Q-3

× a) $P(x) = \theta \sigma^{\theta} x^{-\theta-1}$.

$$L(x) = \theta\sigma^{\theta} x_1^{-\theta-1} \cdot \theta\sigma^{\theta} x_2^{-\theta-1} \cdots \theta\sigma^{\theta} x_n^{-\theta-1}$$
$$= \prod_{i=1}^{n} \theta\sigma^{\theta} x_i^{-\theta-1}$$

$$l(x) = \log(L(x))$$
$$= \log\left(\prod_{i=1}^{n} \theta\sigma^{\theta} x_i^{-\theta-1}\right)$$

$$= \sum_{i=1}^{n} \log\theta + \sum_{i=1}^{n} \theta\log\sigma - (\theta+1)\sum_{i=1}^{n}\log x_i$$

$$l(x) = N\log\theta + N\theta\log\sigma - (\theta+1)\sum_{i=1}^{n}\log x_i$$

$$\frac{\partial l}{\partial \sigma} = \frac{N\theta}{\sigma} \quad -① $$

$$\frac{\partial l}{\partial \theta} = \frac{N}{\theta} + N\log\sigma - \sum_{i=1}^{n}\log(x_i) \quad -②$$

Here $\partial l/\partial\sigma = 0 \Rightarrow \sigma$ is infinity,

that means sigma is a montonously increasing function

so we take $\sigma = x$, as in question given $x \geq \sigma$,

so maximum value of $\sigma = x_i$

Rearranging equation ② and equation to zero.

$$\frac{N}{\theta} = \sum_{i=1}^{n}\log(x_i) - N\log\sigma$$

$$\theta = \frac{N}{\sum_{i=1}^{n}\log(x_i) - N\log\sigma}$$

For maximum value of σ to occur we need, $\sum \ln(x_i) - N\log\sigma$ to be minimum, $\log\sigma$ should be maximum

from previous expression maximum σ when

$\boxed{\sigma = x_i}$

③ b.

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

$$L(x) = \prod_{i=1}^{N} \frac{1}{x_i\sigma\sqrt{2\pi}} \exp\left(\frac{-(\ln x_i - \mu)^2}{2\sigma^2}\right)$$

$$\log(L(x)) = l(x)$$

$$= \log\left(\prod_{i=1}^{N} \frac{1}{x_i\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}\right)\right)$$

$$l = \sum_{i=1}^{N} \log\left(\sigma x_i\sqrt{2\pi}\right) - \left(\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}\right)$$

$$\frac{\partial l}{\partial \mu} = 0$$

$$\Rightarrow$$

$$= 0 + \sum_{i=1}^{N} \frac{2}{2\sigma^2}\left(\ln(x_i) - \mu\right) \times \cancel{\ } = 0$$

$$\frac{1}{\sigma^2}\left[\sum_{i=1}^{N} \ln(x_i) - \sum_{i=1}^{N} \mu\right] = 0$$

$$\sum_{i=1}^{N} \ln(x_i) = N\mu$$

$$\mu = \frac{\sum_{i=1}^{N} \ln(x_i)}{N}$$

$$\frac{\partial l}{\partial \sigma^2} = 0 \quad \frac{\partial l}{\partial \sigma^2} \Rightarrow \left[\sum_{i=1}^{N} \frac{1}{2} \log(2\pi\sigma^2 x_i^2) - \frac{(\ln(x_i) - \mu)^2}{2\sigma^2} \right] = 0$$

$$\frac{1}{2} \times \frac{\cancel{2\pi x_i^2}}{\cancel{2\pi}\sigma^2 x_i^2} - \frac{-(\ln(x_i) - \mu)^2}{2\sigma^4}$$

$$\sum_{i=1}^{n} \frac{1}{\cancel{2}\sigma^2} = \sum_{i=1}^{n} \frac{(\ln(x_i) - \mu)^2}{\cancel{2}\sigma^4}$$

$$n \cdot \sigma^2 = \sum_{i=1}^{n} (\ln(x_i) - \mu)^2$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} (\ln(x_i) - \mu)^2}{n}$$

from the question

$$x_i = \{13, 2, 16, 5, 11, 16, 18, 5, 8, 15\}$$

$$\mu = \lambda = \sum_{i=1}^{N} \ln(x_i) / N$$

$$= \frac{\ln(13) + \ln(2) + \ln(16) + \ln(5) + \ln(11) + \ln(16) + \ln(18) + \ln(5) + \ln(8) + \ln(15)}{10}$$

$$= 22.097/10 = \underline{2.209}$$

$$\sigma^2 = \frac{(\ln(13) - 2.209)^2 + (\ln(2) - 2.209)^2 + (\ln(16) - 2.209)^2 + (\ln(5) - 2.209)^2 + (\ln(11) - 2.209)^2 + (\ln(16) - 2.209)^2 + (\ln(18) - 2.209)^2 + (\ln(5) - 2.209)^2 + (\ln(8) - 2.209)^2 + (\ln(15) - 2.2)^2}{10}$$

$\sigma^2 = 0.4544$

$\sigma = \sqrt{\sigma^2}$

$= 0.6741$

3) i.

$f(x:\theta) = \frac{x}{\theta^2} e^{-x/\theta}$ → probability density function

Assume ~~iden~~ i,i,d conditions

$L(x:\theta) = \frac{x_1 e^{-x_1/\theta}}{\theta^2} \cdot \frac{x_2 e^{-x_2/\theta}}{\theta^2} \cdots \frac{x_n e^{-x_n/\theta}}{\theta^2}$

$= \prod_{i=1}^{N} \frac{x_i e^{-x_i/\theta}}{\theta^2}$

$l(x:\theta) = \log(L(x))$

$= \sum_{i=1}^{N} \log x_i - \sum_{i=1}^{N} \frac{x_i}{\theta} - \sum_{i=1}^{N} 2\log\theta$

$= \sum_{i=1}^{N} \log x_i - \sum_{i=1}^{N} \frac{x_i}{\theta} - 2N\ln\theta$

$\partial l/\partial\theta = \frac{1}{\theta^2} \sum_{i=1}^{N} x_i - \frac{2N}{\theta} = 0$

$\frac{1}{\theta^2} \sum_{i=1}^{N} x_i = \frac{2N}{\theta}$

$$\boxed{\theta = \frac{\sum_{i=1}^{N} x_i}{2N}}$$