) $\vec{U} = xi - yj + zk$. Here S is a closed surface consisting of (i) surface of cylinder, (ii) circular plate $x^2 + y^2 = a^2$ at $z = 0$ and (iii) circular plate $x^2 + y^2 = a^2$ at $z = b$.

$$\int\int_S \vec{U}d\vec{s} = \int\int_{(i)} (xi - yj + zk).\frac{xi + yj}{a}ds + \int\int_{(ii)} (xi - yj - 0k).(-k)\, ds + \int\int_{(iii)} (xi - yj + bk).kds$$

converting to cylindrical polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$ so that $ds = r\, d\theta\, dz$, we get

$$\iint_{(i)} (xi - yj + zk)\cdot\left(\frac{xi + yj}{a}\right)ds = \iint_i \frac{1}{a}(x^2 - y^2)\, ds$$

$$= \frac{1}{a}\int_0^b\int_0^{2\pi} (a^2\cos^2\theta - a^2\sin^2\theta)\,(a\, d\theta dz) = a^2\int_0^b dz\int_0^{2\pi} (\cos 2\theta)\, d\theta$$

$$= a^2 b\,(0) = 0$$

Also, $$\iint_{(ii)} (xi - yj - 0k)\cdot(-k\, ds) = 0$$

and $$\iint_{(iii)} (xi - yj - bk)\cdot(kds) = b\iint ds = b\pi a^2.$$

$$\int\int_V\int \nabla.\vec{U} = \int\int\int\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(xi - yj + zk)dxdydz$$

$$= \int\int\int (1 - 1 + 1)dxdydz = \text{Volume of the cylinder} = \pi a^2 b$$

Hence divergence theorem verified.

EXAMPLE 21.22. Find $\iint_S \vec{F}.\hat{n}\,ds$ where $\vec{F} = (2x+3z)i - (xz+y)j + (y^2+2z)k$ and S is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3.

[GGSIPU II Sem End Term 2007]

SOLUTION: Here S is the surface of the sphere $(x-3)^2 + (y+1)^2 + (z-2)^2 = 9$

Unit vector $\hat{n}$ along the normal to the above surface is given by

$$\hat{n} = \frac{2(x-3)i + 2(y+1)j + 2(z-2)k}{\sqrt{4(x-3)^2 + 4(y+1)^2 + 4(z-2)^2}} = \frac{1}{3}[(x-3)i + (y+1)j + (z-2)k].$$

Here $\vec{F} = (2x+3z)\,i - (xz+y)\,j + (y^2+2z)\,k.$

Since S is a closed surface, we can use divergence theorem and get

$$\iint_S F.\hat{n}ds = \iiint_V \nabla.\vec{F}\,dv = \iiint_V \left[\frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}(y^2+2z)\right]dxdydz$$

$$= \iiint_V (2-1+2)dxdydz = 3 \text{ Volume of the sphere} = 3 \,.\, \frac{4}{3}\pi.3^3 = 108\pi$$ **Ans.**

EXAMPLE 21.23. Let $\vec{F}(x,y,z) = x^2i + y^2j + z^2k$ be the vector field defined over the surface of a cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$. Evaluate the surface integral $\iint_S \vec{F}\cdot\overline{ds}$.

SOLUTION: The integral $\iint_S \vec{F}\cdot\overline{ds}$ can be calculated directly considering the six faces separately, hence six integrals. However, the Gauss theorem is applicable here, hence using the theorem, we get

$$\iint_S \vec{F}\cdot\overline{ds} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \iiint_V \nabla\cdot(x^2i + y^2j + z^2k)dv = \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[2xz + 2yz + z^2\right]_0^1 dx\,dy = \int_0^1\int_0^1 (2x+2y+1)\,dx\,dy$$

$$= \int_0 \left[2xy + y^2 + y\right]_0^1 dx = \int_0^1 (2x+2)\,dx = \left[x^2 + 2x\right]_0^1 = 3.$$ **Ans.**

EXAMPLE 21.24. Let $\vec{F} = -yi + xj - xyzk$ be the vector field defined over the part of the cone $z = \sqrt{x^2+y^2}$ for $x^2 + y^2 = 9$. Verify Stoke's theorem.

SOLUTION: By Stoke's theorem $\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S (\nabla\times F)\cdot\hat{n}\,ds$

$$\text{LHS} = \int_C \vec{F}\cdot\overrightarrow{dr} = \int_C (-yi + xj - xyzk)\cdot(i\,dx + j\,dy + k\,dz) = \int_C -y\,dx + x\,dy - xyz\,dz$$

where $C : x^2 + y^2 = 9,\ z = 3$ or $x = 3\cos\theta,\ y = 3\sin\theta,\ z = 3$

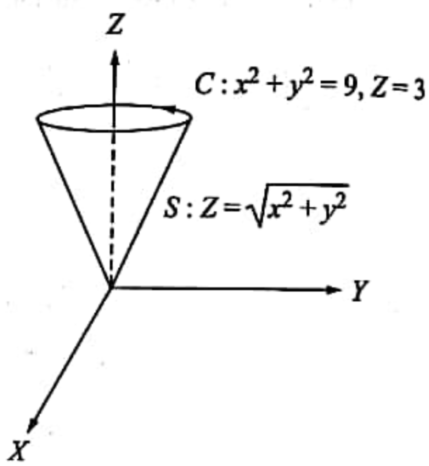
$$\therefore \quad \text{LHS} = \int_0^{2\pi} -3\sin\theta\,(-3\sin\theta)\,d\theta + 3\cos\theta\,(3\cos\theta)\,d\theta - 0$$

$$= \int_0^{2\pi} (9\sin^2\theta + 9\cos^2\theta)\,d\theta = 9\int_0^{2\pi} d\theta = 18\pi.$$

For the R.H.S. the surface S is $\phi(x, y, z) = z - \sqrt{x^2+y^2} = 0$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = -\frac{x}{\sqrt{x^2+y^2}}i - \frac{y}{\sqrt{x^2+y^2}}j + k = -\frac{x}{z}i - \frac{y}{z}j + k$$

$$|\nabla\phi| = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} = \sqrt{2}$$



$\therefore$ Unit vector along the outward drawn normal

$$\hat{n} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}(-xi - yj + zk).$$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -xyz \end{vmatrix} = -xzi + yzj + 2k$$

$$\therefore \quad (\nabla\times F)\cdot\hat{n} = \frac{1}{z\sqrt{2}}(-xi - yj + zk)\cdot(-xzi + yzj + 2k)$$

$$= \frac{1}{z\sqrt{2}}(x^2z - y^2z + 2z) = \frac{1}{\sqrt{2}}(x^2 - y^2 + 2)$$

$$\therefore \quad \text{R.H.S.} = \iint_S (\nabla\times F)\cdot\hat{n}\,ds = \iint_S \frac{1}{\sqrt{2}}(x^2-y^2+2)\cdot\sqrt{2}\,dx\,dy = \iint_S (x^2-y^2+2)\,dx\,dy$$

Converting to polar coordinates the above surface integrals becomes

$$= \int_0^{2\pi}\int_0^3 (r^2\cos^2\theta - r^2\sin^2\theta + 2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r^3\cos 2\theta\,dr\,d\theta + \int_0^{2\pi}\int_0^3 2\,r\,dr\,d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^3 + 2\pi\left[r^2\right]_0^3 = 0 + 2\pi(9) = 18\pi.$$

Hence Verified

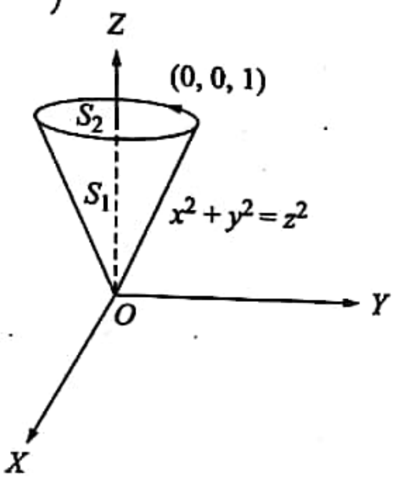
EXAMPLE 21.25. Verify Gauss divergence theorem over the piecewise closed surface consisting of the curved surface S_1 of the cone $z = \sqrt{x^2+y^2}$ for $x^2+y^2 \le 1$ and the flat cap S_2 consisting of the circular disc $x^2+y^2 \le 1$ in the plane $z = 1$. The vector field is $\vec{F} = xi + yj + 3k$.

SOLUTION: The unit vector along the outward drawn normal to the surface S_1: $(\sqrt{x^2+y^2} - z = 0)$ is

$$\hat{n}_1 = \left(\frac{xi}{\sqrt{x^2+y^2}} + \frac{yj}{\sqrt{x^2+y^2}} - 1k\right)\frac{1}{\sqrt{\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1\right)}} = \left(\frac{xi}{z} + \frac{yj}{z} - k\right)\frac{1}{\sqrt{2}}.$$

Then $\vec{F}\cdot\vec{n}_1 = \frac{1}{\sqrt{2}}(xi+yj+zk)\cdot\left(\frac{x}{z}i+\frac{y}{z}j-k\right)=\frac{1}{\sqrt{2}}\left(\frac{x^2}{z}+\frac{y^2}{z}-z\right)=0$

Hence $\iint_{S_1}\vec{F}\cdot\hat{n}_1\,ds = 0$



The unit vector along the outward drawn normal to the surface S_2 is $\hat{n}_2 = k$ hence $\vec{F}\cdot\hat{n}_2 = (xi + yj + zk)\cdot k = z$ and on S_2, $z = 1$ hence

$$\iint_{S_2}\vec{F}\cdot\hat{n}_2\,ds = \iint_{S_2} z\,ds = \iint_{S_2} 1\,ds = \text{area of } S_2 = \pi.$$

Therefore $\iint_{S_2}\vec{F}\cdot\hat{n}\,ds = \iint_{S_1}\vec{F}\cdot\hat{n}_1\,ds + \iint_{S_2}\vec{F}\cdot\hat{n}_2\,ds = 0+\pi=\pi.$

Next, for the volume integral $\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(xi+yj+zk)=3.$

$\therefore$ $\iiint_V \nabla\cdot F\,dv = \iiint_V 3\,dv$ (volume of the cone with height 1 and radius 1)

$= 3\frac{1}{3}\pi 1^2(1)=\pi$. Hence the theorem is verified.

EXAMPLE 21.26. Use Gauss divergence theorem to evaluate $\iint_S(\vec{F}\cdot\hat{n})\,ds$ where $F = x^2zi + yj - xz^2k$ and S is the bounding of the region bounded by the parabolid $z = x^2+y^2$ and the plane $z = 4y$.

SOLUTION: $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V(\nabla\cdot\vec{F})\,dv = \iiint_V(2xz+1-2xz)\,dv = \iiint_V dv$

$$= \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}\int_{z=x^2+y^2}^{4y} dz\,dx\,dy$$

(because the projection of S on the XY-plane is $x^2+y^2=4y$.)

$$= \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}(4y-x^2-y^2)\,dx\,dy$$

$$= 2\int_{y=0}^{4}\left[x(4y-y^2)-\frac{x^3}{3}\right]_0^{\sqrt{4y-y^2}} dy$$

$$= 2\int_0^4\left[(4y-y^2)^{3/2}-\frac{1}{3}(4y-y^2)^{3/2}\right]dy = \frac{4}{3}\int_0^4(4y-y^2)^{3/2}\,dy$$

$$= \frac{4}{3}\int_0^4[4-(y-2)^2]^{3/2}\,dy. \quad (\text{Set } y-2 = 2\sin t)$$

$$= \frac{4}{3}(32)\int_0^{\pi/2}\sin^4 y\,dy = \frac{4}{3}(32)\left(\frac{3}{4}\cdot\frac{1}{2}\frac{\pi}{2}\right) = 8\pi.$$ **Ans.**